THE κ RING OF THE MODULI OF CURVES OF COMPACT TYPE: I

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Abstract. The subalgebra of the tautological ring of the moduli of curves of compact type generated by the κ classes is studied in all genera. Relations, constructed via the virtual geometry of the moduli of stable quotients, are used to obtain minimal sets of generators. Bases and Betti numbers of the κ rings are computed. A universality property relating the higher genus κ rings to the genus 0 rings is stated and proved in a sequel. The λ_g-formula for Hodge integrals arises as the simplest consequence.

Contents

1. Introduction

1.1. **Curves of compact type.** Let C be a reduced and connected curve over $\mathbb{C}$ with at worst nodal singularities. The associated *dual graph* Γ_C has vertices corresponding to the irreducible components of C and edges corresponding to the nodes. The curve C is of *compact type* if Γ_C is a tree. Alternatively, C is of compact type if the Picard variety of line bundles of fixed multidegree on C is compact.

Standard marked points $p_1, \ldots, p_n$ on C must be distinct and lie in the nonsingular locus. The pointed curve $(C, p_1, \ldots, p_n)$ is *stable* if the

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line bundle $\omega_C(p_1 + \ldots + p_n)$ is ample. Stability implies the condition $2g - 2 + n > 0$ holds. Let

$$M^c_{g,n} \subset \overline{M}_{g,n}$$

denote the open subset of genus g, n-pointed stable curves of compact type. The complement

$$\overline{M}_{g,n} \setminus M^c_{g,n} = \delta_0$$

is the irreducible divisor of stable curves with a non-disconnecting node.

Since every nonsingular curve is of compact type, the inclusion

$$M_{g,n} \subset M^c_{g,n}$$

is obtained. While the Jacobian map

$$M_{g,n} \to A_g$$

from the moduli of nonsingular curves to the moduli of principally polarized Abelian varieties does not extend to $\overline{M}_{g,n}$, the extension

$$M_{g,n} \subset M^c_{g,n} \to A_g$$

is easily defined.

1.2. κ **classes.** The κ classes in the Chow ring[1] $A^*(\overline{M}_{g,n})$ are defined by the following construction. Let

$$\epsilon : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

be the universal curve viewed as the $(n+1)$-pointed space, let

$$\mathbb{L}_{n+1} \to \overline{M}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \ \in A^1(\overline{M}_{g,n+1})$$

be the Chern class. The κ classes, first defined by Mumford, are

$$\kappa_i = \epsilon_*(\psi^{i+1}_{n+1}) \ \in A^i(\overline{M}_{g,n}), \quad i \geq 0 \ .$$

The simplest is κ_0 which equals $2g-2+n$ times the unit in $A^0(\overline{M}_{g,n})$. The convention

$$\kappa_{-1} = \epsilon_*(\psi^0_{n+1}) = 0$$

is often convenient.

[1]Since the moduli spaces here are Deligne-Mumford stacks, we will always take Chow rings with $\mathbb{Q}$ coefficients.

The κ classes on $M_{g,n}$ and $M^c_{g,n}$ are defined via restriction from $\overline{M}_{g,n}$. Define the κ rings

$$\begin{aligned} \kappa^*(M_{g,n}) &\subset A^*(M_{g,n}),\\ \kappa^*(M^c_{g,n}) &\subset A^*(M^c_{g,n}),\\ \kappa^*(\overline{M}_{g,n}) &\subset A^*(\overline{M}_{g,n}), \end{aligned}$$

to be the $\mathbb{Q}$-subalgebras generated by the κ classes. Of course, the κ rings are graded by degree.

Since κ_i is a tautological class[2], the κ rings are subalgebras of the corresponding tautological rings. For unpointed nonsingular curves, the κ ring equals the tautological ring,

$$\kappa^*(M_g) = R^*(M_g) \ .$$

The topic of the paper is the compact type case where the inclusion

$$\kappa^*(M^c_{g,n}) \subset R^*(M^c_{g,n})$$

is proper even in degree 1.

1.3. **Results.** We present here several results about the rings $\kappa^*(M^c_{g,n})$. The first two yield a minimal set of generators in the $n > 0$ case.

Theorem 1. *$\kappa^*(M^c_{g,n})$ is generated over $\mathbb{Q}$ by the classes*

$$\kappa_1, \kappa_2, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor}.$$

Theorem 2. *If $n > 0$, there are no relations among*

$$\kappa_1, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor} \in \kappa^*(M^c_{g,n})$$

in degrees $\leq g - 1 + \lfloor \frac{n}{2} \rfloor$.

Since $\kappa^*(M^c_{g,n}) \subset R^*(M^c_{g,n})$, the socle and vanishing results for the tautological ring [5, 10] imply

$$\kappa^{2g-3+n}(M^c_{g,n}) = \mathbb{Q}, \qquad \kappa^{>2g-3+n}(M^c_{g,n}) = 0 \ . \tag{1}$$

By Theorem 2, all the interesting relations among the κ classes lie in degrees $g + \lfloor \frac{n}{2} \rfloor$ to $2g - 3 + n$.

By Theorem 1, the classes $\kappa_1, \ldots, \kappa_{g-1}$ generate $\kappa^*(M^c_g)$. Since M^c_g is excluded in Theorem 2, the possibility of a relation among the κ classes in degree $g - 1$ is left open. However, no lower relations exist.

[2]A discussion of tautological classes is presented in Section 5.1.

Proposition 1. *There are no relations among $\kappa_1, \ldots, \kappa_{g-1} \in \kappa^*(M_g^c)$ in degrees $\leq g-2$ and at most a single relation in degree $g-1$.*

The structure of $\kappa^*(M_g)$ has been studied for many years [17]. Faber [2] conjectured the classes $\kappa_1, \ldots, \kappa_{\lfloor \frac{g}{3} \rfloor}$ form a minimal set of generators for $\kappa^*(M_g)$. The result was proven in cohomology by Morita [16], and a second proof, via admissible covers and valid in Chow, was given by Ionel [12]. A uniform view of M_g, M_g^c, and $\overline{M}_g$ was proposed in [4], but very few results in the latter two cases have been obtained.

1.4. **Relations.** Theorem 1 is proven by finding sufficiently many geometric relations among the κ classes. The method uses the virtual geometry of the moduli space of stable quotients introduced in [15] and reviewed in Section 2. Nonstandard moduli spaces of pointed curves of compact type are required for the construction.

Following the notation of [15], let $\overline{M}_{g,n|d}$ be the moduli space of genus g stable curves with markings

$$\{p_1 \ldots, p_n\} \ \cup \ \{\widehat{p}_1, \ldots, \widehat{p}_d\} \in C$$

lying the nonsingular locus and satisfying the conditions

(i) the points p_i are distinct,
(ii) the points $\widehat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^{n} p_i + \epsilon \sum_{j=1}^{d} \widehat{p}_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\widehat{p}_j$ and $\widehat{p}_{j'}$ to coincide. The moduli space $\overline{M}_{g,n|d}$ is a nonsingular, irreducible, Deligne-Mumford stack.[3]

Denote the open locus of curves of compact type by

$$M_{g,n|d}^c \subset \overline{M}_{g,n|d} \ .$$

Consider the universal curve

$$\pi : U \to M_{g,n|d}^c.$$

[3]In fact, $\overline{M}_{g,n|d}$ is a special case of the moduli of pointed curves with weights studied by [11, 13].

The morphism π has sections $\sigma_1, \ldots, \sigma_d$ corresponding to the markings $\widehat{p}_1, \ldots, \widehat{p}_d$. Let

$$\sigma \subset U$$

be the divisor obtained from the union of the d sections. The two rank d bundles on $M^c_{g,n|d}$,

$$\mathbb{A}_d = \pi_*(\mathcal{O}_\sigma), \qquad \mathbb{B}_d = \pi_*(\mathcal{O}_\sigma(\sigma)),$$

play important roles in the geometry.

The new relations studied here arise from the vanishing of the Chern classes of the virtual bundle $\mathbb{A}^*_d - \mathbb{B}_d$ on $M^c_{g,n|d}$ after push-forward via the proper forgetful map

$$\epsilon^c : M^c_{g,n|d} \to M^c_{g,n}.$$

Theorem 3. *For all $k > n$,*

$$\epsilon^c_* \left(c_{2g-2+k}(\mathbb{A}^*_d - \mathbb{B}_d)\right) = 0 \ \in A^*(M^c_{g,n}).$$

The proofs of Theorem 3 and richer variants are given in Section 3. The ϵ^c push-forwards are calculated by simple rules explained in Section 3.5. In particular, we will see Theorem 3 yields relations purely among the κ classes on the moduli space $M^c_{g,n}$.

Theorem 1 is proven for $M^c_{g,n}$ in Section 4 by examining the relations of Theorem 3. The coefficient of κ_i for $i > g - 1 + \lfloor \frac{n}{2} \rfloor$ is shown to be nonzero. The method yields an effective evaluation of the relations. Theorem 2 and Proposition 1 are proven in Section 5 by intersection calculations in the tautological ring.

1.5. **Genus 0.** The strategy of Theorem 3 does *not* generate all the relations in $\kappa^*(M^c_g)$. The first example of failure, occurring in genus 5, is discussed in Section 6.

Since all genus 0 curves are of compact type,

$$M^c_{0,n} = \overline{M}_{0,n}.$$

For emphasis here, we will use the notation $M^c_{0,n}$. The following universality property, motivated by the relations of Theorem 3, gives considerable weight to the genus 0 case.

Let $x_1, x_2, x_3, \ldots$ be variables with x_i of degree i. Let

$$f \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$$

be *any* graded homogeneous polynomial.

Theorem 4. *If* $f(\kappa_i) = 0 \in \kappa^*(M^c_{0,n})$, *then*

$$f(\kappa_i) = 0 \in \kappa^*(M^c_{g,n-2g})$$

for all genera g *for which* $n - 2g \geq 0$.

Variants of Theorem 3 likely to provide all relations in $\kappa^*(M^c_{0,n})$. A precise statement is given in Section 6.2. The proof of Theorem 4, obtained by stable map techniques, is given in the sequel [20].

1.6. λ_g**-formula.** The rank g Hodge bundle over the moduli space of curves

$$\mathbb{E} \rightarrow \overline{M}_{g,n}$$

has fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. Let

$$\lambda_k = c_k(\mathbb{E})$$

be the Chern classes. Since λ_g vanishes when restricted to δ_0, we obtain a well-defined evaluation

$$\phi : A^*(M^c_{g,n}) \rightarrow \mathbb{Q}$$

given by integration

$$\phi(\gamma) = \int_{\overline{M}_{g,n}} \overline{\gamma} \cdot \lambda_g \ ,$$

where $\overline{\gamma}$ is any lift of $\gamma \in A^*(M^c_{g,n})$ to $A^*(\overline{M}_{g,n})$.

A discussion of the evaluation ϕ and the associated Gorenstein conjecture for the tautological ring can be found in [5, 19]. For background on integrating the classes λ_i on the moduli space of curves, see [3].

The evaluation ϕ is determined on $R^*(M^c_{g,n})$ by the λ_g-formula for descendent integrals,

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_g = \binom{2g-3+n}{a_1, \ldots, a_n} \cdot \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

discovered in [7] and proven in [5]. Theorem 4 is much stronger. The λ_g-formula is a direct consequence of Theorem 4 in the special case where f has degree equal to

$$\dim_{\mathbb{C}}(M^c_{0,n}) = n - 3.$$

Conjecture 1 may be view as an extension of the λ_g-formula from $\mathbb{Q}$ to cycle classes of all intermediate degrees.

1.7. **Bases and Betti numbers.** Let $P(d)$ be the set of partitions of d, and let

$$P(d,k) \subset P(d)$$

be the set of partitions of d into at most k parts. Let $|P(d,k)|$ be the cardinality. To a partition[4]

$$\mathbf{p} = (p_1, \ldots, p_\ell) \in P(d,k),$$

we associate a κ monomial by

$$\kappa_{\mathbf{p}} = \kappa_{p_1} \cdots \kappa_{p_\ell} \in \kappa^d(M^c_{0,n}) \ .$$

Theorem 5. *A $\mathbb{Q}$-basis of $\kappa^d(M^c_{0,n})$ is given by*

$$\{\kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, n-2-d) \ \} \ .$$

For example, if $d \leq \lfloor \frac{n}{2} \rfloor$-1, then $n-2-d \geq d$ and

$$P(d, n-2-d) = P(d).$$

Hence, Theorem 5 agrees with Theorem 2. The Betti number calculation,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) \ = \ |P(d, n-2-d)| \ ,$$

is implied by Theorem 5. The proof of Theorem 5 is given in Section 6.2.

The relations of Theorem 3 and variants provide an indirect approach for multiplication in the canonical basis of $\kappa^*(M^c_{0,n})$ determined by Theorem 5.

Question 1. *Does there exist a direct calculus for multiplication in the canonical basis of $\kappa^*(M^c_{0,n})$?*

1.8. **Universality.** The universality of Theorem 4 expresses the higher genus structures as canonical *ring* quotients,

$$\kappa^*(M^c_{0,2g+n}) \stackrel{\iota_{g,n}}{\to} \kappa^*(M^c_{g,n}) \to 0 \ .$$

Theorem 6. *If $n > 0$, then $\iota_{g,n}$ is an isomorphism.*

The rings $\kappa^*(M^c_{g,n})$ for $n > 0$ are determined by Theorem 6. For example,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,n}) \ = \ |P(d, 2g-2+n-d)|$$

[4]The parts of $\mathbf{p}$ are posititve and satisfy $p_1 \geq \ldots \geq p_\ell$.

by Theorem 5 for $n > 0$. The proof of Theorem 6 is presented in Section 6.3 via intersection calculations.

The quotient $\iota_{g,0}$ is not always an isomorphism. For example, a nontrivial kernel appears for $\iota_{5,0}$.

Question 2. *What is the kernel of $\iota_{g,0}$?*

Universality appears to be special to the moduli of compact type curves. No similar phenomena have been found for M_g or $\overline{M}_g$.

1.9. **Acknowledgments.** Theorem 3 was motivated by the study of stable quotients developed in [15]. Discussions with A. Marian and D. Oprea were very helpful. Easy exploration of the relations of Theorem 3 was made possible by code written by C. Faber. Conversation with C. Faber played an important role.

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2. Stable quotients

2.1. **Stability.** Our relations in $\kappa(M^c_{g,n})$ will be obtained from the virtual geometry of the moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{P}^1, d)$. We start by reviewing basic definitions and results of [15].

Let C be a curve[5] with distinct markings $p_1, \ldots, p_n$ in the nonsingular locus C^{ns}. Let q be a quotient of the rank N trivial bundle C,

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0.$$

If the quotient subsheaf Q is locally free at the nodes and markings of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let $(C, p_1, \ldots, p_n)$ be a pointed curve equipped with a quasi-stable quotient q. The data $(C, p_1, \ldots, p_n, q)$ determine a *stable quotient* if the

[5]All curves here are reduced and connected with at worst nodal singularities.

$\mathbb{Q}$-line bundle

$$\omega_C(p_1 + \ldots + p_n) \otimes (\wedge^r S^*)^{\otimes \epsilon} \tag{2}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 + n \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbb{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

A stable quotient $(C, p_1, \ldots, p_n, q)$ yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r, N)$. We will only require the $\mathbb{G}(1, 2) = \mathbb{P}^1$ case for the proof Theorem 3.

2.2. **Isomorphism.** Let $(C, p_1, \ldots, p_n)$ be a pointed curve. Two quasi-stable quotients

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0, \quad \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q'}{\to} Q' \to 0 \tag{3}$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, p_1, \ldots, p_n, q) \to (C', p'_1, \ldots, p'_n, q')$$

is an isomorphism of curves

$$\phi : C \stackrel{\sim}{\to} C'$$

satisfying

(i) $\phi(p_i) = p'_i$ for $1 \leq i \leq n$,
(ii) the quotients q and $\phi^*(q')$ are strongly isomorphic.

Quasi-stable quotients (3) on the same curve C may be isomorphic without being strongly isomorphic.

The following result is proven in [15] by Quot scheme methods from the perspective of geometry relative to a divisor.

Theorem 7. *The moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{G}(r, N), d)$ parameterizing the data*

$$(C,\ p_1, \ldots, p_n,\ 0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0),$$

with $\mathrm{rank}(S) = r$ *and* $\deg(S) = -d$*, is a separated and proper Deligne-Mumford stack of finite type over* $\mathbb{C}$*.*

2.3. **Structures.** Over the moduli space of stable quotients, there is a universal curve

$$\pi : U \to \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \tag{4}$$

with n sections and a universal quotient

$$0 \to S_U \to \mathbb{C}^N \otimes \mathcal{O}_U \stackrel{q_U}{\to} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is equipped with two basic types of maps. If $2g-2+n > 0$, then the stabilization of $(C, p_1, \ldots, p_m)$ determines a map

$$\nu : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \overline{M}_{g,n}$$

by forgetting the quotient. For each marking p_i, the quotient is locally free over p_i, and hence determines an evaluation map

$$\mathrm{ev}_i : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \mathbb{G}(r,N).$$

The general linear group $\mathbf{GL}_N(\mathbb{C})$ acts on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ via the standard action on $\mathbb{C}^N \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathbf{GL}_N(\mathbb{C})$-equivariant.

2.4. **Obstruction theory.** The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \mathcal{M}_{g,n}.$$

The moduli of stable quotients with fixed underlying curve

$$(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$$

is simply an open set of the Quot scheme. The following result of [15] is obtained from the standard deformation theory of the Quot scheme.

Theorem 8. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on* $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ *relative to* ν^A *given by* $R\mathcal{H}om(S,Q)$*.*

An absolute 2-term obstruction theory on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is obtained from Theorem 8 and the smoothness of $\mathcal{M}_{g,n}$, see [1, 8]. The analogue of Theorem 8 for the Quot scheme of a *fixed* nonsingular curve was observed in [14].

The $\mathbf{GL}_N(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathbf{GL}_N(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_{g,n}(\mathbb{G}(r,N),d)]^{vir} \in A_*^{\mathbf{GL}_N(\mathbb{C})}(\overline{Q}_{g,n}(\mathbb{G}(r,N),d)).$$

3. Construction of the relations

3.1. $\mathbb{C}^*$-equivariant geometry. Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $[0,1]$ on the respective basis elements. Let

$$\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2),$$

and let $0,\infty \in \mathbb{P}^1$ be the $\mathbb{C}^*$-fixed points corresponding the eigenspaces of weight 0 and 1 respectively.

There is an induced $\mathbb{C}^*$-action on $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. Since the virtual dimension of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ is $2g-2+2d+n$,

$$[\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{vir} \in A^{\mathbb{C}^*}_{2g-2+2d+n}(\overline{Q}_{g,n}(\mathbb{P}^1,d)),$$

see [15]. The $\mathbb{C}^*$-action lifts canonically[6] to the universal curve

$$\pi : U \to \overline{Q}_{g,n}(\mathbb{P}^1,d).$$

and to the universal subsheaf S_U. The higher direct image $R^1\pi_*(S_U)$ is a vector bundle of rank $g+d-1$ with top Chern class

$$\mathsf{e}(R^1\pi_*(S_U)) \in A^{g+d-1}_{\mathbb{C}^*}(\overline{Q}_{g,n}(\mathbb{P}^1,d)).$$

3.2. Relations. The relations of Theorem 3 will be obtained by studying the class

$$\Phi_{g,n,d} = \left(\mathsf{e}(R^1\pi_*(S_U)) \ \cup \ \prod_{i=1}^n \mathrm{ev}_i^*([\infty]) \right) \ \cap \ [\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{vir} \ .$$

on the moduli space of stable quotients. A dimension calculation shows

$$\Phi_{g,n,d} \in A^{\mathbb{C}^*}_{g-1+d}(\overline{Q}_{g,n}(\mathbb{P}^1,d)) \ .$$

Let $2g-2+n>0$, and consider the proper morphism

$$\nu : \overline{Q}_{g,n}(\mathbb{P}^1,d) \to \overline{M}_{g,n}.$$

[6] The particular $\mathbb{C}^*$-lift to S_U plays an important role in the calculation.

Let [1] denote the trivial bundle with $\mathbb{C}^*$-weight 1, and let $\mathsf{e}([1])$ be the $\mathbb{C}^*$-equivariant first Chern class. The class

$$\nu_* \left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right) \in A_{g-1+d-k}(\overline{M}_{g,n}) \tag{5}$$

certainly vanishes in the non-equivariant limit for $k > 0$.

We will calculate the push-forward (5) via $\mathbb{C}^*$-localization to find relations. Theorem 3 will be obtained after restriction to the moduli space

$$M^c_{g,n} \subset \overline{M}_{g,n}$$

of curves of compact type.

3.3. **$\mathbb{C}^*$-fixed loci.** Since $\Phi_{g,n,d}\ \mathsf{e}([1])^k$ is a $\mathbb{C}^*$-equivariant class, we may calculate the non-equivariant limit of the push-forward (5) by the virtual localization formula [8] as applied in [15]. We will be interested in the restriction of $\nu_* \left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)$ to $M^c_{g,n}$.

The first step is to determine the $\mathbb{C}^*$-fixed loci of $\overline{Q}_{g,n}(\mathbb{P}^1, d)$. The full list of $\mathbb{C}^*$-fixed loci is indexed by decorated graphs described in [15]. However, we will see most loci do not contribute to the localization calculation of

$$\nu_* \left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$$

by our specific choices of $\mathbb{C}^*$-lifts.

The *principal* component of the $\mathbb{C}^*$-fixed point locus

$$\overline{Q}_{g,n}(\mathbb{P}^1, d)^{\mathbb{C}^*} \subset \overline{Q}_{g,n}(\mathbb{P}^1, d)$$

is defined as follows. Consider

$$\overline{M}_{g,n|d}\ /\ S_d \tag{6}$$

where the symmetric group acts by permutation of the d nonstandard markings. Given an element

$$[C, p_1, \ldots, p_n, \widehat{p}_1, \ldots, \widehat{p}_d] \in \overline{M}_{g,n|d}\ ,$$

there is a canonically associated sequence

$$0 \to \mathcal{O}_C(-\sum_{j=1}^{d} \widehat{p}_j) \to \mathcal{O}_C \to Q \to 0. \tag{7}$$

By including $\mathcal{O}_C$ as the *second* factor of $\mathbb{C}^2 \otimes \mathcal{O}_C$, we obtain a stable quotient from (7). The corresponding S_d-invariant morphism

$$\iota : \overline{M}_{g,n|d} \to \overline{Q}_{g,n}(\mathbb{P}^1, d)$$

surjects onto the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$.

Let $F \subset \overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$ be a component of the $\mathbb{C}^*$-fixed locus, and let $[C, p_1, \ldots, p_n, q] \in F$ be a generic element of F:

(i) If an irreducible component of C lying over $0 \in \mathbb{P}^1$ has genus $h > 0$, then $\mathsf{e}(R^1\pi_*(S_U))$ yields the class λ_h by the contribution formulas of [15]. Since

$$\lambda_h|_{M^c_{h,*}} = 0$$

by [21], such loci F have vanishing contribution to

$$\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}\ .$$

(ii) If an irreducible component of C lying over $0 \in \mathbb{P}^1$ is incident to more than a single irreducible component dominating $\mathbb{P}^1$, then $\mathsf{e}(R^1\pi_*(S_U))$ vanishes on F by the 0 weight space in $\mathbb{C}^2$ associated to $0 \in \mathbb{P}^1$.

(iii) If $p_i \in C$ lies over $0 \in \mathbb{P}^1$, then $\text{ev}_i^*([\infty])$ vanishes on F.

By the vanishings (i-iii) together with the stability conditions, we conclude the principal locus (6) is the *only* $\mathbb{C}^*$-fixed component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ which contributes to $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$.

3.4. **Proof of Theorem 3.** The contribution of the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ to the push-forward $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$ is obtained from the localization formulas of [15] together with an analysis of $\mathsf{e}(R^1\pi_*(S_U))$.

For $[C, p_1, \ldots, p_n, \widehat{p}_1, \ldots, \widehat{p}_d] \in \overline{M}_{g,n|d}$, the long exact sequence associated to (7) yields

$$0 \to \mathbb{C} \otimes \mathcal{O}_C \to \mathcal{O}_{\widehat{p}_1+\ldots+\widehat{p}_d} \to H^1(C,S) \to H^1(C,\mathcal{O}_C) \to 0\ .$$

We conclude

$$\mathsf{e}(R^1\pi_*(S_U)) = \frac{\mathsf{e}(\mathbb{E}^* \otimes [1])\ \mathsf{e}(\mathbb{A}_d \otimes [1])}{\mathsf{e}([1])}$$

on the principal component. The evaluation

$$\prod_{i=1}^n \text{ev}_i^*([\infty]) \cdot \mathsf{e}([1])^k = \mathsf{e}([-1])^n \cdot \mathsf{e}([1])^k$$

is immediate.

By [15], the full localization contribution of the principal component is therefore

$$\frac{\mathsf{e}(\mathbb{E}^*\otimes[1])\ \mathsf{e}(\mathbb{A}_d\otimes[1])}{\mathsf{e}([1])}\mathsf{e}([-1])^n\mathsf{e}([1])^k\cdot\frac{\mathsf{e}(\mathbb{E}^*\otimes[-1])}{\mathsf{e}([-1])}\frac{1}{\mathsf{e}(\mathbb{B}_d\otimes[-1])}\ .$$

Using the Mumford relation $c(\mathbb{E})\cdot c(\mathbb{E}^*)=1$, we conclude, in the non-equivariant limit,

$$\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}=(-1)^{3g-3+d+n+k}\ \epsilon^c_*\left(c_{2g-2+k}(\mathbb{A}^*_d-\mathbb{B}_d)\right)\ .$$

Since the non-equivariant limit of $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$ vanishes, the proof of Theorem 3 is complete. □

3.5. **Evaluation rules.**

3.5.1. *Chern classes.* Associated to each nonstandard marking $\widehat{p}_j$, there is cotangent line bundle

$$\widehat{\mathbb{L}}_j\rightarrow M^c_{g,n|d}\ .$$

Let $\widehat{\psi}_j=c_1(\widehat{\mathbb{L}}_j)$ be the first Chern class.

The nonstandard markings are allowed by the stability conditions to be coincident. The *diagonal*

$$D_{ij}\subset M^c_{g,n|d}$$

is defined to be the locus where $\widehat{p}_i=\widehat{p}_j$. Let

$$S_{ij}=\{\ \ell\mid\ell\neq i,j\ \}\ \cup\ \{\star\}\ .$$

The basic isomorphism

$$D_{ij}\stackrel{\sim}{=}M^c_{g,n|S_{ij}}\ .$$

gives the diagonal geometry a recursive structure compatible with the cotangent line classes,

$$\widehat{\psi}_\ell|_{D_{ij}}=\widehat{\psi}_\ell\ ,$$

$$\widehat{\psi}_i|_{D_{ij}}=\widehat{\psi}_j|_{D_{ij}}=\widehat{\psi}_\star\ .$$

The intersection of distinct diagonals leads to smaller diagonals

$$D_{ij}\cap D_{jk}=D_{ijk}$$

in the obvious sense. The self-intersection is determined by

$$[D_{ij}]^2=-\widehat{\psi}_\star|_{D_{ij}}\ . \tag{8}$$

For convenience, let

$$\Delta_i = D_{1,i} + D_{2,i} + \ldots + D_{i-1,i}$$

with the convention $\Delta_1 = 0$.

The Chern classes of $\mathbb{A}_d$ and $\mathbb{B}_d$ are easily obtained inductively from the sequences

$$0 \to \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(-\sigma_d) \to \mathcal{O}_\sigma \to \mathcal{O}_{\sigma_d} \to 0,$$

$$0 \to \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(\sigma_1 + \ldots + \sigma_{d-1}) \to \mathcal{O}_\sigma(\sigma) \to \mathcal{O}_{\sigma_d}(\sigma) \to 0$$

on the universal curve U over $M^c_{g,n}$. We find

$$(9) \qquad c(\mathbb{A}_d) = \prod_{j=1}^{d}(1 - \Delta_i),$$

$$c(\mathbb{B}_d) = \prod_{j=1}^{d}(1 - \widehat{\psi}_j + \Delta_i) \ ,$$

see [15] for similar calculations.

3.5.2. *Push-forward.* From the Chern class formulas (9) and the diagonal intersection rules of Section 3.5.1,

$$\epsilon^c_*(c_{2g-2+k}(\mathbb{A}^*_d - \mathbb{B}_d)) \in A^*(M^c_{g,n})$$

is canonically a sum of push-forwards of the type

$$\epsilon^c_*\left(\widehat{\psi}_1^{j_1+1} \cdots \widehat{\psi}_s^{j_s+1}\right) \in A^*(M^c_{g,n})$$

along the forgetful maps

$$\epsilon^c_* : M^c_{g,n|s} \to M^c_{g,n}$$

associated to the various diagonals.

Lemma 1. $\epsilon_*\left(\widehat{\psi}_1^{j_1+1} \cdots \widehat{\psi}_s^{j_s+1}\right) = \kappa_{j_1} \cdots \kappa_{j_d}$ *in* $A^*(M^c_{g,n})$.

Proof. There are forgetful maps

$$\gamma_j : M^c_{g,n|s} \to M^c_{g,n|1} = M^c_{g,n+1},$$

associated to each nonstandard marking where the isomorphism on the right follows from the definition stability. Taking the fiber product over $M^c_{g,n}$ of all the γ_j yields a birational morphism

$$\gamma : M^c_{g,n|s} \to M^c_{g,n+1} \times_{M^c_{g,n}} M^c_{g,n+1} \times_{M^c_{g,n}} \cdots \times_{M^c_{g,n}} M^c_{g,n+1} \ .$$

The morphism γ is a small resolution. The exceptional loci are at most codimension 2 in $M^c_{g,n|s}$. Hence,

$$\mu^*(\psi_j) = \widehat{\psi}_j$$

for each nonstandard marking. We see

$$\mu_*\left(\widehat{\psi}_1^{j_1+1}\cdots\widehat{\psi}_s^{j_s+1}\right) = \psi_1^{j_1+1}\cdots\psi_s^{j_s+1}.$$

The result then follows after push-forward to $M^c_{g,n}$ by the definition of the κ classes. □

By Lemma 1, the relations of Theorem 3 are purely among the κ classes in $A^*(M^c_{g,n})$.

3.5.3. *Example.* The $d = 1$ case of Theorem 3 immediately yields the relations

$$\forall\ k > n, \quad \kappa_{2g-2+k} = 0 \ \in A^*(M^c_{g,n})$$

implied also by the vanishing results (1).

More interesting relations occur for $d = 2$. By the Chern class calculation (9),

$$c(\mathbb{A}_2^* - \mathbb{B}_2) = \frac{1+\Delta_1}{1-\widehat{\psi}_1+\Delta_1}\cdot\frac{1+\Delta_2}{1-\widehat{\psi}_2+\Delta_2}\ .$$

Using the series expansion

$$\frac{1+x}{1-y+x} = 1 + \sum_{r\geq 0} y(y-x)^r \tag{10}$$

and the diagonal intersection rules, we obtain

$$c(\mathbb{A}_2^* - \mathbb{B}_2) = \left(1+\sum_{r\geq 0}\widehat{\psi}_1^{r+1}\right)\cdot\left(1+\sum_{r\geq 0}\widehat{\psi}_2(\widehat{\psi}_2^r - (2^r-1)\widehat{\psi}_2^{r-1}\Delta_2)\right)$$

In genus 3 with $n = 0$, the $k = 1$ case of Theorem 3 concerns

$$c_5(\mathbb{A}_2^* - \mathbb{B}_2) = \sum_{r_1+r_2=5}\widehat{\psi}_1^{r_1}\widehat{\psi}_2^{r_2} - \sum_{r=1}^{4}(2^r-1)\widehat{\psi}_\star^4\Delta_2\ .$$

The push-forward is easily evaluated

$$\begin{aligned}\epsilon^c_*(c_5(\mathbb{A}_2^* - \mathbb{B}_2)) &= 4\kappa_3 + \kappa_1\kappa_2 + \kappa_2\kappa_1 + 4\kappa_3 - (1+3+7+15)\kappa_3\\ &= -18\kappa_3 + 2\kappa_1\kappa_2\ .\end{aligned}$$

We obtain the nontrivial relation

$$-18\kappa_3 + 2\kappa_1\kappa_2 = 0 \quad \in A^*(M_3^c) \ .$$

3.6. **Richer relations.** The proof of Theorem 3 naturally yields a richer set of relations among the κ classes. The universal curve

$$\pi : U \to M^c_{g,n|d}$$

carries the basic divisor classes

$$s = c_1(S_U^*), \qquad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U and the π-relative dualizing sheaf.

Proposition 2. *For all $a_i, b_i \geq 0$ and $k > n$,*

$$\epsilon_*\left(\prod_{i=1}^m \pi_*(s^{a_i}\omega^{b_i}) \cdot c_{2g-2+k}(\mathbb{A}_d^* - \mathbb{B}_d)\right) = 0 \ \in A^*(M^c_{g,n}).$$

The proof of Proposition 2 exactly follows the proof Theorem 3. We leave the details to the reader. By the rules of Section 3.5, the relations of Proposition 2 are also purely among the κ classes.

4. Evaluation of the relations

4.1. **Overview.** Our goal here is to explicitly evaluate the relations of Theorem 3 as polynomials in the κ classes. By examining the coefficients, we will obtain a proof of Theorem 1.

4.2. **Term counts.** Consider the total Chern class

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^d \frac{1+\Delta_i}{1-\widehat{\psi}_i + \Delta_i} \ . \tag{11}$$

After substituting

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i},$$

we may expand the right side of (11) fully. The resulting expression is a formal series in the $d + \binom{d}{2}$ variables[7]

$$\widehat{\psi}_1, \ldots, \widehat{\psi}_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \ .$$

[7]The sign on the diagonal variables is chosen because of the self-intersection formula (8).

Let M_r^d denote the coefficient in degree r,

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i, -D_{ij}).$$

Lemma 2. *After setting all the variables to 1,*

$$\sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)\ t^r \ = \ \frac{1}{1-dt}.$$

Proof. After setting the variables to 1 in (11), we find

$$c_t(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^{d} \frac{1-(i-1)t}{1-it},$$

which is a telescoping product. □

Lemma 2 may be viewed counting the number of terms in the expansion of (11),

$$M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1) \ = \ d^r \ .$$

The simple answer will play a crucial role in the analysis.

4.3. **Connected counts.** A monomial in the diagonal variables

$$D_{12}, D_{13}, \ldots, D_{d-1,d} \tag{12}$$

determines a set partition of $\{1, \ldots, d\}$ by the diagonal associations. For example, the monomial $3D_{12}^2 D_{1,3} D_{56}^3$ determines the set partition

$$\{1,2,3\} \ \cup \ \{4\} \ \cup \ \{5,6\}$$

in the $d = 6$ case. A monomial in the variables (12) is *connected* if the corresponding set partition consists of a single part with d elements.

A monomial in the variables

$$\widehat{\psi}_1, \ldots, \widehat{\psi}_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d}$$

is connected if the corresponding monomial in the diagonal variables obtained by setting all $\widehat{\psi}_i = 1$ is connected. Let C_r^d be the summand of $M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)$ consisting of the contributions of only the connected monomials.

Lemma 3. *We have*

$$\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_r^d \ t^r \frac{z^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!}\right) \ .$$

Proof. By a standard application of Wick, the connected and disconnected counts are related by exponentiation,

$$\exp(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_r^d\ t^r \frac{z^d}{d!}) = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)\ t^r \frac{z^d}{d!}\ .$$

The right side is then evaluated by Lemma 2 □

4.4. C_r^d **for** $r \leq d$**.** We may write the series inside the logarithm in Lemma 3 in the following form,

$$F(t,z) = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!} = \frac{1}{1 - tz\frac{d}{dz}}\ e^z\ .$$

Expanding the exponential of the differential operator by order in t yields,

$$F(t,z) = e^z + tze^z + t^2(z^2+z)e^z + \\ t^3(z^3+3z^2+z)e^z + t^4(z^4+6z^3+7z^2+z)e^z + \dots\ .$$

We have proven the following result.

Lemma 4. $F(t,z) = e^z \cdot \sum_{r=0}^{\infty} t^r p_r(z)$ *where*

$$p_r(z) = \sum_{s=0}^{r} c_{r,s} z^{r-s}$$

is a degree r *polynomial.*

By Lemma 4 and the coefficient evaluation $c_{r,0} = 1$, we see

$$\log(F(t,z)) = z + \log(\frac{1}{1-tz}) + \dots$$

where the dots stand for terms of the form $t^r z^d$ with $r > d$. We obtain the following result.

Proposition 3. *The only nonvanishing* C_r^d *for* $r \leq d$ *are* $C_0^1 = 1$ *and*

$$\sum_{r=1}^{\infty} C_r^r\ t^r \frac{z^r}{r!} = -\log(1-tz).$$

4.5. **Evaluation.** Let g and n be fixed. We are interested in calculating

$$R_{g,n}(t,z) = \sum_{d=1}^{\infty} \epsilon^c_* \left(c_t(\mathbb{A}^*_d - \mathbb{B}_d)\right) \frac{z^d}{d!} \ .$$

By the straightforward application of the evaluation rules of Section 3.5, we find

$$R_{g,n}(t,z) = \exp\left(\sum_{d=1}^{\infty} \sum_{r\geq d}^{\infty} (-1)^{d-1} C^d_r \kappa_{r-d}\ t^r z^d \right) \ . \tag{13}$$

We rewrite (13) after separating out the $r=d$ terms using Proposition 3 and the evaluation $\kappa_0 = 2g-2+n$,

$$\begin{aligned} R_{g,n}(t,-z) &= \exp\left(- \sum_{d=1}^{\infty} \sum_{r\geq d}^{\infty} C^d_r \kappa_{r-d}\ t^r z^d \right) \\ &= (1-tz)^{2g-2+n} \exp\left(- \sum_{d=1}^{\infty} \sum_{r>d}^{\infty} C^d_r \kappa_{r-d}\ t^r z^d \right) \ . \end{aligned}$$

The $t^r z^d$ coefficient of $R_{g,n}$ is a valid relation in $A^*(M^c_{g,n})$ if

$$r > 2g-2+n.$$

The above formula, taken together with Lemma 3, provides a very effective approach to the relations of Theorem 3.

4.6. **Proof of Theorem 1.** The generating series for the coefficients of the singleton $\kappa_{\ell>0}$ in the $t^{d+\ell} z^d$ terms of $R_{g,n}(t,-z)$ is

$$R^{\ell}_{g,n}(t,-z) \ = \ -(1-tz)^{2g-2+n} \sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d t^{\ell} \ . \tag{14}$$

In order to analyze the right side of (14), we will use Lemma 4. For $\ell \geq 0$, let

$$G_{\ell}(t,z) = \sum_{d=1}^{\infty} c_{d+\ell,\ell}\ (tz)^d \ . \tag{15}$$

By Lemma 4 and Proposition 3,

$$\begin{aligned}\sum_{\ell\geq 0}\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d t^\ell &= \log\left(\sum_{\ell\geq 0} G_\ell(t,z)\ t^\ell\right)\\ &= \log\left(\frac{1}{1-tz}+\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right)\\ &= \log\left(\frac{1}{1-tz}\right)+\log\left(1+(1-zt)\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right)\ .\end{aligned}$$

So for $\ell > 0$,

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \text{Coeff}_\ell\left(\log\left(1+(1-zt)\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right)\right)\ . \tag{16}$$

Here, Coeff_ℓ extracts all the terms of the form $t^{*+\ell}z^*$ and divides by t^ℓ.

The behavior of the coefficients $c_{r,s}$ is easily determined by induction on s.

Lemma 5. *For $r \geq s$, $c_{r,s} = f_s(r)$ where $f_s(r)$ is a polynomial of degree $2s$ with leading term*

$$f_s(r) = \frac{1}{2^s s!} r^{2s} + \dots\ .$$

For example, $f_0(r) = 1$ and

$$f_1(r) = \frac{1}{2}r^2 + \frac{1}{2}r\ .$$

We leave the elementary proof of Lemma 5 to the reader

From (15) and Lemma 5, we conclude for $\ell > 0$,

$$G_\ell(t,z) = \frac{1}{2^\ell \ell!}\frac{(2\ell)!}{(1-tz)^{2\ell+1}} + \sum_{i=0}^{2\ell}\frac{\tilde{c}_{i,\ell}}{(1-tz)^i}$$

for $\tilde{c}_{i,\ell} \in \mathbb{Q}$. Then by (16),

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \text{Coeff}_\ell\left(\log\left(1+\sum_{\ell\geq 1}\frac{(2\ell)!!}{(1-tz)^{2\ell}}t^\ell\right)\right)\dots, \tag{17}$$

where the dots stand for finitely many terms of the form $(1-tz)^{-j}$ where $j < 2\ell$. By Proposition 4 proven in Section 4.7 below,

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \frac{\alpha_\ell}{(1-zt)^{2\ell}} + \dots \tag{18}$$

with $\alpha_\ell \neq 0$.

We now return to the coefficients of the singleton $\kappa_{\ell>0}$ in the $t^{d+\ell}z^d$ terms of $R_{g,n}(t,-z)$. By (14),

$$R^\ell_{g,n}(t,-z) \;=\; -\alpha_\ell(1-tz)^{2g-2+n-2\ell}t^\ell + \dots \tag{19}$$

where the dots stand for finitely many terms of the form $(1-tz)^j t^\ell$ where $j > 2g-2+n-2\ell$. If

$$2g-2+n-2\ell < 0, \tag{20}$$

then the coefficient of $(tz)^d t^\ell$ in $R^\ell_{g,n}$ will be nonzero for all large d. Once

$$d+\ell > 2g-2+n,$$

the corresponding κ relation is valid by Theorem 3. If (20) is satisfied, κ_ℓ lies in the subring of $\kappa^*(M^c_{g,n})$ generated by $\kappa_1,\dots,\kappa_{\ell-1}$. □

4.7. **Series analysis.** Define the double factorial by

$$(2\ell)!! = \frac{(2\ell)!}{2^\ell \ell!} = (2\ell-1)\cdot(2\ell-3)\cdots 1$$

and let

$$\phi(x) = 1 + \sum_{\ell\geq 1}(2\ell)!!\ x^\ell = 1 + x + 3x^2 + 15x^3 + \dots$$

be the generating series. Define $\alpha_\ell \in \mathbb{Q}$ for $\ell > 0$ by

$$\log(\phi) = \sum_{\ell\geq 1}\alpha_\ell x^\ell\ .$$

Series expansion yields the first terms

$$\log(\phi(x)) = x + \frac{5}{2}x^2 + \frac{37}{3}x^3 + \frac{353}{4}x^4 + \dots\ .$$

To complete the proof of Theorem 3, we must prove the following result.

Proposition 4. *$\alpha_\ell \neq 0$ for all $\ell > 0$.*

Let $x = y^2$. Then $\phi(x(y))$ satisfies the differential equation

$$y^2\frac{d}{dy}(y\phi) = \phi - 1\ .$$

Equivalently,

$$y^3\frac{d}{dy}\log(\phi) + y^2 - 1 = -\frac{1}{\phi}$$

Changing variables back to x, we find

$$2x^2\frac{d}{dx}\log(\phi)+x-1=-\frac{1}{\phi} \tag{21}$$

Let β_ℓ denote the coefficients of the inverse series,

$$\begin{aligned}\phi(x)^{-1} &= 1+\sum_{\ell\geq 0}\beta_\ell x^\ell\\ &= 1-x-2\alpha_1x^2-4\alpha_2x^3-6\alpha_3x^4-\dots\ ,\end{aligned}$$

where the second equality is obtained from (21).

Lemma 6. *$\beta_\ell \neq 0$ for all $\ell > 0$.*

Proof. Series expansion yields

$$\phi(x)^{-1}=1-x-2x^2-10x^3-74x^4-\dots\ .$$

We will establish the following two properties for $\ell > 0$ by joint induction:

(i) $\beta_\ell < 0$,
(ii) $|\beta_\ell| \leq (2\ell)!!$.

By inspection, the conditions hold in the base case $\ell = 1$.

Let $\ell > 1$ and assume conditions (i)-(ii) hold for all $\ell' < \ell$. Since $\phi\cdot\phi^{-1} = 1$,

$$\begin{aligned}(2\ell)!!+\beta_\ell &= -\sum_{k=1}^{\ell-1}(2k)!!\cdot\beta_{\ell-k}\\ &\leq \sum_{k=1}^{\ell-1}(2k)!!\cdot(2\ell-2k)!!,\end{aligned} \tag{22}$$

where the second line uses (ii). For $\frac{\ell}{2}\leq k\leq \ell-1$,

$$\begin{aligned}(2k)!!\cdot(2\ell-2k)!! &= (2\ell)!!\frac{1}{2\ell-1}\ \frac{3}{2\ell-3}\ \cdots\frac{2\ell-2k-1}{2k+1}\\ &\leq (2\ell)!!\frac{1}{2\ell-1}.\end{aligned}$$

By putting the two above inequalities together, we obtain

$$(2\ell)!!+\beta_\ell\leq(\ell-1)\cdot(2\ell)!!\frac{1}{2\ell-1}<(2\ell)!!\ .$$

Hence, $\beta_\ell < 0$. Since also

$$(2\ell)!!+\beta_\ell>0$$

by the first equality of (22) and (i), we see $|\beta_\ell| < (2\ell)!!$. □

Lemma 6 and the relation

$$-2\ell\alpha_\ell = \beta_{\ell+1}$$

together complete the proof of Proposition 4.

5. Independence

5.1. **Tautological classes.** The moduli space $M^c_{g,n}$ has an algebraic stratification by topological type. The push-forward of the κ and ψ classes from the strata generate the *tautological ring*

$$R^*(M^c_{g,n}) \subset A^*(M^c_{g,n}) ,$$

see [9]. Following the Gorenstein philosophy explained in [4], we will study the independence of

$$\kappa_1, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor} \in R^*(M^c_{g,n})$$

through degree $g-1+\lfloor \frac{n}{2} \rfloor$ by pairing with strata classes.

5.2. **Case $n = 1$.** We first prove Theorem 2 for $M^c_{g,1}$. By stability, $g \geq 1$. To each partition $\mathbf{p} \in P(d)$, we associate a κ monomial,

$$\kappa_{\mathbf{p}} = \kappa_{p_1}\kappa_{p_2}\cdots\kappa_{p_\ell} \ \in R^*(M^c_{g,1}) \ .$$

Theorem 2 is equivalent to the independence of the $|P(g-1)|$ monomials

$$\{\ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g-1)\ \}$$

in $R^*(M^c_{g,1})$.

To each partition $\mathbf{p} \in P(g-1)$ of length ℓ, we associate a codimension $g-1$ stratum $S_{\mathbf{p}} \subset M^c_{g,1}$ by the following construction. Start with a chain of elliptic curves E_i of length $\ell+1$ with the marking on the first,

$$E_1^* - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ . \tag{23}$$

The asterisk indicates the marking. Since $\ell \leq g-1$, such a chain does not exceed genus g. Next, we add elliptic tails[8] to the first ℓ elliptic components. To the curve E_i, we add $p_i - 1$ elliptic tails. Let C be the resulting curve. The total genus of C is

$$\ell + 1 + (g-1) - \ell = g \ .$$

[8]An elliptic tail is an unmarked elliptic curve meeting the rest of the curve in exactly 1 point.

The number of nodes of C is

$$\ell + (g-1) - \ell = g-1 \ .$$

Hence, C determines a codimension $g-1$ stratum $S_{\mathbf{p}} \subset M^c_{g,1}$.

The moduli in $S_{\mathbf{p}}$ is found mainly on the first ℓ components of the original chain (23). Each such E_i has p_i+1 moduli parameters. All other components (including $E_{\ell+1}$) are elliptic tails with 1 moduli parameter each.

The λ_g-evaluation on $R^*(M^c_{g,1})$ discussed in Section 1.6 yields a pairing on partitions $\mathbf{p},\mathbf{q} \in P(g-1)$,

$$\mu_g(\mathbf{p},\mathbf{q}) = \int_{\overline{M}_{g,1}} \kappa_{\mathbf{p}} \cdot [S_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Lemma 7. *For all $g \geq 1$, the matrix μ_g is nonsingular.*

Proof. To evaluate the pairing, we first restrict λ_g to $S_{\mathbf{q}}$ by distributing a λ_1 to each elliptic component. To pair $\kappa_{\mathbf{p}}$ with the class $[S_{\mathbf{q}}] \cdot \lambda_g$, we must distribute the factors κ_{p_i} to the components E_j of $S_{\mathbf{q}}$ in all possible ways. By the dimension constraints imposed by the moduli parameters of the components of $S_{\mathbf{q}}$, we immediately conclude

$$\mu_g(\mathbf{p},\mathbf{q}) = 0$$

unless $\ell(\mathbf{p}) \geq \ell(\mathbf{q})$. Moreover, if $\ell(\mathbf{p}) = \ell(\mathbf{q})$, the pairing vanishes unless $\mathbf{p}=\mathbf{q}$.

We have already shown μ_g to be upper-triangular with respect to the length partial ordering on $P(g-1)$. To establish the nonsingularity of μ_g, we must show the diagonal entries $\mu_g(\mathbf{p},\mathbf{p})$ do not vanish. Since $\mu_g(\mathbf{p},\mathbf{p})$ is a product of factors of the form

$$\int_{\overline{M}_{1,p+1}} \kappa_p \lambda_1 = \frac{1}{24},$$

the required nonvanishing holds. □

By Lemma 7, the κ monomials of degree $g-1$ are independent. The proof of Theorem 2 for $M^c_{g,1}$ is complete.

5.3. **Case** $n=2$**.** We now consider Theorem 2 for $M^c_{g,2}$. By stability, $g \geq 1$. We must prove the independence of the $|P(g)|$ monomials

$$\{ \ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g) \ \}$$

in $R^*(M^c_{g,2})$.

To each partition $\mathbf{p} \in P(g)$ of length ℓ, we associate a codimension $g-1$ stratum $T_{\mathbf{p}} \subset M^c_{g,1}$ by the following construction. Start with a chain of elliptic curves E_i of length ℓ with the markings on the first and last,

$$E_1^* - E_2 - E_3 - \ldots - E_\ell^* \ . \tag{24}$$

Since $\ell \leq g$, such a chain does not exceed genus g. Next, we add elliptic tails to the ℓ elliptic components of (24). To the curve E_i, we add $p_i - 1$ elliptic tails. Let C be the resulting curve. The total genus of C is

$$\ell + g - \ell = g$$

The number of nodes of C is

$$\ell - 1 + g - \ell = g - 1 \ .$$

Hence, C determines a codimension $g-1$ stratum $T_{\mathbf{p}} \subset M^c_{g,1}$.

As before, the λ_g-evaluation on $R^*(M^c_{g,2})$ yields a pairing on partitions $\mathbf{p}, \mathbf{q} \in P(g)$,

$$\nu_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_{g,2}} \kappa_{\mathbf{p}} \cdot [T_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Lemma 8. *For all $g \geq 1$, the matrix ν_g is nonsingular.*

The proof is identical to the proof of Lemma 7. We leave the details to the reader. The proof of Theorem 2 for $M^c_{g,2}$ is complete.

5.4. **Proof of Theorem 2.** To complete the proof of Theorem 2, we must consider the case $n \geq 3$ and prove the independence of the monomials

$$\{ \ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g - 1 + \lfloor \frac{n}{2} \rfloor) \ \}$$

in $R^*(M^c_{g,n})$.

We will relate the question to the established cases with 1 and 2 markings. Let

$$\widehat{g} = g + \lfloor \frac{n-1}{2} \rfloor, \quad \widehat{n} = n - 2\lfloor \frac{n-1}{2} \rfloor \ .$$

If n is odd, then $\widehat{n} = 1$. If n is even, $\widehat{n} = 2$. Note

$$\widehat{g} - 1 + \lfloor \frac{\widehat{n}}{2} \rfloor = g - 1 + \lfloor \frac{n}{2} \rfloor \ .$$

To start, assume $\widehat{n} = 1$. We have constructed strata classes in $M^c_{\widehat{g},1}$ which show the independence of the monomials

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(\widehat{g} - 1) \}$$

in $R^*(M^c_{\widehat{g},1})$. For each $\mathbf{q} \in P(\widehat{g} - 1)$, the stratum

$$S_{\mathbf{q}} \subset M^c_{\widehat{g},1}$$

consists of a configuration of $\widehat{g}$ elliptic curves. We construct a corresponding stratum

$$S'_{\mathbf{q}} \subset M^c_{g,n}$$

by the following method. Choose any subset[9] of $\lfloor \frac{n-1}{2} \rfloor$ elliptic components of $S_{\mathbf{q}}$. For each elliptic component E selected, replace E with a rational component carrying 2 additional markings.[10] The construction trades $\lfloor \frac{n-1}{2} \rfloor$ genus for $2\lfloor \frac{n-1}{2} \rfloor$ markings.

Theorem 2 is implied by the nonsingularity of the λ_g-pairing between the κ monomials of degree $\widehat{g} - 1$ and the strata classes $[S'_{\mathbf{q}}]$. The proof of the nonsingularity is identical to the proof of Lemma 7.

The $\widehat{n} = 2$ case proceeds by exactly the same method. Again, elliptic components of the strata

$$T_{\mathbf{q}} \subset M^c_{\widehat{g},2}$$

are traded for rational components with 2 additional markings. Theorem 2 is deduced by nonsingularity of the λ_g-pairing. □

5.5. **Proof of Proposition 1.** Consider M^c_g for $g \geq 2$. Let

$$P^*(g-1) \subset P(g-1) \ \setminus \ \{(1, \ldots, 1)\}$$

be the subset excluding the longest partition. We will first prove the independence of the monomials

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P^*(g-1) \}$$

in $R^*(M^c_g)$. The result shows there can be at most a single κ relation in degree $g - 1$.

To each partition $\mathbf{p} \in P^*(g-1)$ of length $\ell \leq g - 2$, we associate a codimension $g - 2$ stratum $U_{\mathbf{p}} \subset M^c_g$ by the following construction. Start with a chain of curves of length $\ell + 1$,

$$X - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ ,$$

[9]The particular choice of subset is not important.

[10]The particular markings chosen are not important.

where X has genus 2 and all the E_i are elliptic curves. Since $\ell \leq g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the first ℓ components. Since p_1 is the greatest part of $\mathbf{p}$, $p_1 \geq 2$. To the curve X, we add $p_1 - 2$ elliptic tails. To the curve E_i, we add $p_i - 1$ elliptic tails for $2 \leq i \leq \ell$. Let C be the resulting curve. The total genus of C is

$$2 + \ell + (g-1) - \ell - 1 = g$$

The number of nodes of C is

$$\ell + (g-1) - \ell - 1 = g - 2 \ .$$

Hence, C determines a codimension $g-2$ stratum $U_{\mathbf{p}} \subset M_g^c$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P^*(g-1)$,

$$\omega_g(\mathbf{p},\mathbf{q}) = \int_{\overline{M}_g} \kappa_{\mathbf{p}} \cdot [U_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

The argument of Lemma 7 yields the following result.

Lemma 9. *For all $g \geq 2$, the matrix ω_g is nonsingular.*

The independence of the κ monomials in degrees at most $g-2$ is easier and proven in a similar way. To each partition $\mathbf{p} \in P(g-2)$ of length ℓ, we associate a codimension $g-1$ stratum $U'_{\mathbf{p}} \subset M_g^c$ by the following construction. Start with a chain of elliptic curves of length $\ell + 2$,

$$E_0 - E_1 - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ .$$

Since $\ell \leq g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the components. To E_i, for $1 \leq i \leq \ell$, we add $p_i - 1$ elliptic tails. To E_0 and $E_{\ell+1}$, we add nothing. Let C be the resulting curve. The total genus of C is

$$\ell + 2 + (g-2) - \ell = g$$

The number of nodes of C is

$$\ell + 1 + (g-2) - \ell = g - 1 \ .$$

Hence, C determines a codimension $g-1$ stratum $U'_{\mathbf{p}} \subset M_g^c$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P(g-2)$,

$$\omega'_g(\mathbf{p},\mathbf{q}) = \int_{\overline{M}_g} \kappa_{\mathbf{p}} \cdot [U'_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Again, the argument of Lemma 7 yields the required result.

Lemma 10. *For all $g \geq 2$, the matrix ω'_g is nonsingular.*

Together, Lemmas 9 and 10 complete the proof of Proposition 1. □

6. Universality of genus 0

6.1. **Genus 5.** Do the relations of Theorem 3 generate the entire ideal of relations in $\kappa^*(M_g^c)$? Since Proposition 2 contains the relations of Theorem 3, we may ask the same question of the richer system. The answer to these questions is no. The first example occurs in $\kappa^6(M_5^c)$.

There are 11 κ monomials of degree 6. By the evaluation rules of Section 3.5, the κ relations in codimension 6 generated by Proposition 2 are the *same* for all the rings

$$\kappa^*(M_5^c),\ \kappa^*(M_{4,2}^c),\ \kappa^*(M_{3,4}^c),\ \kappa^*(M_{2,6}^c),\ \kappa^*(M_{1,8}^c),\ \kappa^*(M_{0,10}^c)\ .$$

On $M_{0,10}^c$, there are 4 basic types[11] of boundary divisors determined by the point splittings

$$8+2,\quad 7+3,\quad 6+4,\quad 5+5.$$

The pairings of these divisors with the κ monomials

$$\kappa_6,\ \kappa_5\kappa_1,\ \kappa_4\kappa_2,\ \kappa_3^2$$

on $M_{0,10}^c$ are easily seen to determine a nonsingular 4×4 matrix. Hence, the number of independent κ relations in $\kappa^6(M_{0,10}^c)$ is at most 7. In fact, Proposition 2 generates 7 independent relations.

The number of divisor classes in $R^*(M_5^c)$ is 3 given by κ_1 and the 2 boundary divisors with genus splittings $4+1$ and $3+2$. The Gorenstein conjecture for M_5^c predicts $R^6(M_5^c)$ to have rank 3. The rank of $R^6(M_5^c)$ can be proven to be 3 via an application[12] of Getzler's relation [6, 18]. Therefore, there *must* be at least 8 relations among the κ monomials of degree 6 in M_5^c. We have proven the method of Proposition 2 does not yield all the κ relations in $R^6(M_5^c)$.

[11]There are several actual divisors of each type depending on the marking distribution. We select one of each type.

[12]We thank C. Faber for pointing out the argument.

6.2. **Genus 0.** In [20], a set of relations obtained from the virtual geometry of the moduli space of stable maps is proven to generate all the κ relations in the rings $\kappa^*(M^c_{0,n})$.

Question 3. *Does Proposition 2 generate all the κ relations in the rings $\kappa^*(M^c_{0,n})$?*

The answer to Question 3 is affirmative at least for $n \leq 12$. We list below the Betti polynomials $B_n(t)$ of $\kappa^*(M^c_{0,n})$ for low n.

$$\begin{aligned}
B_3 &= 1 \\
B_4 &= 1+t \\
B_5 &= 1+t+t^2 \\
B_6 &= 1+t+2t^2+t^3 \\
B_7 &= 1+t+2t^2+2t^3+t^4 \\
B_8 &= 1+t+2t^2+3t^3+3t^4+t^5 \\
B_9 &= 1+t+2t^2+3t^3+4t^4+3t^5+t^6 \\
B_{10} &= 1+t+2t^2+3t^3+5t^4+5t^5+4t^6+t^7 \\
B_{11} &= 1+t+2t^2+3t^3+5t^4+6t^5+7t^6+4t^7+t^8 \\
B_{12} &= 1+t+2t^2+3t^3+5t^4+7t^5+9t^6+8t^7+5t^8+t^9
\end{aligned}$$

From the table of Betti numbers, a formula is easily guessed. Let

$$P(d,k) \subset P(d)$$

be the subset of partitions of d of length at most k, and let $|P(d,k)|$ be the order. We see

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) = |P(d, n-d-2)|$$

holds in all the above cases.

Theorem 5. *A $\mathbb{Q}$-basis of $\kappa^d(M^c_{0,n})$ is given by*

$$\{\kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, n-2-d)\ \}\ .$$

Proof. In order for $P(d, n-d-2)$ to be nonempty, we must have

$$d \leq n-3.$$

We first prove the independence of the κ monomials associated to $P(d, n-d-2)$ by intersection with strata classes in $R^{n-3-d}(M^c_{0,n})$. To each partition

$$\mathbf{p} \in P(d, n-d-2),$$

we associate a codimension $n-3-d$ stratum $V_{\mathbf{p}} \subset M^c_{0,n}$ by the following construction. We write the parts of $\mathbf{p}$ as

$$(p_1, \ldots, p_\ell, p_{\ell+1}, \ldots, p_{n-d-2})$$

where $p_{\ell+\delta} = 0$ for $\delta > 0$. Start with a chain of rational curves of length $n-d-2$,

$$R_1 - R_2 - R_3 - \ldots - R_{n-d-2} \ .$$

Next, we add markings[13] to the components:

- $p_1 + 2$ markings to R_1,
- $p_i + 1$ markings to R_i for $2 \leq i \leq n-d-3$,
- $p_{n-d-2} + 2$ markings to R_{n-d-2},

Let C be the resulting curve. The total number of markings of C is

$$2 + d + n - d - 2 = n \ .$$

The number of nodes of C is $n-3-d$. Hence, C determines a codimension $n-3-d$ stratum $V_{\mathbf{p}} \subset M^c_{0,n}$.

A simple analysis following the strategy of the proof of Lemma 7 shows the paring on $P(d, n-d-2)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot [V]_{\mathbf{q}}$$

is upper-triangular and nonsingular. We conclude the κ monomials associated to $P(d, n-d-2)$ are linearly independent.

The strata of $M^c_{0,n}$ are indexed by marked trees. Given a marked tree Γ with $n-d-2$ vertices, the associated stratum

$$S_\Gamma \subset M^c_{0,n}$$

parameterizes curves C with marked dual graph Γ. In other words, C is a tree of marked rational components

$$R_1, \ldots, R_{n-2-d} \ .$$

[13] The particular markings chosen are not important.

To S_Γ, we associate a partition $\mathbf{q}(\Gamma) \in P(d, n-d-2)$ by the following construction. Let $\mathsf{m}(R_i)$ and $\mathsf{n}(R_i)$ denote the numbers of markings and nodes incident to R_i. Let

$$q_i = \mathsf{m}(R_i) + \mathsf{n}(R_i) - 3.$$

By stability, $q_i \geq 0$. After reordering by size,

$$\mathbf{q}(\Gamma) = (q_1, \ldots, q_{n-d-2}) \in P(d, n-d-2)\ .$$

Let $\mathbf{p} \in P(d)$. The intersection of $\kappa_{\mathbf{p}}$ with a stratum class S is obtained by distributing the factors κ_{p_i} to the components of S. We conclude

$$\int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot S_\Gamma = \int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot V_{\mathbf{q(\Gamma)}} \tag{25}$$

for all $\mathbf{p} \in P(d)$.

By Poincaré duality[14], the dimension of $\kappa^d(M^c_{0,n})$ is the rank of the intersection pairing

$$\kappa^d(M^c_{0,n}) \times A^{n-3-d}(M^c_{0,n}) \to \mathbb{Q}.$$

The classes of strata generate $A^{n-3-d}(M^c_{0,n})$. Moreover, only the special strata $V_{\mathbf{q}}$ need by considered by (25). So,

$$\dim_{\mathbb{Q}}\ \kappa^d(M^c_{0,n}) \leq |P(d, n-d-2)|\ .$$

The independence property together with the above dimension estimate yields the basis result. □

[14]For $M^c_{0,n}$, singular cohomology and Chow agree.

6.3. **Proof of Theorem 6.**

6.3.1. *Bound.* By Theorem 4 (proven in [20]), we have a surjection

$$\kappa^d(M^c_{0,2g+n}) \stackrel{\iota_{g,n}}{\to} \kappa^d(M^c_{g,n}) \to 0.$$

By Theorem 5, to prove $\iota_{g,n}$ is an isomorphism, we need only establish

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,n}) \geq |P(d, 2g-2+n-d)|$$

for $n > 0$. We will obtain the bound by refining the argument for Theorem 2.

6.3.2. *Dual graph types.* A dual graph of type $A(g_1, \ldots, g_r)$ with $g_i \geq 1$ is a chain of r vertices of genera $g_1, \ldots, g_r$ with 2 markings on the ends. The corresponding curves are of the form:

$$C^*_{g_1} - C_{g_2} - \ldots - C^*_{g_r} \ .$$

If $r = 1$, the unique vertex carries both markings.

A dual graph of type $B(g_1, \ldots, g_r | h_1, \ldots, h_{r-1})$ with $g_i, h_j \geq 1$ is comb of $2r-1$ vertices with 1 marking. The corresponding curves are of the form:

$$\begin{array}{ccccccccc} C^*_{g_1} & - & C_{g_2} & - & \ldots & - & C_{g_{r-1}} & - & C_{g_r} \\ | & & | & & & & | & & \\ C_{h_1} & & C_{h_2} & & \ldots & & C_{h_{r-1}} \ . & & \end{array}$$

There are $r-1$ vertices of valence 3 and r vertices of valence 1. The marking is included in the valence count.

6.3.3. *Case* $n = 1$. Let $\mathbf{p} \in P(d)$ be a partition of length $\ell = a + b$ with parts[15]

$$(p_1, \ldots, p_a, p'_1, \ldots, p'_b),$$

where the p_i are odd and the p'_j are even. We see

$$d + \ell = b \mod 2 \ .$$

[15]All parts of $\mathbf{p}$ here are positive.

If $d+\ell$ is odd, then $b = 2r-1$ for $r>0$. Let $\Gamma_{\mathbf{p}}$ be the dual graph obtained by the following construction:

$$\begin{array}{rcl} \Gamma_{\mathbf{p}} & = & A\left(\frac{p_1+1}{2}, \ldots, \frac{p_a+1}{2}\right) \\ & & | \\ & & B\left(\frac{p'_1}{2}, \ldots, \frac{p'_{r-1}}{2}, \frac{p'_r}{2}+1 \mid \frac{p'_{r+1}}{2}+1, \ldots, \frac{p'_{2r-1}}{2}+1\right) \ , \end{array}$$

where the graphs are attached at the first marking of A and the unique marking of B. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the second marking of A). The genus of $\Gamma_{\mathbf{p}}$ is easily calculated,

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2r - 1 = d + \ell \ . \tag{26}$$

If $a = 0$, then $\Gamma_{\mathbf{p}}$ consists just of B, but the genus and marking results are the same.

The dual graph $\Gamma_{\mathbf{p}}$ determines a stratum in $M^c_{g(\Gamma_{\mathbf{p}}),1}$ which is a product of the moduli spaces,

$$\prod_{v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})} M^c_{g(v),\mathrm{val}(v)} \ \rightarrow \ M^c_{g(\Gamma_{\mathbf{p}}),1} \ .$$

The socle dimensions of $M^c_{g(v),\mathrm{val} v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is even, then b must be even. If $b>0$, then

$$b = 2r - 1 + 1$$

for $r > 0$. Let

$$\begin{array}{rcl} \Gamma_{\mathbf{p}} & = & A\left(\frac{p_1+1}{2}, \ldots, \frac{p_a+1}{2}\right) - C^*_{\frac{p'_{2r}}{2}} - E \\ & & | \\ & & B\left(\frac{p'_1}{2}, \ldots, \frac{p'_{r-1}}{2}, \frac{p'_r}{2}+1 \mid \frac{p'_{r+1}}{2}+1, \ldots, \frac{p'_{2r-1}}{2}+1\right) \ . \end{array}$$

where the graphs A and B are attached at the markings. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (on $C^*_{\frac{p'_{2r}}{2}}$) and an elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2r + 2 - 1 = d + \ell + 1 \ . \tag{27}$$

If $a = 0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

If $d + \ell$ is even and $b = 0$, let

$$\Gamma_{\mathbf{p}} \;=\; A\left(\frac{p_1+1}{2}, \ldots, \frac{p_a+1}{2}\right) - E \ .$$

The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the first marking of A) and ends in the elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2 - 1 = d + \ell + 1 \ . \tag{28}$$

The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

We now turn to the proof of Theorem 6 in the $n = 1$ case. We will prove

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,1}) \geq |P(d, 2g-1-d)| \tag{29}$$

by intersecting κ monomials with tautological classes.

Let $\mathbf{p} \in P(d, 2g-1-d)$ be a partition of length ℓ. Let $\Gamma_{\mathbf{p}}$ be the dual graph of genus $g(\Gamma_{\mathbf{p}})$ obtained by the above constructions. Since

$$2g - 1 \geq d + \ell \ ,$$

equations (26)-(28) imply

$$g - g(\Gamma_{\mathbf{p}}) = \delta \geq 0 \ .$$

We associate to $\mathbf{p}$ a class $w_{\mathbf{p}} \in R^{2g-2-d}(M^c_{g,1})$ by the following construction. Let $v^* \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ be the vertex which carries the marking. Increase the genus of v^* by δ. The resulting graph determines a stratum

$$W_{\mathbf{p}} \subset M^c_{g,1}$$

of codimension $2g(\Gamma_{\mathbf{p}}) - 2 - d$. Let

$$w_{\mathbf{p}} = \psi_1^{2\delta} \cdot [W_{\mathbf{p}}] \in R^{2g-2-d}(M^c_{g,1}) \ .$$

The pairing on $P(d, 2g-1-d)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{\overline{M}_{g,1}} \kappa_{\mathbf{p}} \cdot w_{\mathbf{q}} \tag{30}$$

is upper-triangular. The diagonal elements are nonvanishing because

$$\int_{\overline{M}_h} \kappa_{2h-3}\lambda_h = \frac{2^{2h-1}-1}{2^{2h-1}} \frac{|B_{2h}|}{(2h)!} \neq 0 \ ,$$

$$\int_{\overline{M}_{h,1}} \psi_1^k \kappa_{2h-2-k}\lambda_h = \binom{2h-1}{k} \int_{\overline{M}_h} \kappa_{2h-3}\lambda_h \ \neq 0$$

by [5]. Here, B_{2h} is the Bernoulli number. Hence, the pairing (30) is nonsingular and the bound (29) is established.

6.3.4. *Case* $n = 2$. We will need an additional dual graph type. A dual graph of type $\widetilde{B}(g_1,\ldots,g_r|h_1,\ldots,h_{r-1})$ with $g_i, h_j \geq 1$ is comb of $2r-1$ vertices with 3 markings. The corresponding curves are of the form:

$$\begin{array}{ccccccccc} C^*_{g_1} & - & C_{g_2} & - & \ldots & - & C_{g_{r-1}} & - & C^{**}_{g_r} \\ | & & | & & & & | & & \\ C_{h_1} & & C_{h_2} & & \ldots & & C_{h_{r-1}}\,. & & \end{array}$$

There are r vertices of valence 3 and $r-1$ vertices of valence 1. The marking is included in the valence count.

As before, let $\mathbf{p} \in P(d)$ be a partition of length $\ell = a+b$ with parts

$$(p_1,\ldots,p_a,p'_1,\ldots,p'_b),$$

where the p_i are odd and the p'_j are even.

If $d+\ell$ is even, then b must be even. If $b>0$, then

$$b = 2r-1+1$$

for $r>0$. Let

$$\begin{array}{rcl} \widetilde{\Gamma}_{\mathbf{p}} & = & A\left(\frac{p_1+1}{2},\ldots,\frac{p_a+1}{2}\right) - C_{\frac{p'_{2r}}{2}+1} \\ & & | \\ & & \widetilde{B}\left(\frac{p'_1}{2},\ldots,\frac{p'_{r-1}}{2},\frac{p'_r}{2} \mid \frac{p'_{r+1}}{2}+1,\ldots,\frac{p'_{2r-1}}{2}+1\right)\ . \end{array}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has a two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d+a+2r = d+\ell\ . \tag{31}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is even and $b=0$, let

$$\widetilde{\Gamma}_{\mathbf{p}} = A\left(\frac{p_1+1}{2},\ldots,\frac{p_a+1}{2}\right) .$$

The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings. The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) = d+a = d+\ell \ . \tag{32}$$

The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v\in \mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is odd, then $b=2r-1$ for $r>0$. Let

$$\begin{aligned}\widetilde{\Gamma}_{\mathbf{p}} \ =\ & A\left(\frac{p_1+1}{2},\ldots,\frac{p_a+1}{2}\right) - E\\ & \Big| \\ & \widetilde{B}\left(\frac{p'_1}{2},\ldots,\frac{p'_{r-1}}{2},\frac{p'_r}{2} \mid \frac{p'_{r+1}}{2}+1,\ldots,\frac{p'_{2r-1}}{2}+1\right) ,\end{aligned}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d+a+2(r-1)+2 = d+\ell+1 \ . \tag{33}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v\in\mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

The proof of Theorem 6 now follows the $n=1$ case. Let

$$\mathbf{p}\in P(d,2g-d)$$

be a partition of length ℓ. Let $\widetilde{\Gamma}_{\mathbf{p}}$ be the dual graph of genus $g(\widetilde{\Gamma}_{\mathbf{p}})$ obtained by the above constructions. Since

$$2g \geq d+\ell ,$$

we see $g-g(\widetilde{\Gamma}_{\mathbf{p}})=\delta\geq 0$.

We associate to $\mathbf{p}$ a class $\widetilde{w}_{\mathbf{p}}\in R^{2g-1-d}(M^c_{g,2})$ by the following construction. Let $v^*\in\mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ be the vertex which carries the first marking. Increase the genus of v^* by δ. The resulting graph determines a stratum

$$\widetilde{W}_{\mathbf{p}}\subset M^c_{g,2}$$

of codimension $2g(\widetilde{\Gamma}_{\mathbf{p}}) - 1 - d$. Let

$$\widetilde{w}_{\mathbf{p}} = \psi_1^{2\delta} \cdot [\widetilde{W}_{\mathbf{p}}] \in R^{2g-1-d}(M^c_{g,2}) \ .$$

The pairing on $P(d, 2g-d)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{\overline{M}_{g,2}} \kappa_{\mathbf{p}} \cdot \widetilde{w}_{\mathbf{q}}$$

is upper-triangular and nonsingular as before. Hence,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,2}) \geq |P(d, 2g-d)|,$$

which is the required bound.

6.3.5. *Case $n \geq 3$.* The higher pointed cases are easily reduced to the 1 or 2 pointed cases depending upon the parity of n. The trading of genera for markings follows the proof of Theorem 2 in Section 5.4. We leave the details to the reader. □

References

[1] K. Behrend, B. Fantechi, *The intrinsic normal cone,* Invent. Math. **128** (1997), 45–88.

[2] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, 109–129, Aspects Math., Vieweg, Braunschweig, 1999.

[3] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173 – 199.

[4] C. Faber and R. Pandharipande (with an appendix by D. Zagier), *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J. **48** (2000), 215-252.

[5] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the λ_g conjecture*, Annals of Math. **157** (2003), 97 –124.

[6] E. Getzler, *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov-Witten invariants*, JAMS **10** (1997), 973–998.

[7] E. Getzler and R. Pandharipande, *Virasoro constraints and Chern classes of the Hodge bundle*, Nuclear Phys. **B530** (1998), 701-714.

[8] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[9] T. Graber and R. Pandharipande, *Constructions of nontautological classes on moduli spaces of curves*, Michigan Math J. **51** (2003), 93–109.

[10] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke. Math. J. **130** (2005), 1–37.

[11] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), 316–352.

[12] E. Ionel, *Relations in the tautological ring of M_g*, Duke Math. J. **129** (2005), 157–186.

[13] A. Losev and Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. **48** (2000), 443–472.

[14] A. Marian, D. Oprea, *Virtual intersections on the Quot scheme and Vafa-Intriligator formulas*, Duke Math. J. **136** (2007), 81–113.

[15] A. Marian, D. Oprea, and R. Pandharipande, *The moduli space of stable quotients*, arXiv:0904.2992.

[16] S. Morita, *Generators for the tautological algebra of the moduli space of curves*, Topology **42** (2003), 787–819.

[17] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry* (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271-328.

[18] R. Pandharipande, *A geometric construction of Getzler's elliptic relation*, Math. Ann. **313** (1999), 715 – 729.

[19] R. Pandharipande, *Three questions in Gromov-Witten theory*, Proceedings of the ICM (Beijing 2002), Vol. II, 503-512.

[20] R. Pandharipande, *The κ ring of the moduli of curves of compact type: II*, arXiv:0906.2658.

[21] G. van der Geer, *Cycles on the moduli space of Abelian varieties*, Moduli of curves and abelian varieties, 65–89, Aspects Math., Vieweg, Braunschweig, 1999.

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