MAPS, SHEAVES, AND $K3$ SURFACES

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ABSTRACT. The conjectural equivalence of curve counting on Calabi-Yau 3-folds via stable maps and stable pairs is discussed. By considering Calabi-Yau 3-folds with $K3$ fibrations, the correspondence naturally connects curve and sheaf counting on $K3$ surfaces. New results and conjectures (with D. Maulik) about descendent integration on $K3$ surfaces are announced. The recent proof of the Yau-Zaslow conjecture is surveyed .

1. Counting curves

1.1. **Calabi-Yau 3-folds.** A *Calabi-Yau 3-fold* is a nonsingular projective variety[1] X of dimension 3 with trivial first Chern class

$$\wedge^3 T_X \stackrel{\sim}{=} \mathcal{O}_X.$$

Often the triviality of the fundamental group

$$\pi_1(X) = 1$$

is included in the definition. However, for our purposes, X need not be simply connected.

1.2. **Maps.** Let C be a complete curve with at worst simple nodes as singularities. We do not require C to be connected. The arithmetic genus g of C is defined by the Riemann-Roch formula,

$$\chi(C, \mathcal{O}_C) = 1 - g.$$

We view an algebraic map

$$f : C \to X$$

[1] All varieties here are defined over $\mathbb{C}$.

which is not constant on *any* connected component of C as parameterizing a subcurve of X. Let

$$\beta = f_*[C] \in H_2(X, \mathbb{Z})$$

be the homology class represented by f. Since f is nonconstant, $\beta \neq 0$.

An automorphism of f is an automorphism of the domain

$$\epsilon : C \to C$$

satisfying $f \circ \epsilon = f$. A map f is *stable* [33] if the automorphism group $\text{Aut}(f)$ is finite. Infinite automorphisms can come only from contracted rational and elliptic irreducible components of C incident to too few nodes.

Let $\overline{M}_g(X, \beta)^\bullet$ denote the moduli space of stable maps[2] from genus g curves to X representing the class β. The moduli space $\overline{M}_g(X, \beta)^\bullet$ is a projective Deligne-Mumford stack [5, 16, 33]. Certainly, $\overline{M}_g(X, \beta)^\bullet$ may be singular, non-reduced, and disconnected.

The most important structure carried by $\overline{M}_g(X, \beta)^\bullet$ is the obstruction theory [2, 4, 38] governing deformations of maps. The Zariski tangent space at $[f] \in \overline{M}_g(X, \beta)^\bullet$ has dimension

$$\dim_{\mathbb{C}}(T_{[f]}) = 3g - 3 + h^0(C, f^* T_X).$$

The first term on the right corresponds to deformations of the complex structure of C and the second term to deformations of the map with C fixed.[3] The obstruction space is

$$\text{Obs}_{[f]} = H^1(C, f^* T_X).$$

Formally, we may view the moduli space $\overline{M}_g(X, \beta)^\bullet$ as being cut out by $\dim_{\mathbb{C}}(\text{Obs}_{[f]})$ equations in the tangent space. Hence, we expect the

[2] Usually $\overline{M}_g(X, \beta)$ denotes the moduli space of stable maps with connected domains. The bullet in our notation indicates the possibility of disconnected domains.

[3] We assume, for the given interpretation of terms, the domain C has no infinitesimal automorphisms.

dimension of $\overline{M}_g(X,\beta)^\bullet$ to be

$$\begin{aligned}
\dim_{\mathbb{C}}^{expected}\Big(\overline{M}_g(X,\beta)^\bullet\Big) &= 3g-3+h^0(C,f^*T_X)-h^1(C,f^*T_X)\\
&= 3g-3+\chi(C,f^*T_X)\\
&= 3g-3+\int_C c_1(T_X)+\mathrm{rank}_{\mathbb{C}}(T_X)(1-g)\\
&= 0.
\end{aligned}$$

The third line is by Riemann-Roch. The Calabi-Yau 3-fold condition is imposed in the fourth line.

Since all curves in Calabi-Yau 3-folds are expected to move in 0-dimensional families, we can hope to count them. While $\overline{M}_g(X,\beta)^\bullet$ may have large positive dimensional components, the obstruction theory provides a virtual class

$$[\overline{M}_g(X,\beta)^\bullet]^{vir} \in H_0(\overline{M}_g(X,\beta),\mathbb{Q})$$

in exactly the expected dimension.

Gromov-Witten theory is the curve counting defined via integration against the virtual class of $\overline{M}_g(X,\beta)^\bullet$. The Gromov-Witten invariants of X are

$$N^\bullet_{g,\beta} = \int_{[\overline{M}_g(X,\beta)^\bullet]^{vir}} 1 \ \in \mathbb{Q}.$$

For fixed nonzero $\beta \in H_2(X,\mathbb{Z})$, let

$$\mathsf{Z}_{GW,\beta}(u) = \sum_g N^\bullet_{g,\beta}\ u^{2g-2} \ \in \mathbb{Q}((u)).$$

be the partition function.[4] Since $\overline{M}_g(X,\beta)^\bullet$ is empty for g sufficient negative, $\mathsf{Z}_{GW,\beta}(u)$ is a Laurent series.

The Gromov-Witten invariants $N^\bullet_{g,\beta}$ should be viewed as regularized curve counts. The integrals $N^\bullet_{g,\beta}$ are symplectic invariants. A natural idea is to relate the Gromov-Witten invariants to strict symplecting curve counts after perturbing the almost complex structure J. However, analytic difficulties arise. A complete understanding of the symplectic geometry here has not yet been obtained.

[4]Sometimes $\mathsf{Z}_{GW,\beta}(u)$ as defined here is called the *reduced* partition function since the constant map contribution are absent. The constant contributions, calculated in [13], will not arise in our discussion.

1.3. **Sheaves.** We may also approach curve counting in a Calabi-Yau 3-fold X via a gauge/sheaf theoretic approach [52, 53, 54].

We would like to construct a moduli space parameterizing divisors on curves in X. If the subcurve

$$\iota : C \subset X$$

is nonsingular, a divisor determines a line bundle $L \to C$ together with a section $s \in H^0(C, L)$. The associated torsion sheaf

$$\iota_*(L) = F$$

on X has 1-dimensional support and section $s \in H^0(X, F)$. However, for a compact moduli space, we must allow the support curve C to acquire singularities and nonreduced structure. The line bundle L must also be allowed to degenerate.

A *pair* (F, s) consists of a sheaf F on X supported in dimension 1 together with a section $s \in H^0(X, F)$. A pair (F, s) is *stable* if

(i) the sheaf F is pure,
(ii) the section $\mathcal{O}_X \xrightarrow{s} F$ has 0-dimensional cokernel.

Purity here simply means every nonzero subsheaf of F has support of dimension 1. As a consequence, the scheme theoretic support $C \subset X$ of F is a Cohen-Macaulay curve. The support of the cokernel (ii) is a finite length subscheme $Z \subset C$. If the support C is nonsingular, then the stable pair (F, s) is uniquely determined by $Z \subset C$. However, for general C, the subscheme Z does not determine F and s.

The discrete invariants of a stable pair are the holomorphic Euler characteristic $\chi(F) \in \mathbb{Z}$ and the class[5] $[F] \in H_2(X, \mathbb{Z})$. The moduli space $P_n(X, \beta)$ parameterizes stable pairs satisfying

$$\chi(F) = n, \quad [F] = \beta.$$

After appropriate choices [52], pair stability coincides with stability arising from geometric invariant theory [35]. The moduli space $P_n(X, \beta)$ is a therefore a projective scheme.

[5] $[F]$ is the sum of the classes of the irreducible 1-dimensional curves on which F is supported weighted by the generic length of F on the curve.

To define invariants, a virtual cycle is required. The usual deformation theory of pairs is problematic, but the fixed-determinant deformation theory[6] of the associated *complex* in the derived category

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \ \in \ D^b(X)$$

is shown in [25, 52] to define a perfect obstruction theory for $P_n(X,\beta)$ of virtual dimension zero. A virtual cycle is then obtained by [2, 4, 38]. The resulting regularized counts are

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1 \ \in \mathbb{Z}.$$

Let

$$\mathsf{Z}_{P,\beta}(q) = \sum_{n\in\mathbb{Z}} P_{n,\beta}\ q^n \ \in \mathbb{Q}((q)).$$

be the partition function. Since $P_n(X,\beta)$ is empty for n sufficient negative, $\mathsf{Z}_{P,\beta}(q)$ is a Laurent series.

The Gromov-Witten invariants $N^\bullet_{g,\beta}$ are $\mathbb{Q}$-valued since $\overline{M}_g(X,\beta)^\bullet$ is a Deligne-Mumford stack, but the stable pairs invariants $P_{n,\beta}$ are $\mathbb{Z}$-valued since $P_n(X,\beta)$ is a scheme.

2. Correspondence

2.1. **Two counts.** We have seen there are at least two regularized counting strategies for curves in Calabi-Yau 3-folds. While the Gromov-Witten approach may appear closer to a pure enumerative invariant since no auxiliary line bundles play a role, in fact the two theories are equivalent!

Conjecture 1. *For all Calabi-Yau 3-folds X and nonzero curve classes $\beta \in H_2(X,\mathbb{Z})$,*

$$\mathsf{Z}_{GW,\beta}(u) = \mathsf{Z}_{P,\beta}(q)$$

after the variable change $-e^{iu} = q$.

[6] Every fixed-determinant deformation of the complex (to any order) is quasi-isomorphic to a complex arising from a flat deformation of a stable pair [52]. However, the obstruction theory obtained from derived category deformations differs from the classical deformation theory of pairs.

Actually, the variable change $-e^{iu} = q$ is not a priori well-defined for Laurent series. The issue is addressed by the following rationality property.

Conjecture 2. *For all Calabi-Yau 3-folds X and nonzero curve classes $\beta \in H_2(X, \mathbb{Z})$, the series $\mathsf{Z}_{P,\beta}(q)$ is the Laurent expansion of a rational function invariant under $q \leftrightarrow 1/q$.*

In rigid cases, Conjecture 1 implies the contributions of multiple covers in Gromov-Witten theory exactly match the contributions of the divisor choices on thickened curves in the theory of stable pairs. In geometries with moving curves, the meaning of Conjecture 1 is more subtle.

2.2. **Other counts.** There are other geometric approaches to curve counting on Calabi-Yau 3-folds. On the map side, a new theory of stability has been very recently put forward by B. Kim, A. Kresch, and Y.-G. Oh [29] generalizing the well-known theory of admissible covers for dimension 1 targets. On the sheaf side, the older Donaldson-Thomas theory of ideal sheaf counts [12, 56] is very natural to pursue.

While Conjectures 1 and 2 as stated above are from [52], the relation between Gromov-Witten theory and sheaf counting was first discovered in the context of Donaldson-Thomas theory in [42, 43]. Stable pairs appear to be the closest sheaf enumeration to Gromov-Witten theory. The precise relationship of [29] to the other theories has yet to be discovered, but an equivalence almost surely holds.

2.3. **Evidence.** There are three interesting directions which provide evidence for Conjectures 1 and 2.

The first is the study of local Calabi-Yau toric surfaces.[7] Both the Gromov-Witten and pairs invariants can be calculated by the virtual localization formula [22]. On the Gromov-Witten side, the topological vertex of [1, 40] evaluates the localization formula. On the stable pairs

[7] A local Calabi-Yau toric surface is the total space of the canonical bundle of any nonsingular, projective, toric Fano surface.

side, the evaluation is given by box counting [53]. Conjectures 1 and 2 hold.[8] However, the toric examples are necessarily non-compact. The relevance of toric calculations to compact Calabi-Yau 3-folds is not clear.

The second direction is progress towards a geometric proof of Conjecture 2. The obstruction theory for $P_n(X,\beta)$ is self-dual.[9] By results of K. Behrend [3], there exists a constructable function

$$\chi^B : P_n(X,\beta) \to \mathbb{Z}.$$

with integral[10] equal to the pairs invariant,

$$\int_{P_n(X,\beta)} \chi^B = P_{n,\beta}.$$

If $P_n(X,\beta)$ is nonsingular, then $\chi^B = (-1)^{\dim_{\mathbb{C}}(P_n(X,\beta))}$ is constant and

$$P_{n,\beta} = (-1)^{\dim_{\mathbb{C}}(P_n(X,\beta))}\chi_{\text{top}}(P_n(X,\beta)).$$

In [53], properties of χ^B together with an essential application of Serre duality imply Conjecture 2 for irreducible[11] curves classes $\beta \in H_2(X,\beta)$. In remarkable recent work of Y. Toda [57], using variants of Bridgeland's stability conditions, wall-crossing formulas, and Serre duality, the rationality of the closely related series

$$\mathsf{Z}^{\chi}_{P,\beta}(q) = \sum_{n\in\mathbb{Z}} \chi_{\text{top}}(P_n(X,\beta))q^n$$

[8]In the forthcoming paper [44], the most general local Calabi-Yau toric geometry involving the 3-leg vertex is analysed for the Gromov-Witten/Donaldson-Thomas correspondence. It is likely the same path of argument will apply to stable pairs theory also.

[9]The obstruction theory is equipped with a pairing identifying the tangent space with the dual of the obstruction space [3].

[10]The integral is defined by

$$\int_{P_n(X,\beta)} \chi^B = \sum_{n\in\mathbb{Z}} n \cdot \chi_{\text{top}}\Big((\chi^B)^{-1}(n)\Big)$$

where χ_{top} on the right is the usual Euler characteristic.

[11]A class β is irreducible if all 1-dimensional subschemes representing β are reduced and irreducible.

has been proven for *all* nonzero classes $\beta \in H_2(X, \beta)$. A proper inclusion of χ^B into Toda's argument should soon lead to a complete proof of Conjecture 2.

The third direction, curve counting on $K3$ surfaces, will be discussed in Sections 3 and 4. The topic contains a mix of classical and quantum geometry. While there has been recent progress, many beautiful open questions remain.

3. Curve counting on $K3$ surfaces

3.1. **Reduced virtual class.** Let S be a $K3$ surface, and let

$$\beta \in \mathrm{Pic}(S) = H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$$

be an nonzero effective curve class.[12] By the virtual dimension formula,

$$\begin{aligned} \dim_{\mathbb{C}}^{expected}\left(\overline{M}_g(S, \beta)^{\bullet}\right) &= 3g - 3 + \chi(C, f^*T_S) \\ &= 3g - 3 + 2(1-g) \\ &= g - 1. \end{aligned}$$

Let $[f] \in \overline{M}_g(S, \beta)^{\bullet}$ be a stable map. There is canonical surjection

$$\mathrm{Obs}_{[f]} \to \mathbb{C} \to 0 \tag{1}$$

obtained from the the composition

$$H^1(C, f^*T_S) \stackrel{\sim}{=} H^1(C, f^*\Omega_S) \xrightarrow{df} H^1(C, \omega_C) \stackrel{\sim}{=} \mathbb{C},$$

where the first isomorphism uses

$$\wedge^2 T_S \stackrel{\sim}{=} \mathcal{O}_S.$$

The trivial quotient (1) forces the vanishing of $[\overline{M}_g(S, \beta)^{\bullet}]^{vir}$. However, the obstruction theory can be modified to reduce the obstruction space to the kernel of (1). A reduced virtual class

$$[\overline{M}_g(S, \beta)^{\bullet}]^{red} \in H_{2g}(\overline{M}_g(S, \beta)^{\bullet}, \mathbb{Q}), \tag{2}$$

in dimension 1 greater than expected, is therefore defined.

[12] By Poincaré duality, there is a canonical isomorphism $H_2(S, \mathbb{Z}) \stackrel{\sim}{=} H^2(S, \mathbb{Z})$, so we may view curves classes as taking values in either theory.

By constructing trivial quotients of $\mathrm{Obs}_{[f]}$ for each connected component of the domain, the reduced virtual class (2) is easily seen to be supported on the locus of curves with connected domains. Hence, we need only consider

$$\overline{M}_g(S,\beta) \subset \overline{M}_g(S,\beta)^{\bullet}.$$

We can also consider stable maps from r-pointed curves. The pointed moduli space $\overline{M}_{g,r}(S,\beta)$ has a reduced virtual class of dimension $g+r$.

3.2. **Descendents.** The reduced Gromov-Witten theory of S is defined via integration against $[\overline{M}_{g,r}(S,\beta)]^{red}$. Let

$$\mathrm{ev}_i : \overline{M}_{g,r}(S,\beta) \to S,$$

$$L_i \to \overline{M}_{g,r}(S,\beta)$$

denote the evaluation maps and cotangent lines bundles associated to the r marked points. Let $\gamma_1,\ldots,\gamma_m$ be a basis of $H^*(S,\mathbb{Q})$, and let

$$\psi_i = c_1(L_i) \in \overline{M}_{g,r}(S,\beta).$$

The *descendent* fields, denoted in the brackets by $\tau_k(\gamma_j)$, correspond to the classes $\psi_i^k \cup \mathrm{ev}_i^*(\gamma_j)$ on the moduli space of maps. Let

$$\Big\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r}) \Big\rangle_{g,\beta}^{red} = \int_{[\overline{M}_{g,r}(S,\beta)]^{red}} \prod_{i=1}^{r} \psi_i^{k_i} \cup \mathrm{ev}_i^*(\gamma_{l_i}) \tag{3}$$

denote the descendent Gromov-Witten invariants. Of course (3) vanishes if the integrand does not match the dimension of the reduced virtual class.

The reduced Gromov-Witten theory is invariant under deformations of S which preserve β as an algebraic class. A standard argument[13] shows the invariant (3) depends *only* on the norm

$$\langle \beta,\beta\rangle = \int_S \beta \cup \beta$$

and the divisibility of $\beta \in H^2(S,\mathbb{Z})$.

[13] The group of isometries of the $K3$ lattice $U^3 \oplus E_8(-1)^2$ acts transitively on elements with fixed norm and divisibility. The dependence of the reduced Gromov-Witten on only the norm and divisibility then follows from the global Torelli Theorem. See [9] for a slightly different point of view on the same result.

Let us now specialize, for the remainder of Section 3.2 , to an elliptically fibered $K3$ surface

$$\nu : S \to \mathbb{P}^1$$

with a section. We assume the section and fiber classes

$$\mathbf{s}, \mathbf{f} \in H^2(S, \mathbb{Z})$$

span $\text{Pic}(S)$. The cone of effective curve classes is

$$V = \{ m\mathbf{s} + n\mathbf{f} \mid m \geq 0,\ n \geq 0,\ (m,n) \neq (0,0) \}.$$

Since the norm of $d\mathbf{s} + dk\mathbf{f}$ is $2d^2(k-1)$, effective classes with all divisibilities $d \geq 1$ and norms at least $-2d^2$ can be found on S. Elementary arguments show the integrals (3) vanish in all other cases.[14]

A natural descendent potential function for the reduced theory of $K3$ surfaces is defined by

$$\mathsf{F}^S_{g,m}\big(\tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r})\big) = \sum_{n=0}^{\infty} \Big\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r}) \Big\rangle^{red}_{g, m\mathbf{s}+n\mathbf{f}} q^{m(n-m)}$$

for $g \geq 0$ and $m \geq 1$. The following Conjecture is made jointly with D. Maulik [48].

Conjecture 3. $\mathsf{F}^S_{g,m}\big(\tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r})\big)$ *is the Fourier expansion in q of a quasi-modular form of level m^2 with pole at $q = 0$ of order at most m^2.*

By the ring of quasi-modular forms of level m^2 with possible poles at $q = 0$, we mean the algebra generated by the Eisenstein series[15] E_2 over the ring of modular forms of level m^2. We have been able to prove

[14] See, for example, Lemma 2 of [47].

[15] The Eisenstein series E_{2k} is the modular form defined by the equation

$$-\frac{B_{2k}}{4k} E_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where B_{2n} is the $2n^{th}$ Bernoulli number and $\sigma_n(k)$ is the sum of the k^{th} powers of the divisors of n,

$$\sigma_k(n) = \sum_{i|n} i^k.$$

Conjecture 3 in the primitive case $m = 1$ by relations in the moduli of curves [14], degeneration methods [45], and the elliptic curve results of [49, 50, 51]. The $m > 1$ case appears to require new techniques.

Let $[p] \in H^4(S, \mathbb{Z})$ denote the Poincaré dual of a point. The simplest of the $K3$ series is the count of genus g curves passing through g points,

$$\mathsf{F}^S_{g,1}(\tau_0(p)\cdots\tau_0(p)) = \eta^{-24} \left(-\frac{1}{24}\, q\frac{d}{dq}E_2\right)^g$$

calculated[16] by J. Bryan and C. Leung [9]. Here

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

is Dedekind's function. Similar calculations in genus 1 for $m = 2$ have been done in [37].

3.3. λ_g **integrals for** $K3$ **surfaces.** A connection to the enumerative geometry of Calabi-Yau 3-folds holds for special integrals in the reduced Gromov-Witten theory of $K3$ surfaces. Let

$$R_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g \tag{4}$$

for effective curve classes $\beta \in H^2(S, \mathbb{Z})$. Here, the integrand λ_g is the top Chern class of the Hodge bundle

$$\mathbb{E}_g \to \overline{M}_g(S, \beta)$$

with fiber $H^0(C, \omega_C)$ over moduli point

$$[f : C \to S] \in \overline{M}_g(S, \beta).$$

See [13, 22] for a discussion of Hodge classes in Gromov-Witten theory.

The integrals (4) arise from the following 3-fold geometry. Let

$$\pi : X \to \mathbb{P}^1$$

be a $K3$-fibered Calabi-Yau 3-fold with

$$\iota : S \xrightarrow{\sim} \pi^{-1}(0) \subset X.$$

[16]Our indexing conventions differ slightly from those adopted in [9].

Assume further the family of $K3$ surfaces determined by X is transverse to the Noether-Lefschetz divisor in the moduli of $K3$ surface along which β is an algebraic class. Then, the moduli space

$$\overline{M}_g(S,\beta) \subset \overline{M}_g(X,\iota_*\beta)$$

is a connected component. The integral (4) is precisely the contribution of $\overline{M}_g(S,\beta)$ to the Gromov-Witten theory of X [47]. The discussion here may be viewed as an algebraic analogue of the twistor construction of [9].

The definition of the BPS counts[17] associated to the Hodge integrals (4) is straightforward. Let $\alpha \in \mathrm{Pic}(S)$ be a effective primitive class The Gromov-Witten potential $F_\alpha(u,v)$ for classes proportional to α is

$$F_\alpha = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} R_{g,m\alpha}\ u^{2g-2} v^{m\alpha}.$$

The BPS counts $r_{g,m\alpha}$ are uniquely defined by the following equation:

$$F_\alpha = \quad \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} r_{g,m\alpha}\ u^{2g-2} \sum_{d>0} \frac{1}{d} \left(\frac{\sin(du/2)}{u/2} \right)^{2g-2} v^{dm\alpha}.$$

We have defined BPS counts for both primitive and divisible classes.

The string theoretic calculations of S. Katz, A. Klemm and C. Vafa [27] via heterotic duality yield two conjectures.

Conjecture 4. *The BPS count $r_{g,\beta}$ depends upon β only through the norm $\langle \beta, \beta \rangle$.*

Assuming the validity of Conjecture 4, let $r_{g,h}$ denote the BPS count associated to a class β satisfying

$$\langle \beta, \beta \rangle = 2h - 2.$$

Conjecture 4 is rather surprising from the point of view of Gromov-Witten theory. The invariants $R_{g,\beta}$ depend a priori upon both the norm and the divisibility of β.

[17]BPS state counts can be extracted from Gromov-Witten theory via [20, 21]. The counts $r_{g,\beta}$ are conjecturally *integers*.

Conjecture 5. *The BPS counts $r_{g,h}$ are uniquely determined by the following equation:*

$$\sum_{g=0}^{\infty}\sum_{h=0}^{\infty}(-1)^g r_{g,h}(\sqrt{z}-\frac{1}{\sqrt{z}})^{2g}q^h = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{20}(1-zq^n)^2(1-z^{-1}q^n)^2}.$$

As a consequence of Conjecture 5, $r_{g,h}$ vanishes if $g > h$ and

$$r_{g,g} = (-1)^g(g+1).$$

The first values are tabulated below:

$r_{g,h}$	$h=0$	1	2	3	4
$g=0$	1	24	324	3200	25650
1		-2	-54	-800	-8550
2			3	88	1401
3				-4	-126
4					5

Conjectures 4 and 5 provide a complete solution for λ_g integrals in the reduced Gromov-Witten theory of $K3$ surfaces. The answer is compatible with Conjecture 3 as expected since Hodge integrals may be expressed in terms of descendent integrals [13].

3.4. **Stable pairs on $K3$ surfaces.** Let S be a $K3$ surface with an irreducible class $\beta \in H^2(S,\mathbb{Z})$ satisfying

$$\langle \beta, \beta \rangle = 2h-2,$$

and let $P_n(S,h)$ denote the associated moduli space of pairs on S. Consider again the $K3$-fibered Calabi-Yau 3-fold

$$\pi : X \to \mathbb{P}^1.$$

A deformation argument in [54] proves

$$P_n(S,h) \subset P_n(X,\iota_*\beta) \tag{5}$$

is a connected component of the moduli space of stable pairs of X. Moreover, $P_n(S,h)$ is a *nonsingular* projective variety [28, 54] of dimension $n+2h-1$.

Let Ω_P be the cotangent bundle of the moduli space $P_n(S,h)$. The self-dual obstruction theory on $P_n(S,h)$ induced from the inclusion (5) has obstruction bundle Ω_P. Hence, the contribution of $P_n(S,h)$ to the stable pairs invariants of X is

$$\begin{aligned} \mathsf{Z}^S_{P,h}(y) &= \sum_n \int_{P_n(S,h)} c_{n+2h-1}(\Omega_P)\ y^n \\ &= \sum_n (-1)^{n+2h-1} e(P_n(S,h))\ y^n. \end{aligned}$$

Here, we have written the stable pairs partition function in the variable y instead of the traditional q since the latter will be reserved for the Fourier expansions of modular forms.[18]

Fortunately, the topological Euler characteristics of $P_n(S,h)$ have been calculated by T. Kawai and K. Yoshioka. By Theorem 5.80 of [28],

$$\sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} e(P_n(S,h))\ y^n q^h =$$

$$\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2} .$$

For our pairs invariants, we require the signed Euler characteristics,

$$\sum_{h=0}^{\infty} \mathsf{Z}^S_h(y)\ q^h = \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n+2h-1} e(P_n(S,h))\ y^n q^h.$$

Therefore, $\sum_{h=0}^{\infty} \mathsf{Z}^S_{P,h}(y)\ q^h$ equals

$$-\left(\sqrt{-y} - \frac{1}{\sqrt{-y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2} .$$

3.5. **Correspondence.** We are now in a position to check whether the Katz-Klemm-Vafa predictions for the λ_g integrals in the reduced Gromov-Witten theory of S are compatible with the above stable pairs calculations via the maps/pairs correspondence of Conjecture 1.

[18] The conflicting uses of q seem impossible to avoid. The possibilities for confusion are great.

In the β irreducible case, the Gromov-Witten partition function takes the form

$$\sum_{h=0}^{\infty} \mathsf{Z}^S_{GW,h}(u)\ q^h = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} r_{g,h}\ u^{2g-2}\left(\frac{\sin(u/2)}{u/2}\right)^{2g-2} q^h.$$

Afte substituting $-e^{iu} = y$, we find

$$\sum_{h=0}^{\infty} \mathsf{Z}^S_{GW,h}(y)\ q^h = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} (-1)^{g-1} r_{g,h}\left(\sqrt{-y} - \frac{1}{\sqrt{-y}}\right)^{2g-2} q^h.$$

By Conjecture 5, $\sum_{h=0}^{\infty} \mathsf{Z}^S_{GW,h}(y)\ q^h$ equals

$$-\left(\sqrt{-y} - \frac{1}{\sqrt{-y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2}$$

which is $\sum_{h=0}^{\infty} \mathsf{Z}^S_{P,h}(y)\ q^h$.

The maps/pairs correspondence of Conjecture 1 therefore works perfectly assuming the Katz-Klemm-Vafa prediction for the reduced Gromov-Witten theory. But can the Katz-Klemm-Vafa prediction for stable maps be proven? The answer is yes in genus 0. The proof is our last topic.

4. The Yau-Zaslow conjecture

4.1. **Genus 0.** The genus 0 parts of Conjectures 4 and 5 for $K3$ surfaces were predicted earlier by S.-T. Yau and E. Zaslow [59].

Conjecture 4′. The BPS count $r_{0,\beta}$ depends upon β only through the norm $\langle \beta, \beta \rangle$.

Let $r_{0,m,h}$ denote the genus 0 BPS count associated to a class β of divisibility m satisfying

$$\langle \beta, \beta \rangle = 2h - 2.$$

Assuming Conjecture 4′ holds, we define

$$r_{0,h} = r_{0,m,h}$$

independent[19] of m.

[19] Independence of m holds when $2m^2$ divides $2h-2$. Otherwise, no such class β exists and $r_{0,m,h}$ is defined to vanish.

Conjecture 5′. The BPS counts $r_{0,h}$ are uniquely determined by

$$\sum_{h\geq 0} r_{0,h}\, q^{h-1} = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-24}. \tag{6}$$

A mathematical derivation of the Yau-Zaslow conjectures for primitive classes β via Euler characteristics of compactified Jacobians following [59] can be found in [6, 10, 15]. The Yau-Zaslow formula (6) was proven via Gromov-Witten theory for primitive classes β by J. Bryan and C. Leung [9]. An early calculation by A. Gathmann [19] for a class β of divisibility 2 was important for the correct formulation of the conjectures. Conjectures $4'$ and $5'$ have been proven in the divisibility 2 case by J. Lee and C. Leung [36] and B. Wu [58].

The main result of the paper [32] with A. Klemm, D. Maulik, and E. Scheidegger is a proof of Conjectures $4'$ and $5'$ in all cases.

Theorem 1. *The Yau-Zaslow conjectures hold for all nonzero effective classes $\beta \in \mathrm{Pic}(S)$ on a $K3$ surface S.*

The proof, using the connection to Noether-Lefschetz theory [47], mirror symmetry, and modular form identities, is surveyed in Sections 4.2 -4.5.

4.2. **Noether-Lefschetz theory.**

4.2.1. *$K3$ lattice.* Let S be a $K3$ surface. The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S,\mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \tag{7}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (7) is even.

4.2.2. *Lattice polarization.* A primitive[20] class $L \in \mathrm{Pic}(S)$ is a *quasi-polarization* if

$$\langle L, L \rangle > 0 \quad \text{and} \quad \langle L, [C] \rangle \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasi-polarization is base point free and determines a birational morphism

$$S \to \tilde{S}$$

contracting A-D-E configurations of (-2)-curves on S. Hence, every quasi-polarized $K3$ surface is algebraic.

Let Λ be a fixed rank r primitive[21] embedding

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r-1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix} .$$

The sign is chosen so $\Delta(\Lambda) > 0$.

A Λ-*polarization* of a $K3$ surface S is a primitive embedding

$$j : \Lambda \to \mathrm{Pic}(S)$$

satisfying two properties:

[20] A class in $H^2(S, \mathbb{Z})$ of divisibility 1 is *primitive*.

[21] An embedding of lattices is primitive if the quotient is torsion free.

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S,\mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ,

(ii) $\mathrm{Im}(j)$ contains a quasi-polarization.

By (ii), every Λ-polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1,20,1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M \subset Q \subset \mathbb{P}\big((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}\big).$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on Λ. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of Λ-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda/\Gamma_\Lambda.$$

We refer the reader to [11] for a detailed discussion.

4.2.3. *Families.* Let X be a compact 3-dimensional complex manifold equipped with holomorphic line bundles

$$L_1, \ldots, L_r \ \rightarrow X$$

and a holomorphic map

$$\pi : X \rightarrow C$$

to a nonsingular complete curve.

The tuple $(X, L_1, \ldots, L_r, \pi)$ is a *1-parameter family of nonsingular Λ-polarized $K3$ surfaces* if

(i) the fibers $(X_\xi, L_{1,\xi}, \ldots, L_{r,\xi})$ are Λ-polarized $K3$ surfaces via

$$v_i \mapsto L_{i,\xi}$$

for every $\xi \in C$,

(ii) there exists a $\lambda^\pi \in \Lambda$ which is a quasi-polarization of all fibers of π simultaneously.

The family π yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

Let $\lambda^\pi = \lambda_1^\pi v_1 + \cdots + \lambda_r^\pi v_r$. A vector $(d_1, \ldots, d_r)$ of integers is *positive* if

$$\sum_{i=1}^{r} \lambda_i^\pi d_i > 0.$$

If $\beta \in \mathrm{Pic}(X_\xi)$ has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then β has positive degree with respect to the quasi-polarization if and only if $(d_1, \ldots, d_r)$ is positive.

4.2.4. *Noether-Lefschetz divisors.* Noether-Lefschetz numbers are defined in [47] by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_\Lambda$. We briefly review the definition of the Noether-Lefschetz divisors.

Let $(\mathbb{L}, \iota)$ be a rank $r+1$ lattice $\mathbb{L}$ with an even symmetric bilinear form $\langle , \rangle$ and a primitive embedding

$$\iota : \Lambda \to \mathbb{L}.$$

Two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ are isomorphic if there is an isometry which restricts to identity on Λ. The first invariant of the data $(\mathbb{L}, \iota)$ is the discriminant $\Delta \in \mathbb{Z}$ of $\mathbb{L}$.

An additional invariant of $(\mathbb{L}, \iota)$ can be obtained by considering any vector $v \in \mathbb{L}$ for which

$$\mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v. \tag{8}$$

The pairing

$$\langle v, \cdot \rangle : \Lambda \to \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be quotient defined via the injection $\Lambda \to \Lambda^*$ obtained from the pairing $\langle , \rangle$ on Λ. The group G is abelian of order equal to the discriminant $\Delta(\Lambda)$. The image

$$\delta \in G/\pm$$

of δ_v is easily seen to be independent of v satisfying (8). The invariant δ is the *coset* of $(\mathbb{L}, \iota)$

By elementary arguments, two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ of rank $r+1$ are isomorphic if and only if the discriminants and cosets are equal.

Let $v_1, \ldots, v_r$ be an integral basis of Λ as before. The pairing of $\mathbb{L}$ with respect to an extended basis $v_1, \ldots, v_r, v$ is encoded in the matrix

$$\mathbb{L}_{h,d_1,\ldots,d_r} = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2h-2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \ldots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\ldots,d_r}).$$

The coset $\delta(h, d_1, \ldots, d_r)$ is represented by the functional

$$v_i \mapsto d_i.$$

The Noether-Lefschetz divisor $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces S for which $(\mathrm{Pic}(S), j)$ has rank $r+1$, discriminant Δ, and coset δ. By the Hodge index theorem, $P_{\Delta,\delta}$ is empty unless $\Delta > 0$.

Let $h, d_1, \ldots, d_r$ determine a positive discriminant

$$\Delta(h, d_1, \ldots, d_r) > 0.$$

The Noether-Lefschetz divisor $D_{h,(d_1,\ldots,d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$D_{h,(d_1,\ldots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \ldots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}]$$

where the multiplicity $m(h, d_1, \ldots, d_r | \Delta, \delta)$ is the number of elements β of the lattice $(\mathbb{L}, \iota)$ of type (Δ, δ) satisfying

$$\langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, v_i \rangle = d_i. \tag{9}$$

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \ldots, d_r)$ so only finitely many divisors appear in the above sum.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ has an alternate definition. The tautological line bundle $\mathcal{O}(-1)$ is Γ-equivariant on the period domain M_Λ and descends to the *Hodge line bundle*

$$\mathcal{K} \to \mathcal{M}_\Lambda.$$

We define $D_{h,(d_1,\ldots,d_r)} = \mathcal{K}^*$. See [47] for an alternate view of degenerate intersection.

If $\Delta(h, d_1, \ldots, d_r) < 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ on $\mathcal{M}_\Lambda$ is defined to vanish by the Hodge index theorem.

4.2.5. *Noether-Lefschetz numbers.* Let Λ be a lattice of discriminant $l = \Delta(\Lambda)$, and let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. The Noether-Lefschetz number $NL^\pi_{h,d_1,\ldots,d_r}$ is the classical intersection product

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \int_C \iota^*_\pi [D_{h,(d_1,\ldots,d_r)}]. \tag{10}$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $SL_2(\mathbb{Z})$. There is a canonical representation [7] associated to Λ,

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G]).$$

The full set of Noether-Lefschetz numbers $NL^\pi_{h,d_1,\ldots,d_r}$ defines a vector valued modular form

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi^\pi_\gamma(q) v_\gamma \in \mathbb{C}[[q^{\frac{1}{2l}}]] \otimes \mathbb{C}[G],$$

of weight $\frac{22-r}{2}$ and type ρ^*_Λ by results[22] of Borcherds and Kudla-Millson [7, 34]. The Noether-Lefschetz numbers are the coefficients[23] of the components of Φ^π,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \Phi^\pi_\gamma \left[\frac{\Delta(h, d_1, \ldots, d_r)}{2l} \right]$$

where $\delta(h, d_1, \ldots, d_r) = \pm\gamma$. The modular form results significantly constrain the Noether-Lefschetz numbers.

[22]While the results of the papers [7, 34] have considerable overlap, we will follow the point of view of Borcherds.

[23]If f is a series in q, $f[k]$ denotes the coefficient of q^k.

4.2.6. *Refinements.* If $d_1, \ldots, d_r$ do not simultaneously vanish, refined Noether-Lefschetz divisors are defined. If $\Delta(h, d_1, \ldots, d_r) > 0$,

$$D_{m,h,(d_1,\ldots,d_r)} \subset D_{h,(d_1,\ldots,d_r)}$$

is defined by requiring the class $\beta \in \mathrm{Pic}(S)$ to satisfy (9) and have divisibility $m > 0$. If $\Delta(h, d_1, \ldots, d_r) = 0$, then

$$D_{m,h,(d_1,\ldots,d_r)} = D_{h,(d_1,\ldots,d_r)}$$

if $m > 0$ is the greatest common divisor of $d_1, \ldots, d_r$ and 0 otherwise.

Refined Noether-Lefschetz numbers are defined by

$$NL^{\pi}_{m,h,(d_1,\ldots,d_r)} = \int_C \iota^*_\pi[D_{m,h,(d_1,\ldots,d_r)}]. \tag{11}$$

The full set of Noether-Lefschetz numbers $NL^{\pi}_{h,(d_1,\ldots,d_r)}$ is easily shown in [32] to determine the refined numbers $NL^{\pi}_{m,h,(d_1,\ldots,d_r)}$.

4.3. **Three theories.** The main geometric idea in the proof of Theorem 1 is the relationship of three theories associated to a 1-parameter family

$$\pi : X \to C$$

of Λ-polarized $K3$ surfaces:

(i) the Noether-Lefschetz numbers of π,
(ii) the genus 0 Gromov-Witten invariants of X,
(iii) the genus 0 reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin. For (ii), we view the theory in terms the Gopakumar-Vafa invariants [20, 21].

Let $n^X_{0,(d_1,\ldots,d_r)}$ denote the Gopakumar-Vafa invariant of X in genus 0 for π-vertical curve classes of degrees $d_1, \ldots, d_r$ with respect to the line bundles $L_1, \ldots, L_r$. Let $r_{0,m,h}$ denote the reduced $K3$ invariant. The following result is proven[24] in [47] by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π.

[24]The result of the [47] is stated in the rank $r = 1$ case, but the argument is identical for arbitrary r.

Theorem 2. *For degrees $(d_1, \ldots, d_r)$ positive with respect to the quasi-polarization λ^π,*

$$n^X_{0,(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty}\sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

4.4. **The STU model.** The STU model[25] is a particular nonsingular projective Calabi-Yau 3-fold X equipped with a fibration

$$\pi : X \to \mathbb{P}^1. \tag{12}$$

Except for 528 points $\xi \in \mathbb{P}^1$, the fibers

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered $K3$ surfaces. The 528 singular fibers X_ξ have exactly 1 ordinary double point singularity each.

The 3-fold X is constructed as a nonsingular anticanonical section of the nonsingular projective toric 4-fold Y defined by 10 rays with primitives

$$\begin{array}{lll}
\rho_1 = (1,0,2,3) & \rho_2 = (-1,0,2,3) & \\
\rho_3 = (0,1,2,3) & \rho_4 = (0,-1,2,3) & \\
\rho_5 = (0,0,2,3) & \rho_6 = (0,0,-1,0) & \rho_7 = (0,0,0,-1) \\
\rho_8 = (0,0,1,2) & \rho_9 = (0,0,0,1) & \rho_{10} = (0,0,1,1).
\end{array}$$

The Picard rank of Y is 6. The fibration (12) is obtained from a nonsingular toric fibration

$$\pi^Y : Y \to \mathbb{P}^1.$$

The image of

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(X_\xi)$$

determines a rank 2 sublattice of each fiber $\mathrm{Pic}(X_\xi)$ with intersection form

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[25]The model has been studied in physics since the 80's. The letter S stands for the dilaton and T and U label the torus moduli in the heterotic string. The STU model was an important example for the duality between type IIA and heterotic strings formulated in [26] and has been intensively studied [23, 24, 30, 31, 41].

Let $L_1, L_2 \to X$ denote line bundles which span the standard basis of the form Λ after restriction.

Strictly speaking, the tuple (X, L_1, L_2, π) is not a 1-parameter family of Λ-polarized $K3$ surfaces. The only failing is the 528 singular fibers of π. Let

$$\epsilon : C \xrightarrow{2-1} \mathbb{P}^1$$

be a hyperelliptic curve branched over the 528 points of $\mathbb{P}^1$ corresponding to the singular fibers of π. The family

$$\epsilon^*(X) \to C$$

has 3-fold double point singularities over the 528 nodes of the fibers of the original family. Let

$$\widetilde{\pi} : \widetilde{X} \to C$$

be obtained from a small resolution

$$\widetilde{X} \to \epsilon^*(X).$$

Let $\widetilde{L}_i \to \widetilde{X}$ be the pull-back of L_i by ϵ. The data

$$(\widetilde{X}, \widetilde{L}_1, \widetilde{L}_2, \widetilde{\pi})$$

determine a 1-parameter family of Λ-polarized $K3$ surfaces, see Section 5.3 of [47]. The simultaneous quasi-polarization is obtained from the projectivity of X.

4.5. **Proof of Theorem 1.** Theorem 1 is proven in [32] by studying Theorem 2 applied to the STU model. There are four basic steps:

(i) The modular form [7, 34] determining the intersections of the base curve with the Noether-Lefschetz divisors is calculated. For the STU model, the modular form has vector dimension 1 and is proportional to the product $E_4 E_6$ of Eisenstein series.

(ii) Theorem 2 is used to show the 3-fold BPS counts $n^{\widetilde{X}}_{0,(d_1,d_2)}$ then *determine* all the reduced $K3$ invariants $r_{0,m,h}$. Strong use is made of the rank 2 lattice of the STU model.

(iii) The BPS counts $n^{\widetilde{X}}_{0,(d_1,d_2)}$ are calculated via mirror symmetry. Since the STU model is realized as a Calabi-Yau complete intersection in a nonsingular toric variety, the genus 0 Gromov-Witten invariants are obtained after proven mirror transformations from hypergeometric series [17, 18, 39]. The Klemm-Lerche-Mayr identity, proven in [32], shows the invariants $n^{\widetilde{X}}_{0,(d_1,d_2)}$ are themselves related to modular forms.

(iv) Theorem 1 then follows from the Harvey-Moore identity which simultaneously relates the modular structures of

$$n^{\widetilde{X}}_{0,(d_1,d_2)}, \quad r_{0,m,h}, \quad \text{and} \quad NL^{\widetilde{\pi}}_{m,h,(d_1,d_2)}$$

in the form specified by Theorem 2.

The Harvey-Moore identity of part (iv) is simple to state. Let

$$f(\tau) = \frac{E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c(n)q^n$$

where $q = \exp(2\pi i \tau)$. Then,

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \frac{q_1}{q_1 - q_2} + E_4(\tau_2) - \sum_{d,k,\ell>0} \ell^3 c(k\ell)\; q_1^{kd} q_2^{\ell d}. \tag{13}$$

Equation (13) was conjectured in [23] and proven by D. Zagier — the proof is presented in Section 4 of [32].

The strategy of the proof of the Yau-Zaslow conjectures is special to genus 0. Much less is known in higher genus. For genus 1, the Katz-Klemm-Vafa conjectures follow for all classes on $K3$ surfaces from the Yau-Zaslow conjectures via the boundary relation for λ_1 in the moduli of elliptic curves. In genus 2 and 3, A. Pixton [55] has proven the Katz-Klemm-Vafa formula for primitive classes using boundary relations for λ_2 and λ_3 on $\overline{M}_2$ and $\overline{M}_3$ respectively. New ideas will be required for a complete proof of Conjectures 4 and 5.

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Many of the ideas discussed here are valid in more general contexts. For example, the stable maps/pairs correspondence is conjectured in [52] for all 3-folds — the Calabi-Yau condition is not necessary. The $K3$ study can be pursued along similar lines for abelian surfaces, see [8] for a start. The Enriques surface is a close cousin [46].

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