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Integrality for Calabi-Yau 4-folds

Genus 0

Let X be a nonsingular, projective, Calabi-Yau 4-fold. For genus g, curve class $\beta \in H_2(X, \mathbb{Z})$, and cohomology insertions $\gamma_i \in H^*(X, \mathbb{Z})$, let

$$N_{g,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \mathrm{ev}_1^*(\gamma_1) \cdots \mathrm{ev}_n^*(\gamma_n) \ \in \mathbb{Q}$$

denote the associated *primary* Gromov-Witten invariant. Since

$$\text{vir dim } \overline{M}_g(X, \beta) = (\dim X - 3)(1-g) = 1-g,$$

Gromov-Witten theory vanishes for $g \geq 2$.

The constant map contributions are easily determined in terms of the classical cohomology of X. For $D \in H^2(X, \mathbb{Z})$, the genus 1 invariant

$$N_{1,0}(D) = -\frac{1}{24} \int_X c_3(X) \cup D$$

may not be integral.

We measure the degree of β with respect to a fixed ample polarization L on X,

$$\deg(\beta) = \int_\beta c_1(L).$$

All effective curve classes satisfy $\deg(\beta) > 0$. We are only interested here in invariants for classes $\deg(\beta) > 0$.

Integrality in genus 0 is expressed in the following generalization of the Aspinwall-Morrison formula. The invariants $n_{0,\beta}(\gamma_1, \ldots, \gamma_n)$ are uniquely defined by

$$\sum_{\beta > 0} N_{0,\beta}(\gamma_1, \ldots, \gamma_n) q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \sum_{d=1}^{\infty} d^{-3+n} q^{d\beta}. \qquad (1)$$

Conjecture 0: *The invariants* $n_{0,\beta}(\gamma_1,\ldots,\gamma_n)$ *are integers.*

Meeting invariants

For nonzero classes $\beta_1,\beta_2 \in H_2(X,\mathbb{Z})$, let $\triangle_{\beta_1,\beta_2}$ denote the virtual boundary divisor

$$\triangle_{\beta_1,\beta_2} \xrightarrow{\iota} \overline{M}_{0,0}(X,\beta_1+\beta_2)$$

corresponding to reducible nodal curves with degree splitting of type (β_1,β_2). In the balanced case $\beta_1=\beta_2$, an ordering is taken in $\triangle_{\beta_1,\beta_2}$, and ι is of degree 2.

The virtual dimension of $\triangle_{\beta_1,\beta_2}$ is 0. Let

$$M_{\beta_1,\beta_2} = \int_{[\triangle_{\beta_1,\beta_2}]^{vir}} 1 \ \in \mathbb{Q}$$

be the associated Gromov-Witten invariant. Let $S_1,\ldots,S_s$ be a basis of $H^4(X,\mathbb{Z})$ mod torsion. Let

$$g^{ij} = \int_X S_i \cup S_j$$

be the intersection form. By the splitting axiom of Gromov-Witten theory,

$$M_{\beta_1,\beta_2} = \sum_{i,j} N_{0,\beta_1}(S_i)\ g^{ij}\ N_{0,\beta_2}(S_j).$$

For $\beta_1,\beta_2 \in H_2(X,\mathbb{Z})$, we define invariants m_{β_1,β_2} virtually enumerating rational curves of class β_1 meeting rational curves of class β_2. The meeting invariants are uniquely determined by the following rules.

(i) The invariants are symmetric,

$$m_{\beta_1,\beta_2} = m_{\beta_2,\beta_1}.$$

(ii) If either $\deg(\beta_1) \leq 0$ or $\deg(\beta_2) \leq 0$, then $m_{\beta_1,\beta_2} = 0$.

(iii) If $\beta_1 \neq \beta_2$, then,

$$m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i)\ g^{ij}\ n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_2}.$$

(iv) In case of equality,

$$m_{\beta,\beta} = n_{0,\beta}(c_2(T_X)) + \sum_{i,j} n_{0,\beta}(S_i)\ g^{ij}\ n_{0,\beta}(S_j) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}.$$

The conjectural integrality of the invariants $n_{0,\beta}(\gamma)$ implies the integrality of the meeting invariants m_{β_1,β_2}. By an elementary argument from the definitions,

$$m_{\beta_1,\beta_2} = 0$$

if either β_1 or β_2 is not effective.

The rules (i-iv) are all derived from geometric consideration. Rules (i) and (ii) are clearly motivated. Rules (iii) and (iv) are understood to apply only when (ii) does not apply.

Rule (iii)

Ideally, there are nonsingular 1-dimensional families F_i of embeded rational curves in X of class β_i. Let

$$\pi_i : S_i \to F_i$$

denote the respective universal family with map

$$\iota_i : S_i \to X.$$

Since $\beta_1 \neq \beta_2$, the families F_1 and F_2 are distinct. Ideally, the surfaces S_i are nonsingular and the maps π_i are smooth except for finitely many 1-nodal fibers. The meeting number m_{β_1,β_2} is related to the intersection

$$\iota_{1*}(S_1) \cap \iota_{2*}(S_2) \subset X. \tag{2}$$

The intersection (2) is not transverse (even ideally). A fiber of π_1 may be a component of a reducible fiber of π_2 or vice versa.

The meeting number m_{β_1,β_2} is defined to count the ideal number of isolated points of the intersection (2). Hence,

$$m_{\beta_1,\beta_2} + \delta = \int_X \iota_{1*}[S_1] \cap \iota_{2*}[S_2]$$

where the correction δ is determined by the non-transeversal intersection loci.

The number of times a fiber of π_1 occurs as a component of a reducible fiber of π_2 is simply $m_{\beta_1,\beta_2-\beta_1}$. Similarly, the opposite event occurs $m_{\beta_1-\beta_2,\beta_2}$ times. The contribution to δ of each non-transversal is easily determined. Let

$$C \subset X$$

be a fiber of π_1 and a component of a reducible fiber of π_2. Then

$$\delta(C) = \int_C c_1(E)$$

where

$$0 \to N_{S_1/C} \oplus N_{S_2/C} \to N_{X/C} \to E \to 0$$

is the normal bundle sequence. Certainly $N_{S_1/C}$ is trivial and $N_{S_2/C}$ has degree -1. By the Calabi-Yau condition, $N_{X/C}$ is of degree -2. Hence,

$$\delta(C) = -1.$$

Rule (iii) is obtained by expanding the intersection (2) via the Künneth decomposition of the diagonal. We have

$$\begin{aligned} m_{\beta_1,\beta_2} &= \int_X \iota_*[S_1] \cap \iota_*[S_2] - \delta \\ &= \sum_{i,j} n_{0,\beta_1}(S_i)\ g^{ij}\ n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_2}. \end{aligned}$$

Of course, for a typical 4-fold X, none of the ideal assumptions hold. The ideal geometry is taken here as motivation for the formulas.

Rule (iv)

In case of equality, the meeting number is more subtle. While the surface S_β is ideally nonsingular and

$$\iota : S_\beta \to X$$

is ideally an immersion, ι is not (even ideally) an embedding. The correct interpretation of $m_{\beta,\beta}$ is twice the number of ideal double points of ι. The factor of 2 arises from ordering.

The double point formula yields a calculation of $m_{\beta,\beta}$ as a correction to the self-intersection,

$$m_{\beta,\beta} = \int_X \iota(S_\beta) \cap \iota(S_\beta) - \int_S \frac{c(T_X)}{c(T_S)}$$

where $c(T_X)$ and $c(T_S)$ denote the total Chern classes of the respective bundles. Expanding the correction term (and using the Calabi-Yau condition) we find

$$\int_S \frac{c(T_X)}{c(T_S)} = \int_S c_2(T_X) + c_1(T_S)^2 - c_2(T_S).$$

Certainly,

$$n_{0,\beta}(c_2(T_X)) = \int_S c_2(T_X).$$

There is a decomposition

$$c_1(T_S) = -\psi + c_1(F_\beta)$$

where ψ is the cotangent line on S viewed as the 1-pointed space. Hence

$$\int_S c_1(T_S)^2 = \int_S \psi^2 + 4\chi(F_\beta).$$

An elementary geometric argument shows

$$\int_S \psi^2 = -\frac{1}{2} \sum_{\beta+\beta_2=\beta} m_{\beta_1,\beta_2}$$

where the right side is number of reducible fibers of π_β. Since

$$\int_S c_2(T_S) = \chi(S) = 2\chi(F_\beta) + \frac{1}{2} \sum_{\beta+\beta_2=\beta} m_{\beta_1,\beta_2},$$

only a calculation of $\chi(F_\beta)$ remains. The formula

$$\chi(F_\beta) = -n_{0,\beta}(c_2(T_X)) + \sum_{\beta+\beta_2=\beta} m_{\beta_1,\beta_2}$$

is obtained via a Grothendieck-Riemann-Roch calculation applied to the deformation theoretic characterization of T_F.

Rule (iv) is obtained by expanding the self-intersection of $\iota_*[S_\beta]$ via the Künneth decomposition of the diagonal and putting together the above surface calculations. We have

$$\begin{aligned} m_{\beta,\beta} &= \int_X \iota_*[S_\beta] \cap \iota_*[S_\beta] - \int_S \frac{c(T_X)}{c(T_S)} \\ &= \sum_{i,j} n_{0,\beta}(S_i)\ g^{ij}\ n_{0,\beta}(S_j) + n_{0,\beta}(c_2(T_X)) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}. \end{aligned}$$

Genus 1

We need only consider invariants $N_{1,\beta}$ with *no* insertions since the virtual dimension is 0. The invariants $n_{1,\beta}$ are uniquely defined by

$$\begin{aligned} \sum_{\beta>0} N_{1,\beta}\ q^\beta \quad = \quad & \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} \\ & + \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(T_X))\ \log(1-q^\beta) \\ & - \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2}\ \log(1-q^{\beta_1+\beta_2}). \end{aligned} \tag{3}$$

The function σ is defined by

$$\sigma(d) = \sum_{i|d} i.$$

The number of automorphism-weighted connected degree d étale covers of an elliptic curve is $\sigma(d)/d$.

Conjecture 1: *The invariants $n_{1,\beta}$ are integers.*

Example I

Let X be the local Calabi-Yau determined by the total space of the rank 2 bundle

$$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathbb{P}^2.$$

X is the simplest toric Calabi-Yau 4-fold with nontrivial Gromov-Witten theory. By localization,

$$N_{0,d}(P) = \frac{(-1)^d}{2d^2}\binom{2d}{d}$$

where P is the point class on $\mathbb{P}^2$. The genus 1 series is

$$\begin{aligned} F_1 &= \sum_{d>0} N_{1,d} q^d \\ &= \frac{1}{12}\log\left(\sum_{d\geq 0}(-1)^d\binom{2d}{d}q^d\right) \\ &= -\frac{1}{24}\log(1+4q) \end{aligned}$$

The integral invariants $n_{0,d}(P)$ and $n_{1,d}$ can be easily calculated from the Gromov-Witten invariants by the defining formulas (1) and (3).

	1	2	3	4	5	6	7	8	9	10
$n_{0,d}(P)$	-1	1	-1	2	-5	13	-35	100	-300	925
$n_{1,d}$	0	0	-1	2	-8	27	-90	314	-1140	4158

The underlying moduli space of maps (with the point condition imposed) for the invariants $n_{0,1}(P)$ and $n_{0,2}(P)$ are projective spaces of dimension 1 and 4 respectively. It appears when the underlying moduli space is $\mathbb{P}^k$, the invariant is $(-1)^k$ reminiscent of Seiberg-Witten theory for surfaces.

The genus 1 invariants vanish in case there are no embedded genus 1 curves. The underlying moduli space for $n_{1,3}$ is $\mathbb{P}^9$.

Example II

Let Y be the local Calabi-Yau determined by the total space of the rank 2 bundle

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\to\mathbb{P}^1\times\mathbb{P}^1.$$

Appropriate localization formulas for Y in genus 0 and 1 yield

$$N_{0,(d_1,d_2)}(P)=\sum_{m\in P(d_1)}\sum_{n\in P(d_2)}\frac{(-1)^{d_1+d_2}}{\mathfrak{z}(m)\mathfrak{z}(n)}\cdot$$
$$\int_{\overline{M}_{0,\ell(m)+\ell(n)+1}}\frac{1}{\prod_{i=1}^{\ell(m)}(1+m_i\psi_i)\ \prod_{j=1}^{\ell(n)}(1-n_j\psi_{\ell(m)+j})}$$

and

$$N_{1,(d_1,d_2)}=\sum_{m\in P(d_1)}\sum_{n\in P(d_2)}\frac{(-1)^{d_1+d_2}}{\mathfrak{z}(m)\mathfrak{z}(n)}\cdot$$
$$\int_{\overline{M}_{1,\ell(m)+\ell(n)}}\frac{1}{\prod_{i=1}^{\ell(m)}(1+m_i\psi_i)\ \prod_{j=1}^{\ell(n)}(1-n_j\psi_{\ell(m)+j})}.$$

Here, P is the point class on $\mathbb{P}^1\times\mathbb{P}^1$, and $P(d)$ denotes the set of partitions of d. For $p\in P(d)$, the length is denoted by $\ell(p)$. The function

$$\mathfrak{z}(p)=|\mathrm{Aut}(p)|\cdot\prod_{i=1}^{\ell(p)}p_i$$

is the usual factor.

The above localization sums admit exact evaluations. In genus 0,

$$N_{0,(d_1,d_2)}(P)=\frac{1}{(d_1+d_2)^2}\binom{d_1+d_2}{d_1}^2.$$

The genus 1 series is

$$\begin{aligned} F_1 &= \sum_{(d_1,d_2)\neq(0,0)} N_{1,(d_1,d_2)} q_1^{d_1} q_2^{d_2} \\ &= \frac{1}{12}\log\left(\sum_{d_1\geq 0}\sum_{d_2\geq 0}\binom{d_1+d_2}{d_1}^2 q_1^{d_1}q_2^{d_2}\right). \end{aligned}$$

The integral invariants $n_{0,(d_1,d_2)}(P)$ and $n_{1,(d_1,d_2)}$ can be easily calculated from the Gromov-Witten invariants by the defining formulas (1) and (3).

$n_{0,(d_1,d_2)}(P)$	0	1	2	3	4	5	6
0	*	1	0	0	0	0	0
1	1	1	1	1	1	1	1
2	0	1	2	4	6	9	12
3	0	1	4	11	25	49	87
4	0	1	6	25	76	196	440
5	0	1	9	49	196	635	1764
6	0	1	12	87	440	1764	5926

$n_{1,(d_1,d_2)}$	0	1	2	3	4	5	6
0	*	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	1	2	5	8	14
3	0	0	2	10	28	68	144
4	0	0	5	28	112	350	922
5	0	0	8	68	350	1370	4426
6	0	0	14	144	922	4426	17220

For $n_{0,(1,d)}(P)$ the underlying moduli space is $\mathbb{P}^{2d}$. The elliptic invariants vanish in classes in which there are no embedded elliptic curves. For $n_{1,(2,2)}$ the moduli space is $\mathbb{P}^8$.

Example III

Let Z be the local Calabi-Yau determined by the total space of the rank 1 bundle

$$\mathcal{O}_{\mathbb{P}^3}(-4) \to \mathbb{P}^3.$$

Unfortunately, a closed formula for the Gromov-Witten invariants is not yet available. However, Mayr has calculated up to degree 6 via the localization formula.

	1	2	3	4	5	6
$N_{0,d}(L)$	-20	-825	$-\frac{612560}{9}$	$-\frac{29946585}{4}$	$-\frac{4825194504}{5}$	$-\frac{412709577260}{3}$
$N_{1,d}$	$-\frac{25}{3}$	$-\frac{2425}{6}$	$-\frac{204700}{9}$	$\frac{688375}{12}$	492685322	$\frac{1433052348850}{9}$

Here, L is the line class on $\mathbb{P}^3$. The integer invariants are tabulated below. The genus 0 invariants $n_{0,d}(L)$ were already calculated by Mayr.

	1	2	3	4	5	6
$n_{0,d}(L)$	-20	-820	-68060	-7486440	-965038900	-137569841980
$n_{1,d}$	0	0	11200	3747900	963762432	225851278400

Futher directions

The local toric Calabi-Yau 4-folds X and Y above are particularly easy to study since they are exactly solvable. The local toric Calabi-Yau Z is is already more complicated. More difficult still is the sextic 4-fold in $\mathbb{P}^5$. Hopefully, the last two example will admit solutions via the holomorphic anomaly equation.

Another direction is to find formulas for integral invariants for higher dimensional Calabi-Yau spaces. The genus 0 formulas are clear. There is no doubt that genus 1 formulas can be written using the ingredients already present in dimension 4. As the dimension increases, meeting invariants for arbitrary numbers of rational curves will appear.

Also, integral invariants for all dimensional targets without the Calabi-Yau assumption surely exist.