Notes on classes related to equivalent divisors

R. Pandharipande

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I summarize here a discussion with D. Zvonkine concerning three classes on the moduli of pointed curves related to linearly equivalent divisors: the virtual class of rubber maps, the pull-back of the 0-section of the relative Jacobian, and the intrinsic virtual class via Thom-Porteous (proposed by Zvonkine).

1 Three classes

Let $\overline{M}_{g,n}$ be the moduli space of stable curves. Let $A = (a_1, \ldots, a_n)$ be integers satisfying $\sum_{i=1}^{n} a_i = 0$. Let

$$Z_{g,A} \subset M_{g,A}$$

be the locus parameterizing curves $[C, p_1, \ldots, p_n]$ satisfying

$$\mathcal{O}_C\left(\sum_i a_i p_i\right) \stackrel{\sim}{=} \mathcal{O}_C \ .$$

We consider three compactifications of $Z_{g,A}$.

1.1 Rubber

The first is by the moduli space of relative maps to rubber. Let μ be the partition obtained by collecting the strictly positive parts of A. Let ν be the paritions obtained by taking the absolute values of the strictly negative parts of A. Certainly

$$|\mu| = |\nu| \ .$$

Finally, let I be the subset of 0 elements of A. The moduli space

$$\overline{M}_{g,I}(\mathbb{P}^1, \mu, \nu)^\sim$$

parameterizes stable relative maps to rubber with ramifications profiles μ, ν over $0, \infty \in \mathbb{P}^1$ respectively. The tilde indicates a rubber target. The natural morphism

$$\rho : \overline{M}_{g,I}(\mathbb{P}^1, \mu, \nu)^\sim \to \overline{M}_{g,n}$$

is proper and has image containing $Z_{g,A}$. Because of the possibility of contracted curves, the image of ρ is typically larger than the closure $\overline{Z}_{g,A} \subset \overline{M}_{g,n}$.

The virtual dimension of $\overline{M}_{g,I}(\mathbb{P}^1, \mu, \nu)^\sim$ is $2g - 3 + n$, hence

$$R_{g,A} = \rho_*[\overline{M}_{g,I}(\mathbb{P}^1, \mu, \nu)^\sim]^{vir} \in A^g(\overline{M}_{g,n}) \ .$$

In [FP], the class $R_{g,A}$ was proven to lie in the tautological ring. Though the proof is constructive, the method yields rather complicated inductive formulas.

1.2 Jacobians

The second is *not* a full compactification, but lies in the moduli space of curves of compact type $M^{ct}_{g,n} \subset \overline{M}_{g,n}$. Let

$$\pi^{ct} : C^{ct} \to M^{ct}_{g,n}$$

be the universal curve. The data A determine a line bundle L on C^{ct} of relative degree 0,

$$L = \sum_i a_i[P_i]$$

where $P_i \subset C^{ct}$ is the section corresponding to the marking p_i.

By twisting L by components of the π^{ct} inverse image of the boundary divisors of $M^{ct}_{g,n}$, we easily construct a line bundle $\tilde{L}$ which is degree 0 on *every* component of *every* fiber of π^{ct}. The bundle $\tilde{L}$ is not unique — the differences are given by twisting by the π^{ct} inverse images of the boundary divisors of $M^{ct}_{g,n}$.

Via $\tilde{L}$, we obtain a morphism from the moduli of curves of compact type to the degree 0 relative Jacobian,

$$\phi : M^{ct}_{g,n} \to \text{Jac}^0_{g,n}.$$

Certainly the closure of $Z_{g,A}$ in $M^{ct}_{g,n}$ lies in the ϕ inverse image of the 0-section $S \subset \mathrm{Jac}^0_{g,n}$ of the relative Jacobian. We define

$$J_{g,A} = \phi^{-1}[S] \in A^g(M^{ct}_{g,n}) \ .$$

The class $J_{g,A}$ is independent of the choice of $\tilde{L}$.

1.3 Thom-Porteous

We now consider the the universal curve

$$\pi : C \to \overline{M}_{g,n}$$

over the complete moduli space. As above, we have a line bundle L on C of relative degree 0,

$$L = \sum_i a_i[P_i]$$

where $P_i \subset C$ is the section corresponding to the marking p_i. Twisting, as before, by the components of the π inverse image of *reducible* boundary divisors of $\overline{M}_{g,n}$, we obtain a line bundle $\tilde{L}$ which is degree 0 on on *every* component of *every* fiber of π^{ct}. The degree of $\tilde{L}$ need not be 0 on components of fibers over the non-compact type boundary.

Zvonkine proposes a third locus in $V_{g,A} \subset \overline{M}_{g,n}$ defined by the property:

$$[C] \in V_{g,A} \quad \leftrightarrow \quad H^0(C, \tilde{L}_C) \neq 0 \ .$$

Of course $V_{g,A}$ has a natural scheme structure by the following construction. Let

$$[E^0 \xrightarrow{f} E^1]$$

be a 2-term resolution by vector bundles of the complex $R\pi_*(\tilde{L})$ in $D^b_{coh}(\overline{M}_{g,n})$ with

$$\mathrm{rk}(E^0) = e_0, \quad \mathrm{rk}(E^1) = e_1 \ , \quad e_0 - e_1 = 1 - g.$$

By definition, $V_{g,A}$ is the locus where the rank of f is at most $e_0 - 1$. Hence $V_{g,A}$ has a scheme structure obtained from degeneracy locus point of view. The expected dimension of $V_{g,A}$ is

$$(e_0 - (e_0 - 1))(e_1 - (e_0 - 1)) = g \ .$$

So $V_{g,A}$ has a virtual class

$$TP_{g,A} = [V_{g,A}]^{vir} \in A^g(\overline{M}_{g,n})$$

given by the Thom-Porteous formula

$$TP_{g,A} = c_g(E^1 - E^0) = c_g(-R\pi_*(\tilde{L})) \ .$$

By the usual GRR calculation, $TP_{g,A}$ lies in the tautological ring.

The scheme structure and virtual class of $V_{g,A}$ depend neither upon the choice of $\tilde{L}$ nor upon the choice of the 2-term resolution of $R\pi_*(\tilde{L})$.

2 Comparisons

2.1 Compact type

It is natural to ask how these three constructions are related. The question on the compact type locus has a simple answer:

$$R_{g,A} = J_{g,A} = TP_{g,A} \in A^g(M^{ct}_{g,n}) \ .$$

The first equality has been essentially proven by [CMW] — the authors only study the rational tails locus, but the proof appears fine to me over compact type. The second equality is obtained by usual techniques on the relative Jacobian (as observed by Zvonkine). Finally, Hain has a rather simple formula for the class of $J_{g,A} =\in A^g(M^{ct}_{g,n})$, see [H].

2.2 Stable curves

The geometry of stable curves is much more interesting. Set theoretically, we have

$$\overline{Z}_{g,A} \ \subset \ \rho\left(\overline{M}_{g,I}(\mathbb{P}^1, \mu, \nu)^{\sim}\right) \ \subset \ V_{g,A}$$

with all inclusions (in general) proper. A natural question to ask is whether equality hold on the level of classes:

$$R_{g,A} \stackrel{?}{=} TP_{g,A} \in A^g(\overline{M}_{g,n}) \ .$$

With Zvonkine, we have shown this not be the case. The simplest explanation for the failure is the existence of components of expected dimension of $V_{g,A}$ which are *not* surjected upon by ρ.

An interesting question is to find a formula (perhaps recursive) for the difference $R_{g,A} - TP_{g,A}$. An outcome would be a much more effective approach to the classes $R_{g,A}$.

2.3 Example

Consider $\overline{M}_{2,3}$ with $A = (-1, -1, 2)$. We describe a codimension 2 locus W of $V_{g,A}$ which is not surjected upon by ρ.

Generically W parameterizes curves with 2 components

$$C = C_1 \cup C_0$$

of genus 1 and 0 respectively. The components meet at two points

$$C_1 \cap C_0 = \{n_1, n_2\}$$

which are nodes of the full curve C. Finally the marking distribution is

$$p_1, p_2 \in C_1, \quad p_3 \in C_0 \ .$$

Certainly W is codimension 2 in $\overline{M}_{2,3}$. Moreover, we have the inclusion $W \subset V_{2,A}$. The line bundle $\tilde{L}_C$ has restrictions

$$\tilde{L}_{C_1} = \mathcal{O}_{C_1}(-p_1 - p_2), \quad \tilde{L}_{C_0} = \mathcal{O}_{C_0}(2p_3),$$

so has a section which is nonzero on C_0 and zero on C_1.

Finally, by inspection, we can prove the image of ρ in W is characterized by the nontrivial condition

$$\mathcal{O}_{C_1}(n_1 + n_2 - p_1 - p_2) \stackrel{\sim}{=} \mathcal{O}_{C_1} \ .$$

[CMW] Cavalieri, Marcus, and Wise, *Polynomial families of tautological classes on* $M^{rt}_{g,n}$.

[FP] Faber and Pandharipande, *Relative maps and tautological classes.*

[H] Hain, *Normal functions and the geometry of the moduli of curves.*