Open problems (for AGNES)

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April 2010

Below are a few basic questions and speculations related to the moduli spaces of curves, $K3$ surfaces, maps, and sheaves presented in the problem session of the AGNES conference in Amherst (April 2010).

(i) *On the virtual class:*

Let X be a nonsingular, projective variety over $\mathbb{C}$. Let $\overline{M}_g(X,\beta)$ be the moduli space of stable maps and let

$$\pi\colon \overline{M}_g(X,\beta) \to \overline{M}_g$$

be the forgetful morphism, see [5] for background. The moduli space of stable maps carries a virtual class $[\overline{M}_g(X,\beta)]^{\mathrm{vir}}$ obtained from deformation theory [1, 2, 13]. Tautological classes in the Chow and cohomology rings of $\overline{M}_g$ are defined efficiently in [7]. For a discussion of the properties (many conjectural) of tautological classes, see also [6, 21].

Q1. Does $\pi_*[\overline{M}_g(X,\beta)]^{\mathrm{vir}} \in H^*(\overline{M}_g)$ lie in the tautological ring in cohomology?

Q2. When does $\pi_*[\overline{M}_g(X,\beta)]^{\mathrm{vir}} \in A^*(\overline{M}_g)$ lie in the tautological ring in Chow?

I would guess the answer to **Q1** is yes. If X is a curve, an affirmative answer to **Q1** follows from the results of [7]. We know $\pi_*[\overline{M}_g(X,\beta)]^{\mathrm{vir}}$ does not always lie in the tautological ring in Chow — counterexamples can be found when X is a curve. A wild speculation, motivated by the Bloch-Beilinson conjecture, is that the answer to **Q2** is yes when X is defined over over $\overline{\mathbb{Q}}$.

(ii) *On the Virasoro constraints:*

The spaces $\overline{M}_g(X, \beta)$ determine the Gromov–Witten invariants of X. These are conjectured to satisfy the Virasoro constraints [4]. Virasoro constraints are known to hold now in many, but not all, cases [8, 14]. A very interesting variety for which the Virasoro constraints are unknown is the Enriques surface.

Q3. Prove the Virasoro constraints in case X is an Enriques surface.

A study of the Gromov-Witten theory of the Enriques surface, closely related to modular forms, has been started in [18]. The Enriques surfaces is perhaps the most basic variety where new techniques are required to establish the Virasoro constraints.

(iii) *On the moduli of sheaves:*

Let X be a nonsingular, projective 3-fold. The Gromov-Witten theory of X, defined via $\overline{M}_g(X, \beta)$, is conjecturally [15] equivalent to the Donaldson-Thomas theory of X. The latter is defined via the moduli of ideal sheaves of curves in X [3, 25], or more recently, in terms of the moduli spaces of stable pairs [22].

Q4. Prove the GW/DT correspondence for 3-folds.

The toric cases of **Q4** are known [16]. Algebraic cobordism results [12] suggest the possibility of reducing to the toric case using degeneration methods.

Donaldson–Thomas invariants are defined only in dimension 3 because a virtual fundamental class for the moduli space of sheaves is required. Deformations are given by $\mathrm{Ext}^1(E, E)$, obstructions by $\mathrm{Ext}^2(E, E)$, and to define the virtual fundamental class we need (roughly) the vanishing

$$\mathrm{Ext}^i(E,E) = 0 \ \text{ for } \ i > 2 \ .$$

On 3-folds, the vanishing can often be obtained using Serre duality and stability. However, there are parallel examples of enumerative computations in higher dimensions in Gromov-Witten theory [11, 23]. Moreover, many aspects of Joyce's counting theory are valid in higher dimensions [9].

Q5. Define Donaldson–Thomas invariants in dimensions > 3.

(iv) *On the moduli of* $K3$ *surfaces:*

Let M^{K3}_{2n} denote the moduli space of polarized $K3$ surfaces (S,L) of degree $L^2 = 2n$. Little appears to be known about the cycle theory of M^{K3}_{2n}.

Q6. What is the analogue of the tautological ring for M^{K3}_{2n}?

A natural guess for **Q6** is the subring generated by the classes of the Noether–Lefschetz loci. The Noether-Lefschetz loci parameterize $K3$ surfaces with higher rank Picard lattices.

Q7. Do the Noether-Lefschetz divisors span $\mathrm{Pic}(M^{K3}_{2n}) \otimes_{\mathbb{Z}} \mathbb{Q}$?

Let X be a compact Calabi-Yau 3-fold expressed as $K3$-fibration over $\mathbb{P}^1$,

$$\pi : X \to \mathbb{P}^1 \ .$$

Given an ample line bundle L on X, the family π determines a morphism of the base $\mathbb{P}^1$ to the moduli of polarized $K3$ surfaces. Via [19], the Gromov-Witten theory of X in π-fiber classes is calculated in terms of the Noether–Lefschetz numbers of π and the Katz-Klemm-Vafa [10] conjecture concerning λ_g integrals in the reduced Gromov-Witten theory of a fixed $K3$ surface. The KKV conjecture is proven for all classes in genus 0 in [24] and all genera in primitive classes in [20].

Q8. Prove the Katz-Klemm-Vafa conjecture for in all genera and in all classes on $K3$ surfaces.

A solution to **Q8** would provide a large class of exact formulas for higher genus Gromov-Witten invariants of compact Calabi-Yau 3-folds. Unlike the local toric cases, mathematical results for higher genus Gromov-Witten invariants have been difficult to obtain, see [26] for the genus 1 theory of the quintic 3-fold.

Q9. Find effective mathematical methods for calculating the higher genus Gromov-Witten invariants of compact Calabi-Yau 3-folds.

Effective methods for the Enriques Calabi-Yau in genus $g \leq 2$ have been found in [18]. Complete, but less effective, techniques for the quintic are explained in [17]. At present, the holomorphic anomaly equation in topological string theory is more effective than the higher genus mathematical methods.

I was partially supported by the NSF through DMS-0500187. Thanks are due to P. Hacking and J. Tevelev for organizing the Algebraic Geometry Northeastern Series (AGNES) conference. The present article was adapted from P. Hacking's notes of my lecture.

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