ALGEBRAIC COBORDISM REVISITED

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Abstract. We define a cobordism theory in algebraic geometry based on normal crossing degenerations with double point singularities. The main result is the equivalence of double point cobordism to the theory of algebraic cobordism previously defined by Levine and Morel. Double point cobordism provides a simple, geometric presentation of algebraic cobordism theory. As a corollary, the Lazard ring given by products of projective spaces rationally generates all nonsingular projective varieties modulo double point degenerations.

Double point degenerations arise naturally in relative Donaldson-Thomas theory. We use double point cobordism to prove all the degree 0 conjectures in Donaldson-Thomas theory: absolute, relative, and equivariant.

Introduction

0.1. **Overview.** A first idea for defining cobordism in algebraic geometry is to naively follow the well-known presentation of complex cobordism. Let X be a variety over a field k. We form the free abelian group on projective morphisms

$$f : Y \to X$$

from smooth k-varieties Y to X (modulo isomorphism over X). We impose the relations

$$[\pi^{-1}(0) \to X] = [\pi^{-1}(\infty) \to X] \tag{0.1}$$

obtained from projective morphisms

$$g : Y \to X \times \mathbb{P}^1,$$

where $\pi = p_2 \circ g$ and the varieties Y, $\pi^{-1}(0)$, and $\pi^{-1}(\infty)$ are all smooth. Unfortunately, the resulting theory bears no structural resemblance to complex cobordism. Further discussion of the difficulities here can be found in [22, Remark 1.2.9].

A successful theory of algebraic cobordism has been constructed in [18, 21, 22] from Quillen's axiomatic perspective, see also [16, 17, 19,

Date: 12 September 2008.

20]. The result Ω_* is the universal oriented Borel-Moore homology theory of schemes, yielding the universal oriented Borel-Moore cohomology theory Ω^* for the subcategory of smooth schemes (see [22, Definitions 1.1.2, 5.1.3] for the precise definitions). The construction yields, in principle, a presentation for algebraic cobordism, but the relations are considerably more complicated than desired.

A second idea for defining algebraic cobordism geometrically is to impose relations obtained by fibers of π with normal crossing singularities. The simplest of these are the *double point degenerations* — where the fiber is a union of two smooth transverse divisors. We prove the cobordism theory obtained from double point degenerations *is* algebraic cobordism.

Algebraic cobordism may thus be viewed both functorially and geometrically. In practice, the different perspectives are very useful. We prove several conjectural formulas concerning the virtual class of the Hilbert scheme of points of a 3-fold as an application.

0.2. **Double point degenerations.** Let k be a field of characteristic 0. Let $\mathbf{Sch}_k$ be the category of separated schemes of finite type over k, and let $\mathbf{Sm}_k$ be the full subcategory of smooth quasi-projective k-schemes. Since we will use resolution of singularities, weak factorization, and Bertini's Theorem, the characteristic 0 hypothesis will be required for the entire paper.

For $X \in \mathbf{Sch}_k$, let $\mathcal{M}(X)$ denote the set of isomorphism classes over X of projective morphisms

$$f : Y \to X \tag{0.2}$$

with $Y \in \mathbf{Sm}_k$. The set $\mathcal{M}(X)$ is a monoid under disjoint union of domains and is graded by the dimension of Y over k. Let $\mathcal{M}_*(X)^+$ denote the graded group completion of $\mathcal{M}(X)$.

Alternatively, $\mathcal{M}_n(X)^+$ is the free abelian group generated by morphisms (0.2) where Y is irreducible and of dimension n over k. Let

$$[f : Y \to X] \in \mathcal{M}_*(X)^+$$

denote the element determined by the morphism.

Let $Y \in \mathbf{Sm}_k$ be of pure dimension. A morphism

$$\pi : Y \to \mathbb{P}^1$$

is a *double point degeneration* over $0 \in \mathbb{P}^1$ if $\pi^{-1}(0)$ can be written as

$$\pi^{-1}(0) = A \cup B$$

where A and B are smooth codimension one closed subschemes of Y, intersecting transversely. The intersection

$$D = A \cap B$$

is the *double point locus* of π over $0 \in \mathbb{P}^1$. We also allow A, B or D to be empty.

Let $N_{A/D}$ and $N_{B/D}$ denote the normal bundles of D in A and B respectively. Since $O_D(A+B)$ is trivial,

$$N_{A/D} \otimes N_{B/D} \cong O_D.$$

Hence, as $O_D \oplus N_{A/D} \cong N_{A/D} \otimes (O_D \oplus N_{B/D})$, the projective bundles

$$\mathbb{P}(O_D \oplus N_{A/D}) \to D \ \text{ and } \ \mathbb{P}(O_D \oplus N_{B/D}) \to D \tag{0.3}$$

are isomorphic. Let

$$\mathbb{P}(\pi) \to D$$

denote either of (0.3).

0.3. **Double point relations.** Let $X \in \mathbf{Sch}_k$, and let p_1 and p_2 denote the projections to the first and second factors of $X \times \mathbb{P}^1$ respectively.

Let $Y \in \mathbf{Sm}_k$ be of pure dimension. Let

$$g : Y \to X \times \mathbb{P}^1$$

be a projective morphism for which the composition

$$\pi = p_2 \circ g : Y \to \mathbb{P}^1 \tag{0.4}$$

is a double point degeneration over $0 \in \mathbb{P}^1$. Let

$$[A \to X],\ [B \to X],\ [\mathbb{P}(\pi) \to X] \in \mathcal{M}(X)^+$$

be obtained from the fiber $\pi^{-1}(0)$ and the morphism $p_1 \circ g$.

Definition 0.1. Let $\zeta \in \mathbb{P}^1(k)$ be a regular value of π. We call the map g a *double point cobordism* with degenerate fiber over 0 and smooth fiber over ζ. The associated *double point relation* over X is

$$[Y_\zeta \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X] \tag{0.5}$$

where $Y_\zeta = \pi^{-1}(\zeta)$.

The relation (0.5) depends not only on the morphism g and the point ζ, but also on the choice of decomposition of the fiber

$$\pi^{-1}(0) = A \cup B.$$

We view (0.5) as an analog of the classical relation of rational equivalence of algebraic cycles, see [22, Theorem 1.2.19] for a more precise statement.

Let $\mathcal{R}_*(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by *all* double point relations over X. Since (0.5) is a homogeneous element of $\mathcal{M}_*(X)^+$, $\mathcal{R}_*(X)$ is a graded subgroup of $\mathcal{M}_*(X)^+$.

Definition 0.2. For X in $\mathbf{Sch}_k$, *double-pont cobordism* $\omega_*(X)$ is defined by

$$\omega_*(X) = \mathcal{M}_*(X)^+/\mathcal{R}_*(X).$$

Naive cobordism (0.1) may be viewed as a special case of a double point relation. Indeed, let $Y \in \mathbf{Sm}_k$ be of pure dimension. Let

$$g : Y \to X \times \mathbb{P}^1$$

be a projective morphism with $\pi = p_2 \circ g$ smooth over $0, \infty \in \mathbb{P}^1$. We may view π as a double point degeneration over $0 \in \mathbb{P}^1$ with

$$\pi^{-1}(0) = A \cup \emptyset.$$

The associated double point relation is

$$[Y_\infty \to X] - [Y_0 \to X] \in \mathcal{R}(X).$$

0.4. **Algebraic cobordism.** Let $\Omega_*(X)$ be the theory of algebraic cobordism defined in [22]. The central object of the paper is relation of Ω_* with double-point cobordism ω_*.

Theorem 1. *For $X \in \mathbf{Sch}_k$, there is a canonical isomorphism*

$$\omega_*(X) \cong \Omega_*(X).$$

We actually prove a stronger result, giving an isomorphism $\omega_* \cong \Omega_*$ of oriented Borel-Moore functors of geometric type (see below).

Theorem 1 may be viewed as a geometric presentation of $\Omega_*(X)$ via the simplest possible cobordisms. A homomorphism

$$\omega_*(X) \to \Omega_*(X) \tag{0.6}$$

is obtained immediately from the definitions once the double point relations are shown to hold in $\Omega_*(X)$. The inverse is more difficult to construct.

The idea for constructing the inverse uses oriented Borel-Moore functors of geometric type on $\mathbf{Sch}_k$. Every oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ canonically defines an oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$ [22, Remark 4.1.10, Rroposition 5.2.6], but the converse is false. The theory Ω_* is the universal Borel-Moore homology theory of $\mathbf{Sch}_k$. By the construction in [22] via generators and relations, Ω_* is also the universal oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$. The universal mapping property of the latter will provide an inverse to (0.6) once we establish the following result.

Theorem 2. *ω_* determines an oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$.*

The proof of Theorem 2 is the technical heart of the paper. Besides several elementary structures and properties obviously possessed by ω_*, an oriented Borel-Moore functor of geometric type A has:

(i) A first Chern class operator $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$ for each line bundle $L \to X$.

(ii) A formal group law $F_A(u, v) \in A_*(k)[[u, v]]$, with

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M)) = \tilde{c}_1(L \otimes M) \ .$$

The key geometric step in the argument is the construction of a formal group law for ω_* in Section 8.

0.5. **Algebraic cobordism over a point.** We write $\Omega_*(k)$ and $\omega_*(k)$ for $\Omega_*(\mathrm{Spec}\,(k))$ and $\omega_*(\mathrm{Spec}\,(k))$ respectively. Let $\mathbb{L}_*$ be the Lazard ring [15]. The canonical map

$$\mathbb{L}_* \to \Omega_*(k)$$

classifying the formal group law for Ω_* is proven to be an isomorphism in [22, Theorem 4.3.7]. By Quillen's theorem,

$$\mathbb{L}_n \cong MU^{-2n}(\mathrm{pt}),$$

and the known generators of $MU^*(\mathrm{pt})_{\mathbb{Q}}$ [33, Theorem, pg. 110, Chap. VII], we see $\Omega_*(k) \otimes \mathbb{Q}$ is generated as a $\mathbb{Q}$-algebra by the classes of projective spaces. The following result is then a consequence of Theorem 1.

Corollary 3. *We have*

$$\omega_*(k) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\lambda} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times ... \times \mathbb{P}^{\lambda_{\ell(\lambda)}}]$$

where the sum is over all partitions λ.

0.6. **Donaldson-Thomas theory.** Corollary 3 is directly applicable to the Donaldson-Thomas theory of 3-folds.

Let X be a smooth projective 3-fold over $\mathbb{C}$, and let $\mathrm{Hilb}(X, n)$ be the Hilbert scheme of n points. Viewing the Hilbert scheme as the moduli space of ideal sheaves $I_0(X, n)$, a natural 0-dimensional virtual Chow class can be constructed

$$[\mathrm{Hilb}(X, n)]^{vir} \in A_0(\mathrm{Hilb}(X, n), \mathbb{Z}),$$

see [25, 26, 34]. The degree 0 Donaldson-Thomas invariants are defined by

$$N_{n,0}^X = \int_{[\mathrm{Hilb}(X,n)]^{vir}} 1.$$

Let

$$\mathsf{Z}(X,q) = 1 + \sum_{n\geq 1} N_{n,0}^X \; q^n$$

be the associated partition function.

Conjecture 1. [25] $\mathsf{Z}(X,q) = M(-q)^{\int_X c_3(T_X \otimes K_X)}$.

Here, $M(q)$ denotes the MacMahon function,

$$M(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n},$$

the generating function of 3-dimensional partitions [32].

For a nonsingular divisor $S \subset X$, a relative Donaldson-Thomas theory[1] is defined via the moduli space of relative ideal sheaves $I_0(X/S,n)$. The degree 0 relative invariants,

$$N_{n,0}^{X/S} = \int_{[I_0(X/S,n)]^{vir}} 1,$$

determine a relative partition function

$$\mathsf{Z}(X/S,q) = 1 + \sum_{n\geq 1} N_{n,0}^{X/S} \; q^n.$$

Let $\Omega_X[S]$ denote the locally free sheaf of differential forms of X with logarithmic poles along S. Let

$$T_X[-S] = \Omega_X[S]^{\vee},$$

denote the dual sheaf of tangent fields with logarithmic zeros. Let

$$K_X[S] = \Lambda^3 \Omega_X[S]$$

denote the logarithmic canonical class.

Conjecture 2. [26] $\mathsf{Z}(X/S,q) = M(-q)^{\int_X c_3(T_X[-S] \otimes K_X[S])}$.

We prove Conjectures 1 and 2, as well as an equivariant version of Conjecture 1 proposed in [5], using Corollary 3 to reduce to toric cases previously calculated in [25, 26]. The use of Corollary 3 relies on that fact that the double point relations naturally arise in degree 0 Donaldson-Thomas theory.

[1] See [25, 29] for a detailed discussion.

0.7. **Double point relations in DT theory.** Let $Y \in \mathbf{Sm}_{\mathbb{C}}$ be a 4-dimensional projective variety, and let

$$\pi : Y \to \mathbb{P}^1$$

be a double point degeneration over $0 \in \mathbb{P}^1$. Let

$$\pi^{-1}(0) = A \cup B, \ D = A \cap B.$$

The degeneration formula in relative Donaldson-Thomas theory yields

$$\mathsf{Z}(Y_\zeta) = \mathsf{Z}(A/D) \cdot \mathsf{Z}(B/D) \tag{0.7}$$

for a π-regular value $\zeta \in \mathbb{P}^1$, see [26].

Since the deformation to the normal cone of $D \subset A$ is a double point degeneration,

$$\mathsf{Z}(A) = \mathsf{Z}(A/D) \cdot \mathsf{Z}(\mathbb{P}(O_D \oplus N_{A/D})/D). \tag{0.8}$$

On the right, the divisor $D \subset \mathbb{P}(O_D \oplus N_{A/D})$ is included with normal bundle $N_{A/D}$. Similarly,

$$\mathsf{Z}(B) = \mathsf{Z}(B/D) \cdot \mathsf{Z}(\mathbb{P}(O_D \oplus N_{B/D})/D) \tag{0.9}$$

where the divisor $D \subset \mathbb{P}(O_D \oplus N_{A/D})$ is included with normal bundle $N_{B/D}$.

Since $N_{A/B} \otimes N_{B/D} \cong O_D$, the deformation of $\mathbb{P}(O_D \oplus N_{A/D})$ to the normal cone of $D \subset \mathbb{P}(O_D \oplus N_{A/D})$ yields

$$\mathsf{Z}(\mathbb{P}(\pi)) = \mathsf{Z}(\mathbb{P}(O_D \oplus N_{A/D})/D) \cdot \mathsf{Z}(\mathbb{P}(O_D \oplus N_{B/D})/D).$$

When combined with equations (0.7)-(0.9), we find

$$\mathsf{Z}(Y_\zeta) \cdot \mathsf{Z}(A)^{-1} \cdot \mathsf{Z}(B)^{-1} \cdot \mathsf{Z}(\mathbb{P}(\pi)) = 1 \tag{0.10}$$

which is the double point relation (0.5) over Spec($\mathbb{C}$) in multiplicative form.

0.8. **Gromov-Witten speculations.** Let X be a nonsingular projective variety over $\mathbb{C}$. Gromov-Witten theory concerns integration against the virtual class,

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in H_*(\overline{M}_{g,n}(X,\beta),\mathbb{Q}),$$

of the moduli space of stable maps to X.

There are two main techniques available in Gromov-Witten theory: localization [12, 14] and degeneration [7, 13, 23, 24, 27]. Localization is most effective for toric targets — all the Gromov-Witten data of products of projective spaces are accessible by localization. The degeneration formula yields Gromov-Witten relations precisely for double point degenerations.

By Corollary 3, *all* varieties are linked to products of projective spaces by double point degenerations. We can expect, therefore, that many aspects of the Gromov-Witten theory of arbitrary varieties will follow the behavior found in toric targets. An example is the following speculation about the virtual class — which, at present, appears out of reach of Corollary 3.

Speculation. *The push forward $\epsilon_*[\overline{M}_{g,n}(X,\beta)]^{vir}$ via the canonical map*

$$\epsilon : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$$

lies in the tautological ring

$$RH^*(\overline{M}_{g,n},\mathbb{Q}) \subset H^*(\overline{M}_{g,n},\mathbb{Q}).$$

See [9, 30] for a discussion of similar (and stronger) statements. In particular, a definition of the tautological ring can be found there.

Gromov-Witten theory is most naturally viewed as an aspect of symplectic geometry. The construction of a parallel symplectic cobordism theory based on double point degenerations appears to be a natural path to follow.

0.9. **An outline of the paper.** The construction of algebraic cobordism Ω_* is recalled in Section 1. Oriented Borel-Moore functors of geometric type and the universality of Ω_* are discussed in Section 2 following [22, Sections 2.1, 2.2 and 5.1]. The elementary properties of ω_* and the construction of the map

$$\omega_* \to \Omega_*$$

are treated in Section 3. We begin the construction of Chern classes for ω_* in Section 4 by defining the Chern class operators for globally generated line bundles. Our next goal is the construction of the formal group law for ω_*. The required technical preparation is presented in Sections 5-7. The formal group law is constructed in Section 8. We use the formal group law to extend the Chern class operators to arbitrary bundles in Section 9. The proof of Theorem 2 is completed in Section 10. Theorem 1 and Corollary 3 are proven in Section 11. We apply our results in Section 12 to give a new proof of a result of Fulton on Euler characteristics, and conclude by presenting the proofs of the conjectures in Donaldson-Thomas theory in Section 13.

0.10. **Acknowledgments.** We thank D. Maulik, A. Okounkov, and B. Totaro for useful discussions about double point degenerations, Gromov-Witten theory, and algebraic cobordism. We thank W. Fulton for pointing out the connection to his characterization of the sheaf Euler characteristic. Finally, we would like to thank the referee for several useful suggestions.

A proof of Conjecture 1 was announced in March 2005 by J. Li. Li's method is to show $\mathsf{Z}(X,q)$ depends only upon the Chern numbers of X by an explicit (topological) study of the cones defining the virtual class. The result is then obtained from the toric calculations of [26] via the complex cobordism class. A proof of Conjecture 1 in case X is a Calabi-Yau 3-fold via a study of self-dual obstruction theories appears in [2, 3]. Our proof is direct and algebraic, but depends upon the construction of relative Donaldson-Thomas theory (which is required in any case for the calculations of [26]).

M. L. was supported by the Humboldt Foundation through the Wolfgang Paul Program and the NSF via grants DMS-0140445 and DMS-0457195. R. P. was supported by the Packard foundation and the NSF via grant DMS-0500187. The research was partially pursued during a visit of R. P. to the Instituto Superior Técnico in Lisbon in the fall of 2005.

1. Algebraic cobordism theory Ω_*

1.1. **Construction.** Algebraic cobordism theory is constructed in [22], and the fundamental properties of Ω_* are verified there. We recall the construction of Ω_* here.

1.2. $\underline{\Omega}_*$**.** For $X \in \mathbf{Sch}_k$, $\underline{\Omega}_n(X)$ is generated (as an abelian group) by *cobordism cycles*

$$(f : Y \to X, L_1, \ldots, L_r),$$

where f is a projective morphism, $Y \in \mathbf{Sm}_k$ is irreducible of dimension $n + r$ over k, and the L_i are line bundles on Y. We identify two cobordism cycles if they are isomorphic over X up to reorderings of the line bundles L_i.

We will impose several relations on cobordism cycles. To start, two basic relations are imposed (see [22, 2.4.2, 2.4.3]):

I. If there exists a smooth quasi-projective morphism $\pi : Y \to Z$ and line bundles $M_1, \ldots, M_{s>\dim_k Z}$ on Z with $L_i \cong \pi^* M_i$ for $i = 1, \ldots, s \leq r$, then

$$(f : Y \to X, L_1, \ldots, L_r) = 0.$$

II. If $s : Y \to L$ is a section of a line bundle with smooth associated divisor $i : D \to Y$, then

$$(f : Y \to X, L_1, \dots, L_r, L) = (f \circ i : D \to X, i^*L_1, \dots, i^*L_r).$$

The graded group generated by cobordism cycles modulo relations I and II is denoted $\underline{\Omega}_*(X)$.

Relation II yields as a special case the naive cobordism relation. Let

$$\pi : Y \to X \times \mathbb{P}^1$$

be a projective morphism with $Y \in \mathbf{Sm}_k$ for which $p_2 \circ \pi$ is transverse to the inclusion $\{0, \infty\} \to \mathbb{P}^1$. Let $L_1, \dots, L_r$ be line bundles on Y, and let

$$i_0 : Y_0 \to Y, \quad i_\infty : Y_\infty \to Y$$

be the inclusions of the fibers over $0, \infty$. Then

$$(p_1 \circ \pi : Y_0 \to X, i_0^*L_1, \dots, i_0^*L_r) = (p_1 \circ \pi : Y_\infty \to X, i_\infty^*L_1, \dots, i_\infty^*L_r)$$

in $\underline{\Omega}_*(X)$.

Several structures are easily constructed on $\underline{\Omega}_*$. For a projective morphism $g : X \to X'$, define

$$g_* : \underline{\Omega}_*(X) \to \underline{\Omega}_*(X')$$

by the rule

$$g_*(f : Y \to X, L_1, \dots, L_r) = (g \circ f : Y \to X', L_1, \dots, L_r).$$

For a smooth quasi-projective morphism $g : X \to X'$ of relative dimension d, define

$$g^* : \underline{\Omega}_*(X') \to \underline{\Omega}_{*+d}(X)$$

by the rule

$$g^*(f : Y \to X', L_1, \dots, L_r) = (p_2 : Y \times_{X'} X \to X, p_1^*L_1, \dots, p_1^*L_r).$$

External products $\underline{\Omega}_*(X) \otimes \underline{\Omega}_*(X) \to \underline{\Omega}_*(X \times_k X')$ are defined by

$$\begin{aligned}&(f : Y \to X, L_1, \dots, L_r) \otimes (f' : Y' \to X', M_1, \dots, M_s)\\ &\mapsto (f \times f' : Y \times_k Y' \to X \times_k X', p_1^*L_1, \dots, p_1^*L_r, p_2^*M_1, \dots, p_2^*M_s).\end{aligned}$$

The *Chern class operator* $\tilde{c}_1(L) : \underline{\Omega}_n(X) \to \underline{\Omega}_{n-1}(X)$ is defined by the following formula:

$$\tilde{c}_1(L)((f : Y \to X, L_1, \dots, L_r)) = (f : Y \to X, L_1, \dots, L_r, f^*L).$$

1.3. Ω_*. Contrary to the purely topological theory of complex cobordism, relations I and II do not suffice to define Ω_*. One needs to impose the formal group law.

A (commutative, rank one) *formal group law* over a commutative ring R is a power series $F(u,v) \in R[[u,v]]$ satisfying the formal relations of identity, commutativity and associativity:

(i) $F(u,0) = F(0,u) = u$,
(ii) $F(u,v) = F(v,u)$,
(iii) $F(F(u,v),w) = F(u,F(v,w))$.

The *Lazard ring* $\mathbb{L}$ is defined by the following construction [15]. Start with the polynomial ring

$$\mathbb{Z}[\{A_{ij}, i,j \geq 1\}],$$

and form the power series

$$\tilde{F}(u,v) = u + v + \sum_{i,j\geq 1} A_{ij}u^iv^j.$$

Relation (i) is already satisfied. Relations (ii) and (iii) give polynomial relations on the A_{ij}. $\mathbb{L}$ is the quotient of $\mathbb{Z}[\{A_{ij}\}]$ by these relations. Letting a_{ij} be the image of A_{ij} in $\mathbb{L}$, the universal formal group law is

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^iv^j \in \mathbb{L}[[u,v]].$$

We grade $\mathbb{L}$ by giving a_{ij} degree $i+j-1$. If we give u and v degrees -1, then has $F_{\mathbb{L}}(u,v)$ total degree -1.

To construct Ω_*, we take the functor $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*$ and impose the relations

$$\begin{aligned} F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))&(f: Y \to X, L_1, \ldots, L_r) \\ &= \tilde{c}_1(L\otimes M)(f: Y \to X, L_1, \ldots, L_r) \end{aligned}$$

for each pair of line bundles L, M on X. Since as $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute, the expression $F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))$ makes sense as a formal infinite sum of operators. The defining relation I for $\underline{\Omega}_*$ shows the *a priori* infinite sum $F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(f: Y \to X, L_1, \ldots, L_r)$, is actually a finite sum. Thus, the above relation is well-defined.

The structures of projective push-forward, smooth pull-back, external products and first Chern class for $\underline{\Omega}_*$ extend $\mathbb{L}_*$-linearly to these structures on Ω_*.

2. Oriented Borel-Moore functors of geometric type

2.1. **Universality.** Algebraic cobordism Ω_* is the universal theory in the setting of oriented Borel-Moore functors of geometric type. We recall the definitions from [22, §2.1 and §2.2] here for the reader's convenience.

2.2. **Notation.** Let $X \in \mathbf{Sch}_k$. A *divisor* D on X will be understood to be Cartier unless otherwise stated. The line bundle associated to the locally free sheaf $\mathcal{O}_X(D)$ is denoted $O_X(D)$.

Let $\mathcal{E}$ be a rank n locally free sheaf $\mathcal{E}$ on X. Let

$$q : \mathbb{P}(\mathcal{E}) \to X$$

denote the projective bundle $\mathrm{Proj}_X(\mathrm{Sym}^*(\mathcal{E}))$ of rank one *quotients*[2] of $\mathcal{E}$ with tautological quotient invertible sheaf

$$q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}.$$

We let $O(1)_{\mathcal{E}}$ denote the line bundle on $\mathbb{P}(\mathcal{E})$ with sheaf of sections $\mathcal{O}(1)_{\mathcal{E}}$. The subscript $_{\mathcal{E}}$ is omitted if the context makes the meaning clear. The notation $\mathbb{P}_X(\mathcal{E})$ is used to emphasize the base scheme X.

2.3. **Oriented Borel-Moore functors with product.** Let $\mathbf{Sch}'_k$ be the subcategory of $\mathbf{Sch}_k$, with the same objects, but with morphisms given by the projective morphisms of $\mathbf{Sch}_k$. Let $\mathbf{Ab}_*$ denote the category of graded abelian groups.

Definition 2.1. An *oriented Borel-Moore functor with product* on $\mathbf{Sch}_k$ consists of the following data:

(D1) An additive functor $H_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*$.

(D2) For each smooth quasi-projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$ of pure relative dimension d, a homomorphism of graded abelian groups
$$f^* : H_*(X) \to H_{*+d}(Y).$$

(D3) For each line bundle L on X, a homomorphism of graded abelian groups
$$\tilde{c}_1(L) : H_*(X) \to H_{*-1}(X).$$

(D4) For each pair (X, Y) in $\mathbf{Sch}_k$, a bilinear graded pairing
$$\begin{aligned} \times : H_*(X) \times H_*(Y) &\to H_*(X \times Y) \\ (\alpha, \beta) &\mapsto \alpha \times \beta \end{aligned}$$

[2] $\mathbb{P}(\mathcal{E})$ will denote projectivization by one dimension quotients for the entire paper *except* for Section 13 where projectivization by subspaces will be used.

which is commutative, associative, and admits a distinguished element $1 \in H_0(\mathrm{Spec}(k))$ as a unit.

The pairing in (D4) is the *external product*. The data (D1)-(D4) are required to satisfy eight conditions:

(A1) Let $f : Y \to X$ and $g : Z \to Y$ be smooth quasi-projective morphisms in $\mathbf{Sch}_k$ of pure relative dimension. Then,

$$(f \circ g)^* = g^* \circ f^* .$$

Moreover, $\mathrm{Id}_X^* = \mathrm{Id}_{H_*(X)}$.

(A2) Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{Sch}_k$ where f is projective and g is smooth and quasi-projective of pure relative dimension. In the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z , \end{tikzcd}$$

f' is projective and g is smooth and quasi-projective of pure relative dimension. Then,

$$g^* f_* = f'_* g'^* .$$

(A3) Let $f : Y \to X$ be projective. Then,

$$f_* \circ \tilde{c}_1(f^* L) = \tilde{c}_1(L) \circ f_*$$

for all line bundles L on X.

(A4) Let $f : Y \to X$ be smooth and quasi-projective of pure relative dimension. Then,

$$\tilde{c}_1(f^* L) \circ f^* = f^* \circ \tilde{c}_1(L) .$$

for all line bundles L on X.

(A5) For all line bundles L and M on $X \in \mathbf{Sch}_k$,

$$\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L) .$$

Moreover, if L and M are isomorphic, then $\tilde{c}_1(L) = \tilde{c}_1(M)$.

(A6) For projective morphisms f and g,

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times .$$

(A7) For smooth quasi-projective morphisms f and g of pure relative dimension,

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times .$$

(A8) For $X, Y \in \mathbf{Sch}_k$,

$$(\tilde{c}_1(L)(\alpha)) \times \beta = \tilde{c}_1(p_1^*(L))\big(\alpha \times \beta\big),$$

for $\alpha \in H_*(X)$, $\beta \in H_*(Y)$, and all line bundles L on X.

Let H_* be an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$. The external products make $H_*(k)$ into a graded, commutative ring with unit $1 \in H_0(k)$. For each X, the external product

$$H_*(k) \otimes H_*(X) \to H_*(X)$$

makes $H_*(X)$ into a graded $H_*(k)$-module. The pull-back and push-forward maps are $H_*(k)$-module homomorphisms.

2.4. **Geometric type.** Let R_* be a graded commutative ring with unit. An *oriented Borel-Moore R_*-functor with product* on $\mathbf{Sch}_k$ is an oriented Borel-Moore functor with product H_* on $\mathbf{Sch}_k$ together with a graded ring homomorphism

$$R_* \to H_*(k).$$

An oriented Borel-Moore functor on $\mathbf{Sch}_k$ of *geometric type* (see [22, Definition 2.2.1]) is an oriented Borel-Moore $\mathbb{L}_*$-functor A_* with product on $\mathbf{Sch}_k$ satisfying the following three additional axioms:

(Dim) For $Y \in \mathbf{Sm}_k$ and line bundles $L_1, \ldots, L_{r>\dim_k(Y)}$ on Y,

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y) = 0 \in A_*(Y)\,.$$

(Sect) For $Y \in \mathbf{Sm}_k$ and a section $s \in H^0(Y, L)$ of a line bundle L transverse to the zero section of L,

$$\tilde{c}_1(L)(1_Y) = i_*(1_Z),$$

where $i : Z \to Y$ is the closed immersion of the zero subscheme of s.

(FGL) For $Y \in \mathbf{Sm}_k$ and line bundles L, M on Y,

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y) \in A_*(Y)\,,$$

where $F_A \in A_*(k)[[u, v]]$ is the image of the universal formal group law $F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]]$ under the homomorphism

$$\phi_A : \mathbb{L}_* \to A_*(k)$$

giving the $\mathbb{L}_*$-structure.

Ω_* has a natural structure of an oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$. In fact, more is true.

Theorem 2.2 ([22, Theorem 2.4.13]). *The oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$ determined by Ω_* is universal.*

2.5. **The classifying map.** Let A_* be an oriented Borel-Moore functor of geometric type on $\mathbf{Sch}_k$. Universality yields a canonical natural transformation

$$\vartheta_A : \Omega_* \to A_*$$

which is compatible with projective push-forward, smooth pull-backs, Chern class operators $\tilde{c}_1(L)$, and external products.

In addition, ϑ_A is compatible with the $\mathbb{L}_*$-structures:

$$\phi_A = \vartheta_A(k) \circ \phi_\Omega,$$

hence $\vartheta_A(k)$ defines a morphism of formal group laws

$$\vartheta_A(k) : (\Omega_*(k), F_\Omega) \to (A_*(k), F_A).$$

In fact, $(\Omega_*(k), F_\Omega)$ is the universal formal group law.

Theorem 2.3 ([22, Theorem 4.3.7])**.** *The homomorphism*

$$\phi_\Omega : \mathbb{L}_* \to \Omega_*(k)$$

is an isomorphism.

For $X \in \mathbf{Sm}_k$ of dimension d, we often regrade $\Omega_*(X)$ by codimension,

$$\Omega^n(X) = \Omega_{d-n}(X).$$

We refer the reader to [22, Definition 1.1.2, Proposition 5.2.1] for the definition of an oriented Borel-Moore cohomology theory and the relation to oriented Borel-Moore homology theories. The restriction to $\mathbf{Sm}_k$ of an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ defines an oriented Borel-Moore cohomology theory on $\mathbf{Sm}_k$, after regrading as above. We recall a main result of [22].

Theorem 2.4 ([22, Theorem 7.1.2])**.** *Let k be a field of characteristic 0. Then $X \mapsto \Omega^*(X)$ is the universal oriented Borel-Moore cohomology theory on $\mathbf{Sm}_k$.*

Fix an embedding $\sigma : k \to \mathbb{C}$. Complex cobordism $MU^*(-)$ defines an oriented Borel-Moore cohomology theory MU^{2*}_σ on $\mathbf{Sm}_k$ by

$$X \mapsto MU^{2*}(X(\mathbb{C})).$$

By Theorem 2.4, we obtain a natural transformation

$$\vartheta^{MU,\sigma} : \Omega^* \to MU^{2*}_\sigma.$$

In particular, we obtain a graded ring homomorphism

$$\vartheta^{MU,\sigma}_{\mathrm{pt}} : \Omega^*(k) \to MU^{2*}(\mathrm{pt}).$$

The formal group law for MU^* is also the Lazard ring (after multiplying the degrees by 2, see [31]), so by Theorem 2.3, we have obtained the following result.

Corollary 2.5 ([22, Corollary 1.2.13]). *The map*

$$\vartheta_{\mathrm{pt}}^{MU,\sigma} : \Omega^*(k) \to MU^{2*}(\mathrm{pt})$$

is an isomorphism of graded rings.

2.6. **Borel-Moore homology.** The notion of an oriented Borel-Moore functor of geometric type is weaker than that of an oriented Borel-Moore homology theory [22, Definition 5.1.3]. The two notions are related as follows:

(i) An oriented Borel-Moore homology theory has pull-back maps for all l.c.i. morphisms, while an oriented Borel-Moore functor of geometric type has pull-back maps for smooth quasi-projective morphisms only. In particular, if A is an oriented Borel-Moore homology theory, then pulling-back the external products by the diagonal map makes $A_*(X)$ a graded commutative ring for all X smooth over k.

(ii) The Chern class operators for an oriented Borel-Moore homology theory are not given as part of the data, but are rather defined by the following explicit formula. Let $L \to X$ be a line bundle with zero-section $s : X \to L$. Then,

$$\tilde{c}_1(L)(x) = s^*(s_*(x)).$$

(iii) An oriented Borel-Moore homology theory A satisfies the *projective bundle formula.* Let $E \to X$ be a rank $n+1$ vector bundle with associated $\mathbb{P}^n$-bundle

$$q : \mathbb{P}(E) \to X.$$

Let $\xi = \mathcal{O}(1)$ be the tautological quotient line bundle on $\mathbb{P}(E)$. Then,

$$\sum_{j=0}^{n} \tilde{c}_1(\xi)^j \circ q^* : \oplus_{j=0}^{n} A_{*-n+j}(X) \to A_*(\mathbb{P}(E))$$

is an isomorphism.

(iv) An oriented Borel-Moore homology theory A satisfies the *extended homotopy property.* Let

$$p : V \to X$$

be a principal homogenous space for a vector bundle $E \to X$ of rank n. Then,

$$p^* : A_*(X) \to A_{*+n}(V)$$

is an isomorphism.

(v) By applying the projective bundle formula twice, we obtain

$$A_*(\mathbb{P}^n \times \mathbb{P}^m) = A_*(k)[u,v]/(u^{n+1}, v^{m+1})$$

with $u = \tilde{c}_1(\mathcal{O}(1,0))(1_{\mathbb{P}^n \times \mathbb{P}^m})$ and $v = \tilde{c}_1(\mathcal{O}(0,1))(1_{\mathbb{P}^n \times \mathbb{P}^m})$. Expressing $\tilde{c}_1(\mathcal{O}(1,1))(1_{\mathbb{P}^n \times \mathbb{P}^m})$ in terms of u and v and taking the limit over n, m yields a power series

$$F_A(u,v) \in A_*(k)[[u,v]].$$

The series $F_A(u,v)$ defines a formal group law with coefficients in $A_*(k)$, and the resulting homomorphism $\mathbb{L}_* \to A_*(k)$ makes A_* an oriented Borel-Moore functor of geometric type (see [22, Remarks 4.1.10, Proposition 5.2.6]).

We will directly show double-point cobordism ω_* determines an oriented Borel-Moore functor of geometric type instead of an oriented Borel-Moore homology theory. Since the former has less structure, fewer properties of ω_* will be required. However, since Ω_* is the universal oriented Borel-Moore homology theory, Theorem 1 will imply ω_* is also an oriented Borel-Moore homology theory and, in particular, ω_* admits pull-back maps for arbitrary l.c.i. morphisms.

2.7. **Characteristic.** As stated in Section 0.2, our base field k is of characteristic 0. Some of the results quoted above are also true for k of positive characteristic. Some are true for such k if we assume that resolution of singularities holds for varieties of finite type over k, while others use, in addition, the weak factorization theorem of Abramovich-Karu-Matsuki-Włodarczyk [1].

The construction of Ω_* sketched above makes sense over an arbitrary field, as does the universality of Ω_* as an oriented Borel-Moore functor of geometric type in Theorem 2.2. If resolution of singularities is assumed, then Ω_* is the universal oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ and Ω^* is the universal Borel-Moore cohomology theory on $\mathbf{Sm}_k$. However, the identification of $\mathbb{L}_*$ with $\Omega_*(k)$ in Theorem 2.3 requires both resolution of singularities and weak factorization.

Although our definition of ω_* makes sense in arbitrary characteristic, our approach to the construction of Chern class operators relies on Bertini's Theorem for sections of globally generated line bundles on smooth varieties. We do not know if $\omega_* \cong \Omega_*$ holds in arbitrary characteristic.

3. The functor ω_*

3.1. **Push-forward, pull-back, and external products.** We start the proof of Theorem 2 by observing several basic properties of ω_*. The assignment

$$X \mapsto \omega_*(X)$$

carries the following structures:

Projective push-forward. Let $g : X \to X'$ be a projective morphism in $\mathbf{Sch}_k$. A map

$$g_* : \mathcal{M}_*(X)^+ \to \mathcal{M}_*(X')^+$$

is defined by

$$g_*([f : Y \to X]) = [g \circ f : Y \to X'].$$

By the definition of double point cobordism, g_* descends to a functorial push-forward

$$g_* : \omega_*(X) \to \omega_*(X')$$

satisfying

$$(g_1 \circ g_2)_* = g_{1*} \circ g_{2*}.$$

Smooth pull-back. Let $g : X' \to X$ be a smooth quasi-projective morphism in $\mathbf{Sch}_k$ of pure relative dimension d. A map

$$g^* : \mathcal{M}_*(X)^+ \to \mathcal{M}_{*+d}(X')^+$$

is defined by

$$g^*([f : Y \to X]) = [p_2 : Y \times_X X' \to X'].$$

Since the pull-back by $g \times \mathrm{Id}_{\mathbb{P}^1}$ of a double point cobordism over X is a double point cobordism over X', g^* descends to a functorial pull-back

$$g^* : \omega_*(X) \to \omega_{*+d}(X')$$

satisfying

$$(g_1 \circ g_2)^* = g_2^* \circ g_1^*.$$

External product. A double point cobordism $\pi : Y \to X \times \mathbb{P}^1$ over X gives rise to a double point cobordism

$$Y \times Y' \to X \times X' \times \mathbb{P}^1$$

for each $[Y' \to X'] \in \mathcal{M}(X')$. Hence, the external product

$$[f : Y \to X] \times [f' : Y' \to X'] = [f \times f' : Y \times_k Y' \to X \times_k X']$$

on $\mathcal{M}_*(-)^+$ descends to an external product on ω_*.

Multiplicative unit. The class $[\mathrm{Id} : \mathrm{Spec}(k) \to \mathrm{Spec}(k)] \in \omega_0(k)$ is a unit for the external product on ω_*.

3.2. **Borel-Moore functors with product.** A *Borel-Moore functor with product* on $\mathbf{Sch}_k$ consists of the structures (D1), (D2), and (D4) of Section 2.3 satisfying axioms (A1),(A2), (A6), and (A7). A Borel-Moore functor with product is simply an oriented Borel-Moore functor with product *without* Chern class operations.

Lemma 3.1. *Double point cobordism ω_* is a Borel-Moore functor with product.*

Proof. The structures (D1), (D2), and (D4) have been constructed in Section 3.1 . Axioms (A1), (A2), (A6), and (A7) follow easily from the definitions. ☐

3.3. **Double point relations.** We will show the double point relations are satisfied in Ω_*. Then, a natural transformation of Borel-Moore functors with product is obtained,

$$\omega_* \to \Omega_* \ .$$

In what follows, we will be working in Ω_* rather than $\underline{\Omega}_*$, unless expressly noted. We write $[f : Y \to X]$ for the image in Ω_* of

$$1 \otimes [f : Y \to X] \in \mathbb{L}_* \otimes \underline{\Omega}_* \ .$$

The class $[f : Y \to X]$ may also be considered as an element of ω_*. If needed, we will distinguish the two meanings explicity by using the notation $[f : Y \to X]_\omega$ or $[f : Y \to X]_\Omega$.

Let $F(u, v) \in \Omega_*(k)[[u, v]]$ be the formal group law for Ω_*. By definition,

$$F(u, v) = u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j$$

with $a_{i,j} \in \Omega_{i+j-1}$. Let $F^{1,1}(u, v) = \sum_{i,j\geq 1} a_{i,j} u^{i-1} v^{j-1}$. We have

$$F(u, v) = u + v + uv \cdot F^{1,1}(u, v).$$

Let $Y \in \mathbf{Sm}_k$. Let E_1, E_2 be smooth divisors on Y, intersecting transversely in Y, with sum $E = E_1 + E_2$. Let

$$i_D : D = E_1 \cap E_2 \to Y$$

be the inclusion of the intersection. Let $O_D(E_1)$, $O_D(E_2)$ be the restrictions to D of the line bundles $O_Y(E_1)$, $O_Y(E_2)$. Define an element $[E \to Y] \in \Omega_*(Y)$ by

$$\begin{aligned}[E \to Y] = [E_1 \to Y] &+ [E_2 \to Y] \\ &+ i_{D*}\Big(F^{1,1}\big(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2))\big)(1_D)\Big)\end{aligned}$$

(see [22, Definition 3.1.5] for the definition of $[E \to Y]$ for an arbitrary strict normal crossing divisor E). The following result is proven in [22, Definition 2.4.5 and Proposition 3.1.9] as a consequence of the formal group law for Ω_*.

Lemma 3.2. *Let $F \subset Y$ be a smooth divisor linearly equivalent to E, then*

$$[F \to Y] = [E \to Y] \in \Omega_*(Y).$$

If the additional condition

$$O_D(E_1) \cong O_D(E_2)^{-1}$$

is satisfied, a direct evaluation is possible. Let $\mathbb{P}_D \to D$ be the $\mathbb{P}^1$-bundle $\mathbb{P}(O_D \oplus O_D(E_1))$.

Lemma 3.3. *We have*

$$F^{1,1}\big(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2))\big)(1_D) = -[\mathbb{P}_D \to D] \in \Omega_*(D).$$

Proof. Both sides of the formula depend only upon the line bundles $O_D(E_1)$ and $O_D(E_2)$. To prove the Lemma, we may replace E with any $E' = E'_1 + E'_2$ on any Y', so long as $E'_1 \cap E'_2 = D$ and $O_{Y'}(E'_i)$ restricts to $O_D(E_i)$ on D.

The surjection $O_D \oplus O_D(E_1) \to O_D(E_1)$ defines a section $s : D \to \mathbb{P}_D$ with normal bundle $O_D(E_1)$. Let Y' be the deformation to the normal cone of the closed immersion s. By definition, Y' is the blow-up of $\mathbb{P}_D \times \mathbb{P}^1$ along $s(D) \times 0$. The blow-up of $\mathbb{P}_D$ along D is $\mathbb{P}_D$ and the exceptional divisor $\mathbb{P}$ of $Y' \to P_D \times \mathbb{P}^1$ is also $\mathbb{P}_D$.

The composition $Y' \to \mathbb{P}_D \times \mathbb{P}^1 \to \mathbb{P}^1$ has fiber Y'_0 over $0 \in \mathbb{P}^1$ equal to $\mathbb{P}_D \cup \mathbb{P}$. The intersection $\mathbb{P}_D \cap \mathbb{P}$ is $s(D)$ and the line bundles $O_{Y'}(\mathbb{P})$, $O_{Y'}(\mathbb{P}_D)$ restrict to $O_D(E_1)$, $O_D(E_2)$ on $s(D)$ respectively. Thus, we may use $E' = Y'_0$, $E'_1 = \mathbb{P}_D$, and $E'_2 = \mathbb{P} \cong \mathbb{P}_D$.

By Lemma 3.2, we have the relation $[Y'_\infty \to Y'] = [Y'_0 \to Y']$ in $\Omega_*(Y')$. By definition, $[Y'_0 \to Y']$ is the sum

$$\begin{aligned}[Y'_0 \to Y] = [\mathbb{P}_D \to Y'] + [\mathbb{P} \to Y'] \\ + i_{D*}\Big(F^{1,1}\big(\tilde{c}_1(O_D(\mathbb{P}_D)), \tilde{c}_1(O_D(\mathbb{P}))\big)(1_D)\Big).\end{aligned}$$

Pushing forward the relation $[Y'_\infty \to Y] = [Y'_0 \to Y']$ to $\Omega_*(D)$ by the composition

$$Y' \to \mathbb{P}_D \times \mathbb{P}^1 \xrightarrow{p_1} \mathbb{P}_D \to D$$

yields the relation

$$[\mathbb{P}_D \to D] = [\mathbb{P}_D \to D] + [\mathbb{P} \to D] + F^{1,1}\big(\tilde{c}_1(O_D(\mathbb{P}_D)), \tilde{c}_1(O_D(\mathbb{P}))\big)(1_D)$$

in $\Omega_*(D)$. Since $\mathbb{P} \cong \mathbb{P}_D$ as a D-scheme, the proof is complete. □

Corollary 3.4. *Let $\pi : Y \to \mathbb{P}^1$ be a double point degeneration over $0 \in \mathbb{P}^1$. Let*

$$\pi^{-1}(0) = A \cup B.$$

Suppose the fiber $Y_\infty = \pi^{-1}(\infty)$ is smooth. Then

$$[Y_\infty \to Y] = [A \to Y] + [B \to Y] - [\mathbb{P}(\pi) \to Y] \in \Omega_*(Y).$$

Sending $[f : Y \to X] \in \mathcal{M}_n^+(X)$ to the class $[f : Y \to X]_\Omega \in \Omega_n(X)$ defines a natural transformation $\mathcal{M}_*^+ \to \Omega_*$ of Borel-Moore functors with product on $\mathbf{Sch}_k$.

Proposition 3.5. *The map $\mathcal{M}_*^+ \to \Omega_*$ descends to a natural transformation*

$$\vartheta : \omega_* \to \Omega_*$$
$$\vartheta_X([f : Y \to X]_\omega) = [f : Y \to X]_\Omega$$

of Borel-Morel functors with product on $\mathbf{Sch}_k$. Moreover, ϑ_X is surjective for each $X \in \mathbf{Sch}_k$.

Proof. Let $\pi : Y \to X \times \mathbb{P}^1$ be a double point degeneration over X. We obtain a canonical double point degeneration

$$\pi' = (\mathrm{Id}, p_2 \circ \pi) : Y \to Y \times \mathbb{P}^1.$$

Certainly

$$\pi = (p_1 \circ f, \mathrm{Id}) \circ g.$$

Since $\mathcal{M}_*^+ \to \Omega_*$ is compatible with projective push-forward, the first assertion reduces to Lemma 3.4.

The surjectivity follows from the fact that the canonical map

$$\mathcal{M}_*(X)^+ \to \Omega_*(X)$$

is surjective by [22, Lemma 2.5.11]. $\square$

We will prove Theorem 1 by showing ϑ is an isomorphism. The strategy of the proof is to show that ω_* admits first Chern class operators and a formal group law, making ω_* into an oriented Borel-Moore functor of geometric type. We then use the universality of Ω_* given by Theorem 2.2 to determine an inverse $\Omega_* \to \omega_*$ to ϑ.

4. Chern classes I

Let $X \in \mathbf{Sch}_k$, and let $L \to X$ be a line bundle generated by global sections. We will define a *first Chern class operator*

$$\tilde{c}_1(L) : \omega_*(X) \to \omega_{*-1}(X).$$

A technical Lemma is required for the definition.

Let $[f : Y \to X] \in \mathcal{M}(X)^+$ with Y irreducible of dimension n. For $s \in H^0(Y, f^*L)$, $s \neq 0$, let

$$i_s : H_s \to Y$$

be the inclusion of the zero subscheme of s; note that H_s depends only on the point $[s]$ of $\mathbb{P}(H^0(Y, f^*L))$ defined by s. Let

$$U \subset \mathbb{P}(H^0(Y, f^*L)) = \{[s] \mid H_s \text{ is smooth and of codimension 1 in } Y\}.$$

Lemma 4.1. *We have*

(i) *U is an open subscheme of $\mathbb{P}(H^0(Y, f^*L))$ with $U(k)$ non-empty.*
(ii) *For $[s_1], [s_2] \in U(k)$, $[H_{s_1} \to X] = [H_{s_2} \to X] \in \omega_{n-1}(X)$.*

Proof. Since L is globally generated, so is f^*L. Then (i) follows from Bertini's theorem (using the characteristic 0 assumption for k).

Let $\mathcal{H} \subset Y \times \mathbb{P}(H^0(Y, f^*L))$ be the universal Cartier divisor. Let $y \in Y$ be a closed point with ideal sheaf $\mathfrak{m}_y \subset \mathcal{O}_{Y,y}$. Since f^*L is globally generated, the fiber of $\mathcal{H} \to Y$ over y is the hyperplane

$$\mathbb{P}(H^0(Y, f^*L \otimes \mathfrak{m}_y)) \subset \mathbb{P}(H^0(Y, f^*L)).$$

Hence, $\mathcal{H}$ is smooth over k.

For (ii), let

$$i : \mathbb{P}^1 \to \mathbb{P}(H^0(Y, f^*L))$$

be a linearly embedded $\mathbb{P}^1$ with $i(0) = s_1$. By Bertini's theorem, the pull-back

$$\mathcal{H}_i = \mathcal{H} \times_{\mathbb{P}(H^0(Y, f^*L))} \mathbb{P}^1$$

is smooth for general i. Clearly $\mathcal{H}_i \to X \times \mathbb{P}^1$ gives a naive cobordism between $[H_{s_1} \to X]$ and $[H_{i(t)} \to X]$ for all k-valued points t in a dense open subset of $\mathbb{P}^1$. Since i is general, we have

$$[H_{s_1} \to X] = [H_s \to X] \in \omega_{n-1}(X)$$

for all k-valued points s in a dense open subset of U. The same result for s_2 completes the proof. □

For L globally generated, we can therefore define the homomorphism

$$\tilde{c}_1(L) : \mathcal{M}_*(X)^+ \to \omega_{*-1}(X)$$

by sending $[f : Y \to X]$ to $[H_s \to X]$ for H_s smooth and codimension 1 in Y.

Lemma 4.2. *The map $\tilde{c}_1(L)$ descends to*

$$\tilde{c}_1(L) : \omega_*(X) \to \omega_{*-1}(X)$$

Proof. Let $\pi : W \to X \times \mathbb{P}^1$ be a double point cobordism with degenerate fiber over $0 \in \mathbb{P}^1$ and smooth fiber over $\infty \in \mathbb{P}^1$. Hence,

$$W_0 = A \cup B$$

with A, B smooth divisors intersecting transversely in the double point locus $D = A \cap B$. The double point relation is

$$[W_\infty \to X] = [A \to X] + [B \to X] - [\mathbb{P}(\pi) \to X]. \tag{4.1}$$

Let $i_s : H_s \to W$ be the divisor of a general section s of $(p_1 \circ \pi)^*L$. As in the proof of lemma 4.1, we may assume H_s, $H_s \cap W_\infty$, $H_s \cap D$, $H_s \cap A$ and $H_s \cap B$ are smooth divisors on W, W_∞, A, B, and D respectively. Then

$$\pi \circ i_s : H_s \to X \times \mathbb{P}^1$$

is again a double point cobordism. The associated double point relation

$$[H_s \cap W_\infty \to X] = [H_s \cap S \to X] + [H_s \cap T \to X] - [\mathbb{P}(\pi \circ i_s) \to X].$$

is obtained by applying $\tilde{c}_1(L)$ term-wise to relation (4.1) and therefore $\tilde{c}_1(L)$ descends. □

Thus, we have established the existence of first Chern class operators, axiom (D3) of Definition 2.1, for globally generated line bundles. Axioms (A3), (A4), (A5) and (A8) for an oriented Borel-Moore functor with product are easily checked for our definition of $\tilde{c}_1(L)$, assuming all line bundles in question are globally generated. In particular, the operators $\tilde{c}_1(L)$ for globally generated line bundles L on X are $\omega_*(k)$-linear and commute pairwise.

Lemma 4.3. *Let $X \in \mathbf{Sch}_k$, and let*

$$L_1, \ldots, L_{r > \dim_k X} \to X$$

be globally generated line bundles. Then,

$$\prod_{i=1}^{r} \tilde{c}_1(L_i) = 0$$

as an operator on $\omega_*(X)$.

Proof. Let $[f : Y \to X] \in \mathcal{M}(X)^+$. By Bertini's theorem, H_{f^*s} is smooth for a general choice of section $s \in H^0(X, L)$. Thus

$$\tilde{c}_1(L)(f) = [f : H_{f^*s} \to X].$$

By induction, $\prod_i \tilde{c}_1(L_i)(f)$ is represented by the restriction of f to $\cap_{i=1}^r H_{f^*s_i}$. But set-theoretically, $\cap_{i=1}^r H_{f^*s_i} = f^{-1}(\cap_{i=1}^r H_{s_i})$. Since the sections s_i are general, the intersection $\cap_{i=1}^r H_{s_i}$ is empty, whence the result. □

Let $F(u_1, \ldots, u_r) \in \omega_*(k)[[u_1, \ldots, u_r]]$ be a power series and let $L_1, \ldots, L_r$ be globally generated on $X \in \mathbf{Sch}_k$. By Lemma 4.3, the expression $F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))$ is well defined as an operator on $\omega_*(X)$.

Lemma 4.3 is condition (Dim) for an oriented Borel-Moore functor of geometric type in case all the line bundles in question are globally generated.

Chern classes for arbitrary line bundle will be constructed in Section 9. The axioms (FGL) and (Sect) will be verified in Section 9 and Section 10, and we will then complete the story by proving the remaining axioms for arbitrary (not necessarily globally generated) line bundles.

5. Extending the double point relation

5.1. **The blow-up relation.** Before we construct the formal group law and the rest of the Chern class operators for ω_*, we describe two useful relations which are consequences of the basic double point cobordism relation.

The first is the blow-up relation. Let $F \to X$ be a closed embedding in $\mathbf{Sm}_k$ with conormal bundle $\eta = \mathcal{I}_F/\mathcal{I}_F^2$ of rank n. Let

$$\mu : X_F \to X$$

be the blow-up of X along F. Let $\mathbb{P}_F$ be the $\mathbb{P}^{n-1}$-bundle $\mathbb{P}(\eta) \to F$. Let

$$\mathbb{P}_1 = \mathbb{P}(\eta \oplus O_F) \to F$$
$$\mathbb{P}_2 = \mathbb{P}_{\mathbb{P}_F}(O_{\mathbb{P}_F} \oplus O(1)) \to \mathbb{P}_F.$$

We consider $\mathbb{P}_1$ and $\mathbb{P}_2$ as X-schemes by the composition of the structure morphisms with the inclusion $F \to X$.

Lemma 5.1. *We have*

$$[X_F \to X] = [\mathrm{Id} : X \to X] - [\mathbb{P}_1 \to X] + [\mathbb{P}_2 \to X] \in \omega_*(X).$$

Proof. The Lemma follows the double point relation obtained from the deformation to the normal cone of $F \to X$. Indeed, let

$$\pi : Y \to X \times \mathbb{P}^1$$

be the blow-up along $F \times 0$ with structure morphism

$$\pi_2 = p_2 \circ \pi : Y \to \mathbb{P}^1.$$

The fiber $\pi^{-1}(\infty)$ is just X, and

$$\pi^{-1}(0) = X_F \cup \mathbb{P}_1,$$

with X_F and $\mathbb{P}_1$ intersecting transversely along the exceptional divisor $\mathbb{P}_F$ of μ. The normal bundle of $\mathbb{P}_F$ in $\mathbb{P}_1$ is $O(1)$. Thus the associated double point relation is

$$[\mathrm{Id} : X \to X] = [X_F \to X] + [\mathbb{P}_1 \to X] - [\mathbb{P}_2 \to X]$$

in $\omega_*(X)$. □

Via Proposition 3.5, one obtains the blow-up relation

$$[X_F \to X] = [\mathrm{Id} : X \to X] - [\mathbb{P}_1 \to X] + [\mathbb{P}_2 \to X]$$

in $\Omega_*(X)$ as well, first proved by Nenashev [28].

5.2. **The extended double point relation.** Let $Y \in \mathbf{Sm}_k$. Let $A, B, C \subset Y$ be smooth divisors such that $A + B + C$ is a reduced strict normal crossing divisor. Let

$$D = A \cap B, \quad E = A \cap B \cap C.$$

As before, we let $O_D(A)$ denote the restriction of $O_Y(A)$ to D, and use a similar notation for the restrictions of bundles to E. Let

$$\begin{aligned}
\mathbb{P}_1 &= \mathbb{P}(O_D(A) \oplus O_D) \to D \\
\mathbb{P}_E &= \mathbb{P}(O_E(-B) \oplus O_E(-C)) \to E \\
\mathbb{P}_2 &= \mathbb{P}_{\mathbb{P}_E}(O \oplus O(1)) \to \mathbb{P}_E \to E \\
\mathbb{P}_3 &= \mathbb{P}(O_E(-B) \oplus O_E(-C) \oplus O_E) \to E.
\end{aligned}$$

We consider $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ as Y-schemes by composing the structure morphisms with the inclusions $D \to Y$ and $E \to Y$.

Lemma 5.2. *Suppose C is linearly equivalent to $A + B$ on Y. Then,*

$$[C \to Y] = [A \to Y] + [B \to Y] - [\mathbb{P}_1 \to Y] + [\mathbb{P}_2 \to Y] - [\mathbb{P}_3 \to Y]$$

in $\omega_*(Y)$

Proof. Let $Y_1 \to Y$ be the blow-up of Y along $(A \cup B) \cap C$. Since $(A \cup B) \cap C$ is a Cartier divisor on both $A \cup B$ and C, the proper transforms of both $A \cup B$ and C define closed immersions

$$A \cup B \to Y_1, \quad C \to Y_1$$

lifting the inclusions $A \cup B \to Y$ and $C \to Y$. We denote the resulting closed subschemes of Y_1 by A_1, B_1 and C_1.

Let f be a rational function on Y with $\mathrm{Div}(f) = A + B - C$. We obtain a morphism $f : Y_1 \to \mathbb{P}^1$ satisfying

$$f^{-1}(0) = A_1 \cup B_1, \quad f^{-1}(\infty) = C_1.$$

However, Y_1 is singular, unless $E = \emptyset$. Indeed, if A, B and C are defined near a point x of E by local parameters a, b and c, then locally analytically near $x \in A_1 \cap B_1 \subset Y_1$,

$$Y_1 \cong E \times \mathrm{Spec}\,(k[a,b,c,z]/(ab - cz))\,.$$

Here, the exceptional divisor of $Y_1 \to Y$ is defined by the ideal (c), A_1 is defined by (a,z) and B_1 is defined by (b,z). The singular locus of Y_1 is isomorphic to E. We write E_1 for the singular locus of Y_1.

Let $\mu_2 : Y_2 \to Y_1$ be the blow-up of Y_1 along A_1. Since $A_1 \subset Y_1$ is a Cartier divisor off of the singular locus E_1, the blow-up μ_2 is an isomorphism over $Y_1 \setminus E_1$. In our local description of Y_1, we see that $A_1 \cap B_1$ is the Cartier divisor on B_1 defined by (a), hence the proper transform of B_1 to Y_2 is isomorphic to B. Also, since

$$b(a,z) = (ab, zb) = (zb, zc) = z(b,c),$$

the strict transform of A_1 by μ_2 is identified with the blow-up A_E of A along E. In particular, since E has codimension 2 in A with normal bundle $O_E(B) \oplus O_E(C)$, we have the identification

$$\mu_2^{-1}(E_1) = \mathbb{P}(O_E(-B) \oplus O_E(-C)).$$

In addition, Y_2 is smooth. Indeed, the singular locus of Y_2 is contained in

$$\mu_2^{-1}(E_1) \subset \mu_2^{-1}(A_1) = A_E.$$

Since A_E is a smooth Cartier divisor on Y_2, Y_2 is itself smooth, as claimed.

The morphism $\pi : Y_2 \to \mathbb{P}^1$ defined by $\pi = f \circ \mu_2$ is a double point degeneration over $0 \in \mathbb{P}^1$. with

$$\pi^{-1}(0) = A_E \cup B$$

and double point locus $A_E \cap B = A \cap B = D$.

Since $\pi^{-1}(\infty) = C$, we obtain the following double point relation

$$[C \to Y] = [A_E \to Y] + [B \to Y] - [\mathbb{P}(O_D(A) \oplus O_D) \to Y].$$

in $\omega_*(Y)$. Inserting the blow-up formula from Lemma 5.1 completes the proof. □

6. Pull-backs in ω_*

6.1. **Pull-backs.** The most difficult part of the construction of Ω_* is the extension of the pull-back maps from smooth quasi-projective morphisms to l.c.i. morphisms. We cannot hope to reproduce the full theory for ω_* directly. Fortunately, only smooth pull-backs for ω_* are required for the construction of an oriented Borel-Moore functor of geometric type. However, our discussion of the formal group law for

ω_* will require more than just smooth pull-backs. The technique of *moving by translation* gives us sufficiently many pull-back maps for ω_*.

6.2. **Moving by translation.** We consider pull-back maps in the following setting. Let G_1 and G_2 be general linear groups over k

$$G_i = \mathrm{GL}_{n_i}/k;\ n_i \geq 0, i = 1, 2.$$

Let $Y \in \mathbf{Sm}_k$ admit a $G_1 \times G_2$-action, and let $B \in \mathbf{Sm}_k$ admit a transitive G_2-action. Let

$$p : Y \to B$$

be a smooth quasi-projective morphism equivariant with respect to $G_1 \times G_2 \to G_2$. Let

$$s : B \to Y$$

be a section of p satisfying three conditions:

(i) s is equivariant with respect to the inclusion $G_2 = \mathrm{Id} \times G_2 \subset G_1 \times G_2$,
(ii) $G_1 = G_1 \times \mathrm{Id} \subset G_1 \times G_2$ acts trivially on $s(B)$,
(iii) $G_1 \times G_2$ acts transitively on $Y \setminus s(B)$.

We will assume the above conditions hold throughout Section 6.2.

A special case in which all the hypotheses are verified occurs when $G_1 = 1$, Y admits a transitive G_2-action, and

$$p : Y \to Y, \quad s : Y \to Y$$

are both the identity.

Given $A, B, C \in \mathbf{Sm}_k$ and morphisms $f : A \to C$, $g : B \to C$, we say f and g are *transverse* if $A \times_C B$ is smooth, and, for all irreducible components $A' \subset A$, $B' \subset B$ with $f(A')$ and $g(B')$ contained in the same irreducible component $C' \subset C$, we have

$$\dim_k A' \times_{C'} B' = \dim_k A' + \dim_k B' - \dim_k C',$$

or $A' \times_{C'} B' = \emptyset$. For $y \in Y \in \mathbf{Sm}_k$, we denote the tangent space to Y at y by $T_y(Y)$.

Lemma 6.1. *Let $i : Z \to Y$ be a morphism in $\mathbf{Sm}_k$ transverse to $s : B \to Y$. Take $C \in \mathbf{Sm}_k$ and let $f : W \to Y \times C$ be a projective morphism in $\mathbf{Sm}_k$.*

(1) *For all $g = (g_1, g_2)$ in a nonempty open set*

$$U(i, f) \subset G_1 \times G_2,$$

the morphisms $(g \cdot i) \times \mathrm{Id}_C$ and f are transverse.

(2) *If $C = \mathrm{Spec}\,(k)$, then for $g, g' \in U(i, f)$,*

$$[Z \times_{g \cdot i} W \to Z] = [Z \times_{g' \cdot i} W \to Z] \in \omega_*(Z).$$

Proof. Let $G = G_1 \times G_2$. Consider the map

$$\mu : G \times Z \to Y$$

defined by $\mu(g, z) = g \cdot i(z)$. We first prove μ is smooth. In fact, we will check μ is a submersion at each point (g, z).

If $i(z) \in Y \setminus s(B)$, then $G \times z \to Y$ is smooth [3] and surjective by condition (iii), hence μ is a submersion at (g, z) for all g.

Suppose $i(z) \in s(B)$. The map $G_2 \times z \to s(B)$ is smooth and surjective since G_2 acts transitively on B, so the image of $T_{(g,z)}(G \times z)$ contains

$$T_{i(z)}(s(B)) \subset T_{i(z)}(Y).$$

Since i is transverse to s, $g \cdot i$ is transverse to s for all g and the composition

$$T_z Z \xrightarrow{d(g\cdot i)} T_{g\cdot i(z)}(Y) \to T_{g\cdot i(z)}(Y)/T_{g\cdot i(z)}(s(B))$$

is surjective. Thus

$$T_{(g,z)}(G \times Z) = T_{(g,z)}(G \times z) \oplus T_{(g,z)}(g \times Z) \xrightarrow{d\mu} T_{g\cdot i(z)}(Y)$$

is surjective, and μ is a submersion at (g, z).

The smoothness of μ clearly implies the smoothness of

$$\mu \times \mathrm{Id}_C : G \times Z \times C \to Y \times C.$$

Hence $(G \times Z \times C) \times_\mu W$ is smooth over k, and the projection

$$(G \times Z \times C) \times_\mu W \to G \times Z \times C$$

is a well-defined element of $\mathcal{M}(G \times Z \times C)$. Consider the projection

$$\pi : (G \times Z \times C) \times_\mu W \to G.$$

Since the characteristic is 0, the set of regular values of π contains a nonempty Zariski open dense subset

$$U(i, f) \subset G.$$

Since G is an open subscheme of the affine space $\mathbb{A}^{n_1^2+n_2^2}$, the set of k-points of $U(i, f)$ is dense in $U(i, f)$. Any k-point $g = (g_1, g_2)$ in $U(i, f)$ satisfies claim (1) of the Lemma.

For $g \in U(i, f)$, denote the element of $\mathcal{M}(Z \times C)$ corresponding to

$$(Z \times C) \times_{g\cdot i \times \mathrm{Id}_C} W \to Z \times C$$

by $(g \cdot i)^*(f)$.

For (2), let $g, g' \in U(i, f)$ be two k-points. As we have seen, $U(i, f)$ is an open subset of an affine space $\mathbb{A}^N$. The pull-back $\pi^{-1}(\ell_{g,g'})$ of the

[3] Since k has characteristic 0 and G acts transitively on $Y \setminus s(B)$, the orbit map is smooth.

line $\ell_{g,g'}$ through g and g' will be a closed subscheme of $(G \times Z) \times_\mu W$ which smooth and projective over an open neighborhood $U \subset \ell_{g,g'}$ containing g and g'. Then

$$(U \times Z) \times_\mu W \to U \tag{6.1}$$

provides a naive cobordism proving

$$[Z \times_{g\cdot i} W \to Z] = [Z \times_{g'\cdot i} W \to Z] \in \omega_*(Z). \tag{6.2}$$

Technically, the naive cobordism (6.1) has been constructed only over an open set $U \subset \mathbb{P}^1$. By taking a closure followed by a resolution of singularities, the family (6.1) can be extended appropriately over $\mathbb{P}^1$. The relation is (6.2) unaffected. □

Let $i : Z \to Y$ be a morphism in $\mathbf{Sm}_k$ of pure codimension d transverse to $s : B \to Y$. We define

$$i^* : \mathcal{M}_*(Y)^+ \to \omega_{*-d}(Z) \tag{6.3}$$

using (2) of Lemma 6.1 by

$$i^*[f : W \to Y] = [(g \cdot i)^*(f)]$$

for $g \in U(i, f)$.

Proposition 6.2. *The pull-back* (6.3) *descends to a well-defined* $\omega_*(k)$*-linear pull-back*

$$i^* : \omega_*(Y) \to \omega_{*-d}(Z).$$

Proof. The $\mathcal{M}_*(k)^+$-linearity of the map

$$i^* : \mathcal{M}_*(Y)^+ \to \omega_{*-d}(Z)$$

is evident from the construction.

Given a double point cobordism $f : W \to Y \times \mathbb{P}^1$ over $0 \in \mathbb{P}^1$, we will show the pull-back of f by $(g \cdot i) \times \mathrm{Id}_{\mathbb{P}^1}$ gives a double point cobordism for all g in a dense open set of $U(i, f)$.

Applying (1) of Lemma 6.1 with $C = \mathbb{P}^1$ yields an open subscheme

$$U_1 \subset G_1 \times G_2$$

for which $(g\cdot i)\times \mathrm{Id}_{\mathbb{P}^1}$ pulls W back to a smooth scheme $(g\cdot i)\times\mathrm{Id}_{\mathbb{P}^1}(W)$, with a projective map to $Z \times \mathbb{P}^1$. Similarly, applying Lemma 6.1 to the smooth fiber $W_\infty \to Y$, we find a subset $U_2 \subset U_1$ for which the fiber $(g\cdot i)\times\mathrm{Id}_{\mathbb{P}^1}(W)_\infty$ is smooth. Finally, if $W_0 = A \cup B$, applying Lemma 6.1 to $A \to Y$, $B \to Y$ and $A \cap B \to Y$ yields an open subscheme $U_3 \subset U_2$ for which $(g \cdot i) \times \mathrm{Id}_{\mathbb{P}^1}(W)$ gives the double point relation

$$\begin{aligned}(g \cdot i)^*([W_\infty \to Y]) = \\ (g \cdot i)^*([A \to Y]) + (g \cdot i)^*([B \to Y]) - (g \cdot i)^*([\mathbb{P}(f) \to Y]),\end{aligned}$$

as desired. $\square$

Lemma 6.3. *Let $L \to Y$ be a globally generated line bundle on Y. Then,*

$$i^* \circ \tilde{c}_1(L) = \tilde{c}_1(i^*L) \circ i^*.$$

Proof. Since i^*L is globally generated on Z, $\tilde{c}_1(i^*L)$ is well-defined. Let $[f : W \to Y] \in \mathcal{M}(Y)$ and take $g \in G_1 \times G_2$ so $g \cdot i : Z \to Y$ is transverse to f. For a general section s of f^*L, the divisor of s,

$$H_s \to W,$$

is also transverse to $g \cdot i$. Hence,

$$i^* \circ \tilde{c}_1(L)([W \to Y]) = [Z \times_{g \cdot i} H_s \to Z].$$

Let $H_{p_1^*(s)}$ be the divisor of $p_1^*(s)$ on $Z \times_{g \cdot i} W$ where p_1 is projection to the first factor. Then,

$$\tilde{c}_1(i^*L) \circ i^*([W \to Y]) = [H_{p_1^*(s)} \to Z].$$

The isomorphism (as Z-schemes)

$$Z \times_{g \cdot i} H_s \cong H_{p_1^*(s)}$$

yields the Lemma. $\square$

6.3. **Examples.** There are two main applications of pull-backs constructed in Section 6.2.

First, let $Y = \prod_i \mathbb{P}^{N_i}$ be a product of projective spaces. Let

$$G_1 = 1, \quad G_2 = \prod_i \mathrm{GL}_{N_i+1}.$$

Let $p : Y \to Y$ and $s : Y \to Y$ both be the identity. For each morphism

$$i : Z \to \prod_i \mathbb{P}^{N_i}$$

in $\mathbf{Sm}_k$ of codimension d, Proposition 6.2 gives us a well-defined $\omega_*(k)$-linear pull-back

$$i^* : \omega_*(\prod_i \mathbb{P}^{N_i}) \to \omega_{*-d}(Z).$$

Second, let Y be the total space of a line bundle L on $B = \prod_i \mathbb{P}^{N_i}$ with projection p and zero-section s,

$$p : L \to B, \quad s : B \to L.$$

Here, $G_1 = \mathrm{GL}_1$ acts by scaling L, and $G_2 = \prod_i \mathrm{GL}_{N_i+1}$ acts by symmetries on B. For each morphism

$$i : Z \to L$$

in $\mathbf{Sm}_k$ which is transverse to the zero-section, Proposition 6.2 gives us an $\omega_*(k)$-linear pull-back

$$i^* : \omega_*(L) \to \omega_{*-d}(Z).$$

6.4. **Independence.** The pull-backs constructed in Section 6.2 can be used to prove several independence statements.

A *linear embedding* of $\mathbb{P}^{N-j} \to \mathbb{P}^N$ is an inclusion as linear subspace. A *multilinear embedding*

$$\prod_{i=1}^{m} \mathbb{P}^{N_i - j_i} \to \prod_{i=1}^{m} \mathbb{P}^{N_i}$$

is a product of linear embeddings (for $m = 2$, we call this a *bi-linear* embedding). For fixed j_i, any two multilinear embeddings are related by a naive cobordism. The classes

$$M_{j_1,\dots,j_m} = \left[\prod_{i=1}^{m} \mathbb{P}^{N_i - j_i} \to \prod_{i=1}^{m} \mathbb{P}^{N_i} \right] \in \omega_*(\prod_{i=1}^{m} \mathbb{P}^{N_i}) \tag{6.4}$$

are therefore well-defined, independent of the choice of multi-linear embedding.

Proposition 6.4. *The classes*

$$\{ M_{j_1,\dots,j_m} \mid 0 \le j_i \le N_i \ \} \subset \omega_*(\prod_{i=1}^{m} \mathbb{P}^{N_i})$$

are independent over $\omega_*(k)$.

Proof. Let $J = (j_1, \dots, j_m)$ be a multi-index. There is a partial ordering defined by

$$J \le J'$$

if $j_i \le j_i'$ for all $1 \le i \le m$. Let

$$\alpha = \sum_J a_J M_J \in \omega_*(\prod_{i=1}^{m} \mathbb{P}^{N_i})$$

where $a_J \in \omega_*(k)$.

If the a_J are not all zero, let $J_0 = (j_1, \dots, j_m)$ be a minimal multi-index for which $a_J \neq 0$. If we take a pull-back by a multi-linear embedding

$$i : \prod_i \mathbb{P}^{j_i} \to \prod_i \mathbb{P}^{N_i},$$

then
$$i^*(\alpha) = a_{J_0} \cdot \left[\prod_{i=1}^m \mathbb{P}^0 \to \prod_{i=1}^m \mathbb{P}^{j_i} \right] \in \omega_*(\prod_{i=1}^m \mathbb{P}^{j_i}).$$
Let $q : \prod_{i=1}^m \mathbb{P}^{j_i} \to \mathrm{Spec}\,(k)$ be the structure morphism. Pushing-forward to $\omega_*(k)$ gives $q_*(i^*(\alpha)) = a_{J_0} \neq 0$ and hence $\alpha \neq 0$. □

Let $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^m$ be the hypersurface defined by the vanishing of a general section of $O(1,1)$. More generally, for $0 \leq i \leq n$, let
$$H_{n,m}^{(i)} \subset \mathbb{P}^n \times \mathbb{P}^m$$
be the (smooth) subscheme defined by the vanishing of i general sections of $O(1,1)$. Taking linear embeddings $\mathbb{P}^{m-j} \to \mathbb{P}^n$, we may consider
$$H_{n,m-j}^{(i)} \subset \mathbb{P}^n \times \mathbb{P}^m$$
for $0 \leq j \leq m$. The proof of the following result is similar to the proof of proposition 6.4 and is left to the reader.

Lemma 6.5. *The classes* $[H_{n,m-j}^{(i)} \to \mathbb{P}^n \times \mathbb{P}^m] \in \omega_*(\mathbb{P}^n \times \mathbb{P}^m)$ *for* $0 \leq i \leq n$, $0 \leq j \leq m$ *are independent over* $\omega_*(k)$.

If classes $H_{n,j}^{(i)}$ are taken for $i > n$, we have a partial independence result.

Proposition 6.6. *If the identity*
$$\sum_{i=0}^{n+2m} \sum_{j=0}^{m} \alpha_{i,j} \cdot [H_{n+m,m-j}^{(i)} \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = 0 \in \omega_*(\mathbb{P}^{n+m} \times \mathbb{P}^m)$$
holds for $\alpha_{i,j} \in \omega_*(k)$, *then* $\alpha_{i,j} = 0$ *for* $0 \leq i + j \leq n + m$, $0 \leq j \leq m$.

Proof. We argue by induction. Consider all pairs (i,j) satisfying
$$0 \leq i + j \leq n + m, \quad 0 \leq j \leq m$$
for which $\alpha_{i,j} \neq 0$. Of these, take the ones with minimal sum $i + j$, and of these, take the one with minimal j, denote the resulting pair by (a,b). Note that $a \leq a + b \leq n + m$.

Take the pull-back of the identity by a bi-linear embedding
$$i : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^{n+m} \times \mathbb{P}^m.$$
Then, for each pair (i,j) with $i + j > a + b$,
$$i^*[H_{n+m,m-j}^{(i)} \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = 0,$$
since $H_{n+m,m-j}^{(i)}$ has codimension $i + j$. Similarly
$$i^*[H_{n+m,m-j}^{(i)} \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = 0$$

if $j > b$. Thus the identity in question pulls back to

$$\alpha_{a,b} \cdot [H_{a,0}^{(a)} \to \mathbb{P}^a \times \mathbb{P}^b] = 0$$

Since $H_{a,0}^{(a)} = \mathrm{Spec}\,(k)$, pushing-forward to a point yields $\alpha_{a,b} = 0$. $\square$

Let $Y_{N,M}$ be the total space of the bundle $O(1,-1)$ on $\mathbb{P}^N \times \mathbb{P}^M$, and let $Y_{i,j} \to Y_{N,M}$ be the closed immersion induced by a bi-linear embedding

$$\mathbb{P}^i \times \mathbb{P}^j \to \mathbb{P}^n \times \mathbb{P}^m.$$

Proposition 6.7. *If the identity*

$$\sum_{i=0}^{N} \sum_{j=0}^{M} \alpha_{i,j} \cdot [Y_{N-i,M-j} \to Y_{N,M}] = 0 \in \omega_*(Y_{N,M})$$

holds for $\alpha_{i,j} \in \omega_*(k)$, then $\alpha_{i,j} = 0$ for $0 \le i+j \le N$, $0 \le j \le M$.

Proof. The proof is similar to that of Proposition 6.6. Consider all pairs (i,j) satisfying

$$0 \le i+j \le N, \quad 0 \le j \le m$$

for which $\alpha_{i,j} \neq 0$. Of these, take the ones with minimal sum $i+j$, and of these, take the one with minimal j, denote the resulting pair by (a,b). Note that $a \le a+b \le N$.

Let $s_0, \dots, s_N$ be sections of $H^0(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{O}(1,1))$. Since

$$N+1 \ge a+b+1 > \dim_k \mathbb{P}^a \times \mathbb{P}^b,$$

we may choose the s_i so as to have no common zeros. Hence $s_0, \dots, s_N$ define a morphism

$$f : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^N.$$

Let $g : \mathbb{P}^b \to \mathbb{P}^M$ be a linear embedding. We obtain a morphism

$$h = (f, g \circ p_2) : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^N \times \mathbb{P}^M$$

satisfying $h^*(O(1,-1)) \cong O(1,0)$.

A non-zero section $s \in H^0(\mathbb{P}^a \times \mathbb{P}^b, O(1,0))$ with smooth divisor defines a lifting

$$(h,s) : \mathbb{P}^a \times \mathbb{P}^b \to Y_{N,M}$$

of h which is transverse to the zero-section

$$\mathbb{P}^N \times \mathbb{P}^M \to Y_{N,M}.$$

We may therefore apply Proposition 6.2 as explained in the second example of Section 6.3 to give a well-defined $\omega_*(k)$-linear pull-back map

$$(h,s)^* : \omega_*(Y_{N,M}) \to \omega_*(\mathbb{P}^a \times \mathbb{P}^b).$$

We have

$$(h,s)^*([Y_{N-i,M-j} \to Y_{N,M}]) = [H^{(i)}_{a,b-j} \to \mathbb{P}^a \times \mathbb{P}^b].$$

Hence,

$$(h,s)^*([Y_{N-i,M-j} \to Y_{N,M}]) = 0$$

if $i+j > a+b$ or $j > b$ for dimensional reasons. Also,

$$\begin{aligned} (h,s)^*([Y_{N-a,M-b} \to Y_{N,M}]) &= [H^{(a)}_{a,0} \to \mathbb{P}^a \times \mathbb{P}^b] \\ &= [\mathrm{Spec}\,(k) \to \mathbb{P}^a \times \mathbb{P}^b]. \end{aligned}$$

If we pull-back the identity stated in the Proposition by $(h,s)^*$ and then push-forward to the point, we see that $\alpha_{a,b} = 0$. □

7. Admissible towers

7.1. **Overview.** We would like to construct a formal group law over $\omega_*(k)$ using the method of Quillen described in Section 2.6 (v). For Quillen's construction, the classes

$$\Big\{[\mathbb{P}^i \times \mathbb{P}^j \to \mathbb{P}^n \times \mathbb{P}^m]\Big\}_{0\le i\le n,\ 0\le j\le m} \subset \omega_*(\mathbb{P}^n \times \mathbb{P}^m) \tag{7.1}$$

are required to constitute an $\omega_*(k)$-basis. However, Lemma 6.4 only establishes independence. We circumvent the problem by proving a weak version of the generation of $\omega_*(\mathbb{P}^n \times \mathbb{P}^m)$ by the classes (7.1).

Let Y be in $\mathbf{Sm}_k$. An *admissible projective bundle* over Y is a morphism of the form

$$\mathbb{P}(\oplus_i L_i) \to Y$$

where the L_i are line bundles on Y. An *admissible tower* over Y is a morphism $\mathbb{P} \to Y$ which factorizes

$$\mathbb{P} = \mathbb{P}_n \to \mathbb{P}_{n-1} \to \ldots \to \mathbb{P}_1 \to \mathbb{P}_0 = Y$$

as a sequence of admissible projective bundles, i.e., $\mathbb{P}_{i+1} \to \mathbb{P}_i$ is an admissible projective bundle over $\mathbb{P}_i$ for each $i = 0, \ldots, n-1$. We call n the *length* of the admissible tower $\mathbb{P} \to Y$. In particular, the identity map $Y \to Y$ is an admissible tower of length 0.

We proceed to prove that the $\omega_*(k)$-span of classes (7.1) contains the classes of all admissible towers over $\mathbb{P}^n \times \mathbb{P}^m$.

7.2. **Twisting.** Our main decomposition result for admissible towers $[\mathbb{P} \to Y]$ is based on "twisting modifications" in the various steps of the tower.

Take $Y \in \mathbf{Sm}_k$. Let E be a vector bundle on Y, let L be a line bundle on Y, and let H a smooth divisor on Y. Denote $L \otimes O_Y(H)$ by $L(H)$ and let E_H, L_H and $L(H)_H$ denote the restrictions to H. The projections

$$E \oplus L \oplus L(H) \to E \oplus L,$$
$$E \oplus L \oplus L(H) \to E \oplus L(H)$$

give closed immersions

$$\mathbb{P}(E \oplus L) \to \mathbb{P}(E \oplus L \oplus L(H)),$$
$$\mathbb{P}(E \oplus L(H)) \to \mathbb{P}(E \oplus L \oplus L(H)).$$

The projective bundle

$$\mathbb{P}(E_H \oplus L_H \oplus L(H)_H) \to H$$

has a closed immersion over $H \to Y$,

$$\mathbb{P}(E_H \oplus L_H \oplus L(H)_H) \to \mathbb{P}(E \oplus L \oplus L(H)).$$

The subvarieties $\mathbb{P}(E \oplus L)$, $\mathbb{P}(E \oplus L(H))$, and $\mathbb{P}(E_H \oplus L_H \oplus L(H)_H)$ are smooth divisors in $\mathbb{P}(E \oplus L \oplus L(H))$ and the union

$$\mathbb{P}(E \oplus L(H)) + \mathbb{P}(E \oplus L) + \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) \subset \mathbb{P}(E \oplus L \oplus L(H))$$

has strict normal crossing singularities. Indeed, as a Y-scheme the union is locally isomorphic to

$$Y \times H_1 + Y \times H_2 + H \times \mathbb{P}^N \subset Y \times \mathbb{P}^N,$$

where $H_1, H_2 \subset \mathbb{P}^N$ are distinct hyperplanes.

We also have the bundles

$$\mathbb{P}(E_H \oplus L_H) \to H, \quad \mathbb{P}(E_H) \to H,$$

with closed immersions into $\mathbb{P}(E \oplus L \oplus L(H))$ over $H \to Y$. The intersections

$$\mathbb{P}(E \oplus L) \cap \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) = \mathbb{P}(E_H \oplus L_H),$$
$$\mathbb{P}(E \oplus L) \cap \mathbb{P}(E \oplus L(H)) \cap \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) = \mathbb{P}(E_H)$$

are easily calculated.

Lemma 7.1. *The linear equivalence*

$$\mathbb{P}(E \oplus L(H)) \sim \mathbb{P}(E \oplus L) + \mathbb{P}(E_H \oplus L_H \oplus L(H)_H)$$

holds on $\mathbb{P}(E \oplus L \oplus L(H))$.

Proof. Let P denote $\mathbb{P}(E \oplus L \oplus L(H))$, and let $q : P \to Y$ be the structure morphism. As $\mathbb{P}(E \oplus L) \subset P$ is given by the vanishing of the composition

$$q^*(L(H)) \to q^*(E \oplus L \oplus L(H)) \to O_P(1),$$

we find $O_P(\mathbb{P}(E \oplus L)) \cong q^*(L(H))^\vee(1)$. Similarly,

$$O_P(\mathbb{P}(E \oplus L(H))) \cong q^*(L)^\vee(1),$$
$$O_P(\mathbb{P}(E_H \oplus L_H \oplus L(H)_H)) \cong q^*(O_Y(H)).$$

The linear equivalence of the Lemma is now easily obtained. □

Let H be a smooth divisor on $Y \in \mathbf{Sm}_k$. Let

$$\mathbb{P} = \mathbb{P}_n \to \mathbb{P}_{n-1} \to \ldots \to \mathbb{P}_1 \to \mathbb{P}_0 = Y$$

be the factorization of an admissible tower $\mathbb{P} \to Y$ as a sequence of admissible projective bundles. Fix an $i \leq n-1$ and write the bundle $\mathbb{P}_{i+1} \to \mathbb{P}_i$ as

$$\mathbb{P}(\oplus_{j=1}^r L_j) \to \mathbb{P}_i$$

for line bundles L_j on $\mathbb{P}_i$. We write $L_i(H)$ for $L_i(\pi_i^* H)$, where $\pi_i : \mathbb{P}_i \to Y$ is the projection.

Lemma 7.2. *There exists an admissible tower $\mathbb{P}' \to Y$ which factors as*

$$\mathbb{P}' = \mathbb{P}'_n \to \mathbb{P}'_{n-1} \to \ldots \to \mathbb{P}'_{i+1} \to \mathbb{P}_i \to \ldots \mathbb{P}_1 \to \mathbb{P}_0 = Y$$

with $\mathbb{P}'_{i+1} \to \mathbb{P}_i$ given by the bundle

$$\mathbb{P}(\oplus_{j=1}^{r-1} L_j \oplus L_r(H)) \to \mathbb{P}_i,$$

and admissible towers $Q_0 \to H$, $Q_1 \to H$, $Q_2 \to H$, $Q_3 \to H$ satisfying

$$[\mathbb{P}' \to Y] = [\mathbb{P} \to Y] + \sum_{\ell=0}^{3} (-1)^\ell i_{H*}([Q_i \to H]) \ \in \omega_*(Y).$$

Proof. If $X \in \mathbf{Sm}_k$ is irreducible and $E \to X$ is a vector bundle,

$$\mathrm{Pic}(\mathbb{P}(E)) = \mathrm{Pic}(X) \oplus \mathbb{Z} \cdot [O(1)].$$

In particular, if $E \to F$ is a surjection of vector bundles on X, the restriction map

$$\mathrm{Pic}(\mathbb{P}(E)) \to \mathrm{Pic}(\mathbb{P}(F))$$

is surjective. Hence, if $\mathbb{P}_{\mathbb{P}(F)} \to \mathbb{P}(F)$ is an admissible projective bundle, then there is an admissible projective bundle $\mathbb{P}_{\mathbb{P}(E)} \to \mathbb{P}(E)$ and an isomorphism of projective bundles over $\mathbb{P}(F)$

$$\mathbb{P}_{\mathbb{P}(F)} \cong \mathbb{P}(F) \times_{\mathbb{P}(E)} \mathbb{P}_{\mathbb{P}(E)}.$$

By induction on the length of an admissible tower, the same holds for each admissible tower $\mathbb{P} \to \mathbb{P}(F)$.

Let $E = \oplus_{i=1}^{r-1} L_i$, and let $L = L_r$. Consider the admissible projective bundle

$$\hat{\mathbb{P}}_{i+1} = \mathbb{P}(E \oplus L_r \oplus L_r(H)) \to \mathbb{P}_i$$

and the closed immersions

$$\begin{aligned} i_0 &: \mathbb{P}(E \oplus L) \to \hat{\mathbb{P}}_{i+1} \\ i_1 &: \mathbb{P}(E \oplus L(H)) \to \hat{\mathbb{P}}_{i+1}. \end{aligned}$$

By our remarks above, we may extend i_0 to a closed embedding of admissible towers over Y,

$$\tilde{i}_0 : \mathbb{P} \to \hat{\mathbb{P}},$$

where $\hat{\mathbb{P}} \to Y$ admits a factorization

$$\hat{\mathbb{P}} = \hat{\mathbb{P}}_n \to \hat{\mathbb{P}}_{n-1} \to \ldots \to \hat{\mathbb{P}}_{i+1} \to \mathbb{P}_i \to \ldots \to \mathbb{P}_1 \to \mathbb{P}_0 = Y$$

Let $\tilde{i}_1 : \mathbb{P}' \to \hat{\mathbb{P}}$ be the pull-back $\mathbb{P}(E \oplus L(H)) \times_{\mathbb{P}_i} \hat{\mathbb{P}}$, and let $\hat{\mathbb{P}}_H \to H$ be the pull-back of $\hat{\mathbb{P}} \to Y$ via $H \to Y$. By Lemma 7.1, we have the linear equivalence

$$\mathbb{P}' \sim \mathbb{P} + \hat{\mathbb{P}}_H$$

on $\hat{\mathbb{P}}$.

Since $\mathbb{P}(E \oplus L) + \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) + \mathbb{P}(E \oplus L(H))$ is a reduced strict normal crossing divisor on $\mathbb{P}(E \oplus L \oplus L(H))$, the sum $\mathbb{P} + \hat{\mathbb{P}}_H + \mathbb{P}'$ is a reduced strict normal crossing divisor on $\hat{\mathbb{P}}$. Since

$$\begin{aligned} &\mathbb{P}(E \oplus L) \cap \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) = \mathbb{P}(E_H \oplus L_H), \\ &\mathbb{P}(E \oplus L) \cap \mathbb{P}(E_H \oplus L_H \oplus L(H)_H) \cap \mathbb{P}(E \oplus L(H)) = \mathbb{P}(E_H) \end{aligned}$$

are both admissible projective bundles over $\mathbb{P}_i \times_Y H$,

$$D = \mathbb{P} \cap \hat{\mathbb{P}}_H, \quad F = \mathbb{P} \cap \hat{\mathbb{P}}_H \cap \mathbb{P}'$$

are both admissible towers over H. Let

$$\begin{aligned} Q_0 &= \hat{\mathbb{P}}_H \\ Q_1 &= \mathbb{P}_D(O_D(\mathbb{P}) \oplus O_D) \\ Q_2 &= \mathbb{P}_{\mathbb{P}_F(O_F(-H) \oplus O_F(-\mathbb{P}'))}(O \oplus O(1)) \\ Q_3 &= \mathbb{P}_F(O_F(-H) \oplus O_F(-\mathbb{P}') \oplus O_F). \end{aligned}$$

Each $Q_i \to H$ is an admissible tower. Lemma 5.2 completes the proof. □

7.3. **Generation.** Let $\omega_*(k)' \subset \omega_*(k)$ be the subgroup generated by classes of admissible towers over $\mathrm{Spec}\,(k)$. Clearly, $\omega_*(k)'$ is a subring.

Let $H_1, \ldots, H_s$ be divisors on some $Y \in \mathbf{Sm}_k$ for which the associated invertible sheaves $\mathcal{O}_Y(H_i)$ are generated by global sections. Let

$$I = (i_1, \ldots, i_s)$$

be a multi-index with i_r non-negative for all r. Let

$$[H^I \to Y] \in \omega_*(Y)$$

denote the class of the closed immersion $H^I \to Y$, where H^I is the closed subscheme of codimension $\sum_r i_r$ defined by the simultaneous vanishing of i_1 sections of $\mathcal{O}_Y(H_1)$, i_2 sections of $\mathcal{O}_Y(H_2)$, $\ldots$, and i_s sections of $\mathcal{O}_Y(H_2)$. By definition,

$$[H^{(0,\ldots,0)} \to Y] = [Y \to Y].$$

For a general choice of sections, H^I is smooth. By naive cobordisms, $[H^I \to Y]$ is independent of the choice of sections.

The subvarieties H^I may not be irreducible; let $H^I_1, \ldots H^I_{n_I}$ be the irreducible components of H^I.

Lemma 7.3. *If the restrictions of the invertible sheaves $\mathcal{O}_Y(H_i)$ generate $\mathrm{Pic}(H^I_j)$ for every H^I_j, then the classes of admissible towers over Y lie in the $\omega_*(k)'$-span of $[H^I_j \to Y]$ in $\omega_*(Y)$.*

Proof. Given an admissible tower $\mathbb{P} \to Y$, we must find an identity

$$[\mathbb{P} \to Y] = \sum_{I,j} a_{I,j} \cdot [H^I_j \to Y] \ \in \omega_*(Y)$$

with $a_{I,j} \in \omega_*(k)'$.

We may assume Y is irreducible and the divisors H_i are smooth. If Y has dimension 0, then every line bundle on Y is trivial. By induction on the length of the tower, every admissible tower $\mathbb{P} \to Y$ is the pull-back of an admissible tower $\mathbb{P}' \to \mathrm{Spec}\,(k)$ by the structure morphism $Y \to \mathrm{Spec}\,(k)$. The result is proven in case $\dim_k Y = 0$.

We proceed by induction on $\dim_k Y$. Let $\omega_*(Y)'$ be the subgroup generated by the push-forward to Y of classes of the form $[\mathbb{P}' \to H^I_j]$, where $\mathbb{P}' \to H^I_j$ is an admissible tower and $I \neq (0, \ldots, 0)$. Since such H^I_j satisfy the hypotheses of the Lemma and have dimension strictly less than Y, the push-forwards to Y of the classes $[\mathbb{P}' \to H^I_j]$ lie in the $\omega_*(k)'$-span of the classes $[H^I_j \to Y]$.

Let $\mathbb{P} \to Y$ be an admissible tower of length n which factors as

$$\mathbb{P} \to Q \to Y$$

where $\mathbb{P} \to Q$ is an admissible tower of length $n-i$ and $Q \to Y$ is an admissible tower of length $i < n$ isomorphic to a pull-back

$$Q \cong Q_0 \times_k Y \to Y$$

of an admissible tower $Q_0 \to \mathrm{Spec}\,(k)$ of length i. By twisting, we will prove the condition

$$[\mathbb{P} \to Y] - [\mathbb{P}' \to Y] \in \omega_*(Y)' \tag{7.2}$$

is satisfied for an admissible tower $\mathbb{P}' \to Y$ of length n which admits a factorization $\mathbb{P}' \to Q' \to Y$ as above where $Q' \to Y$ is an admissible tower of length $i+1$ of the form

$$Q' \cong Q'_0 \times_k Y \to Y$$

for an admissible tower $Q'_0 \to \mathrm{Spec}\,(k)$ of length $i+1$.

The construction of $\mathbb{P}' \to Y$ satisfying (7.2) follows directly from Lemma 7.2. Indeed, suppose

$$\mathbb{P}_{i+1} \to \mathbb{P}_i = Q$$

is of the form $\mathbb{P}_Q(\oplus_i L_i) \to Q$. Since $Q = Q_0 \times_k Y$, we have

$$\mathrm{Pic}(Q) = \mathrm{Pic}(Q_0) \oplus \mathrm{Pic}(Y).$$

We can write each L_i as

$$L_i \cong p_1^* L_i^0 \otimes p_2^* M_i$$

for suitable line bundles L_i^0 on Q_0, and M_i on Y. By Lemma 7.2 and our induction hypothesis, the class $[\mathbb{P} \to Y]$ is equivalent modulo $\omega_*(Y)'$ to a class $[\tilde{\mathbb{P}} \to Y]$, where $\tilde{\mathbb{P}} \to Y$ is an admissible tower of length n which factors as

$$\tilde{\mathbb{P}} \to \tilde{\mathbb{P}}_{i+1} \to Q \to Y$$

and where $\tilde{\mathbb{P}}_{i+1} = \mathbb{P}(\oplus_{i \neq j} L_i \oplus L_j(H_\ell))$ for any choice of j and ℓ we like. Since the H_ℓ generate $\mathrm{Pic}(Y)$, there are non-negative integers m_{ij} such that

$$M_i(\sum_j m_{ij} H_j) \cong M_{i'}(\sum_j m_{i'j} H_j)$$

for all i, i'. Let $L = M_1(\sum_j m_{1j} H_j)$. After several such applications of Lemma 7.2, we may replace $\mathbb{P}$ with an admissible tower

$$\mathbb{P}' \to \mathbb{P}'_{i+1} \to Q \to Y,$$

where

$$\mathbb{P}'_{i+1} \cong \mathbb{P}(\oplus_i\ p_1^* L_i^0 \otimes p_2^* M_i(\sum_j m_{ij} H_j)) \cong \mathbb{P}(\oplus_i\ p_1^* L_i^0 \otimes p_2^* L).$$

But as $\mathbb{P}(E) \cong \mathbb{P}(E \otimes M)$ for $E \to T$ a vector bundle and $M \to T$ a line bundle, we have

$$\mathbb{P}'_{i+1} \cong \mathbb{P}(\oplus_i\, p_1^* L_i^0).$$

Thus, $\mathbb{P}'_{i+1} \to Q \to Y$ is the pullback to Y of an admissible tower $Q'_0 \to Q_0 \to \mathrm{Spec}\,(k)$, and we obtain condition (7.2).

Repeated application of (7.2) yields the relation

$$[\mathbb{P} \to Y] - [Q \to Y] \ \in \omega_*(Y)'$$

where

$$Q \cong Y \times_k Q_0 \to Y$$

for an admissible tower $Q_0 \to \mathrm{Spec}\,(k)$ of length n. □

Recall the "multi-linear" classes (6.4) $M_{j_1,\ldots,j_m} \in \omega_*(\prod_{i=1}^m \mathbb{P}^{N_i})$.

Corollary 7.4. *Let $\mathbb{P} \to \prod_{i=1}^m \mathbb{P}^{N_i}$ be an admissible tower. Then,*

$$[\mathbb{P} \to \prod_{i=1}^m \mathbb{P}^{N_i}] = \sum_{J=(j_1,\ldots,j_m)} a_J \cdot M_J \ \in \omega_*(\prod_{i=1}^m \mathbb{P}^{N_i})$$

for unique elements $a_J \in \omega_*(k)$. The a_J are in fact in $\omega_*(k)' \subset \omega_*(k)$.

Proof. For existence, we apply Lemma 7.3 with $Y = \prod_{i=1}^m \mathbb{P}^{N_i}$ and the divisors H_i defined by the pull-backs of hyperplanes in $\mathbb{P}^{N_i}$ via the projections $Y \to \mathbb{P}^{N_i}$. Uniqueness follows from Proposition 6.4. □

Corollary 7.5. *Let $\mathbb{P} \to H_{n,m}$ be an admissible tower. Then,*

$$[\mathbb{P} \to H_{n,m}] = \sum_{i,j} a_{i,j} \cdot [H_{n-i,m-j} \to H_{n,m}]$$

for elements $a_{i,j} \in \omega_*(k)'$.

Here, $H_{n-i,m-j} \to H_{n,m}$ is induced by the bi-linear embedding

$$\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m.$$

The sum in Corollary 7.5 is over

$$0 \leq i \leq n, \quad 0 \leq j \leq m, \quad i + j < n + m$$

for dimension reasons.

Proof. We apply Lemma 7.3 with $Y = H_{n,m}$ and divisors $H_1 = H_{n-1,m}$, $H_2 = H_{n,m-1}$. If $n \geq m$, the projection

$$p_2 : H_{n,m} \to \mathbb{P}^m$$

expresses $H_{n,m}$ as a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^m$. Hence, H_1 and H_2 generate $\mathrm{Pic}(H_{n,m})$. Since

$$H_1^{(i)} \cdot H_2^{(j)} = H_{n-i,m-j},$$

the hypotheses of Lemma 7.3 are satisfied and yield the desired result. □

Proposition 7.6. *Let $\mathbb{P} \to H_{n,m}$ be an admissible tower. Then,*

$$i_{H_{n,m}*}([\mathbb{P} \to H_{n,m}]) = \sum_{(i,j)\neq(0,0)} a_{i,j} \cdot [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m]$$

for unique elements $a_{i,j} \in \omega_*(k)$, $0 \leq i \leq n$, $0 \leq j \leq m$.

Proof. If $m = 0$, then $H_{n,m}$ is a hyperplane in $\mathbb{P}^n$, and the result follows from Corollary 7.4. The same argument is valid for $n = 0$.

We proceed by induction on (n, m). Only existence is required since uniqueness follows from Proposition 6.4; we will show in fact that the $a_{i,j}$ are in $\omega_*(k)' \subset \omega_*(k)$. By Corollary 7.5, we need only construct relations of the form

$$i_{H_{n,m}*}(a \cdot [H_{n,m} \to H_{n,m}]) = \sum_{(i,j)\neq(0,0)} a_{i,j} \cdot [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m].$$

with $a_{i,j} \in \omega_*(k)'$, for every $a \in \omega_*(k)'$. Since

$$i_{H_{n,m}*}(a \cdot [H_{n,m} \to H_{n,m}]) = a \cdot i_{H_{n,m}*}([H_{n,m} \to H_{n,m}]),$$

the case $a = 1$ suffices.

We have the linear equivalence on $\mathbb{P}^n \times \mathbb{P}^m$,

$$H_{n,m} \sim \mathbb{P}^{n-1} \times \mathbb{P}^m + \mathbb{P}^n \times \mathbb{P}^{m-1}.$$

By the extended double point relation of Lemma 5.2, there are admissible towers $\mathbb{P}_1 \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, $\mathbb{P}_2 \to H_{n-1,m-1}$ and $\mathbb{P}_3 \to H_{n-1,m-1}$ for which

$$\begin{aligned}
[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] \quad = \quad & [\mathbb{P}^{n-1} \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m] \\
& +[\mathbb{P}^n \times \mathbb{P}^{m-1} \to \mathbb{P}^n \times \mathbb{P}^m] \\
& -[\mathbb{P}_1 \to \mathbb{P}^n \times \mathbb{P}^m] \\
& +[\mathbb{P}_2 \to \mathbb{P}^n \times \mathbb{P}^m] \\
& -[\mathbb{P}_3 \to \mathbb{P}^n \times \mathbb{P}^m].
\end{aligned}$$

By induction, the classes $[\mathbb{P}_2 \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}]$ and $[\mathbb{P}_3 \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}]$ are expressible as

$$[\mathbb{P}_\ell \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}] = \sum_{i,j} a^\ell_{i,j} \cdot [\mathbb{P}^{n-i-1} \times \mathbb{P}^{m-j-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}],$$

for $a^\ell_{i,j} \in \omega_*(k)'$ for $\ell = 2, 3$. By Corollary 7.4, a similar expression is obtained in case $\ell = 1$. □

8. The formal group law over $\omega_*(k)$

We use the classical method of Quillen to construct a formal group law over $\omega_*(k)$. Proposition 7.6 replaces the projective bundle formula.

By Proposition 7.6 applied to the admissible tower $\mathrm{Id} : H_{n,m} \to H_{n,m}$, there are unique elements $a_{i,j}^{n,m} \in \omega_{i+j-1}(k)$ for which the identity

$$(8.1) \quad [H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = \sum_{(i,j)\neq(0,0)} a_{i,j}^{n,m} \cdot [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m]$$

holds in $\omega_*(\mathbb{P}^n \times \mathbb{P}^m)$. For convenience, we set $a_{0,0}^{n,m} = 0$.

Lemma 8.1. *If $N \geq n$, $M \geq m$, then*

$$a_{i,j}^{N,M} = a_{i,j}^{n,m}$$

for $0 \leq i \leq n$, $0 \leq j \leq m$.

Proof. We pull back the relation (8.1) for N, M by a bi-linear embedding

$$i : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N \times \mathbb{P}^M;$$

see Section 6.3 for the pull-back construction. We find

$$i^*([H_{N,M} \to \mathbb{P}^N \times \mathbb{P}^M]) = [H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m]$$
$$i^*([\mathbb{P}^{N-i} \times \mathbb{P}^{M-j} \to \mathbb{P}^N \times \mathbb{P}^M]) = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m]$$

for $0 \leq i \leq n$ and $0 \leq j \leq m$. Since i^* is $\omega_*(k)$-linear, the result follows from the uniqueness of the $a_{i,j}^{n,m}$. □

By Lemma 8.1, we may define $a_{i,j} \in \omega_*(k)$ by

$$a_{i,j} = \lim_{N\to\infty, M\to\infty} a_{i,j}^{N,M}.$$

Following the convention

$$[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m] = 0$$

if $i > n$ or if $j > m$, we write $a_{i,j}$ for $a_{i,j}^{n,m}$ in relation (8.1).

Taking $n = 0$ and noting $H_{0,m} = \mathbb{P}^{m-1}$ linearly embeds in $\mathbb{P}^m$, we find

$$a_{0,1} = 1, \quad a_{0,j>1} = 0.$$

As the exchange of factors $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^n$ sends $H_{n,m}$ to $H_{m,n}$, we obtain the symmetry

$$a_{i,j} = a_{j,i}.$$

Let $F_\omega(u, v) \in \omega_*(k)[[u, v]]$ be the power series

$$F_\omega(u, v) = u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j.$$

Proposition 8.2. *Let L_1 and L_2 be globally generated line bundles on $X \in \mathbf{Sch}_k$. Then, $L_1 \otimes L_2$ is globally generated and*

$$\tilde{c}_1(L_1 \otimes L_2) = F_\omega(\tilde{c}_1(L_1), \tilde{c}_1(L_2)).$$

Proof. The Proposition follows from the equation

$$\tilde{c}_1(L_1 \otimes L_2)(1_Y) = F_\omega(\tilde{c}_1(L_1), \tilde{c}_1(L_2))(1_Y). \tag{8.2}$$

for all globally generated L_1, L_2 on all $Y \in \mathbf{Sm}_k$. Indeed, if $[f : Y \to X] \in \mathcal{M}(X)^+$, then

$$f_*(1_Y) = [f : Y \to X] \in \omega_*(X).$$

By (A3), we have

$$\tilde{c}_1(L)([f : Y \to X]) = \tilde{c}_1(f_*(1_Y)) = f_*(\tilde{c}_1(f^*L)(1_Y))$$

for all globally generated L on X, which verifies the claim.

Since L_1 and L_2 are globally generated, we have morphisms

$$f_i : Y \to \mathbb{P}^{n_i}$$

with $L_i \cong f_i^*(O(1))$ for $i = 1, 2$. Thus,

$$L_1 \otimes L_2 \cong (f_1 \times f_2)^*(O(1,1)).$$

By the functoriality result Lemma 6.3, we need only prove (8.2) in case

$$Y = \mathbb{P}^n \times \mathbb{P}^m,\ L_1 = O(1,0),\ L_2 = O(0,1),\ L_1 \otimes L_2 = O(1,1).$$

Since

$$\tilde{c}_1(O(1,1))(1_{\mathbb{P}^n \times \mathbb{P}^m}) = [H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m]$$
$$\tilde{c}_1(O(1,0))^i \circ \tilde{c}_1(O(0,1))^j(1_{\mathbb{P}^n \times \mathbb{P}^m}) = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m],$$

the defining relation (8.1) for the $a_{i,j}$ becomes

$$\tilde{c}_1(O(1,1))(1_{\mathbb{P}^n \times \mathbb{P}^m}) = F_\omega(\tilde{c}_1(O(1,0), \tilde{c}_1(O(0,1))(1_{\mathbb{P}^n \times \mathbb{P}^m}),$$

as desired. □

Proposition 8.3. *$F_\omega(u, v)$ defines a formal group law over $\omega_*(k)$.*

Proof. Of the axioms for formal group laws, the first two have already been established:

(i) $F(u, 0) = F(0, u) = u$,
(ii) $F(u, v) = F(v, u)$.

The last axiom

(iii) $F(F(u, v), w) = F(u, F(v, w))$.

will now be proven.

Let $G_1(u, v, w) = F(F(u, v), w)$ and $G_2(u, v, w) = F(u, F(v, w))$. For $\ell = 1, 2$, write

$$G_\ell(u, v, w) = \sum_{i,j,k} a^\ell_{i,j,k} u^i v^j w^k.$$

For globally generated line bundles L_1, L_2, L_3 on $X \in \mathbf{Sch}_k$,

$$G_1(\tilde{c}_1(L_1), \tilde{c}_1(L_2), \tilde{c}_1(L_3)) = F(\tilde{c}_1(L_1 \otimes L_2), \tilde{c}_1(L_3)) = \tilde{c}_1(L_1 \otimes L_2 \otimes L_3)$$

by Proposition 8.2. A similar equation holds for G_2. Thus

$$G_1(\tilde{c}_1(L_1), \tilde{c}_1(L_2), \tilde{c}_1(L_3)) = G_2(\tilde{c}_1(L_1), \tilde{c}_1(L_2), \tilde{c}_1(L_3)) \tag{8.3}$$

as operators on $\omega_*(X)$.

Specializing to $X = \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^r$, we find

$$G_\ell(\tilde{c}_1(O(1, 0, 0), \tilde{c}_1(O(0, 1, 0)), \tilde{c}_1(O(0, 0, 1))(1_X)$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{r} a^\ell_{i,j,k} \cdot [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \times \mathbb{P}^{r-k} \to \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^r]$$

for $\ell = 1, 2$. By Proposition 6.4 and (8.3), we have

$$a^1_{i,j,k} = a^2_{i,j,k}$$

for $0 \leq i \leq n$, $0 \leq j \leq m$, $0 \leq k \leq r$. As n, m and r were arbitrary, the proof is complete. □

9. Chern classes II

9.1. **Definition.** Because $F_\omega(u, v)$ is a formal group law, there exists an inverse power series $\chi_\omega(u) \in \omega_*(k)[[u]]$ characterized by the identity

$$F_\omega(u, \chi_\omega(u)) = 0.$$

We let $F^-_\omega(u, v)$ be the difference in our group law,

$$F^-_\omega(u, v) = F_\omega(u, \chi_\omega(v)).$$

Using $F^-_\omega(u, v)$, we can extend the definition of $\tilde{c}_1(L)$ given in Section 4 for globally generated L to arbitrary line bundles.

Lemma 9.1. *Let L, M, N be line bundles on $Y \in \mathbf{Sm}_k$ where*

$$L,\ M,\ L \otimes N,\ M \otimes N$$

are globally generated. Then,

$$F^-_\omega(\tilde{c}_1(L), \tilde{c}_1(M)) = F^-_\omega(\tilde{c}_1(L \otimes N), \tilde{c}_1(M \otimes N))$$

as operators on $\omega_*(Y)$.

Proof. We first assume N is globally generated. Then

$$\begin{aligned}\tilde{c}_1(L \otimes N) &= F_\omega(\tilde{c}_1(L), \tilde{c}_1(N)) \\ \tilde{c}_1(M \otimes N) &= F_\omega(\tilde{c}_1(M), \tilde{c}_1(N))\end{aligned}$$

by Proposition 8.2. The result then follows from the power series identity

$$F_\omega^-(F_\omega(u, w), F_\omega(v, w)) = F_\omega^-(u, v).$$

In general, since Y is quasi-projective, there is a very ample line bundle N' such that $N'' = N' \otimes N^{-1}$ is very ample. Then

$$\begin{aligned}F_\omega^-(\tilde{c}_1(L), \tilde{c}_1(M)) &= F_\omega^-(\tilde{c}_1(L \otimes N'), \tilde{c}_1(M \otimes N')) \\ &= F_\omega^-(\tilde{c}_1(L \otimes N \otimes N''), \tilde{c}_1(M \otimes N \otimes N'')) \\ &= F_\omega^-(\tilde{c}_1(L \otimes N), \tilde{c}_1(M \otimes N)),\end{aligned}$$

completing the proof. $\square$

Let L be an arbitrary line bundle on $X \in \mathbf{Sch}_k$. Define the operator

$$\tilde{c}_1(L) : \mathcal{M}_*(X)^+ \to \omega_{*-1}(X)$$

by the following construction. Let $Y \in \mathbf{Sm}_k$ be irreducible. Let

$$[f : Y \to X] \in \mathcal{M}(X)^+. \tag{9.1}$$

Let M be a very ample line bundle on Y for which $f^*(L) \otimes M$ is also very ample. Define

$$\tilde{c}_1(L)([f : Y \to X]) = f_*\Big(F_\omega^-\big(\tilde{c}_1(f^*(L) \otimes M), \tilde{c}_1(M)\big)(1_Y)\Big).$$

By Lemma 9.1, $\tilde{c}_1(L)([f])$ is independent of the choice of M. Since $\mathcal{M}_*(X)^+$ is the free abelian group with generators (9.1), $\tilde{c}_1(L)$ is defined on $\mathcal{M}_*(X)^+$. Also, by Lemma 9.1 and Definition 2.1(A3), we see $\tilde{c}_1(L)$ agrees with the definition given in Section 4 in case L is globally generated.

Let $X \in \mathbf{Sch}_k$, and let $\pi : Y \to X \times \mathbb{P}^1$ be a double point degeneration over $0 \in \mathbb{P}^1$. Let

$$Y_0 = A \cup B \to X$$

be the fiber over 0, and let $Y_\infty \to X$ be a regular fiber. The associated double point relation is

$$[Y_\infty \to X] = [A \to X] + [B \to X] - [\mathbb{P}(\pi) \to X] \ \in \omega_*(X).$$

Lemma 9.2. *Let L be a line bundle on X. Then,*

$$\tilde{c}_1(L)([Y_\infty \to X]) = \tilde{c}_1(L)\Big([A \to X] + [B \to X] - [\mathbb{P}(\pi) \to X]\Big).$$

Proof. The various classes $\tilde{c}_1(L)([W \to X])$ are defined by operating on $\omega_*(W)$ and then pushing forward to X. Hence, we may replace X with Y, L with $\pi^* p_1^* L$, and π with

$$(\mathrm{Id}_Y, p_2 \circ \pi) : Y \to Y \times \mathbb{P}^1.$$

Since $Y \in \mathbf{Sm}_k$, we may choose a very ample line bundle M for which $L \otimes M$ is also very ample. Then, by the definition of $\tilde{c}_1(L)$ given above, we have

$$\tilde{c}_1(L) = F_\omega^-(\tilde{c}_1(L \otimes M), \tilde{c}_1(M)),$$

as a map from $\mathcal{M}_*(Y)^+$ to $\omega_{*-1}(Y)$. The result follows from Lemmas 4.2 and 4.3. □

By Lemma 9.2, the operator $\tilde{c}_1(L) : \mathcal{M}_*(X)^+ \to \omega_{*-1}(X)$ descends to

$$\tilde{c}_1(L) : \omega_*(X) \to \omega_{*-1}(X).$$

Hence, we have constructed first Chern class operators on ω_* for arbitrary line bundles.

Lemma 9.3. *Let $Y \in \mathbf{Sm}_k$, and let*

$$L_1, \ldots, L_{r > \dim_k Y} \to Y$$

be line bundles. Then,

$$\prod_{i=1}^{r} \tilde{c}_1(L_i) = 0$$

as an operator on $\omega_*(Y)$.

Proof. Since Y quasi-projective, $\tilde{c}_1(L_i) = F_\omega^-(\tilde{c}_1(L_i \otimes M), \tilde{c}_1(M))$ for any choice of very ample line bundle M on Y for which $L_i \otimes M$ is very ample. Since

$$F_\omega^-(u, v) = u - v \mod (u, v)^2,$$

Lemma 4.3 implies the result. □

Axioms (A3), (A4), (A5) and (A8) for globally generated L immediately imply these axioms for arbitrary L. Similarly, the functoriality of Lemma 6.3 extends to arbitrary line bundles L.

Proposition 9.4. *Let L and M be line bundles on $X \in \mathbf{Sch}_k$. Then,*

$$\tilde{c}_1(L \otimes M) = F_\omega(\tilde{c}_1(L), \tilde{c}_1(M)).$$

Proof. By the definition of Chern classes and Lemma 9.3, the operator

$$F_\omega(\tilde{c}_1(L), \tilde{c}_1(M)) : \omega_*(X) \to \omega_{*-1}(X)$$

is well-defined.

Since $\omega_*(X)$ is generated by the classes $f_*(1_Y)$ for

$$[f : Y \to X] \in \mathcal{M}(X)^+,$$

property (A3) can be used to reduce to the case of $X \in \mathbf{Sm}_k$.

Take very ample line bundles N_1, N_2 on X with $L \otimes N_1$ and $M \otimes N_2$ very ample. Then,

$$L \otimes M \otimes N_1 \otimes N_2, \quad N_1 \otimes N_2$$

are also very ample. The Proposition follows from Proposition 8.2 and the power series identity

$$F_\omega(F_\omega^-(u_1, v_1), F_\omega^-(u_2, v_2)) = F_\omega^-(F_\omega(u_1, u_2), F_\omega(v_1, v_2)),$$

after taking

$$u_1 = \tilde{c}_1(L \otimes N_1), \ v_1 = \tilde{c}_1(N_1),$$
$$u_2 = \tilde{c}_1(M \otimes N_2), \ v_2 = \tilde{c}_1(N_2).$$

□

9.2. **Proof of Theorem 2.** Double point cobordism theory ω_* was shown in Section 3.2 to define a Borel-Moore functor with product: structures (D1), (D2), and (D4) satisfying axioms (A1), (A2), (A6), and (A7).

We have added first Chern classes (D3) and verified axioms (A3), (A4), (A5), and (A8). Hence, ω_* is oriented.

The formal group law defined by Proposition 8.3 yields a canonical ring homomorphism

$$\mathbb{L}_* \to \omega_*(k).$$

Hence, ω_* is $\mathbb{L}_*$-functor.

In order for ω_* to be an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type, the axioms of Section 2.4 must be satisfied. Axiom (Dim) is Lemma 9.3, and axiom (FGL) is Proposition 9.4. The proof of Theorem 2 will be completed by establishing the remaining axiom (Sect).

10. Axiom (Sect)

10.1. **The difference series.** Since the Chern class operator $\tilde{c}_1(L)$ for a general line bundle L is defined using the difference F_ω^- in our formal group law, we will require a universal construction of F_ω^- along the lines of our construction of F_ω.

The variety $Y_{n,m}$, defined in Section 6.4, is the total space of the line bundle $O(1, -1)$ on $\mathbb{P}^n \times \mathbb{P}^m$ with projection π and zero-section s,

$$\pi : Y_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m, \quad s : \mathbb{P}^n \times \mathbb{P}^m \to Y_{n,m}.$$

Let $S_{n,m} \subset Y_{n,m}$ be the image of the zero section.

For $0 \leq i \leq n$ and $0, \leq j \leq m$, a closed immersion

$$Y_{i,j} \to Y_{n,m}$$

is induced by a choice of bi-linear embedding $\mathbb{P}^i \times \mathbb{P}^j \to \mathbb{P}^n \times \mathbb{P}^m$.

Lemma 10.1. *For $n, m \geq 0$,*

$$(10.1) \quad [S_{n,m} \to Y_{n,m}] = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j}^{n,m} \cdot [Y_{n-i,m-j} \to Y_{n,m}] \ \in \omega_*(Y_{n,m})$$

for $b_{i,j}^{n,m} \in \omega_{i+j-1}(k)$.

Proof. If $n = m = 0$, then $Y_{n,m} = \mathbb{A}^1$ with $S_{n,m} \to Y_{n,m}$ given by the inclusion of 0. Clearly $[0 \to \mathbb{A}^1] = 0$ in $\omega_0(\mathbb{A}^1)$, whence the result.[4]

We proceed by induction on (n, m). We give the argument for the induction from $(n, m-1)$ to (n, m). The induction from $(n-1, m)$ to (n, m) is similar and is left to the reader.

We have the linear equivalence

$$S_{n,m} + Y_{n,m-1} \sim Y_{n-1,m}$$

on $Y_{n,m}$. Clearly $S_{n,m} + Y_{n,m-1} + Y_{n-1,m}$ is a reduced strict normal crossing divisor on $Y_{n,m}$. By Lemma 5.2, we obtain the relation

$$\begin{aligned}[S_{n,m} \to Y_{n,m}] = [Y_{n-1,m} &\to Y_{n,m}] - [Y_{n,m-1} \to Y_{n,m}] \\ &+ [\mathbb{P}_1 \to Y_{n,m}] - [\mathbb{P}_2 \to Y_{n,m}] + [\mathbb{P}_3 \to Y_{n,m}]\end{aligned}$$

where $\mathbb{P}_1 \to S_{n,m-1}$ is an admissible $\mathbb{P}^1$-bundle, $\mathbb{P}_2 \to S_{n-1,m-1}$ is an admissible tower, and $\mathbb{P}_3 \to S_{n-1,m-1}$ is an admissible $\mathbb{P}^2$-bundle.

We apply Lemma 7.3 to $\mathbb{P}_1 \to S_{n,m-1}$ with generators $S_{n-1,m-1}$ and $S_{n,m-2}$ for $\mathrm{Pic}(S_{n,m-1})$. Similarly, we apply Lemma 7.3 to $\mathbb{P}_2 \to S_{n-1,m-1}$ and $\mathbb{P}_3 \to S_{n-1,m-1}$. We find

$$\begin{aligned}[S_{n,m} \to Y_{n,m}] = [Y_{n-1,m} &\to Y_{n,m}] - [Y_{n,m-1} \to Y_{n,m}] \\ &+ \sum_{i=0}^{n} \sum_{j=1}^{m} c_{i,j} \cdot [S_{n-i,m-j} \to Y_{n,m}]\end{aligned}$$

with $c_{i,j} \in \omega_*(k)$.

[4]Consider the morphism $\pi : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ determined by (Id, i) where

$$i : \mathbb{A}^1 \to \mathbb{P}^1$$

is the inclusion obtained by omitting $0 \in \mathbb{P}^1$. The projective morphism π is a double point degeneration over $0 \in \mathbb{P}^1$,

$$\pi^{-1}(0) = \emptyset \cup \emptyset.$$

The associated double point cobordism shows $[\mathrm{Spec}\,(k) \to \mathbb{A}^1] = 0$ in $\omega_*(\mathbb{A}^1)$ for every closed point.

Since $S_{n-i,m-j} \to Y_{n,m}$ factors through $S_{n-i,m-j} \to Y_{n-i,m-j}$, the induction hypothesis finishes the proof. $\square$

For $0 \leq i+j \leq n$, $0 \leq j \leq m$, the elements $b_{i,j}^{n,m}$ on the right side of (10.1) are uniquely determined by Proposition 6.7.

Lemma 10.2. *If $N \geq n$, $M \geq m$, then*

$$b_{i,j}^{n,m} = b_{i,j}^{N,M}$$

for $0 \leq i+j \leq n$, $0 \leq j \leq m$.

Proof. The bi-linear embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N \times \mathbb{P}^M$ induces a closed embedding

$$i : Y_{n,m} \to Y_{N,M}$$

which satisfies the conditions of the second example of Section 6.3. Thus, we have a well-defined $\omega_*(k)$-linear pull-back

$$i^* : \omega_*(Y_{N,M}) \to \omega_{*-d}(Y_{n,m})$$

with $d = N - n + M - m$. Clearly

$$i^*([S_{N,M} \to Y_{N,M}]) = [S_{n,m} \to Y_{n,m}],$$
$$i^*([Y_{N-i,M-j} \to Y_{N,M}]) = [Y_{n-i,m-j} \to Y_{n,m}],$$

so the uniqueness statement implies the result. $\square$

By Lemma 10.2, we may define $b_{i,j} \in \omega_*(k)$ by

$$b_{i,j} = \lim_{n,m\to\infty} b_{i,j}^{n,m}.$$

By the proof of Lemma 10.1, $b_{0,0} = 0$, $b_{1,0} = 1$, and $b_{0,1} = -1$.

Lemma 10.3. $F_\omega^-(u,v) = \sum_{i,j} b_{i,j} u^i v^j$.

Proof. Let $n, m \geq 0$, and let $N = n + 2m$, $M = m$. Let $a = n + m$, $b = m$. We will use the morphism

$$(h,s) : \mathbb{P}^a \times \mathbb{P}^a \to Y_{N,M}$$

constructed in the proof of Proposition 6.7, using a suitably general choice of sections $s_0, \ldots, s_N \in H^0(\mathbb{P}^{n+m} \times \mathbb{P}^m, \mathcal{O}(1,1))$, $s \in H^0(\mathbb{P}^{n+m} \times \mathbb{P}^m, \mathcal{O}(1,0))$.

Since s is a non-zero section of $H^0(\mathbb{P}^{n+m} \times \mathbb{P}^m, \mathcal{O}(1,0))$, the pull-back by (h,s) of the inclusion $S_{N,M} \hookrightarrow Y_{N,M}$ is a bi-linear embedding

$$(h,s)^{-1}(S_{N,M}) = \mathbb{P}^{n+m-1} \times \mathbb{P}^m \to \mathbb{P}^{n+m} \times \mathbb{P}^m;$$

we have already seen in the proof of Proposition 6.7 that the pull-back by (h,s) of the inclusion $Y_{N-i,M-j} \hookrightarrow Y_{N,M}$ is the map

$$(h,s)^{-1}(Y_{N-i,M-j}) = H_{n+m,m-j}^{(i)} \to \mathbb{P}^{n+m} \times \mathbb{P}^m.$$

The relation (10.1) for (N, M) therefore pulls back under (h, s) to

$$[\mathbb{P}^{n+m-1} \times \mathbb{P}^m \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = \sum_{i,j} b_{i,j}^{N,M} \cdot [H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m].$$

By Lemma 10.2 , we have $b_{i,j}^{N,M} = b_{i,j}$ for

$$0 \leq i + j \leq N = n + 2m, \quad 0 \leq j \leq M = m.$$

Since $H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m$ has codimension $i + j$ and is empty if $j > m$,

$$[\mathbb{P}^{n+m-1} \times \mathbb{P}^m] = \sum_{i=0}^{n+m} \sum_{j=0}^{m} b_{i,j} \cdot [H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m].$$

Consider the formal group law determined by ω_*. The difference F^-_ω admits a power series expansion,

$$F^-_\omega(u, v) = \sum_{i,j} \tilde{b}_{i,j} u^i v^j,$$

where $\tilde{b}_{i,j} \in \omega_*(k)$. Certainly,

$$\tilde{c}_1(O(1, 0))(1_{\mathbb{P}^{n+m} \times \mathbb{P}^m}) = F^-_\omega(\tilde{c}_1(O(1, 1)), \tilde{c}_1(O(0, 1))(1_{\mathbb{P}^{n+m} \times \mathbb{P}^m}).$$

Since

$$[\mathbb{P}^{n+m-1} \times \mathbb{P}^m] = \tilde{c}_1(O(1, 0))(1_{\mathbb{P}^{n+m} \times \mathbb{P}^m}),$$
$$[H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = \tilde{c}_1(O(1, 1))^i \tilde{c}_1(O(0, 1))^j (1_{\mathbb{P}^{n+m} \times \mathbb{P}^m}),$$

we find

$$[\mathbb{P}^{n+m-1} \times \mathbb{P}^m] = \sum_{i,j} \tilde{b}_{i,j} \cdot [H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m].$$

Therefore,

$$\sum_{i,j} (\tilde{b}_{i,j} - b_{i,j}) \cdot [H^{(i)}_{n+m,m-j} \to \mathbb{P}^{n+m} \times \mathbb{P}^m] = 0.$$

By Proposition 6.6, $b_{i,j} = \tilde{b}_{i,j}$ for $0 \leq i + j \leq n + m$, $0 \leq j \leq m$. As n and m were arbitrary, the proof is complete. $\square$

10.2. **Proof of Theorem 2.** We now complete the last step in the proof of Theorem 2.

Proposition 10.4. *Double point cobordism ω_* satisfies axiom (Sect).*

Proof. Let $Y \in \mathbf{Sm}_k$ be of dimension d. Let L be a line bundle on Y with transverse section $s \in H^0(Y, L)$. Let $D \subset Y$ be the smooth divisor associated to s.

Let M be a very ample line bundle on Y for which $L \otimes M$ is also very ample. Let

$$f : Y \to \mathbb{P}^n, \quad g : Y \to \mathbb{P}^m$$

be closed embeddings satisfying

$$L \otimes M \cong f^*O(1), \quad M \cong g^*O(1).$$

Certainly, $d \leq n$, $d \leq m$.

Let $h = (f, g) : Y \to \mathbb{P}^n \times \mathbb{P}^m$. The section s defines a lifting

$$(h, s) : Y \to Y_{n,m}$$

which satisfies the conditions of the second example of Section 6.3. We obtain a well-defined $\omega_*(k)$-linear pull-back

$$(h, s)^* : \omega_*(Y_{n,m}) \to \omega_{*-n-m-1+d}(Y).$$

By construction, $(h, s)^*([S_{n,m} \to Y_{n,m}]) = [D \to Y]$.

Since $\tilde{c}_1(\pi^*O(1, 0))^i \tilde{c}_1(\pi^*O(0, 1))^j(1_{Y_{n,m}}) = [Y_{n-i,m-j} \to Y_{n,m}]$ and

$$(h, s)^*(\pi^*O(1, 0)) = L \otimes M, \ (h, s)^*(\pi^*O(0, 1)) = M,$$

Lemma 10.2, Lemma 10.3, and the naturality of $\tilde{c}_1$ given by Lemma 6.3 yield

$$(h, s)^*(\sum_{i,j} b^{n,m}_{i,j}[Y_{n-i,m-j} \to Y_{n,m}]) = F^-_\omega(\tilde{c}_1(L \otimes M), \tilde{c}_1(M))(1_Y).$$

The "error terms" arising from any inequalities $b^{n,m}_{i,j} \neq b_{i,j}$ vanish because

$$(h, s)^*([Y_{n-i,m-j} \to Y_{n,m}]) = 0$$

if $i + j > n \geq d$ or if $j > m$ for dimensional reasons.

Applying $(h, s)^*$ to the relation (10.1) yields the identity

$$[D \to Y] = F^-_\omega(\tilde{c}_1(L \otimes M), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L)(1_Y),$$

which verifies axiom (Sect). □

11. Theorem 1 and Corollary 3

11.1. **Proofs.** *Proof of Theorem 1.* For clarity, we write $[f : Y \to X]_\omega$ for

$$[f : Y \to X] \in \omega_*(X)$$

and $[f : Y \to X]_\Omega$ for the associated class in $\Omega_*(X)$. Similarly, let

$$1^\omega_Y = [\mathrm{Id}_Y]_\omega, \quad 1^\Omega_Y = [\mathrm{Id}_Y]_\Omega.$$

By Proposition 3.5, there is natural transformation

$$\vartheta : \omega_* \to \Omega_*$$

of Borel-Moore functors on $\mathbf{Sch}_k$,

$$\vartheta_X([f : Y \to X]_\omega) = [f : Y \to X]_\Omega \in \Omega_*(X).$$

Moreover, ϑ_X is surjective for every $X \in \mathbf{Sch}_k$.

By Theorems 2 and 2.2, there is a natural transformation

$$\tau : \Omega_* \to \omega_*$$

of oriented Borel-Moore functors of geometric type. Let $Y \in \mathbf{Sm}_k$, and let

$$p : Y \to \mathrm{Spec}\,(k)$$

be the structure map. Since

$$1_Y^\Omega = p^*(1), \quad 1_Y^\omega = p^*(1),$$

and τ respects the unit and smooth pull-back,

$$\tau(1_Y^\Omega) = 1_Y^\omega.$$

Hence,

$$\begin{aligned} \tau_X([f : Y \to X]_\Omega) &= \tau_X(f_*(1_Y^\Omega)) \\ &= f_*(\tau_Y(1_Y^\Omega)) \\ &= f_*(1_Y^\omega) \\ &= [f : Y \to X]_\omega. \end{aligned}$$

Therefore $\tau \circ \vartheta = \mathrm{Id}_\omega$, so ϑ is an isomorphism. □

Proof of Corollary 3. We temporarily include the choice of base field in our notation, writing $\omega_*^k(X)$ for the value on $X \in \mathbf{Sch}_k$ of the oriented Borel-Moore functor ω_* on $\mathbf{Sch}_k$.

If $L \to L'$ is an extension of fields, a natural map

$$- \times_L L' : \omega_*^L(X) \to \omega_*^{L'}(X_{L'})$$

is defined by sending $[f : Y \to X] \in \mathcal{M}_*(X)^+$ with $X \in \mathbf{Sch}_L$ to the base extension $[f_{L'} : Y_{L'} \to X_{L'}]$ with $X_{L'} \in \mathbf{Sch}_{L'}$. Since all the generators and relations defining $\omega_*^k(k)$ are given by smooth varieties of finite type over k, we have

$$\omega_*^k(k) \cong \varinjlim_L \omega_*^L(L)$$

as L runs over fields finitely generated over $\mathbb{Q}$, where we use the natural maps defined above to define both the colimit and the map of the colimit to $\omega_*^k(k)$. The proof of Corollary 3 is reduced to the case of a field finitely generated over $\mathbb{Q}$. We may therefore assume k admits an embedding $\sigma : k \hookrightarrow \mathbb{C}$. We now revert to our original notation.

By Corollary 2.5, the canonical homomorphism

$$\vartheta^{MU,\sigma} : \Omega^*(k) \to MU^{2*}(\mathrm{pt})$$

is an isomorphism of graded rings. Since $MU^{2*}(\mathrm{pt})$ has a rational basis given by the classes of products of projective spaces, the Corollary follows from Theorem 1. □

11.2. **Curves.** In the coefficient ring $\Omega_*(k)$ of algebraic cobordism, the relation

$$[C] = (1-g)[\mathbb{P}^1] \tag{11.1}$$

holds for every smooth irreducible curve C of genus g (see [22, Remark 1.2.9]). The genus of a curve is certainly invariant modulo naive cobordism (0.1), hence naive cobordism does not suffice to give the relation (11.1).

We do not know a simple proof that the double point relations imply (11.1) for arbitrary C. In the proof of the isomorphism

$$\Omega_*(\mathbb{C}) \cong \mathbb{L}_*$$

in [22, Theorem 4.3.7], even for the case of curves, an extended relation which takes into account degenerations to effective strict normal crossing divisors having components of multiplicity > 1 is needed. The extended relation is furnished by the formal group law.

If C is a smooth *plane* curve of genus g a direct argument for (11.1) using double point relations is available. Let f be the degree d equation of C. Let g and h be the equations of generic curves of degrees $d-1$ and 1 respectively. Let $Y \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the nonsingular surface defined by $x_0 \cdot f + x_1 \cdot gh$ where $[x_0, x_1]$ are coordinates on $\mathbb{P}^1$. The double point relation obtained from the fibers over 0 and ∞ of the projection

$$\pi : Y \to \mathbb{P}^1$$

inductively yields (11.1) from the trivial degree d=1 case.

Double point cobordism is related to rational equivalence since the base of cobordism

$$\pi : Y \to X \times \mathbb{P}^1$$

has a $\mathbb{P}^1$-factor. A double point cobordism theory ω_*^{alg} related to algebraic equivalence can be defined by considering cobordisms

$$\pi : Y \to X \times B$$

with arbitrary smooth bases B of dimension one. We easily see

$$\omega_*(k) \cong \omega_*^{alg}(k) \cong \mathbb{L}_*.$$

Relation (11.1) is obtained directly in $\omega_*^{alg}(k)$ by considering a 1-parameter degeneration of C to the boundary of the moduli space of curves.

12. A theorem of Fulton

Let χ be a $\mathbb{C}$-valued function on the set of isomorphism classes of smooth projective varieties over k normalized by

(i) $\chi(\mathrm{Spec}\,(k)) = 1$

and satisfying additivity for disjoint union,

(ii) $\chi(X \amalg Y) = \chi(X) + \chi(Y)$.

Suppose further that the relation

(iii) $\chi(C) = \chi(A) + \chi(B) - \chi(A \cap B)$

holds whenever $A, B, C \subset Y$ are smooth divisors satisfying the linear equivalence

$$A + B \sim C$$

in an ambient smooth projective variety Y over k and $A \cap B$ is a transverse intersection.

As a first application, we use Theorem 1 to give a new proof of the following result of Fulton.

Theorem 12.1 ([11])**.** *Let k be an algebraically closed field of characteristic 0. If χ satisfies (i-iii), then χ is the sheaf Euler characteristic,*

$$\chi(Y) = \sum_{i=0}^{\dim Y} (-1)^i \dim_k H^i(Y, \mathcal{O}_Y).$$

The sheaf Euler characteristic is easily seen to satisfy the required conditions (i-iii). The main point of Theorem 12.1 is uniqueness. We will prove a stronger result — we will assume k has characteristic 0, but will not require k to be algebraically closed.

For $X \in \mathbf{Sch}_k$, consider the subgroup $I(X) \subset \omega_*(X)$ generated by elements of the form

$$f_*([A \to Y] + [B \to Y] - [A \cap B \to Y] - [C \to Y])$$

where $f : Y \to X$ is in $\mathcal{M}(X)$ and A, B, C are smooth divisors on Y satisfying condition (iii) of Theorem 12.1. Since $I(X)$ is not a graded subgroup of $\omega_*(X)$, we consider as well a variant that is graded. Let

$$I_*(X) \subset \omega_*(X)$$

be generated by elements of the form

$$f_*([A \to Y] + [B \to Y] - [\mathbb{P}^1 \times (A \cap B) \to Y] - [C \to Y])$$

with $f : Y \to X$, A, B, C as above. Here,

$$\mathbb{P}^1 \times (A \cap B) \to Y$$

is the projection $\mathbb{P}^1 \times (A \cap B) \to A \cap B$ followed by the inclusion $A \cap B \to Y$. Let

$$\overline{\omega}_*(X) = \omega_*(X)/I_*(X).$$

Lemma 12.2. *The following results hold:*

(i) *$X \mapsto \overline{\omega}_*(X)$ inherits the structure of an oriented Borel-Moore functor of geometric type from ω_*. In particular, $\overline{\omega}_*(X)$ is a $\overline{\omega}_*(k)$-module.*

(ii) $\omega_*(k)/I(k) = \overline{\omega}_*(k)/([\mathbb{P}^1] - [\mathrm{Spec}\,(k)]) \cdot \overline{\omega}_*(k)$.

(iii) $\omega_*(X)/I(X) = \overline{\omega}_*(X) \otimes_{\overline{\omega}_*(k)} (\omega_*(k)/I(k))$.

Proof. For (i), the only non-evident point to check is the descent of the first Chern class endomorphisms $\tilde{c}_1(L)$ on $\omega_*(X)$ to $\overline{\omega}_*(X)$. We may assume L is globally generated. Then, given $f : Y \to X$ and A, B, C on Y as above, a general section s of f^*L has smooth divisor $i : E \to Y$ intersecting A, B, C and $A \cap B$ transversely, so

$$\begin{aligned}\tilde{c}_1(L)(f_*([A \to Y] + [B \to Y] - [\mathbb{P}^1 \times (A \cap B) \to Y] - [C \to Y]))\\ = (f \circ i)_*([A \cap E \to E] + [B \cap E \to E]\\ - [\mathbb{P}^1 \times (A \cap B \cap E) \to E] - [C \cap E \to Y]).\end{aligned}$$

For (ii) and (iii), we need only verify

$$([\mathbb{P}^1] - 1) \cdot [\mathrm{Id}_W] \in I(W) \tag{12.1}$$

for each $W \in \mathbf{Sm}_k$. Indeed, given $f : Y \to X$ in $\mathcal{M}(X)$ and A, B, C on Y as above, we have the identity

$$\begin{aligned}([A \to Y] + [B \to Y] - [\mathbb{P}^1 \times (A \cap B) \to Y] - [C \to Y]))\\ - ([A \to Y] + [B \to Y] - [(A \cap B) \to Y] - [C \to Y]))\\ = i_{A \cap B*}(([\mathbb{P}^1] - 1) \cdot [\mathrm{Id}_{A \cap B}]).\end{aligned}$$

Taking $W = A \cap B$, this shows that (12.1) implies $I_*(X) \subset I(X)$ for all $X \in \mathbf{Sch}_k$, and in fact

$$I_*(X) + ([\mathbb{P}^1] - 1) \cdot \omega_*(X) = I(X)$$

for all $X \in \mathbf{Sch}_k$, from which (ii) and (iii) follow directly.

Finally, (12.1) is obtained by taking the generator of $I(W)$ with $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times W$, f the projection to W and

$$A = \mathbb{P}^1 \times 0 \times W,\ B = 0 \times \mathbb{P}^1 \times W,\ C = \Delta \times W,$$

with $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ the diagonal, where we use the linear equivalence

$$\Delta \times W \sim \mathbb{P}^1 \times 0 \times W + 0 \times \mathbb{P}^1 \times W$$

on $\mathbb{P}^1 \times \mathbb{P}^1 \times W$,

□

Lemma 12.3. *Let D be smooth and projective over k, and let L be a line bundle over D. Let χ be as in Theorem 12.1. Then,*

$$\chi(\mathbb{P}(O_D \oplus L)) = \chi(D) = \chi(\mathbb{P}^1 \times D)\ .$$

Proof. We need only check the first identity. As in the proof of Lemma 3.3, we have the double point degeneration

$$\pi : Y \to \mathbb{P}^1$$

with $\pi^{-1}(0) = \mathbb{P}(O_D \oplus L) \cup_D \mathbb{P}(O_D \oplus L)$ and $\pi^{-1}(1) = \mathbb{P}(O_D \oplus L)$. By condition (iii), we have

$$\chi(\mathbb{P}(O_D \oplus L)) = 2\chi(\mathbb{P}(O_D \oplus L)) - \chi(D)$$

or $\chi(D) = \chi(\mathbb{P}(O_D \oplus L))$. □

Lemma 12.4. *Let $\chi : \mathcal{M}(k) \to \mathbb{C}$ be as in Theorem 12.1. Then, χ descends to a group homomorphism $\chi : \omega_*(k)/I(k) \to \mathbb{C}$.*

Proof. Since χ is additive, χ defines a group homomorphism

$$\chi : \mathcal{M}_*(k)^+ \to \mathbb{C}.$$

Let $\pi : Y \to \mathbb{P}^1$ be a double-point cobordism with $Y_0 = A \cup B$, Y_∞ smooth. By Lemma 12.3, we have

$$\chi(Y_\infty) = \chi(A) + \chi(B) - \chi(\mathbb{P}(\pi)),$$

so χ descends to a group homomorphism

$$\chi : \omega_*(k) \to \mathbb{C}$$

annihilating $I(k)$ by assumption. □

Lemma 12.5. *Let $F_{\bar{\omega}}$ be the formal group law of $\overline{\omega}$. Then,*

$$F_{\overline{\omega}}(u, v) = u + v - [\mathbb{P}^1]uv.$$

Proof. It suffices to check the universal examples

$$O_{\mathbb{P}^n \times \mathbb{P}^m}(1, 1) = O_{\mathbb{P}^n}(1) \boxtimes O_{\mathbb{P}^m}(1).$$

The linear equivalence $H_{n,m} \sim \mathbb{P}^n \times \mathbb{P}^{m-1} + \mathbb{P}^{n-1} \times \mathbb{P}^m$ on $\mathbb{P}^n \times \mathbb{P}^m$ gives the relation

$$\begin{aligned}[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = [\mathbb{P}^n \times \mathbb{P}^{m-1} \to \mathbb{P}^n \times \mathbb{P}^m] + [\mathbb{P}^{n-1} \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m] \\ - [\mathbb{P}^1] \cdot [\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^n \times \mathbb{P}^n]\end{aligned}$$

in $\overline{\omega}_*(\mathbb{P}^n \times \mathbb{P}^m)$. Since

$$\begin{aligned}\tilde{c}_1(O_{\mathbb{P}^n\times\mathbb{P}^m}(1,1))(1_{\mathbb{P}^n\times\mathbb{P}^m}) &= [H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m]\\ \tilde{c}_1(O_{\mathbb{P}^n\times\mathbb{P}^m}(0,1))(1_{\mathbb{P}^n\times\mathbb{P}^m}) &= [\mathbb{P}^n \times \mathbb{P}^{m-1} \to \mathbb{P}^n \times \mathbb{P}^m]\\ \tilde{c}_1(O_{\mathbb{P}^n\times\mathbb{P}^m}(1,0))(1_{\mathbb{P}^n\times\mathbb{P}^m}) &= [\mathbb{P}^{n-1} \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m]\end{aligned}$$

and

$$\tilde{c}_1(O_{\mathbb{P}^n\times\mathbb{P}^m}(0,1))\circ\tilde{c}_1(O_{\mathbb{P}^n\times\mathbb{P}^m}(1,0))(1_{\mathbb{P}^n\times\mathbb{P}^m}) = [\mathbb{P}^{n-1}\times\mathbb{P}^{m-1} \to \mathbb{P}^n\times\mathbb{P}^m],$$

the projective bundle formula shows that we have

$$F_{\overline{\omega}}(u,v) = u + v - [\mathbb{P}^1]uv \mod (u^{n+1}, v^{m+1}).$$

Since n, m were arbitrary, the results is proven. □

Lemma 12.6. *The ring homomorphism $\phi : \mathbb{Z}[t] \to \overline{\omega}_*(k)$ sending t to $-[\mathbb{P}^1]$ is surjective. In addition, the canonical ring homomorphism $\mathbb{Z} \to \omega_*(k)/I(k)$ is an isomorphism.*

Proof. The homomorphism $\mathbb{Z} \to \omega_*(k)/I(k)$ is split by $Y \mapsto \chi(\mathcal{O}_Y)$, hence the second assertion follows from the first and (2) of Lemma 12.2.

For the first assertion, write the universal group law as

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^iv^j.$$

By Lemma 12.5, the canonical homomorphism $\phi_{\overline{\omega}} : \mathbb{L} \to \bar{\omega}_*(k)$ classifying $F_{\overline{\omega}}$ sends a_{11} to $-[\mathbb{P}^1]$ and all other a_{ij} to zero. By Theorem 1 and the isomorphism $\mathbb{L}_* \to \Omega_*(k)$ [22, Theorem 4.3.7],

$$\mathbb{L}_* \to \overline{\omega}_*(k)$$

is surjective, completing the proof. □

Proof of Theorem 12.1. Let $\chi : \mathcal{M}(k) \to \mathbb{C}$ be given. By Lemma 12.4, χ descends to a homomorphism

$$\chi : \omega_*(k)/I(k) \to \mathbb{C}$$

with $\chi([\mathrm{Spec}\,(k)]) = 1$. Since $\omega_*(k)/I(k) \cong \mathbb{Z}$ by Lemma 12.6, there is at most one such χ, hence $\chi(Y)$ equals the sheaf Euler characteristic. □

The proof improves Fulton's result slightly (still assuming k has characteristic 0). We may replace replace $\mathbb{C}$ with any abelian group A,

$$\chi : \mathcal{M}(k) \to A.$$

If χ satisfies conditions (ii) and (iii), then

$$\chi(Y) = \chi([\mathrm{Spec}\,(k)]) \cdot \left(\sum_{i=0}^{\dim Y} (-1)^i \dim_k H^i(Y, \mathcal{O}_Y) \right) \in A$$

for all smooth projective Y over k.

In fact, we can prove more. Denote the localization of $\overline{\omega}_*$ at $[\mathbb{P}^1]$ by

$$\widetilde{\omega}_* = \overline{\omega}_*[[\mathbb{P}^1]^{-1}].$$

Let $\mathbb{L} \to \mathbb{Z}[t]$ be the homomorphism classifying the group law

$$u + v + tuv.$$

For $X \in \mathbf{Sch}_k$, let $G_0(X)$ denote the Grothendieck group of coherent sheaves following the notation of [22]. Let $\mathbf{QSch}_k \subset \mathbf{Sch}_k$ denote the full subcategory of quasi-proejctive k-schemes.

Theorem 12.7. *There are natural isomorphisms for $X \in \mathbf{QSch}_k$:*

$$\begin{aligned} \overline{\omega}_*(X) &\cong \Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t] \\ \widetilde{\omega}_*(X) &\cong G_0(X)[t, t^{-1}] \\ \omega_*(X)/I(X) &\cong G_0(X). \end{aligned}$$

Proof. We have already seen that the formal group law for $\overline{\omega}_*$ is

$$u + v - [\mathbb{P}^1]uv.$$

The canonical morphism $\Omega_* \to \bar{\omega}_*$ therefore factors through

$$\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[t] \to \overline{\omega}_* \ , \tag{12.2}$$

with t mapping to $-[\mathbb{P}^1]$. The map (12.2) is clearly surjective.

Injectivity is obtained from the formal group law

$$u + v - [\mathbb{P}^1]uv$$

of $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[t]$. Let $f : Y \to X$, A, B, C be as in condition (iii) of Theorem 12.1. As operators on $\Omega_*(Y) \otimes_{\mathbb{L}} \mathbb{Z}[t]$,

$$\tilde{c}_1(O_Y(C)) = \tilde{c}_1(O_Y(A)) + \tilde{c}_1(O_Y(B)) - [\mathbb{P}^1]\tilde{c}_1(O_Y(A)) \circ \tilde{c}_1(O_Y(B)).$$

Evaluating on 1_Y, using the Gysin relations, and pushing forward to X gives the relation

$$[C \to X] = [A \to X] + [B \to X] - [\mathbb{P}^1] \cdot [A \cap B \to X]$$

in $\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t]$. In other words, $I_*(X) = 0$ in $\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t]$. Since $\omega_* = \Omega_*$, we conclude (12.2) is injective and hence an isomorphism.

The definition of $\widetilde{\omega}_*$ and isomorphism (12.2) together yield

$$\tilde{\omega}_*(X) \cong \Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t, t^{-1}].$$

In case $X \in \mathbf{Sm}_k$, the natural map

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t, t^{-1}] \to K_0(X)[t, t^{-1}]$$

is an isomorphism by [22], where $K_0(X)$ is the Grothendieck group of locally free sheaves. For the general case $X \in \mathbf{QSch}_k$, Dai [6] has shown that the natural map $\Omega_*(X) \to G_0(X)[t, t^{-1}]$ induces an isomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[t, t^{-1}] \to G_0(X)[t, t^{-1}],$$

proving the second isomorphism of the Theorem.

Since $\omega_*(X)/I(X) \cong \overline{\omega}_*(X)/([\mathbb{P}^1]-1)$, the third isomorphism follows from the second. $\square$

By Theorem 12.7, we have a presentation of $G_0(X)$ for $X \in \mathbf{QSch}_k$ as

$$G_0(X) \cong \mathcal{M}(X)^+/{<}f_*([A \to Y]+[B \to Y]-[A \cap B \to Y]-[C \to Y]){>}$$

for $f : Y \to X \in \mathcal{M}(X)$, A, B, C as in condition (iii) of Theorem 12.1. Strangely, only the relation of linear equivalence of smooth divisors on smooth varieties is used!

13. Donaldson-Thomas theory

13.1. **Proof of Conjecture 1.** Let $\mathbb{Q}[[q]]^* \subset \mathbb{Q}[[q]]$ denote the multiplicative group of power series with constant term 1. Define a group homomorphism

$$\mathsf{Z} : (\mathcal{M}_3(\mathrm{Spec}\,(\mathbb{C}))^+, +) \to (\mathbb{Q}[[q]]^*, \cdot)$$

on generators by the partition function for degree 0 Donaldson-Thomas theory defined in Section 0.6,

$$\mathsf{Z}([Y]) = \mathsf{Z}(Y, q).$$

We use here the abbreviated notation

$$[Y] = [Y \to \mathrm{Spec}\,(\mathbb{C})] \in \mathcal{M}_3(\mathrm{Spec}\,(\mathbb{C})).$$

Since double point relations hold in Donaldson-Thomas theory (0.10), the homomorphism Z descends to $\omega_*(\mathbb{C})$,

$$\mathsf{Z} : \omega_*(\mathbb{C}) \to \mathbb{Q}[[q]]^*.$$

By Corollary 3, the class $[Y] \in \omega_3(\mathbb{C})$ is expressible rationally in terms of the classes

$$[\mathbb{P}^3],\ [\mathbb{P}^2 \times \mathbb{P}^1],\ [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1].$$

Hence,

$$r[Y] = s_3[\mathbb{P}^3] + s_{21}[\mathbb{P}^2 \times \mathbb{P}^1] + s_{111}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1] \ \in \omega_*(\mathbb{C})$$

for integers $r \neq 0$, s_3, s_{21}, and s_{111}. Therefore

$$\mathsf{Z}(Y,q)^r = \prod_{|\lambda|=3} \mathsf{Z}(\mathbb{P}^\lambda, q)^{s_\lambda}. \tag{13.1}$$

Conjecture 1 has been proven for 3-dimensional products of projective spaces in [25, 26]. The right side of (13.1) can therefore be evaluated:

$$\begin{aligned}\prod_{|\lambda|=3} \mathsf{Z}(\mathbb{P}^\lambda, q)^{s_\lambda} &= \prod_{|\lambda|=3} M(-q)^{s_\lambda \int_{\mathbb{P}^\lambda} c_3(T_{\mathbb{P}^\lambda}\otimes K_{\mathbb{P}^\lambda})} \\ &= M(-q)^{\sum_{|\lambda|=3} s_\lambda \int_{\mathbb{P}^\lambda} c_3(T_{\mathbb{P}^\lambda}\otimes K_{\mathbb{P}^\lambda})}\end{aligned}$$

Since algebraic cobordism respects Chern numbers[5],

$$\mathsf{Z}(Y,q)^r = M(-q)^{r\int_Y c_3(T_Y\otimes K_Y)}.$$

Finally, since $Z(Y,0)=1$ and $M(0)=1$,

$$\mathsf{Z}(Y,q) = M(-q)^{\int_Y c_3(T_Y\otimes K_Y)},$$

completing the proof. □

13.2. **Conjecture** $1'$**.** Next, we consider an equivariant version of Conjecture 1 proposed in [5].

Let X be a smooth quasi-projective 3-fold over $\mathbb{C}$ equipped with an action of an algebraic torus T with *compact* fixed locus X^T. If X^T is compact, $\mathrm{Hilb}(X,n)^T$ is also compact, and

$$N^X_{n,0} = \int_{[\mathrm{Hilb}(X,n)^T]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})} \in \mathbb{Q}(\mathbf{t})$$

is well-defined [12]. Here

$$\mathbf{t} = \{t_1,\ldots,t_{\mathrm{rk}(T)}\}$$

is a set of generators of the T-equivariant cohomology of a point. Let

$$\mathsf{Z}(X,q,\mathbf{t}) = 1 + \sum_{n\geq 1} N^X_{n,0}\, q^n$$

be the equivariant partition function.

Since X^T is compact, the right side of the equality of Conjecture 1 is also well-defined via localization,

$$\int_X c_3(T_X\otimes K_X) = \int_{X^T} \frac{c_3(T_X\otimes K_X)}{e(\mathrm{Norm})} \in \mathbb{Q}(\mathbf{t}).$$

[5]Either use the operations $\vartheta^{CF} : \Omega_*(k) \to \mathrm{CH}_*(k)[\mathbf{t}] = \mathbb{Z}[\mathbf{t}]$ constructed in [22, Example 4.1.26], or reduce to the case of k admitting an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and recall that complex cobordism respects Chern numbers [33, Theorem, pg. 117, Chap. VII].

Conjecture $1'$**.** [5] $\mathsf{Z}(X, q, \mathbf{t}) = M(-q)^{\int_X c_3(T_X \otimes K_X)}$.

We will prove Conjecture $1'$ before proving Conjecture 2 for relative Donldson-Thomas theory.

13.3. **Local geometries.** Let M be a smooth projective variety over $\mathbb{C}$ of pure dimension at most 3. Let

$$N \to M$$

be a vector bundle of satisfying

$$\mathrm{rk}(N) = 3 - \dim_{\mathbb{C}} M.$$

The space total space N may be viewed as a *local* neighborhood[6] of M in a 3-fold embedding. If

$$N = \bigoplus_{i=1}^{r} N_i$$

is a direct sum decomposition, an r-dimensional torus T acts canonically on the total space N by scaling the factors of N. Since $N^T = M$, the fixed locus is compact.

We will first prove Conjecture $1'$ for the local geometry N. In case M has dimension 0 or 1, Conjecture $1'$ has been proven in [25, 26] and [29] respectively. If Y has dimension 3, Conjecture $1'$ reduces to Conjecture 1. Only the dimension 2 case remains.

13.4. **Proof of Conjecture** $1'$ **for local surfaces.** The proof relies upon a double point cobordism theory for local geometries. To abbreviate the discussion, we focus our attention on the double point cobordism theory for local surfaces over Spec$\,(\mathbb{C})$.

Consider the free group $M_{2,1}(\mathbb{C})^+$ generated by pairs $[S, L]$ where S is smooth, irreducible, projective surface and

$$L \to S$$

is a line bundle. The subscript $(2, 1)$ captures the dimension of S and the rank of L. We will define a double point cobordism theory $\omega_{2,1}^{alg}(\mathbb{C})$ as a quotient of $M_{2,1}(\mathbb{C})^+$ by double point relations.

Double point relations are easily defined in the local setting. Let C be a smooth projective curve with a base point $0 \in C$. Let

$$\pi : \mathcal{S} \to C$$

be a projective morphism determining a double point degeneration with

$$\mathcal{S}_0 = A \cup B,$$

[6]There is no algebraic tubular neighborhood result even formally.

and let

$$\mathcal{L} \to \mathcal{S}$$

be a line bundle. For each regular value $\zeta \in C$ of π, define an associated double point relation by

$$[\mathcal{S}_\zeta, \mathcal{L}_\zeta] - [A, \mathcal{L}_A] - [B, \mathcal{L}_B] + [\mathbb{P}(\pi), \mathcal{L}_{\mathbb{P}(\pi)}]. \tag{13.2}$$

Here, subscripts denote restriction (or, in the case of $\mathcal{L}_{\mathbb{P}(\pi)}$, pull-back).

Let $\mathcal{R}^{alg}_{2,1}(\mathbb{C}) \subset M_{2,1}(\mathbb{C})^+$ be the subgroup generated by all double point relations. Double point cobordism theory for local surfaces is defined by

$$\omega^{alg}_{2,1}(\mathbb{C}) = M_{2,1}(\mathbb{C})^+/\mathcal{R}^{alg}_{2,1}(\mathbb{C}).$$

Lemma 13.1. *Double point cobordism theory $\omega^{alg}_{2,1}(\mathbb{C})$ for local surfaces is generated (over $\mathbb{Q}$) by elements of the following form:*

(i) $[\mathbb{P}^2, O_{\mathbb{P}^2}]$,
(ii) $[\mathbb{P}^1 \times \mathbb{P}^1, L]$,
(iii) $[F_1, L]$,

where F_1 is the blow-up of $\mathbb{P}^2$ in a point.

Proof. There is a natural group homomorphism

$$\iota : \omega_2(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \omega^{alg}_{2,1}(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined by $\iota([S]) = [S, O_S]$. By Corollary 3, the image of ι is generated by

$$[\mathbb{P}^2, O_{\mathbb{P}^2}], \quad [\mathbb{P}^1 \times \mathbb{P}^1, O_{\mathbb{P}^1 \times \mathbb{P}^1}].$$

Let $[S, O_S(C)] \in M_{2,1}(\mathbb{C})^+$ where $C \subset S$ is smooth divisor. Consider the deformation to the normal cone of $C \subset S$,

$$\pi : \mathcal{S} \to \mathbb{P}^1$$

with degenerate fiber

$$\mathcal{S}_0 = S \cup \mathbb{P}(O_C \oplus O_C(C)).$$

Since $\mathcal{S}$ is the blow-up of $S \times \mathbb{P}^1$ along $C \times 0$, there is a canonical morphism

$$\nu : \mathcal{S} \to S$$

obtained from blow-down and projection. Let $\mathcal{L} \to \mathcal{S}$ be defined by

$$\mathcal{L} = \nu^*(O_S(C + D)) \otimes O_{\mathcal{S}}(-\mathbb{P}(O_C \oplus O_C(C))).$$

where D is a Cartier divisor on S. The double point relation associated to $\mathcal{L} \to \mathcal{S}$ is

$$[S, O_S(C+D)] - [S, O_S(D)] - [\mathbb{P}(O_C \oplus O_C(C)), L'] + [\mathbb{P}(\pi), L''] \tag{13.3}$$

where L' and L'' are line bundles.

Let $\Gamma \subset \omega^{alg}_{2,1}(\mathbb{C})$ be the subgroup generated by $\mathrm{Im}(\iota)$ and elements of the form $[P, L]$ where P is a $\mathbb{P}^1$-bundle over a smooth projective curve. If D is taken to be 0 in (13.3), we find $[S, O_S(C)] \in \Gamma$. For general a Cartier divisor D,

$$[S, O_S(C + D)] \in \Gamma \iff [S, O_S(D)] \in \Gamma.$$

Since, for any D, there exists smooth curves C, C' for which

$$O_S(C + D) \cong O_S(C'),$$

we find $\Gamma = \omega^{alg}_{2,1}(\mathbb{C})$.

Since the double point cobordisms here are allowed to have arbitrary curves as bases, elementary degenerations show elements of type (ii) and (iii) generate the classes $[P, L]$ of Γ. □

The computation of the degree 0 equivariant vertex in [25, 26] proves Conjecture 1′ for the toric generators (i-iii) of Lemma 13.1. Conjecture 1′ then follows for local surfaces by an argument parallel to the proof of Conjecture 1. □

13.5. **Proof of Conjecture 1′.** Let T be an r-dimensional torus acting on a smooth quasi-projective 3-fold X with compact fixed locus X^T. The 1-dimensional subtori of T are described by elements of the lattice $\mathbb{Z}^r$. Since 1-dimensional tori $T_1 \subset T$ with equal fixed loci

$$X^{T_1} = X^T$$

determine a Zariski dense subset of $\mathbb{Z}^r$, Conjecture 1′ is implied by the rank 1 case.

We assume T is a 1-dimensional torus. If the T-action on X is trivial, Conjecture 1′ reduces to Conjecture 1. We assume the T-action is nontrivial. The components of the fixed locus

$$X^T = \bigcup_i X^T_i$$

are of dimension 0, 1, or 2. Certainly

$$\mathsf{Z}(X, q, t) = \prod_i \mathsf{Z}(X_i, q, t) \tag{13.4}$$

where

$$\mathsf{Z}(X_i, q, t) = \sum_n q^n \int_{[\mathrm{Hilb}(X,n)^T_i]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})}$$

and $\mathrm{Hilb}(X, n)^T_i \subset \mathrm{Hilb}(X, n)^T$ is locus supported on X^T_i. We will prove

$$\mathsf{Z}(X_i, q, t) = M(-q)^{\int_{X^T_i} \frac{c_3(T_X \otimes K_X)}{e(\mathrm{Norm}_i)}} \tag{13.5}$$

where Norm_i is the normal bundle of $X_i^T \subset X$. Conjecture $1'$ follows from (13.4) and (13.5).

Equality (13.5) is proven separately for each possible dimension of X_i^T. The dimension 1 case is the most delicate.

Dim 0. If $X_i^T = p$ is a point, then by Theorem 2.4 of [4], the T-action on X is analytically equivalent in a Euclidean neighborhood of p to the T-action on the tangent space $T_p(X)$. The T-action at a point $u \in U$ of a Euclidean neighborhood is defined only locally at $1 \in T$. Equality (13.5) in the dimension 0 case follows from the degree 0 vertex evaluation of [25, 26].

Dim 2. If $X_i^T = S$ is a surface, the T-weight on the normal bundle of $S \subset X$ may be assumed positive. The Bialynicki-Birula stratification [4] provides a T-equivariant Zariski neighborhood of S determined by a T-equivariant affine bundle

$$S_+ \to S$$

of rank 1 with a T-fixed section. In the rank 1 case, S_+ is the total space of a T-equivariant line bundle over S. Equality (13.5) in the dimension 2 case follows from Conjecture $1'$ for local surfaces.

If $X_i^T = C$ is a curve, there are three possibilities. Let N_C be the rank 2 normal bundle of $C \subset X$. The T-representation on the fiber of N_C has nontrivial weights w_1 and w_2.

Dim 1, weights of opposite sign. If the weights w_1 and w_2 have opposite signs, then there is a canonical T-equivariant splitting

$$N = N_+ \oplus N_-$$

as a sum of line bundles. The Bialynicki-Birula stratification yields quasi-projective surfaces

$$C_+, C_- \subset X$$

corresponding to the positive and negative normal directions. Since the affine bundles

$$C_\pm \to C$$

are of rank 1 with T-fixed sections, there are T-equivariant isomorphisms

$$\phi_\pm : C_\pm \to N_\pm$$

where the total spaces of the line bundles occur on the right.

Let $p \in C$. By Theorem 2.4 of [4], the T-action on a Euclidean neighborhood $U_X \subset X$ of $p \in X$ is analytically equivalent to the T-action on a Euclidean neighborhood $U_N \subset N_C$ of $p \in N_C$. Certainly the images of $C_\pm$ are the intersections of U with $N_\pm$.

Since the T-action on N_C has weights of opposite sign, the T-equivariant automorphism group of U over C which fixes $U \cap N_\pm$ pointwise is trivial. In particular, there is a unique T-equivariant isomorphism

$$U_X \to U_N$$

compatible with $\phi_\pm$. Patching together the isomorphisms yields an T-equivariant analytic isomorphism between X and N_C defined in a Euclidean neighborhood of C. Equality (13.5) in the 1-dimensional opposite sign case then follows from Conjecture $1'$ for local curves proved in [29].

If the weights w_1 and w_2 are of the same sign, we may assume the weights to be positive. The Biaylnicki-Birula stratification yields a T-equivariant Zariski neighborhood of C determined by a T-equivariant affine bundle

$$C_+ \to C$$

of rank 2. We will see C_+ need not be the total space of a T-equivariant rank 2 vector bundle on C.

The weights w_1 and w_1 are *related* if there exists an integer $k \geq 2$ for which either

$$w_1 \cong kw_2 \quad \text{or} \quad kw_1 \cong w_2.$$

Dim 1, related weights of same sign. Without loss of generality, we may assume the relation is $w_1 = kw_2$.

Let $\mathbb{C}^2$ be a T-representation with weights w_1 and w_2,

$$t \cdot (z_1, z_2) = (t^{w_1} z_1, t^{w_2} z_2).$$

The T-equivariant automorphism group G of $\mathbb{C}^2$ is given by 2×2 upper triangular matrices,

$$\gamma_{\begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{pmatrix}} \Big(z_1, z_2 \Big) = \Big(\lambda_1 z_1 + \delta z_2^k, \lambda_2 z_2 \Big). \tag{13.6}$$

Every Zariski locally trivial G-torsor τ on C yields an T-equivariant affine bundle

$$A_\tau \to C$$

of rank 2 over C with a T-equivariant section. The bundle A_τ is obtained by the G-action (13.6). The family of homomorphisms

$$\rho_\xi : G \to G$$

for $\xi \in \mathbb{C}$ defined by

$$\rho_\xi \begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \xi \cdot \delta \\ 0 & \lambda_2 \end{pmatrix}$$

is a algebraic deformation of the identity ρ_1 to the the diagonal projection

$$\rho_0 : G \to (\mathbb{C}^*)^2.$$

For each G-torsor τ, let τ_ξ be the G-torsor induced by ρ_ξ. Then, the algebraic family A_{τ_ξ} of G-torsors is a T-equivariant deformation of A_τ to A_{τ_0}. The latter is the total space of a T-equivariant vector bundle on C.

Bialynicki-Birula proves the T-equivariant affine bundle

$$C_+ \to C$$

is obtained from a G-torsor as above. Since C_+ is T-equivariantly deformation equivalent to the total space of a rank 2 vector bundle over C, equality (13.5) follows from the local curve case together with the deformation invariance of the virtual class.

Dim 1, unrelated weights of the same sign. If w_1 and w_2 are not related,

$$C_+ \to C$$

is the total space of a T-equivariant rank 2 vector bundle over C, see Section 3 of [4]. Equality (13.5) then follows from Conjecture $1'$ for local curves. $\square$

13.6. **Proof of Conjecture 2.** Let X be a smooth projective 3-fold over $\mathbb{C}$, and let Let $S \subset X$ be a smooth surface. Let

$$\mathbb{P} = \mathbb{P}(O_S \oplus O_S(S)).$$

Let $S_+, S_- \subset \mathbb{P}$ denote the sections with respective normal bundles $O_S(S)$, $O_S(-S)$ corresponding to the quotients $O_S(S)$, O_S.

We will study the Donaldson-Thomas theory of $\mathbb{P}/S_-$ by localization. A 1-dimensional scaling torus T acts on $\mathbb{P}$ with

$$\mathbb{P}^T = S_+ \cup S_-$$

and normal weights t and $-t$ along S_+ and S_- respectively. The components of the T-fixed loci of $I_n(\mathbb{P}/S_-, 0)$ lie over either S_- or S_+.

A Donaldson-Thomas theory of *rubber* naturally arises on the fixed loci of $I_n(\mathbb{P}/S_-,0)$ over S_-. Let

$$\mathsf{W}_- = 1 + \sum_{n\geq 1} q^n \int_{[I_n(\mathbb{P}/S_-\cup S_+,0)^{\sim}]^{vir}} \frac{1}{-t-\Psi_+}$$

denote the rubber contributions. Here, $I_n(\mathbb{P}/S_-\cup S_+,0)^{\sim}$ denotes the rubber moduli space, and Ψ_+ denotes the cotangent line associated to target degeneration. However, since the virtual dimension of the rubber space $I_n(\mathbb{P}/S_-\cup S_+,0)^{\sim}$ is -1,

$$\mathsf{W}_- = 1.$$

A discussion of virtual localization in relative Donaldson-Thomas theory and rubber moduli spaces can be found in [26]. See [27] for a construction of Ψ_+.

A local neighborhood of $S_+ \subset \mathbb{P}$ is given by the total space

$$\mathbb{P}_+ = \mathbb{P}\setminus S_-$$

of the line bundle

$$O_S(S) \to S_+.$$

Hence, the contributions over S_+ are determined by Conjecture $1'$ for local surfaces,

$$\mathsf{W}_+ = M(-q)^{\int_{\mathbb{P}_+} c_3(T_{\mathbb{P}_+}\otimes K_{\mathbb{P}_+})}.$$

The equivariant integral in the exponent is easily computed

$$\int_{\mathbb{P}_+} c_3(T_{\mathbb{P}_+}\otimes K_{\mathbb{P}_+}) = \int_{\mathbb{P}} c_3(T_{\mathbb{P}}[-S_-]\otimes K_{\mathbb{P}}[S_-]).$$

The product of the localization contributions over S_- and S_+ yields the partition function,

$$\begin{aligned}\mathsf{Z}(\mathbb{P}/S_-,q) &= \mathsf{W}_-\cdot \mathsf{W}_+ \\ &= M(-q)^{\int_{\mathbb{P}} c_3(T_{\mathbb{P}}[-S_-]\otimes K_{\mathbb{P}}[S_-])}.\end{aligned}$$

Conjecture 2 for $\mathbb{P}/S_-$ is proven.

Deformation to the normal cone of $S\subset X$ yields

$$\mathsf{Z}(X/S,q) = \mathsf{Z}(X,q)\cdot \mathsf{Z}(\mathbb{P}/S,q)^{-1}. \tag{13.7}$$

Then, Conjecture 1 for $\mathsf{Z}(X,q)$ and Conjecture 2 for $\mathsf{Z}(\mathbb{P}/S,q)$ imply Conjecture 2 for $\mathsf{Z}(X/S,q)$. □

References

[1] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk, *Torification and factorization of birational maps.* J. Amer. Math. Soc. **15** (2002), 531–572.

[2] K. Behrend, *Donaldson-Thomas invariants via microlocal geometry*, math.AG/0507523.

[3] K. Behrend and B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra Number Theory **2** (2008), no. 3, 313–345.

[4] A. Bialynicki-Birula, *Some theorems on the actions of algebraic groups*, Ann. of Math. **98** (1973), 480–497.

[5] J. Bryan and R. Pandharipande, *The local Gromov-Witten theory of curves.* With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande. J. Amer. Math. Soc. **21** (2008), no. 1, 101–136.

[6] S. Dai, *Isomorphisms between algebraic cobordism and K-theory over singular schemes.* Northeastern University Ph.D. thesis. 2007, http://www.math.uiuc.edu/K-theory/0911/

[7] Y. Eliashberg, A. Givental, H. Hofer, *Introduction to symplectic field theory*, GAFA 2000, 560–673.

[8] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173-199.

[9] C. Faber and R. Pandharipande, *Relative maps and tautological classes*, JEMS **7** (2005), 13–49.

[10] W. Fulton, *Intersection theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**, Springer-Verlag, Berlin-New York, 1984.

[11] W. Fulton, *A note on the arithmetic genus.* Amer. J. Math. **101** (1979), no. 6, 1355–1363.

[12] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[13] E. Ionel and T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. **157** (2003), 45–96.

[14] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math. **129**, Birkhuser: Boston, MA, 1995.

[15] M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251-274.

[16] M. Levine, *Algebraic cobordism*, Proceedings of the ICM (Beijing 2002), Vol II., 57–66.

[17] M. Levine, *A survey of algebraic cobordism*, Proceedings of the International Conference on Algebra, Algebra Colloq. **11** (2004), 79–90.

[18] M. Levine, *Algebraic cobordism II*, preprint 2002, www.math.uiuc.edu/K-theory/0577.

[19] M. Levine and F. Morel, *Cobordisme algébrique I*, C. R. Acad. Sci. Paris Sr. I Math. **332** (2001), 723–728.

[20] M. Levine and F. Morel, *Cobordisme algébrique II*, C. R. Acad. Sci. Paris Sr. I Math. **332** (2001), 815–820.

[21] M. Levine and F. Morel, *Algebraic cobordism I*, preprint 2002, www.math.uiuc.edu/K-theory/0547.

[22] M. Levine and F. Morel, *Algebraic Cobordism*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2007.
[23] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, Invent. Math. **145** (2001), no. 1, 151–218.
[24] J. Li, *A degeneration formula for Gromov-Witten invariants*, J. Diff. Geom. **60** (2002), 199–293.
[25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, Compos. Math. **142** (2006), no. 5, 1263–1285..
[26] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, Compos. Math. **142** (2006), no. 5, 1286–1304..
[27] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology **45** (2006), no. 5, 887–918.
[28] A. Nenashev, *A blow-up relation in algebraic cobordism*, preprint 2005, http://www.math.uiuc.edu/K-theory/0758/.
[29] A. Okounkov and R. Pandharipande, *Local Donaldson-Thomas theory of curves*, math.AG/0512573.
[30] R. Pandharipande, *Three questions in Gromov-Witten theory*, Proceedings of the ICM (Beijing 2002), Vol II., 503–512.
[31] D. G. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971) 29–56.
[32] R. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1999.
[33] R. Stong, *Note on cobordism theory*, Mathematical notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.
[34] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations*, JDG **54** (2000), 367–438.

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