THE κ RING OF THE MODULI OF CURVES OF COMPACT TYPE: II

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ABSTRACT. The subalgebra of the tautological ring of the moduli of curves of compact type generated by the κ classes is studied. Relations, constructed via the virtual geometry of the moduli of stable maps, are used to prove universality results relating the κ rings in genus 0 to higher genus. Predictions for κ classes of the Gorenstein conjecture are proven.

CONTENTS

1. INTRODUCTION

1.1. κ **classes.** Let $\overline{M}_{g,n}$ be the moduli space of genus g, n-pointed stable curves. The κ classes in the Chow ring $A^*(\overline{M}_{g,n})$ with $\mathbb{Q}$-coefficients are defined by the following construction. Let

$$\epsilon : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

be the universal curve viewed as the $(n+1)$-pointed space, let

$$\mathbb{L}_{n+1} \to \overline{M}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \ \in A^1(\overline{M}_{g,n+1})$$

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be the Chern class. The κ classes are

$$\kappa_i = \epsilon_*(\psi_{n+1}^{i+1}) \ \in A^i(\overline{M}_{g,n}), \quad i \geq 0 \ .$$

The simplest is κ_0 which equals $2g-2+n$ times the unit in $A^0(\overline{M}_{g,n})$.

The κ classes on the moduli space of curves of compact type

$$M_{g,n}^c \subset \overline{M}_{g,n}$$

are defined by restriction. The κ ring

$$\kappa^*(M_{g,n}^c) \subset A^*(M_{g,n}^c),$$

is the $\mathbb{Q}$-subalgebra generated by the κ classes. The κ rings are graded by degree.

By the results of [11], $\kappa^*(M_{g,n}^c)$ is generated as a $\mathbb{Q}$-algebra by

$$\kappa_1, \kappa_2, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor}.$$

Moreover, there are no relation of degree less than or equal to $g-1+\lfloor \frac{n}{2} \rfloor$ if $n > 0$.

1.2. **Universality.** Let $x_1, x_2, x_3, \ldots$ be variables with x_i of degree i, and let

$$f \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$$

be *any* graded homogeneous polynomial. The following universality property was stated in [11].

Theorem 1. *If* $f(\kappa_i) = 0 \in \kappa^*(M_{0,n}^c)$, *then*

$$f(\kappa_i) = 0 \in \kappa^*(M_{g,n-2g}^c)$$

for all genera g *for which* $n - 2g \geq 0$.

By Theorem 1, the higher genus κ rings are canonically quotients of the genus 0 rings,

$$\kappa^*(M_{0,2g+n}^c) \stackrel{\iota_{g,n}}{\to} \kappa^*(M_{g,n}^c) \to 0.$$

Theorem 1 is our main result here.

1.3. **Bases.** Let $P(d)$ be the set of partitions of d, and let

$$P(d,k) \subset P(d)$$

be the set of partitions of d into at most k parts. Let $|P(d,k)|$ be the cardinality. To a partition[1]

$$\mathbf{p} = (p_1, \ldots, p_\ell) \in P(d,k),$$

we associate a κ monomial by

$$\kappa_{\mathbf{p}} = \kappa_{p_1} \cdots \kappa_{p_\ell} \in \kappa^d(M^c_{g,n}) \ .$$

In [11], two basic facts about the κ rings of the moduli space of curves of compact type are derived from Theorem 1:

- the canonical quotient,
$$\kappa^*(M^c_{0,2g+n}) \stackrel{\iota_{g,n}}{\to} \kappa^*(M^c_{g,n}) \to 0$$
is an isomorphism for $n > 0$,
- a $\mathbb{Q}$-basis of $\kappa^d(M^c_{g,n})$ is given by
$$\{\kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, 2g-2+n-d) \ \}$$
for $n > 0$.

The main tools used in [11] are the virtual geometry of the moduli space of stable quotients [9] and the intersection theory of strata classes in the tautological ring $R^*(M^c_{g,n})$.

By Theorem 5 of [11], proven unconditionally,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) = |P(d, n-2-d)| \ .$$

Hence, Theorem 1 is a consequence of the following result.

Proposition 1. *The space of relations among κ monomials of degree d valid in* all *the rings*

$$\{\kappa^*(M^c_{g,n}) \mid 2g-2+n = \zeta \ \}$$

is of rank at least $|P(d)| - |P(d, \zeta - d)|$.

Proposition 1 is proven in Sections 2 - 4 by constructing universal relations in $\kappa^*(M^c_{g,n})$ via the virtual geometry of the moduli space of stable maps. The interplay between stable quotients and stable maps is an interesting aspect of the study of $\kappa^*(M^c_{g,n})$.

[1] The parts of $\mathbf{p}$ are positive and satisfy $p_1 \geq \ldots \geq p_\ell$.

1.4. **Gorenstein conjecture.** The rank g Hodge bundle over the moduli space of curves

$$\mathbb{E} \to \overline{M}_{g,n}$$

has fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. Let

$$\lambda_k = c_k(\mathbb{E})$$

be the Chern classes. Since λ_g vanishes when restricted to

$$\delta_0 = \overline{M}_{g,n} \setminus M^c_{g,n} \ ,$$

we obtain a well-defined evaluation

$$\phi : A^*(M^c_{g,n}) \to \mathbb{Q}$$

given by integration

$$\phi(\gamma) = \int_{\overline{M}_{g,n}} \overline{\gamma} \cdot \lambda_g \ ,$$

where $\overline{\gamma}$ is any lift of $\gamma \in A^*(M^c_{g,n})$ to $A^*(\overline{M}_{g,n})$.

The tautological rings $R^*(M^c_{g,n}) \subset A^*(M^c_{g,n})$ have been conjectured in [4, 10] to be Gorenstein algebras with socle in degree $2g-3+n$,

$$\phi : R^{2g-3+n}(M^c_{g,n}) \xrightarrow{\sim} \mathbb{Q} \ .$$

As a consequence of Theorem 1 and the intersection calculations of [11], we obtain the following result.

Theorem 2. *If $n > 0$ and $\xi \in \kappa^d(M^c_{g,n}) \neq 0$, the linear function*

$$L_\xi : R^{2g-3+n-d}(M^c_{g,n}) \to \mathbb{Q}$$

defined by the socle evaluation

$$L_\xi(\gamma) = \phi(\gamma \cdot \xi)$$

is non-trivial.

Theorem 2, discussed in Section 5.1, may be viewed as significant evidence for the Gorenstein conjecture for all $M^c_{g,n}$ with $n > 0$.

1.5. **Acknowledgments.** The results here on the κ rings were motivated by the study of stable quotients in [9]. Discussions with A. Marian and D. Oprea were very helpful. The methods developed with C. Faber in [5] played an important role.

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2. κ AND ψ

2.1. ψ **classes.** Consider the cotangent line classes

$$\psi_{n+1}, \ldots, \psi_{n+\ell} \in A^1(M^c_{g,n+\ell})$$

at the last ℓ marked points. Let

$$\epsilon^c : M^c_{g,n+\ell} \to M^c_{g,n}$$

be the proper forgetful map. For each partition $\mathbf{p} \in P(d)$ of length ℓ, we associate the class

$$\epsilon^c_* \left(\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell} \right) \in A^d(M^c_{g,n}) \ .$$

The relation between the above push-forwards of ψ monomials and the κ classes is easily obtained. For $\mathbf{p} = (d)$, we have

$$\epsilon^c_*(\psi_{n+1}^{1+d}) = \kappa_d$$

by definition. The standard cotangent line comparison formulas yield the length 2 case,

$$\epsilon^c_*(\psi_{n+1}^{1+p_1}\psi_{n+2}^{1+p_2}) = \kappa_{p_1}\kappa_{p_2} + \kappa_{p_1+p_2} \ .$$

The full formula, due to Faber, is

$$\epsilon^c_* \left(\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell} \right) = \sum_{\sigma \in S_\ell} \kappa_{\sigma(\mathbf{p})} \ , \tag{1}$$

where the sum is over the symmetric group S_ℓ. For $\sigma \in S_\ell$, let

$$\sigma = \gamma_1 \ldots \gamma_r$$

be the canonical cycle decomposition (including the 1-cycles), and let $\sigma(\mathbf{p})_i$ be the sum of the parts of $\mathbf{p}$ with indices in the cycle γ_i. Then,

$$\kappa_{\sigma(\mathbf{p})} = \kappa_{\sigma(\mathbf{p})_1} \cdots \kappa_{\sigma(\mathbf{p})_r} \ .$$

A discussion of (1) can be found in [1].

Lemma 1. *The sets of classes in* $A^d(M^c_{g,n})$ *defined by*

$$\{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}\quad \text{and}\quad \{\ \kappa_{\mathbf{p}}\ |\ \mathbf{p}\in P(d)\ \}$$

are related by an invertible linear transformation independent of g *and* n*.*

Proof. Formula (1) defines a universal transformation independent of g and n. Since the transformation is triangular in the partial ordering of $P(d)$ by length (with 1's on the diagonal), the invertibility is clear. □

2.2. **Bracket classes.** Let $\mathbf{p}\in P(d)$ be a partition of length ℓ. Let

$$\langle\mathbf{p}\rangle = \epsilon^c_*\left[\prod_{i=1}^{\ell}\frac{1}{1-p_i\psi_{n+i}}\right]^{\ell+d}\ \in A^d(M^c_{g,n})\ . \tag{2}$$

The superscript in the inhomogeneous expression $\left[\prod_{i=1}^{\ell}\frac{1}{1-p_i\psi_{n+i}}\right]^{\ell+d}$ indicates the summand in $A^{\ell+d}(M^c_{g,n+\ell})$.

We can easily expand definition (2) to express the class $\langle\mathbf{p}\rangle$ linearly in terms of the classes

$$\{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}\ .$$

Since the string and dilation equation must be used to remove the ψ^0_{n+i} and ψ^1_{n+i} factors, the transformation depends upon g and n only through $2g-2+n$.

Lemma 2. *The sets of classes in* $A^d(M^c_{g,n})$ *defined by*

$$\{\ \langle\mathbf{p}\rangle\ |\ \mathbf{p}\in P(d)\ \}\quad \text{and}\quad \{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}$$

are related by an invertible linear transformation depending only upon $2g-2+n$*.*

Proof. Only the invertibility remains to be established. The result exactly follows from the proof of Proposition 3 in [5]. □

By Lemma 1 and 2, the bracket classes lie in the κ ring,

$$\langle\mathbf{p}\rangle\in\kappa^d(M^c_{g,n})\ .$$

We will prove Proposition 1 in the following equivalent form.

Proposition 2. *The space of relations among the classes*

$$\{ \ \langle \mathbf{p} \rangle \ | \ \mathbf{p} \in P(d) \ \}$$

valid in all *the rings*

$$\{\kappa^*(M^c_{g,n}) \ | \ 2g-2+n = \zeta \ \}$$

is of rank at least $|P(d)| - |P(d, \zeta - d)|$.

3. Relations via stable maps

3.1. **Moduli of stable maps.** Let $\overline{M}_{g,n+m}(\mathbb{P}^1, d)$ denote the moduli of stable maps[2] to $\mathbb{P}^1$ of degree d, and let

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}^1, d) \to \overline{M}_{g,n}$$

be the morphism forgetting the map and the last m markings. The moduli space

$$M^c_{g,n+m}(\mathbb{P}^1, d) \subset \overline{M}_{g,n+m}(\mathbb{P}^1, d)$$

is defined by requiring the domain curve to be of compact type. The restriction

$$\nu^c : M^c_{g,n+m}(\mathbb{P}^1, d) \to M^c_{g,n}$$

is proper and equivariant with respect to the symmetries of $\mathbb{P}^1$.

We will find relations in $A^*(M^c_{g,n})$ by localizing ν^c push-forwards which vanish geometrically. A complete analysis in the socle $A^{2g-3}(M^c_g)$ was carried out in [5], but much more will be required for Theorem 1. While the relations in $A^*(M^c_{g,n})$ via stable quotients [11] are more elegantly expressed, the ranks of the relations via stable maps appear easier to compute.

3.2. **Relations.**

3.2.1. *Indexing.* Let $d \leq 2g-3+n$, and let

$$\delta = 2g - 3 + n - d \ .$$

We will construct a series of relations $I(g, d, \alpha)$ in $A^d(M^c_{g,n})$ where

$$\alpha = (\alpha_1, \ldots, \alpha_m)$$

is a (non-empty) vector of non-negative integers satisfying two conditions:

(i) $|\alpha| = \sum_{i=1}^m \alpha_i \leq d - 2 - \delta$,

[2] Stable maps were defined in [8], see [6] for an introduction.

(ii) $\alpha_i > 0$ for $i > 1$.

By condition (i), $d - 2 - \delta \geq 0$ so

$$d > g - 1 + \lfloor \frac{n}{2} \rfloor \ .$$

Condition (ii) implies α_1 is the only integer permitted to vanish. The relation $I(g, d, \alpha)$ will be a variant of the equations considered in [5].

3.2.2. *Formulas.* Let Γ denote the data type

$$(p_1, \ldots, p_m) \cup \{p_{m+1}, \ldots, p_\ell\}, \tag{3}$$

satisfying

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The first part of Γ is an ordered m-tuple $(p_1, \ldots, p_m)$. The second part $\{p_{m+1}, \ldots, p_\ell\}$ is an unordered set. Let $\text{Aut}(\{p_{m+1}, \ldots, p_\ell\})$ be the group which permutes equal parts. The group of automorphisms $\text{Aut}(\Gamma)$ equals $\text{Aut}(\{p_{m+1}, \ldots, p_\ell\})$.

Theorem 3. *For all α satisfying (i-ii),*

$$\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{j=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1, \ldots, p_\ell \rangle$$
$$= 0 \ \in A^d(M^c_{g,n}),$$

where the sum is over all Γ of type (3).

The bracket $\langle p_1, \ldots, p_\ell \rangle \in A^d(M^c_{g,n})$ denotes the class associated to the partition defined by the union of all the parts p_i of Γ.

3.3. **Proof of Theorem 3.**

3.3.1. *Torus actions.* The first step is to define the appropriate torus actions. Let

$$\mathbb{P}^1 = \mathbb{P}(V)$$

where $V = \mathbb{C} \oplus \mathbb{C}$. Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{4}$$

Let $\mathsf{p}_1, \mathsf{p}_2$ be the fixed points $[1, 0], [0, 1]$ of the corresponding action on $\mathbb{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbb{P}(V)$ is uniquely determined by the weights $[l_1, l_2]$ of the fiber representations at the fixed points

$$L_1 = L|_{\mathsf{p}_1}, \quad L_2 = L|_{\mathsf{p}_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbb{P}}$ has weights $[1, -1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V)}(-1)$ with weights $[1, 0]$, $[0, 1]$ respectively.

Over the moduli space of stable maps $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$, we have

$$\pi : U \to \overline{M}_{g,n+m}(\mathbb{P}(V), d), \quad \mu : U \to \mathbb{P}(V)$$

where U is the universal curve and μ is the universal map. The representation (4) canonically induces $\mathbb{C}^*$-actions on U and $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ compatible with the maps π and μ. The $\mathbb{C}^*$-equivariant virtual class

$$[\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \in A^{\mathbb{C}^*}_{2g+2d-2+n+m}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$

will play an important role.

3.3.2. *Equivariant classes.* Three types of equivariant Chow classes on $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ will be considered here:

- The linearization $[0, 1]$ on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ defines an $\mathbb{C}^*$-action on the rank $d + g - 1$ bundle

  $$\mathbb{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}(V)}(-1))$$

  on $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$. Let

  $$c_{top}(\mathbb{R}) \in A^{g+d-1}_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$

  be the top Chern class.
- For each marking i, let $\psi_i \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ denote the first Chern class of the canonically linearized cotangent line corresponding to i.
- Denote the i^{th} evaluation morphism by

  $$\mathrm{ev}_i : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V).$$

  With $\mathbb{C}^*$-linearization $[1, 0]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let

  $$\rho_i = c_1(\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d)\ .$$

  With $\mathbb{C}^*$-linearization $[0, -1]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let

  $$\widetilde{\rho}_i = c_1(\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d)\ .$$

In the non-equivariant limit, $\rho_i^2 = 0$. Our notation here closely follows [5].

3.3.3. *Vanishing integrals.* The forgetful morphism

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \to \overline{M}_{g,n}$$

is $\mathbb{C}^*$-equivariant with respect to the trivial action on $\overline{M}_{g,n}$. As in Section 3.2.1, let

$$d \leq 2g - 3 + n, \quad \delta = 2g - 3 + n - d,$$

and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ satisfy

(i) $|\alpha| = \sum_{i=1}^{m} \alpha_i \leq d - 2 - \delta$,
(ii) $\alpha_i > 0$ for $i > 1$.

Let $I(g, d, \alpha)$ be the $\mathbb{C}^*$-equivariant push-forward

$$\nu_* \left(\rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^{m} \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^{n} \widetilde{\rho}_j \ c_{top}(\mathbb{R}) \ \cap \ [\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \right) .$$

The degree of the class

$$\rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^{m} \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^{n} \widetilde{\rho}_j \ c_{top}(\mathbb{R})$$

is easily computed to be

$$d - 1 - \delta - |\alpha| + m + |\alpha| + n + d + g - 1 = \\ g + 2d - 2 + n + m - \delta \ .$$

Since the cycle dimension of the virtual class is $2g + 2d - 2 + n + m$, the push-forward $I(g, d, \alpha)$ has cycle dimension

$$\begin{aligned} 2g + 2d - 2 + n + m - (g + 2d - 2 + n + m - \delta) &= g + \delta \\ &= 3g - 3 + n - d \ . \end{aligned}$$

Equivalently, $I(g, d, \alpha) \in A^d_{\mathbb{C}^*}(\overline{M}_{g,n})$. Since the class ρ_{n+1} appears with exponent

$$d - \delta - |\alpha| \geq 2,$$

$I(g, d, \alpha)$ vanishes in the non-equivariant limit.

3.3.4. *Localization terms.* The virtual localization formula of [7] calculates $I(g,d,\alpha)$ in terms of tautological classes on the moduli space $\overline{M}_{g,n}$. To prove Theorem 3, we will calculate the restriction of the localization formula to $M^c_{g,n}$.

The localization formula expresses $I(g,d,\alpha)$ as a sum over connected decorated graphs Γ indexing the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$. The vertices of the graphs lie over the fixed points $\mathsf{p}_1, \mathsf{p}_2 \in \mathbb{P}(V)$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbb{P}^1$ and are labelled with degrees (which sum over the graph to d). Finally, the graphs carry $n+m$ markings on the vertices. The valence $\text{val}(v)$ of a vertex $v \in \Gamma$ counts both the incident edges and markings. The edge valence of v counts only the incident edges.

Only a very restricted subset of graphs will yield non-vanishing contributions to $I(g,d,\alpha)$ in the non-equivariant limit. If a graph Γ contains a vertex lying over p_1 of edge valence greater than 1, then the contribution of Γ to vanishes by our choice of linearization on the bundle $\mathbb{R}$. A vertex over p_1 of edge valence greater than 1 yields a trivial Chern root of $\mathbb{R}$ (with trivial weight 0) in the numerator of the localization formula to force the vanishing.

By the above vanishing, only *comb* graphs Γ contribute to $I(g,d,\alpha)$. Comb graphs contain $\ell \leq d$ vertices lying over p_1 each connected by a distinct edge to a unique vertex lying over p_2.

If Γ contains a vertex over p_1 of positive genus, then the restriction to $M^c_{g,n}$ of the contribution of Γ to $I(g,d,\alpha)$ vanishes by the following argument. Let v be a genus $g(v) > 0$ vertex lying over p_1. The integrand term $c_{top}(\mathbb{R})$ yields a factor $c_{g(v)}(\mathbb{E}^*)$ with trivial $\mathbb{C}^*$-weight on the genus $g(v)$ moduli space corresponding to the vertex v. Since

$$\lambda_{g(v)}|_{M^c_{g(v),\text{val}(v)}} = 0$$

by [12], the required vanishing holds.

The linearizations of the classes ρ_i and $\widetilde{\rho}_j$ place restrictions on the marking distribution. Since the class $\widetilde{\rho}_j$ is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[0,-1]$, the first n markings must lie on the unique vertex over over p_2. Since the class ρ_i is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[1,0]$, the last m markings must lie on vertices over p_1.

Finally, we claim the last m markings of Γ must lie on distinct vertices over p_1 for nonvanishing contribution to $I(g,d,\alpha)$. Let v be a vertex over p_1 (with $g(v)=0$). If v carries at least two markings, the fixed locus corresponding to Γ contains a product factor $\overline{M}_{0,r+1}$ where r is the number of markings incident to v. The classes $\psi_{n+i}^{\alpha_i}$ carry trivial $\mathbb{C}^*$-weight. Moreover, as each $\alpha_i > 0$ for $i > 1$, we see the sum of the α_i as i ranges over the set of markings incident to v is at least $r-1$. Since the sum exceeds the dimension of $\overline{M}_{0,r+1}$, the graph contribution to $I(g,d,\alpha)$ vanishes.

The proof of the main result about the localization terms for $I(g,d,\alpha)$ is now complete.

Proposition 3. *The restriction of $I(g,d,\alpha)$ to $M^c_{g,n}$ is expressed via the virtual localization formula as a sum over genus g, degree d, marked comb graphs Γ satisfying:*

(i) *all vertices over p_1 are of genus 0,*
(ii) *the unique vertex over p_2 carries all of the first n markings,*
(iii) *the last m markings all lie over p_1,*
(iv) *each vertex over p_1 carries at most 1 of the last m markings.*

3.3.5. *Formulas.* The precise contributions of allowable graphs Γ to the non-equivariant limit of $I(g,d,\alpha)$ are now calculated.

Let Γ be a genus g, degree d, comb graph with $n+m$ markings satisfying conditions (i-iv) of Proposition 3. By condition (iv), Γ must have $\ell \geq m$ edges. Γ may be described uniquely by the data

$$(p_1,\ldots,p_m) \cup \{p_{m+1},\ldots,p_\ell\}, \tag{5}$$

satisfying:

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The elements of the ordered m-tuple $(p_1,\ldots,p_m)$ correspond to the degree assignments of the edges incident to the vertices marked by the last m markings. The elements of the unordered partition $\{p_{m+1},\ldots,p_\ell\}$ correspond to the degrees of edges incident to the unmarked vertices over p_1. The group of graph automorphisms is

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\{p_{m+1},\ldots,p_\ell\})\ .$$

By a direct application of the virtual localization formula of [7], we find the contribution of the graph (5) to the normalized[3] push-forward

$$(-1)^{g+1+|\alpha|+n+m} \cdot I(g,d,\alpha)$$

equals

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1,\ldots,p_\ell\rangle \ .$$

Hence, the vanishing of $I(g,d,\alpha)$ yields the relation

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1,\ldots,p_\ell\rangle = 0 \ ,$$

where the sum is over all graphs (5). □

Question 1. *Are the relations of Theorem 3 equivalent to relations constructed in Section 3 of* [9] *via stable quotients?*

4. Rank analysis

4.1. **Matrix of relations.** Theorem 3 yields relations in $\kappa^d(M^c_{g,n})$, indexed by $\alpha = (\alpha_1,\ldots,\alpha_m)$ satisfying conditions (i-ii) of Section 3.2.1 with

$$\delta = 2g-3+n-d \geq 0.$$

We rewrite the relation obtained from the vanishing of $I(g,d,\alpha)$ as

$$\sum_{\mathbf{p}\in P(d)} \mathsf{C}^{\mathbf{p}}_{\alpha} \ \langle \mathbf{p}\rangle = 0 \ . \tag{6}$$

The coefficients are

$$\mathsf{C}^{\mathbf{p}}_{\alpha} = \frac{1}{|\mathrm{Aut}(\mathbf{p})|} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \sum_{\phi} \prod_{i=1}^{m} p_{\phi(i)}^{-\alpha_i} \prod_{j\in \mathrm{Im}(\phi)^c} (-p_j)^{-1} \ ,$$

where the sum is over all injections

$$\phi : \{1,\ldots,m\} \to \{1,\ldots,\ell\}$$

and

$$\mathrm{Im}(\phi)^c \subset \{1,\ldots,\ell\}$$

is the complement of the image of ϕ.

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[3]The parallel equation on page 106 of [5] has a sign error in the normalization. Instead of $(-1)^{g+1}I(g,d,\alpha)$ there, the normalization should be $(-1)^{g+1+|\alpha|+\ell(\alpha)}I(g,d,\alpha)$. The sign change makes no difference.

To prove Proposition 2, we will show the system (6) is of rank at least $|P(d)| - |P(d, \delta+1)|$. The claim is empty unless $0 \leq \delta \leq d-2$.

4.2. **Ordering.** For $0 \leq \delta \leq d-2$, define the subset $P_\delta(d) \subset P(d)$ by removing partitions of length at most $\delta+1$,

$$P_\delta(d) = P(d) \setminus P(d, \delta+1) \ .$$

We order $P_\delta(d)$ by the following rules

- longer partitions appear before shorter partitions,
- for partitions of the same length, we use the lexicographic ordering with larger parts[4] appearing before smaller parts.

For example, the ordered list of the 10 elements of $P_0(6)$ is

$$(1^6),\ (2,1^4),\ (3,1^3),\ (2^2,1^2),\ (4,1^2),\ (3,2,1),\ (2^3),\ (5,1),\ (4,2),\ (3,3) \ .$$

Given a partition $\mathbf{p} \in P(d)$, let $\widehat{\mathbf{p}}$ be the partition obtained removing all parts equal to 1. For example,

$$\widehat{(1^6)} = \emptyset, \quad \widehat{(3,2,1)} = (3,2) \ .$$

Let $\mathbf{p}^-$ be the partition obtained by lowering all the parts of $\mathbf{p}$ by 1,

$$(1^6)^- = \emptyset, \quad (3,2,1)^- = (2,1) \ .$$

If $\mathbf{p}$ has length ℓ, then

$$\mathbf{p}^- \in P(d-\ell).$$

To each partition $\mathbf{p} \in P_\delta(d)$, we associate data $\alpha[\mathbf{p}]$ satisfying conditions (i)-(ii) with respect to δ by the following rules. The special designation

$$\alpha[(1^d)] = (0)$$

is given. Otherwise

$$\alpha[\mathbf{p}] = \mathbf{p}^- \ .$$

We note condition (i) of Section 3.2.1,

$$|\alpha[\mathbf{p}]| \leq d-2-\delta \ ,$$

is satisfied in all cases.

Let $\mathsf{M}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$M_\delta(d)[\mathbf{p},\mathbf{q}] = \mathsf{C}^{\mathbf{q}}_{\alpha[\mathbf{p}]} \ .$$

[4]Remember the parts of $\mathbf{p} = (p_1, \ldots, p_\ell)$ are ordered by $p_1 \geq \ldots \geq p_\ell$.

The rank of the system (6) is at least

$$|P_\delta(d)| = |P(d)| - |P(d, \delta+1)|$$

by the following nonsingularity result proven in Sections 4.3 - 4.6 below.

Proposition 4. *For $0 \le \delta \le d-2$, the matrix $\mathsf{M}_\delta(d)$ is nonsingular.*

Proposition 4 implies Proposition 2 and thus Theorem 1. Moreover, Proposition 4 provides a new approach to [5].

4.3. **Scaling.** Let $\mathsf{X}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$\begin{aligned} \mathsf{X}_\delta(d)[(1)^d, \mathbf{q}] &= (-1)^{\ell(\mathbf{q})-1} d \\ \mathsf{X}_\delta(d)[\mathbf{p} \neq (1)^d, \mathbf{q}] &= \sum_\phi (-1)^{\ell(\mathbf{q})-\ell(\widehat{\mathbf{p}})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} q_{\phi(i)}^{-\widehat{p}_i+2} , \end{aligned}$$

where the sum is over all injections

$$\phi : \{1, \ldots, \ell(\widehat{\mathbf{p}})\} \to \{1, \ldots, \ell(\mathbf{q})\} .$$

For example, $\mathsf{X}_0(6)$ is

$$\begin{pmatrix} -6 & 6 & -6 & -6 & 6 & 6 & 6 & -6 & -6 & -6 \\ -6 & 5 & -4 & -4 & 3 & 3 & 3 & -2 & -2 & -2 \\ -6 & \frac{9}{2} & -\frac{10}{3} & -3 & \frac{9}{4} & \frac{11}{6} & \frac{3}{2} & -\frac{6}{5} & -\frac{3}{4} & -\frac{2}{3} \\ 30 & -20 & 12 & 12 & -6 & -6 & -6 & 2 & 2 & 2 \\ -6 & \frac{17}{4} & -\frac{28}{9} & -\frac{5}{2} & \frac{33}{16} & \frac{49}{36} & \frac{3}{4} & -\frac{26}{25} & -\frac{5}{16} & -\frac{2}{9} \\ 30 & -18 & 10 & 9 & -\frac{9}{2} & -\frac{11}{3} & -3 & \frac{6}{5} & \frac{3}{4} & \frac{2}{3} \\ -120 & 60 & -24 & -24 & 6 & 6 & 6 & 0 & 0 & 0 \\ -6 & \frac{33}{8} & -\frac{82}{27} & -\frac{9}{4} & \frac{129}{64} & \frac{251}{216} & \frac{3}{8} & -\frac{126}{125} & -\frac{9}{64} & -\frac{2}{27} \\ 30 & -17 & \frac{28}{3} & \frac{15}{2} & -\frac{33}{8} & -\frac{49}{18} & -\frac{3}{2} & \frac{26}{25} & \frac{5}{16} & \frac{2}{9} \\ 30 & -16 & 8 & \frac{13}{2} & -3 & -2 & -\frac{3}{2} & \frac{2}{5} & \frac{1}{4} & \frac{2}{9} \end{pmatrix} .$$

The matrix $\mathsf{X}_\delta(d)$ is obtained from $\mathsf{M}_\delta(d)$ by dividing each column corresponding to $\mathbf{q}$ by

$$\frac{1}{|\mathrm{Aut}(\mathbf{q})|} \prod_{i=1}^{\ell(\mathbf{q})} \frac{q_i^{q_i-1}}{q_i!} .$$

Hence, $\mathsf{X}_\delta(d)$ is nonsingular if and only if $\mathsf{M}_\delta(d)$ is nonsingular.

4.4. **Elimination.** Our strategy for proving Proposition 4 is to find an upper-triangular square matrix $\mathsf{Y}_0(d)$ for which the product

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) \tag{7}$$

is lower-triangular with ± 1's on the diagonal. Since $\mathsf{X}_\delta(d)$ for

$$0 \leq \delta \leq d-2$$

occurs as an upper left minor of $\mathsf{X}_0(d)$, the lower-triangularity of the product (7) will establish Proposition 4 for the full range of δ values.

We define $\mathsf{Y}_0(d)$ to be the square matrix indexed by the ordered set $P_0(d)$ given by the following rules. The upper left corner is

$$\mathsf{Y}_0(d)[(1^d),(1^d)] = \frac{1}{d}$$

If at least one of $\{\mathbf{p}, \mathbf{q}\}$ is not equal to (1^d), then the matrix elements are

$$\mathsf{Y}_0(d)[\mathbf{p},\mathbf{q}] = \\ \frac{1}{|\mathrm{Aut}(\mathbf{p})|} \frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|} \sum_\theta \prod_{i=1}^{\ell(\mathbf{q})} \binom{q_i}{p_{i[1]}, \ldots, p_{i[\ell_i]}} q_i^{\ell_i - 2} \prod_{j=1}^{\ell_i} p_{ij}^{p_{ij}-1} ,$$

where the sum is over all functions

$$\theta : \{1, \ldots, \ell(\mathbf{p})\} \to \{1, \ldots, \ell(\mathbf{q})\}$$

with

$$\theta^{-1}(i) = \{i[1], \ldots, i[\ell_i]\}$$

satisfying

$$q_i = \sum_{j=1}^{\ell_i} p_{i[j]} .$$

For example, $\mathsf{Y}_0(6)$ is

$$\begin{pmatrix} \frac{1}{6} & 1 & 3 & \frac{1}{2} & 16 & 3 & \frac{1}{6} & 125 & 16 & \frac{9}{2} \\ 0 & 1 & 6 & 1 & 48 & 9 & \frac{1}{2} & 500 & 64 & 18 \\ 0 & 0 & 3 & 0 & 36 & 3 & 0 & 450 & 36 & 9 \\ 0 & 0 & 0 & \frac{1}{2} & 12 & 6 & \frac{1}{2} & 300 & 60 & 18 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 320 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 180 & 36 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \end{pmatrix}.$$

By the conditions on θ in the definition, $\mathsf{Y}_0(d)$ is easily seen to be upper-triangular.

4.5. **Generating functions.** Let $\mathbb{Q}[t]$ denote the polynomial ring in infinitely many variables

$$t = \{t_1, t_2, t_3, \ldots\} \ .$$

Define a $\mathbb{Q}$-linear function

$$\langle\ \rangle : \mathbb{Q}[t] \to \mathbb{Q}$$

by the equations $\langle 1 \rangle = 1$ and

$$\langle t_{d_1} t_{d_2} \cdots t_{d_k} \rangle = (d_1 + d_2 + \ldots + d_k)^{k-3} \ .$$

We may extend $\langle\ \rangle$ uniquely to define a x-linear function:

$$\langle\ \rangle : \mathbb{Q}[t][[x]] \to \mathbb{Q}[[x]].$$

For each non-negative integer i, let

$$Z_i(t,x) = \sum_{j>0} x^j t_j \frac{j^{j-i}}{j!} \in \mathbb{Q}[t][[x]].$$

Applying the bracket, we define

$$\mathsf{F}_{\alpha_1,\ldots,\alpha_m} = \langle \exp(-Z_1) \cdot Z_{\alpha_1} \cdots Z_{\alpha_m} \rangle \ \in \mathbb{Q}[[x]].$$

Lemma 3. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a non-empty sequence of non-negative integers satisfying $\alpha_i > 0$ for $i > 1$. The series*

$$\mathsf{F}_{\alpha_1,\ldots,\alpha_m} \in \mathbb{Q}[[x]]$$

is a polynomial *of degree at most $1 + \sum_{i=1}^m \alpha_i$ in x.*

Lemma 4. *Let* $\alpha_1 \geq 0$. *Then,*

$$\mathsf{F}_{\alpha_1} = \frac{(-1)^{\alpha_1}}{(1+\alpha_1)(1+\alpha_1)!} x^{1+\alpha_1} + \ldots$$

where the dots stand for lower order terms.

Lemma 3 can be proven by various methods. A proof via localization on moduli space is given in [5] in Section 1.7. □

Lemma 4 is more interesting. The integral

$$J_{1+\alpha_1} = \int_{\overline{M}_{0,1}(\mathbb{P}^1, 1+\alpha_1)} \rho_1 \psi_1^{\alpha_1} \ c_{top}(\mathbb{R}) \tag{8}$$

can be evaluated by exactly following[5] the localization analysis of Section 3.3. We find

$$J_{1+\alpha_1} = (-1)^{\alpha_1} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \ p_1^{-\alpha_1} \prod_{i=2}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ (1+\alpha_1)^{\ell-3}$$

where the sum is over all 1-pointed comb graphs (5) of total degree $1+\alpha_1$. We conclude $J_{1+\alpha_1}$ equals, up to the factor of $(-1)^{\alpha_1}$, the leading $x^{1+\alpha_1}$ coefficient of $\langle \exp(-Z_1) \cdot Z_{\alpha_1} \rangle$.

To calculate the integral (8), we use well-known equations in Gromov-Witten theory. Certainly

$$J_1 = 1 \ . \tag{9}$$

By two applications of the divisor equation,

$$k^2 J_k = \int_{\overline{M}_{0,3}(\mathbb{P}^1, k)} \rho_1 \psi_1^{k-1} \rho_2 \rho_3 \ c_{top}(\mathbb{R})$$

By the topological recursion relation [2] applied to the right side,

$$k^2 J_k = \int_{\overline{M}_{0,2}(\mathbb{P}^1, k-1)} \rho_1 \psi_1^{k-2} \rho_2 \ c_{top}(\mathbb{R}) \ \cdot \ \int_{\overline{M}_{0,3}(\mathbb{P}^1, 1)} \rho_1 \rho_2 \rho_3 \ c_{top}(\mathbb{R}) \ .$$

We obtain the recursion

$$\begin{aligned} k^2 J_k &= (k-1) J_{k-1} J_1 \\ &= (k-1) J_{k-1} \end{aligned}$$

which we can easily solve

$$J_k = \frac{1}{k \cdot k!}$$

[5]The equivariant lifts are taken just as in Section 3.3.2.

starting with the initial condition (9). □

The case where the α data is empty will arise naturally. We define

$$\mathsf{F}_\emptyset = \langle \exp(-Z_1) \rangle.$$

The following result is derived from Lemma 3 by the relation

$$x \frac{d}{dx} \mathsf{F}_\emptyset = -\mathsf{F}_0.$$

Lemma 5. $\mathsf{F}_\emptyset = 1 - x$.

4.6. **Product.** We will now prove the basic identity

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) = \mathsf{L}_0(d) \tag{10}$$

where $\mathsf{L}_0(d)$ is lower triangular with diagonal entries all ± 1.

We first address the special upper left corner. The product on the left side of (10) is

$$\mathsf{L}_0(d)[(1^d), (1^d)] = (-1)^{d-1} d \cdot \frac{1}{d} = (-1)^{d-1} \ ,$$

a diagonal entry of the specified form.

Next assume $\mathbf{p} \neq (1^d)$. Then, the matrix elements are

$$\mathsf{L}_0(d)[\mathbf{p}, \mathbf{q}] = \frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|} \sum_{\gamma} \prod_{i=1}^{\ell(\mathbf{q})} \mathrm{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i}) \ q_i \ q_i! \ , \tag{11}$$

where the sum is over all functions

$$\gamma : \{1, \ldots, \ell(\widehat{\mathbf{p}})\} \to \{1, \ldots, \ell(\mathbf{q})\} \ .$$

In case $\gamma^{-1}(i) = \{i[1], \ldots, i[\ell_i]\}$ is nonempty, we define

$$\mathsf{F}_{\gamma^{-1}(i)} = \mathsf{F}_{\widehat{p}_{i[1]}-1, \ldots, \widehat{p}_{i[\ell_i]}-1} \ .$$

If $\gamma^{-1}(i) = \emptyset$, then

$$\mathsf{F}_\emptyset = \langle \exp(-Z_1) \rangle = 1 - x.$$

Equation (11) is obtained from a simple unravelling of the definitions.

If $q_i > 1$, $\mathrm{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i})$ vanishes unless $\gamma^{-1}(i)$ is nonempty by Lemma 5 and unless

$$q_i \leq 1 - \ell_i + \sum_{j=1}^{\ell_i} \widehat{p}_{i[j]} \tag{12}$$

by Lemma 3. Inequality (12) for all parts $q_i > 1$ implies

$$\ell(\mathbf{q}) \geq \ell(\mathbf{p}) \ .$$

Moreover, if equality of length holds, then inequality (12) implies either $\mathbf{q}$ precedes $\mathbf{p}$ in the ordering of $P_0(d)$ or $\mathbf{q} = \mathbf{p}$.

We conclude the matrix $\mathsf{L}_0(d)$ is lower-triangular when the first coordinate $\mathbf{p}$ is not $(1)^d$. The diagonal elements for $\mathbf{p} \neq (1^d)$ are

$$\mathsf{L}_0(d)[\mathbf{p},\mathbf{p}] = \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i - 1} \cdot (-1)^{\ell(\mathbf{p}) - \ell(\widehat{\mathbf{p}})}$$

by Lemmas 4 and 5.

To complete the proof of the lower-triangularity of $\mathsf{L}_0(d)$, we must show the vanishing of $\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)]$. The matrix elements are

$$\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)] = \frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|} \sum_{\tilde{\gamma}} \prod_{i=1}^{\ell(\mathbf{q})} \mathrm{Coeff}(\widetilde{F}_{\tilde{\gamma}^{-1}(i)}, x^{q_i})\ q_i\ q_i! \ ,$$

where the sum is over all functions

$$\tilde{\gamma} : \{1\} \to \{1, \ldots, \ell(\mathbf{q})\} \ .$$

In case $\tilde{\gamma}^{-1}(i) = \{1\}$ is nonempty, we define

$$\widetilde{\mathsf{F}}_{\tilde{\gamma}^{-1}(i)} = \mathsf{F}_0 \ .$$

If $\tilde{\gamma}^{-1}(i) = \emptyset$, then

$$\widetilde{\mathsf{F}}_{\emptyset} = \langle \exp(-Z_1) \rangle = 1 - x.$$

Let $q_1 > 1$ be the largest part of $\mathbf{q}$. Then

$$\mathrm{Coeff}(\widetilde{F}_{\tilde{\gamma}^{-1}(1)}, x^{q_1}) = 0$$

by Lemmas 3 and 5. Hence,

$$\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)] = 0,$$

and the lower-triangularity of $\mathsf{L}_0(d)$ is fully proven.

The proof of Proposition 4 is complete. Following the implications back, the proof of Theorem 1 is also complete. □

Since we know explicitly the diagonal elements of the triangular matrices $\mathsf{Y}_0(d)$ and $\mathsf{L}_0(d)$, the product

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) = \mathsf{L}_0(d)$$

yields a simple formula for the determinant,

$$\det(\mathsf{X}_{0,d}) = (-1)^{d-1} \prod_{\mathbf{p} \in P_0(d) \setminus \{(1^d)\}} \left(\frac{|\mathrm{Aut}(\widehat{\mathbf{p}})|}{\prod_{i=1}^{\ell(\mathbf{p})} p_i^{p_i - 2}}\ (-1)^{\ell(\mathbf{p})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i} \right) .$$

5. Gorenstein conjecture

5.1. **Proof of Theorem 2.** If $n > 0$, the pairing

$$\kappa^d(M^c_{g,n}) \times R^{2g-3+n-d}(M^c_{g,n}) \to \mathbb{Q}$$

is shown to have rank at least $|P(d, 2g-2+n-d)|$ in Section 6.3 of [11]. Since

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,n}) = |P(d, 2g-2+n-d)|$$

by Theorem 1 and [11], Theorem 2 follows. □

5.2. **Further directions.** Perhaps the universality of Theorem 1 extends to larger subrings of $R^*(M^c_{g,n})$. A natural place to start is the ring

$$S^*(M^c_{g,n}) \subset R^*(M^c_{g,n})$$

generated by all the κ and ψ classes.

Question 2. *Is $S^*(M^c_{g,n})$ canonically a subring of $S^*(M^c_{0,2g+n})$?*

At least the condition $n > 0$ must be imposed in Question 2. How to include the strata classes in a universality statement is not clear.

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