INSTITUTO SUPERIOR TÉCNICO

Consider the Cohomology

$H^*(\mathrm{Hilb}(\mathbb{C}^2, n))$

natural to view all together

$$\bigoplus_{n=0}^{\infty} H^*(\mathrm{Hilb}(\mathbb{C}^2, n)) \qquad (*)$$

A central result by Grojnowski, Nakajima

is the identification of

$(*)$ with Fock Space

Fock Space is freely generated over $\mathbb{C}$ by commuting creation operators

$$\alpha_{-k} \quad k \in \mathbb{Z}_{>0}$$

acting on the vacuum vector v_ϕ

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be a partition as before

A basis of Fock space is given by

$$|\mu\rangle = \frac{1}{|\mathrm{Aut}\,\mu|} \frac{1}{\prod \mu_i} \prod \alpha_{-\mu_i} \, v_\phi$$

INSTITUTO
SUPERIOR
TÉCNICO

Natural isomorphism.

$$\bigoplus_{n \geq 0} H^*(\mathrm{Hilb}(\mathbb{C}^2, n)) \cong \mathcal{F}$$

$$\frac{1}{\prod \mu_i} [\, C_\mu \,] \longleftarrow |\mu\rangle$$

Cycle with length
partition μ.

INSTITUTO SUPERIOR TÉCNICO

Fock space also carries annihilation operators α_k $k \in \mathbb{Z}_{>0}$ which kill the vacuum

$$\alpha_k \cdot v_\phi = 0$$

and satisfy the basic commutation relations $[\alpha_k, \alpha_l] = k \, \delta_{k+l}$

We can construct further operators on Fock space from α_k

For example $E = \sum_{k>0} \alpha_{-k} \alpha_k$

energy operator

$$E \, |\mu\rangle = |\mu| \, |\mu\rangle$$

$$|\mu| = \sum \mu_i$$

INSTITUTO SUPERIOR TÉCNICO

We have discussed the classical cohomology of $Hilb(\mathbb{C}^2, n)$

What is the quantum cohomology?

M manifold

$$\alpha, \beta \in H^*(M)$$

$$\Longrightarrow \quad \alpha \cup \beta \in H^*(M) \qquad \text{cup product}$$

M is nonsingular alg variety (symplectic manifold)

$$\Longrightarrow \quad \alpha \star \beta \in H^*(M)[[q]]$$

quantum product

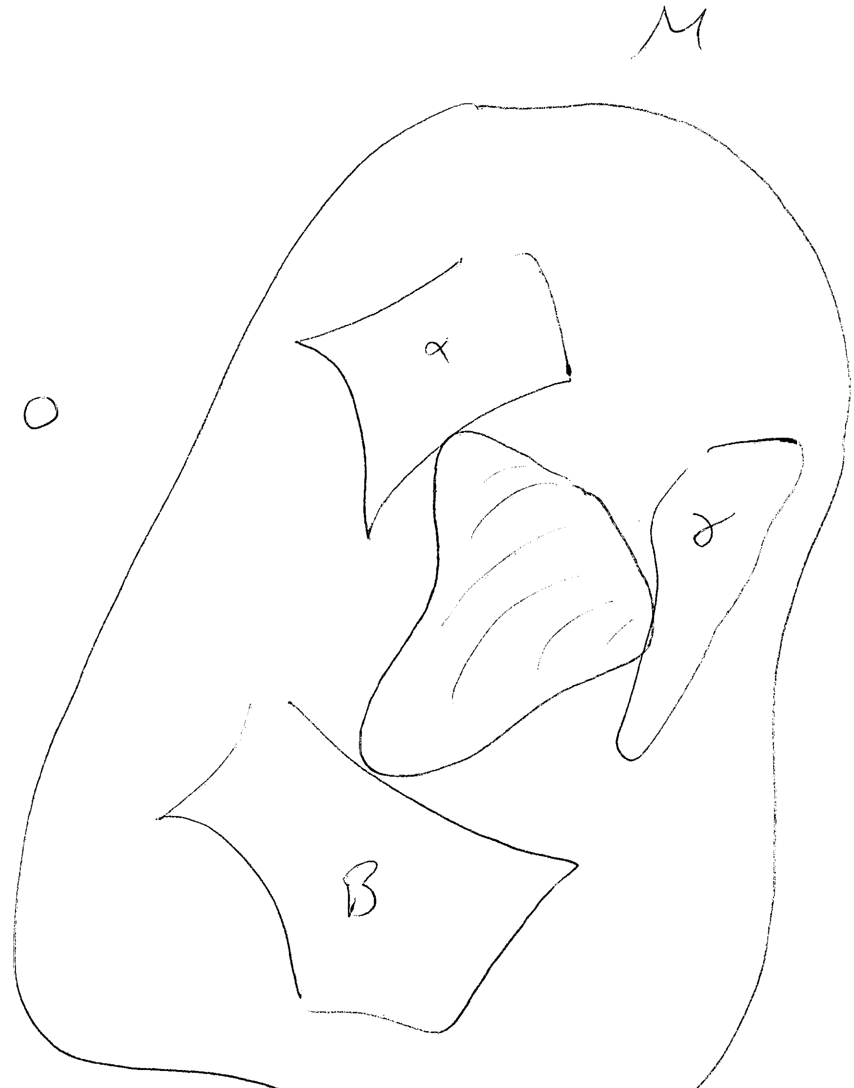
Definition follows ideas of Gromov, Witten, ...

$$\alpha * \beta = \sum_d \sum_i N_d(\alpha, \beta, \gamma_i)\, \gamma_i^\vee \; q^d$$

$\gamma_1, \dots, \gamma_N$ basis of $H^*(M)$

$N_d(\alpha, \beta, \gamma)$

counts the number of genus 0 curves of degree d meeting the three cycles



INSTITUTO SUPERIOR TÉCNICO

What is the quantum Cohomology of $\mathrm{Hilb}(\mathbb{C}^2, n)$?

D divisor class $- |2, \underbrace{1, \dots, 1}_{n-2}\rangle$

$*D : \mathcal{F} \to \mathcal{F}$ $\mathbb{Q}[[q]]$ coeffs

What is the matrix?

(turns out that $*D$ also determines the classical product)

Aside $\mathrm{Hilb}(\mathbb{C}^2, n)$ non compact

$\Rightarrow$ need to consider everything I say equivariantly with respect to $(\mathbb{C}^*)^2$-action weights t_1, t_2

INSTITUTO
SUPERIOR
TÉCNICO

Answer $\quad * D : \mathcal{F} \to \mathcal{F}$

$$* D = \frac{t_1 + t_2}{2} \sum_{d>0} \left(d \, \frac{(-q)^d + 1}{(-q)^d - 1} - \frac{(-q)+1}{(-q)-1} \right) \alpha_{-d} \alpha_d$$

$$+ \frac{1}{2} \sum_{k,\ell>0} t_1 t_2 \, \alpha_{k+\ell} \alpha_{-k} \alpha_{-\ell} - \alpha_{-k-\ell} \alpha_k \alpha_\ell$$

[Okounkov – P]

Comes from Hodge integral
Calculation

Why? Deep relationship

M_g — moduli spaces of sheaves on
Surfaces and 3 fold

GW/DT Correspondence [MNOP]

$$e^{iu} = -q$$

INSTITUTO SUPERIOR TÉCNICO

QDE $\quad q \frac{d}{dq} \Psi = D * \Psi \quad \Psi \in \mathbb{F}[[q]]$

$$(*D)_{\substack{q=0 \\ t_1 = 1 \\ t_2 = -\theta}} \cong \mathcal{H}_{\text{Calogero-Sutherland}}$$

Quantum Mech System
describing particles moving
on the torus $|z_i| = 1$
interacting via $\frac{1}{|z_i - z_j|^2}$

Exactly Solvable
Eigenfunction Jack polys.

$q \frac{d}{dq} \Psi = D * \Psi \quad$ nonstationary perturbation
Has integrability features
Ration Solutions when
$t_1 + t_2 \in \mathbb{Z}$

INSTITUTO SUPERIOR TÉCNICO

Quantum Cohomology of $Hilb(\mathbb{C}^2, n)$ is the beginning of several lines of inquiry.

motivates GW/DT correspondence [MNOP]

Key step in proof of GW/DT for torics [MOOP]

Basic role in the Study of descendent vertex for stable pair and rationality [Thomas-P] [Pixton-P]