

Three questions in Gromov-Witten theory

R. Pandharipande *

Abstract

Three conjectural directions in Gromov-Witten theory are discussed: Gorenstein properties, BPS states, and Virasoro constraints. Each points to basic structures in the subject which are not yet understood.

2000 Mathematics Subject Classification: 14N35, 14H10 .

Keywords and Phrases: Gromov-Witten theory, moduli of curves.

1. Introduction

Let X be a nonsingular projective variety over \mathbb{C} . Gromov-Witten theory concerns integration over $\overline{M}_{g,n}(X, \beta)$, the moduli space of maps from genus g , n -pointed curves to X representing the class $\beta \in H_2(X, \mathbb{Z})$. While substantial progress in the mathematical study of Gromov-Witten theory has been made in the past decade, several fundamental questions remain open. My goal here is to describe three conjectural directions:

- (i) Gorenstein properties of tautological rings,
- (ii) BPS states for threefolds,
- (iii) Virasoro constraints.

Each points to basic structures in Gromov-Witten theory which are not yet understood. New ideas in the subject will be required for answers to these questions.

2. Gorenstein properties of tautological rings

The study of the structure of the entire Chow ring of the moduli space of pointed curves $\overline{M}_{g,n}$ appears quite difficult at present. As the principal motive is to understand cycle classes obtained from algebro-geometric constructions, we

*Department of Mathematics, Princeton University, Princeton, NJ 08544, USA.

may restrict inquiry to the system of *tautological rings*, $R^*(\overline{M}_{g,n})$. The tautological system is defined to be the set of smallest \mathbb{Q} -subalgebras of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}),$$

satisfying the following three properties:

- (i) $R^*(\overline{M}_{g,n})$ contains the cotangent line classes ψ_1, \dots, ψ_n where

$$\psi_i = c_1(L_i),$$

the first Chern class of the i th cotangent line bundle.

- (ii) The system is closed under push-forward via all maps forgetting markings:

$$\pi_* : R^*(\overline{M}_{g,n}) \rightarrow R^*(\overline{M}_{g,n-1}).$$

- (iii) The system is closed under push-forward via all gluing maps:

$$\pi_* : R^*(\overline{M}_{g_1, n_1 \cup \{*\}}) \otimes_{\mathbb{Q}} R^*(\overline{M}_{g_2, n_2 \cup \{\bullet\}}) \rightarrow R^*(\overline{M}_{g_1+g_2, n_1+n_2}),$$

$$\pi_* : R^*(\overline{M}_{g_1, n_1 \cup \{*, \bullet\}}) \rightarrow R^*(\overline{M}_{g_1+1, n_1}).$$

Natural algebraic constructions typically yield Chow classes lying in the tautological ring. See [7], [18] for further discussion.

Consider the following basic filtration of the moduli space of pointed curves:

$$\overline{M}_{g,n} \supset M_{g,n}^c \supset M_{g,n}^{rt} \supset C_{g,n}.$$

Here, $M_{g,n}^c$ denotes the moduli of pointed curves of compact type, $M_{g,n}^{rt}$ denotes the moduli of pointed curves with rational tails, and $C_{g,n}$ denote the moduli of pointed curves with a fixed stabilized complex structure C_g . The choice of C_g will play a role below.

The tautological rings for the elements of the filtration are defined by the images of $R^*(\overline{M}_{g,n})$ in the associated quotient sequence:

$$R^*(\overline{M}_{g,n}) \rightarrow R^*(M_{g,n}^c) \rightarrow R^*(M_{g,n}^{rt}) \rightarrow R^*(C_{g,n}) \rightarrow 0 \quad (2.1)$$

Remarkably, the tautological rings of the strata are conjectured to resemble cohomology rings of compact manifolds.

A finite dimension graded algebra R is Gorenstein with socle in degree s if there exists an evaluation isomorphism,

$$\phi : R^s \xrightarrow{\sim} \mathbb{Q},$$

for which the bilinear pairing induced by the ring product,

$$R \times R \rightarrow R \xrightarrow{\phi} \mathbb{Q},$$

is nondegenerate. The cohomology rings $H^*(M, \mathbb{Q})$ of compact manifolds are Gorenstein algebras.

Conjecture 1. *The tautological rings of the filtration of $\overline{M}_{g,n}$ are finite dimensional Gorenstein algebras.*

The Gorenstein structure of $R^*(M_g)$ with socle in degree $g - 2$ was discovered by Faber in his study of the Chow rings of M_g in low genus. The general conjecture is primarily motivated by Faber's original work and can be found in various stages in [5], [19], and [7].

The application of the conjecture to the stratum $C_{g,n}$ takes a special form due to the choice of the underlying curve C_g . The conjecture is stated for a nonsingular curve C_g defined over $\overline{\mathbb{Q}}$ or, alternatively, for the tautological ring in $H^*(C_{g,n}, \mathbb{Q})$. The tautological ring of $C_{g,n}$ in Chow is *not* Gorenstein for all C_g by recent results of Green and Griffiths.

Two main questions immediately arise if the tautological rings are Gorenstein algebras:

- (i) Can the ring structure be described explicitly?
- (ii) Are the tautological rings associated to embedded compact manifolds in the moduli space of pointed curves?

The tautological ring structures are implicitly determined by the conjectural Gorenstein property and the Virasoro constraints [10].

As the moduli space of curves may be viewed as a special case of the moduli space of maps, a development of these ideas may perhaps be pursued more fully in Gromov-Witten theory. It is possible to define a tautological ring for $\overline{M}_{g,n}(X, \beta)$ in the context of the virtual class by *assuming* the Gorenstein property, but no structure has been yet been conjectured. Again, the Virasoro constraints in principle determine the tautological rings.

3. BPS states for threefolds

Let X be a nonsingular projective variety over \mathbb{C} of dimension 3. Let $\{\gamma_a\}_{a \in A}$ be a basis of $H^*(X, \mathbb{Z})$ modulo torsion. Let $\{\gamma_a\}_{a \in D_2}$ and $\{\gamma_a\}_{a \in D_{>2}}$ denote the classes of degree 2 and degree greater than 2 respectively. The Gromov-Witten invariants of X are defined by integration over the moduli space of stable maps (against the virtual fundamental class):

$$\langle \gamma_{a_1}, \dots, \gamma_{a_n} \rangle_{g, \beta}^X = \int_{[\overline{M}_{g,n}(X, \beta)]^{vir}} \text{ev}_1^*(\gamma_{a_1}) \dots \text{ev}_n^*(\gamma_{a_n}), \quad (3.1)$$

where ev_i is the i th evaluation map. As the moduli spaces are Deligne-Mumford stacks, the Gromov-Witten invariants take values in \mathbb{Q} .

Let $\{t_a\}$ be a set of variables corresponding to the classes $\{\gamma_a\}$. The Gromov-Witten potential $F^X(t, \lambda)$ of X may be written,

$$F^X = F_{\beta=0}^X + \tilde{F}^X, \quad (3.2)$$

as a sum of constant and nonconstant map contributions.

The constant map contribution $F_{\beta=0}^X$ may be further divided by genus:

$$F_{\beta=0}^X = F_{\beta=0}^0 + F_{\beta=0}^1 + \sum_{g \geq 2} F_{\beta=0}^g.$$

The genus 0 constant contribution records the classical intersection theory of X :

$$F_{\beta=0}^0 = \lambda^{-2} \sum_{a_1, a_2, a_3 \in A} \frac{t_{a_3} t_{a_2} t_{a_1}}{3!} \int_X \gamma_{a_1} \cup \gamma_{a_2} \cup \gamma_{a_3}.$$

The genus 1 constant contribution is obtained from a virtual class calculation:

$$F_{\beta=0}^1 = \sum_{a \in D_2} t_a \langle \gamma_a \rangle_{g=1, \beta=0}^X = - \sum_{a \in D_2} \frac{t_a}{24} \int_X \gamma_a \cup c_2(X).$$

Similarly, the genus $g \geq 2$ contributions are

$$F_{\beta=0}^g = \langle 1 \rangle_{g, \beta=0}^X = (-1)^g \frac{\lambda^{2g-2}}{2} \int_X (c_3(X) - c_1(X) \cup c_2(X)) \cdot \int_{\overline{M}_g} \lambda_{g-1}^3.$$

The Hodge integrals which arise here have been computed in [6]:

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}| |B_{2g-2}|}{2g \cdot 2g-2} \frac{1}{(2g-2)!},$$

where B_{2g} and B_{2g-2} are Bernoulli numbers. The constant map contributions to F^X are therefore completely understood.

The second term in (3.2) is the nonconstant map contribution:

$$\tilde{F}^X = \sum_{g \geq 0} \sum_{\beta \neq 0} F_{\beta}^g.$$

Since the virtual dimension of the moduli space $\overline{M}_g(X, \beta)$ is

$$\int_{\beta} c_1(X) + 3g - 3 + 3 - 3g = \int_{\beta} c_1(X),$$

the classes β satisfying $\int_{\beta} c_1(X) < 0$ do not contribute to the potential F^X . Therefore, \tilde{F}^X may be divided into two sums:

$$\begin{aligned} \tilde{F}^X &= \sum_{g \geq 0} \sum_{\beta \neq 0, \int_{\beta} c_1(X) = 0} F_{\beta}^g \\ &+ \sum_{g \geq 0} \sum_{\beta \neq 0, \int_{\beta} c_1(X) > 0} F_{\beta}^g. \end{aligned}$$

In case $\beta \neq 0$, we will write the series $F_{\beta}^g(t, \lambda)$ in the following form:

$$F_{\beta}^g = \lambda^{2g-2} q^{\beta} \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \dots, a_n \in D_{>2}} t_{a_n} \cdots t_{a_1} \langle \gamma_{a_1} \cdots \gamma_{a_n} \rangle_{g, \beta}^X.$$

The degree 2 variables $\{t_a\}_{a \in D_2}$ are formally suppressed in q via the divisor equation:

$$q^\beta = \prod_{a \in D_2} q_a^{\int_\beta \gamma_a}, \quad q_a = e^{t_a}.$$

Cohomology classes of degree 0 or 1 do not appear in nonvanishing Gromov-Witten invariants (3.1) for curve classes $\beta \neq 0$.

We will define new invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$ for every genus g , curve class $\beta \neq 0$, and classes $\gamma_{a_1}, \dots, \gamma_{a_n}$. Our primary interest will be in the case where the following conditions hold:

- (i) $\deg(\gamma_{a_i}) > 2$ for all i .
- (ii) $n + \int_\beta c_1(X) = \sum_{i=1}^n \deg(\gamma_{a_i})$.

The invariants will be defined to satisfy the divisor equation (which allows for the extraction of degree 2 classes γ_a) and defined to vanish if degree 0 or 1 classes are inserted or if condition (ii) is violated. If $\int_\beta c_1(X) = 0$, then n_β^g is well-defined without cohomology insertions.

The new invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$ are defined via Gromov-Witten theory by the following equation:

$$\begin{aligned} \tilde{F}^X &= \sum_{g \geq 0} \sum_{\beta \neq 0, \int_\beta c_1(X) = 0} n_\beta^g \lambda^{2g-2} \sum_{d > 0} \frac{1}{d} \left(\frac{\sin(d\lambda/2)}{\lambda/2} \right)^{2g-2} q^{d\beta} \\ &+ \sum_{g \geq 0} \sum_{\beta \neq 0, \int_\beta c_1(X) > 0} \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \dots, a_n \in D_{>2}} t_{a_n} \cdots t_{a_1} \\ &\quad \cdot n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n}) \lambda^{2g-2} \left(\frac{\sin(\lambda/2)}{\lambda/2} \right)^{2g-2 + \int_\beta c_1(X)} q^\beta. \end{aligned}$$

The above equation uniquely determines the invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$.

Conjecture 2. *For all nonsingular projective threefolds X ,*

- (i) *the invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$ are integers,*
- (ii) *for fixed β , the invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$ vanish for all sufficiently large genera g .*

If X is a Calabi-Yau threefold, the Gopakumar-Vafa conjecture is recovered [15], [16]. Here, the invariants n_β^g arise as BPS state counts in a study of Type IIA string theory on X via M-theory. The outcome is a physical deduction of the conjecture in the Calabi-Yau case.

Gopakumar and Vafa further propose a mathematical construction of the Calabi-Yau invariants n_β^g using moduli spaces of sheaves on X . The invariants n_β^g should arise as multiplicities of special representations of \mathfrak{sl}_2 in the cohomology of the moduli space of sheaves. The local Calabi-Yau threefold consisting of a curve C together with a rank 2 normal bundle N satisfying $c_1(N) = \omega_C$ should be the most basic case. Here the BPS states n_d^g should be found in the cohomology of an appropriate moduli space of rank d bundles on C . A mathematical development of

the proposed connection between integrals over the moduli of stable maps and the cohomology of the moduli of sheaves has not been completed. However, evidence for the program can be found both in local and global calculations in several cases [1], [20], [21].

The conjecture for arbitrary threefolds is motivated by the Calabi-Yau case together with the degeneracy calculations of [29]. Evidence can be found, for example, in the low genus enumerative geometry of \mathbf{P}^3 [9], [29]. If the conjecture is true, the invariants $n_\beta^g(\gamma_{a_1}, \dots, \gamma_{a_n})$ of \mathbf{P}^3 may be viewed as *defining* an integral enumerative geometry of space curves for all g and β . Classically the enumerative geometry of space curves does not admit a uniform description.

The conjecture does not determine the Gromov-Witten invariants of threefolds. A basic related question is to find some means to calculate higher genus invariants of Calabi-Yau threefolds. The basic test case is the quintic hypersurface in \mathbf{P}^4 . There are several approaches to the genus 0 invariants of the quintic: Mirror symmetry, localization, degeneration, and Grothendieck-Riemann-Roch [2],[3],[8],[11],[23]. But, the higher genus invariants of the quintic are still beyond current string theoretic and geometric techniques. The best tool for the higher genus Calabi-Yau case, the holomorphic anomaly equation, is not well understood in mathematics. On the other hand, all the invariants of \mathbf{P}^3 may be in principle calculated by virtual localization [17].

4. Virasoro constraints

Let X be nonsingular projective variety over \mathbb{C} of dimension r . Let $\{\gamma_a\}$ be basis of $H^*(X, \mathbb{C})$ homogeneous with respect to the Hodge decomposition,

$$\gamma_a \in H^{p_a, q_a}(X, \mathbb{C}),$$

The descendent Gromov-Witten invariants of X are:

$$\langle \tau_{k_1}(\gamma_{a_1}) \cdots \tau_{k_n}(\gamma_{a_n}) \rangle_{g, \beta}^X = \int_{[\overline{M}_{g, n}(X, \beta)]^{vir}} \psi_1^{k_1} \text{ev}_1^*(\gamma_{a_1}) \cdots \psi_n^{k_n} \text{ev}_n^*(\gamma_{a_n}).$$

Let $\{t_a^k\}$ be a set of variables. Let $F^X(t, \lambda)$ be the generating function of the descendent invariants:

$$F^X = \sum_{g \geq 0} \lambda^{2g-2} \sum_{\beta \in H_2(X, \mathbb{Z})} q^\beta \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{a_1 \dots a_n \\ k_1 \dots k_n}} t_{k_n}^{a_n} \cdots t_{k_1}^{a_1} \langle \tau_{k_1}(\gamma_{a_1}) \cdots \tau_{k_n}(\gamma_{a_n}) \rangle_{g, \beta}^X.$$

The partition function Z^X is formed by exponentiating F^X :

$$Z^X = \exp(F^X). \tag{4.1}$$

We will now define formal differential operators $\{L_k\}_{k \geq -1}$ in the variables t_k^a satisfying the Virasoro bracket,

$$[L_k, L_\ell] = (k - \ell)L_{k+\ell}.$$

The definitions of the operators L_k will depend only the following three structures of $H^*(X, \mathbb{C})$:

- (i) the intersection pairing $g_{ab} = \int_X \gamma_a \cup \gamma_b$,
- (ii) the Hodge decomposition $\gamma_a \in H^{p_a, q_a}(X, \mathbb{C})$,
- (iii) the action of the anticanonical class $c_1(X)$.

The formulas for the operators L_k are:

$$\begin{aligned}
 L_k = & \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \left([b_a + m]_i^k (C^i)_a^b \tilde{t}_m^a \partial_{b, m+k-i} \right. \\
 & \left. + \frac{\lambda^2}{2} (-1)^{m+1} [-b_a - m]_i^k (C^i)^{ab} \partial_{a, m} \partial_{b, k-m-i-1} \right) \\
 & + \frac{\lambda^{-2}}{2} (C^{k+1})_{ab} t_0^a t_0^b \\
 & + \frac{\delta_{k0}}{48} \int_X ((3-r)c_r(X) - 2c_1(X)c_{r-1}(X)),
 \end{aligned}$$

where the Einstein convention for summing over the repeated indices $a, b \in A$ is followed.

Several terms require definitions. For each class γ_a , a half integer b_a is obtained from the Hodge decomposition,

$$b_a = p_a + (1-r)/2.$$

The combinatorial factor $[x]_i^k$ is defined by:

$$[x]_i^k = e_{k+1-i}(x, x+1, \dots, x+k),$$

where e_k is the k th elementary symmetric function. The matrix C_a^b is determined by the action of the anticanonical class,

$$C_a^b \gamma_b = c_1(X) \cup \gamma_a.$$

The indices of C are lowered and raised by the metric g_{ab} and its inverse g^{ab} . The terms \tilde{t}_m^a and $\partial_{a, m}$ are defined by:

$$\begin{aligned}
 \tilde{t}_m^a &= t_m^a - \delta_{a0} \delta_{m1}, \\
 \partial_{a, m} &= \partial / \partial t_m^a,
 \end{aligned}$$

where both are understood to vanish if $m < 0$.

Conjecture 3. *For all nonsingular projective varieties X , $L_k(Z^X) = 0$.*

The conjecture for varieties X with only (p, p) cohomology was made by Eguchi, Hori, and Xiong [4]. The full conjecture involves ideas of Katz. In case X is a point, the constraints specialize to the known Virasoro formulation of Witten's conjecture [22], [30] (see also [25]). After the point, the simplest varieties occur in two basic families: curves C_g of genus g and projective spaces \mathbf{P}^n of dimension n . A proof of the Virasoro constraints for target curves C_g is presented in sequence of papers [26], [27], [28]. Givental has recently proven the Virasoro

constraints for the projective spaces \mathbf{P}^n [12], [13], [14]. The two families of varieties are quite different in flavor. Curves are of dimension 1, but have non- (p, p) cohomology, non-semisimple quantum cohomology, and do not always carry torus actions. Projective spaces cover all target dimensions, but have algebraic cohomology, semisimple quantum cohomology, and always carry torus actions.

The Virasoro constraints are especially appealing from the point of view of algebraic geometry as *all* nonsingular projective varieties are covered. While many aspects of Gromov-Witten theory may be more naturally pursued in the symplectic category, the Virasoro constraints appear to require more than a symplectic structure to define. For example, the bracket

$$[L_1, L_{-1}] = 2L_0,$$

depends upon formulas expressing the Chern numbers,

$$\int_X c_r(X), \quad \int_X c_1(X)c_{r-1}(X),$$

in terms of the Hodge numbers $h^{p,q}$ of X (see [24]).

The Virasoro constraints may be a shadow of a deeper connection between the Gromov-Witten theory of algebraic varieties and integrable systems. In case the target is the point or the projective line, precise connections have been made to the KdV and Toda hierarchies respectively. The connections are proven by explicit formulas for the descendent invariants in terms of matrix integrals (for the point) and vacuum expectation in $\Lambda^{\frac{\infty}{2}} V$ (for the projective line) [22], [25], [27]. The extent of the relationship between Gromov-Witten theory and integrable system is not known. In particular, an understanding of the surface case would be of great interest. Perhaps a link to integrable systems can be found in the circle of ideas involving Hilbert schemes of points, Heisenberg algebras, and Göttsche's conjectures concerning the enumerative geometry of linear series.

Finally, one might expect Virasoro constraints to hold in the context of Gromov-Witten theory relative to divisors in the target X . For the relative theory of 1 dimensional targets X , Virasoro constraints have been found and play a crucial role in the proof of the Virasoro constraints for the absolute theory of X [28].

References

- [1] J. Bryan and R. Pandharipande, *BPS states of curves in Calabi-Yau 3-folds*, *Geom. Topol.* **5** (2001), 287–318.
- [2] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory*, *Nuclear Physics* **B359** (1991), 21–74.
- [3] T. Coates and A. Givental, *Quantum Riemann-Roch, Lefschetz, and Serre*, *math.AG/0110142*.
- [4] T. Eguchi, K. Hori, and C.-S. Xiong, *Quantum cohomology and Virasoro algebra*, *Phys. Lett.* **B402** (1997), 71–80.

- [5] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, in Moduli of Curves and Abelian Varieties (The Dutch Intercity Seminar on Moduli), C. Faber and E. Looijenga, eds., 109–129, Aspects of Mathematics E33, Vieweg: Wiesbaden, 1999.
- [6] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.
- [7] C. Faber and R. Pandharipande, *Logarithmic series and Hodge integrals in the tautological ring*, with an appendix by Don Zagier, Michigan Math. J. **48** (2000), 215–252.
- [8] A. Gathmann, *Relative Gromov-Witten invariants and the mirror formula*, math.AG/0202002.
- [9] E. Getzler, *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), 973–998.
- [10] E. Getzler and R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle*, Nucl. Phys. **B530** (1998), 701–714.
- [11] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices **13** (1996), 613–663.
- [12] A. Givental, *Semisimple Frobenius structures at higher genus*, math.AG/0008067.
- [13] A. Givental, *Gromov-Witten invariants and quantization of quadratic hamiltonians*, math.AG/0108100.
- [14] A. Givental, in preparation.
- [15] R. Gopakumar and C. Vafa, *M-theory and topological strings I*, hep-th/9809187.
- [16] R. Gopakumar and C. Vafa, *M-theory and topological strings II*, hep-th/9812127.
- [17] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.
- [18] T. Graber and R. Pandharipande, *A non-tautological algebraic class on $\overline{M}_{2,22}$* , math.AG/0104057.
- [19] R. Hain and E. Looijenga, *Mapping class groups and moduli spaces of curves*, in Proceedings of Symposia in Pure Mathematics: Algebraic Geometry Santa Cruz 1995, J. Kollár, R. Lazarsfeld, D. Morrison, eds., Volume 62, Part 2, 97–142.
- [20] S. Hosono, M.-H. Saito, and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface*, Adv. Theor. Math. Phys. **1** (1999), 177–208.
- [21] S. Katz, A. Klemm, and C. Vafa, *M-theory, topological strings, and spinning black holes*, hep-th/9910181
- [22] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), 1–23.
- [23] B. Lian, K. Liu, and S.-T. Yau, *Mirror principle I*, Asian J. Math. **1** (1997), 729–763.
- [24] A. Libgober and J. Wood, *Uniqueness of the complex structure on Kähler manifolds of certain homology types*, J. Diff. Geom. **32** (1990), 139–154.
- [25] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers*,

- and matrix models, I*, math.AG/0101147 .
- [26] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, math.AG/0204305.
 - [27] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of \mathbf{P}^1* , in preparation.
 - [28] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, in preparation.
 - [29] R. Pandharipande, *Hodge integrals and degenerate contributions*, Comm. Math. Phys. **208** (1999), 489-506.
 - [30] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243-310.