# ABELIAN HURWITZ-HODGE INTEGRALS

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ABSTRACT. Hodge classes on the moduli space of admissible covers with monodromy group $G$ are associated to irreducible representations of $G$. We evaluate all linear Hodge integrals over moduli spaces of admissible covers with abelian monodromy in terms of multiplication in an associated wreath group algebra. In case $G$ is cyclic and the representation is faithful, the evaluation is in terms of double Hurwitz numbers. In case $G$ is trivial, the formula specializes to the well-known result of Ekedahl-Lando-Shapiro-Vainshtein for linear Hodge integrals over the moduli space of curves in terms of single Hurwitz numbers.

CONTENTS



## 0. INTRODUCTION

0.1. **Moduli of covers.** Let $\mathcal{M}_{g,n}$ be the moduli space of nonsingular, connected, genus $g$ curves over $\mathbb{C}$ with $n$ distinct points. Let $G$ be a finite group. Given an element $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$, we will consider principal $G$-bundles,

$$
(1) \qquad \begin{array}{ccc} G \longrightarrow & & P \\ & & \Big\downarrow \pi \\ & & C \setminus \{p_1, \ldots, p_n\} \ , \end{array}
$$

over the punctured curve. Denote the $G$-action on the fibers of $\pi$ by

$$\tau : G \times P \to P.$$

The monodromy defined by a positively oriented loop around the $i^{th}$ puncture determines a conjugacy class $\gamma_i \in \text{Conj}(G)$. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the $n$-tuple of monodromies. The moduli space

*Date*: December 23, 2008.

of covers $\mathcal{A}_{g,\gamma}(G)$ parameterizes $G$-bundles (1) with the prescribed monodromy conditions. There is a canonical morphism

$$\epsilon : \mathcal{A}_{g,\gamma}(G) \to \mathcal{M}_{g,n}$$

obtained from the base of the $G$-bundle. Both $\mathcal{A}_{g,\gamma}(G)$ and $\mathcal{M}_{g,n}$ are nonsingular Deligne-Mumford stacks.

A compactification $\mathcal{A}_{g,\gamma}(G) \subset \overline{\mathcal{A}}_{g,\gamma}(G)$ by *admissible covers* was introduced by Harris and Mumford in [15]. An admissible cover

$$[\pi, \tau] \in \overline{\mathcal{A}}_{g,\gamma}(G)$$

is a degree $|G|$ finite map of complete curves

$$\pi : D \to (C, p_1, \ldots, p_n)$$

together with a $G$-action

$$\tau : G \times D \to D$$

on the fibers of $\pi$ satisfying the following properties:

(i) $D$ is a possibly disconnected nodal curve,
(ii) $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$ is a stable curve,
(iii) $\pi$ maps the nonsingular points to nonsingular points and nodes to nodes,

$$\pi(D^{ns}) \subset C^{ns}, \quad \pi(D^{sing}) \subset C^{sing},$$

(iv) $[\pi, \tau]$ restricts to a principal $G$-bundle over the punctured nonsingular locus

$$\pi^{open} : D^{open} \to C^{ns} \setminus \{p_1, \ldots, p_n\}$$

with monodromy $\gamma$,
(v) distinct branches of a node $\eta \in D^{sing}$ map to distinct branches of $\pi(\eta) \in C^{sing}$ with equal ramification orders over $\pi(\eta)$,
(vi) the monodromies of the $G$-bundle $\pi^{open}$ determined by the two branches of $C$ at $\eta \in C^{sing}$ lie in opposite conjugacy classes.

Harris and Mumford originally considered only symmetric group $\Sigma_d$ monodromy, but the natural setting for the construction is for all finite $G$.

An admissible cover may be alternatively viewed as a principal $G$-bundle over the stack quotient[1] $[D/G]$ inducing a stable map to the classifying space

$$f : [D/G] \to \mathcal{B}G. \tag{2}$$

Then, $\overline{\mathcal{A}}_{g,\gamma}(G)$ is simply a moduli space of stable maps [2, 5] [2],

$$\overline{\mathcal{A}}_{g,\gamma}(G) \cong \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G).$$

The deformation theory of stable maps endows $\overline{\mathcal{A}}_{g,\gamma}(G)$ with a canonical nonsingular Deligne-Mumford stack structure. We take the stable maps perspective here.

There are two flavors of such stable map theories. If the base $C$ is required to be connected as above, we write $\overline{\mathcal{M}}^{\circ}_{g,\gamma}(\mathcal{B}G)$. If disconnected bases $C$ are allowed, we write $\overline{\mathcal{M}}^{\bullet}_{g,\gamma}(\mathcal{B}G)$. In the

[1] $[D/G]$ differs from $C$ only by possible stack structure at the markings $p_i$ and the nodes. In both cases, the order of the isotropy group is the order of the local monodromy in $G$.

[2] We do not trivialize the marked gerbes on the domain in the definition of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$.

disconnected case, the genus $g$ may be negative. If the superscript is omitted, the connected case is assumed.

Our results are restricted to abelian groups $G$. Here, Conj(G) is the set of elements of $G$. Of course, the cyclic groups $\mathbb{Z}_a$ will play the most important role. In case $G$ is trivial, there is no extra monodromy data, and the moduli space of maps $\overline{\mathcal{M}}_{g,(0,\dots,0)}(\mathcal{B}\mathbb{Z}_1)$ specializes to $\overline{\mathcal{M}}_{g,n}$.

0.2. **Hodge integrals.** Let $R$ be an irreducible $\mathbb{C}$-representation of $G$. If $G$ is abelian, $R$ is a character

$$\phi^R : G \to \mathbb{C}^*.$$

By associating to each map $[f] \in \overline{\mathcal{M}}_{g,\gamma}(G)$ presented as (2) above the $R$-summand of the $G$-representation $H^0(D, \omega_D)$, we obtain a vector bundle

$$\mathbb{E}^R \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \ .$$

The rank of $\mathbb{E}^R$ is locally constant and determined by the orbifold Riemann-Roch formula discussed in Section 1. The *Hodge classes* on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are Chern classes of $\mathbb{E}^R$,

$$\lambda_i^R = c_i(\mathbb{E}^R) \in H^{2i}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The $i^{th}$ cotangent line bundle $L_i$ on the moduli space of curves has fiber

$$L_i|_{(C,p_1,\dots,p_n)} = T^*_{p_i}(C).$$

Descendent classes on $\overline{\mathcal{M}}_{g,n}$ are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Descendent classes $\bar{\psi}_i$ on the space of stable maps are defined by pull-back via the morphism

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$$

to the moduli space of curves,

$$\bar{\psi}_i = \epsilon^*(\psi_i) \in H^2(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The *Hodge integrals* over $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are the top intersection products of the classes $\{\lambda_i^R\}_{R\in Irr(G)}$ and $\{\bar{\psi}_j\}_{1\leq j\leq n}$. Linear Hodge integrals are of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \lambda_i^R \cdot \prod_{j=1}^{n} \bar{\psi}_j^{m_j}.$$

The term *Hurwitz-Hodge integral* was used in [3] to emphasize the role of the covering spaces.

0.3. **Hurwitz numbers.** Let $g$ be a genus and let $\nu$ and $\mu$ be two (unordered) partitions of $d \geq 1$. Let $\ell(\nu)$ and $\ell(\mu)$ denote the lengths of the respective partitions. A Hurwitz cover of $\mathbb{P}^1$ of genus $g$ with ramifications $\nu$ and $\mu$ over $0, \infty \in \mathbb{P}^1$ is a morphism

$$\pi : C \to \mathbb{P}^1$$

satisfying the following properties:

(i) $C$ is a nonsingular, connected, genus $g$ curve,
(ii) the divisors $\pi^{-1}(0), \pi^{-1}(\infty) \subset C$ have profiles equal to the partitions $\nu$ and $\mu$ respectively,
(iii) the map $\pi$ is simply ramified over $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$.

By condition (ii), the degree of $\pi$ must be $d$. Two covers

$$\pi : C \to \mathbb{P}^1, \ \pi' : C' \to \mathbb{P}^1$$

are isomorphic if there exists an isomorphism of curves $\phi : C \to C'$ satisfying $\pi' \circ \phi = \pi$. Each cover $\pi$ has an naturally associated automorphism group $\text{Aut}(\pi)$.

By the Riemann-Hurwitz formula, the number of simple ramification points of $\pi$ over $\mathbb{C}^*$ is

$$r_g(\nu, \mu) = 2g - 2 + \ell(\nu) + \ell(\mu).$$

Let $U_r \subset \mathbb{C}^*$ be a fixed set of $r_g(\nu, \mu)$ distinct points. The set of $r_g(\nu, \mu)^{th}$ roots of unity is the standard choice. The *double Hurwitz number* $H_g(\nu, \mu)$ is a weighted count of the distinct Hurwitz covers $\pi$ of genus $g$ with ramifications $\nu$ and $\mu$ over $0, \infty \in \mathbb{P}^1$ and simple ramification over $U_r$. Each such cover is weighted by $1/|\text{Aut}(\pi)|$. The count $H_g(\nu, \mu)$ does not depend upon the location of the points of $U_r$.

There are two flavors of Hurwitz numbers. The connected case defined above will be denoted $H_g^\circ(\nu, \mu)$. If $C$ is allowed to be disconnected, the Hurwitz count is denoted $H_g^\bullet(\nu, \mu)$. Again, the absence of a superscript indicates the connected theory.

Disconnected Hurwitz numbers are easily expressed as products in the center $\mathcal{Z}\Sigma_d$ of the group algebra of $\Sigma_d$,

$$H_g^\bullet(\nu, \mu) = \frac{1}{d!} \left( C_\nu T^{r_g(\nu,\mu)} C_\mu \right)_{[\text{Id}]} . \tag{3}$$

Here, $C_\nu$ and $C_\mu$ are the sums in the group algebra of all elements of $\Sigma_d$ with cycle types $\nu$ and $\mu$ respectively, and $T$ is the sum of all transpositions. The subscript denotes the coefficient of the identity $[\text{Id}]$.

Multiplication in $\mathcal{Z}\Sigma_d$ is diagonalized by the representation basis. Hurwitz numbers can be written as sums over characters of $\Sigma_d$ and conveniently expressed as matrix elements in the infinite wedge representation. The latter formalism naturally connects Hurwitz numbers to integrable systems [20, 21, 24].

0.4. **Formula for $\mathbb{Z}_a$.** The formula for linear Hodge integrals is simplest in case the monodromy group is $\mathbb{Z}_a$ and the representation $U$ is given by

$$\phi^U : \mathbb{Z}_a \to \mathbb{C}^*, \quad \phi^U(1) = e^{\frac{2\pi i}{a}}.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector[3] of *nontrivial* elements of $\mathbb{Z}_a$,

$$\gamma_i \in \{1, \ldots, a-1\}.$$

Let $\mu$ be a partition of $d \geq 1$ with parts $\mu_j$ and length $\ell$,

$$\sum_{j=1}^{\ell} \mu_j = d.$$

Let $\gamma - \mu$ denote the vector of elements of $\mathbb{Z}_a$ defined by

$$\gamma - \mu = (\gamma_1, \ldots, \gamma_n, -\mu_1, \ldots, -\mu_\ell).$$

[3]The length $n$ may be taken to be 0 in which case $\gamma = \emptyset$.

While the parts of $\mu$ are unordered, an ordering is chosen for $\gamma - \mu$. The vector $\gamma - \mu$ may contain trivial parts. We will consider Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$.

For nonemptiness, the parity condition

$$d - \sum_{i=1}^{n} \gamma_i = 0 \mod a \tag{4}$$

is required. Non-negativity,

$$d - \sum_{i=1}^{n} \gamma_i \geq 0,$$

and boundedness,

$$\forall i \neq j, \ \ \gamma_i + \gamma_j \leq a$$

will also be imposed. If $\gamma = \emptyset$, non-negativity and boundedness are satisfied.

An automorphism of a partition is an element of the permutation group preserving equal parts. Let $|\text{Aut}(\gamma)|$ and $|\text{Aut}(\mu)|$ denote the orders of the automorphism groups.[4] Let $\gamma_+$ be the partition of $d$ determined by adjoining $\frac{d-\sum_{i=1}^{n}\gamma_i}{a}$ parts of size $a$,

$$\gamma_+ = (\gamma_1, \ldots, \gamma_n, a, \ldots, a).$$

A calculation shows

$$r_g(\gamma_+, \mu) = 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^{n} \frac{\gamma_i}{a}.$$

Let the monodromy group $\mathbb{Z}_a$ and representation $\phi^U$ be specified as above. Our main result for linear $\mathbb{Z}_a$-Hodge integrals is the following formula.

**Theorem 1.** *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be nontrivial monodromies in $\mathbb{Z}_a$ satisfying the parity, non-negativity, and boundedness conditions with respect to the partition $\mu$. Then,*

$$H_g(\gamma_+, \mu) =$$

$$\frac{r_g(\gamma_+,\mu)!}{|\text{Aut}(\gamma)|\,|\text{Aut}(\mu)|} a^{1-g-\sum_{i=1}^{n}\frac{\gamma_i}{a}+\sum_{j=1}^{\ell}\langle\frac{\mu_j}{a}\rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor\frac{\mu_j}{a}\rfloor}}{\lfloor\frac{\mu_j}{a}\rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i\lambda_i^U}{\prod_{j=1}^{\ell}(1-\mu_j\bar{\psi}_j)}.$$

The integer and fractional parts of a rational number are denoted in the above formula by

$$q = \lfloor q \rfloor + \langle q \rangle, \ \ q \in \mathbb{Q}.$$

The cotangent lines in the denominator on the far right are associated to the stack points of the stable map domain corresponding to the parts of $\mu$.

Theorem 1 is proven by virtual localization on the moduli space of stable maps to the stack $\mathbb{P}^1[a]$ with $\mathbb{Z}_a$-structure at 0 following the arguments of [9, 12]. The space of stable maps to $\mathbb{P}^1[a]$ is discussed in Section 1, and the proof is given in Section 2. The formula is easily seen to determine *all* linear $\mathbb{Z}_a$-Hodge integrals with respect to $U$ in terms of double Hurwitz numbers. In fact, the set of evaluations with $\gamma = \emptyset$ is sufficient. Conversely, every double Hurwitz number is realized for $a$ sufficiently large.

[4] Here, $\gamma$ is considered as a partition by forgetting the ordering of the elements.

For the disconnected formula, we assume $\gamma = \emptyset$ and the parity condition $d = 0 \pmod{a}$.[5] Then, Theorem 1 holds in exactly the same form,

$$H_g^{\bullet}(\emptyset_+, \mu) = \frac{r_g(\emptyset_+, \mu)!}{|\mathrm{Aut}(\mu)|} a^{1-g+\sum_{j=1}^{\ell}\langle \frac{\mu_j}{a}\rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !} \int_{\overline{\mathcal{M}}^{\bullet}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^U}{\prod_{j=1}^{\ell}(1-\mu_j \bar{\psi}_j)} . \tag{5}$$

The ELSV formula [6] for linear Hodge integrals on the moduli space of curves arises from the $a = 1$ specialization of Theorem 1,

$$H_g(\mu) = \frac{(2g-2+d+\ell)!}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{\mu_j^{\mu_j}}{\mu_j!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\sum_{i=0}^{g}(-1)^i \lambda_i}{\prod_{j=1}^{\ell}(1-\mu_j \psi_j)} .$$

For $a = 1$, we must have $\gamma = \emptyset$.

The conditions $\gamma$ allow for greater freedom in the $a > 1$ case. For example, the proof of Theorem 1 yields a remarkable vanishing property. The monodromy conditions $\gamma$ satisfy negativity if

$$d - \sum_{i=1}^{n} \gamma_i < 0$$

and strong negativity if

$$d - n - \frac{d - \sum_{i=1}^{n} \gamma_i}{a} < 0.$$

Strong negativity is easily seen to imply negativity.

**Theorem 2.** *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be nontrivial monodromies in $\mathbb{Z}_a$ satisfying the parity condition with respect to the partition $\mu$. In addition, let $\gamma$ satisfy at least one of the following two conditions:*

*(i) negativity and boundedness, or*

*(ii) strong negativity.*

*Then, a vanishing result for Hurwitz-Hodge integrals holds:*

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^U}{\prod_{j=1}^{\ell}(1-\mu_j \bar{\psi}_j)} = 0.$$

A few examples of Theorems 1 and 2 where alternative approaches to the integrals are available are presented in Section 3.

0.5. **Abelian $G$.** Since any faithful representation $R$ of $\mathbb{Z}_a$ differs from $U$ by an automorphism of $\mathbb{Z}_a$, Theorem 1 determines linear Hodge integrals with respect to $R$. Representations of $\mathbb{Z}_a$ with kernels require an additional analysis.

Let $G$ be an abelian group with group law written additively. Consider an irreducible representation $R$,

$$\phi^R : G \to \mathbb{C}^*,$$

[5] If $\gamma \neq \emptyset$, the non-negativity condition may satisfied globally but be violated on connected components.

with associated exact sequence

$$0 \to K \to G \xrightarrow{\phi^R} \mathrm{Im}(\phi^R) \cong \mathbb{Z}_a \to 0. \tag{6}$$

The homomorphism $\phi^R$ induces a canonical morphism

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a).$$

The morphism $\rho$ satisfies

$$\rho^*(\lambda_i^U) = \lambda_i^R$$

and has the same degree over each component of $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$. Therefore, linear Hodge integrals with respect to $R$ can be calculated by multiplying the formula of Theorem 1 by the degree of $\rho$.

In Section 4, the solution for arbitrary $G$ and $R$ is cast in a more appealing way. When

$$\phi^R(\gamma) = -\mu \in \mathbb{Z}_a,$$

Hodge integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^R}{\prod_{j=1}^{\ell}(1-\mu_j\bar{\psi}_j)}$$

are expressed in terms of Hurwitz numbers for $K_d$, the wreath product of $K$ with the symmetric group $\Sigma_d$. Since the infinite wedge formalism for $\Sigma_d$ extends to a Fock space formalism for the wreath product $K_d$, there is again a connection to integrable systems [25].

Conjugacy classes in $K_d$ are indexed by Conj$(K)$-weighted partitions of $d$,

$$\overline{\mu} = \{(\mu_1, \kappa_1), \ldots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}.$$

Here, $\mu$ is a partition of $d$ with parts $\mu_j$, the weights $\kappa_i \in$ Conj$(K)$ are conjugacy classes in $K$, and $\overline{\mu}$ is an unordered set of pairs. Let Aut$(\overline{\mu})$ denote the automorphism group of $\overline{\mu}$. Let $C_{\overline{\mu}} \in \mathcal{Z}K_d$ be the element of the group algebra associated to the conjugacy class $\overline{\mu}$. The transposition element $T \in \mathcal{Z}K_d$ is associated to conjugacy class of $K_d$ indexed by

$$\overline{\tau} = \{(2,0),(1,0),\ldots,(1,0)\}$$

where all the Conj$(K)$-weights are 0.

The wreath product $K_d$ has a forgetful map to $\Sigma_d$ which sends elements of cycle type $\overline{\mu}$ to elements of type $\mu$. The *$K_d$-Hurwitz number* $H_{g,K}(\overline{\nu},\overline{\mu})$ counts the degree $d|K|$-fold covers of $\mathbb{P}^1$ with monodromy in $K_d$ given by $\overline{\nu}$ and $\overline{\mu}$ at $0,\infty \in \mathbb{P}^1$ and $\overline{\tau}$ at all the points of

$$U_{r_g(\nu,\mu)} \subset \mathbb{P}^1.$$

Since $K \subset K_d$ is contained in the center, any such cover has a canonical $K$-action which defines a $K$-bundle over a punctured Hurwitz cover counted by $H_g(\nu,\mu)$. The connectivity requirement we place on covers counted by $H_{g,K}(\overline{\nu},\overline{\mu})$ is *not* that the $d|K|$-fold cover is connected, but only that the associated Hurwitz $d$-fold cover is connected. Similarly, $g$ is the genus of the $d$-fold cover.

The natural extension of formula (3) for disconnected Hurwitz covers for the wreath product $K_d$ is

$$H^{\bullet}_{g,K}(\overline{\nu},\overline{\mu}) = \frac{1}{|K_d|}\left(C_{\overline{\nu}} T^{r_g(\nu,\mu)} C_{\overline{\mu}}\right)_{[\mathrm{Id}]},$$

where the product on the right takes place in the group algebra of $K_d$.

Select an element $x \in G$ with $\phi^R(x) = 1$. Let $k = ax \in K$. Denote by $-\overline{\mu}$ the $\ell(\mu)$-tuple of elements of $G$ defined by:

$$-\overline{\mu} = (\kappa_1 - \mu_1 x, \kappa_2 - \mu_2 x, \ldots, \kappa_{\ell(\mu)} - \mu_{\ell(\mu)} x).$$

Although the parts of $\overline{\mu}$ are unordered, an ordering is chosen for $-\overline{\mu}$. The parity condition is now

$$\sum_{j=1}^{\ell} \kappa_j - \mu_j x = 0 \in G.$$

Denote by $\emptyset_+(k)$ the conjugacy class given by

$$\emptyset_+(k) = \{\underbrace{(a,-k),\ldots,(a,-k)}_{d/a \text{ times}}\}.$$

**Theorem 3.** *For weighted-partitions $\overline{\mu}$ satisfying the parity condition,*

$$H_{g,K}(\emptyset_+(k),\overline{\mu}) = \frac{r_g(\emptyset_+,\mu)!}{|\mathrm{Aut}(\overline{\mu})|} a^{1-g+\sum_{j=1}^{\ell}\langle\frac{\mu_j}{a}\rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor\frac{\mu_j}{a}\rfloor}}{\lfloor\frac{\mu_j}{a}\rfloor!} \int_{\overline{\mathcal{M}}_{g,-\overline{\mu}}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^R}{\prod_{j=1}^{\ell}(1-\mu_j\bar{\psi}_j)}.$$

Theorem 3 determines all linear Hurwitz-Hodge integrals for $G$ and holds in exactly the same form for the disconnected theories $H^{\bullet}_{g,K}(\emptyset_+(k),\overline{\mu})$ and $\overline{\mathcal{M}}^{\bullet}_{g,-\overline{\mu}}(\mathcal{B}G)$.

0.6. **Future directions.** The ELSV formula has two immediate applications in Gromov-Witten theory. The first is the determination of descendent integrals over $\overline{\mathcal{M}}_{g,n}$ via asymptotics to remove the Hodge classes [18, 21]. The second is the exact evaluation of the vertex integrals in the localization formula for $\mathbb{P}^1$ in [22, 23]. The latter requires the Hodge classes.

Since $\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$ is a finite map, a geometric approach to the descendent integrals is not strictly necessary [16]. However, for the calculation of the Gromov-Witten theory of target curves with orbifold structure [17], Theorem 3 is essential. The results may be viewed as a first step for orbifolds along the successful line of exact Hodge integral formulas which have culminated in the topological and equivariant vertices in ordinary Gromov-Witten theory.

Hurwitz-Hodge integrals can be viewed as pairings of tautological classes

$$\epsilon_*(\lambda_i^R) \in H^{2i}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

against the descendents $\psi_i$. Given an action

$$\alpha : G \times \{1,\ldots,k\} \to \{1,\ldots,k\}$$

on a set with $k$ elements, there is a second map to the moduli space of curves. Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \ \ \mathcal{D} \to \mathcal{C}$$

be the universal domain curve and the universal $G$-bundle respectively. A second universal curve

$$\mathcal{D}^{\alpha} = \mathcal{D} \times_G \{1,\ldots,k\} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

is obtained by the mixing construction. We obtain

$$\epsilon^{\alpha} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g^{\alpha},n^{\alpha}},$$

where $g^\alpha$ and $n^\alpha$ are the genus and the number of distinguished sections[6] of the universal curve $\mathcal{D}^\alpha$. Two questions immediately arise:

(i) Do the classes $\epsilon^\alpha_*(\lambda^R_i)$ lie in the tautological ring of $\overline{\mathcal{M}}_{g^\alpha,n^\alpha}$?
(ii) Do the pairings of $\epsilon^\alpha_*(\lambda^R_i)$ against the descendents of $\overline{\mathcal{M}}_{g^\alpha,n^\alpha}$ admit simple evaluations?

The answer to (i) is known [11] to be false for $g = 1$, but may be true for $g = 0$. See [8] for positive results related to (i) for the standard action of the symmetric group $\Sigma_k$ in the $g = 0$ case.

0.7. **Acknowledgments.** We thank J. Bryan, R. Cavalieri, T. Graber, C. Faber, D. Maulik, A. Okounkov, Y. Ruan, and R. Vakil for related conversations.

P.J. was partially supported by RTG grant DMS-0602191 at the University of Michigan. R.P. was partially supported by DMS-0500187. H.-H. T. thanks the Institut Mittag-Leffler for hospitality and support during a visit in Spring 2007. The paper was furthered at a lunch in Kyoto while the last two authors were visiting RIMS in January 2008. Section 3.4 was added after discussions at the Banff workshop on *Recent progress on the moduli of curves* in March 2008.

## 1. Stable relative maps

1.1. **Definitions.** For $a \geq 1$, let $\mathbb{P}^1[a]$ be the projective line with a single stack point of order $a$ at $0$. Let

$$\langle \zeta_a \rangle \subset \mathbb{C}^*, \quad \zeta_a = e^{\frac{2\pi i}{a}}$$

be the group of $a^{th}$-roots of unity. Locally at 0, $\mathbb{P}^1[a]$ is the quotient stack $\mathbb{C}/\langle \zeta_a \rangle$. Alternatively, $\mathbb{P}^1[a]$ is the $a^{th}$-root stack of $\mathbb{P}^1$ along the divisor 0.

Let $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ be the stack of stable relative maps to $(\mathbb{P}^1[a], \infty)$ where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a vector of nontrivial elements

$$1 \leq \gamma_i \leq a - 1, \quad \gamma_i \in \mathbb{Z}_a,$$

and $\mu$ is a partition of $d \geq 1$ with parts $\mu_j$ and length $\ell$. The moduli space parametrizes maps

$$[\ f : (C, p_1, \ldots, p_n) \to \mathbb{P}^1[a]\ ] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$$

for which

(i) the domain $C$ is a nodal curve of genus $g$ with stack structure at $p_i$ determined by $\gamma_i$,
(ii) relative conditions over $\infty \in \mathbb{P}^1[a]$ are given by the partition $\mu$.

The isotropy group of $p_i \in C$ is the subgroup of $\mathbb{Z}_a$ generated by $\gamma_i$. Let $a_i$ denote the order of $\gamma_i$. The domain $C$, called a *twisted curve*, may have additional stack structure at the nodes, see [2].

We recall the Riemann-Roch formula for twisted curves.[7] Let $C$ be a twisted curve whose non-singular stack points are $p_1, ..., p_n$ with cyclic isotropy groups $I_1, \ldots, I_n$. The group $I_i$ is identified with the $a_i^{th}$-roots of unity via the action on $T_{p_i}C$,

$$I_i \xrightarrow{\sim} \langle \zeta_{a_i} \rangle \subset \mathbb{C}^*, \quad \zeta_{a_i} = e^{\frac{2\pi i}{a_i}}.$$

[6]We suppress the ordering issues here.

[7]See Theorem 7.2.1 of [1] for precisely our situation.

Let $E$ be a locally free sheaf over the stack $C$. Then, $I_i$ acts on the restriction $E|_{p_i}$. Let

$$E|_{p_i} = \bigoplus_{0\le s\le a_i-1} V_s^{\oplus e_s}$$

be the direct sum decomposition, where $V_s$ is the irreducible representation of $\mathbb{Z}_{a_i}$ associated to the character

$$\phi^s : I_i \to \mathbb{C}^*, \quad \phi^s(\zeta_{a_i}) = \zeta_{a_i}^s.$$

The *age* of $E$ at $p_i$ is defined by

$$\mathrm{age}_{p_i}(E) = \sum_{0\le s\le a_i-1} e_s \frac{s}{a_i} .$$

The Riemann-Roch formula for twisted curves is given as follows:

$$\chi(C,E) = \mathrm{rk}(E)(1-g) + \deg(E) - \sum_{i=1}^{n} \mathrm{age}_{p_i}(E). \tag{7}$$

The virtual dimension of $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ is calculated by the Riemann-Roch formula (7). Let

$$[\, f : (C,p_1,\ldots,p_n) \to \mathbb{P}^1[a]\,] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu).$$

Certainly, $\deg\big(f^*T_{\mathbb{P}^1[a]}(-\infty)\big) = d/a$. By the quotient presentation of $\mathbb{P}^1[a]$, the character of $f^*T_{0,\mathbb{P}^1[a]}$ at $p_i$ is

$$\zeta_{a_i} \mapsto \zeta_{a_i}^{\frac{\gamma_i a_i}{a}} = \zeta_a^{\gamma_i}.$$

Therefore, $\mathrm{age}_{p_i}\big(f^*T_{\mathbb{P}^1[a]}(-\infty)\big) = \frac{\gamma_i}{a}$ and

$$\begin{aligned} \mathrm{vdim}\,\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu) &= 3g-3+n+\ell+\chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) \\ &= 3g-3+n+\ell+1-g+\frac{d}{a}-\sum_{i=1}^{n}\frac{\gamma_i}{a} \\ &= 2g-2+n+\ell+\frac{d}{a}-\sum_{i=1}^{n}\frac{\gamma_i}{a} . \end{aligned}$$

To simplify notation, let $r$ denote the above virtual dimension. Since $r$ must be an integer, $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ is empty unless the parity condition $d = \sum_{i=1}^n \gamma_i \pmod a$ holds.

1.2. **Hurwitz numbers.** We now impose the non-negativity condition,

$$d - \sum_{i=1}^{n} \gamma_i \ge 0.$$

Let $H_{g,a}(\gamma,\mu)$ denote the weighted count of degree $d$ representable maps from nonsingular, connected, genus $g$ twisted curves with stack points of type $\gamma$ to $\mathbb{P}^1[a]$ with profile $\mu$ over $\infty$ and simple ramification over $r$ fixed points in $\mathbb{P}^1[a] \setminus \{0,\infty\}$.

**Lemma 1.** *$H_{g,a}(\gamma,\mu)$ is well-defined and equal to $|\mathrm{Aut}(\gamma)| \cdot H_g(\gamma_+,\mu)$.*

Given a stack map $[f : C \to \mathbb{P}^1[a]] \in \mathcal{M}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ satisfying the simple ramification condition over the $r$ points, the associated coarse map

$$f^c : C^c \to \mathbb{P}^1$$

is a usual Hurwitz covering counted by $H_g(\gamma_+, \mu)$. The representability condition implies the point $p_i$ has ramification profile $\gamma_i$ over 0 for the coarse map. Conversely, we have the following result.

**Lemma 2.** *Let $C^c$ be a nonsingular curve and let $f^c : C^c \to \mathbb{P}^1$ be a nonconstant map. Then, there is a unique (up to isomorphism) twisted curve $(C, p_1, \ldots, p_m)$ and a representable morphism $f : C \to \mathbb{P}^1[a]$ whose induced map between coarse curves is $f^c$.*

*Proof.* Since the natural map $\mathbb{P}^1[a] \to \mathbb{P}^1$ is an isomorphism over $\mathbb{P}^1[a] \setminus [0/\mathbb{Z}_a]$, we may consider the composite

$$C^c \setminus (f^c)^{-1}(0) \xrightarrow{f^c} \mathbb{P}^1 \setminus \{0\} \xrightarrow{\sim} \mathbb{P}^1[a] \setminus \{[0/\mathbb{Z}_a]\} \subset \mathbb{P}^1[a].$$

The Lemma follows by applying Lemma 7.2.6 of [2]. □

To proceed, we need to identify the ramification profile of $f^c$ over $0$. Since $\mathbb{P}^1[a]$ is a root stack, we may use classification results on maps to root stacks proven in [4]. According to Theorem 3.3.6 of [4], maps considered in our stack Hurwitz problem are in bijective correspondence with maps $f^c : C^c \to \mathbb{P}^1$ from a coarse curve $C^c$ satisfying

$$(f^c)^*[0] = \sum_{i=1}^{n} \gamma_i[\bar{p}_i] + aD, \tag{8}$$

where $\bar{p}_1, ..., \bar{p}_n \in C^c$ are distinct points and $D \subset C^c$ is a divisor consisting of $\frac{d-\sum_{i=1}^n \gamma_i}{a}$ additional distinct points.

The proof of Lemma 1 is complete. The factor $|\text{Aut}(\gamma)|$ occurs since the stack points of $C$ are labelled while the corresponding ramification points on the Hurwitz covers enumerated by $H_g(\gamma_+, \mu)$ are not. □

1.3. **Branch maps.** There exists a basic branch morphism for stable maps,

$$\text{br} : \overline{\mathcal{M}}_g(\mathbb{P}^1, \mu) \to \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1),$$

constructed in [9]. By composing with the coarsening map, we obtain

$$\text{br} : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \to \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

To proceed, we impose the boundedness condition,

$$\forall i \neq j, \;\; \gamma_i + \gamma_j \leq a.$$

**Lemma 3.** *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$\text{Im(br)} \subset \left( d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} \right) [0] + \text{Sym}^r(\mathbb{P}^1) \subset \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

*Proof.* Let $f : C \to \mathbb{P}^1[a]$ be a Hurwitz cover counted by $H_{g,a}(\gamma, \mu)$. The expression

$$E = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a}$$

is the order of $[0]$ in $\text{br}([f])$. The claim of the Lemma is simply that the minimum order of $[0]$ in $\text{br}(f)$ is achieved at such Hurwitz covers $f$.

The proof requires checking all possible degenerations of $f$ over 0. If the stack points $p_1, \ldots, p_n$ do not bubble off the domain, the claim follows easily as in the coarse case. We leave the details to the reader.

A more interesting calculus is encountered if a subset of stack points $p_1, \ldots, p_l$ bubbles off the domain together over $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$. We do the analysis for a single bubble. We can assume the bubble is of genus 0 since higher genus increases the branching order. The multi-bubble calculation is identical.

The genus 0 bubble is attached to the rest of the curve in $m$ stack points of type

$$\delta_1, \ldots, \delta_m \in \mathbb{Z}_a, \;\; 1 \leq \delta_j \leq a$$

on the noncollapsed side. The parity condition

$$\sum_{i=1}^{l} \gamma_i - \sum_{j=1}^{m} \delta_j = ka \tag{9}$$

must be satisfied with $k \in \mathbb{Z}$.

The branch contribution over 0 of the bubbled map is at least

$$E' = \sum_{i=l+1}^{n} (\gamma_i - 1) + \sum_{j=1}^{m} (\delta_j - 1) + 2m - 2 + \frac{d - \sum_{i=l+1}^{n} \gamma_i - \sum_{j=1}^{m} \delta_j}{a}(a-1).$$

All the terms on the right are obtained from the ramifications on the noncollapsed side except for the $2m$ from the $m$ nodes of the bubble and the $-2$ from the bubble itself, see [9]. Rewriting using the parity condition (9), we find

$$E' = E + l + m - 2 - k.$$

By connectedness and bubble stability, we have

$$m \geq 1, \;\; l + m \geq 3.$$

If $k \leq 0$, we conclude $E' > E$. If $k \geq 0$, then $k \leq l - 2$ by the boundedness condition and the positivity of $\delta_1$. Again, $E' > E$. $\square$

By Lemma 3, we may view the branch map with restricted image,

$$\text{br}_0 : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \to \text{Sym}^r(\mathbb{P}^1).$$

The proof of Lemma 3 shows the maps $f : C \to \mathbb{P}^1[a]$ satisfying $[0] \notin \text{br}_0(f)$ have no contraction over 0 and coarse profile exactly $\gamma_+$. The usual nonsingularity and Bertini arguments [9] then imply the following result.

**Lemma 4.** *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$H_{g,a}(\gamma, \mu) = \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \text{br}_0^*(H^r),$$

*where* $H \in H^2(\text{Sym}^r(\mathbb{P}^1), \mathbb{Q})$ *is the hyperplane class.*

## 2. LOCALIZATION

2.1. **Fixed loci.** The standard $\mathbb{C}^*$-action on $\mathbb{P}^1$, defined by $\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$, lifts canonically to $\mathbb{C}^*$-actions on $\mathbb{P}^1[a]$ and $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ . We will evaluate the integral

$$(10) \qquad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r)$$

by virtual localization for relative maps [10, 13] following [9, 12]. We assume the parity, non-negativity, and boundedness conditions.

The first step is to define a lift of the $\mathbb{C}^*$-action to the integrand. Certainly the $\mathbb{C}^*$-action lifts canonically to $\mathrm{Sym}^r(\mathbb{P}^1)$. A lift of $H^r$ can be defined by choosing the $\mathbb{C}^*$-fixed point $r[0] \in \mathrm{Sym}^r(\mathbb{P}^1)$. The tangent weights at $[0/\mathbb{Z}_a], \infty \in \mathbb{P}^1[a]$ are $\frac{t}{a}$ and $-t$ respectively. The equivariant Euler class of the normal bundle to $r[0]$ in $\mathrm{Sym}^r(\mathbb{P}^1)$ has weight $r!t^r$.

The second step is to identify the $\mathbb{C}^*$-fixed locus $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)^{\mathbb{C}^*} \subset \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$. The components of the $\mathbb{C}^*$-fixed locus lie over the $r+1$ points of $\mathrm{Sym}^r(\mathbb{P}^1)^{\mathbb{C}^*}$. By our lifting of $H^r$, we need only consider

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)^{\mathbb{C}^*} \cap \mathrm{br}_0^{-1}(r[0]).$$

Because of the strong restriction on the branching, the maps

$$[f : C \to \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_0^{\mathbb{C}^*}$$

have a very simple structure:

(i) $C = C_0 \cup \coprod_{j=1}^{\ell} C_j$,

(ii) $f|_{C_0}$ is a constant map from a genus $g$ curve to $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$,

(iii) the coarse map $f^c|_{C_j} : C_j^c \to \mathbb{P}^1$ is a $\mathbb{C}^*$-fixed Galois cover of degree $\mu_j$ for $j > 0$,

(iv) $C_0$ meets $C_j$ at a node $q_j$.

The stack structure at $q_j \in C_j$ is easily determined using the relationship between stack Hurwitz covers of $\mathbb{P}^1[a]$ and ordinary Hurwitz covers of $\mathbb{P}^1$ discussed in Section 1.2. The stack structure at $q_j \in C_j$ is of type $\mu_j \in \mathbb{Z}_a$. The stack structure at $q_j \in C_0$ where $C_j$ is attached is of the *opposite* type $-\mu_j \in \mathbb{Z}_a$. The map

$$f|_{C_0} : (C, p_1, \ldots, p_n, q_1, \ldots, q_\ell) \to [0/\mathbb{Z}_a]$$

is an element of $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$.

The $\mathbb{C}^*$-fixed locus may be identified with a quotient of a fibered product,

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} \cong \left(\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times_{(\bar{I}\mathcal{B}\mathbb{Z}_a^\ell)} P_1 \times \ldots \times P_\ell\right)_{/\mathrm{Aut}(\mu)},$$

where $\bar{I}\mathcal{B}\mathbb{Z}_a$ is the rigidified inertia stack of $\mathcal{B}\mathbb{Z}_a$ and $P_j$ is the moduli stack of $\mathbb{C}^*$-fixed Galois covers of degree $\mu_j$. By the standard multiplicity obtained from gluing stack $\mathbb{Z}_a$-bundles, the projection

$$(11) \qquad \overline{\mathcal{M}}_0^{\mathbb{C}^*} \to \left(\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \ldots \times P_\ell\right)_{/\mathrm{Aut}(\mu)}$$

has degree $\prod_{j=1}^{\ell} \frac{a}{b_j}$ where $b_j$ is the order of $\mu_j \in \mathbb{Z}_a$.

Fortunately, the residue integral over $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$ in the virtual localization formula for (10) is pulled-back via (11). Instead of integrating over $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$, we will integrate over

$$\widetilde{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \ldots \times P_\ell$$

and multiply by

$$\frac{1}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j}.$$

2.2. **Virtual normal bundle.** The virtual localization formula for (10) with our choice of equivariant lifts takes the following form:

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{1}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j} \int_{\widetilde{\mathcal{M}}_0^{\mathbb{C}^*}} \frac{r!\, t^r}{e(\mathrm{Norm}^{vir})}. \tag{12}$$

The equivariant Euler class of the virtual normal bundle is

$$\frac{1}{e(\mathrm{Norm}^{vir})} = \frac{e(H^1(C, f^* T_{\mathbb{P}^1[a]}(-\infty)))}{e(H^0(C, f^* T_{\mathbb{P}^1[a]}(-\infty)))} \frac{1}{\prod_{j=1}^{\ell} e(N_j)}, \tag{13}$$

see [10]. The last product is over the nodes of $C$, and $N_j$ is the equivariant line bundle associated to the smoothing of $q_j$. The terms in (13) are computed via the normalization sequence of the domain $C$. The various contributions over the components $C_0, C_1, \ldots, C_\ell$ are computed separately.

First consider the collapsed component $C_0$. The space $H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ is identified with the subspace of $T_{\mathbb{P}^1[a]}(-\infty)|_{[0/\mathbb{Z}_a]}$ consisting of vectors invariant under the action of the image of the monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$. Therefore, $H^0$ vanishes unless the monodromy representation is trivial, in which case $H^0$ is 1-dimensional with weight $\frac{t}{a}$.

The trivial monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$ is possible only if

$$\gamma = \emptyset \quad \text{and} \quad \forall j,\ \mu_j = 0 \mod a\ .$$

Even then, the locus with trivial monodromy is just a component[8] of $\overline{\mathcal{M}}_{g,(0,\ldots,0)}(\mathcal{B}\mathbb{Z}_a)$. The trivial monodromy representation locus will play a slightly special role throughout the calculation. But, in the final formula, no different treatment is required.

The space $H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ yields the vector bundle

$$\mathbb{B} = (\mathbb{E}^U)^\vee$$

over $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ whose rank may be calculated by the orbifold Riemann-Roch formula. Over the component of the fixed locus where the monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$ is trivial, the rank of $\mathbb{B}$ is $g$. Otherwise, the rank is

$$r_{\mathbb{B}} = g - 1 + \sum_{i=1}^{n} \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \bmod a} \left(1 - \left\langle \frac{\mu_j}{a} \right\rangle\right). \tag{14}$$

[8]If $g > 0$, there will typically be other components as well.

The $H^1 - H^0$ contribution from the collapsed component to the localization formula is

$$\sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} c_i(\mathbb{B}) = \sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U. \tag{15}$$

For the component where the monodromy representation is trivial, an additional factor of $\frac{a}{t}$ must be inserted in (15).

Next consider the $H^1 - H^0$ contribution from the $\mathbb{C}^*$-fixed Galois covers. Since

$$\deg(f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\mu_j}{a},$$

we have

$$H^k(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = H^k\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right).$$

The $H^0$ weights are

$$\frac{t}{\mu_j}, 2\frac{t}{\mu_j}, ..., \left\lfloor \frac{\mu_j}{a} \right\rfloor \frac{t}{\mu_j},$$

where the weight 0 is omitted.[9] The group $H^1$ vanishes. The $H^1 - H^0$ contribution is

$$t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !}.$$

Finally, consider the $H^1 - H^0$ contribution from the nodal point $q_j$. If $\mu_j \neq 0 \,(\mathrm{mod}\, a)$, then $q_j$ is a stack point and

$$H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j}) = 0$$

as there is no invariant section. If $\mu_j = 0 \,(\mathrm{mod}\, a)$ then $H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$ is 1-dimensional and contributes a factor $\frac{t}{a}$. Certainly, $H^1$ vanishes here for dimension reasons.

The contribution from smoothing the node $q_j$ is the tensor product of the tangent lines of the two branches incident to $q_j$,

$$e(N_j) = \frac{1}{b_j}\left(-\bar{\psi}_j + \frac{t}{\mu_j}\right).$$

After putting the component calculations together in (13), we obtain the following expression for for $1/e(\mathrm{Norm}^{vir})$:

$$\left(\sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U\right) \cdot \prod_{j=1}^{\ell} \left(t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !} \frac{1}{\frac{1}{b_j}\left(-\bar{\psi}_j + \frac{t}{\mu_j}\right)}\right) \cdot \prod_{j=1}^{\ell} \left(\frac{t}{a}\right)^{\delta_{0,\langle \frac{\mu_j}{a} \rangle}}.$$

Regrouping of terms yields

$$\frac{\prod_{j=1}^{\ell} b_j \mu_j}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0,\langle \frac{\mu_j}{a} \rangle}}} \left(\prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !}\right) \left(\sum_{i=0}^{r_{\mathbb{B}}} t^{r_{\mathbb{B}}-i} (-a)^i \lambda_i^U\right) \cdot t^{-\sum_{j=1}^{\ell} \lfloor \frac{\mu_j}{a} \rfloor} \prod_{j=1}^{\ell} \frac{t^{\delta_{0,\langle \frac{\mu_j}{a} \rangle}}}{(t - \mu_j \bar{\psi}_j)}. \tag{16}$$

For the component with trivial monodromy representation, a factor of $\frac{a}{t}$ must be inserted in the formulas for $1/e(\mathrm{Norm}^{vir})$.

[9] The 0 weight is from reparameterization of the domain $C_j$ and is not in the virtual normal bundle.

2.3. **Proof of Theorem 1.** Putting the calculations of Section 2.2 together and passing to the non-equivariant limit, we obtain the following evaluation

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{r!}{|\mathrm{Aut}(\mu)|} \frac{a^\ell}{a^{r_{\mathbb{B}}+\sum_{j=1}^\ell \delta_{0,\langle \frac{\mu_j}{a}\rangle}}} \prod_{j=1}^\ell \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_i}{a} \rfloor !} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1-\mu_j \bar{\psi}_j)}.$$

On the right side, we have included the fundamental class factors

$$\prod_{j=1}^\ell \frac{1}{\mu_j}$$

of the moduli spaces $P_j$. For the component with trivial monodromy representation, a factor of $a$ must be inserted in the formula.

We can simplify the integral evaluation by using the calculation (14) of $r_{\mathbb{B}}$,

$$\begin{aligned} & r_{\mathbb{B}} + \sum_{i=1}^\ell \delta_{0,\langle \frac{\mu_j}{a}\rangle} - \ell \\ &= g-1+\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \bmod a} \left(1-\left\langle \frac{\mu_j}{a} \right\rangle\right) + \left( \sum_{\mu_j = 0 \bmod a} 1 \right) - \ell \\ &= g-1+\sum_{i=1}^n \frac{\gamma_i}{a} - \sum_{j=1}^\ell \left\langle \frac{\mu_j}{a} \right\rangle. \end{aligned}$$

The above calculation is not valid for the component with trivial monodromy since $r_{\mathbb{B}} = g$ not $g-1$. The discrepancy is exactly fixed by the extra factor $a$ required for the trivial monodromy case. We conclude

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{r!}{|\mathrm{Aut}(\mu)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{j=1}^\ell \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^\ell \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_i}{a} \rfloor !} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1-\mu_j \bar{\psi}_j)}. \tag{17}$$

holds uniformly. Theorem 1 is then obtained from Lemmas 1 and 4. □

In degenerate cases, unstable integrals may appear on the right side of the formula in Theorem 1. The unstable integrals come in two forms and are defined by the localization contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i\geq 0} (-a)^i \lambda_i^U}{(1-x\bar{\psi}_1)} = \frac{1}{a} \cdot \frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i\geq 0} (-a)^i \lambda_i^U}{(1-x\bar{\psi}_1)(1-y\bar{\psi}_2)} = \frac{1}{a} \cdot \frac{1}{x+y}.$$

With the above definitions, Theorem 1 holds in all cases.

The disconnected formula (5) follows easily from the connected case by the usual combinatorics of distributing ramification points to the components of Hurwitz covers.

2.4. **Proof of Theorem 2.** Suppose $\gamma$ satisfies the parity and strong negativity condition with respect to $\mu$. Since

$$\delta = d - n - \frac{d - \sum_{i=1}^{n} \gamma_i}{a} < 0,$$

the virtual dimension $r$ of $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ is greater than $2g - 2 + d + \ell$. As a consequence, we immediately obtain the vanishing

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}^*(H^r) = 0 \tag{18}$$

since $H^r = 0 \in H^*(\mathrm{Sym}^{2g-2+d+\ell}(\mathbb{P}^1), \mathbb{Q})$.

We may nevertheless calculate (18) by localization with the lift

$$H^r = (2g - 2 + d + \ell)[0] \cdot t^{-\delta}$$

which does *not* vanish equivariantly. The analysis is identical to the calculations of Sections 2.1-2.3. We find the integral (18) is proportional (with nonzero factor) to

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)},$$

and therefore conclude the vanishing.

Assume now strong negativity does not hold, but $\gamma$ satisfies the parity, negativity, and boundedness condition. By the proof of Lemma 3, using the boundedness condition, the maps

$$f : C \to \mathbb{P}^1[a]$$

which satisfy $[0] \notin \mathrm{br}_0(f)$ have no contraction over 0 and coarse profile determined by $\gamma$. By the negativity condition, no such maps exists. Hence, $[0]$ is always in $\mathrm{br}_0(f)$. Therefore,

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = 0$$

and we conclude as above.□

## 3. Examples

3.1. $\mathbb{Z}_2$ **example.** The Hodge bundle $\mathbb{E}^U$ has a very simple interpretation in the $\mathbb{Z}_2$ case. Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2), \ \ \mathcal{D} \to \mathcal{C}$$

be the universal domain curve and the universal $\mathbb{Z}_2$-bundle. Let

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_g, \ \ \tilde{\epsilon} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_{g-1+\frac{n}{2}}$$

be the maps to moduli obtained from $\mathcal{C}$ and $\mathcal{D}$ respectively. The exact sequence

$$0 \to \epsilon^*(\mathbb{E}_g) \to \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}) \to \mathbb{E}^U \to 0.$$

exhibits $\mathbb{E}^U$ as the $K$-theoretic difference of the pulled-back Hodge bundles. If $g = 0$, then the situation[10] is even simpler,

$$\mathbb{E}^U \cong \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}). \tag{19}$$

[10]The map $\epsilon$ is not well-defined here for stability reasons.

Consider the case of Theorem 1 where $g = 0$, $\gamma = (1, 1)$, and $\mu = (1, 1)$. The statement is

$$H_0((1,1),(1,1)) = \frac{2}{2!2!}2^1 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1-2\lambda_1^U}{(1-\bar{\psi}_1)(1-\bar{\psi}_2)}.$$

The double Hurwitz number on the left is $\frac{1}{2}$. Expansion of the right side yields:

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1-2\lambda_1^U}{(1-\bar{\psi}_1)(1-\bar{\psi}_2)} &= \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{1}{(1-\psi_1)(1-\psi_2)} - 2\int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U \\ &= 1 - 2\int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U.\end{aligned}$$

To evaluate the last integral, we note the map

$$\tilde{\epsilon} : \overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_{1,1},$$

where the first branch point is selected for the marking on the elliptic curve, is of degree 6. Moreover, $\lambda_1^U$ is the pull-back of $\lambda_1$ under $\tilde{\epsilon}$ by (19). Hence,

$$1 - 2\int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U = 1 - 2\cdot 6\cdot \frac{1}{24} = \frac{1}{2}.$$

3.2. **Vanishing example.** The simplest example of the vanishing of Theorem 2 occurs for $\mathbb{Z}_2$. Let $g = 0$,

$$\gamma = (\underbrace{1,\ldots,1}_{n})$$

and $\mu = (1)$. By the parity condition, $n$ must be odd. Boundedness holds. For the negativity condition, we require $n \geq 2$. By Theorem 2 (i),

$$\int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i\geq 0}(-2)^i\lambda_i^U}{1-\bar{\psi}_1}$$

vanishes for all odd $n \geq 3$.

We now use the identification of $\lambda_i^U$ with the Chern classes of the Hodge bundle $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ whose fiber over

$$f : [D/\mathbb{Z}_2] \to \mathcal{B}\mathbb{Z}_2$$

is simply given by the space of differential forms on the genus $\frac{n-1}{2}$ curve $D$. The Chern roots of $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ can be identified by the vanishing sequence at a Weierstrass point of $D$. The Weierstrass point can be chosen to lie above the marking corresponding to the single part of $\mu$. The Chern roots of $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ are then $L, 3L, \ldots, (n-2)L$ where $L$ is the Chern class of the cotangent line of the Weierstrass point. The class $L$ on $\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)$ is $\frac{1}{2}\bar{\psi}_1$. Expanding the Chern roots, we find

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i\geq 0}(-2)^i\lambda_i^U}{1-\bar{\psi}_1} &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\prod_{i=1}^{\frac{n-1}{2}}(1-(2i-1)\bar{\psi}_1)}{(1-\bar{\psi}_1)} \\ &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \prod_{i=2}^{\frac{n-1}{2}}(1-(2i-1)\bar{\psi}_1) \\ &= 0,\end{aligned}$$

where the last integral vanishes for dimension reasons.

3.3. $\mathbb{Z}_\infty$ **example.** An interesting feature of Theorem 1 is the possibility of studying the behavior for large $a$. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ determine a partition of $d$,

$$d = \sum_{i=1}^{n} \gamma_i.$$

Let $\mu = (d)$ consist of a single part. For $a > d$, the rank of the Hodge bundle

$$\mathbb{E}^U \to \overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$$

is 0 by (14). Since the parity, non-negativity, and boundedness conditions hold for $a > d$, we may apply Theorem 1 to conclude

$$\begin{aligned} H_0(\gamma,(d)) &= \frac{(n-1)!}{|\mathrm{Aut}(\gamma)|}\, a \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{1}{1-d\bar{\psi}_1} \\ &= \frac{(n-1)!}{|\mathrm{Aut}(\gamma)|}\, d^{n-2}, \end{aligned}$$

which is a well-known formula for genus 0 double Hurwitz numbers.

3.4. **1-point series.** If $\mu = (d)$ consists of a single part, the entire generating series for double Hurwitz numbers has been computed [11] in [14]:

$$\sum_{g\geq 0} t^{2g}(-1)^g H_g(\nu,(d)) = \frac{r!\, d^{r-1}}{|\mathrm{Aut}(\nu)|} \prod_{k\geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\nu)-\delta_{k,1}}, \tag{20}$$

where $r = r_g(\nu,(d))$ and $m_k(\nu)$ is the number of times $k$ appears as a part of $\nu$. Single part double Hurwitz numbers are considerably simpler because such covers are automatically connected and the only characters with nonzero evaluation on the $d$-cycle are exterior powers of the standard $(d-1)$-dimensional representation.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector of nontrivial elements of $\mathbb{Z}_a$ satisfying the boundedness condition. We will consider degrees $d$ for which the parity and non-negativity conditions are satisfied. Then,

$$d - \sum_{i=1}^{n} \gamma_i = ab$$

for an integer $b \geq 0$. Consider the generating series

$$F_\gamma(t,z) = \sum_{g=0}^{\infty} \sum_{l=-\infty}^{g} t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{2g-2+\ell(\gamma)+l} \lambda_{g-l}^U$$

where $\bar{\psi}_0$ is the class corresponding to the point with monodromy $-d$.

[11] We write Theorem 3.1 of [14] in terms of $\sin$ instead of $\sinh$ and divide by $|\mathrm{Aut}(\nu)|$ since we do not mark ramifications in our definition of Hurwitz numbers.

The double Hurwitz number formula of Theorem 1 is

$$\begin{aligned} H_g(\gamma_+,(d)) &= \frac{r!}{|\mathrm{Aut}(\gamma)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a}+\langle \frac{d}{a}\rangle} \frac{d^{\lfloor \frac{d}{a}\rfloor}}{\lfloor \frac{d}{a}\rfloor !} \sum_{l=-\infty}^{g} d^{r-b-1+l}(-a)^{g-l}\int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l}\lambda_{g-l}^U \\ &= (-1)^g \frac{ad^{r-1}r!\left(\frac{d}{a}\right)^{\left\lfloor \frac{\sum\gamma_i}{a}\right\rfloor}}{|\mathrm{Aut}(\gamma)|\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}\sum_{l=-\infty}^{g}\left(\frac{-d}{a}\right)^l \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l}\lambda_{g-l}^U \end{aligned}$$

or, equivalently,

$$\sum_{g\geq 0}(-1)^g t^{2g} H_g(\gamma_+,(d)) = \frac{ad^{r-1}r!}{|\mathrm{Aut}(\gamma)|\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}\left(\frac{d}{a}\right)^{\left\lfloor \frac{\sum\gamma_i}{a}\right\rfloor} F_\gamma(t,-d/a)$$

where $r=r_g(\gamma_+,(d))$. After combining with (20), we obtain

$$F_\gamma(t,-d/a) = \frac{1}{a}\frac{\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{b!}\left(\frac{a}{d}\right)^{\left\lfloor \frac{\sum\gamma_i}{a}\right\rfloor}\prod_{k\geq 1}\left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma_+)-\delta_{k,1}}. \tag{21}$$

for $b\geq 0$.

**Theorem 4.** $F_\gamma(t,z)$ *equals*

$$\frac{1}{a}\frac{\left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{\left(-z-\sum\frac{\gamma_i}{a}\right)!}(-z)^{-\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor}\left(\frac{\sin(at/2)}{at/2}\right)^{-z-\sum\frac{\gamma_i}{a}}\prod_{k\geq 1}\left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma)-\delta_{k,1}}.$$

*Proof.* Using the standard polynomial expansion

$$\frac{\left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{\left(-z-\sum\frac{\gamma_i}{a}\right)!} = \left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)\cdots\left(-z-\sum\frac{\gamma_i}{a}+1\right),$$

we see the $t^{2g}$ coefficients of both sides of Theorem 4 are Laurent polynomials in $z$. Equation (21) shows Theorem 4 holds for all evaluations of the form $z=-d/a$ where

$$d-\sum_{i=1}^n \gamma_i = ab$$

and $b$ is a non-negative integer. Since there are infinitely many such evaluations, the coefficient Laurent polynomials in $z$ must be equal for all $t^{2g}$. □

If we specialize Theorem 4 to the case where $\gamma=\emptyset$, we obtain

$$\frac{1}{a}+\sum_{g>0}\sum_{l=0}^{g} t^{2g}z^l\int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)}\bar{\psi}_1^{2g-2+l}\lambda_{g-l}^U = \frac{1}{a}\left(\frac{at/2}{\sin(at/2)}\right)^z\frac{t/2}{\sin(t/2)} \tag{22}$$

If $\gamma=\emptyset$ and $a=1$ we recover

$$1+\sum_{g>0}\sum_{l=0}^{g} t^{2g}z^l\int_{\overline{\mathcal{M}}_{g,1}}\psi_1^{2g-2+l}\lambda_{g-l} = \left(\frac{t/2}{\sin(t/2)}\right)^{z+1} \tag{23}$$

first calculated in [7].

In (22), the term $\lambda_g^U$ vanishes for dimensional reasons except over the trivial monodromy component, where it agrees with the usual $\lambda_g$. Indeed, setting $z = 0$ in (22) yields

$$\frac{1}{a} + \sum_{g>0} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)} \psi_1^{2g-2} \lambda_g^U = \frac{1}{a} \frac{t/2}{\sin(t/2)}$$

which is the expected contribution from (23) with a factor of $1/a$ coming from the automorphisms.

## 4. Abelian groups

4.1. **Pull-back.** For an abelian group $G$ and irreducible representation $R$, recall the sequence (6),

$$0 \to K \to G \stackrel{\phi^R}{\to} \mathrm{Im}(\phi^R) \cong \mathbb{Z}_a \to 0.$$

By construction $R \cong \phi^{R*}(U)$. The homomorphism $\phi^R$ induces a canonical map

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$$

by sending a principal $G$-bundle to its quotient by $K$.

**Lemma 5.** $\mathbb{E}^R \cong \rho^*(\mathbb{E}^U)$.

*Proof.* Recall $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$ is the bundle whose fiber over

$$[f] : [D/H] \to \mathcal{B}H \in \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$$

is $H^0(D, \omega_D)$. The latter can be understood as the space of 1-forms $\alpha$ on the normalization $\tilde{D}$ of $D$ with possible simple poles with opposite residues at the two preimages of each node $q_i$.

Let $\tilde{\rho}$ be the map between the universal principal $G$- and $\mathbb{Z}_a$-curves that induces $\rho$. We obtain

$$d\tilde{\rho} : \rho^*(\mathbb{E}) \to \mathbb{E}$$

by pulling-back differential forms. An easy verification shows $\tilde{\rho}$ is well-defined even at points in the moduli space $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ for which the $G$-curve is nodal.

The map $d\tilde{\rho}$ is injective on each fiber since the pull-back of a nonzero differential form by a finite surjective map is nonzero. Certainly $d\tilde{\rho}$ carries the subbundle $\rho^*(\mathbb{E}^U)$ to the subbundle $\mathbb{E}^R$. These bundles have the same dimension by the Riemann-Roch formula for twisted curves. Hence, $d\tilde{\rho}$ is an isomorphism. □

The map $\rho$ does not preserve the isotropy groups at the marked points. However, since the classes $\bar{\psi}_i$ are pulled-back from $\overline{\mathcal{M}}_{g,n}$,

$$\rho^*(\bar{\psi}) = \bar{\psi}.$$

By Lemma 5, we concluded the integrand in Theorem 3 is exactly the integrand of Theorem 1 pulled-back via $\rho$.

4.2. **Degree.** The degree of $\rho$ is determined by the following result.

**Lemma 6.** *We have*

$$\deg(\rho) = \begin{cases} 0 & \sum_i \gamma_i \neq 0 \\ |K|^{2g-1} & \sum_i \gamma_i = 0 \end{cases} .$$

*Proof.* Consider a nonsingular curve $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$. Let

$$\Gamma = \pi_1(C \setminus \{p_1, \ldots, p_n\}) = \left\langle \Gamma_i, A_j, B_j \middle| \prod_{i=1}^{n} \Gamma_i \prod_{j=1}^{g} [A_j, B_j] \right\rangle,$$

where $\Gamma_i$ is a loop around $p_i$ and the loops $A_j, B_j$ are the standard generators of $\pi_1(C)$.

The elements of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ lying above $[C, p_1, \ldots p_n]$ are in bijective correspondence with the homomorphisms[12] $\varphi : \Gamma \to G$ with

$$\varphi(\Gamma_i) = \gamma_i. \tag{24}$$

Since $G$ is abelian, $\varphi([A_j, B_j]) = 0$. Hence, the parity condition

$$\sum_{i=1}^{n} \gamma_i = 0 \tag{25}$$

must be satisfied for $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ to be nonempty.

If the parity condition holds, then the images of $A_j$ and $B_j$ are completely unconstrained. There are $|G|^{2g}$ homomorphisms $\phi$ satisfying (24). Stated in terms of homomorphisms, the map $\rho$ corresponds to composing $\varphi : \Gamma \to G$ with $\phi^R : G \to \mathbb{Z}_a$. Since there are $|K|$ elements of $G$ in the preimage of any element of $\mathbb{Z}_a$, there are $|K|^{2g}$ elements in a generic fiber of $\rho$. Since $G$ is abelian, a cover in $\mathcal{M}_{g,\gamma}(\mathcal{B}G)$ has automorphism group $G$. A cover in the image of $\rho$ only has automorphism group $\mathbb{Z}_a$. Thus, the degree of $\rho$ is $|K|^{2g-1}$. □

Although $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ may have several components, Lemma 6 implies the degree of $\rho$ is the same over each component. In the nonabelian case, the situation is much more complicated. For example, let $\eta$ be the conjugacy class of a 3-cycle in $\Sigma_3$, let

$$s : \Sigma_3 \to \mathbb{Z}_2$$

be the sign representation, and let

$$\rho : \overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3) \to \overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$$

be the map induced by $s$. The space $\overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$ consists of two components: one with trivial monodromy, and one with nontrivial monodromy. There are covers in $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$ lying above the nontrivial monodromy component. If $t_1 \neq t_2 \in \Sigma_3$ are two transpositions, then $[t_1, t_2]$ is a 3-cycle. On the other hand, there are no elements of $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$ lying above the trivial monodromy component. All the monodromy in such a cover would lie in the abelian group $\mathbb{Z}_3 = \ker(s)$, and there are no such covers with nontrivial monodromy about the one marked point by (25). As the formula in Theorem 1 considers all components of $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ at once, a more nuanced approach would be required to understand Hurwitz-Hodge integrals for nonabelian groups, even for 1-dimensional representations.

[12]Composition in $\Gamma$ is written multiplicatively while composition in $G$ is additive.

In the disconnected case $\rho : \overline{\mathcal{M}}^{\bullet}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}^{\bullet}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$, Lemma 6 has a few minor complications:

(i) The monodromy condition $\sum_i \gamma_i = 0 \in G$ cannot be checked globally, but must be verified separately on each domain component.
(ii) The number of components matters. For disconnected curves with $h$ components, each of which satisfies the monodromy requirements, the degree of $\rho$ is $|K|^{2g-2+h}$.

When $\rho$ is nonzero, the degree $|K|^{2g-2+h}$ is independent of $G$ and the monodromy conditions (25). The only role these conditions play is to determine when the degree is nonzero.

4.3. **Wreath Hurwitz numbers.** The wreath product $K_d$ is defined by

$$K_d = \{(k, \sigma) \mid k = (k_1, \ldots, k_d) \in K^d, \sigma \in \Sigma_d\},$$
$$(k, \sigma)(k', \sigma') = (k + \sigma(k'), \sigma\sigma').$$

Conjugacy classes of $K_d$ are determined by their cycle types [19]. Since $K$ is abelian, for each $m$-cycle $(i_1 i_2 \cdots i_m)$ of $\sigma$, the element $k_{i_m} + k_{i_{m-1}} + \cdots + k_{i_1}$ is well-defined. The resulting Conj$(K)$-wieghted partition of $d$ is the called the *cycle type* of $(k, \sigma)$. Two elements of $K_d$ are conjugate exactly when they have the same cycle type.

We index the conjugacy classes of $K_d$ by Conj$(K)$-weighted partitions of $d$. Let

$$\overline{\nu} = \{(\nu_1, \iota_1), \ldots, (\nu_{\ell(\nu)}, \iota_{\ell(\mu)})\},$$
$$\overline{\mu} = \{(\mu_1, \kappa_1), \ldots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}$$

be two such partitions. Let $\nu^*$ be the partition with parts $\nu_j$ with a partial labelling given by $\iota_j$. Then

$$\text{Aut}(\nu^*) = \text{Aut}(\overline{\nu}).$$

The Hurwitz number $H_g(\nu^*, \mu^*)$ counts cover with the additional labelling data,

$$H_g(\nu^*, \mu^*) = \frac{|\text{Aut}(\nu)|}{|\text{Aut}(\nu^*)|} \frac{|\text{Aut}(\mu)|}{|\text{Aut}(\mu^*)|} H_g(\nu, \mu).$$

**Lemma 7.** *$H_{g,K}(\overline{\nu}, \overline{\mu})$ is the count of the covers $\pi : C \to \mathbb{P}^1$ enumerated by $H_g(\nu^*, \mu^*)$ with multiplicity $m_\pi$. The multiplicity $m_\pi$ is the automorphism-weighted count of principal $K$-bundles on $C \setminus \pi^{-1}(\{0, \infty\})$ with monodromy $\iota_i$ at $p_i \in \pi^{-1}(0)$ and $\kappa_j$ at $q_j \in \pi^{-1}(\infty)$.*

*Proof.* Let $\pi' : D \to \mathbb{P}^1$ be a cover counted by $H_{g,K}(\overline{\nu}, \overline{\mu})$. By definition, $\pi'$ is a $d|K|$-fold cover of $\mathbb{P}^1$ with monodromies $\overline{\nu}, \overline{\mu}$ and $\overline{\tau}$ over $0, \infty$ and the points of $U_r$ respectively.

Each such cover has an associated cover $\pi : C \to \mathbb{P}^1$ counted by $H_g(\nu^*, \mu^*)$. Algebraically, the cover is obtained by the forgetful map from $K_d \to \Sigma_d$. Geometrically, the cover is obtained by taking the quotient of $D$ by the diagonal subgroup $K \subset K_d$. There is a natural map $f : D \to C$. Away from the preimages of $0, \infty$ and $U_r$, the map $f$ is a principal $K$-bundle.

Consider the point $p_i \in \pi^{-1}(0)$ corresponding to a cycle $\nu_i$ which is labelled with $\iota_i \in K$. A small loop winding once around $p_i$ on $C$ has an image that winds $\nu_i$ times around $0$. But we know that the monodromy for $\pi' : D \to \mathbb{P}^1$ around $0$ is given by $\overline{\nu}$. By the definition of the cycle type, the monodromy of $f$ around $p_i$ is $\iota_i$. An identical argument shows the monodromy at $q_i$ over $\infty$ is $\kappa_i$ and the monodromy around all preimages of a point in $U_r$ is zero.

The above process is reversible. We start with a $d$-fold cover $\pi' : C \to \mathbb{P}^1$ counted by $H_g(\nu^*, \mu^*)$ and a principal $K$-bundle $f : D \to C$ with monodromy $\iota_i$ around $p_i$ and $\kappa_i$ around $q_i$. Then, the composition $\pi = \pi' \circ f$ is a cover counted by $H_{g,K}(\overline{\nu}, \overline{\mu})$. □

In other words, if $\rho' : \overline{\mathcal{M}}_{g,\iota\cup\kappa}(\mathcal{B}K) \to \overline{\mathcal{M}}_{g,\ell(\lambda)+\ell(\mu)}$ is the natural map, then

$$H_{g,K}(\overline{\nu}, \overline{\mu}) = \deg(\rho') H_g(\nu^*, \mu^*).$$

4.4. **Proof of Theorem 3.** By Lemma 5, we can compute the integral in Theorem 3 by computing the analogous Hurwitz-Hodge integral (appearing in Theorem 1) over $\overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a)$ and multiplying by the degree of

$$\rho : \overline{\mathcal{M}}_{g,-\overline{\mu}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a).$$

On the other hand, by Lemma 7, we can calculate $H_{g,K}(\emptyset_+(k), \overline{\mu})$ by computing $H_g(\emptyset_+, \mu)$, multiplying by the degree of

$$\rho' : \overline{\mathcal{M}}_{g,(-k)^{d/a}\cup\kappa}(\mathcal{B}K) \to \overline{\mathcal{M}}_{g,d/a+\ell(\mu)},$$

and correcting for the difference in the sizes of the automorphism groups $\mathrm{Aut}(\mu)$ and

$$\mathrm{Aut}(\overline{\mu}) = \mathrm{Aut}(\mu^*).$$

Thus, to deduce Theorem 3 from Theorem 1, we need only check that the degrees of $\rho$ and $\rho'$ agree. By Lemma 6, the degrees agree when nonzero. The last step is to check the parity condition (25) is the same for $\rho$ and $\rho'$. For $\rho$, the parity condition is

$$0 = \sum_{j=1}^{\ell} (-\overline{\mu})_j = \sum_{j=1}^{\ell} (\kappa_j - \mu_j x) = \sum_{j=1}^{\ell} \kappa_j - dx.$$

For $\rho'$, the parity condition is

$$0 = -\frac{d}{a}k + \sum_{j=1}^{\ell} \kappa_j.$$

Since $ax = k$, the conditions are equivalent. □

As in the faithful case, unstable integrals may appear on the right side of the formula in Theorem 3. These unstable terms are defined in a completely analogous manner, and extend Theorem 3 to all contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}G)} \frac{\sum_{i\geq 0}(-a)^i \lambda_i^R}{(1 - x\bar{\psi}_1)} = \frac{1}{|G|} \cdot \frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}G)} \frac{\sum_{i\geq 0}(-a)^i \lambda_i^R}{(1 - x\bar{\psi}_1)(1 - y\bar{\psi}_2)} = \frac{1}{|G|} \cdot \frac{1}{x+y}.$$

Alternatively, using a theory of stable maps relative to a stack divisor[13] at $\infty$, Theorem 3 could be proven in a manner closely parallel to the proof of Theorem 1.

[13]We avoid the foundational discussion of this theory.

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