

The κ classes on the moduli spaces of curves

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In the past few years, substantial progress has been made in the understanding of the algebra of κ classes on the moduli spaces of curves. My goal here is to provide a short introduction to the new results. Along the way, I will discuss several open questions. The article accompanies my talk at *A celebration of algebraic geometry* at Harvard in honor of the 60th birthday of J. Harris.

A. Moduli spaces of curves

Let \mathcal{M}_g be the moduli space of compact nonsingular curves over \mathbb{C} of genus g . We view \mathcal{M}_g as a nonsingular Deligne-Mumford stack of dimension $3g - 3$. We will consider the spaces

$$(1) \quad \mathcal{M}_g \subset \mathcal{M}_g^c \subset \overline{\mathcal{M}}_g .$$

Here, \mathcal{M}_g^c is the moduli space of curves of compact type (curves with no cycles in the dual graph). While \mathcal{M}_g and \mathcal{M}_g^c are open, the moduli space of stable curves $\overline{\mathcal{M}}_g$ is compact. The Chow rings of the moduli spaces (1) are well-defined. Because of stack considerations, we will take the Chow rings with \mathbb{Q} -coefficients.

For the investigation of the κ classes, the moduli spaces of curves with markings

$$\mathcal{M}_{g,n}^{rt} \subset \mathcal{M}_{g,n}^c \subset \overline{\mathcal{M}}_{g,n} .$$

play an essential role and should be treated on the same footing. The moduli space $\mathcal{M}_{g,n}^{rt}$ of curves with rational tails is the inverse image of \mathcal{M}_g under the forgetful map

$$\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_g .$$

B. κ classes

The κ classes in the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ are defined by the following geometry. Let

$$\epsilon : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$$

be the universal curve viewed as the $(n+1)$ -pointed space, let

$$\mathbb{L}_{n+1} \rightarrow \overline{\mathcal{M}}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \in A^1(\overline{\mathcal{M}}_{g,n+1})$$

be the Chern class. The κ classes, first defined by Mumford, are

$$\kappa_i = \epsilon_*(\psi_{n+1}^{i+1}) \in A^i(\overline{\mathcal{M}}_{g,n}), \quad i \geq 0 .$$

The simplest is κ_0 which equals $2g-2+n$ times the unit in $A^0(\overline{\mathcal{M}}_{g,n})$. The convention

$$\kappa_{-1} = \epsilon_*(\psi_{n+1}^0) = 0$$

is often convenient.

The κ classes on $\mathcal{M}_{g,n}^{rt}$ and $\mathcal{M}_{g,n}^c$ are defined via restriction from $\overline{\mathcal{M}}_{g,n}$. Define the κ rings

$$\begin{aligned} \kappa^*(\mathcal{M}_{g,n}^{rt}) &\subset A^*(\mathcal{M}_{g,n}^{rt}), \\ \kappa^*(\mathcal{M}_{g,n}^c) &\subset A^*(\mathcal{M}_{g,n}^c), \\ \kappa^*(\overline{\mathcal{M}}_{g,n}) &\subset A^*(\overline{\mathcal{M}}_{g,n}), \end{aligned}$$

to be the \mathbb{Q} -subalgebras generated by the κ classes. Of course, the κ rings are graded by degree.

Since κ_i is a tautological class the κ rings are subalgebras of the corresponding tautological rings. For unpointed nonsingular curves, the κ ring equals the tautological ring by definition [10],

$$\kappa^*(\mathcal{M}_g) = R^*(\mathcal{M}_g) .$$

Otherwise, the inclusion of the κ ring in the tautological ring is usually proper.

C. M_g and the Faber-Zagier conjecture

Consider first the ring $\kappa^*(\mathcal{M}_g)$. The basic non-vanishing and vanishing results,

$$\kappa^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}, \quad \kappa^{>g-2}(\mathcal{M}_g) = 0$$

have been known now for some time [2, 7]. Moreover, there are no relations [1] among the κ classes of degree less than or equal to $\lfloor \frac{g}{3} \rfloor$, and the classes

$$\kappa_1, \dots, \kappa_{\lfloor \frac{g}{3} \rfloor}$$

generate $\kappa^*(\mathcal{M}_g)$ [6, 9]. These properties had all been conjectured earlier by Faber [2].

We define a set of relations as follows. Let

$$\mathbf{p} = \{ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \dots \}$$

be a variable set indexed by positive integers not congruent to 2 mod 3. Let

$$\begin{aligned} \Psi(t, \mathbf{p}) = & (1 + tp_3 + t^2p_6 + t^3p_9 + \dots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i \\ & + (p_1 + tp_4 + t^2p_7 + \dots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i. \end{aligned}$$

Define the constants $C^r(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) t^r \mathbf{p}^{\sigma}.$$

Here, σ denotes a partition of size $|\sigma|$ which avoids all parts congruent to 2 mod 3. If $\sigma = 1^{n_1} 3^{n_3} 4^{n_4} \dots$, then $\mathbf{p}^{\sigma} = p_1^{n_1} p_3^{n_3} p_4^{n_4} \dots$ as usual. Let

$$\gamma = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) \kappa_r t^r \mathbf{p}^{\sigma}.$$

Theorem 1. *In $\kappa^r(\mathcal{M}_g)$, the relation*

$$[\exp(-\gamma)]_{t^r \mathbf{p}^{\sigma}} = 0$$

holds when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \pmod{2}$.

The relations of Theorem 1, called the *FZ relations*, were conjectured to hold several years ago by Faber and Zagier from low genus data and a study of the Gorenstein quotient of $\kappa^*(\mathcal{M}_g)$. Guessing the full structure here was certainly a remarkable feat. Theorem 1 was proven by myself and A. Pixton [13] last year using the geometry of stable quotients [8].

To the best of our knowledge, a relation in $\kappa^*(\mathcal{M}_g)$ which is not in the span of the FZ relations has not yet been found. In particular, all relations obtained from the various geometrical constructions attempted in the past [2] appear to be covered by Theorem 1. Whether Theorem 1 exhausts all relations in $\kappa^*(\mathcal{M}_g)$ is a very interesting question.

Q1. Are all relations among the κ classes in $\kappa^*(\mathcal{M}_g)$ generated by Theorem 1?

Theorem 1 only provides finitely many relations in $\kappa^r(\mathcal{M}_g)$ for fixed g and r , and thus may be calculated completely. When the relations yield a Gorenstein ring with socle in $\kappa^{g-2}(\mathcal{M}_g)$, no further relations are possible. However, the relations of Theorem 1 do not always yield such a Gorenstein ring (failing first in genus 24 as checked by Faber). For $g < 24$, Faber's calculations show Theorem 1 does provide all relations in $\kappa^*(\mathcal{M}_g)$. For higher genus $g \geq 24$, either Theorem 1 fails to provide all the relations in $\kappa^*(\mathcal{M}_g)$ or $\kappa^*(\mathcal{M}_g)$ is not Gorenstein.

Q2. Is $\kappa^*(\mathcal{M}_g)$ a Gorenstein ring with socle in degree $g - 2$?

Faber's original conjecture [2] asserts an affirmative answer to Q2. Questions Q1 and Q2 can not both have an affirmative answer in genus 24. Which assertion is false?

The main actors in the FZ relations are the functions

$$A(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{z}{72}\right)^i,$$

$$B(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left(\frac{z}{72}\right)^i$$

written here in the variable $z = 72t$. The function B is determined from A by the differential equation

$$-\frac{1}{2}A + zA + 6z^2 \frac{dA}{dz} = \frac{1}{2}B.$$

The main hypergeometric differential equation satisfied by A is

$$(2) \quad 36z^2 \frac{d^2}{dz^2} A + (72z - 6) \frac{d}{dz} A + 5A = 0 .$$

A more open ended question is following.

Q3. What is the meaning of the function A ? Does A or the differential equation (2) occur elsewhere in mathematics?

The statement of Theorem 1 contains a peculiar mod 2 condition. As a consequence, the relations in $\kappa^*(\mathcal{M}_g)$ given by Theorem 1 also hold in $\kappa^*(\mathcal{M}_{g-2})$. Note κ_0 specializes to $2g - 2$ in genus g and $2g - 6$ in genus $g - 2$! In our proof [13] of Theorem 1, the mod 2 condition arises (after a considerable amount of work) from the 2 fixed points of \mathbb{C}^* on \mathbb{P}^1 .

Q4. Is there a simple explanation of the mod 2 condition?

We can also ask about relations on the moduli space $\mathcal{M}_{g,n}^{rt}$ with markings. A. Pixton has a precise proposal for the $n = 1$ case in the form of Theorem 1. Work here is just starting and will be reported by Pixton elsewhere.

D. Universality for $M_{g,n}^c$

Consider next the ring $\kappa^*(\mathcal{M}_{g,n}^c)$. The basic non-vanishing and vanishing results,

$$\kappa^{2g-3+n}(\mathcal{M}_{g,n}^c) \cong \mathbb{Q}, \quad \kappa^{>2g-3+n}(\mathcal{M}_{g,n}^c) = 0$$

have been proven in [4, 5]. The classes

$$\kappa_1, \kappa_2, \dots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor}$$

generate $\kappa^*(\mathcal{M}_{g,n}^c)$. If $n > 0$, there are no relations of degree less than or equal to $g - 1 + \lfloor \frac{n}{2} \rfloor$. In $\kappa^*(\mathcal{M}_g^c)$, there are no relations of degree less than $g - 1$ (whether degree $g - 1$ relations can occur is not known). The proofs of the above generation and freeness results can be found in [12].

A surprising feature about the κ rings in the compact type case is the following universality result proven in [12].

Theorem 2. *Let $g > 0$ and $n > 0$, then the assignment $\kappa_i \mapsto \kappa_i$ extends to a ring isomorphism*

$$\iota : \kappa^*(M_{g-1,n+2}^c) \cong \kappa^*(M_{g,n}^c) .$$

In other words, the relations among the κ classes in the above cases are genus independent. By composing the isomorphisms ι of Theorem 2, we obtain isomorphisms

$$\iota : \kappa^*(M_{0,2g+n}^c) \cong \kappa^*(M_{g,n}^c)$$

so long as $n > 0$. Hence, universality reduces all questions about the κ rings to genus 0.

In [12], calculations of the relations, bases, and Betti numbers of the ring $\kappa^*(M_{g,n>0}^c)$ are obtained using the genus 0 reduction. Let $P(d)$ be the set of partitions of d , and let

$$P(d, k) \subset P(d)$$

be the set of partitions of d into at most k parts. Let $|P(d, k)|$ be the cardinality. To a partition with positive parts $\mathbf{p} = (p_1, \dots, p_\ell)$ in $P(d, k)$, we associate a κ monomial by

$$\kappa_{\mathbf{p}} = \kappa_{p_1} \cdots \kappa_{p_\ell} \in \kappa^d(M_{g,n}^c) .$$

Theorem 3. For $n > 0$, a \mathbb{Q} -basis of $\kappa^d(M_{g,n}^c)$ is given by

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, 2g - 2 + n - d) \} .$$

The Betti number calculation,

$$\dim_{\mathbb{Q}} \kappa^d(M_{g,n}^c) = |P(d, 2g - 2 + n - d)| ,$$

is of course implied by Theorem 3.

Q5. Is there an analogue of Schubert calculus in the κ ring with respect to the basis of Theorem 3?

The tautological rings $R^*(M_{g,n}^c)$ have been conjectured in [3, 11] to be Gorenstein algebras with socle in degree $2g - 3 + n$,

$$\phi : R^{2g-3+n}(M_{g,n}^c) \xrightarrow{\sim} \mathbb{Q} .$$

The following result [12] shows the socle evaluation is as non-trivial as possible on the κ ring.

Theorem 4. If $n > 0$ and $\xi \in \kappa^d(M_{g,n}^c) \neq 0$, the linear function

$$L_{\xi} : R^{2g-3+n-d}(M_{g,n}^c) \rightarrow \mathbb{Q}$$

defined by the socle evaluation

$$L_{\xi}(\gamma) = \phi(\gamma \cdot \xi)$$

is non-trivial.

All of the above results for the κ rings in the compact type case require at least 1 marked point. In the unpointed case $n = 0$, half of the universality still holds: the assignment $\kappa_i \mapsto \kappa_i$ extends to a surjection

$$\iota_g : \kappa^*(M_{0,2g}^c) \rightarrow \kappa^*(M_g^c) .$$

However, a non-trivial kernel is possible. The first kernel occurs in genus $g = 5$.

Q6. What is the kernel of $\kappa^*(M_{0,2g}^c) \rightarrow \kappa^*(M_g^c)$?

In genus 5, the kernel of ι_5 is related to Getzler's relation in $\overline{M}_{1,4}$, see [12] for a discussion. A complete answer to Q6 will likely involve sequences of special relations in the tautological ring.

A natural question to ask is whether the κ relations in the compact type case can be put in a form parallel to Theorem 1. An affirmative answer has been found by A. Pixton.

We define a set of relations as follows. Let

$$\mathbf{p} = \{ p_1, p_2, p_3, \dots \}$$

be a variable set indexed by all positive integers. Let

$$\begin{aligned} \Psi(t, \mathbf{p}) = (1 + tp_2 + t^2p_4 + t^3p_6 + \dots) \sum_{i=0}^{\infty} (2i-1)!! t^i \\ + (p_1 + tp_3 + t^2p_5 + \dots) \end{aligned}$$

where $(2i-1)!! = \frac{(2i)!}{2^i i!}$ as usual. Define the constants $C^r(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) t^r \mathbf{p}^{\sigma} .$$

Here, σ denotes any partition with positive parts. Let

$$\gamma = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) \kappa_r t^r \mathbf{p}^{\sigma} .$$

Theorem 5. [Pixton] *In $\kappa^r(\mathcal{M}_{g,n}^c)$, the relation*

$$[\exp(-\gamma)]_{t^r \mathbf{p}^{\sigma}} = 0$$

holds when $2g - 2 + n + |\sigma| < 2r$.

Pixton's proof uses the Gorenstein criterion of Theorem 4 to check the relations. Furthermore, Pixton proves the relations of Theorem 5 generate *all* the relations among κ classes in $\kappa^*(\mathcal{M}_{g,n}^c)$ in case $n > 0$.

E. Moduli of stable curves

The socle evaluations of polynomials in the κ classes are known for all three cases

$$\kappa^*(\mathcal{M}_{g,n}^{rt}), \quad \kappa^*(\mathcal{M}_{g,n}^c), \quad \kappa^*(\overline{M}_{g,n}) .$$

The κ evaluations can be transformed to Hodge integrals. The relevant evaluations of the latter are summarized in [3]. The socle evaluation in the stable curve case is obtained from the proof of Witten's conjecture [14].

The socle evaluations imply relations in the top degree

$$\kappa^{g-2+n}(\mathcal{M}_{g,n}^{rt}), \quad \kappa^{2g-3+n}(\mathcal{M}_{g,n}^c), \quad \kappa^{3g-3+n}(\overline{M}_{g,n}) .$$

The investigation of the κ ring concerns relations in all degrees. In the stable curve case, no uniform results are known at the moment for κ relations above the socle.

Q7. Is there a formula for relations in $\kappa^*(\overline{M}_{g,n})$ parallel to Theorem 1 and Theorem 5?

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