Relations in the tautological ring of the moduli space of curves

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Abstract

The virtual geometry of the moduli space of stable quotients is used to obtain Chow relations among the κ classes on the moduli space of nonsingular genus g curves. In a series of steps, the stable quotient relations are rewritten in successively simpler forms. The final result is the proof of the Faber-Zagier relations (conjectured in 2000).

Contents

0 Introduction

0.1 Tautological classes

For $g \geq 2$, let $\mathcal{M}_g$ be the moduli space of nonsingular, projective, genus g curves over $\mathbb{C}$, and let

$$\pi : \mathcal{C}_g \to \mathcal{M}_g \tag{1}$$

be the universal curve. We view $\mathcal{M}_g$ and $\mathcal{C}_g$ as nonsingular, quasi-projective, Deligne-Mumford stacks. However, the orbifold perspective is sufficient for most of our purposes.

The relative dualizing sheaf ω_π of the morphism (1) is used to define the cotangent line class

$$\psi = c_1(\omega_\pi) \in A^1(\mathcal{C}_g, \mathbb{Q}) \ .$$

The κ classes are defined by push-forward,

$$\kappa_r = \pi_*(\psi^{r+1}) \in A^r(\mathcal{M}_g) \ .$$

The *tautological ring*

$$R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g, \mathbb{Q})$$

is the $\mathbb{Q}$-subalgebra generated by all of the κ classes. Since

$$\kappa_0 = 2g - 2 \in \mathbb{Q}$$

is a multiple of the fundamental class, we need not take κ_0 as a generator. There is a canonical quotient

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \xrightarrow{q} R^*(\mathcal{M}_g) \longrightarrow 0 \ .$$

We study here the ideal of relations among the κ classes, the kernel of q.

We may also define a tautological ring $RH^*(\mathcal{M}_g) \subset H^*(\mathcal{M}_g, \mathbb{Q})$ generated by the κ classes in cohomology. Since there is a natural factoring

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \xrightarrow{q} R^*(\mathcal{M}_g) \xrightarrow{c} RH^*(\mathcal{M}_g)$$

via the cycle class map c, algebraic relations among the κ classes are also cohomological relations. Whether or not there exist *more* cohomological relations is not yet settled.

There are two basic motivations for the study of the tautological rings $R^*(\mathcal{M}_g)$. The first is Mumford's conjecture, proven in 2002 by Madsen and Weiss [11],

$$\lim_{g\to\infty} H^*(\mathcal{M}_g, \mathbb{Q}) \ = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots],$$

determining the *stable* cohomology of the moduli of curves. While the κ classes do not exhaust $H^*(\mathcal{M}_g, \mathbb{Q})$, there are no other stable classes. The study of $R^*(\mathcal{M}_g)$ undertaken here is from the opposite perspective — we are interested in the ring of κ classes for fixed g.

The second motivation is from a large body of cycle class calculations on $\mathcal{M}_g$ (often related to Brill-Noether theory). The answers invariably lie in the tautological ring $R^*(\mathcal{M}_g)$. The first definition of the tautological rings by Mumford [14] was at least partially motivated by such algebro-geometric cycle constructions.

0.2 Faber-Zagier conjecture

Faber and Zagier have conjectured a remarkable set of relations among the κ classes in $R^*(\mathcal{M}_g)$. Our main result is a proof of the Faber-Zagier relations, stated as Theorem 1 below, by a geometric construction involving the virtual class of the moduli space of stable quotients.

To write the Faber-Zagier relations, we will require the following notation. Let the variable set

$$\mathbf{p} = \{\ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots\ \}$$

be indexed by positive integers *not* congruent to 2 modulo 3. Define the series

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i$$
$$+ (p_1 + tp_4 + t^2 p_7 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i \ .$$

Since Ψ has constant term 1, we may take the logarithm. Define the constants $C_r^{\mathsf{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ t^r \mathbf{p}^{\sigma} \ .$$

The above sum is over all partitions σ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. The empty partition is included in the sum. To the partition $\sigma = 1^{n_1}3^{n_3}4^{n_4}\cdots$, we associate the monomial $\mathbf{p}^\sigma = p_1^{n_1}p_3^{n_3}p_4^{n_4}\cdots$. Let

$$\gamma^{\mathsf{FZ}} = \sum_\sigma \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^\sigma\ .$$

For a series $\Theta \in \mathbb{Q}[\kappa][[t,\mathbf{p}]]$ in the variables κ_i, t, and p_j, let $[\Theta]_{t^r\mathbf{p}^\sigma}$ denote the coefficient of $t^r\mathbf{p}^\sigma$ (which is a polynomial in the κ_i).

Theorem 1. *In $R^r(\mathcal{M}_g)$, the Faber-Zagier relation*

$$\left[\exp(-\gamma^{\mathsf{FZ}})\right]_{t^r\mathbf{p}^\sigma} = 0$$

holds when $g-1+|\sigma| < 3r$ and $g \equiv r+|\sigma|+1 \mod 2$.

The dependence upon the genus g in the Faber-Zagier relations of Theorem 1 occurs in the inequality, the modulo 2 restriction, and via $\kappa_0 = 2g-2$. For a given genus g and codimension r, Theorem 1 provides only *finitely* many relations. While not immediately clear from the definition, the $\mathbb{Q}$-linear span of the Faber-Zagier relations determines an ideal in $\mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\ldots]$ — the matter is discussed in Section 6 and a subset of the Faber-Zagier relations generating the same ideal is described.

As a corollary of our proof of Theorem 1 via the moduli space of stable quotients, we obtain the following stronger boundary result. If $g-1+|\sigma| < 3r$ and $g \equiv r+|\sigma|+1 \mod 2$, then

$$\left[\exp(-\gamma^{\mathsf{FZ}})\right]_{t^r\mathbf{p}^\sigma} \in R^*(\partial\overline{\mathcal{M}}_g)\ . \tag{2}$$

Not only is the Faber-Zagier relation 0 on $R^*(\mathcal{M}_g)$, but the relation is equal to a tautological class on the boundary of the moduli space $\overline{\mathcal{M}}_g$. A precise conjecture for the boundary terms has been proposed in [18].

0.3 Gorenstein rings

By results of Faber [3] and Looijenga [10], we have

$$\dim_{\mathbb{Q}} R^{g-2}(\mathcal{M}_g) = 1, \quad R^{>g-2}(\mathcal{M}_g) = 0. \tag{3}$$

A canonical parameterization of $R^{g-2}(\mathcal{M}_g)$ is obtained via integration. Let

$$\mathbb{E} \to \mathcal{M}_g$$

be the *Hodge bundle* with fiber $H^0(C,\omega_C)$ over the moduli point $[C] \in \mathcal{M}_g$. Let λ_k denote the k^{th} Chern class of $\mathbb{E}$. The linear map

$$\epsilon : \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \longrightarrow \mathbb{Q}, \qquad f(\kappa) \stackrel{\epsilon}{\longmapsto} \int_{\overline{\mathcal{M}}_g} f(\kappa) \cdot \lambda_g \lambda_{g-1}$$

factors through $R^*(\mathcal{M}_g)$ and determines an isomorphism

$$\epsilon : R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

via the non-trivial evaluation

$$\int_{\overline{M}_g} \kappa_{g-2} \lambda_g \lambda_{g-1} = \frac{1}{2^{2g-1}(2g-1)!!} \frac{|B_{2g}|}{2g} \ . \tag{4}$$

A survey of the construction and properties of ϵ can be found in [5].

The evaluations under ϵ of all polynomials in the κ classes are determined by the following formulas. First, the Virasoro constraints for surfaces [7] imply a related evaluation previously conjectured in [3]:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g \lambda_{g-1} = \frac{(2g+n-3)!(2g-1)!!}{(2g-1)! \prod_{i=1}^{n} (2\alpha_i - 1)!!} \int_{\overline{M}_g} \kappa_{g-2} \lambda_g \lambda_{g-1}, \tag{5}$$

where $\alpha_i > 0$. Second, a basic relation (due to Faber) holds:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g \lambda_{g-1} = \sum_{\sigma \in \mathsf{S}_n} \int_{\overline{M}_g} \kappa_\sigma \lambda_g \lambda_{g-1} \ . \tag{6}$$

The sum on the right is over all elements of the symmetric group $\mathbb{S}_n$,

$$\kappa_\sigma = \kappa_{|c_1|} \ldots \kappa_{|c_r|}$$

where $c_1, \ldots, c_r$ is the set partition obtained from the cycle decomposition of σ, and

$$|c_i| = \sum_{j \in c_i} (\alpha_j - 1) \ .$$

Relation (6) is triangular and can be inverted to express the ϵ evaluations of the κ monomials in terms of (5).

Computations of the tautological rings in low genera led Faber to formulate the following conjecture in 1991.

Conjecture 1. *For all $g \geq 2$ and all $0 \leq k \leq g-2$, the pairing*

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \xrightarrow{\epsilon \circ \cup} \mathbb{Q} \tag{7}$$

is perfect.

The pairing (7) is the ring multiplication $\cup$ of $R^*(\mathcal{M}_g)$ composed with ϵ. A perfect pairing identifies the first vector space with the dual of the second. If Faber's conjecture is true in genus g, then $R^*(\mathcal{M}_g)$ is a Gorenstein local ring.

Let $\mathcal{I}_g \subset R^*(\mathcal{M}_g)$ be the ideal determined by the kernel of the pairing (7) in Faber's conjecture. Define the *Gorenstein quotient*

$$R^*_{\mathrm{G}}(\mathcal{M}_g) = \frac{R^*(\mathcal{M}_g)}{\mathcal{I}_g} \ .$$

If Faber's conjecture is true for g, then $\mathcal{I}_g = 0$ and $R^*_{\mathrm{G}}(\mathcal{M}_g) = R^*(\mathcal{M}_g)$.

The pairing (7) can be evaluated directly on polynomials in the κ classes via (4)-(6). The Gorenstein quotient $R^*_{\mathrm{G}}(\mathcal{M}_g)$ is completely determined by the κ evaluations and the ranks (3). The ring $R^*_{\mathrm{G}}(\mathcal{M}_g)$ can therefore be studied as a purely algebro-combinatorial object.

Faber and Zagier conjectured the relations of Theorem 1 from a concentrated study of the Gorenstein quotient $R^*_{\mathrm{G}}(\mathcal{M}_g)$. The Faber-Zagier relations were first written in 2000 and were proven to hold in $R^*_{\mathrm{G}}(\mathcal{M}_g)$ in 2002. The validity of the Faber-Zagier relations in $R^*(\mathcal{M}_g)$ has been an open question since then.

0.4 Other relations?

By substantial computation, Faber has verified Conjecture 1 holds for genus $g < 24$. Moreover, his calculations show the Faber-Zagier set yields *all* relations among κ classes in $R^*(\mathcal{M}_g)$ for $g < 24$. However, he finds the Faber-Zagier relations of Theorem 1 do *not* yield a Gorenstein quotient in genus 24. Let

$$\mathsf{FZ}_g \subset \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]$$

be the ideal determined by the Faber-Zagier relations of Theorem 1, and let

$$R^*_{\mathrm{FZ}}(\mathcal{M}_g) = \frac{\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]}{\mathsf{FZ}_g} \ .$$

Faber finds a mismatch in codimension 12,

$$R^{12}_{\mathrm{FZ}}(\mathcal{M}_{24}) \neq R^{12}_{\mathrm{G}}(\mathcal{M}_{24}) \ . \tag{8}$$

Exactly 1 more relation holds in the Gorenstein quotient.

To the best of our knowledge, a relation in $R^*(\mathcal{M}_g)$ which is not in the span of the Faber-Zagier relations of Theorem 1 has not yet been found. The following prediction is consistent with all present calculations.

Conjecture 2. *For all $g \geq 2$, the kernel of*

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \xrightarrow{q} R^*(\mathcal{M}_g) \longrightarrow 0$$

is the Faber-Zagier ideal FZ_g.

Conjectures 1 and 2 are both true for $g < 24$. By the inequality (8), Conjectures 1 and 2 can *not* both be true for all g. Which is false?

Finally, we note the above discussion might have a different outcome if the tautological ring $RH^*(\mathcal{M}_g)$ in cohomology is considered instead. Perhaps there are more relations in cohomology? These questions provide a very interesting line of inquiry.

0.5 Plan of the paper

We start the paper in Section 1 with a modern treatment of Faber's classical construction of relations among the κ classes. The result, in Wick form, is stated as Theorem 2 of Section 1.2. While the outcome is an effective source of relations, their complexity has so far defied a complete analysis.

After reviewing stable quotients on curves in Section 2, we derive an explicit set of κ relations from the virtual geometry of the moduli space of stable quotients in Section 3. The resulting equations are more tractable than those obtained by classical methods. In a series of steps, the stable quotient relations are transformed to simpler and simpler forms. The first step, Theorem 5, comes almost immediately from the virtual localization formula [8] applied to the moduli space of stable quotients. After further analysis in Section 4, the simpler form of Proposition 10 is found. A change of variables is applied in Section 5 that transforms the relations to Proposition 15. Our final result, Theorem 1, establishes the previously conjectural set of tautological relations proposed more than a decade ago by Faber and Zagier. The proof of Theorem 1 is completed in Section 6.

A natural question is whether Theorem 1 can be extended to yield explicit relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. A precise conjecture of exactly such an extension is given in [18]. There is no doubt that our methods here can also be applied to investigate tautological relations in $\overline{\mathcal{M}}_{g,n}$. Whether the simple form of [18] will be obtained remains to be seen. A different method, valid only in cohomology, of approaching the conjecture of [18] is pursued in [17].

0.6 Acknowledgements

We first presented our proof of the Faber-Zagier relations in a series of lectures at Humboldt University in Berlin during the conference *Intersection theory on moduli space* in 2010. A detailed set of notes, which is the origin of the current paper, is available [16]. We thank G. Farkas for the invitation to speak there.

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1 Classical vanishing relations

1.1 Construction

Faber's original relations in his article *Conjectural description of the tautological ring* [3] are obtained from a very simple geometric construction. As before, let

$$\pi : \mathcal{C}_g \to \mathcal{M}_g$$

be the universal curve over the moduli space, and let

$$\pi^d : \mathcal{C}_g^d \to \mathcal{M}_g$$

be the map associated to the d^{th} fiber product of the universal curve. For every point $[C, p_1, \ldots, p_d] \in \mathcal{C}_g^d$, we have the restriction map

$$H^0(C, \omega_C) \to H^0(C, \omega_C|_{p_1+\ldots+p_d}) \ . \tag{9}$$

Since the canonical bundle ω_C has degree $2g-2$, the restriction map is injective if $d > 2g-2$. Let

$$\Omega_d \to \mathcal{C}_g^d$$

be the rank d bundle with fiber $H^0(C, \omega_C|_{p_1+\ldots+p_d})$ over the moduli point $[C, p_1, \ldots, p_d] \in \mathcal{C}_g^d$. If $d > 2g-2$, the restriction map (9) yields an exact sequence over $\mathcal{C}^d$,

$$0 \to \mathbb{E} \to \Omega_d \to Q_{d-g} \to 0$$

where $\mathbb{E}$ is the rank g Hodge bundle and Q_{d-g} is the quotient bundle of rank $d-g$. We see

$$c_k(Q_{d-g}) = 0 \in A^k(\mathcal{C}_g^d) \quad \text{for} \quad k > d-g \ .$$

After cutting the vanishing Chern classes $c_k(Q_{d-g})$ down with cotangent line and diagonal classes in $\mathcal{C}_g^d$ and pushing-forward via π_*^d to $\mathcal{M}_g$, we arrive at Faber's relations in $R^*(\mathcal{M}_g)$.

1.2 Wick form

From our point of view, at the center of Faber's relations in [3] is the function

$$\Theta(t,x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} (1+it) \ \frac{(-1)^d}{d!} \frac{x^d}{t^d} \ .$$

The differential equation

$$t(x+1)\frac{d}{dx}\Theta + (t+1)\Theta = 0$$

is easily found. Hence, we obtain the following result.

Lemma 1. $\Theta = (1+x)^{-\frac{t+1}{t}}$.

We introduce a variable set $\mathbf{z}$ indexed by pairs of integers

$$\mathbf{z} = \{ \ z_{i,j} \mid i \geq 1, \ \ j \geq i-1 \ \} \ .$$

For monomials

$$\mathbf{z}^\sigma = \prod_{i,j} z_{i,j}^{\sigma_{i,j}},$$

we define

$$\ell(\sigma) = \sum_{i,j} i\sigma_{i,j}, \quad |\sigma| = \sum_{i,j} j\sigma_{i,j} \ .$$

Of course $|\mathrm{Aut}(\sigma)| = \prod_{i,j} \sigma_{i,j}!$.

The variables $\mathbf{z}$ are used to define a differential operator

$$\mathcal{D} = \sum_{i,j} z_{i,j} \ t^j \left(x \frac{d}{dx} \right)^i \ .$$

After applying $\exp(\mathcal{D})$ to Θ, we obtain

$$\begin{aligned} \Theta^{\mathcal{D}} &= \exp(\mathcal{D}) \ \Theta \\ &= \sum_{\sigma} \sum_{d=0}^{\infty} \prod_{i=1}^{d} (1+it) \ \frac{(-1)^d}{d!} \frac{x^d}{t^d} \ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{z}^\sigma}{|\mathrm{Aut}(\sigma)|} \end{aligned}$$

where σ runs over all monomials in the variables $\mathbf{z}$. Define constants $C_r^d(\sigma)$ by the formula

$$\log(\Theta^{\mathcal{D}}) = \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_r^d(\sigma) \ t^r \frac{x^d}{d!} \mathbf{z}^\sigma \ .$$

By an elementary application of Wick's formula (as explained in Section 1.3.2 below), the t dependence of $\log(\Theta^{\mathcal{D}})$ has at most simple poles.

Finally, we consider the following function,

$$\gamma^{\mathsf{F}} = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} + \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_r^d(\sigma) \ \kappa_r t^r \frac{x^d}{d!} \mathbf{z}^\sigma \ . \qquad (10)$$

The Bernoulli numbers appear in the first term,

$$\sum_{k=0}^{\infty} B_k \frac{u^k}{k!} = \frac{u}{e^u - 1} \ .$$

Denote the $t^r x^d \mathbf{z}^\sigma$ coefficient of $\exp(-\gamma^{\mathsf{F}})$ by

$$\left[\exp(-\gamma^{\mathsf{F}}) \right]_{t^r x^d \mathbf{z}^\sigma} \in \mathbb{Q}[\kappa_{-1}, \kappa_0, \kappa_1, \kappa_2, \ldots] \ .$$

Our form of Faber's equations is the following result.

Theorem 2. *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[\exp(-\gamma^{\mathsf{F}})\Big]_{t^r x^d \mathbf{z}^\sigma} = 0$$

holds when $r > -g + |\sigma|$ and $d > 2g - 2$.

In the tautological ring $R^*(\mathcal{M}_g)$, the standard conventions

$$\kappa_{-1} = 0, \qquad \kappa_0 = 2g - 2$$

are followed. For fixed g and r, Theorem 2 provides infinitely many relations by increasing d. The variables $z_{i,j}$ efficiently encode both the cotangent and diagonal operations studied in [3]. In particular, the relations of Theorem 2 are equivalent to a mixing of all cotangent and diagonal operations studied there. The proof of Theorem 2 is presented in Section 1.3.

While Theorem 2 has an appealingly simple geometric origin, the relations do not seem to fit the other forms we will see later. In particular, we do not know how to derive Theorem 1 from Theorem 2. Extensive computer calculations by Faber suggest the following.

Conjecture 3. *For all $g \geq 2$, the relations of Theorem 2 are equivalent to the Faber-Zagier relations.*

In particular, despite significant effort, the relation in $R^{12}_{\mathsf{G}}(\mathcal{M}_{24})$ which is missing in $R^{12}_{\mathsf{FZ}}(\mathcal{M}_{24})$ has *not* been found via Theorem 2. Other geometric strategies have so far also failed to find the missing relation [19, 20].

1.3 Proof of Theorem 2

1.3.1 The Chern roots of Ω_d

Let $\psi_i \in A^1(\mathcal{C}^d_g, \mathbb{Q})$ be the first Chern class of the relative dualizing sheaf ω_π pulled back from the i^{th} factor,

$$\mathcal{C}^d_g \to \mathcal{C}_g \ .$$

For $i \neq j$, let $D_{ij} \in A^1(\mathcal{C}^d_g, \mathbb{Q})$ be the class of the diagonal $\mathcal{C}_g \subset \mathcal{C}^2_g$ pulled-back from the product of the i^{th} and j^{th} factors,

$$\mathcal{C}^d_g \to \mathcal{C}^2_g \ .$$

Finally, let

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i} \ \in A^1(\mathcal{C}_g^d, \mathbb{Q})$$

following the convention $\Delta_1 = 0$. The Chern roots of Ω_d,

$$\begin{aligned} c_t(\Omega_d) &= \prod_{i=1}^{d} 1 + (\psi_i - \Delta_i)t \qquad (11) \\ &= (1+\psi_1 t) \cdot \big(1 + (\psi_2 - D_{12})t\big) \cdots \left(1 + \Big(\psi_d - \sum_{i=1}^{d-1} D_{id}\Big)t\right) \end{aligned}$$

are obtained by a simple induction, see [3].

We may expand the right side of (11) fully. The resulting expression is a polynomial in the $d + \binom{d}{2}$ variables.

$$\psi_1, \ldots, \psi_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \ .$$

The sign on the diagonal variables is chosen because of the self-intersection formula

$$(-D_{ij})^2 = \psi_i(-D_{ij}) = \psi_j(-D_{ij}) \ .$$

Let M_r^d denote the coefficient in degree r,

$$c_t(\Omega_d) = \sum_{r=0}^{\infty} M_r^d(\psi_i, -D_{ij}) \ t^r.$$

Lemma 2. *After setting all the variables to 1,*

$$\sum_{r=0}^{\infty} M_r^d(\psi_i = 1, -D_{ij} = 1) \ t^r \ = \ \prod_{i=1}^{d} (1 + it).$$

Proof. The results follows immediately from the Chern roots (11). □

Lemma 2 may be viewed counting the number of terms in the expansion of the total Chern class $c_t(\Omega_d)$.

1.3.2 Connected counts

A monomial in the diagonal variables

$$D_{12}, D_{13}, \ldots, D_{d-1,d} \tag{12}$$

determines a set partition of $\{1, \ldots, d\}$ by the diagonal associations. For example, the monomial $3D_{12}^2 D_{1,3} D_{56}^3$ determines the set partition

$$\{1,2,3\} \,\cup\, \{4\} \,\cup\, \{5,6\}$$

in the $d = 6$ case. A monomial in the variables (12) is *connected* if the corresponding set partition consists of a single part with d elements.

A monomial in the variables

$$\psi_1, \ldots, \psi_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \tag{13}$$

is connected if the corresponding monomial in the diagonal variables obtained by setting all $\psi_i = 1$ is connected. Let S_r^d be the summand of the evaluation $M_r^d(\psi_i = 1, -D_{ij} = 1)$ consisting of the contributions of only the connected monomials.

Lemma 3. *We have*

$$\sum_{d=1}^{\infty} \sum_{r=0}^{d} S_r^d \ t^r \frac{x^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} (1+it) \frac{x^d}{d!}\right) .$$

Proof. By a standard application of Wick's formula, the connected and disconnected counts are related by exponentiation,

$$\exp\left(\sum_{d=1}^{\infty} \sum_{r=0}^{d} S_r^d \ t^r \frac{x^d}{d!}\right) = 1 + \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} M_r^d(\psi_i = 1, -D_{ij} = 1) \ t^r \frac{x^d}{d!} .$$

The right side is then evaluated by Lemma 2. $\square$

Since a connected monomial in the variables (13) must have at least $d-1$ factors of the variables $-D_{ij}$, we see $S_r^d = 0$ if $r < d-1$. Using the self-intersection formulas, we obtain

$$\sum_{d=1}^{\infty} \sum_{r=0}^{d} \pi_*^d\big(c_r(\Omega_d)\big) \ t^r \frac{x^d}{d!} = \exp\left(\sum_{d=1}^{\infty} \sum_{r=0}^{d} S_r^d (-1)^{d-1} \kappa_{r-d} \ t^r \frac{x^d}{d!}\right) . \tag{14}$$

To account for the alternating factor $(-1)^{d-1}$ and the κ subscript, we define the coefficients C_r^d by

$$\sum_{d=1}^{\infty} \sum_{r \geq -1}^{d} C_r^d \ t^r \frac{x^d}{d!} = \log \left(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} (1+it) \frac{(-1)^d}{t^d} \frac{x^d}{d!} \right) \ .$$

The vanishing $S^d_{r<d-1} = 0$ implies the vanishing $C^d_{r<-1} = 0$.

The formula for the total Chern class of the Hodge bundle $\mathbb{E}$ on $\mathcal{M}_g$ follows immediately from Mumford's Grothendieck-Riemann-Roch calculation [14],

$$c_t(\mathbb{E}) = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} \ .$$

Putting the above results together yields the following formula:

$$\sum_{d=1}^{\infty} \sum_{r \geq 0} \pi_*^d \big(c_r(Q_{d-g}) \big) \ t^{r-d} \frac{x^d}{d!} =$$
$$\exp \left(- \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} - \sum_{d=1}^{\infty} \sum_{r \geq -1} C_r^d \ \kappa_r t^r \frac{x^d}{d!} \right) \ .$$

1.3.3 Cutting

For $d > 2g-2$ and $r > d-g$, we have the vanishing

$$c_r(Q_{d-g}) = 0 \in A^r(\mathcal{C}_g^d, \mathbb{Q}) \ .$$

Before pushing-forward via π^d, we will cut $c_r(Q_{d-g})$ with products of classes in $A^*(\mathcal{C}_g^d, \mathbb{Q})$. With the correct choice of cutting classes, we will obtain the relations of Theorem 2.

Let (a, b) be a pair of integers satisfying $a \geq 0$ and $b \geq 1$. We define the cutting class

$$\phi[a,b] = (-1)^{b-1} \sum_{|I|=b} \psi_I^a D_I \tag{15}$$

where $I \subset \{1, \ldots, d\}$ is subset of order b, $D_I \in A^{b-1}(\mathcal{C}_g^d, \mathbb{Q})$ is the class of the corresponding small diagonal, and ψ_I is the cotangent line at the point indexed by I. The class ψ_I is well-defined on the small diagonal indexed by

I. The degree of $\phi[a,b]$ is $a+b-1$. The number of terms on the right side of (15) is a degree b polynomial in d,

$$\binom{d}{b} = \frac{d^b}{b!} + \ldots + (-1)^{b-1}\frac{d}{b}$$

with no constant term.

The sign $(-1)^{b-1}$ in definition (15) is chosen to match the sign conventions of the Wick analysis in Section 1.3.2. For example,

$$\phi[0,2] = \sum_{i<j}(-D_{ij}) \ , \qquad \phi[0,3] = \sum_{i<j<k}(-D_{ij})(-D_{jk}).$$

The *number of terms* means the evaluation at $\psi_I = 1$ and $-D_{ij} = -1$.

A better choice of cutting class is obtained by the following observation. For every pair of integers (i,j) with $i \geq 1$ and $j \geq i-1$, we can find a unique linear combination

$$\Phi[i,j] = \sum_{a+b-1=j} \lambda_{a,b} \cdot \phi[a,b], \quad \lambda_{a,b} \in \mathbb{Q}$$

for which the evaluation of $\Phi[i,j]$ at $\psi_I = 1$ and $-D_{ij} = -1$ is d^i. By definition, $\Phi[i,j]$ is of pure degree j.

1.3.4 Full Wick form

We repeat the Wick analysis of Section 1.3.2 for the Chern class of Q_{d-g} cut by the classes $\Phi[i,j]$ in order to write a formula for

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi^d_* \left(\exp\left(\sum_{i,j} z_{i,j}t^j\Phi[i,j]\right) \cdot c_r(Q_{d-g})t^r \right) \ \frac{1}{t^d}\frac{x^d}{d!}$$

where the sum in the argument of the exponential is over all $i \geq 1$ and $j \geq i-1$. The variable set $\mathbf{z}$ introduced in Section 1.2 appears here. Since $\Phi[i,j]$ yields d^i after evaluation at $\psi_I = 1$ and $-D_{ij} = -1$ and is of pure degree j, we conclude

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi^d_* \left(\exp\left(\sum_{i,j} z_{i,j}t^j\Phi[i,j]\right) \cdot c_r(Q_{d-g})t^r \right) \ \frac{1}{t^d}\frac{x^d}{d!} = \exp(-\gamma^{\mathsf{F}}) \ . \quad (16)$$

Let $d > 2g - 2$. Since $c_s(Q_{d-g}) = 0$ for $s > d - g$, the $t^r x^d \mathbf{z}^\sigma$ coefficient of (16) vanishes if

$$r + d - |\sigma| > d - g$$

which is equivalent to $r > -g + |\sigma|$. The proof of Theorem 2 is complete. $\square$

2 Stable quotients

2.1 Stability

Our proof of the Faber-Zagier relations in $R^*(M_g)$ will be obtained from the virtual geometry of the moduli space of stable quotients. We start by reviewing the basic definitions and results of [13].

Let C be a curve which is reduced and connected and has at worst nodal singularities. We require here only unpointed curves. See [13] for the definitions in the pointed case. Let q be a quotient of the rank N trivial bundle C,

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0.$$

If the quotient subsheaf Q is locally free at the nodes and markings of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let C be a curve equipped with a quasi-stable quotient q. The data (C, q) determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C \otimes (\wedge^r S^*)^{\otimes \epsilon} \qquad (17)$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbf{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

A stable quotient (C, q) yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r, N)$. We will only require the $\mathbb{G}(1, 2) = \mathbf{P}^1$ case for the proof Theorem 1.

2.2 Isomorphism

Let C be a curve. Two quasi-stable quotients

$$\mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0, \quad \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q'} Q' \to 0 \tag{18}$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, q) \to (C', q')$$

is an isomorphism of curves

$$\phi : C \xrightarrow{\sim} C'$$

such that the quotients q and $\phi^*(q')$ are strongly isomorphic. Quasi-stable quotients (18) on the same curve C may be isomorphic without being strongly isomorphic.

The following result is proven in [13] by Quot scheme methods from the perspective of geometry relative to a divisor.

Theorem 3. *The moduli space of stable quotients $\overline{Q}_g(\mathbb{G}(r,N),d)$ parameterizing the data*

$$(C,\ 0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0),$$

with $\operatorname{rank}(S) = r$ *and* $\deg(S) = -d$*, is a separated and proper Deligne-Mumford stack of finite type over* $\mathbb{C}$*.*

2.3 Structures

Over the moduli space of stable quotients, there is a universal curve

$$\pi : U \to \overline{Q}_g(\mathbb{G}(r,N),d) \tag{19}$$

with a universal quotient

$$0 \to S_U \to \mathbb{C}^N \otimes \mathcal{O}_U \xrightarrow{q_U} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\overline{Q}_g(\mathbb{G}(r,N),d)$ is equipped with two basic types of maps. If $2g-2>0$, then the stabilization of C determines a map

$$\nu : \overline{Q}_g(\mathbb{G}(r,N),d) \to \overline{M}_g$$

by forgetting the quotient.

The general linear group $\mathbf{GL}_N(\mathbb{C})$ acts on $\overline{Q}_g(\mathbb{G}(r,N),d)$ via the standard action on $\mathbb{C}^N \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathbf{GL}_N(\mathbb{C})$-equivariant.

2.4 Obstruction theory

The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_g(\mathbb{G}(r,N),d) \to \mathcal{M}_g.$$

The moduli of stable quotients with fixed underlying curve $[C] \in \mathcal{M}_g$ is simply an open set of the Quot scheme of C. The following result of [13, Section 3.2] is obtained from the standard deformation theory of the Quot scheme.

Theorem 4. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on the moduli space $\overline{Q}_g(\mathbb{G}(r,N),d)$ relative to ν^A given by $RHom(S,Q)$.*

More concretely, for the stable quotient,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0,$$

the deformation and obstruction spaces relative to ν^A are $\mathrm{Hom}(S,Q)$ and $\mathrm{Ext}^1(S,Q)$ respectively. Since S is locally free, the higher obstructions

$$\mathrm{Ext}^k(S,Q) = H^k(C, S^* \otimes Q) = 0, \quad k>1$$

vanish since C is a curve. An absolute 2-term obstruction theory on the moduli space $\overline{Q}_g(\mathbb{G}(r,N),d)$ is obtained from Theorem 4 and the smoothness of $\mathcal{M}_g$, see [1, 2, 7]. The analogue of Theorem 4 for the Quot scheme of a *fixed* nonsingular curve was observed in [12].

The $\mathbf{GL}_N(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathbf{GL}_N(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_g(\mathbb{G}(r,N),d)]^{vir} \in A_*^{\mathbf{GL}_N(\mathbb{C})}(\overline{Q}_g(\mathbb{G}(r,N),d)).$$

For the construction of the Faber-Zagier relation, we are mainly interested in the open stable quotient space

$$\nu : Q_g(\mathbf{P}^1, d) \longrightarrow \mathcal{M}_g$$

which is simply the corresponding relative Hilbert scheme. However, we will require the full stable quotient space $\overline{Q}_g(\mathbf{P}^1, d)$ to prove the Faber-Zagier relations can be completed over $\mathcal{M}_g$ with tautological boundary terms.

3 Stable quotients relations

3.1 First statement

Our relations in the tautological ring $R^*(\mathcal{M}_g)$ obtained from the moduli of stable quotients are based on the function

$$\Phi(t, x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it} \; \frac{(-1)^d}{d!} \frac{x^d}{t^d} \; . \tag{20}$$

Define the coefficients $\widetilde{C}_r^d$ by the logarithm,

$$\log(\Phi) = \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \; t^r \frac{x^d}{d!} \; .$$

Again, by an application of Wick's formula in Section 3.3, the t dependence has at most a simple pole. Let

$$\widetilde{\gamma} = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} + \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \kappa_r t^r \frac{x^d}{d!} \; . \tag{21}$$

Denote the $t^r x^d$ coefficient of $\exp(-\widetilde{\gamma})$ by

$$\big[\exp(-\widetilde{\gamma})\big]_{t^r x^d} \in \mathbb{Q}[\kappa_{-1}, \kappa_0, \kappa_1, \kappa_2, \ldots] \; .$$

In fact, $[\exp(-\widetilde{\gamma})]_{t^r x^d}$ is homogeneous of degree r in the κ classes.

The first form of the tautological relations obtained from the moduli of stable quotients is given by the following result.

Proposition 4. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp(-\widetilde{\gamma})\right]_{t^r x^d} = 0$$

holds when $g - 2d - 1 < r$ and $g \equiv r + 1 \mod 2$.

For fixed r and d, if Proposition 4 applies in genus g, then Proposition 4 applies in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$. The genus shifting mod 2 property is present also in the Faber-Zagier relations.

3.2 K-theory class $\mathbb{F}_d$

For genus $g \geq 2$, we consider as before

$$\pi^d : \mathcal{C}_g^d \to \mathcal{M}_g \ ,$$

the d-fold product of the universal curve over M_g. Given an element

$$[C, p_1, \ldots, p_d] \in \mathcal{C}_g^d \ ,$$

there is a canonically associated stable quotient

$$0 \to \mathcal{O}_C(-\sum_{j=1}^{d} p_j) \to \mathcal{O}_C \to Q \to 0. \tag{22}$$

Consider the universal curve

$$\epsilon : U \to \mathcal{C}_g^d$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (22). Let

$$\mathbb{F}_d = -R\epsilon_*(S_U^*) \in K(\mathcal{C}_g^d)$$

be the class in K-theory. For example,

$$\mathbb{F}_0 = \mathbb{E}^* - \mathbb{C}$$

is the dual of the Hodge bundle minus a rank 1 trivial bundle.

By Riemann-Roch, the rank of $\mathbb{F}_d$ is

$$r_g(d) = g - d - 1.$$

However, $\mathbb{F}_d$ is not always represented by a bundle. By the derivation of [13, Section 4.6],

$$\mathbb{F}_d = \mathbb{E}^* - \mathbb{B}_d - \mathbb{C}, \tag{23}$$

where $\mathbb{B}_d$ has fiber $H^0(C, \mathcal{O}_C(\sum_{j=1}^d p_j)|_{\sum_{j=1}^d p_j})$ over $[C, p_1, \ldots, p_d]$.

The Chern classes of $\mathbb{F}_d$ can be easily computed. Recall the divisor $D_{i,j}$ where the markings p_i and p_j coincide. Set

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i},$$

with the convention $\Delta_1 = 0$. Over $[C, p_1, \ldots, p_d]$, the virtual bundle $\mathbb{F}_d$ is the formal difference

$$H^1(\mathcal{O}_C(p_1 + \ldots + p_d)) - H^0(\mathcal{O}_C(p_1 + \ldots + p_d)).$$

Taking the cohomology of the exact sequence

$$0 \to \mathcal{O}_C(p_1 + \ldots + p_{d-1}) \to \mathcal{O}_C(p_1 + \ldots + p_d) \to \mathcal{O}_C(p_1 + \ldots + p_d)|_{\widehat{p}_d} \to 0,$$

we find

$$c(\mathbb{F}_d) = \frac{c(\mathbb{F}_{d-1})}{1 + \Delta_d - \psi_d}.$$

Inductively, we obtain

$$c(\mathbb{F}_d) = \frac{c(\mathbb{E}^*)}{(1 + \Delta_1 - \psi_1) \cdots (1 + \Delta_d - \psi_d)}.$$

Equivalently, we have

$$c(-\mathbb{B}_d) = \frac{1}{(1 + \Delta_1 - \psi_1) \cdots (1 + \Delta_d - \psi_d)}. \tag{24}$$

3.3 Proof of Proposition 4

Consider the proper morphism

$$\nu : Q_g(\mathbf{P}^1, d) \to M_g.$$

Certainly the class

$$\nu_*\left(0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir}\right) \in A^*(\mathcal{M}_g, \mathbb{Q}), \tag{25}$$

where 0 is the first Chern class of the trivial bundle, vanishes if $c > 0$. Proposition 4 is proven by calculating (25) by localization. We will find Proposition 4 is a subset of the much richer family of relations of Theorem 5 of Section 3.4.

Let the torus $\mathbb{C}^*$ act on a 2-dimensional vector space $V \stackrel{\sim}{=} \mathbb{C}^2$ with diagonal weights $[0, 1]$. The $\mathbb{C}^*$-action lifts canonically to $\mathbf{P}(V)$ and $Q_g(\mathbf{P}(V), d)$. We lift the $\mathbb{C}^*$-action to a rank 1 trivial bundle on $Q_g(\mathbf{P}(V), d)$ by specifying fiber weight 1. The choices determine a $\mathbb{C}^*$-lift of the class

$$0^c \cap [Q_g(\mathbf{P}(V), d)]^{vir} \in A_{2d+2g-2-c}(Q_g(\mathbf{P}(V), d), \mathbb{Q}).$$

The push-forward (25) is determined by the virtual localization formula [7]. There are only two $\mathbb{C}^*$-fixed loci. The first corresponds to a vertex lying over $0 \in \mathbf{P}(V)$. The locus is isomorphic to

$$\mathcal{C}_g^d \ / \ \mathbb{S}_d$$

and the associated subsheaf (22) lies in the first factor of $V \otimes \mathcal{O}_C$ when considered as a stable quotient in the moduli space $Q_g(\mathbf{P}(V), d)$. Similarly, the second fixed locus corresponds to a vertex lying over $\infty \in \mathbf{P}(V)$.

The localization contribution of the first locus to (25) is

$$\frac{1}{d!}\pi^d_*\left(c_{g-d-1+c}(\mathbb{F}_d)\right) \quad \text{where} \quad \pi^d : \mathcal{C}_g^d \to \mathcal{M}_g \ .$$

Let $c_-(\mathbb{F}_d)$ denote the total Chern class of $\mathbb{F}_d$ evaluated at -1. The localization contribution of the second locus is

$$\frac{(-1)^{g-d-1}}{d!}\pi^d_*\Big[c_-(\mathbb{F}_d)\Big]^{g-d-1+c}$$

where $[\gamma]^k$ is the part of γ in $A^k(\mathcal{C}_g^d, \mathbb{Q})$.

Both localization contributions are found by straightforward expansion of the vertex formulas of [13, Section 7.4.2]. Summing the contributions yields

$$\pi^d_*\Big(c_{g-d-1+c}(\mathbb{F}_d) \ + \ (-1)^{g-d-1}\Big[c_-(\mathbb{F}_d)\Big]^{g-d-1+c}\Big) \ = \ 0 \qquad \text{in} \quad R^*(\mathcal{M}_g)$$

for $c > 0$. We obtain the following result.

Lemma 5. *For $c > 0$ and $c \equiv 0 \mod 2$,*

$$\pi^d_*\Big(c_{g-d-1+c}(\mathbb{F}_d)\Big) = 0 \quad in \quad R^*(\mathcal{M}_g) \ .$$

For $c > 0$, the relation of Lemma 5 lies in $R^r(\mathcal{M}_g)$ where

$$r = g - 2d - 1 + c \ .$$

Moreover, the relation is trivial unless

$$g - d - 1 \equiv g - d - 1 + c = r - d \mod 2 \ . \tag{26}$$

We may expand the right side of (24) fully. The resulting expression is a polynomial in the $d + \binom{d}{2}$ variables.

$$\psi_1, \ldots, \psi_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \ .$$

Let $\widetilde{M}^d_r$ denote the coefficient in degree r,

$$c_t(-\mathbb{B}_d) = \sum_{r=0}^{\infty} \widetilde{M}^d_r(\psi_i, -D_{ij}) \ t^r.$$

Let $\widetilde{S}^d_r$ be the summand of the evaluation $\widetilde{M}^d_r(\psi_i = 1, -D_{ij} = 1)$ consisting of the contributions of only the connected monomials.

Lemma 6. *We have*

$$\sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \widetilde{S}^d_r \ t^r \frac{x^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it} \ \frac{x^d}{d!}\right) \ .$$

Proof. As before, by Wick's formula, the connected and disconnected counts are related by exponentiation,

$$\exp\left(\sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \widetilde{S}^d_r \ t^r \frac{x^d}{d!}\right) = 1 + \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \widetilde{M}^d_r(\widehat{\psi}_i = 1, -D_{ij} = 1) \ t^r \frac{x^d}{d!} \ .$$

□

Since a connected monomial in the variables ψ_i and $-D_{ij}$ must have at least $d-1$ factors of the variables $-D_{ij}$, we see $\widetilde{S}^d_r = 0$ if $r < d-1$. Using the self-intersection formulas, we obtain

$$\sum_{d=1}^{\infty} \sum_{r \geq 0} \pi^d_* \big(c_r(-\mathbb{B}_d) \big) \ t^r \frac{x^d}{d!} = \exp \left(\sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \widetilde{S}^d_r (-1)^{d-1} \kappa_{r-d} \ t^r \frac{x^d}{d!} \right) \ . \tag{27}$$

To account for the alternating factor $(-1)^{d-1}$ and the κ subscript, we define the coefficients $\widetilde{C}^d_r$ by

$$\sum_{d=1}^{\infty} \sum_{r \geq -1} \widetilde{C}^d_r \ t^r \frac{x^d}{d!} = \log \left(1 + \sum_{d=1}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \ \frac{(-1)^d}{t^d} \frac{x^d}{d!} \right) \ .$$

The vanishing $\widetilde{S}^d_{r<d-1} = 0$ implies the vanishing $\widetilde{C}^d_{r<-1} = 0$.

Again using Mumford's Grothendieck-Riemann-Roch calculation [14],

$$c_t(\mathbb{E}^*) = -\sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} \ .$$

Putting the above results together yields the following formula:

$$\sum_{d=1}^{\infty} \sum_{r \geq 0} \pi^d_* \big(c_r(\mathbb{F}_d) \big) \ t^{r-d} \frac{x^d}{d!} =$$
$$\exp \left(-\sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} - \sum_{d=1}^{\infty} \sum_{r \geq -1} \widetilde{C}^d_r \kappa_r t^r \frac{x^d}{d!} \right) \ .$$

The restrictions on g, d, and r in the statement of Proposition 4 are obtained from (26). □

3.4 Extended relations

The universal curve

$$\epsilon : U \to Q_g(\mathbf{P}^1, d)$$

carries the basic divisor classes

$$s = c_1(S^*_U), \quad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U of the moduli of stable quotients and the ϵ-relative dualizing sheaf. Following [13, Proposition 5], we can obtain a much larger set of relations in the tautological ring of $\mathcal{M}_g$ by including factors of $\epsilon_*(s^{a_i}\omega^{b_i})$ in the integrand:

$$\nu_*\left(\prod_{i=1}^{n} \epsilon_*(s^{a_i}\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir}\right) = 0 \text{ in } A^*(\mathcal{M}_g, \mathbb{Q})$$

when $c > 0$. We will study the associated relations where the a_i are always 1. The b_i then form the parts of a partition σ.

To state the relations we obtain, we start by extending the function $\widetilde{\gamma}$ of Section 3.1,

$$\begin{aligned} \gamma^{\mathsf{SQ}} &= \sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1} \\ &+ \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} \widetilde{C}_r^d \kappa_{r+|\sigma|}\ t^r \frac{x^d}{d!}\ \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ . \end{aligned}$$

Let $\overline{\gamma}^{\mathsf{SQ}}$ be defined by a similar formula,

$$\begin{aligned} \overline{\gamma}^{\mathsf{SQ}} &= \sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}(-t)^{2i-1} \\ &+ \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} \widetilde{C}_r^d \kappa_{r+|\sigma|}\ (-t)^r \frac{x^d}{d!}\ \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ . \end{aligned}$$

The sign of t in $t^{|\sigma|}$ does not change in $\overline{\gamma}^{\mathsf{SQ}}$. The κ_{-1} terms which appear will later be set to 0.

The full system of relations are obtained from the coefficients of the functions

$$\exp(-\gamma^{\mathsf{SQ}}), \qquad \exp(-\sum_{r=0}^{\infty}\kappa_r t^r p_{r+1})\cdot\exp(-\overline{\gamma}^{\mathsf{SQ}})$$

Theorem 5. *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[\exp(-\gamma^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^{\sigma}} = (-1)^g \Big[\exp(-\sum_{r=0}^{\infty}\kappa_r t^r p_{r+1})\cdot\exp(-\overline{\gamma}^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^{\sigma}}$$

holds when $g - 2d - 1 + |\sigma| < r$.

Again, we see the genus shifting mod 2 property. If the relation holds in genus g, then the *same* relation holds in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$.

In case $\sigma = \emptyset$, Theorem 5 specializes to the relation

$$\begin{aligned}\Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} &= (-1)^g \Big[\exp(-\widetilde{\gamma}(-t,x))\Big]_{t^r x^d} \\ &= (-1)^{g+r} \Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} ,\end{aligned}$$

nontrivial only if $g \equiv r+1 \mod 2$. If the mod 2 condition holds, then we obtain the relations of Proposition 4.

Consider the case $\sigma = (1)$. The left side of the relation is then

$$\Big[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(-\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} .$$

The right side is

$$(-1)^g \Big[\exp(-\widetilde{\gamma}(-t,x)) \cdot \left(-\kappa_0 t^0 + \sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+1} (-t)^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} .$$

If $g \equiv r+1 \mod 2$, then the large terms cancel and we obtain

$$-\kappa_0 \cdot \Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} = 0 .$$

Since $\kappa_0 = 2g-2$ and

$$(g - 2d - 1 + 1 < r) \quad \Longrightarrow \quad (g - 2d - 1 < r),$$

we recover most (but not all) of the $\sigma = \emptyset$ equations.

If $g \equiv r \mod 2$, then the resulting equation is

$$\Big[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(\kappa_0 - 2\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} = 0$$

when $g - 2d < r$.

3.5 Proof of Theorem 5

3.5.1 Partitions, differential operators, and logs.

We will write partitions σ as $(1^{n_1}2^{n_2}3^{n_3}\ldots)$ with

$$\ell(\sigma) = \sum_i n_i \quad \text{and} \quad |\sigma| = \sum_i i n_i \ .$$

The empty partition $\emptyset$ corresponding to $(1^0 2^0 3^0 \ldots)$ is permitted. In all cases, we have

$$|\mathrm{Aut}(\sigma)| = n_1! n_2! n_3! \cdots \ .$$

In the infinite set of variables $\{p_1, p_2, p_3, \ldots\}$, let

$$\Phi^{\mathbf{p}}(t,x) = \sum_\sigma \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \ \frac{(-1)^d}{d!} \frac{x^d}{t^d} \ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \ ,$$

where the first sum is over all partitions σ. The summand corresponding to the empty partition equals $\Phi(t,x)$ defined in (20).

The function $\Phi^{\mathbf{p}}$ is easily obtained from Φ,

$$\Phi^{\mathbf{p}}(t,x) = \exp\left(\sum_{i=1}^{\infty} p_i t^i x \frac{d}{dx}\right) \ \Phi(t,x) \ .$$

Let D denote the differential operator

$$D = \sum_{i=1}^{\infty} p_i t^i x \frac{d}{dx} \ .$$

Expanding the exponential of D, we obtain

$$\begin{aligned} \Phi^{\mathbf{p}} &= \Phi + D\Phi + \frac{1}{2} D^2\Phi + \frac{1}{6} D^3 \Phi + \ldots \\ &= \Phi \left(1 + \frac{D\Phi}{\Phi} + \frac{1}{2}\frac{D^2\Phi}{\Phi} + \frac{1}{6}\frac{D^3\Phi}{\Phi} + \ldots\right) \ . \end{aligned} \tag{28}$$

Let $\gamma^* = \log(\Phi)$ be the logarithm,

$$D\gamma^* = \frac{D\Phi}{\Phi} \ .$$

After applying the logarithm to (28), we see

$$\begin{aligned}\log(\Phi^{\mathbf{p}}) &= \gamma^* + \log\left(1 + D\gamma^* + \frac{1}{2}(D^2\gamma^* + (D\gamma^*)^2) + \ \dots\right) \\ &= \gamma^* + D\gamma^* + \frac{1}{2}D^2\gamma^* + \dots\end{aligned}$$

where the dots stand for a universal expression in the $D^k\gamma^*$. In fact, a remarkable simplification occurs,

$$\log(\Phi^{\mathbf{p}}) = \exp\left(\sum_{i=1}^{\infty} p_i t^i x \frac{d}{dx}\right)\ \gamma^* \ .$$

The result follows from a general identity.

Proposition 7. *If f is a function of x, then*

$$\log\left(\exp\left(\lambda x \frac{d}{dx}\right)\ f\right) = \exp\left(\lambda x \frac{d}{dx}\right)\ \log(f) \ .$$

Proof. A simple computation for monomials in x shows

$$\exp\left(\lambda x \frac{d}{dx}\right)\ x^k = (e^{\lambda} x)^k \ .$$

Hence, since the differential operator is additive,

$$\exp\left(\lambda x \frac{d}{dx}\right)\ f(x) = f(e^{\lambda} x) \ .$$

The Proposition follows immediately. □

We apply Proposition 7 to $\log(\Phi^{\mathbf{p}})$. The coefficients of the logarithm may be written as

$$\begin{aligned}\log(\Phi^{\mathbf{p}}) &= \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} \widetilde{C}^d_r(\sigma)\ t^r \frac{x^d}{d!}\mathbf{p}^{\sigma} \\ &= \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} \widetilde{C}^d_r\ t^r \frac{x^d}{d!} \exp\left(\sum_{i=1}^{\infty} d p_i t^i\right) \\ &= \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} \widetilde{C}^d_r\ t^r \frac{x^d}{d!}\ \frac{d^{\ell(\sigma)} t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|} \ .\end{aligned}$$

We have expressed the coefficients $\widetilde{C}^d_r(\sigma)$ of $\log(\Phi^{\mathbf{p}})$ solely in terms of the coefficients $\widetilde{C}^d_r$ of $\log(\Phi)$.

3.5.2 Cutting classes

Let $\theta_i \in A^1(U, \mathbb{Q})$ be the class of the i^{th} section of the universal curve

$$\epsilon : U \to \mathcal{C}_g^d \tag{29}$$

The class $s = c_1(S_U^*)$ on the universal curve over $Q_g(\mathbf{P}^1, d)$ restricted to the $\mathbb{C}^*$-fixed locus $\mathcal{C}_g^d/\mathbb{S}_d$ and pulled-back to (29) yields

$$s = \theta_1 + \ldots + \theta_d \ \in A^1(U, \mathbb{Q}).$$

We calculate

$$\epsilon_*(s\ \omega^b) = \psi_1^b + \ldots + \psi_d^b \quad \in A^b(\mathcal{C}_g^d, \mathbb{Q})\ . \tag{30}$$

3.5.3 Wick form

We repeat the Wick analysis of Section 3.3 for the vanishings

$$\nu_* \left(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) = 0 \ \text{ in } A^*(\mathcal{M}_g, \mathbb{Q})$$

when $c > 0$. We start by writing a formula for

$$\sum_{d=1}^{\infty} \sum_{r \geq 0} \pi_*^d \left(\exp\left(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i) \right) \cdot c_r(\mathbb{F}_d) t^r \right) \ \frac{1}{t^d} \frac{x^d}{d!}\ .$$

Applying the Wick formula to equation (30) for the cutting classes, we see

$$\sum_{d=1}^{\infty} \sum_{r \geq 0} \pi_*^d \left(\exp\left(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i) \right) \cdot c_r(\mathbb{F}_d) t^r \right) \ \frac{1}{t^d} \frac{x^d}{d!} = \exp(-\widetilde{\gamma}^{\mathsf{SQ}}) \tag{31}$$

where $\widetilde{\gamma}^{\mathsf{SQ}}$ is defined by

$$\widetilde{\gamma}^{\mathsf{SQ}} = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} + \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d(\sigma) \kappa_r\ t^r \frac{x^d}{d!}\ \mathbf{p}^{\sigma}\ .$$

We follow here the notation of Section 3.5.1,

$$\Phi^{\mathbf{P}}(t, x) = \sum_{\sigma} \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it}\ \frac{(-1)^d}{d!} \frac{x^d}{t^d}\ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ ,$$

$$\log(\Phi^{\mathbf{p}}) = \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d(\sigma)\ t^r \frac{x^d}{d!} \mathbf{p}^\sigma \ .$$

In the Wick analysis, the class $\epsilon_*(s\omega^b)$ simply acts as dt^b.

Using the expression for the coefficents $\widetilde{C}_r^d(\sigma)$ in terms of $\widetilde{C}_r^d$ derived in Section 3.5.1, we obtain the following result from (31).

Proposition 8. *We have*

$$\sum_{d=1}^{\infty} \sum_{r\geq 0} \pi_*^d \left(\exp\Big(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i) \Big) \cdot c_r(\mathbb{F}_d) t^r \right) \ \frac{1}{t^d} \frac{x^d}{d!} = \exp(-\gamma^{\mathsf{SQ}}) \ .$$

3.5.4 Geometric construction

We apply $\mathbb{C}^*$-localization on $Q_g(\mathbf{P}^1, d)$ to the geometric vanishing

$$\nu_* \left(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) = 0 \ \text{ in } A^*(\mathcal{M}_g, \mathbb{Q}) \tag{32}$$

when $c > 0$. The result is the relation

$$\pi_* \Big(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot c_{g-d-1+c}(\mathbb{F}_d) + \\ (-1)^{g-d-1} \Big[\prod_{i=1}^{\ell} \epsilon_* \left((s-1)\omega^{b_i} \right) \cdot c_-(\mathbb{F}_d) \Big]^{g-d-1+\sum_i b_i + c} \Big) = 0 \tag{33}$$

in $R^*(\mathcal{M}_g)$. After applying the Wick formula of Proposition 8, we immediately obtain Theorem 5.

The first summand in (33) yields the left side

$$\Big[\exp(-\gamma^{\mathsf{SQ}}) \Big]_{t^r x^d \mathbf{p}^\sigma}$$

of the relation of Theorem 5. The second summand produces the right side

$$(-1)^g \Big[\exp(-\sum_{r=0}^{\infty} \kappa_r t^r p_{r+1}) \cdot \exp(-\widehat{\gamma}^{\mathsf{SQ}}) \Big]_{t^r x^d \mathbf{p}^\sigma} \ . \tag{34}$$

Recall the localization of the virtual class over $\infty \in \mathbf{P}^1$ is

$$\frac{(-1)^{g-d-1}}{d!}\pi^d_*\Big[c_-(\mathbb{F}_d)\Big]^{g-d-1+c} .$$

Of the sign prefactor $(-1)^{g-d-1}$,

- $(-1)^{-1}$ is used to move the term to the right side,
- $(-1)^{-d}$ is absorbed in the $(-t)$ of the definition of $\widehat{\gamma}^{\mathsf{SQ}}$,
- $(-1)^g$ remains in (34).

The -1 of $s-1$ produces the the factor $\exp(-\sum_{r=0}^{\infty}\kappa_r t^r p_{r+1})$.

Finally, a simple dimension calculation (remembering $c>0$) implies the validity of the relation when $g-2d-1+|\sigma|<r$. $\square$

4 Analysis of the relations

4.1 Expanded form

Let $\sigma = (1^{a_1}2^{a_2}3^{a_3}\ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We can directly write the corresponding tautological relation in $R^r(\mathcal{M}_g)$ obtained from Theorem 5.

A *subpartition* $\sigma' \subset \sigma$ is obtained by selecting a nontrivial subset of the parts of σ. A *division* of σ is a disjoint union

$$\sigma = \sigma^{(1)} \cup \sigma^{(2)} \cup \sigma^{(3)} \ldots \tag{35}$$

of subpartitions which exhausts σ. The subpartitions in (35) are unordered. Let $\mathcal{S}(\sigma)$ be the set of divisions of σ. For example,

$$\begin{aligned}\mathcal{S}(1^1 2^1) &= \{\ (1^1 2^1),\ (1^1)\cup(2^1)\ \}\ ,\\ \mathcal{S}(1^3) &= \{\ (1^3),\ (1^2)\cup(1^1)\ \}\ .\end{aligned}$$

We will use the notation $\sigma^\bullet$ to denote a division of σ with subpartitions $\sigma^{(i)}$. Let

$$m(\sigma^\bullet) = \frac{1}{|\mathrm{Aut}(\sigma^\bullet)|}\frac{|\mathrm{Aut}(\sigma)|}{\prod_{i=1}^{\ell(\sigma^\bullet)}|\mathrm{Aut}(\sigma^{(i)})|}.$$

Here, $\mathrm{Aut}(\sigma^\bullet)$ is the group permuting equal subpartitions. The factor $m(\sigma^\bullet)$ may be interpreted as counting the number of different ways the disjoint union can be made.

To write explicitly the $\mathbf{p}^\sigma$ coefficient of $\exp(\gamma^{\mathsf{SQ}})$, we introduce the functions

$$F_{n,m}(t,x) = -\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+m} t^{s+m} \frac{d^n x^d}{d!}$$

for $n, m \geq 1$. Then,

$$|\mathrm{Aut}(\sigma)| \cdot \Big[\exp(-\gamma^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^\sigma} = \\ \left[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(\sum_{\sigma^\bullet \in \mathcal{S}(\sigma)} m(\sigma^\bullet) \prod_{i=1}^{\ell(\sigma^\bullet)} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}\right)\right]_{t^r x^d} .$$

Let $\sigma^{*,\bullet}$ be a division of σ with a marked subpartition,

$$\sigma = \sigma^* \cup \sigma^{(1)} \cup \sigma^{(2)} \cup \sigma^{(3)} \ldots , \tag{36}$$

labelled by the superscript $*$. The marked subpartition is permitted to be empty. Let $\mathcal{S}^*(\sigma)$ denote the set of marked divisions of σ. Let

$$m(\sigma^{*,\bullet}) = \frac{1}{|\mathrm{Aut}(\sigma^\bullet)|} \frac{|\mathrm{Aut}(\sigma)|}{|\mathrm{Aut}(\sigma^*)| \prod_{i=1}^{\ell(\sigma^{*,\bullet})} |\mathrm{Aut}(\sigma^{(i)})|}.$$

The length $\ell(\sigma^{*,\bullet})$ is the number of unmarked subpartitions.

Then, $|\mathrm{Aut}(\sigma)|$ times the right side of Theorem 5 may be written as

$$(-1)^{g+|\sigma|} |\mathrm{Aut}(\sigma)| \cdot \Big[\exp(-\widetilde{\gamma}(-t,x)) \cdot \\ \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1}(-t)^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}(-t,x)\right)\Big]_{t^r x^d}$$

To write Theorem 5 in the simplest form, the following definition using the Kronecker δ is useful,

$$m^{\pm}(\sigma^{*,\bullet}) = (1 \pm \delta_{0,|\sigma^*|}) \cdot m(\sigma^{*,\bullet}).$$

There are two cases. If $g \equiv r + |\sigma| \mod 2$, then Theorem 3 is equivalent to the vanishing of

$$|\mathrm{Aut}(\sigma)| \Big[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m^-(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1} t^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \Big]_{t^r x^d}.$$

If $g \equiv r + |\sigma| + 1 \mod 2$, then Theorem 5 is equivalent to the vanishing of

$$|\mathrm{Aut}(\sigma)| \Big[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m^+(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1} t^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \Big]_{t^r x^d}.$$

In either case, the relations are valid in the ring $R^*(\mathcal{M}_g)$ only if the condition $g - 2d - 1 + |\sigma| < r$ holds.

We denote the above relation corresponding to g, r, d, and σ (and depending upon the parity of $g - r - |\sigma|$) by

$$\mathsf{R}(g, r, d, \sigma) = 0$$

The $|\mathrm{Aut}(\sigma)|$ prefactor is included in $\mathsf{R}(g, r, d, \sigma)$, but is only relevant when σ has repeated parts. In case of repeated parts, the automorphism scaled normalization is more convenient.

4.2 Further examples

If $\sigma = (k)$ has a single part, then the two cases of Theorem 5 are the following. If $g \equiv r + k \mod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \kappa_{k-1} t^{k-1} \Big]_{t^r x^d} = 0$$

which is a consequence of the $\sigma = \emptyset$ case. If $g \equiv r + k + 1 \mod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k-1} t^{k-1} + 2F_{1,k} \big) \Big]_{t^r x^d} = 0$$

If $\sigma = (k_1 k_2)$ has two distinct parts, then the two cases of Theorem 5 are as follows. If $g \equiv r + k_1 + k_2 \mod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k_1-1} \kappa_{k_2-1} t^{k_1+k_2-2} + \kappa_{k_1-1} t^{k_1-1} F_{1,k_2} + \kappa_{k_2-1} t^{k_2-1} F_{1,k_1} \big) \Big]_{t^r x^d} = 0 \ .$$

If $g \equiv r + k_1 + k_2 + 1 \mod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k_1-1}\kappa_{k_2-1}t^{k_1+k_2-2} + \kappa_{k_1-1}t^{k_1-1}F_{1,k_2} + \kappa_{k_2-1}t^{k_2-1}F_{1,k_1} + 2F_{2,k_1+k_2} + 2F_{1,k_1}F_{1,k_2}\big)\Big]_{t^r x^d} = 0 \ .$$

In fact, the $g \equiv r + k_1 + k_2 \mod 2$ equation above is not new. The genus g and codimension $r_1 = r - k_2 + 1$ case of partition (k_1) yields

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k_1-1}t^{k_1-1} + 2F_{1,k_1}\big)\Big]_{t^{r_1} x^d} = 0 \ .$$

After multiplication with $\kappa_{k_2-1}t^{k_2-1}$, we obtain

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k_1-1}\kappa_{k_2-1}t^{k_1+k_2-2} + 2\kappa_{k_2-1}t^{k_2-1}F_{1,k_1}\big)\Big]_{t^r x^d} = 0 \ .$$

Summed with the corresponding equation with k_1 and k_2 interchanged yields the above $g \equiv r + k_1 + k_2 \mod 2$ case.

4.3 Expanded form revisited

Consider the partition $\sigma = (k_1 k_2 \cdots k_\ell)$ with distinct parts. Relation $\mathsf{R}(g, r, d, \sigma)$, in the $g \equiv r + |\sigma| \mod 2$ case, is the vanishing of

$$\left[\exp(-\widetilde{\gamma}) \ \cdot \ \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1 - \delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma^*_j - 1} t^{\sigma^*_j - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

since all the factors $m(\sigma^{*,\bullet})$ are 1. In the $g \equiv r + |\sigma| + 1 \mod 2$ case, $\mathsf{R}(g, r, d, \sigma)$ is the vanishing of

$$\left[\exp(-\widetilde{\gamma}) \ \cdot \ \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1 + \delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma^*_j - 1} t^{\sigma^*_j - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

for the same reason.

If σ has repeated parts, the relation $\mathsf{R}(g, r, d, \sigma)$ is obtained by viewing the parts as distinct and specializing the indicies afterwards. For example, the two cases for $\sigma = (k^2)$ are as follows. If $g \equiv r + 2k \mod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k-1}\kappa_{k-1}t^{2k-2} + 2\kappa_{k-1}t^{k-1}F_{1,k}\big)\Big]_{t^r x^d} = 0 \ .$$

If $g \equiv r + 2k + 1 \bmod 2$, we have

$$\Big[\exp(-\widetilde{\gamma}) \cdot \big(\kappa_{k-1}\kappa_{k-1}t^{2k-2} + 2\kappa_{k-1}t^{k-1}F_{1,k} + 2F_{2,2k} + 2F_{1,k}F_{1,k}\big)\Big]_{t^r x^d} = 0 \ .$$

The factors occur via repetition of terms in the formulas for distinct parts.

Proposition 9. *The relation* $\mathsf{R}(g, r, d, \sigma)$ *in the* $g \equiv r + |\sigma| \bmod 2$ *case is a consequence of the relations in* $\mathsf{R}(g, r', d, \sigma')$ *where* $g \equiv r' + |\sigma'| + 1 \bmod 2$ *and* $\sigma' \subset \sigma$ *is a strictly smaller partition.*

Proof. The strategy follows the example of the phenonenon already discussed in Section 4.2.

If $g \equiv r + |\sigma| \bmod 2$, then for every subpartition $\tau \subset \sigma$ of odd length, we have

$$g \equiv r - |\tau| + \ell(\tau) + |\sigma/\tau| + 1 \mod 2$$

where σ/τ is the complement of τ. The relation

$$\prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}(g, r - |\tau| + \ell(\tau), d, \sigma/\tau)$$

is of codimension r.

Let $g \equiv r + |\sigma| \bmod 2$, and let σ have distinct parts. The formula

$$\mathsf{R}(g, r, d, \sigma) = \sum_{\tau \subset \sigma} \left(\frac{2^{\ell(\tau)+2} - 2}{\ell(\tau) + 1}\right) B_{\ell(\tau)+1} \cdot \prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}\Big(g, r - |\tau| + \ell(\tau), d, \sigma/\tau\Big) \quad (37)$$

follows easily by grouping like terms and the Bernoulli identity

$$\sum_{k \geq 1} \binom{n}{2k-1} \left(\frac{2^{2k+1} - 2}{2k}\right) B_{2k} = -\left(\frac{2^{n+2} - 2}{n+1}\right) B_{n+1} \quad (38)$$

for $n > 0$. The sum in (37) is over all subpartitions $\tau \subset \sigma$ of odd length.

The proof of the Bernoulli identity (38) is straightforward. Let

$$a_i = \left(\frac{2^{i+2} - 2}{i+1}\right) B_{i+1} \ , \quad A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Using the definition of the Bernoulli numbers as

$$\frac{x}{e^x-1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} \ ,$$

we see

$$A(x) = \frac{2}{x}\sum_{i=0}^{\infty}(2^i-1)B_r\frac{x^r}{r!} = \frac{2}{x}\left(\frac{2x}{e^{2x}-1} - \frac{x}{e^x-1}\right) = -\left(\frac{2}{1+e^x}\right) \ .$$

The identity (38) follows from the series relation

$$e^x A(x) = -A(x) - 2 \ .$$

Formula (37) is valid for $\mathsf{R}(g,r,d,\sigma)$ even when σ has repeated parts: the sum should be interpreted as running over all odd subsets $\tau \subset \sigma$ (viewing the parts of σ as distinct). □

4.4 Recasting

We will recast the relations $\mathsf{R}(g,r,d,\sigma)$ in case $g \equiv r + |\sigma| + 1 \mod 2$ in a more convenient form. The result will be crucial to the further analysis in Section 5.

Let $g \equiv r + |\sigma| + 1 \mod 2$, and let $\mathsf{S}(g,r,d,\sigma)$ denote the κ polynomial

$$|\mathrm{Aut}|\Big[\exp\left(-\widetilde{\gamma}(t,x) + \sum_{\sigma\neq\emptyset}\left(F_{\ell(\sigma),|\sigma|} + \frac{\delta_{\ell(\sigma),1}}{2}\kappa_{|\sigma|-1}\right)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{t^r x^d \mathbf{p}^\sigma} \ .$$

We can write $\mathsf{S}(g,r,d,\sigma)$ in terms of our previous relations $\mathsf{R}(g,r',d,\sigma')$ satisfying $g \equiv r' + |\sigma'| + 1 \mod 2$ and $\sigma' \subset \sigma$:

If $g \equiv r + |\sigma| + 1 \mod 2$, then for every subpartition $\tau \subset \sigma$ of even length (including the case $\tau = \emptyset$), we have

$$g \equiv r - |\tau| + \ell(\tau) + |\sigma/\tau| + 1 \mod 2$$

where σ/τ is the complement of τ. The relation

$$\prod_i \kappa_{\tau_i - 1}\cdot \mathsf{R}(g, r - |\tau| + \ell(\tau), d, \sigma/\tau)$$

is of codimension r.

In order to express S in terms of R, we define $z_i \in \mathbb{Q}$ by

$$\frac{2}{e^x + e^{-x}} = \sum_{i=0}^{\infty} z_i \frac{x^i}{i!} \ .$$

Let $g \equiv r + |\sigma| + 1$ mod 2, and let σ have distinct parts. The formula

$$\mathsf{S}(g, r, d, \sigma) = \sum_{\tau \subset \sigma} \frac{z_{\ell(\tau)}}{2^{\ell(\tau)+1}} \cdot \prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}\Big(g, r - |\tau| + \ell(\tau), d, \sigma/\tau\Big) \tag{39}$$

follows again grouping like terms and the combinatorial identity

$$\sum_{i \geq 0} \binom{n}{i} \frac{z_i}{2^i + 1} = -\frac{z_n}{2^{n+1}} - \frac{1}{2^n} \tag{40}$$

for $n > 0$. The sum in (39) is over all subpartitions $\tau \subset \sigma$ of even length.

As before, there the identity (40) is straightforward to prove. We see

$$Z(x) = \sum_{i=0}^{\infty} \frac{z_i}{2^{i+1}} \frac{x^i}{i!} = \frac{1}{e^{x/2} + e^{-x/2}} \ .$$

The identity (40) follows from the series relation

$$e^x Z(x) = e^{x/2} - Z(x).$$

Formula (37) is valid for $\mathsf{S}(g, r, d, \sigma)$ even when σ has repeated parts: the sum should be interpreted as running over all even subsets $\tau \subset \sigma$ (viewing the parts of σ as distinct). We have proved the following result.

Proposition 10. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp\left(-\widetilde{\gamma}(t, x) + \sum_{\sigma \neq \emptyset} \left(F_{\ell(\sigma), |\sigma|} + \frac{\delta_{\ell(\sigma),1}}{2} \kappa_{|\sigma|-1} \right) \frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \right) \right]_{t^r x^d \mathbf{p}^\sigma} = 0$$

holds when $g - 2d - 1 + |\sigma| < r$ and $g \equiv r + |\sigma| + 1$ mod 2.

5 Transformation

5.1 Differential equations

The function Φ satisfies a basic differential equation obtained from the series definition:

$$\frac{d}{dx}(\Phi - tx\frac{d}{dx}\Phi) = -\frac{1}{t}\Phi \ .$$

After expanding and dividing by Φ, we find

$$-tx\frac{\Phi_{xx}}{\Phi} - t\frac{\Phi_x}{\Phi} + \frac{\Phi_x}{\Phi} = -\frac{1}{t}$$

which can be written as

$$-t^2 x\gamma^*_{xx} = t^2 x(\gamma^*_x)^2 + t^2\gamma^*_x - t\gamma^*_x - 1 \tag{41}$$

where, as before, $\gamma^* = \log(\Phi)$. Equation (41) has been studied by Ionel in *Relations in the tautological ring* [9]. We present here results of hers which will be useful for us.

To kill the pole and match the required constant term, we will consider the function

$$\Gamma = -t\left(\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}t^{2i-1} + \gamma^*\right) \ . \tag{42}$$

The differential equation (41) becomes

$$tx\Gamma_{xx} = x(\Gamma_x)^2 + (1-t)\Gamma_x - 1 \ .$$

The differential equation is easily seen to uniquely determine Γ once the initial conditions

$$\Gamma(t,0) = -\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}t^{2i}$$

are specified. By Ionel's first result,

$$\Gamma_x = \frac{-1+\sqrt{1+4x}}{2x} + \frac{t}{1+4x} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1}q_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}-1}$$

where the postive integers $q_{k,j}$ (defined to vanish unless $k \geq j \geq 0$) are defined via the recursion

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l}$$

from the initial value $q_{0,0} = 1$.

Ionel's second result is obtained by integrating Γ_x with respect to x. She finds

$$\Gamma = \Gamma(0,x) + \frac{t}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1} c_{k,j}(-x)^j (1+4x)^{-j-\frac{k}{2}}$$

where the coefficients $c_{k,j}$ are determined by

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}$$

for $k \geq 1$ and $k \geq j \geq 0$.

While the derivation of the formula for Γ_x is straightforward, the formula for Γ is quite subtle as the intial conditions (given by the Bernoulli numbers) are used to show the vanishing of constants of integration. Said differently, the recursions for $q_{k,j}$ and $c_{k,j}$ must be shown to imply the formula

$$c_{k,0} = \frac{B_{k+1}}{k(k+1)} \ .$$

A third result of Ionel's is the determination of the extremal $c_{k,k}$,

$$\sum_{k=1}^{\infty} c_{k,k} z^k = \log\left(\sum_{k=1}^{\infty} \frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right) \ .$$

The formula for Γ becomes simpler after the following very natural change of variables,

$$u = \frac{t}{\sqrt{1+4x}} \quad \text{and} \quad y = \frac{-x}{1+4x} \ . \tag{43}$$

The change of variables defines a new function

$$\widehat{\Gamma}(u,y) = \Gamma(t,x) \ .$$

The formula for Γ implies

$$\frac{1}{t}\widehat{\Gamma}(u,y) = \frac{1}{t}\widehat{\Gamma}(0,y) - \frac{1}{4}\log(1+4y) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j} u^k y^j \ . \tag{44}$$

Ionel's fourth result relates coefficients of series after the change of variables (43). Given any series

$$P(t,x) \in \mathbb{Q}[[t,x]],$$

let $\widehat{P}(u,y)$ be the series obtained from the change of variables (43). Ionel proves the coefficient relation

$$\Big[P(t,x)\Big]_{t^r x^d} = (-1)^d \Big[(1+4y)^{\frac{r+2d-2}{2}} \cdot \widehat{P}(u,y)\Big]_{u^r y^d} .$$

5.2 Analysis of the relations of Proposition 4

We now study in detail the simple relations of Proposition 4,

$$\Big[\exp(-\widetilde{\gamma})\Big]_{t^r x^d} = 0 \ \in R^r(\mathcal{M}_g)$$

when $g - 2d - 1 < r$ and $g \equiv r + 1 \mod 2$. Let

$$\widehat{\gamma}(u,y) = \widetilde{\gamma}(t,x)$$

be obtained from the variable change (43). Equations (21), (42), and (44) together imply

$$\widehat{\gamma}(u,y) = \frac{\kappa_0}{4}\log(1+4y) + \sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j$$

modulo κ_{-1} terms which we set to 0.

Applying Ionel's coefficient result,

$$\begin{aligned}
\Big[\exp(-\widetilde{\gamma})\Big]_{t^r x^d} &= \Big[(1+4y)^{\frac{r+2d-2}{2}} \cdot \exp(-\widehat{\gamma})\Big]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r+2d-2}{2} - \frac{\kappa_0}{4}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} .
\end{aligned}$$

In the last line, the substitution $\kappa_0 = 2g - 2$ has been made.

Consider first the exponent of $1 + 4y$. By the assumptions on g and r in Proposition 4,

$$\frac{r - g + 2d - 1}{2} \geq 0$$

and the fraction is integral. Hence, the y degree of the prefactor

$$(1+4y)^{\frac{r-g+2d-1}{2}}$$

is exactly $\frac{r-g+2d-1}{2}$. The y degree of the exponential factor is bounded from above by the u degree. We conclude

$$\left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} = 0$$

is the *trivial* relation unless

$$r \geq d - \frac{r-g+2d-1}{2} = -\frac{r}{2} + \frac{g+1}{2} \ .$$

Rewriting the inequality, we obtain $3r \geq g+1$ which is equivalent to $r > \lfloor \frac{g}{3} \rfloor$. The conclusion is in agreement with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

A similar connection between Proposition 4 and Ionel's relations in [9] has also been found by Shengmao Zhu [21].

5.3 Analysis of the relations of Theorem 5

For the relations of Theorem 5, we will require additional notation. To start, let

$$\gamma^c(u,y) = \sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j \ .$$

By Ionel's second result,

$$\frac{1}{t}\Gamma = \frac{1}{t}\Gamma(0,x) + \frac{1}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^k c_{k,j}(-x)^j (1+4x)^{-j-\frac{k}{2}} \ . \quad (45)$$

Let $c^0_{k,j} = c_{k,j}$. We define the constants $c^n_{k,j}$ for $n \geq 1$ by

$$\left(x\frac{d}{dx}\right)^n \frac{1}{t}\Gamma = \left(x\frac{d}{dx}\right)^{n-1} \left(\frac{-1}{2t} + \frac{1}{2t}\sqrt{1+4x}\right) - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n} t^k c^n_{k,j}(-x)^j (1+4x)^{-j-\frac{k}{2}} \ .$$

Lemma 11. *For $n > 0$, there are constants b_j^n satisfying*

$$\left(x\frac{d}{dx}\right)^{n-1}\left(\frac{1}{2t}\sqrt{1+4x}\right) = \sum_{j=0}^{n-1} b_j^n u^{-1} y^j \ .$$

Moreover, $b_{n-1}^n = -2^{n-2} \cdot (2n-5)!!$ where $(-1)!! = 1$ and $(-3)!! = -1$.

Proof. The result is obtained by simple induction. The negative evaluations $(-1)!! = 1$ and $(-3)!! = -1$ arise from the Γ-regularization. □

Lemma 12. *For $n > 0$, we have $c_{0,n}^n = 4^{n-1}(n-1)!$.*

Proof. The coefficients $c_{0,n}^n$ are obtained directly from the t^0 summand $\frac{1}{4}\log(1+4x)$ of (45). □

Lemma 13. *For $n > 0$ and $k > 0$, we have*

$$c_{k,k+n}^n = (6k)(6k+4)\cdots(6k+4(n-1)) \ c_{k,k}.$$

Proof. The coefficients $c_{k,k+n}^n$ are extremal. The differential operators $x\frac{d}{dx}$ must always attack the $(1+4x)^{-j-\frac{k}{2}}$ in order to contribute $c_{k,k+n}^n$. The formula follows by inspection. □

Consider next the full set of equations given by Theorem 5 in the expanded form of Section 4. The function $F_{n,m}$ may be rewritten as

$$\begin{aligned} F_{n,m}(t,x) &= -\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+m} t^{s+m} \frac{d^n x^d}{d!} \\ &= -t^m \left(x\frac{d}{dx}\right)^n \sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}_s^d \ \kappa_{s+m} t^s \frac{x^d}{d!}. \end{aligned}$$

We may write the result in terms of the constants b_j^n and $c_{k,j}^n$,

$$\begin{aligned} t^{-(m-n)} F_{n,m} = & -\delta_{n,1}\frac{\kappa_{m-1}}{2} \\ & + (1+4y)^{-\frac{n}{2}} \Big(\sum_{j=0}^{n-1} \kappa_{m-1} b_j^n u^{n-1} y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n} \kappa_{k+m} c_{k,j}^n u^{k+n} y^j \Big) \end{aligned}$$

Define the functions $G_{n,m}(u,y)$ by

$$G_{n,m}(u,y) = \sum_{j=0}^{n-1} \kappa_{m-1} b_j^n u^{n-1} y^j - \sum_{k=0}^{\infty} \sum_{j=0}^{k+n} \kappa_{k+m} c_{k,j}^n u^{k+n} y^j \ .$$

Let $\sigma = (1^{a_1} 2^{a_2} 3^{a_3} \ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We assume the parity condition

$$g \equiv r + |\sigma| + 1 \ . \tag{46}$$

Let $G_\sigma^\pm(u,y)$ be the following function associated to σ,

$$G_\sigma^\pm(u,y) = \sum_{\sigma^\bullet \in \mathcal{S}(\sigma)} \prod_{i=1}^{\ell(\sigma^\bullet)} \left(G_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \pm \frac{\delta_{\ell(\sigma^{(i)}),1}}{2} \sqrt{1+4y}\ \kappa_{|\sigma^{(i)}|-1} \right) \ .$$

The relations of Theorem 5 in the the expanded form of Section 4.1 written in the variables u and y are

$$\left[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}} \exp(-\gamma^c) \left(G_\sigma^+ + G_\sigma^- \right) \right]_{u^{r-|\sigma|+\ell(\sigma)} y^d} = 0$$

In fact, the relations of Proposition 10 here take a much more efficient form. We obtain the following result.

Proposition 14. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}} \exp\left(-\gamma^c - \sum_{\sigma \neq \emptyset} G_{\ell(\sigma),|\sigma|} \frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \right) \right]_{u^{r-|\sigma|+\ell(\sigma)} y^d \mathbf{p}^\sigma} = 0$$

holds when $g - 2d - 1 + |\sigma| < r$ and $g \equiv r + |\sigma| + 1$ mod 2.

Consider the exponent of $1+4y$. By the inequality and the parity condition (46),

$$\frac{r - |\sigma| - g + 2d - 1}{2} \geq 0$$

and the fraction is integral. Hence, the y degree of the prefactor

$$(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}} \tag{47}$$

is exactly $\frac{r-|\sigma|-g+2d-1}{2}$. The y degree of the exponential factor is bounded from above by the u degree. We conclude the relation of Theorem 4 is *trivial* unless

$$r-|\sigma|+\ell(\sigma) \geq d - \frac{r-|\sigma|-g+2d-1}{2} = -\frac{r-|\sigma|}{2} + \frac{g+1}{2} \ .$$

Rewriting the inequality, we obtain

$$3r \geq g+1+3|\sigma|-2\ell(\sigma)$$

which is consistent with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

5.4 Another form

A subset of the equations of Proposition 14 admits an especially simple description. Consider the function

$$H_{n,m}(u) = 2^{n-2}(2n-5)!!\ \kappa_{m-1}u^{n-1} + 4^{n-1}(n-1)!\ \kappa_m u^n \\ + \sum_{k=1}^{\infty}(6k)(6k+4)\cdots(6k+4(n-1))c_{k,k}\ \kappa_{k+m}u^{k+n} \ .$$

Proposition 15. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} = 0$$

holds when $3r \geq g+1+3|\sigma|-2\ell(\sigma)$ and $g \equiv r+|\sigma|+1$ mod 2.

Proof. Let $g \equiv r+|\sigma|+1$, and let

$$\frac{3}{2}r - \frac{1}{2}g - \frac{1}{2} - \frac{3}{2}|\sigma| + \ell(\sigma) = \Delta > 0 \ .$$

By the parity condition, δ is an integer. For $0 \leq \delta \leq \Delta$, let

$$\mathsf{E}_\delta(g,r,\sigma) = \left[\exp\left(-\gamma^c + \sum_{\sigma\neq\emptyset} G_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}y^{r-|\sigma|+\ell(\sigma)-\delta}\mathbf{p}^\sigma} \ .$$

The $\delta = 0$ case is special. Only the monomials of $G_{n,m}$ of equal u and y degree contribute to the relations of Proposition 14. By Lemmas 11 - 13, $H_{u,m}(uy)$ is exactly the subsum of $G_{n,m}$ consisting of such monomials. Similarly,

$$\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k y^k$$

is the subsum of γ^c of monomials of equal u and y degree. Hence,

$$\mathsf{E}_0(g,r,\sigma) =$$

$$\left[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k y^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}(uy)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{(uy)^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} =$$

$$\left[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}(u)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} .$$

We consider the relations of Proposition 14 for fixed g, r, and σ as d varies. In order to satisfy the inequalty $g - 2d - 1 + |\sigma| < r$, let

$$d(\widehat{\delta}) = \frac{-r+g+1+|\sigma|}{2} + \widehat{\delta}\ , \quad \text{for}\ \ \widehat{\delta} \geq 0.$$

For $0 \leq \widehat{\delta} \leq \Delta$, relation of Proposition 14 for g, r, σ, and $d(\widehat{\delta})$ is

$$\sum_{i=0}^{\widehat{\delta}} 4^i \binom{\widehat{\delta}}{i} \cdot \mathsf{E}_{\Delta-\widehat{\delta}+i}(g,r,\sigma) = 0\ .$$

As $\widehat{\delta}$ varies, we therefore obtain all the relations

$$\mathsf{E}_\delta(g,r,\sigma) = 0 \tag{48}$$

for $0 \leq \delta \leq \Delta$. The relations of Proposition 15 are obtained when $\delta = 0$ in (48). □

The main advantage of Proposition 15 is the dependence on only the function

$$\sum_{k=1}^{\infty} c_{k,k} z^k = \log\left(\sum_{k=1}^{\infty} \frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right)\ . \tag{49}$$

Proposition 15 only provides finitely many relations for fixed g and r. In Section 6, we show Proposition 15 is equivalent to the Faber-Zagier conjecture.

5.5 Relations left behind

In our analysis of relations obtained from the virtual geometry of the moduli space of stable quotients, twice we have discarded large sets of relations. In Section 3.4, instead of analyzing all of the geometric possibilities

$$\nu_* \left(\prod_{i=1}^{n} \epsilon_*(s^{a_i}\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) = 0 \ \text{ in } A^*(\mathcal{M}_g, \mathbb{Q}) \ ,$$

we restricted ourselves to the case where $a_i = 1$ for all i. And just now, instead of keeping all the relations (48), we restricted ourselves to the $\delta = 0$ cases.

In both instances, the restricted set was chosen to allow further analysis. In spite of the discarding, we will arrive at the Faber-Zagier relations. We expect the discarded relations are all redundant (consistent with Conjecture 2), but we do not have a proof.

6 Equivalence

6.1 Notation

The relations in Proposition 15 have a similar flavor to the Faber-Zagier relations. We start with formal series related to

$$A(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{z}{72}\right)^i ,$$

we insert classes κ_r, we exponentiate, and we extract coefficients to obtain relations among the κ classes. In order to make the similarities clearer, we will introduce additional notation.

If F is a formal power series in z,

$$F = \sum_{r=0}^{\infty} c_r z^r$$

with coefficients in a ring, let

$$\{F\}_\kappa = \sum_{r=0}^{\infty} c_r \kappa_r z^r$$

be the series with κ-classes inserted.

Let A be as above, and let

$$B(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left(\frac{z}{72}\right)^i$$

be the second power series appearing in the Faber-Zagier relations. Let

$$C = \frac{B}{A} ,$$

and let

$$E = \exp(-\{\log(A)\}_\kappa) = \exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k z^k\right).$$

We will rewrite the Faber-Zagier relations and the relations of Proposition 15 in terms of C and E. The equivalence between the two will rely on the principal differential equation satisfied by C,

$$12z^2\frac{dC}{dz} = 1 + 4zC - C^2. \qquad (50)$$

6.2 Rewriting the relations

The relations conjectured by Faber and Zagier are straightforward to rewrite using the above notation:

$$\left[E \cdot \exp\Big(-\Big\{\log\big(1 + p_3 z + p_6 z^2 + \cdots + C(p_1 + p_4 z + p_7 z^2 + \cdots)\big)\Big\}_\kappa\Big)\right]_{z^r p^\sigma} = 0 \qquad (51)$$

for $3r \geq g + |\sigma| + 1$ and $3r \equiv g + |\sigma| + 1 \bmod 2$. The above relation (51) will be denoted $\mathsf{FZ}(r,\sigma)$.

The stable quotient relations of Proposition 15 are more complicated to rewrite in terms of C and E. Define a sequence of power series $(C_n)_{n\geq 1}$ by

$$2^{-n}C_n = 2^{n-2}(2n-5)!!z^{n-1} + 4^{n-1}(n-1)!z^n + \sum_{k=1}^{\infty}(6k)(6k+4)\cdots(6k+4(n-1))c_{k,k}z^{k+n}.$$

We see

$$H_{n,m}(z) = 2^{-n} z^{n-m} \{z^{m-n} C_n\}_\kappa.$$

The series C_n satisfy

$$C_1 = C, \qquad C_{i+1} = \left(12z^2 \frac{d}{dz} - 4iz\right) C_i. \tag{52}$$

Using the differential equation (50), each C_n can be expressed as a polynomial in C and z:

$$C_1 = C, \quad C_2 = 1 - C^2, \quad C_3 = -8z - 2C + 2C^3, \ldots, \ .$$

Proposition 15 can then be rewritten as follows (after an appropriate change of variables):

$$\left[E \cdot \exp\left(- \sum_{\sigma \neq \emptyset} \{z^{|\sigma| - \ell(\sigma)} C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|} \right) \right]_{z^r p^\sigma} = 0 \tag{53}$$

for $3r \geq g + 3|\sigma| - 2\ell(\sigma) + 1$ and $3r \equiv g + 3|\sigma| - 2\ell(\sigma) + 1 \bmod 2$. The above relation (53) will be denoted $\mathsf{SQ}(r, \sigma)$.

The FZ and SQ relations now look much more similar, but the relations in (51) are indexed by partitions with no parts of size 2 mod 3 and satisfy a slightly different inequality. The indexing differences can be erased by observing that the variables p_{3k} are actually not necessary in (51) if we are just interested in the *ideal* generated by a set of relations (rather than the linear span). This observation follows from the identity

$$-\mathsf{FZ}(r, \sigma \sqcup 3a) = \kappa_a \mathsf{FZ}(r - a, \sigma) + \sum_\tau \mathsf{FZ}(r, \tau),$$

where the sum runs over the $\ell(\sigma)$ partitions τ (possibly repeated) formed by increasing one of the parts of σ by $3a$.

If we remove the variables p_{3k} and reindex the others by replacing p_{3k+1} with p_{k+1}, we obtain the following equivalent form of the FZ relations:

$$\Big[E \cdot \exp\big(- \big\{ \log(1 + C(p_1 + p_2 z + p_3 z^2 + \cdots)) \big\}_\kappa \big) \Big]_{z^r p^\sigma} = 0 \tag{54}$$

for $3r \geq g + 3|\sigma| - 2\ell(\sigma) + 1$ and $3r \equiv g + 3|\sigma| - 2\ell(\sigma) + 1 \bmod 2$.

6.3 Comparing the relations

We now explain how to write the SQ relations (53) as linear combinations of the FZ relations (54) with coefficients in $\mathbb{Q}[\kappa_0, \kappa_1, \kappa_2, \ldots]$. In fact, the associated matrix will be triangular with diagonal entries equal to 1.

We start with further notation. For a partition σ, let

$$\mathsf{FZ}_\sigma = \left[\exp\left(-\left\{\log(1 + C(p_1 + p_2 z + p_3 z^2 + \cdots))\right\}_\kappa\right)\right]_{p^\sigma}$$

and

$$\mathsf{SQ}_\sigma = \left[\exp\left(-\sum_{\sigma\neq\emptyset}\{z^{|\sigma|-\ell(\sigma)}C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|}\right)\right]_{p^\sigma}$$

be power series in z with coefficients that are polynomials in the κ classes. The relations themselves are given by

$$\mathsf{FZ}(r,\sigma) = [E \cdot \mathsf{FZ}_\sigma]_{z^r}\ , \quad \mathsf{SQ}(r,\sigma) = [E \cdot \mathsf{SQ}_\sigma]_{z^r}\ .$$

It is straightforward to expand FZ_σ and SQ_σ as linear combinations of products of factors $\{z^a C^b\}$ for $a \geq 0$ and $b \geq 1$, with coefficients that are polynomials in the kappa classes. When expanded, FZ_σ always contains exactly one term of the form

$$\{z^{a_1}C\}_\kappa\{z^{a_2}C\}_\kappa \cdots \{z^{a_m}C\}_\kappa\ . \tag{55}$$

All the other terms involve higher powers of C. If we expand SQ_σ, we can look at the terms of the form (55) to determine what the coefficients must be when writing the SQ_σ as linear combinations of the FZ_σ. For example,

$$\begin{aligned}
\mathsf{SQ}_{(111)} &= -\frac{1}{6}\{C_3\}_\kappa + \frac{1}{2}\{C_2\}_\kappa\{C_1\}_\kappa - \frac{1}{6}\{C_1\}_\kappa^3 \\
&= \frac{4}{3}\kappa_1 z + \frac{1}{3}\{C\}_\kappa - \frac{1}{3}\{C^3\}_\kappa + \frac{1}{2}(\kappa_0 - \{C^2\}_\kappa)\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3 \\
&= \left(\frac{4}{3}\kappa_1 z\right) + \left(\left(\frac{1}{3} + \frac{\kappa_0}{2}\right)\{C\}_\kappa\right) \\
&\qquad + \left(-\frac{1}{3}\{C^3\}_\kappa - \frac{1}{2}\{C^2\}_\kappa\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3\right) \\
&= \frac{4}{3}\kappa_1 z\,\mathsf{FZ}_\emptyset + \left(-\frac{1}{3} - \frac{\kappa_0}{2}\right)\mathsf{FZ}_{(1)} + \mathsf{FZ}_{(111)}\,.
\end{aligned}$$

In general we must check that the terms involving higher powers of C also match up. The matching will require an identity between the coefficients of C_i when expressed as polynomials in C. Define polynomials $f_{ij} \in \mathbb{Z}[z]$ by

$$C_i = \sum_{j=0}^{i} f_{ij} C^j.$$

It will also be convenient to write $f_{ij} = \sum_k f_{ijk} z^k$, so

$$C_i = \sum_{\substack{j,k \geq 0 \\ j+3k \leq i}} f_{ijk} z^k C^j.$$

If we define

$$F = 1 + \sum_{i,j \geq 1} \frac{(-1)^{j-1} f_{ij}}{i!(j-1)!} x^i y^j \in \mathbb{Q}[z][[x,y]],$$

then we will need a single property of the power series F.

Lemma 16. *There exists a power series $G \in \mathbb{Q}[z][[x]]$ such that $F = e^{yG}$.*

Proof. The recurrence (52) for the C_i together with the differential equation (50) satisfied by C yield a recurrence relation for the polynomials f_{ij}:

$$f_{i+1,j} = (j+1) f_{i,j+1} + 4(j-i) z f_{ij} - (j-1) f_{i,j-1}.$$

This recurrence relation for the coefficients of F is equivalent to a differential equation:

$$F_x = -yF_{yy} + 4zyF_y - 4zxF_x + yF.$$

Now, let $G \in \mathbb{Q}[z][[x,y]]$ be $\frac{1}{y}$ times the logarithm of f (as a formal power series). The differential equation for F can be rewritten in terms of G:

$$G_x = -2G_y - yG_{yy} - (G + yG_y)^2 + 4z(G + yG_y) - 4zxG_x + 1.$$

We now claim that the coefficient of $x^k y^l$ in G is zero for all $k \geq 0, l \geq 1$, as desired. For $k = 0$ this is a consequence of the fact that $F = 1 + O(xy)$ and thus $G = O(x)$, and higher values of k follow from induction using the differential equation above. □

We can now write the SQ_σ as linear combinations of the FZ_σ.

Theorem 6. *Let σ be a partition. Then $\mathsf{SQ}_\sigma - \mathsf{FZ}_\sigma$ is a $\mathbb{Q}$-linear combination of terms of the form*

$$\kappa_\mu z^{|\mu|}\,\mathsf{FZ}_\tau,$$

where μ and τ are partitions (μ possibly containing parts of size 0) satisfying $\ell(\tau) < \ell(\sigma)$, $3|\mu| + 3|\tau| - 2\ell(\tau) \leq 3|\sigma| - 2\ell(\sigma)$, and

$$3|\mu| + 3|\tau| - 2\ell(\tau) \equiv 3|\sigma| - 2\ell(\sigma) \mod 2\ .$$

Proof. We will need some additional notation for subpartitions. If σ is a partition of length $\ell(\sigma)$ with parts $\sigma_1, \sigma_2, \ldots$ (ordered by size) and S is a subset of $\{1, 2, \ldots, \ell(\sigma)\}$, then let $\sigma_S \subset \sigma$ denote the subpartition consisting of the parts $(\sigma_i)_{i\in S}$.

Using this notation, we explicitly expand SQ_σ and FZ_σ as sums over set partitions of $\{1, \ldots, \ell(\sigma)\}$:

$$\mathsf{SQ}_\sigma = \frac{1}{|\operatorname{Aut}(\sigma)|} \sum_{P\vdash\{1,\ldots,\ell(\sigma)\}} \prod_{S\in P} \left(\sum_{j,k} -f_{|S|,j,k} \{z^{|\sigma_S|-|S|+k} C^j\}_\kappa \right),$$

$$\mathsf{FZ}_\sigma = \frac{1}{|\operatorname{Aut}(\sigma)|} \sum_{P\vdash\{1,\ldots,\ell(\sigma)\}} \prod_{S\in P} \left((-1)^{|S|}(|S|-1)!\{z^{|\sigma_S|-|S|} C^{|S|}\}_\kappa \right).$$

Matching coefficients for terms of the form (55) tells us what the linear combination must be. We claim

$$\mathsf{SQ}_\sigma = \sum_{\substack{R\vdash\{1,\ldots,\ell(\sigma)\} \\ P\sqcup Q = R \\ k: R\to\mathbb{Z}_{\geq 0}}} \frac{|\operatorname{Aut}(\sigma')|}{|\operatorname{Aut}(\sigma)|} \times \tag{56}$$
$$\prod_{S\in P}\left(-f_{|S|,0,k(S)}\kappa_{|\sigma_S|-|S|+k(S)} z^{|\sigma_S|-|S|+k(S)}\right) \prod_{S\in Q} \left(f_{|S|,1,k(S)}\right) \mathsf{FZ}_{\sigma'},$$

where σ' is the partition with parts $|\sigma_S| - |S| + 1 + k(S)$ for $S \in Q$. Using the vanishing $f_{i,j,k} = 0$ unless $j + 3k \leq i$ and $j + 3k \equiv i \mod 2$, we easily check the above expression for SQ_σ is of the desired type.

Expanding SQ_σ and $\mathsf{FZ}_{\sigma'}$ in (56) and canceling out the terms involving the $f_{i,0,k}$ coefficients, it remains to prove

$$\sum_{\substack{Q\vdash\{1,\ldots,\ell(\sigma)\}\\ k:Q\to\mathbb{Z}_{\geq 0}\\ j:Q\to\overline{\mathbb{N}}}}\prod_{S\in Q}\left(-f_{|S|,j(S),k(S)}\{z^{|\sigma_S|-|S|+k(S)}C^{j(S)}\}_\kappa\right)$$

$$=\sum_{\substack{Q\vdash\{1,\ldots,\ell(\sigma)\}\\ k:Q\to\mathbb{Z}_{\geq 0}}}\prod_{S\in Q}\left(f_{|S|,1,k(S)}\right)\sum_{P\vdash\{1,\ldots,\ell(\sigma')\}}\prod_{S\in P}\Big((-1)^{|S|}(|S|-1)!\{z^{|(\sigma')_S|-|S|}C^{|S|}\}_\kappa\Big)\,.$$

A single term on the left side of the above equation is determined by choosing a set partition Q_{left} of $\{1,\ldots,\ell(\sigma)\}$ and then for each part S of Q_{left} choosing a positive integer $j(S)$ and a nonnegative integer $k_{\text{left}}(S)$. We claim that this term is the sum of the terms of the right side given by choices Q_{right}, k_{right}, P such that Q_{right} is a refinement of Q_{left} that breaks each part S in Q_{left} into exactly $j(S)$ parts in Q_{right}, P is the associated grouping of the parts of Q_{right}, and the $k_{\text{right}}(S)$ satisfy

$$k_{\text{left}}(S)=\sum_{T\subseteq S}k_{\text{right}}(T)\ .$$

These terms all are integer multiples of the same product of $\{z^aC^b\}_\kappa$ factors, so we are left with the identity

$$\frac{(-1)^{j_0-1}}{(j_0-1)!}f_{i_0,j_0,k_0}=\sum_{\substack{P\vdash\{1,\ldots,i_0\}\\ |P|=j_0\\ k:P\to\mathbb{Z}_{\geq 0}\\ |k|=k_0}}\prod_{S\in P}f_{|S|,1,k(S)}. \tag{57}$$

to prove.

But by the exponential formula, identity (57) is simply a restatement of Lemma 16. □

The conditions on the linear combination in Theorem 6 are precisely those needed so that multiplying by E and taking the coefficient of z^r allows us to write any SQ relation as a linear combination of FZ relations. The associated matrix is triangular with respect to the partial ordering of partitions by size, and the diagonal entries are equal to 1. Hence, the matrix is invertible. We conclude the SQ relations are equivalent to the FZ relations.

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