Resolved surface double point

Rahul — September 2005

1

Let (ξ_1, ξ_2) be coordinates on the 2-dimensional algebraic torus T. Consider the standard T-action on $\mathbb{C}^2$:

$$(\xi_1, \xi_2) \cdot (z_1, z_2) \mapsto (\xi_1 z_1, \xi_2 z_2).$$

Let t_1 and t_2 denote the weights of the associated 1-dimensional T-representations. The standard T-action commutes with the $\mathbb{Z}/2\mathbb{Z}$-action generated by negation,

$$(z_1, z_2) \mapsto (-z_1, -z_2).$$

The associated $\mathbb{Z}/2\mathbb{Z}$ quotient,

$$\mathbb{C}^2 / \ \mathbb{Z}/2\mathbb{Z},$$

admits an equivariant T-action and a T-equivariant crepant resolution,

$$S \to \mathbb{C}^2 / \ \mathbb{Z}/2\mathbb{Z},$$

where S is the total space of the cotangent bundle $\mathcal{O}(-2)$ on $\mathbf{P}^1$.

Both $\mathbb{C}^2$ and S carry T-equivariant holomorphic symplectic forms. The T-representations determined by the holomorphic symplectic forms have weight $-t_1 - t_2$.

2

We study here the T-equivariant Gromov-Witten theory of S. The moduli space of (connected) stable maps to S of *positive* degree,

$$\overline{M}_{g,k}(S, d),$$

is compact as the image must cover the zero section $Z \subset S$. Hence, the T-equivariant Gromov-Witten theory of S is well-defined with virtual dimension $g-1+k$. In degree 0, the T-fixed locus is compact and the T-equivariant theory is well-defined by residues.

Since S carries a T-equivariant holomorphic symplectic form, the T-equivariant virtual class of the moduli space of positive degeee maps to S contains a trivial factor of weight t_1+t_2. The trivial factor has weight equal to the *dual* of the representation defined by the holomorphic symplectic form. We are specifically interested here in the T-equivariant Gromov-Witten invariants of S which are proportional to t_1+t_2.

3

The residue definition yields the following evaluations of the T-equivariant Gromov-Witten theory of S proportional to t_1+t_2 in degree 0.

We consider descendents of the identity and ω in in the T-equivariant Gromov-Witten theory of S:

$$\langle \prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega) \rangle_{g,0}^{S,T}.$$

where $\omega = -\frac{1}{2}[Z] \in H_T^*(S,\mathbb{Z})$.

In order for the T-equivariant Gromov-Witten to be proportional to t_1+t_2, the dimension constraint,

$$\sum_{i=1}^{n} r_i + \sum_{j=1}^{m} s_j = g+n,$$

must hold.

The formulas evaluating the T-equivariant Gromov-Witten theory of S proportional to t_1+t_2 in degree 0 separate into two cases. First, for $g>0$,

$$\langle \prod_{i=1}^{n} \tau_{r_i}(1) \cdot \tau_{s_1}(\omega)\tau_{s_2}(\omega) \rangle_{g,0}^{S,T} = -2(t_1+t_2) \int_{\overline{M}_{g,n+2}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{r_i} \cdot \psi_{n+1}^{s_1} \psi_{n+2}^{s_2}.$$

The Hodge integrals are evaluated via Faber's formula,

$$\begin{aligned}\int_{\overline{M}_{g,k}} \lambda_g \lambda_{g-1} \psi_1^{\alpha_1} \cdots \psi_k^{\alpha_k} &= \frac{(2g+k-3)!(2g-1)!!}{(2g-1)! \prod_{i=1}^k (\alpha_i - 1)!!} \frac{|B_{2g}|}{2^{2g-1}(2g-1)!!\ 2g} \\ &= \frac{(2g+k-3)!}{(2g)! \prod_{i=1}^k (\alpha_i - 1)!!} \frac{|B_{2g}|}{2^{2g-1}},\end{aligned}$$

for $\alpha_i > 0$. After the application of the string equation, only Hodge integrals with all descendent indexes $\alpha_i > 0$ need be considered.

Finally, for $g = 0$, a special evaluation is easily calculated from the definitions:

$$\langle \prod_{i=1}^n \tau_{r_i}(1) \cdot \tau_{s_1}(\omega)\tau_{s_2}(\omega)\tau_{s_3}(\omega) \rangle_{0,0}^{S,red} = \frac{1}{2} \binom{n}{r_1, \ldots, r_n, s_1, s_2, s_3} (t_1 + t_2).$$

The above evaluations are the only non-vanishing T-equivariant Gromov-Witten invariant of S proportional to $t_1 + t_2$ in degree 0.

4

An alternative approach to the T-equivariant Gromov-Witten theory of S proportional to $t_1 + t_2$ is very natural in *positive* degree.

Since S carries a holomorphic symplectic form, the (non-equivariant) virtual class vanishes,

$$[\overline{M}_{g,k}(S, d)]^{vir} = 0.$$

After removing a trivial factor, a canonical *reduced* obstruction theory is obtained. The reduced theory defines a non-zero virtual class

$$[\overline{M}_{g,k}(S, d)]^{red}$$

of dimension $g + k$.

We define reduced descendent Gromov-Witten invariants of S in positive degree by the usual integral formula:

$$\langle \prod_{i=1}^n \tau_{r_i}(1) \prod_{j=1}^m \tau_{s_j}(\omega) \rangle_{g,d}^{S,red} = \int_{[\overline{M}_{g,n+m}(S,d)]^{red}} \prod_{i=1}^n \psi_i^{r_i} \cup \prod_{j=1}^m \mathrm{ev}_j^*(\omega) \psi_j^{s_j}.$$

Here, ω is more naturally viewed as the pull-back to S of the Poincaré dual of a point on $\mathbb{P}^1$. The above integral vanishes for dimension reasons unless

$$\sum_{i=1}^{n} r_i + \sum_{j=1}^{m} s_j = g + n.$$

If the above dimension constraint holds, then the T-equivariant Gromov-Witten theory of S proportional to $t_1 + t_2$ is related to the reduced theory:

$$\langle \prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega) \rangle_{g,d}^{S,T} = (t_1 + t_2) \langle \prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega) \rangle_{g,d}^{S,red}.$$

5

We first consider the descendents of ω in the reduced Gromov-Witten theory of S.

Proposition 5.1. *The descendents of ω have the following evaluation for $d > 0$:*

$$\langle \tau_{s_1}(\omega) \ldots \tau_{s_m}(\omega) \rangle_{g,d}^{S,red} = d^{2g+m-3} \prod_{j=1}^{m} \frac{s_j!}{(2s_j+1)!} \left(-\frac{1}{2} \right)^{s_j}.$$

Sketch of proof. We calculate the integral by localization. Let $\mathbb{C}^*$ act on $\mathbf{P^1}$ with tangent weights $[+,-]$ at the fixed points

$$0, \infty \in \mathbf{P^1}.$$

The lift of the $\mathbb{C}^*$-action to $\mathcal{O}(-2)$ is specified by the fiber weights $[-,+]$ over the fixed points. We take ω be the equivariant class of the fixed point 0.

The reduced Gromov-Witten invariants of S are calculated by an equivariant integral over the moduli space of maps to $\mathbf{P}^1$:

$$\langle \tau_{s_1}(\omega) \ldots \tau_{s_m}(\omega) \rangle_{g,d}^{S,red} = \int_{[\overline{M}_{g,m}(\mathbb{P}^1,d)]^{vir}} \tilde{e}\Big(R^1\pi_* f^*(\mathcal{O}(-2)) \Big) \prod_{j=1}^{m} \mathrm{ev}_j^*(\omega) \psi_i^{s_j},$$

where $\tilde{e}$ denote the Euler class modified by the reduced condition.

As usual, the virtual localization formula yields an enormous graph sum with Hodge integrals at the vertices. The main observation is, with the above linearization, the only graphs which contribute consist of a genus g_1 vertex with n marking over 0 and an unmarked genus $g - g_2$ vertex over ∞ connected by a degree d Galois cover. For the vanishing of the other graphs, the dimension contraint of the problem is crucial.

Extracting the d dependence of the localization terms is then straightforward:

$$\langle \tau_{s_1}(\omega) \dots \tau_{s_m}(\omega) \rangle_{g,d}^{S,red} = d^{2g+m-3} \langle \tau_{s_1}(\omega) \dots \tau_{s_m}(\omega) \rangle_{g,1}^{S,red}.$$

A catalogue of the d factors:

(i) -1 for the Galois automorphism,

(ii) $3g_1 - g + m - 2 + 1$ for the vertex over 0,

(iii) 0 for the edge,

(iv) $3g_2 - 2 + 1$ for the vertex over ∞.

The degree 1 integral will be evaluated by a second observation.

We consider a very closely related integral in degree 1: the local theory of a surface at an exceptional $\mathbf{P}^1$:

$$\langle \tau_{s_1}(\omega) \dots \tau_{s_m}(\omega) \rangle_{g,1}^{O(-1)} = \int_{[\overline{M}_{g,n}(\mathbb{P}^1,1)]^{vir}} e\Big(R^1\pi_* f^*(\mathcal{O}(-1))\Big) \prod_{j=1}^{m} \mathrm{ev}_j^*(\omega)\psi_j^{s_j},$$

where e is the usual Euler class. The above integral also vanishes unless

$$\sum_{j=1}^{m} s_j = g.$$

We will prove the equality:

$$\langle \tau_{s_1}(\omega) \dots \tau_{s_m}(\omega) \rangle_{g,1}^{O(-1)} = \langle \tau_{s_1}(\omega) \dots \tau_{s_m}(\omega) \rangle_{g,1}^{S,red}.$$

The proof is by examing the localization graph terms for the two integrals. The linearization conventions for left side are specified by choosing the fiber weights $[-, 0]$ for $\mathcal{O}(-1)$. The lifts for the right side are as before. Then, the localization contribution of the vertex over 0 and the edge are identical.

The vertex over ∞ must be carefully examined. Consider first the left side. The Hodge integral of the vertex over ∞ is

$$\begin{aligned}\int_{\overline{M}_{g_2,1}} \frac{(-1)^{g_2}\lambda_{g_2}(-1)^{g_2}\sum_{j=1}^{g_2}\lambda_j}{1+\psi} &= \int_{\overline{M}_{g_2,1}} \frac{\lambda_{g_2}\sum_{j=1}^{g_2}(-1)^j\lambda_j}{1-\psi} \\ &= \frac{1}{(24)^{g_2}g_2!}.\end{aligned}$$

The latter Hodge integral is well-known. Now consider the right side. The Hodge integral of the vertex over ∞ is

$$\begin{aligned}\int_{\overline{M}_{g_2,1}} \frac{(-1)^{g_2}}{1+\psi} &= \int_{\overline{M}_{g_2,1}} \psi^{3g_2-2} \\ &= \frac{1}{(24)^{g_2}g_2!}.\end{aligned}$$

The proof of the equality is degree 1 is finished.

Finally, the local theory of a surface at an exceptional $\mathbb{P}^1$ is simply evaluated by factorization (using the degeneration formula) and localization (using the 2-point series of descendent integrals on the moduli space of curves). $\square$

6

We now consider descendents of the 1 and ω together in the reduced Gromov-Witten theory of S. The extraction of the d dependence in the proof of Proposition 5.1 still holds.

Proposition 6.1. *Degree d integrals are determined by degree 1 integrals:*

$$\langle\prod_{i=1}^{n}\tau_{r_i}(1)\prod_{j=1}^{m}\tau_{s_j}(\omega)\rangle_{g,d}^{S,red} = d^{2g+m-3}\langle\prod_{i=1}^{n}\tau_{r_i}(1)\prod_{j=1}^{m}\tau_{s_j}(\omega)\rangle_{g,1}^{S,red}.$$

The number of descendents of 1 does not affect the d dependence.

Localization in degree 1 yields a simple formula for the reduced Gromov-Witten invariants of S in terms of descendent integrals over the moduli space of curves.

Proposition 6.2. *We have*

$$\langle \prod_{i\in I} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,1}^{S,red} =$$

$$\sum_{g_1+g_2=g} \sum_{I_1\cup I_2=I} \sum_{k,l\geq 0} (-1)^{g+l} \langle \tau_k \prod_{i\in I_1} \tau_{r_i} \prod_{j=1}^{m} \tau_{s_j}\rangle^{\cdot}_{g_1} \langle \tau_l \prod_{i\in I_2} \tau_{r_i}\rangle^{\cdot}_{g_2}.$$

The notation on the right side has the following meaning. The brackets $\langle , \rangle^{\cdot}_{g}$ denote standard descendent integration on the moduli space of curves,

$$\langle \tau_{a_1} \cdots \tau_{a_n}\rangle^{\cdot}_{g} = \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.$$

Special consideration is given to the (degenerate) moduli spaces $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$:

$$\begin{aligned} \langle \tau_k \rangle^{\cdot}_0 &= \delta_{k,0}, \\ \langle \tau_l \rangle^{\cdot}_0 &= \delta_{l,0}, \\ \langle \tau_k \tau_a \rangle^{\cdot}_0 &= (-1)^a \delta_{k,0}, \\ \langle \tau_l \tau_b \rangle^{\cdot}_0 &= -\delta_{l,0}. \end{aligned}$$

The conventions in the degenerate case are slightly different for the two factors on the right side — the insertions τ_k and τ_l indicate where the conventions should be applied.

7

After a few computations using Proposition 6.1, I have obtained a complete conjecture for the evaluations of descendent integrals in the reduced Gromov-Witten theory of S. First, by the string equation,

$$\langle \tau_0(1) \prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,1}^{S,red} =$$

$$\sum_{i=1}^{n} \langle \tau_{r_i-1}(1) \prod_{i'\neq i} \tau_{r_{i'}}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,1}^{S,red}$$

$$+ \sum_{j=1}^{m} \langle \tau_{s_j-1}(\omega) \prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j'\neq j} \tau_{s_{j'}}(\omega)\rangle_{g,1}^{S,red}$$

we can assume the descendents of 1 which occur have positive subscripts.

Conjecture *For $r_i > 0$,*

$$\langle\prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,1}^{S,red} = \gamma(r,n,s,m)\langle\prod_{i=1}^{n} \tau_{r_i-1}(\omega) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,1}^{S,red},$$

where

$$r = \sum_{i=1}^{n} r_i, \quad s = \sum_{j=1}^{m} s_j,$$

and

$$\begin{aligned}\gamma(r,n,s,m) &= (2r+2s-n+m-3)\cdots(2r+2s-2n+m-2) \\ &= \frac{(2r+2s-n+m-3)!}{(2r+2s-2n+m-3)!},\end{aligned}$$

a product of n consecutive integers.

Propositions 5.1 and 6.1 then yield the following closed formula for the reduced Gromov-Witten theory of S:

$$\langle\prod_{i=1}^{n} \tau_{r_i}(1) \prod_{j=1}^{m} \tau_{s_j}(\omega)\rangle_{g,d}^{S,red} =$$

$$d^{2g+m-3}\gamma(r,n,s,m) \prod_{i=1}^{n} \frac{(r_i-1)!}{(2r_i-1)!}\left(-\frac{1}{2}\right)^{r_i-1} \prod_{j=1}^{m} \frac{s_j!}{(2s_j+1)!}\left(-\frac{1}{2}\right)^{s_j}$$

for $r_i > 0$.

8 October update

The conjecture of Section 7 has been proven by Davesh using localization, degeneration, and Virasoro constraints in two surface geometries.