INTERSECTION THEORY ON MODULI OF DISKS, OPEN KDV AND VIRASORO

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Abstract. We define a theory of descendent integration on the moduli spaces of stable pointed disks. The descendent integrals are proved to be coefficients of the τ-function of an open KdV hierarchy. A relation between the integrals and a representation of half the Virasoro algebra is also proved. The construction of the theory requires an in depth study of homotopy classes of multivalued boundary conditions. Geometric recursions based on the combined structure of the boundary conditions and the moduli space are used to compute the integrals. We also provide a detailed analysis of orientations.

Our open KdV and Virasoro constraints uniquely specify a theory of higher genus open descendent integrals. As a result, we obtain an open analog (governing all genera) of Witten's conjectures concerning descendent integrals on the Deligne-Mumford space of stable curves.

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1. Introduction

1.1. **Moduli of closed Riemann surfaces.** Let C be a connected complex manifold of dimension 1. If C is closed, the underlying topology is classified by the genus g. The moduli space $\mathcal{M}_g$ of complex structures of genus g has been studied since Riemann [18] in the 19^{th}

century. Deligne and Mumford defined a natural compactification

$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g$$

via stable curves (with possible nodal singularities) in 1969. The moduli $\mathcal{M}_{g,l}$ of curves $(C, p_1, \ldots, p_l)$ with l distinct marked points has a parallel treatment with compactification

$$\mathcal{M}_{g,l} \subset \overline{\mathcal{M}}_{g,l} \ .$$

We refer the reader to [4, 8] for the basic theory. The moduli space $\overline{\mathcal{M}}_{g,l}$ is a nonsingular complex orbifold of dimension $3g - 3 + l$.

1.2. **Witten's conjectures.** A new direction in the study of the moduli of curves was opened by Witten [24] in 1992 motivated by theories of 2-dimensional quantum gravity. For each marking index i, a complex *cotangent* line bundle

$$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,l}$$

is defined as follows. The fiber of $\mathbb{L}_i$ over the point

$$[C, p_1, \ldots, p_l] \in \overline{\mathcal{M}}_{g,l}$$

is the complex cotangent[1] space T^*_{C,p_i} of C at p_i. Let

$$\psi_i \in H^2(\overline{\mathcal{M}}_{g,l}, \mathbb{Q})$$

denote the first Chern class of $\mathbb{L}_i$. Witten considered the intersection products of the classes ψ_i. We will follow the standard bracket notation:

$$\big\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \big\rangle_g = \int_{\overline{M}_{g,l}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_l^{a_l}. \tag{1}$$

The integral on the right of (1) is well-defined when the stability condition

$$2g - 2 + l > 0$$

is satisfied, all the a_i are nonnegative integers, and the dimension constraint

$$3g - 3 + l = \sum a_i \tag{2}$$

holds. In all other cases, $\big\langle \prod_{i=1}^l \tau_{a_i} \big\rangle_g$ is defined to be zero. The empty bracket $\big\langle 1 \big\rangle_1$ is also set to zero. The intersection products (1) are often called *descendent integrals.*

By the dimension constraint (2), a unique genus g is determined by the a_i. For brackets without a genus subscript, the genus specified by

[1] By stability, p_i lies in the nonsingular locus of C.

the dimension constraint is assumed (the bracket is set to zero if the specified genus is fractional). The simplest integral is

$$\langle \tau_0^3 \rangle = \langle \tau_0^3 \rangle_0 = 1 \ . \tag{3}$$

Let t_i (for $i \geq 0$) be a set of variables. Let $\gamma = \sum_{i=0}^{\infty} t_i \tau_i$ be the formal sum. Let

$$F_g(t_0, t_1, ...) = \sum_{n=0}^{\infty} \frac{\langle \gamma^n \rangle_g}{n!}$$

be the generating function of genus g descendent integrals (1). The bracket $\langle \gamma^n \rangle_g$ is defined by monomial expansion and multilinearity in the variables t_i. Concretely,

$$F_g(t_0, t_1, ...) = \sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g,$$

where the sum is over all sequences of nonnegative integers $\{n_i\}$ with finitely many nonzero terms. The generating function

$$F = \sum_{g=0}^{\infty} u^{2g-2} F_g \tag{4}$$

arises as a partition function in 2-dimensional quantum gravity. Based on a different physical realization of this function in terms of matrix integrals, Witten [24] conjectured F satisfies two distinct systems of differential equations. Each system determines F uniquely and provides explicit recursions which compute all the brackets (1). Witten's conjectures were proven by Kontsevich [13]. Other proofs can be found in [15, 16].

Before describing the full systems of equations, we recall two basic properties. The first is the *string equation*: for $2g - 2 + l > 0$,

$$\left\langle \tau_0 \prod_{i=1}^{l} \tau_{a_i} \right\rangle_g = \sum_{j=1}^{l} \left\langle \tau_{a_j - 1} \prod_{i \neq j} \tau_{a_i} \right\rangle_g \ .$$

The second property is the *dilaton equation*: for $2g - 2 + l > 0$,

$$\left\langle \tau_1 \prod_{i=1}^{l} \tau_{a_i} \right\rangle_g = (2g - 2 + l) \left\langle \prod_{i=1}^{l} \tau_{a_i} \right\rangle_g \ .$$

The string and dilaton equations may be written as differential operators annihilating $\exp(F)$ in the following way:

$$
\begin{aligned}
(5) \qquad L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{u^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \ , \\
L_0 &= -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16} \ .
\end{aligned}
$$

Both the string and dilaton equations are derived [24] from a comparison result describing the behavior of the ψ classes under pull-back via the forgetful map

$$\pi : \overline{\mathcal{M}}_{g,l+1} \to \overline{\mathcal{M}}_{g,l} \ .$$

The string equation and the evaluation (3) together determine all the genus 0 brackets. The string equation, dilaton equation, and the evaluation

$$(6) \qquad \langle \tau_1 \rangle_1 = \frac{1}{24}$$

determine all the genus 1 brackets. In higher genus, further constraints are needed.

The first differential equations conjectured by Witten are the KdV equations. We define the functions

$$(7) \qquad \langle\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \rangle\rangle = \frac{\partial}{\partial t_{a_1}} \frac{\partial}{\partial t_{a_2}} \cdots \frac{\partial}{\partial t_{a_l}} F.$$

Of course, we have

$$\langle\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \rangle\rangle \Big|_{t_i=0, u=1} = \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \rangle \ .$$

The KdV equations are equivalent to the following set of equations for $n \geq 1$:

$$
\begin{aligned}
&(2n+1) u^{-2} \langle\langle \tau_n \tau_0^2 \rangle\rangle = \\
&\qquad = \langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle.
\end{aligned}
$$

For example, consider the KdV equation for $n = 3$ evaluated at $t_i = 0$. We obtain

$$7 \langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + \frac{1}{4} \langle \tau_2 \tau_0^4 \rangle_0.$$

Use of the string equation yields:

$$7 \langle \tau_1 \rangle_1 = \langle \tau_1 \rangle_1 + \frac{1}{4} \langle \tau_0^3 \rangle_0.$$

Hence, we conclude (6). In fact, the KdV equations and the string equation *together* determine all the products (1) and thus uniquely determine F.

The second system of differential equations for F is determined by a representation of a subalgebra of the Virasoro algebra. Consider the Lie algebra $\mathbf{L}$ of holomorphic differential operators spanned by

$$L_n = -z^{n+1}\frac{\partial}{\partial z}$$

for $n \geq -1$. The bracket is given by $[L_n, L_m] = (n-m)L_{n+m}$.

The equations (5) may be viewed as the beginning of a representation of $\mathbf{L}$ in a Lie algebra of differential operators. In fact, with certain homogeneity restrictions, there is a unique way to extend the assignment of L_{-1} and L_0 to a complete representation of $\mathbf{L}$. For $n \geq 1$, the expression for L_n takes the form

$$\begin{aligned} L_n &= \\ &= -\frac{3\cdot 5\cdot 7\cdots(2n+3)}{2^{n+1}}\frac{\partial}{\partial t_{n+1}} \\ &\quad + \sum_{i=0}^{\infty}\frac{(2i+1)(2i+3)\cdots(2i+2n+1)}{2^{n+1}}t_i\frac{\partial}{\partial t_{i+n}} \\ &\quad + \frac{u^2}{2}\sum_{i=0}^{n-1}(-1)^{i+1}\frac{(-2i-1)(-2i+1)\cdots(-2i+2n-1)}{2^{n+1}}\frac{\partial^2}{\partial t_i\partial t_{n-1-i}}. \end{aligned}$$

The second form of Witten's conjecture is that the above representation of $\mathbf{L}$ annihilates $\exp(F)$:

$$\forall n \geq -1, \quad L_n \exp(F) = 0\ . \tag{8}$$

The system of equations (8) also uniquely determines F.

The KdV equations and the Virasoro constraints provide a very satisfactory approach to the products (1). The aim of our paper is to develop a parallel theory for open Riemann surfaces. An *open Riemann surface* for us is obtained by removing open disks from a closed Riemann surface. See Section 1.3 below for a more detailed discussion. Hence, the terminology *Riemann surface with boundary* is more appropriate. We will use the terms *open* and *with boundary* synonymously.

For the remainder of the paper, a superscript c will signal integration over the moduli of closed Riemann surfaces. For example, we will write the generating series of descendent integrals (4) as

$$F^c(u, t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{2g-2}\sum_{n=0}^{\infty}\frac{\left\langle \gamma^n \right\rangle_g^c}{n!}\ .$$

We will later introduce a generating series F^o of descendent integrals over the moduli of open Riemann surfaces.

1.3. **Moduli of Riemann surfaces with boundary.** Let $\Delta \subset \mathbb{C}$ be the open unit disk, and let $\overline{\Delta}$ be the closure. An *extendable* embedding of the open disk in a closed Riemann surface

$$f : \Delta \to C$$

is a holomorphic map which extends to a holomorphic embedding of an open neighborhood of $\overline{\Delta}$. Two extendable embeddings in C are *disjoint* if the images of $\overline{\Delta}$ are disjoint.

A *Riemann surface with boundary* $(X, \partial X)$ is obtained by removing finitely many disjoint extendably embedded open disks from a connected closed Riemann surface. The boundary ∂X is the union of images of the unit circle boundaries of embedded disks Δ. Alternatively, Riemann surfaces with boundary can be defined as 1 dimensional complex manifolds with finitely many circular boundaries, each with a holomorphic collar structure.

Given a Riemann surface with boundary $(X, \partial X)$, we can canonically construct a *double* via Schwarz reflection through the boundary. The double $D(X, \partial X)$ of $(X, \partial X)$ is a closed Riemann surface. The *doubled genus* of $(X, \partial X)$ is defined to be the usual genus of $D(X, \partial X)$.

On a Riemann surface with boundary $(X, \partial X)$, we consider two types of marked points. The markings of *interior type* are points of $X \setminus \partial X$. The markings of *boundary type* are points of ∂X. Let $\mathcal{M}_{g,k,l}$ denote the moduli space of Riemann surfaces with boundary of doubled genus g with k distinct boundary markings and l distinct interior markings. The moduli space $\mathcal{M}_{g,k,l}$ is defined to be empty unless the stability condition,

$$2g - 2 + k + 2l > 0,$$

is satisfied. The moduli space $\mathcal{M}_{g,k,l}$ may have several connected components depending upon the topology of $(X, \partial X)$ and the cyclic orderings of the boundary markings. Foundational issues concerning the construction of $\mathcal{M}_{g,k,l}$ are addressed in [14].

We view $\mathcal{M}_{g,k,l}$ as a real orbifold of real dimension $3g - 3 + k + 2l$. Of course, $\mathcal{M}_{g,k,l}$ is not compact (in addition to the nodal degenerations present in the moduli of closed Riemann surfaces, new issues involving the boundary approach of interior markings and the meeting of boundary circles arise). Furthermore, $\mathcal{M}_{g,k,l}$ may be not be orientable. Non-orientability presents serious obstacles for the definition of a theory of descendent integration over the moduli spaces of Riemann surfaces with boundary.

We will often refer to connected Riemann surfaces with boundary as *open* Riemann surfaces or *open* geometries (as the interior is open). The *genus* of an open Riemann surface will always be the doubled genus.

1.4. **Descendents.** Since interior marked points have well-defined cotangent spaces, there is no difficulty in defining the cotangent line bundles

$$\mathbb{L}_i \to \mathcal{M}_{g,k,l}$$

for each interior marking, $i = 1, \dots, l$. We do *not* consider the cotangent lines at the boundary points.

Naively, we would like to consider a descendent theory via integration of products of the first Chern classes $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,k,l})$ over a compactification $\overline{\mathcal{M}}_{g,k,l}$ of $\mathcal{M}_{g,k,l}$. Namely,

$$\left\langle \tau_{a_1}\tau_{a_2}\cdots\tau_{a_l}\sigma^k \right\rangle_g^o = \int_{\overline{\mathcal{M}}_{g,k,l}} \psi_1^{a_1}\psi_2^{a_2}\cdots\psi_l^{a_l} \ . \tag{9}$$

when

$$2\sum_{i=1}^{l} a_i = 3g - 3 + k + 2l,$$

and in all other cases $\left\langle \tau_{a_1}\cdots\tau_{a_l}\sigma^k \right\rangle_g^o = 0$. Here, τ_a corresponds to the a^{th} power of a cotangent class ψ^a as before. The new insertion σ corresponds to the addition of a boundary marking.[2] To rigorously define the right-hand side of (9), at least three significant steps must be taken:

(i) A compact moduli space $\overline{\mathcal{M}}_{g,k,l}$ must be constructed. Because degenerations of Riemann surfaces with boundary occur in real codimension one, candidates for $\overline{\mathcal{M}}_{g,k,l}$ are real orbifolds with boundary $\partial\overline{\mathcal{M}}_{g,k,l}$.
(ii) For integration over $\overline{\mathcal{M}}_{g,k,l}$ to be well-defined, boundary conditions of the integrand must be specified along $\partial\overline{\mathcal{M}}_{g,k,l}$. That is, the integrand must be lifted to the relative cohomology group $H^{3g-3+k+2l}(\overline{\mathcal{M}}_{g,k,l}, \partial\overline{\mathcal{M}}_{g,k,l})$.
(iii) Orientation issues must be addressed.

The most challenging aspect of defining open descendent integrals is the specification of boundary conditions (ii). At first glance, one might hope to find a natural lift of ψ_i to $H^2(\overline{\mathcal{M}}_{g,k,l}, \partial\overline{\mathcal{M}}_{g,k,l})$. However, this

[2] The power of σ specifies the number of boundary markings.

does not appear feasible. Rather, consider the bundle

$$E = \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus a_i}. \tag{10}$$

The Euler class of E is given by

$$e(E) = \psi_1^{a_1}\psi_2^{a_2}\cdots\psi_l^{a_l}.$$

So, it suffices to find a natural lift of $e(E)$ to relative cohomology. This is the approach we follow. Such an approach leads to considerable difficulties in proving recursive relations between descendent integrals. Indeed, for closed descendent integrals, the proofs of recursive relations use heavily the factorization of the integrand as a product of cohomology classes [24].

The need to specify boundary conditions and the orientation issues impose serious constraints on the ultimate definition of $\overline{\mathcal{M}}_{g,k,l}$. For $g > 0$, it appears these constraints can only be satisfied if $\overline{\mathcal{M}}_{g,k,l}$ is the compactification of a covering space of $\mathcal{M}_{g,k,l}$ that arises as the moduli space of open Riemann surfaces with an additional structure. The construction of $\overline{\mathcal{M}}_{g,k,l}$ for $g > 0$ will be given in [21]. In the case $g = 0$ treated here, we take $\overline{\mathcal{M}}_{0,k,l}$ to be the space of stable disks studied previously in the context of the Fukaya category.

In this paper, we complete steps (i-iii) in the doubled genus 0 case. The outcome is a fully rigorous theory of descendent integration on the moduli space of disks with interior and boundary markings. Moreover, we prove analogs for $\overline{\mathcal{M}}_{0,k,l}$ of the string and dilation equations as well as the topological recursion relations, which allow us to completely solve the theory.

1.5. **Construction of the descendent theory of pointed disks.** By the Riemann Mapping Theorem, the open geometry of genus 0 is just the disk with a single boundary circle. The simplest moduli space is $\mathcal{M}_{0,3,0}$ parameterizing disks with 3 boundary markings. There are exactly two disks with 3 distinct boundary points (corresponding to the two possible cyclic orders). Thus $\mathcal{M}_{0,3,0}$ is already compact, and it has no boundary. So, we can evaluate the corresponding open descendent integral without reference to boundary conditions. In our definition of open descendent integrals (18), the geometric integral over $\overline{\mathcal{M}}_{0,k,l}$ is multiplied by $2^{\frac{1-k}{2}}$. In particular, for $\mathcal{M}_{0,3,0}$ the power is 2^{-1}. We conclude that

$$\left\langle \sigma^3 \right\rangle_0^o = 1 \ . \tag{11}$$

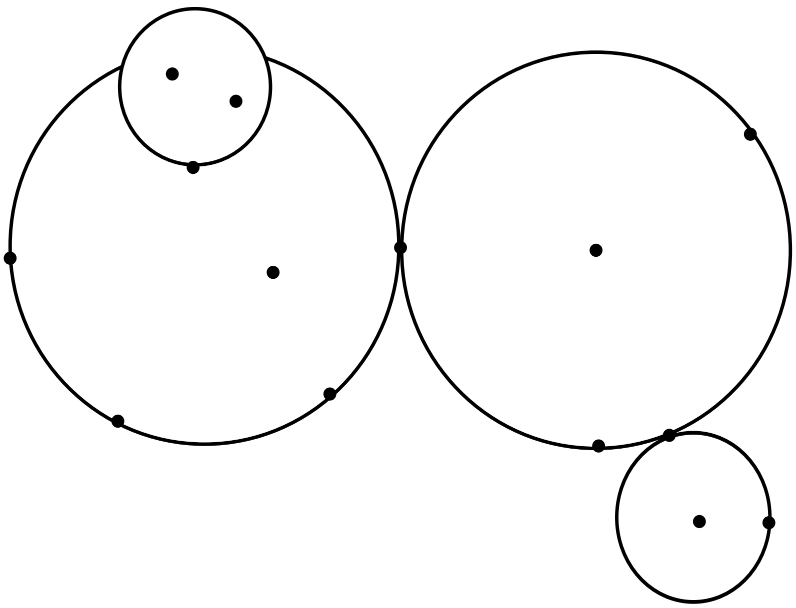

FIGURE 1. A nodal disk with 3 disk components, one sphere component, 5 internal marked points and 6 boundary marked points.

Similarly, the moduli space $\mathcal{M}_{0,1,1}$, parameterizing disks with 1 boundary point and 1 interior point, consists of a single point. It follows that also $\left\langle \tau_0 \sigma \right\rangle^o = 1$.

In general, the compact moduli space $\overline{\mathcal{M}}_{0,k,l}$ of our construction is a compactification of $\mathcal{M}_{0,k,l}$ stemming from ideas very close to Deligne-Mumford stability. It allows for internal sphere bubbles and boundary disk bubbles following the approach familiar from the Fukaya category [5, 14, 19]. See Figure 1.

The boundary conditions we impose for our definition of the descendent integrals are the most delicate aspect of the construction. As mentioned above, our strategy is to lift the Euler class $e(E)$ to the relative cohomology group $H^{k+2l-3}(\overline{\mathcal{M}}_{0,k,l}, \partial\overline{\mathcal{M}}_{0,k,l})$. Here, $E \to \overline{\mathcal{M}}_{0,k,l}$ is the bundle given by equation (10). Such a lift can be given by constructing a non-vanishing section $\mathbf{s}$ of the restriction $E|_{\partial\overline{\mathcal{M}}_{0,k,l}}$. There is no unique construction of such a section $\mathbf{s}$. Rather, we give a construction that is well-defined up to non-vanishing homotopy. It follows that the resulting lift of $e(E)$ to relative cohomology is well-defined. Our construction of $\mathbf{s}$ relies on the decomposition of the boundary $\partial\overline{\mathcal{M}}_{0,k,l}$ into products of moduli spaces of open Riemann surfaces with fewer marked points.

A surprising feature of our construction is that the section $\mathbf{s}$ must be multi-valued. Multiple valued sections are forced on us by a non-trivial monodromy in the geometric constraint defining $\mathbf{s}$. See Remark 3.5. In genus zero, the moduli space $\overline{\mathcal{M}}_{0,k,l}$ is always a smooth manifold. So, the phenomenon of multi-valued sections is not the result of orbifold isotropy groups.

Another unintuitive aspect of the boundary conditions is the complexity of their dependence on the boundary marked points. Indeed, we consider only the cotangent lines $\mathbb{L}_i$ at interior marked points. So, by analogy with the string equation, one would expect a simple geometric recursion to govern the dependence of open descendent integrals on the number of boundary marked points. This is not the case. To the contrary, in Section 1.7 we observe a parallel between the formulas for open descendent integrals on $\overline{\mathcal{M}}_{0,k,l}$ and closed $\lambda_g\lambda_{g-1}$ descendent integrals on $\overline{\mathcal{M}}_{g,l}$, where g is proportional to k. That is, the number of boundary marked points in open genus zero descendent integrals plays a role analogous to the genus in closed $\lambda_g\lambda_{g-1}$ descendent integrals. This is one indication of the complex dependency of the boundary conditions on the boundary marked points.

Our proofs of the open analogs of the string, dilaton, and topological recursion relations all use the boundary conditions in an essential way. The boundary conditions are constructed in Section 3.

1.6. **Differential equations.**

1.6.1. *Partition functions.* Though the resolution of the issues (i-iii) of Section 1.4 for the moduli of pointed disks (the genus 0 case) requires a substantial mathematical development, the evaluation of the theory is remarkably simple. The answer guides the higher genus open cases. We propose here an evaluation of the theory of descendent integration over the moduli of Riemann surfaces with boundary for all g, k, and l. For the genus 0 case, we prove our proposal is correct using our foundational development. The main conjectures of the paper concern the $g > 0$ cases. Even before giving complete definitions resolving (i-iii) for $g > 0$, we are able to conjecture a complete solution.

The solution is again via differential equations for the generating series of descendent invariants. Recall the descendent series for the moduli of closed Riemann surfaces,

$$F^c(u, t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{2g-2} F_g^c(t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{2g-2} \sum_{n=0}^{\infty} \frac{\left\langle \gamma^n \right\rangle_g^c}{n!} ,$$

where $\gamma = \sum_{i=0}^{\infty} t_i \tau_i$. Similarly, we define the open descendent series as

$$F^o(u, s, t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{g-1} F_g^o(t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{g-1} \sum_{n=0}^{\infty} \frac{\left\langle \gamma^n \delta^k \right\rangle_g^o}{n!k!} \ ,$$

where $\gamma = \sum_{i=0}^{\infty} t_i \tau_i$ is as before and $\delta = s\sigma$. The associated partition functions are

$$\mathsf{Z}^c = \exp(F^c) \ , \quad \mathsf{Z}^o = \exp(F^o) \ .$$

We define the full partition function by

$$\mathsf{Z} = \exp(F^c + F^o) \ .$$

1.6.2. *Virasoro constraints.* Let L_n be the differential operators in the variables u and t_i defined in Section 1.2. We define an s extension $\mathcal{L}_n$ of L_n by the following formula:

$$\mathcal{L}_n = L_n + u^n s \frac{\partial^{n+1}}{\partial s^{n+1}} + \frac{3n+3}{4} u^n \frac{\partial^n}{\partial s^n} \ , \tag{12}$$

for $n \geq -1$. Using the relations

$$[L_n, L_m] = (n-m) L_{n+m}$$

and the commutation of L_n with the operators u, s, and $\frac{\partial}{\partial s}$, we easily obtain the Virasoro relation

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m) \mathcal{L}_{n+m}$$

By Witten's conjecture, L_n annihilates Z^c.

Conjecture 1. *The operators $\mathcal{L}_n$ annihilate the full partition function,*

$$\forall n \geq -1, \qquad \mathcal{L}_n \ \mathsf{Z} = 0.$$

The restriction of the full partition function Z to the subspace defined by $t_i = 0$ for all i is easily evaluated,

$$\mathsf{Z}(s, t_0 = 0, t_1 = 0, t_2 = 0, \ldots) = \left\langle \sigma^3 \right\rangle_0^o \frac{s^3}{3!} = \frac{s^3}{3!} \ . \tag{13}$$

By a dimension analysis, the descendent $\left\langle \sigma^3 \right\rangle_0^o$, evaluated by (11), is the only nonzero term which survives the restriction. The Virasoro constraints of Conjecture 1 then determine Z from the restriction (13). In other words, Z^o is uniquely and effectively specified by Conjecture 1, the restriction (13), and Z^c.

Using our construction of the descendent theory of pointed disks, we prove the genus 0 part of Conjecture 1.

Theorem 1.1. *The operators $\mathcal{L}_n$ annihilate the genus zero partition function up to terms of higher genus. That is, for $n \geq -1$, the coefficient of u^{-1} in*

$$\mathcal{L}_n \ \exp(u^{-2}F_0^c + u^{-1}F_0^o)$$

vanishes.

The proof of Theorem 1.1 is presented in Section 5.

1.6.3. *String and dilaton equations.* The string and dilaton equations for F^o are obtained from the operators $\mathcal{L}_{-1}$ and $\mathcal{L}_0$ respectively. The string equation for the open geometry is

$$\frac{\partial F^o}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F^o}{\partial t_i} + u^{-1}s. \tag{14}$$

The dilaton equation is

$$\frac{\partial F^o}{\partial t_1} = \sum_{i=0}^{\infty} \left(\frac{2i+1}{3}\right) t_i \frac{\partial F^o}{\partial t_i} + \frac{2}{3} s \frac{\partial F^o}{\partial s} + \frac{1}{2}.$$

The string equation implies that for $2g - 2 + k + 2l > 0$,

$$\left\langle \tau_0 \prod_{i=1}^{l} \tau_{a_i} \sigma^k \right\rangle_g^o = \sum_j \left\langle \tau_{a_j - 1} \prod_{i \neq j} \tau_{a_i} \sigma^k \right\rangle_g^o.$$

The dilaton equation implies that for $2g - 2 + k + 2l > 0$,

$$\left\langle \tau_1 \prod_{i=1}^{l} \tau_{a_i} \sigma^k \right\rangle_g^o = (g - 1 + k + l) \left\langle \prod_{i=1}^{l} \tau_{a_i} \sigma^k \right\rangle_g^o. \tag{15}$$

The string and dilaton equations for F^c together with the Virasoro relations

$$\mathcal{L}_{-1} \ \mathsf{Z} = \mathcal{L}_0 \ \mathsf{Z} = 0$$

imply the string and dilaton equations for F^o. The following result is therefore a consequence of Theorem 1.1. It is also an important step in the proof.

Theorem 1.2. *The string and dilaton equations hold for F_0^o.*

1.6.4. *KdV equations.* We have already defined (7) double brackets in the compact case. For the open invariants, the definition is parallel:

$$\big\langle\big\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \big\rangle\big\rangle^o = \frac{\partial}{\partial t_{a_1}} \frac{\partial}{\partial t_{a_2}} \cdots \frac{\partial}{\partial t_{a_l}} \frac{\partial^k}{\partial s^k} F^o.$$

Also,

$$\big\langle\big\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \big\rangle\big\rangle_g^o = \frac{\partial}{\partial t_{a_1}} \frac{\partial}{\partial t_{a_2}} \cdots \frac{\partial}{\partial t_{a_l}} \frac{\partial^k}{\partial s^k} F_g^o.$$

We conjecture an analog of Witten's KdV equations in the compact case.

Conjecture 2. *For $n \geq 1$, we have*

$$\begin{aligned}(2n+1)u^{-1}\big\langle\big\langle\tau_n\big\rangle\big\rangle^o &= u\big\langle\big\langle\tau_{n-1}\tau_0\big\rangle\big\rangle^c\big\langle\big\langle\tau_0\big\rangle\big\rangle^o \\ &\quad +2\big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle^o\big\langle\big\langle\sigma\big\rangle\big\rangle^o + 2\big\langle\big\langle\tau_{n-1}\sigma\big\rangle\big\rangle^o \\ &\quad -\frac{u}{2}\big\langle\big\langle\tau_{n-1}\tau_0^2\big\rangle\big\rangle^c .\end{aligned}$$

Together with the string equation (14), the system of differential equations of Conjecture 2 uniquely determines F^o from $\big\langle\sigma^3\big\rangle_0^o$ and F^c. For example, we calculate (using $n=1$):

$$3\big\langle\tau_1\big\rangle_1^o = 2\big\langle\tau_0\sigma\big\rangle_0^o - \frac{1}{2}\big\langle\tau_0^3\big\rangle_0^c = \frac{3}{2} ,$$

so $\big\langle\tau_1\big\rangle = \frac{1}{2}$. In fact, the system is significantly overdetermined. We speculate the differential equations for F^o of Conjecture 2 have a solution if and only if F^c satisfies Witten's KdV equations. The agreement of Conjectures 1 and 2 is certainly not obvious. However, recent work of Buryak [1] proves they are equivalent. Moreover, Buryak proves the consistency of the open KdV equations.

Using our construction of the descendent theory of pointed disks, we prove the genus 0 part of Conjecture 2.

Theorem 1.3. *The open analogs of the KdV equations hold in genus zero. Namely,*

$$(2n+1)\big\langle\big\langle\tau_n\big\rangle\big\rangle_0^o = \big\langle\big\langle\tau_{n-1}\tau_0\big\rangle\big\rangle_0^c\big\langle\big\langle\tau_0\big\rangle\big\rangle_0^o + 2\big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma\big\rangle\big\rangle_0^o$$

for $n \geq 1$.

A complete proposal for a theory of descendent integration in higher genus will be presented in a forthcoming paper by J.S. and R.T. [21]. Via the construction of [21], R.T. has found a combinatorial formula that allows effective calculation of the descendent integrals in arbitrary genus [22]. Several months after the first version of this paper appeared, Conjectures 1 and 2 were proved by A. Buryak and R.T. [2] using the combinatorial formula of [22].

The study here of descendent integration over the moduli of open Riemann surfaces fits into a larger investigation of exact formulas for open Gromov-Witten theory. See [6, 10, 12, 14, 17, 20, 23] for related integration over the moduli space of disk maps. It is an interesting problem to extend the open descendent theory of the present paper to moduli spaces of open stable maps and the Fukaya category. We plan to address this problem in future work.

1.7. **Formulas in genus 0.** Descendent integration over the moduli space of compact genus 0 Riemann surfaces with marked points has a very simple answer,

$$\left\langle \tau_{a_1} \dots \tau_{a_l} \right\rangle_0^c = \binom{l-3}{a_1, \dots, a_l} .$$

The above evaluation is easily derived from the string equation for F^c and the initial value

$$\left\langle \tau_0^3 \right\rangle_0^o = 1 .$$

Alternatively, the evaluation can be derived from the topological recursion relations [24] for F_0^c.

A explicit evaluation also can be obtained for the open invariants

$$\left\langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o$$

in genus 0. Using the string equation for F^o of Theorem 1.2, we can assume $a_i \geq 1$ for all i. By the dimension constraint,

$$-3 + k + 2l = \sum_{i=1}^{l} 2a_i .$$

Theorem 1.4. *We have the evaluation*

$$\left\langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o = \frac{(\sum_{i=1}^{l} 2a_i - l + 1)!}{\prod_{i=1}^{l} (2a_i - 1)!!}$$

in case $a_i \geq 1$ for all i.

The double factorial of an odd positive integer is the product of all odd integers not exceeding the argument,

$$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 .$$

While such double factorials also occur [7] in the formula for $\lambda_g \lambda_{g-1}$ descendent integrals over the moduli space of $\overline{\mathcal{M}}_{g,l}$ of higher genus curves, a direct connection is not known to us.

We derive Theorem 1.4 as a consequence of the following topological recursion relations for the open theory in genus 0.

Theorem 1.5. *For $n > 0$, two topological recursion relations hold for F_0^o:*

(TRR I) $\quad \langle\langle \tau_n \sigma \rangle\rangle_0^o = \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_0^c \, \langle\langle \tau_0 \sigma \rangle\rangle_0^o + \langle\langle \tau_{n-1} \rangle\rangle_0^o \, \left\langle\left\langle \sigma^2 \right\rangle\right\rangle_0^o ,$

(TRR II) $\quad \langle\langle \tau_n \tau_m \rangle\rangle_0^o = \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_0^c \, \langle\langle \tau_0 \tau_m \rangle\rangle_0^o + \langle\langle \tau_{n-1} \rangle\rangle_0^o \, \langle\langle \tau_m \sigma \rangle\rangle_0^o .$

1.8. **Plan of the paper.** In Section 2 we review the moduli space of stable marked disks and discuss stable graphs. In Section 3 we define the canonical boundary conditions and the open descendent integrals. We then define the more subtle special canonical boundary conditions and show they exist. We prove the string and dilaton equations and the topological recursion relations using geometric methods in Section 4. In Sections 5 and 6, we prove the genus 0 open Virasoro relations and KdV equations using the string and dilaton equations and assuming the genus 0 formula of Theorem 1.4. Finally, in Section 7 we prove the genus 0 formula using the open topological recursion relations and the dilaton equation.

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2. Moduli of disks

2.1. **Conventions.** We begin with some useful notations and comments.

Notation 2.1. Throughout this paper the notation $\dim_{\mathbb{C}}$ $(rk_{\mathbb{C}})$ will mean $\frac{\dim_{\mathbb{R}}}{2}$ $(\frac{rk_{\mathbb{R}}}{2})$.

Throughout this paper whenever we say a manifold, unless specified otherwise, we mean a smooth manifold with corners in the sense of [11]. Similarly, notions which relate to manifolds or maps between them are in accordance with that article.

Notation 2.2. We write Δ for the standard unit disk in $\mathbb{C}$, with the standard complex structure.

Notation 2.3. For a set A denote by A° the set

$$\{x^\circ \text{ for } x \in A\}\,.$$

For $l \in \mathbb{N}$, we use the notation $[l]$ to denote $\{1, 2, \ldots, l\}$. We write $[0]$ for the empty set. We also denote by $[l^\circ]$ the set $[l]^\circ$.

Notation 2.4. For a set A write 2^A_{fin} for the collection of finite subsets of A. We say that $B \subseteq 2^A_{fin}$ is a *disjoint subset* if its elements are pairwise disjoint.

Notation 2.5. Put $\mathfrak{L} = 2^{\mathbb{Z}\cup\mathbb{Z}^\circ}_{fin}$. Throughout the article we identify $i \in \mathbb{Z} \cup \mathbb{Z}^\circ$ with $\{i\} \in \mathfrak{L}$, without further mention. We denote by $2^{\mathfrak{L}}_{fin,disj}$ the collection of finite disjoint subsets A of $\mathfrak{L}$, such that $\emptyset \notin A$. For $A \in 2^{\mathfrak{L}}_{fin,disj}$, let $\cup A \in \mathfrak{L}$ denote the union of its elements as sets.

2.2. **Stable disks.** Throughout the paper markings will be taken from $\mathfrak{L}$. We recall the notion of a *stable marked disk*.

Definition 2.6. Given $B, I \in 2^{\mathfrak{L}}_{fin,disj}$ with $B\cap I = \emptyset$ and $B\cup I$ disjoint, we define a $(B, I)-$*marked smooth surface* to be a triple

$$(\Sigma, \{z_i\}_{i\in B}, \{z_i\}_{i\in I})$$

where

(a) Σ is a Riemann surface with boundary.
(b) For each $i \in B$, $z_i \in \partial\Sigma$.
(c) For each $i \in I$, $z_i \in \operatorname{int}\Sigma$.

We call B *the set of boundary labels*. We call I *the set of interior labels*.

We sometimes omit the marked points from our notations. Given a smooth marked surface Σ, we write $B(\Sigma)$ for the set of its boundary labels. We also use $B(\Sigma)$ to denote the set of boundary marked points of Σ. Similarly, we write $I(\Sigma)$ the set of interior labels of Σ, and again, we also write $I(\Sigma)$ for the set of internal marked points of Σ.

Definition 2.7. Given $B, I \in 2^{\mathfrak{L}}_{fin,disj}$ with $B\cap I = \emptyset$ and $B\cup I$ disjoint, a $(B, I)-$*pre-stable marked* genus 0 surface is a tuple

$$\Sigma = \left(\{\Sigma_\alpha\}_{\alpha\in\mathcal{D}\coprod\mathcal{S}}, \sim_B, \sim_I\right),$$

where

(a) $\mathcal{D}$ and $\mathcal{S}$ are finite sets. For $\alpha \in \mathcal{D}$, Σ_α is a smooth marked disk; for $\alpha \in \mathcal{S}$, Σ_α is a smooth marked sphere.
(b) An equivalence relation $\sim_B$ on the set of all boundary marked points, with equivalence classes of size at most 2. An equivalence relation $\sim_I$ on the set of all internal marked points, with equivalence classes of size at most 2.

The two equivalence relations $\sim_B$ and $\sim_I$ taken together are denoted by $\sim$. The above data satisfies

(a) B is the set of labels of points belonging to $\sim_B$ equivalence classes of size 1. I is the set of labels of points belonging to $\sim_I$ equivalence classes of size 1.

(b) The topological space $\coprod_{\alpha\in\mathcal{D}\cup\mathcal{S}}\Sigma_\alpha/\sim$ is connected and simply connected.

(c) The topological space $\coprod_{\alpha\in\mathcal{D}}\Sigma_\alpha/\sim_B$ is connected or empty.

We also write $\Sigma = \coprod_{\alpha\in\mathcal{D}\cup\mathcal{S}}\Sigma_\alpha/\sim$. If $\mathcal{D}$ is empty, Σ is called a *pre-stable marked sphere*. Otherwise it is called a *pre-stable marked disk*. We denote by $\mathcal{M}_B(\Sigma_\alpha)$ the set of labels of boundary marked points of Σ_α which belong to $\sim_B$ equivalence classes of size 1. We define $\mathcal{M}_I(\Sigma_\alpha)$ similarly. The $\sim_B$ (resp. $\sim_I$) equivalence classes of size 2 are called boundary (resp. interior) nodes.

A smooth marked disk D is called *stable* if

$$|B(D)| + 2|I(D)| \geq 3.$$

A smooth marked sphere is *stable* if it has at least 3 marked points. A pre-stable marked genus 0 surface is called a *stable marked genus* 0 *surface* if each of its constituent smooth marked spheres and smooth marked disks are stable.

Notation 2.8. In case $B = A^\circ$ for some A, we denote the marked point z_{i°, for $i^\circ \in B$, by x_i. In this case we also use the notation $(\Sigma, \mathbf{x}, \mathbf{z})$ to denote a stable marked surface, where $\mathbf{x} = \{x_i\}_{i^\circ\in B(\Sigma)}$ and $\mathbf{z} = \{z_i\}_{i\in I(\Sigma)}$.

Definition 2.9. Let Σ, Σ', be stable marked genus 0 surfaces with $B(\Sigma) = B(\Sigma')$ and $I(\Sigma) = I(\Sigma')$. An *isomorphism* $f : \Sigma \to \Sigma'$ is a homeomorphism such that

(a) For each $\alpha \in \mathcal{D}\cup\mathcal{S}$, the restriction $f|_{\Sigma_\alpha}$ maps Σ_α biholomorphically to some $\Sigma_{\alpha'}$ for $\alpha' \in \mathcal{D}'\cup\mathcal{S}'$.

(b) For each $i \in B(\Sigma)\cup I(\Sigma)$ there holds $f(z_i) = z'_i$.

Remark 2.10. The automorphism group of a stable marked genus 0 surface is trivial.

2.3. **Stable graphs.** It is useful to encode some of the combinatorial data of stable marked disks in graphs.

Definition 2.11. A (not necessarily connected, genus 0) *pre-stable graph* Γ is a tuple $\left(V = V^O \cup V^C, E, \ell_I, \ell_B\right)$, where

(a) V^O, V^C, are finite sets.

(b) E is a subset of the set of (unordered) pairs of elements of V.

(c) $\ell_I : V \to 2^{\mathfrak{L}}_{fin,disj}$, $\ell_B : V^O \to 2^{\mathfrak{L}}_{fin,disj}$.

We call the elements of V the *vertices* of Γ, where V^O are the *open vertices*, and V^C are the *closed vertices*. We call the elements of E the *edges* of Γ. An edge between open vertices is called a *boundary edge*.

The other edges are called *interior edges*. We call $\ell_I(v)$ the *interior labels* of v, and $\ell_B(v)$ the *boundary labels*. We demand that Γ satisfies

(a) The graph (V,E) is a forest, namely, a collection of trees.
(b) If $v,u\in V^O$ belong to the same connected component of Γ, they also belong to the same connected component in the subgraph of Γ spanned by V^O.
(c) The sets $\ell_I(v)$ for $v\in V$ and $\ell_B(v)$ for $v\in V^O$ are collectively pairwise disjoint. That is, labels are unique.
(d) (i) For $W\subset V$ spanning a connected component of Γ, the subset $\cup_{v\in W}(\ell_B(v)\cup\ell_I(v))\subset 2^{\mathfrak{L}}_{fin}$ is disjoint.
 (ii) For $i=1,2$, let $W_i\subset V$ span connected components of Γ and let $U_i\subset\cup_{v\in W_i}(\ell_B(v)\cup\ell_I(v))$ be proper subsets that are disjoint. Then $\cup U_1\neq\cup U_2$.

We say that Γ is connected if its underlying graph, (V,E) is connected.

Remark 2.12. Condition (d) is designed to achieve the following:

(a) The operator $\mathcal{B}$ of Definition 3.10 takes stable graphs to stable graphs.
(b) The operator ∂ of Definition 2.18 takes stable graphs to stable graphs.

Part (i) ensures the label sets $\ell_I(v),\ell_B(v)\subset 2^{\mathfrak{L}}_{fin}$, remain disjoint under the above operations. Part (ii) ensures labels remain unique.

Notation 2.13. For a vertex $v\in V$, denote by $E_v\subseteq E$ the set of edges containing v. We denote by E^I_v the set of interior edges of v and by E^I the set of all interior edges of Γ. For $v\in V^O$, denote by E^B_v the set of boundary edges of v. Denote by E^B the set of all boundary edges of Γ. We define

$$B(v)=\ell_B(v)\cup E^B_v,\qquad I(v)=\ell_I(v)\cup E^I_v,$$

and we set $k(v)=|B(v)|,l(v)=|I(v)|$. We also write

$$B(\Gamma)=\cup_{v\in V^O}\ell_B(v),\qquad I(\Gamma)=\cup_{v\in V}\ell_I(v).$$

We define $k(\Gamma)=|B(\Gamma)|,l(\Gamma)=|I(\Gamma)|$. Finally, if $i\in I(\Gamma)$ we define $v_i=v_i(\Gamma)\in V$ to be the unique vertex $v\in V$ with $i\in\ell_I(v)$.

For Γ a pre-stable graph, we write $V(\Gamma),E(\Gamma),\ell^\Gamma_I,\ell^\Gamma_B$, for the sets of vertices, edges, interior labels and boundary labels respectively. Similarly, we write $V^C(\Gamma)$ and so on. We also use analogously defined notation $I^\Gamma(v),B^\Gamma(v)$.

Given a pre-stable graph Γ, we define

$$\varepsilon=\varepsilon_\Gamma:V(\Gamma)\to\{O,C\}$$

by $\varepsilon(v) = O$ if and only if $v \in V^O$. In specifying a stable graph, we may specify ε instead of specifying the partition $V = V^O \cup V^C$.

Remark 2.14. Condition (b) above means that each connected component of closed vertices is a tree rooted in a neighbor of an open vertex. Other vertices in this tree have no open neighbors. The root has a unique open neighbor. This is a combinatorial analog of the geometric condition (c) of Definition 2.7.

Although ℓ_B was defined only for boundary vertices, we sometimes write $\ell_B(v) = \emptyset$ for $v \in V^C$. Similarly, we set $B(\Gamma) = \emptyset$ in case $V^O = \emptyset$.

Definition 2.15. An open vertex v in a pre-stable graph Γ is called *stable* if $k(v) + 2l(v) \geq 3$. A closed vertex v in a pre-stable graph Γ is called *stable* if $l(v) \geq 3$. If all the vertices of Γ are stable we say that Γ is *stable*. We denote by $\mathcal{G}$ the collection of all stable graphs.

To each stable marked genus 0 surface Σ we associate a connected stable graph as follows. We set $V^O = \mathcal{D}$ and $V^C = \mathcal{S}$. For $v \in V$, we set

$$\ell_B(v) = \mathcal{M}_B(\Sigma_v), \qquad \ell_I(v) = \mathcal{M}_I(\Sigma_v).$$

An edge between two vertices corresponds to a node between their corresponding components. One easily checks that the associated stable graph is well defined and satisfies all the requirements of the definitions. Moreover, Σ is a stable marked disk if and only if $V^O \neq \emptyset$. Otherwise it is a stable marked sphere.

Notation 2.16. The graph associated to a stable disk Σ is denoted by $\Gamma(\Sigma)$.

2.4. **Smoothing and boundary.**

Definition 2.17. The *smoothing* of a stable graph Γ at an edge e is the stable graph

$$d_e\Gamma = d_{\{e\}}\Gamma = \Gamma' = (V', E', \ell'_I, \ell'_B)$$

defined as follows. Write $e = \{u, v\}$. The vertex set is given by

$$V' = (V \setminus \{u, v\}) \cup \{uv\}.$$

The new vertex uv is closed if and only if both u and v are closed. Writing

$$E'_{uv} = \{\{w, uv\} | \{w, u\} \in E \text{ or } \{w, v\} \in E \text{ and } w \neq u, v\},$$

we set

$$E' = (E \setminus (E_u \cup E_v)) \cup E'_{uv}.$$

Furthermore,

$$\ell'_I(w) = \ell_I(w), \qquad w \in V' \setminus \{uv\},$$
$$\ell'_I(uv) = \ell_I(u) \cup \ell_I(v),$$

and similarly for ℓ_B.

Observe that there is a natural proper injection $E' \hookrightarrow E$, so we may identify E' with a subset of E. Using the identification, we extend the definition of smoothing in the following manner. Given a set $S = \{e_1, \ldots, e_n\} \subseteq E(\Gamma)$, define the smoothing at S as

$$d_S\Gamma = d_{e_n}(\ldots d_{e_2}(d_{e_1}\Gamma)\ldots).$$

Observe that $d_S\Gamma$ does not depend on the order of smoothings performed.

Note that in case $\Gamma = d_S\Gamma'$, we have a natural identification between $E(\Gamma)$ and $E(\Gamma') \setminus S$.

Definition 2.18. We define the boundary maps

$$\partial : \mathcal{G} \to 2^{\mathcal{G}}, \qquad \partial^! : \mathcal{G} \to 2^{\mathcal{G}}, \qquad \partial^B : \mathcal{G} \to 2^{\mathcal{G}},$$

by

$$\partial\Gamma = \{\Gamma' |\, \exists\emptyset \neq S \subseteq E(\Gamma'),\, \Gamma = d_S\Gamma'\}, \qquad \partial^!\Gamma = \{\Gamma\} \cup \partial\Gamma,$$
$$\partial^B\Gamma = \left\{\Gamma' \,\middle|\, \Gamma' \in \partial\Gamma,\, |E^B(\Gamma')| \geq 1\right\}.$$

Denote also by ∂ the map $2^{\mathcal{G}} \to 2^{\mathcal{G}}$ given by

$$\partial\{\Gamma_\alpha\}_{\alpha\in A} = \bigcup_{\alpha\in A} \partial\Gamma_\alpha.$$

and similarly for $\partial^!, \partial^B$.

2.5. **Moduli and orientations.**

Notation 2.19. For $B, I \in 2^{\mathfrak{L}}_{fin,disj}$ with $B \cap I = \emptyset$ and $B \cup I$ disjoint, denote by $\overline{\mathcal{M}}_{0,B,I}$ the set of isomorphism classes of stable marked disks whose set of boundary labels is B and whose set of interior labels is I. Denote by $\mathcal{M}_{0,B,I}$ the subset of $\overline{\mathcal{M}}_{0,B,I}$ consisting of isomorphism classes of smooth marked disks. We denote by $\overline{\mathcal{M}}_{0,I}$ the set of isomorphism classes of stable marked spheres whose label set is I. Let $\mathcal{M}_{0,I}$ be the set of isomorphism classes of smooth marked spheres with label set I. For $\Gamma \in \mathcal{G}$, denote by $\mathcal{M}_\Gamma$ the set of isomorphism classes of stable marked genus zero surfaces with associated graph Γ. Define

$$\overline{\mathcal{M}}_\Gamma = \coprod_{\Gamma' \in \partial^!\Gamma} \mathcal{M}_{\Gamma'}.$$

We abbreviate $\overline{\mathcal{M}}_{0,k,l} = \overline{\mathcal{M}}_{0,[k^\circ],[l]}$. We may also write $\overline{\mathcal{M}}_{0,k,I}$, $\overline{\mathcal{M}}_{0,B,l}$, with the obvious meanings. Similarly, we abbreviate $\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,[n]}$.

When we say that a stable marked disk belongs to $\overline{\mathcal{M}}_{0,B,I}$, we mean that its isomorphism class is in $\overline{\mathcal{M}}_{0,B,I}$. The same applies for the other sets defined above as well.

Notation 2.20. Given nonnegative integers k, l with $k+2l \geq 3$, denote by $\Gamma_{0,k,l}$ the stable graph with $V^O = \{*\}, V^C = \emptyset$, and with

$$\ell_B(*) = [k^\circ], \qquad \ell_I(*) = [l].$$

Remark 2.21. The above moduli of stable marked disks are smooth manifolds with corners. We have

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} = k + 2l - 3.$$

A stable marked disk with b boundary nodes belongs to a corner of the moduli space $\overline{\mathcal{M}}_{0,k,l}$ of codimension b. Thus $\partial\overline{\mathcal{M}}_{0,k,l}$ consists of stable marked disks with at least one boundary node. That is,

$$\partial\overline{\mathcal{M}}_{0,k,l} = \coprod_{\Gamma \in \partial^B \Gamma_{0,k,l}} \mathcal{M}_\Gamma.$$

In the following, building on the discussion in [5, Section 2.1.2], we describe a natural orientation on the spaces $\overline{\mathcal{M}}_{0,k,l}$ for k odd. We start by recalling a few useful facts and conventions. Let Σ be a genus zero smooth marked surface with boundary and denote by j its complex structure. For $p \in \Sigma$, and $v \in T_p\Sigma$, we follow the convention that $\{v, jv\}$ is a complex oriented basis. The complex orientation Σ induces an orientation of $\partial\Sigma$ by requiring the outward normal at $p \in \partial\Sigma$ followed by an oriented vector in $T_p\partial\Sigma$ to be an oriented basis of $T_p\Sigma$. The orientation of $\partial\Sigma$ gives rise to a cyclic order of the boundary marked points. Denote by $\mathcal{M}^{main}_{0,k,l} \subset \mathcal{M}_{0,k,l}$ the component where the induced cyclic order on the boundary marked points is the usual order on $[k]$. Denote by $\overline{\mathcal{M}}^{main}_{0,k,l}$ the corresponding component of $\overline{\mathcal{M}}_{0,k,l}$.

The fiber of the forgetful map $\overline{\mathcal{M}}_{0,k,l+1} \to \overline{\mathcal{M}}_{0,k,l}$ is homeomorphic to a disk. It inherits the complex orientation from an open dense subset that carries a tautological complex structure. For $k \geq 1$, the fiber of the forgetful map $\overline{\mathcal{M}}_{0,k+1,l} \to \overline{\mathcal{M}}_{0,k,l+1}$ is a closed interval. An open subset of this closed interval comes with a canonical embedding into the boundary of a disk, which induces an orientation on the fiber. The fiber of the forgetful map $\overline{\mathcal{M}}^{main}_{0,k+2,l} \to \overline{\mathcal{M}}^{main}_{0,k,l}$ is homeomorphic to $[0,1]^2$. We fix the orientation of the fiber by identifying the first factor of $[0,1]^2$ with the fiber of the map forgetting the $k+1$ marked point and the second factor with the fiber of the map forgetting the $k+2$ marked point. This orientation is called the *natural orientation* below.

Lemma 2.22. *Let k be odd and l arbitrary. There exists a unique collection of orientations $o_{0,k,l}$ for the spaces $\overline{\mathcal{M}}_{0,k,l}$ with the following properties:*

(a) *In the zero dimensional cases $k=1, l=1$, and $k=3, l=0$, the orientations $o_{0,k,l}$ are positive at each point.*
(b) *$o_{0,k,l}$ is invariant under permutations of interior and boundary labels.*
(c) *$o_{0,k,l+1}$ agrees with the orientation induced from $o_{0,k,l}$ by the fibration $\overline{\mathcal{M}}_{0,k,l+1} \to \overline{\mathcal{M}}_{0,k,l}$ and the complex orientation on the fiber.*
(d) *$o_{0,k+2,l}$ agrees with the orientation induced from $o_{0,k,l}$ by the fibration $\overline{\mathcal{M}}^{main}_{0,k+2,l} \to \overline{\mathcal{M}}^{main}_{0,k,l}$ and the natural orientation on the fiber.*

Remark 2.23. In the preceding lemma, it does not matter which ordering convention we use for the induced orientation on the total space of a fibration. Indeed, the base and fiber are always even dimensional.

Proof of Lemma 2.22. If the orientations $o_{0,k,l}$ exist, properties (a)-(d) imply they are unique. It remains to check existence.

For property (b) to hold, we must show permutations of labels that map the component $\overline{\mathcal{M}}^{main}_{0,k,l}$ to itself are orientation preserving. Indeed, let

$$U = \left\{ (z,w) \;\middle|\; \begin{array}{lll} z=(z_1,\ldots,z_k)\in(S^1)^k, & z_i\neq z_j, & i\neq j \\ w=(w_1,\ldots,w_l)\in(\operatorname{int} D^2)^l, & w_i\neq w_j, & i\neq j \end{array} \right\}.$$

Denote by $U^{main}\subset U$ the subset where the cyclic order of $z_1,\ldots,z_k$, on $S^1=\partial D^2$ with respect to the orientation induced from the complex orientation of D^2 agrees with the standard order of $[k]$. Then

$$\mathcal{M}^{main}_{0,k,l} = U^{main}/PSL_2(\mathbb{R}).$$

Since k is odd, cyclic permutations of the boundary labels preserve the orientation of U^{main} and thus also $\mathcal{M}^{main}_{0,k,l}$ and $\overline{\mathcal{M}}^{main}_{0,k,l}$. Similarly, arbitrary permutations of the interior labels preserve the orientation of $\overline{\mathcal{M}}^{main}_{0,k,l}$.

A direct calculation shows that the orientation on $\overline{\mathcal{M}}_{0,3,1}$ induced by property (c) from $o_{0,3,0}$ agrees with the orientation induced by property (d) from $o_{0,1,1}$. So $o_{0,3,1}$ exists. Existence of $o_{0,k,l}$ satisfying properties (c) and (d) for other k,l, follows from the commutativity of the diagram of forgetful maps

$$\begin{array}{ccc}
\overline{\mathcal{M}}^{main}_{0,k+2,l+1} & \longrightarrow & \overline{\mathcal{M}}^{main}_{0,k+2,l} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}^{main}_{0,k,l+1} & \longrightarrow & \overline{\mathcal{M}}^{main}_{0,k,l}\,.
\end{array}$$

□

For the remainder of the paper, we always consider the spaces $\overline{\mathcal{M}}_{0,k,l}$ for k odd equipped with the orientations $o_{0,k,l}$.

2.6. **Edge labels.** In constructing the boundary conditions for open descendent integrals, it is necessary to be able to refer to the edges of a vertex and their corresponding nodal points unambiguously. We do this as follows.

Notation 2.24. Given a stable graph Γ and an edge

$$e = \{u, v\} \in E(\Gamma),$$

we denote by Γ_e the (not necessarily stable) graph obtained from Γ by removing e. We denote by $\Gamma_{e,u}$ the pre-stable graph which is the connected component of u in Γ_e. We define $\Gamma_{e,v}$ similarly.

Definition 2.25. Denote by $i_v^\Gamma : I(v) \cup B(v) \to \mathfrak{L}$ the map defined by

(a) for $x \in \ell_I(v) \cup \ell_B(v)$, $i_v^\Gamma(x) = x$,
(b) for $e = \{u, v\} \in E$, $i_v^\Gamma(e) = \cup I(\Gamma_{e,u}) \cup \cup B(\Gamma_{e,u})$,

When the graph Γ is clear from the context, we write i_v instead of i_v^Γ.

Remark 2.26. It is easy to see that i_v is actually an injection. Hence, we may identify $I(v) \cup B(v)$ with its image under i_v. In addition, it follows from Definition 2.11 part (d)(i) that

$$i_v(I(v) \cup B(v)) \in 2^{\mathfrak{L}}_{fin,disj}.$$

Definition 2.27. Let $\Gamma \in \mathcal{G}$ and $\Lambda \in \partial^! \Gamma$. In light of Definition 2.17, we have canonical maps

$$\varsigma = \varsigma_{\Lambda,\Gamma} : V(\Lambda) \longrightarrow V(\Gamma), \qquad \iota = \iota_{\Gamma,\Lambda} : E(\Gamma) \longrightarrow E(\Lambda).$$

The map ς is uniquely determined by the condition that for $v \in V(\Lambda)$, there exists a partition $\operatorname{Im} i^\Gamma_{\varsigma(v)} = P_1 \coprod \ldots \coprod P_n$ such that

$$\operatorname{Im} i_v^\Lambda = \{\cup P_i\}_{i=1}^n.$$

If $e = \{u, v\} \in E(\Gamma)$, then $\iota(e)$ is the unique edge of the form $\{\tilde{u}, \tilde{v}\}$ where $\varsigma(\tilde{u}) = u$ and $\varsigma(\tilde{v}) = v$.

Definition 2.28. For $\Gamma \in \mathcal{G}$ and $U \subset V(\Gamma)$, let Γ_U be the stable graph spanned by U with labels added in place of edges connecting U to its complement. Specifically,

$$V(\Gamma_U) = U, \qquad \varepsilon_{\Gamma_U} = \varepsilon_\Gamma|_U, \qquad E(\Gamma_U) = \{\{u,v\} \in E(\Gamma)|u,v \in U\},$$
$$\forall v \in U, \qquad \ell_I^{\Gamma_U}(v) = \ell_I^\Gamma(v) \cup \left\{ i_v^\Gamma(e) \middle| \, e = \{u,v\} \in E^I(\Gamma), u \notin U \right\},$$
$$\ell_B(v) = \ell_B^\Gamma(v) \cup \left\{ i_v^\Gamma(e) \middle| \, e = \{u,v\} \in E^B(\Gamma), u \notin U \right\}.$$

For $v \in V(\Gamma)$, abbreviate $\Gamma_v = \Gamma_{\{v\}}$ and

$$\mathcal{M}_v = \mathcal{M}_{\Gamma_v}.$$

2.7. **Forgetful maps.** We now define forgetful maps for stable graphs. Let Γ be a connected pre-stable graph. Set

$$k = k(\Gamma), \quad l = l(\Gamma), \quad I = I(\Gamma), \quad B = B(\Gamma).$$

In case $V^O = \emptyset$, assume $l \geq 3$. In case $V^O \neq \emptyset$, assume $k+2l \geq 3$. Define the graph $stab(\Gamma)$ as follows. Take any unstable vertex $v \in V^O \cup V^C$.

(a) In case $v \in V^O$ (V^C) has no boundary or interior labels and exactly 2 boundary (interior) edges

$$e_1 = \{v,u\}, \quad e_2 = \{v,w\},$$

remove v and its edges from the graph and add the new boundary (interior) edge $\{u,w\}$.

(b) In case $v \in V^O$ has a single boundary edge $\{v,u\}$, a single boundary label i and no interior edges or labels, remove v and its edge from the graph and add i to $\ell_B(u)$.

(c) In case $v \in V^C$ has a single interior edge $\{v,u\}$ and a single interior label i, remove v and its edge from the graph and add i to $\ell_I(u)$.

(d) In case v has a single edge, and no labels, remove v and its edge from the graph.

Other cases are not possible. We iterate this procedure until we get a stable graph. Note that the process does stop, and that the final result does not depend on the order of the above steps. We extend the definition of *stab* to not necessarily connected graphs by applying it to each component if each component satisfies the assumptions.

Definition 2.29. The graph $stab(\Gamma)$ is called the *stabilization* of Γ.

Notation 2.30. Consider a stable graph Γ such that

$$k(\Gamma) + 2(l(\Gamma) - 1) \geq 3.$$

If $i \notin I(\Gamma)$, we define $for_i(\Gamma) = \Gamma$. If $i \in I(\Gamma)$, we define $for_i(\Gamma)$ to be the graph obtained by removing the label i from the vertex v_i and stabilizing.

Observation 2.31. Let $\Gamma \in \mathcal{G}$. The natural map

$$\prod_{v \in V(\Gamma)} \mathcal{M}_v \to \mathcal{M}_\Gamma,$$

is an isomorphism of smooth manifolds with corners.

We shall use the preceding observation to identify the two moduli spaces throughout the article.

Notation 2.32. Let Γ, Γ', be stable graphs, and assume there exist injective mappings

$$f : V(\Gamma') \longrightarrow V(\Gamma),$$

and

$$f_v : \mathrm{Im}\left(i_v^{\Gamma'}\right) \longrightarrow \mathrm{Im}\left(i_{f(v)}^{\Gamma}\right), \qquad v \in V(\Gamma'),$$

such that

$$S \subset f_v(S), \qquad \forall v \in V(\Gamma'), \quad S \in \mathrm{Im}\left(i_v^{\Gamma'}\right). \tag{16}$$

Such f, f_v, if they exist, are unique. In this case, we say that Γ' is a *stable subgraph* of Γ.

The map f_v induces a forgetful map

$$For_v : \mathcal{M}_{f(v)} \to \mathcal{M}_v, \qquad v \in V(\Gamma').$$

Denote by

$$\pi_{\Gamma,\Gamma'} : \prod_{v \in V(\Gamma)} \mathcal{M}_v \to \prod_{v \in V(\Gamma')} \mathcal{M}_{f(v)}$$

the projection. We define the forgetful map

$$For_{\Gamma,\Gamma'} : \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$$

by

$$For_{\Gamma,\Gamma'} = \left(\prod_{v \in V(\Gamma')} For_v\right) \circ \pi_{\Gamma,\Gamma'}.$$

We abbreviate

$$For_i = For_{\Gamma, for_i(\Gamma)} : \mathcal{M}_\Gamma \longrightarrow \mathcal{M}_{for_i(\Gamma)}.$$

Observation 2.33. If Γ'' is a stable subgraph of Γ' and Γ' is a stable subgraph of Γ, then

$$F_{\Gamma,\Gamma''} = F_{\Gamma',\Gamma''} \circ F_{\Gamma,\Gamma'}.$$

3. Line bundles and relative Euler classes

3.1. **Cotangent lines and canonical boundary conditions.** For $i \in I$, denote by

$$\mathbb{L}_i \to \overline{\mathcal{M}}_{0,B,I}$$

the i^{th} tautological line bundle. The fiber of $\mathbb{L}_i$ over a stable disk Σ is the cotangent line at the i^{th} marked point $T_{z_i}\Sigma$. For any stable graph Γ with $i \in I(\Gamma)$, define

$$\mathbb{L}_i \to \mathcal{M}_\Gamma$$

using the canonical identification of Observation 2.31. This definition of $\mathbb{L}_i \to \mathcal{M}_\Gamma$ agrees with restriction of $\mathbb{L}_i \to \overline{\mathcal{M}}_{0,k,l}$ to $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{0,k,l}$ for $\Gamma \in \partial\Gamma_{0,k,l}$.

Let

$$E = \bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to \overline{\mathcal{M}}_{0,k,l},$$

where a_i, k, l, are non-negative integers such that

$$\operatorname{rk}_{\mathbb{C}} E = \sum_{i\in[l]} a_i = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k,l} = \frac{2l+k-3}{2}, \tag{17}$$

$$k + 2l - 2 > 0.$$

In particular, since $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k,l}$ is an integer, k must be odd. We shall begin by defining the vector space $\mathcal{S}$ of canonical boundary conditions for E. It is a vector subspace of the vector space of multisections of $E|_{\partial\overline{\mathcal{M}}_{0,k,l}}$. See Appendix A for background on multisections.

Consider a stable graph $\Gamma \in \partial\Gamma_{0,k,l}$ corresponding to a codimension one corner of $\overline{\mathcal{M}}_{0,k,l}$. Thus,

$$|E(\Gamma)| = \left|E^B(\Gamma)\right| = 1.$$

Write $V(\Gamma) = \{v_1, v_2\}$. Exactly one of $k(v_1), k(v_2)$, is even. Without loss of generality, it is $k(v_2)$. Let Γ' be the stable graph with no edges and two open vertices v_1', v_2', with

$$\ell_B(v_1') = i_{v_1}(B(v_1)), \qquad \ell_B(v_2') = \ell_B(v_2)$$
$$\ell_I(v_1') = \ell_I(v_1'), \qquad \ell_I(v_2') = \ell_I(v_2).$$

Here, Γ' is stable because of the assumption on the parity of $k(v_2)$. The definition of Γ' implies that $\mathcal{M}_{\Gamma'}$ is the same as $\mathcal{M}_\Gamma$ except that the marked point corresponding to the edge of Γ on the component of v_2 has been forgotten. Let E' be the vector bundle given by

$$E' = \bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to \mathcal{M}_{\Gamma'}.$$

Since the map $For_{\Gamma,\Gamma'}$ does not contract any components of the stable disks in $\mathcal{M}_\Gamma$, we have

$$E|_{\mathcal{M}_\Gamma} \simeq For^*_{\Gamma,\Gamma'}E'.$$

See Observation 3.32 and the preceding discussion for details.

Recall that the boundary ∂X of a manifold with corners X is itself a manifold with corners, equipped with a map

$$i_X : \partial X \to X,$$

which may not be injective. A section s of a bundle $F \to \partial X$ is *consistent* if

$$\forall\ p_1, p_2 \in X, \text{ such that } i_X(p_1) = i_X(p_2) \text{ we have } s(p_1) = s(p_2).$$

Consistency for multisections is similar. For a vector bundle $F \to X$, we write $F|_{\partial X} = i_X^* F$ and similarly for sections of F.

Definition 3.1. A smooth consistent multisection s of $E|_{\partial\overline{\mathcal{M}}_{0,k,l}}$ is called a *canonical multisection* if for each graph $\Gamma \in \partial^B\Gamma_{0,k,l}$ with a single edge,

$$s|_{\mathcal{M}_\Gamma} = For^*_{\Gamma,\Gamma'}s',$$

where s' is a multisection of $E' \to \mathcal{M}_{\Gamma'}$. The vector space of all canonical multisections is denoted by $\mathcal{S}$.

3.2. **Definition of open descendent integrals.**

Notation 3.2. Given a complex vector bundle $F \to X$, where X is a manifold with corners, denote by $C_m^\infty(F)$ the space of smooth multisections. Given a nowhere vanishing smooth consistent multisection

$$\mathbf{s} \in C_m^\infty(F|_{\partial X}),$$

denote by

$$e(F;\mathbf{s}) \in H^*(X, \partial X)$$

the *relative Euler class*. This is by definition the Poincaré dual of the vanishing set of a transverse extension of $\mathbf{s}$ to X. See Appendix A for details.

Theorem 3.3. *When condition* (17) *holds, one can find a nowhere vanishing multisection* $\mathbf{s} \in \mathcal{S}$. *Hence one can define* $e(E;\mathbf{s})$. *Moreover, any two nowhere vanishing multisections of* $\mathcal{S}$ *define the same relative Euler class.*

Definition 3.4. When condition (17) holds, define $e(E;\mathcal{S})$ to be the relative Euler class $e(E,\mathbf{s})$ for any $\mathbf{s} \in \mathcal{S}$. This notation is unambiguous by the preceding theorem. The genus zero *open descendent integrals* are defined by

$$\left\langle \tau_{a_1} \ldots \tau_{a_l} \sigma^k \right\rangle_0^o = 2^{-\frac{k-1}{2}} \int_{\overline{\mathcal{M}}_{0,k,l}} e\left(E, \mathcal{S}\right) \tag{18}$$

when condition (17) holds. Otherwise, they are defined to be zero.

The division by the power of 2 in the preceding definition is only for convenience. When r_0 of the a_i are equal to 0, r_1 of them equal to 1, and so on, we sometimes use the notation $\left\langle \tau_0^{r_0} \tau_1^{r_1} \ldots \sigma^k \right\rangle$ or $\left\langle \tau_0^{r_0} \tau_1^{r_1} \ldots \sigma^k \right\rangle^o$ for the above quantity.

Remark 3.5. A surprising feature of our construction is the use multi-sections rather than sections. The reason for this is that in general one cannot find a non-vanishing section in $\mathcal{S}$. This fact will be transparent later when we calculate intersection numbers. We shall see that often the intersection numbers will not be a multiple of the number of components of $\overline{\mathcal{M}}_{0,k,l}$. However, each component contributes equally to the intersection number, so each component must contribute a non-integer to the intersection number.

But we want also to understand geometrically what happens. Consider the case of $\overline{\mathcal{M}}_{0,5,1}$ and $E = \mathbb{L}_1^{\oplus 2}$. For simplicity we illustrate a section of $\mathbb{L}_1$ as a tangent vector at the interior marked point. Consider Figure 2. We may take the interior marked point to be the center of the disk, as a result of the $PSL_2\left(\mathbb{R}\right)$ equivalence relation. Let $\Gamma \in \partial\Gamma_{0,5,1}$ be the unique stable graph with two vertices, v_1, v_2, both open, and

$$\ell_I(v_2) = \emptyset, \qquad \ell_B(v_2) = \{4, 5\}.$$

Consider a non-vanishing section s of $\mathbb{L}_1$, which is a component of a canonical section of $E|_{\partial\overline{\mathcal{M}}_{0,5,1}}$.

In item (a) of the figure, we depict a stable disk $\Sigma \in \mathcal{M}_\Gamma$, at which s points to the boundary marked point x_3. Note that pointing at x_3 is preserved by the action of $PSL_2\left(\mathbb{R}\right)$. When the bubble of x_4, x_5, approaches x_3, while nothing else changes in the component of z_1, x_1, x_2, x_3, the section keeps on pointing towards x_3 by the definition of canonical boundary conditions. After the bubble reaches x_3, we move to item (b) of the figure. By continuity, the section still points in the direction of x_3, only that now there is a boundary node there. Continuous changes in the component of x_3, x_4, x_5, do not affect s, again by the definition of canonical boundary conditions. In particular, s does not change when x_5 approaches the node. After x_5 reaches the node we pass to item (c). Again continuity guarantees no change in s.

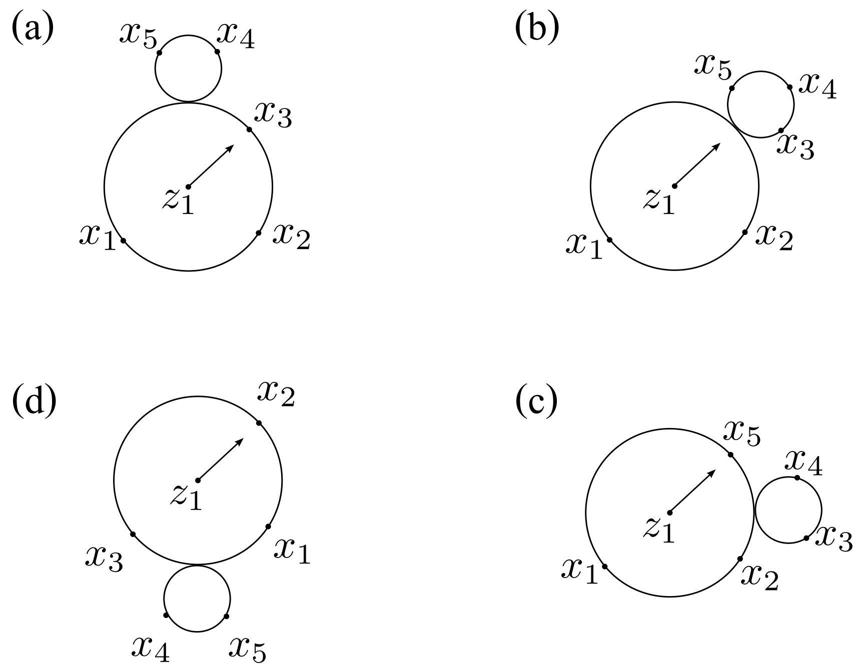


FIGURE 2. A canonical multisection at different boundary points.

Continuing in this manner, we finally reach item (d). When we finish, s points at x_2.

Now, let Σ be the unique marked disk in $\mathcal{M}_\Gamma$ such that if we take the interior marked point z_1 to be the center of the disk as before, we have the angle condition

$$\measuredangle x_1 z_1 x_2 = \measuredangle x_2 z_1 x_3 = \measuredangle x_3 z_1 x_1 = \frac{2\pi}{3}.$$

If a canonical multisection of $E|_{\partial\overline{\mathcal{M}}_{0,5,1}}$ does not vanish at Σ, then without loss of generality we may assume that its first component, s, does not vanish there. Moreover, after possibly multiplying by a complex scalar, we may assume s points at x_3. So, we are in item (a). Using the above reasoning we see that on the surface Σ' of item (d), the section s must point at x_2. As a consequence of the choice of Σ,

$$\Sigma' \simeq \Sigma,$$

which is a contradiction. Of course, this example generalizes beyond $\overline{\mathcal{M}}_{0,5,1}$ and establishes the need for multisections.

3.3. **The base.** In order to prove Theorem 3.3 we need to understand how canonical multisections behave on boundary strata of arbitrary codimension. We encode the relevant combinatorics in an operation on graphs called the base.

Definition 3.6. Let $\Gamma \in \mathcal{G}$. A boundary edge $e = \{u, v\} \in E^B(\Gamma)$ is said to be *illegal* for the vertex v if $k(\Gamma_{e,v})$ is odd. Otherwise it is *legal*. Denote by $E_{legal}(v)$ the set of legal edges of v. Recall that a boundary node in a stable curve $\Sigma \in \mathcal{M}_\Gamma$ corresponds to a boundary edge of Γ, and a component of Σ corresponds to a vertex of Γ. We define a boundary node of Σ to be *legal* for a component Σ_α if the corresponding edge is legal for the corresponding vertex. Otherwise it is *illegal*.

Notation 3.7. Denote by $\mathcal{G}_{odd}$ the set of all $\Gamma \in \mathcal{G}$ such that for every connected component Γ_i of Γ, either $V^O(\Gamma_i) = \emptyset$ or $k(\Gamma_i)$ is odd.

Simple parity considerations show the following.

Observation 3.8. If $\Gamma \in \mathcal{G}_{odd}$ and $e = \{u, v\} \in E^B(\Gamma)$, then e is legal for exactly one of u, v.

Observation 3.9. Let $\Gamma \in \mathcal{G}_{odd}$ and $v \in V^O(\Gamma)$. Then the total number of legal edges and boundary labels of v is an odd number. Moreover, in case $\ell_I(v) \neq \emptyset$, even if we erase from Γ the edges which are illegal for v, the vertex v remains stable.

Proof. Let $e = \{u, v\}$ be a boundary edge of v. If e is legal for v, then it is illegal for u, so $k(\Gamma_{e,u})$ is odd. Otherwise, $k(\Gamma_{e,u})$ is even. Thus

$$\begin{aligned} |\ell_B(v) \cup E_{legal}(v)| &\cong |\ell_B(v)| + \sum_{e=\{u,v\} \in E_{legal}(v)} k(\Gamma_{e,u}) \\ &\cong k(\Gamma) \cong 1 \pmod 2, \end{aligned}$$

which is the first claim of the lemma.

Regarding stability, if v is a closed vertex, it has no illegal edges and the stability is clear. For an open vertex v, we have just seen that $|\ell_B(v) \cup E_{legal}(v)|$ is odd, hence at least 1, and by assumption $\ell_I(v) \neq \emptyset$. Stability follows. □

Definition 3.10. The *base* is an operation on graphs

$$\mathcal{B} : \mathcal{G}_{odd} \to \mathcal{G}_{odd}$$

defined as follows. For $\Gamma \in \mathcal{G}_{odd}$ the graph $\mathcal{B}\Gamma$ is given by

$$V(\mathcal{B}\Gamma) = \{v \in \Gamma \,|\, 2l(v) + |\ell_B(v) \cup E_{legal}(v)| \geq 3\},$$

$$\varepsilon_{\mathcal{B}\Gamma} = \varepsilon_\Gamma|_{V(\mathcal{B}\Gamma)}, \qquad E(\mathcal{B}\Gamma) = \emptyset,$$

$$\ell_I^{\mathcal{B}\Gamma}(v) = i_v^\Gamma\left(I^\Gamma(v)\right), \quad \ell_B^{\mathcal{B}\Gamma}(v) = i_v^\Gamma\left(\ell_B^\Gamma(v) \cup E_{legal}^\Gamma(v)\right), \quad v \in V(\mathcal{B}\Gamma).$$

We abbreviate

$$F_\Gamma = For_{\Gamma,\mathcal{B}\Gamma} : \mathcal{M}_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}.$$

Observation 3.11. A multisection s of

$$E = \bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to \partial\overline{\mathcal{M}}_{0,k,l}$$

is canonical if and only if for each $\Gamma \in \partial^B\Gamma_{0,k,l}$, there exists a multisection $s^{\mathcal{B}\Gamma}$ of

$$\bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to \mathcal{M}_{\mathcal{B}\Gamma}$$

such that $s|_{\mathcal{M}_\Gamma} = F_\Gamma^* s^{\mathcal{B}\Gamma}$.

Proof. The case where Γ has a single edge is exactly the definition. The general case follows from the continuity of s. □

Observation 3.12. Observation 3.9 implies that $I(\Gamma) \subseteq I(\mathcal{B}\Gamma)$. It follows from the definition of $\mathcal{B}$ that there is a canonical inclusion

$$\iota_{\mathcal{B}}^V : V(\mathcal{B}\Gamma) \hookrightarrow V(\Gamma).$$

The following observation is straightforward.

Observation 3.13. Recall Definition 2.27. Let $e = \{u, v\} \in E(\Gamma)$ and let $\Lambda \in \partial\Gamma$. Let $\tilde{e} = \iota_{\Gamma,\Lambda}(e)$ and let $\tilde{u}, \tilde{v} \in V(\Lambda)$ be such that $\varsigma_{\Lambda,\Gamma}(\tilde{u}) = u$, $\varsigma_{\Lambda,\Gamma}(\tilde{v}) = v$ and $\tilde{e} = \{\tilde{u}, \tilde{v}\}$. Then $\tilde{e}$ is illegal for $\tilde{v}$ if and only if e is illegal for v.

The following is a consequence of the preceding observation.

Observation 3.14. We have

$$\mathcal{B} \circ \partial^! = \mathcal{B} \circ \partial^! \circ \mathcal{B}.$$

The key to constructing homotopies between canonical multisections is the following.

Observation 3.15. For $\Gamma \in \partial\Gamma_{0,k,l}$ with k odd, we have

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{B}\Gamma} \le \dim_{\mathbb{C}} \mathcal{M}_{0,k,l} - 1.$$

In addition, for any $v \in V(\mathcal{B}\Gamma)$, we have $\dim_{\mathbb{C}} \mathcal{M}_v \in \mathbb{Z}$. It follows that $\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{B}\Gamma} \in \mathbb{Z}$.

Proof. If Γ has at least one interior edge or two boundary edges, then $\dim_{\mathbb{C}} \mathcal{M}_\Gamma \le \dim_{\mathbb{C}} \mathcal{M}_{0,k,l} - 1$. So, since $\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{B}\Gamma} \le \dim_{\mathbb{C}} \mathcal{M}_\Gamma$, the desired inequality follows. It remains to consider the case that Γ consists of two vertices u, v, connected by a single boundary edge e. Then e is illegal for exactly one of the vertices, say v. The stability of v and the illegality of e for v imply $k(v) \geq 4$. So, dropping e in passing to $\mathcal{B}\Gamma$ does not destabilize v. Thus there is a corresponding vertex v' in $\mathcal{B}\Gamma$ with $k(v') = k(v) - 1$. It follows that

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{B}\Gamma} \leq \dim_{\mathbb{C}} \mathcal{M}_{\Gamma} - \frac{1}{2} = \dim_{\mathbb{C}} \mathcal{M}_{0,k,l} - 1.$$

This completes the proof of the first claim. The integrality follows immediately from Observation 3.9. $\square$

Recall Lemma 2.22. Let k be odd, and let $\Gamma \in \partial^B\Gamma_{0,k,l}$ consist of two open vertices $v_\Gamma^\pm$, connected by a single boundary edge that is legal for v_Γ^+ and illegal for v_Γ^-. In particular, $\mathcal{M}_\Gamma$ is an open subset of $\partial\overline{\mathcal{M}}_{0,k,l}$. Denote by o_Γ the orientation on $\overline{\mathcal{M}}_\Gamma$ induced by $o_{0,k,l}$ and the outward normal vector, ordering the outward normal first. Furthermore, writing $k_\Gamma^+ = k(v_\Gamma^+), k_\Gamma^- = k(v_\Gamma^-) - 1$ and $l_\Gamma^\pm = l(v_\Gamma^\pm)$, we have

$$\mathcal{M}_{\mathcal{B}\Gamma} \simeq \mathcal{M}_{0,k_\Gamma^+,l_\Gamma^+} \times \mathcal{M}_{0,k_\Gamma^-,l_\Gamma^-} \tag{19}$$

where $k_\Gamma^\pm$ are both odd. So we define the orientation $o_{\mathcal{B}\Gamma}$ of $\mathcal{M}_{\mathcal{B}\Gamma}$ to be the product of the orientations $o_{0,k_\Gamma^\pm,l_\Gamma^\pm}$. The choice of isomorphism (19) does not affect o_Γ because of property (b) of $o_{0,k,l}$. The fiber of the map F_Γ is a collection of open intervals in the boundary of a disk, and thus carries an induced orientation, which we call *natural* below.

Lemma 3.16. *The orientation o_Γ agrees with the orientation induced from $o_{\mathcal{B}\Gamma}$ by the fibration $F_\Gamma : \mathcal{M}_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}$ and the natural orientation on the fiber.*

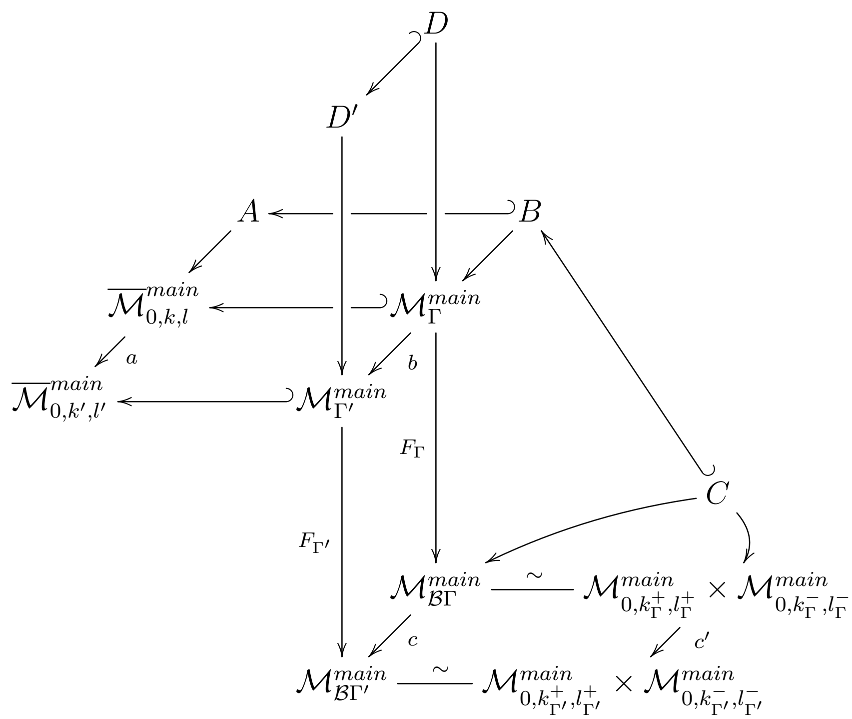
Proof. The claim can be checked explicitly in the three cases when $\dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} = 2$. We use induction on $\dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l}$ to reduce to the two dimensional case. Indeed, assume $\dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} \geq 4$. Since $k_\Gamma^\pm$ are odd and

$$4 \leq \dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} = k_\Gamma^+ + k_\Gamma^- + 2(l_\Gamma^+ + l_\Gamma^-) - 4,$$

either $k_\Gamma^+ + 2l_\Gamma^+ \geq 5$ or $k_\Gamma^- + 2l_\Gamma^- \geq 5$. By property (b) of $o_{0,k,l}$, in case $k_\Gamma^+ + 2l_\Gamma^+ \geq 5$, we may assume $\ell_B(v_\Gamma^+) = \{k_\Gamma^- + 1, \ldots, k\}$. Otherwise, we may assume $\ell_B(v_\Gamma^-) = \{k_\Gamma^+, \ldots, k\}$. Again by property (b), it suffices to prove the claim for o_Γ restricted to $\mathcal{M}_\Gamma^{main} := \mathcal{M}_\Gamma \cap \mathcal{M}_{0,k,l}^{main}$. Denote by $\mathcal{M}_{\mathcal{B}\Gamma}^{main}$ the corresponding component of $\mathcal{M}_{\mathcal{B}\Gamma}$. We choose isomorphism (19) so it induces an isomorphism

$$\mathcal{M}_{\mathcal{B}\Gamma}^{main} \simeq \mathcal{M}_{0,k_\Gamma^+,l_\Gamma^+}^{main} \times \mathcal{M}_{0,k_\Gamma^-,l_\Gamma^-}^{main}$$

such that the k^{th} marked point of $\mathcal{M}_{\mathcal{B}\Gamma}$ corresponds to either the k_Γ^+ boundary marked point of $\mathcal{M}_{0,k_\Gamma^+,l_\Gamma^+}$ or the k_Γ^- boundary marked point of $\mathcal{M}_{0,k_\Gamma^-,l_\Gamma^-}$. If $k_\Gamma^+ + 2l_\Gamma^+ \geq 5$ and $k_\Gamma^+ \geq 3$, or if $k_\Gamma^- + 2l_\Gamma^- \geq 5$ and $k_\Gamma^- \geq 3$, let $(k',l')=(k-2,l)$. Otherwise, let $(k',l')=(k,l-1)$. If $k'=k-2$, let $\Gamma'\in\partial^B\Gamma_{0,k',l'}$ be the graph obtained from Γ by forgetting the boundary markings $k,k-1$. Otherwise, let $\Gamma'=for_l(\Gamma)$. Consider the following commutative diagram.



The spaces A,B and C, are the fibers of the forgetful maps a,b, and c, respectively. So, if $k'=k-2$, then $A\simeq[0,1]^2$. Otherwise, $A\simeq D^2$. The fibers B and C are open subsets of A and the inclusions preserve the natural or complex orientations. The spaces D,D', are the fibers of the maps $F_\Gamma,F_{\Gamma'}$, respectively. Both D and D' are homeomorphic to an open interval, and the open inclusion $D\hookrightarrow D'$ preserves the natural orientations. By our assumption on $\ell_B(v_\Gamma^\pm)$, the map c' is the identity on one of the factors $\mathcal{M}_{0,k_\Gamma^\pm,l_\Gamma^\pm}^{main}$ and the forgetful map on the other.

We say a fibration is oriented if the orientation on the total space is induced by that on the base and fiber. Thus a and c' are oriented by properties (c) and (d) of the orientations $o_{0,k,l}$. By the definition of the orientations o_Γ and $o_{\mathcal{B}\Gamma}$, it follows that b and c are oriented. By induction $F_{\Gamma'}$ is oriented. So the diagram implies F_Γ is oriented as well. $\square$

3.4. **Abstract vertices.** For proving theorems, a refinement of canonical multisections is helpful. The relevant definition, given in Section 3.5, uses the notion of an abstract vertex.

Definition 3.17. An *abstract vertex* v is a triple (ε, k, I), where

(a) $\varepsilon \in \{C, O\}$.
(b) $k \in \mathbb{Z}_{\geq 0}$.
(c) $I = \{i_1, i_2, \ldots, i_l\} \in 2^{\mathfrak{L}}_{fin,disj}$.

We demand that if $\varepsilon = C$, then $k = 0$. We call $k = k(v)$ the number of boundary labels of the abstract vertex and $l = l(v) = |I|$ the number of interior labels. We also use the notation $I(v)$ for I, and we call the elements of $I(v)$ the interior labels of v. An abstract vertex is said to be *open* if $\varepsilon = O$, and otherwise it is *closed*. An abstract vertex is called *stable* if $k + 2l \geq 3$.

Denote by $\mathcal{V}$ the set of all stable abstract vertices.

Notation 3.18. Let $v \in \mathcal{V}$. We define

$$\mathcal{M}_v = \begin{cases} \mathcal{M}_{0,I(v)}, & \varepsilon(v) = C, \\ \mathcal{M}_{0,k(v),I(v)}, & \varepsilon(v) = O. \end{cases}$$

We define $\overline{\mathcal{M}}_v$ similarly.

We turn to the boundary of an abstract vertex, soon to be related with the boundary of a stable graph.

Definition 3.19. Given an abstract vertex $v = (\varepsilon, k, I)$, we define the *boundary* of v, denoted by ∂v, as the collection of abstract vertices $v' = (\varepsilon', k', I') \neq v$ which satisfy

(a) If $\varepsilon = C$, then $\varepsilon' = C$.
(b) $k' \leq k$.
(c) Every element in I' is a union of elements of I.

Definition 3.20. For $\Gamma \in \mathcal{G}$, we define the map

$$\eta = \eta_\Gamma : V(\Gamma) \to \mathcal{V}$$

by

$$\eta(v) = (\varepsilon(v), k(v), i_v(I(v))).$$

Here, i_v is as in Definition 2.25.

Definition 3.21. Let $\Gamma \in \mathcal{G}$ and $v \in V(\Gamma)$. Each bijection

$$B(v) \simeq [k(v)^\circ]$$

induces a natural diffeomorphism

$$\phi_v : \mathcal{M}_v \to \mathcal{M}_{\eta(v)}.$$

For the rest of the article, we fix one such diffeomorphism for each open vertex in each stable graph. For $v \in V^C(\Gamma)$, we have a natural identification $\phi_v : \mathcal{M}_v \to \mathcal{M}_{\eta(v)}$ without making any choices. The maps ϕ_v extend to the boundary of $\overline{\mathcal{M}}_v$, and we use the same notation for the extension:

$$\phi_v : \overline{\mathcal{M}}_v \to \overline{\mathcal{M}}_{\eta(v)}.$$

Remark 3.22. The symmetric group S_k acts naturally on $\mathcal{M}_{0,k,l}$ by sending $\left(\Sigma, \{x_i\}_1^k, \{z_i\}_1^l\right)$ to $\left(\Sigma, \{x_{\sigma\cdot i}\}_1^k, \{z_i\}_1^l\right)$ for $\sigma \in S_k$. In the same way, given $\Gamma \in \mathcal{G}$, the group $\prod_{v\in V(\Gamma)} S_{k(v)}$ acts on $\mathcal{M}_\Gamma$. In particular, for every vertex $v \in V(\Gamma)$, the symmetric group $S_{k(v)}$ acts on $\mathcal{M}_\Gamma$.

Thus, if we had chosen another map ϕ'_v for a vertex $v \in V(\Gamma)$, then ϕ_v, ϕ'_v, would differ by the action of some $\sigma \in S_{k(v)}$. That is,

$$\phi'_v = \phi_v \circ \sigma,$$

where we denote the group element and its action by the same notation.

We also have a map $\nu : \mathcal{V} \to \mathcal{G}$ which takes the abstract vertex $v = (\varepsilon, k, I) \in \mathcal{V}$ to the stable graph $\left(V = V^O \cup V^C, E, \ell_I, \ell_B\right)$ such that

(a) If $\varepsilon = C$, then $V = V^C = \{*\}$. Otherwise $V = V^O = \{*\}$.
(b) $E = \emptyset$.
(c) If $\varepsilon = O$, then $\ell_B(*) = [k^\circ]$.
(d) $\ell_I(*) = I$.

One can easily verify that $\eta \circ \nu = id$.

Notation 3.23. Denote by $\mathcal{V}_{odd} \subset \mathcal{V}$ the set of all abstract vertices v such that either $\varepsilon(v) = C$ or $k(v)$ is odd.

Notation 3.24. Let $\Gamma \in \mathcal{G}_{odd}$ and $i \in I(\Gamma)$. By Observation 3.12, we have $i \in I(\mathcal{B}\Gamma)$. So, we write

$$v_i^*(\Gamma) = \eta(v_i(\mathcal{B}\Gamma)).$$

From Observations 3.9 and 3.13, we immediately obtain the following.

Observation 3.25. Let $\Gamma \in \mathcal{G}_{odd}, i \in I(\Gamma)$, and let $v = v_i^*(\Gamma)$. Then $v \in \mathcal{V}_{odd}$. In addition, for every $\Gamma' \in \partial\Gamma$, either $v_i^*(\Gamma') = v$ or $v_i^*(\Gamma') \in \partial v$. In the latter case,

$$\dim_{\mathbb{C}} \mathcal{M}_{v_i^*(\Gamma')} < \dim_{\mathbb{C}} \mathcal{M}_v.$$

Definition 3.26. Let $\Gamma \in \mathcal{G}_{odd}$. The *base component* of the interior label $i \in I(\Gamma)$ is $\mathcal{M}_{v_i^*(\Gamma)}$. The *base moduli* of Γ is the space $\mathcal{M}_{\mathcal{B}\Gamma}$.

Recall Definition 2.28. Let $\Gamma \in \mathcal{G}_{odd}$, let $v \in V(\mathcal{B}\Gamma)$ and let $i \in I(\Gamma)$. Define

$$\begin{aligned} \Phi_{\Gamma,v} &:= \phi_v \circ For_{\Gamma,\mathcal{B}\Gamma_v} : \mathcal{M}_\Gamma \to \mathcal{M}_{\eta(v)}, \\ \Phi_{\Gamma,i} &:= \Phi_{\Gamma,v_i(\mathcal{B}\Gamma)} : \mathcal{M}_\Gamma \to \mathcal{M}_{v_i^*(\Gamma)}. \end{aligned} \tag{20}$$

We use the same notation for the natural extensions of these maps to the appropriate compactified moduli spaces.

Notation 3.27. Let $v \in \mathcal{V}_{odd}$ be an abstract vertex. We define a map

$$\partial_v : \mathcal{G}_{odd} \to 2^{\mathcal{G}_{odd}}$$

by

$$\partial_v \Gamma = \{ \Lambda \in \partial\Gamma |\, \exists u \in V\left(\mathcal{B}\Lambda\right),\ \eta\left(u\right) = v \}\,.$$

Moreover, for $\Gamma \in \mathcal{G}_{odd}$ we write

$$\partial_v \mathcal{M}_\Gamma = \coprod_{\Lambda \in \partial_v \Gamma} \mathcal{M}_\Lambda.$$

By abuse of notation, we define a map

$$\partial_v : \mathcal{V}_{odd} \longrightarrow 2^{\mathcal{G}_{odd}}$$

by

$$\partial_v u = \partial_v \left(\nu\left(u\right)\right).$$

For $u \in \mathcal{V}_{odd}$ an abstract vertex, we write

$$\partial_v \mathcal{M}_u = \phi_{\nu(u)}\left(\partial_v \mathcal{M}_{\nu(u)}\right) \subset \overline{\mathcal{M}}_u.$$

A crucial property of the base is the following. Fix $\Gamma \in \mathcal{G}_{odd}$, a label $i \in I\left(\Gamma\right)$, and $\Gamma' \in \partial\Gamma$. Write $v = v_i^*(\Gamma)$. Let

$$\Phi_{\Gamma,i}^{\Gamma'} : \mathcal{M}_{\Gamma'} \to \overline{\mathcal{M}}_{\nu(v)}$$

be given by the composition

$$\mathcal{M}_{\Gamma'} \hookrightarrow \overline{\mathcal{M}}_\Gamma \xrightarrow{\Phi_{\Gamma,i}} \overline{\mathcal{M}}_v \xrightarrow{\phi_{\nu(v)}^{-1}} \overline{\mathcal{M}}_{\nu(v)}.$$

The image of $\Phi_{\Gamma,i}^{\Gamma'}$ is a unique stratum $\mathcal{M}_\Lambda \subset \overline{\mathcal{M}}_{\nu(v)}$, where $\Lambda \in \partial(\nu(v))$ or $\Lambda = \nu(v)$. Note that

$$v_i^*(\Gamma') = v_i^*(\Lambda),$$

and denote this abstract vertex by v'. Then by Observation 3.25 we have either $v' = v$ or $v' \in \partial v$ and $\Lambda \in \partial_{v'} v$. See Figure 3.

Observation 3.28. In the scenario described above, the diagram

$$\begin{array}{ccc} \mathcal{M}_{\Gamma'} & \xrightarrow{\Phi_{\Gamma,i}^{\Gamma'}} & \mathcal{M}_\Lambda \\ & {\scriptstyle \Phi_{\Gamma',i}} \searrow \quad \swarrow {\scriptstyle \Phi_{\Lambda,i}} & \\ & \mathcal{M}_{v'} & \end{array}$$

commutes up to the action of $\sigma \in S_{k(v')}$.

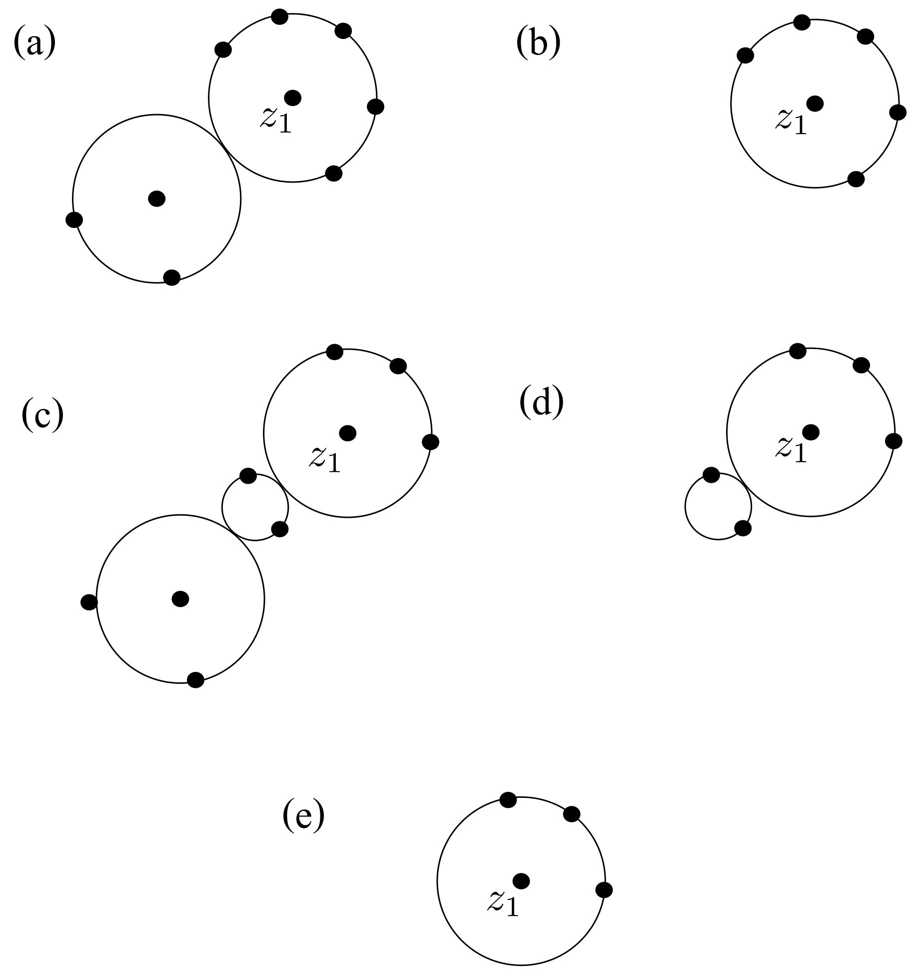


FIGURE 3. (a) shows Γ, (b) shows v, (c) shows Γ', (d) shows Λ, and (e) shows v'.

Notation 3.29. For $v \in \mathcal{V}_{odd}$ write

$$\partial^{eff} v = \{v' \in \mathcal{V}_{odd} | \partial_{v'} v \neq \emptyset\},$$

and

$$\partial_i^{eff} v = \left\{ v' \in \partial^{eff} v \middle| \, i \in I(v) \right\}.$$

For $v' \in \partial^{eff} v$ we define

$$\Phi_{v,v'} : \partial_{v'} \overline{\mathcal{M}}_v \to \mathcal{M}_{v'}$$

by

$$\Phi_{v,v'} = \coprod_{\Lambda \in \partial_{v'} v} \Phi_{\Lambda,v'}.$$

Definition 3.30. Given $\mathcal{C} \subseteq \mathcal{G}_{odd}$, define

$$\mathcal{V}_{\mathcal{C}} = \{\eta\,(v)\,|v \in V\,(\mathcal{B}\Gamma)\,,\ \Gamma \in \mathcal{C}\}.$$

Define

$$\mathcal{V}^i_{\mathcal{C}} = \{v \in \mathcal{V}_{\mathcal{C}}|\, i \in I(v)\}\,.$$

3.5. **Special canonical boundary conditions.** We return to the line bundles $\mathbb{L}_i \to \overline{\mathcal{M}}_{0,k,l}$ in order to define special canonical boundary conditions. We consider only the case k is odd, which is necessary for $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k,l}$ to be an integer.

Denote by $\tilde{\pi}_\Gamma : \mathcal{C}_\Gamma \to \mathcal{M}_\Gamma$ the universal curve. Thus $\tilde{\pi}_\Gamma^{-1}([\Sigma]) = \Sigma$. Denote by $\mathcal{U}_\Gamma \subset \mathcal{C}_\Gamma$ the open subset on which π_Γ is a submersion, and let $\pi_\Gamma = \tilde{\pi}_\Gamma|_{\mathcal{U}_\Gamma}$. Thus $\pi_\Gamma^{-1}([\Sigma])$ is the smooth locus of Σ. For $i \in I(\Gamma)$, denote by $\mu_i : \mathcal{M}_\Gamma \to \mathcal{U}_\Gamma$ the section of π_Γ corresponding to the i^{th} interior marked point. Denote by $\mathbb{L}_\Gamma \to \mathcal{U}_\Gamma$ the vertical cotangent line bundle, which is by definition the cokernel of the map $d\pi_\Gamma^* : T^*\mathcal{M}_\Gamma \to T^*\mathcal{U}_\Gamma$. So, $\mathbb{L}_i = \mu_i^*\mathbb{L}_\Gamma$. Let Γ' be a stable subgraph of Γ. Then the forgetful map $For_{\Gamma,\Gamma'} : \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$ lifts canonically to a map $\widetilde{For}_{\Gamma,\Gamma'} : \mathcal{U}_\Gamma \to \mathcal{U}_{\Gamma'}$. Let $t_{\Gamma,\Gamma'}$ be defined by the following diagram.

$$\begin{array}{ccccc}
\pi_\Gamma^* T^*\mathcal{M}_\Gamma & \xleftarrow{\ \pi_\Gamma^* dFor^*_{\Gamma,\Gamma'}\ } & \pi_\Gamma^* For^*_{\Gamma,\Gamma'} T^*\mathcal{M}_{\Gamma'} & \overset{\sim}{\text{———}} & \widetilde{For}^*_{\Gamma,\Gamma'}\pi_\Gamma^* T^*\mathcal{M}_{\Gamma'} \\
\downarrow{\scriptstyle d\pi_\Gamma^*} & & & & \downarrow{\scriptstyle \widetilde{For}^*_{\Gamma,\Gamma'} d\pi^*_{\Gamma'}} \\
T^*\mathcal{U}_\Gamma & & \xleftarrow{\qquad d\widetilde{For}_{\Gamma,\Gamma,'}\qquad} & & \widetilde{For}^*_{\Gamma,\Gamma'} T^*\mathcal{U}_{\Gamma'} \\
\downarrow & & & & \downarrow \\
\mathbb{L}_\Gamma & & \xleftarrow{\ - - - - t_{\Gamma,\Gamma'} - - - -\ } & & \widetilde{For}^*_{\Gamma,\Gamma'}\mathbb{L}_{\Gamma'}
\end{array}$$

The diagram implies the following.

Observation 3.31. The morphism $t_{\Gamma,\Gamma'}$ is an isomorphism except on components of $\mathcal{U}_\Gamma$ that are contracted by $\widetilde{For}_{\Gamma,\Gamma'}$, where it vanishes identically.

For $i \in I(\Gamma)$, Observation 3.9 implies that $\widetilde{For}_{\Gamma,\mathcal{B}\Gamma}$ and $\widetilde{For}_{\Gamma,\mathcal{B}\Gamma_{v_i(\mathcal{B}\Gamma)}}$ do not contract the component containing the i^{th} interior marked point. The following is an immediate consequence.

Observation 3.32. For $\Gamma \in \mathcal{G}_{odd}$, we have isomorphisms

$$\mathbb{L}_i \simeq F_\Gamma^*\mathbb{L}_i, \qquad \mathbb{L}_i \simeq \Phi^*_{\Gamma,i}\mathbb{L}_i.$$

given by $\mu_i^* t_{\Gamma,\mathcal{B}\Gamma}$ and $\mu_i^* t_{\Gamma,\mathcal{B}\Gamma_{v_i(\mathcal{B}\Gamma)}}$.

Remark 3.33. The natural action of the symmetric group $S_{k(v_i(\Gamma))}$ on $\mathcal{M}_\Gamma$ by permuting the boundary labels and edges lifts canonically to a natural action on the bundle $\mathbb{L}_i \to \mathcal{M}_\Gamma$. The same goes for the natural action of $S_{k(v_i(\Gamma))}$ on $\mathcal{M}_v$ and $\mathcal{M}_{\eta(v)}$. The isomorphisms of Observation 3.32 are equivariant with respect to these actions.

Notation 3.34. Let $\Upsilon \in \mathcal{G}_{odd}$. For a subset $C \subseteq \partial\overline{\mathcal{M}}_\Upsilon$, a vector bundle $E \to C$, a multisection $s \in C_m^\infty(C, E)$, and $\Gamma \in \partial^B\Upsilon$, we write

$$s^\Gamma := s|_{\mathcal{M}_\Gamma \cap C}.$$

Observation 3.11 allows us to generalize the definition of canonical multisection as follows. Let $\Upsilon \in \mathcal{G}_{odd}$, let $C \subseteq \partial\overline{\mathcal{M}}_\Upsilon$, and let

$$\mathcal{C} = \{\Gamma \in \partial^B\Upsilon | C \cap \mathcal{M}_\Gamma \neq \emptyset\}.$$

Definition 3.35. A multisection s of

$$E = \bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to C$$

is called *canonical* if for each $\Gamma \in \mathcal{C}$, there exists a section $s^{\mathcal{B}\Gamma}$ of

$$\bigoplus_{i\in[l]} \mathbb{L}_i^{\oplus a_i} \to \mathcal{M}_{\mathcal{B}\Gamma}$$

such that $s^\Gamma = F_\Gamma^* s^{\mathcal{B}\Gamma}|_{C\cap\mathcal{M}_\Gamma}$.

The following refinement of canonical sections is useful in proofs.

Definition 3.36. A multisection $s \in C_m^\infty(C, \mathbb{L}_i)$ is said to be *pulled back from the base component*, or *pulled back from the base* for short, if for every $v \in \mathcal{V}_\mathcal{C}^i$ there exists $s^v \in C_m^\infty(\mathcal{M}_v, \mathbb{L}_i)$ such that for every $\Gamma \in \mathcal{C}$ with $v_i^*(\Gamma) = v$, we have

$$s^\Gamma = \Phi_{\Gamma,i}^* s^v|_{C\cap\mathcal{M}_\Gamma}.$$

A multisection $s \in C_m^\infty(\mathcal{M}_{0,B,I}, \mathbb{L}_i)$ is said to be *invariant* if it is invariant under the action of the permutation group S_B.

A multisection $s \in C_m^\infty(C, \mathbb{L}_i)$ is *special canonical* if it is pulled from the base, and for every $v \in \mathcal{V}_\mathcal{C}^i$ the multisection s^v is invariant. We write $\mathcal{S}_i = \mathcal{S}_{i,k,l}$ for the vector space of special canonical multisections of $\mathbb{L}_i$ over $C = \partial\overline{\mathcal{M}}_{0,k,l}$. Below, we use the notation s^v as in this definition.

Remark 3.37. It is straightforward to verify that a multisection which is pulled back from the base is consistent.

Remark 3.38. Let $s \in C_m^\infty(C, \mathbb{L}_i)$ be special canonical. By Observation 2.33, the map $\Phi_{\Gamma,i} : \mathcal{M}_\Gamma \to \mathcal{M}_{v_i^*(\Gamma)}$ factors as the composition

$$\mathcal{M}_\Gamma \xrightarrow{F_\Gamma} \mathcal{M}_{\mathcal{B}\Gamma} \xrightarrow{\Phi_{\mathcal{B}\Gamma,i}} \mathcal{M}_{v_i^*(\Gamma)}.$$

So there exists $s^{\mathcal{B}\Gamma} \in C_m^\infty(\mathcal{M}_{\mathcal{B}\Gamma}, \mathbb{L}_i)$ such that

$$s^\Gamma = F_\Gamma^* s^{\mathcal{B}\Gamma}|_{C \cap \mathcal{M}_\Gamma}.$$

It follows from Observation 3.11 that the vector space $\bigoplus \mathcal{S}_i^{\oplus a_i}$ is a subvector space of $\mathcal{S}$. Below, we use the notation $s^{\mathcal{B}\Gamma}$ as in this remark.

3.6. **Forgetful maps, cotangent lines and base.** We introduce notations and formulate the basic properties of pull-backs of cotangent lines by forgetful maps.

Observation 3.39. Let $\Gamma \in \mathcal{G}_{odd}$ and $i \in I(\Gamma)$. Then $\mathcal{B}for_i(\Gamma)$ is a stable subgraph of $for_i(\mathcal{B}\Gamma)$ and $For_{for_i(\mathcal{B}\Gamma),\mathcal{B}for_i(\Gamma)}$ is a diffeomorphism. Indeed, vertices and markings are in one-to-one correspondence. Moreover, the following diagram commutes.

$$\begin{array}{ccccc}
\mathcal{M}_\Gamma & & \xrightarrow{\quad For_i \quad} & & \mathcal{M}_{for_i(\Gamma)} \\
\downarrow{\scriptstyle F_\Gamma} & & & & \downarrow{\scriptstyle F_{for_i(\Gamma)}} \\
\mathcal{M}_{\mathcal{B}\Gamma} & \xrightarrow{For_i} & \mathcal{M}_{for_i(\mathcal{B}\Gamma)} & \xrightarrow{For_{for_i(\mathcal{B}\Gamma),\mathcal{B}for_i(\Gamma)}} & \mathcal{M}_{\mathcal{B}for_i(\Gamma)}
\end{array}$$

This is a consequence of Observation 2.33.

Notation 3.40. Let k be odd, let $I \subseteq [l+1]$ and let $i \in I \cap [l]$. If $l+1 \in I$, denote by $D_i \subset \overline{\mathcal{M}}_{0,k,I}$ the locus where the marked points z_i, z_{l+1}, belong to a sphere component that contains only them and a unique interior node. If $l+1 \notin I$, set $D_i = \emptyset$. For $\Gamma \in \partial\Gamma_{0,k,l+1}$, define $D_i \subset \overline{\mathcal{M}}_\Gamma, D_i \subset \mathcal{M}_{\mathcal{B}\Gamma}$, similarly.

Write

$$\partial D_i = D_i \cap \partial\overline{\mathcal{M}}_{0,k,l+1}.$$

Let $\mathcal{G}_{D_i} \subset \partial^B\Gamma_{0,k,l+1}$ be the subset such that

$$\partial D_i = \coprod_{\Gamma \in \mathcal{G}_{D_i}} \mathcal{M}_\Gamma.$$

Observation 3.41. For $\Gamma \in \partial^B\Gamma_{0,k,l+1} \setminus \mathcal{G}_{D_i}$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{M}_\Gamma & \xrightarrow{For_{l+1}} & \mathcal{M}_{for_{l+1}\Gamma} \\
\downarrow{\scriptstyle \Phi_{\Gamma,i}} & & \downarrow{\scriptstyle \Phi_{for_{l+1}\Gamma,i}} \\
\mathcal{M}_{v_i^*(\Gamma)} & \xrightarrow{For_{l+1}} & \mathcal{M}_{v_i^*(for_{l+1}\Gamma)}
\end{array}$$

Again, this is a consequence of Observation 2.33.

Notation 3.42. Write

$$\mathbb{L}_i' = For_{l+1}^*\mathbb{L}_i \to \overline{\mathcal{M}}_{0,k,I}.$$

For $\Gamma \in \partial\Gamma_{0,k,l+1}$ write

$$\mathbb{L}_i' = For_{l+1}^*\mathbb{L}_i \to \overline{\mathcal{M}}_\Gamma, \qquad \mathbb{L}_i' = For_{l+1}^*\mathbb{L}_i \to \overline{\mathcal{M}}_{\mathcal{B}\Gamma}.$$

Denote by $\mathcal{S}_i' = \mathcal{S}_{i,k,l+1}' \subset C_m^\infty(\partial\overline{\mathcal{M}}_{0,k,l+1}, \mathbb{L}_i')$ the vector space of pull-backs of sections in $\mathcal{S}_{i,k,l}$ by For_{l+1}. Denote by

$$\tilde{t}_i : \mathbb{L}_i' \to \mathbb{L}_i$$

the morphism given by $\tilde{t}_i|_{\mathcal{M}_\Gamma} = \mu_i^* t_{\Gamma, for_i(\Gamma)}$.

Lemma 3.43.

(a) The morphism $\tilde{t}_i$ vanishes transversely exactly at D_i.
(b) For_{l+1} maps D_i diffeomorphically onto $\overline{\mathcal{M}}_{0,k,l}$ carrying the orientation induced on D_i by $\tilde{t}_i$ to the orientation $o_{0,k,l}$ on $\overline{\mathcal{M}}_{0,k,l}$.
(c) The morphism $\tilde{t}_i$ satisfies

$$F_\Gamma^*\tilde{t}_i = \tilde{t}_i,$$

and for $\Gamma \in \partial^B\Gamma_{0,k,l+1} \setminus \mathcal{G}_{D_i}$,

$$\Phi_{\Gamma,i}^*\tilde{t}_i = \tilde{t}_i.$$

Here, we have used the isomorphisms of Observation 3.32 to identify relevant domains and ranges of $\tilde{t}_i$.
(d) The morphism $\tilde{t}_i$ is invariant under permutations of the boundary marked and nodal points as in Remark 3.33.

Proof. Observation 3.31 implies that $\tilde{t}_i$ vanishes exactly at D_i. It follows from the definitions that For_{l+1} maps D_i diffeomorphically onto $\overline{\mathcal{M}}_{0,k,l}$. So the transversality and orientation statements are equivalent to the following claim. Let $p \in \overline{\mathcal{M}}_{0,k,l}$, let $F_p = For_{l+1}^{-1}(p)$ and equip F_p with its complex orientation. Then $\tilde{t}_i|_{F_p}$ vanishes with multiplicity $+1$ at the unique point $\check{p} \in D_i \cap F_p$.

To prove the claim, we construct a map $\alpha : D^2 \to F_p$, that preserves complex orientations and calculate $\tilde{t}_i \circ \alpha$ in an explicit trivialization of $\alpha^*\mathbb{L}_i$. Indeed, let $\Sigma = (\{\Sigma_\alpha\}, \sim)$ be a stable disk representing p. Denote by Σ_0 the component of Σ containing the marked point z_i. Denote by $B_r \subset \mathbb{C}$ the disk of radius r centered at 0. Let $U \subset \Sigma_0$ be an open neighborhood of z_i with local coordinate

$$\xi : U \xrightarrow{\sim} B_2, \qquad \xi(z_1) = 0.$$

For $z \in B_1$, let Σ_0^z be obtained from Σ_0 as follows. If $z \neq 0$, add the marked point $z_{l+1} = \xi^{-1}(z)$. If $z = 0$, replace z_i with a new marked point z_0. Denote by S the marked sphere $(\mathbb{C} \cup \{\infty\}, z_{-1}, z_i, z_{l+1})$ where $z_{-1} = \infty$, $z_i = 0$, and $z_{l+1} = 1$. For $z \neq 0$, let Σ^z be the stable disk $(\{\Sigma_\alpha\}_{\alpha \neq 0} \cup \{\Sigma_0^z\}, \sim)$. For $z = 0$, let

$$\Sigma^z = (\{\Sigma_\alpha\}_{\alpha \neq 0} \cup \{\Sigma_0^z, S\}, \sim_0),$$

where $\sim_0$ is obtained from $\sim$ by adding the relation $z_0 \sim_0 z_{-1}$. Define $\alpha : B_1 \to F_p$ by $\alpha(z) = \Sigma^z$.

For $z \in B_1$, the stable disk Σ^z is the deformation of the stable disk Σ^0 obtained by removing appropriate disks around the nodal points $z_0 \in \Sigma_0$ and $z_{-1} \in S$ and identifying annuli adjacent to the resulting boundaries. More explicitly, denoting by ζ the standard coordinate on $S = \mathbb{C} \cup \{\infty\}$, we glue the surfaces

$$\Sigma^z \setminus \xi^{-1}\left(B_{\sqrt{2|z|}}\right), \qquad B_{\sqrt{3/|z|}} \subset S$$

along the map $\zeta \mapsto \xi^{-1}(z\zeta)$ for $\zeta \in B_{\sqrt{3/|z|}} \setminus B_{\sqrt{2/|z|}}$. Thus we take $d\zeta|_{z_i} \in T^*_{z_i}S \simeq T^*_{z_i}\Sigma^z$ as a frame for $\alpha^*\mathbb{L}_i$. On the other hand, $d\xi|_{z_i} \in T^*_{z_i}\Sigma$ is a frame for $\alpha^*\mathbb{L}'_i$. Since $\xi = z\zeta$, we have $\tilde{t}_i\left(d\xi|_{z_i}\right) = z d\zeta|_{z_i}$. Thus $\tilde{t}_i$ vanishes with multiplicity 1 at $z = 0$, which is the point $\hat{p}$, as claimed. So we have proved parts (a) and (b) of the lemma.

Part (c) follows from Observation 3.39 and part (d) follows from the definition of $\tilde{t}_i$. □

Notation 3.44. Denote by $\mathcal{O}(D_i)$ the line bundle

$$Hom(\mathbb{L}'_i, \mathbb{L}_i) = (\mathbb{L}'_i)^* \otimes \mathbb{L}_i.$$

So $\tilde{t}_i$ is a section of $\mathcal{O}(D_i)$. Write

$$t_i = \tilde{t}_i\big|_{\partial\overline{\mathcal{M}}_{0,k,l+1}}.$$

Lemma 3.43 shows that $\mathcal{O}(D_i)$ is the trivial complex line bundle twisted at D_i as implied by the notation. Moreover, tautologically,

$$\mathbb{L}_i \simeq \mathbb{L}'_i \otimes \mathcal{O}(D_i).$$

The following observation is a consequence of Observations 3.32 and 3.39 and the relevant definitions.

Observation 3.45. For $\Gamma \in \partial\Gamma_{0,k,l+1}$ and $i \in [l]$, we have

$$F_\Gamma^*\mathcal{O}(D_i) \simeq \mathcal{O}(D_i).$$

Observation 3.46. If $s' \in \mathcal{S}'_i$, then $s = s't_i$ belongs to $\mathcal{S}_i$ and vanishes on D_i.

Proof. Using Observation 3.41, we see that for $\Gamma \in \partial\Gamma_{0,k,l} \setminus \mathcal{G}_{D_i}$ we may take

$$s^{v_i^*(\Gamma)} = For_{l+1}^*(s')^{v_i^*(for_{l+1}\Gamma)}\tilde{t}_i.$$

For $\Gamma \in \mathcal{G}_{D_i}$, we take $s^{v_i^*(\Gamma)} = 0$. □

Remark 3.47. Recall Observation 3.11 and Remark 3.38. A multisection $s \in \mathcal{S}_i'$ behaves similarly. Namely, for each $\Gamma \in \partial^B\Gamma_{0,k,l+1}$, there exists

$$s^{\mathcal{B}\Gamma} \in C_m^\infty\left(\mathcal{M}_{\mathcal{B}\Gamma}, \mathbb{L}_i'\right)$$

such that

$$s^\Gamma = F_\Gamma^* s^{\mathcal{B}\Gamma}.$$

This follows from Observation 3.39.

3.7. **Construction of multisections and homotopies.** In this section we prove Theorem 3.3, namely, the open descendent integrals are well defined. In addition we construct special canonical multisections of special types, which we later use to prove the geometric recursions.

Notation 3.48. For a bundle $E \to M$, we denote by 0 its 0-section. Given a multisection s, the notation $s \pitchfork 0$ means that s is transverse to the 0-section. See Appendix A.

Proposition 3.49. *Consider $\mathbb{L}_i \to \overline{\mathcal{M}}_{0,k,l}$, with k odd.*

(a) *For any $p \in \partial\overline{\mathcal{M}}_{0,k,l}$ one can find $s \in \mathcal{S}_i$ which does not vanish at p. Hence, one can choose finitely many such multisections which span the fiber of $\mathbb{L}_i$ over each point of $\partial\overline{\mathcal{M}}_{0,k,l}$.*

(b) *For $i \in [l]$, and*

$$p \in \partial\overline{\mathcal{M}}_{0,k,l+1} \setminus \partial D_i, \qquad q \in \partial D_i,$$

one can find $s \in \mathcal{S}_i$ of the form

$$s = s't_i, \qquad s' \in \mathcal{S}_i',$$

that does not vanish at p, vanishes on ∂D_i and such that $ds|_q$ surjects onto $(\mathbb{L}_i)_q$.

Hence, one can choose finitely many such multisections that span the fiber of $\mathbb{L}_i$ over each point of $\partial\overline{\mathcal{M}}_{0,k,l+1} \setminus \partial D_i$ and such that images of their derivatives span the fiber of $\mathbb{L}_i$ at each point of D_i.

Proof. In both cases the 'hence' part follows immediately from the previous part because of the compactness of $\partial\overline{\mathcal{M}}_{0,k,l}$. We first prove part (a). To construct the special canonical multisection s, it suffices to construct multisections $s^v \in C_m^\infty(\overline{\mathcal{M}}_v, \mathbb{L}_i)$ for each abstract vertex $v \in \mathcal{V}^i_{\partial^B\Gamma_{0,k,l}}$ that have certain properties.

Let $\mathcal{M}_\Gamma$ be the boundary stratum of $\overline{\mathcal{M}}_{0,k,l}$ that contains p, let

$$v^* = v_i^*(\Gamma),$$

and write $k^* = k\left(v^*\right)$. Write

$$\hat{p}_1, \ldots, \hat{p}_{k^*!} \in \mathcal{M}_{v^*}$$

for $\Phi_{\Gamma,i}\left(p\right)$ and its conjugates under the action of S_{k^*} on $\mathcal{M}_{v^*}$.

The properties the multisections $\{s^v\}$ should satisfy are as follows.

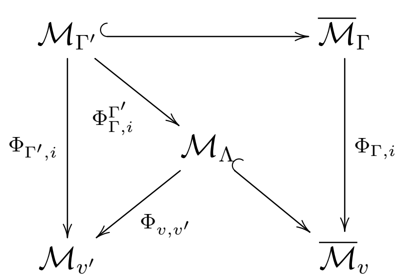
(a) For all $v \in \mathcal{V}^i_{\partial^B\Gamma_{0,k,l}}$ the multisection s^v is invariant.

(b) For all $v, v' \in \mathcal{V}^i_{\partial^B\Gamma_{0,k,l}}$ such that $v' \in \partial_i^{eff} v$, we have

$$s^v|_{\partial_{v'}\mathcal{M}_v} = \Phi^*_{v,v'} s^{v'}.$$

(c) No branch of s^{v^*} vanishes at $\hat{p}_1, \ldots, \hat{p}_{k^*!}$.

The compatibility property (b), the invariance property (a), Observation 3.25, Observation 3.28 and Remark 3.33, imply that the pull-backs of the various s^v to $\partial\overline{\mathcal{M}}_{0,k,l}$ fit together to give a smooth s. The implication depends on the commutativity of the following diagram, in which we use the notation of Observation 3.28.



Consistency of s follows from Remark 3.37. Property (c) implies that s does not vanish at p.

We construct the multisections s^v by induction on $\dim_{\mathbb{C}} \mathcal{M}_v$. Start the induction with $\dim_{\mathbb{C}} \mathcal{M}_v = -1$. Then the multisections s^v exist trivially, since there are no such v.

Assume we have constructed multisections s^u that satisfy properties (a)-(c) for all $u \in \mathcal{V}^i_{\partial^B\Gamma_{0,k,l}}$ such that $\dim_{\mathbb{C}} \mathcal{M}_u \leq m$. Let $v \in \mathcal{V}^i_{\partial^B\Gamma_{0,k,l}}$ be an abstract vertex such that $\dim_{\mathbb{C}} \mathcal{M}_v = m+1$. By induction we have defined $s^{v'}$ for all $v' \in \partial_i^{eff} v$, as for such v' we have

$$\dim_{\mathbb{C}} \mathcal{M}_{v'} < \dim_{\mathbb{C}} \mathcal{M}_v.$$

Define the section s_1 on

$$\partial\overline{\mathcal{M}}_v = \bigcup_{v' \in \partial_i^{eff} v} \partial_{v'}\overline{\mathcal{M}}_v$$

by

$$(21) \qquad s_1|_{\partial_{v'}\overline{\mathcal{M}}_v} = \Phi^*_{v,v'} s^{v'}.$$

The induction hypotheses on compatibility (b) and invariance (a), Observation 3.25, Observation 3.28 and Remark 3.33, imply that the section s_1 thus defined is smooth on $\partial\overline{\mathcal{M}}_v$. Consistency of s_1 follows directly from the defining equation (21). So, we may extend s_1 smoothly to all $\overline{\mathcal{M}}_v$. If $v = v^*$, we make sure that the extension is non-vanishing at $\hat{p}_1, \ldots, \hat{p}_{k^*!} \in \mathcal{M}_{v^*}$. We denote the resulting multisection by s_1 as well. It satisfies the compatibility condition (b) by construction.

Define s^v to be the $S_{k(v)}$ symmetrization of s_1. See Appendix A, Definition A.10. So s^v satisfies the invariance condition (a). But by the induction hypothesis on invariance (a) and Remark 3.33, $s^v|_{\partial\overline{\mathcal{M}}_v} = s_1$. So, s^v satisfies the compatibility condition (b) as well.

For case (b), write

$$For_{l+1}(p) = p', \qquad For_{l+1}(q) = q'.$$

Using case (a), construct a special canonical multisection s_1 of

$$\mathbb{L}_i \to \partial\overline{\mathcal{M}}_{0,k,l}$$

that does not vanish at p'. Construct a second special canonical multisection s_2 of $\mathbb{L}_i \to \partial\overline{\mathcal{M}}_{0,k,l}$ that does not vanish at q'. Denote by s_3 a linear combination of s_1 and s_2 that does not vanish at p', q'. Then $s = s_3 t_i$ satisfies our requirements by Observation 3.46.

□

Another ingredient we need for the proof of Theorem 3.3 is the following transversality theorem.

Theorem 3.50. *Let V be a manifold, let N be a manifold with corners, and let $\mathbb{E} \to N$ be a vector bundle. Denote by $p_N : V \times N \to N$ the projection. Let*

$$F : V \to C^\infty(N, \mathbb{E}), \qquad v \mapsto F_v,$$

satisfy the following conditions:

(a) The section

$$F^{ev} \in C^\infty(V \times N, p_N^*\mathbb{E}), \qquad F^{ev}(v, x) = F_v(x),$$

is smooth.

(b) F^{ev} is transverse to 0.

Then the set

$$\{v \in V \,|\, F_v \pitchfork 0\}$$

is residual.

Remark 3.51. A similar theorem may be found in [9, pp. 79-80] in the more general setting where $C^\infty(N, \mathbb{E})$ is replaced by the space of smooth maps between two manifolds. However, the manifolds considered do not have boundary or corners. In [11], Joyce defines a notion of smooth maps of manifolds with corners that guarantees the existence of fiber products for transverse smooth maps. In Joyce's terminology, a map of manifolds with corners that is smooth in each coordinate chart is called weakly smooth. To be smooth, it must satisfy an additional condition at corners. Since we consider only sections of vector bundles, the section F^{ev} is automatically smooth if it is weakly smooth. Thus $(F^{ev})^{-1}(0)$, being a transverse fiber product, is a manifold with corners, and the proof given in [9] goes through for our case as well.

As a consequence, we have the following theorem on multisection transversality. Relevant operations on multisections are reviewed in Appendix A. See, in particular, Definition A.9 for the definition of summation.

Theorem 3.52. *We continue with the notation of Theorem 3.50 in the special case where V is the vector space $\mathbb{R}^n$. Fix $s_0, \ldots, s_n \in C^\infty_m(N, \mathbb{E})$. Take*

$$F : V \to C^\infty_m(N, \mathbb{E})$$

to be the map

$$(\lambda_i)_{i\in[n]} \mapsto s_0 + \sum \lambda_i s_i.$$

If the multisection

$$F^{ev} \in C^\infty_m(V \times N, p_N^*\mathbb{E}), \qquad F^{ev}(v, x) = F_v(x),$$

is transverse to 0*, then the set*

$$\{v \in V \,|\, F_v \pitchfork 0\}$$

is residual.

Proof. Take $p \in N$. There exists a neighborhood W of p such that each multisection $s_i|_W$ is a weighted combination of m_i sections. Hence $F^{ev}|_{V\times W}$ is a weighted combination of appropriately defined sections $F^{ev}_{W,j}$ for $j = 1, \ldots, \prod_{i=1}^n m_i$. Apply Theorem 3.50 to each section $F_{W,j}$ individually to conclude that the set

$$U_W = \bigcap_j \left\{ v \in V \,|\, F^{ev}_{W,j}(v, -) \pitchfork 0 \right\}$$

is residual. Choose a countable open cover $\{W_l\}$ of N. Then for every

$$v \in U = \bigcap_l U_{W_l}$$

we have $F_v \pitchfork 0$. Moreover, U is residual. The theorem follows. $\square$

Lemma 3.53. *Fix a sequence of non-negative integers*

$$a_1, \ldots, a_l, \qquad 2\sum a_i = k + 2l - 3,$$

and set $E = \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus a_i} \to \overline{\mathcal{M}}_{0,k,l}$.

(a) One can construct special canonical multisections

$$s_{ij} \in \mathcal{S}_i, \qquad i \in [l], \quad j \in [a_i],$$

such that $\mathbf{s} = \oplus s_{ij}$ *vanishes nowhere on* $\partial\overline{\mathcal{M}}_{0,k,l}$. *Hence,* $e\left(E; \mathbf{s}\right)$ *is defined.*

(b) Moreover, we may impose the following further condition on the multisections s_{ij}. *For all abstract vertices* $v \in \mathcal{V}_{\partial^B \Gamma_{0,k,l}}$, *and all*

$$K \subseteq \bigcup_{i \in I(v)} \{i\} \times [a_{ij}],$$

we have

$$\bigoplus_{ab \in K} s^v_{ab} \pitchfork 0.$$

Proof. We begin with the proof of part (a). Let

$$w_{ijk} \in \mathcal{S}_i, \qquad\qquad i \in [l], \quad j \in [a_i], \quad k \in [m_{ij}],$$

be a finite collection of special canonical multisections of the j^{th} copy of $\mathbb{L}_i$, which together span its fiber $\left(\mathbb{L}_i\right)_p$ for all $p \in \partial\overline{\mathcal{M}}_{0,k,l}$. Such multisections exist by Proposition 3.49, case (a). We write

$$J = \{ijk\}_{i \in [l], j \in [a_i], k \in [m_{ij}]}.$$

Apply Theorem 3.52 with

$$N = \partial\overline{\mathcal{M}}_{0,k,l}, \qquad \mathbb{E} = E|_N, \qquad V = V_0 = \mathbb{R}^J,$$

and F given by

$$F_\lambda = \sum_{ijk \in J} \lambda_{ijk} w_{ijk}, \qquad \lambda = \{\lambda_{ijk}\}_{ijk \in J} \in V_0.$$

Let Λ_0 be the set of $\lambda \in V$ such that $F_\lambda \pitchfork 0$. Theorem 3.52 implies that Λ_0 is residual. Dimension counting shows that for each $\lambda \in \Lambda_0$, we have $F_\lambda^{-1}(0) = \emptyset$. Thus for any $\lambda \in \Lambda_0$, we may take

$$s_{ij} = s_{ij}^{\lambda} = \sum_k \lambda_{ijk} w_{ijk}. \tag{22}$$

We turn to the proof of part (b). For an abstract vertex $v \in \mathcal{V}_{\Gamma_{0,k,l}}$, and a set K as in the statement of the lemma, write

$$J_{v,K} = \{abc |\, ab \in K,\ c \in [m_{ab}]\} \subseteq J.$$

Apply Theorem 3.52 with

$$N = \mathcal{M}_v, \qquad \mathbb{E} = \bigoplus_{\{ab \in K\}} \mathbb{L}_a, \qquad V = V_{v,K} = \mathbb{R}^{J_{v,K}},$$

and $F = F_{v,K}$ given by

$$(F_{v,K})_\lambda = \sum_{ijk \in J_{v,K}} \lambda_{ijk} w^v_{ijk}, \qquad \lambda = \{\lambda_{ijk}\}_{ijk \in J_{v,K}} \in V_{v,K}.$$

Let

$$\Lambda_{v,K} = \{\lambda \in V_{v,K} | (F_{v,K})_\lambda \pitchfork 0\}.$$

Theorem 3.52 implies that $\Lambda_{v,K}$ is residual. Denote by $p_{v,K} : V_0 \to V_{v,K}$ the projection. It follows that

$$\Lambda = \Lambda_0 \cap \bigcap_{v,K} p^{-1}_{v,K}\left(\Lambda_{v,K}\right)$$

is residual.

For any $\lambda \in \Lambda$, take $s_{ij} = s^\lambda_{ij}$ as in equation (22). Then for any abstract vertex v and set K, we have

$$\bigoplus_{ab \in K} s^v_{ab} = (F_{v,K})_\lambda \pitchfork 0,$$

as desired. □

Lemma 3.54. *Let $E_1, E_2 \to \overline{\mathcal{M}}_{0,k,l}$ be given by*

$$E_1 = \bigoplus_{i \in [l]} \mathbb{L}^{a_i}_i, \qquad E_2 = \bigoplus_{i \in [l]} \mathbb{L}^{b_i}_i.$$

Put $E = E_1 \oplus E_2$, and assume $\operatorname{rk} E = \frac{k+2l-3}{2}$. Let $\mathcal{C} \subseteq \partial^B \Gamma_{0,k,l}$ and

$$C = \coprod_{\Gamma \in \mathcal{C}} \mathcal{M}_\Gamma \subseteq \partial \overline{\mathcal{M}}_{0,k,l}.$$

Let $\mathbf{s}, \mathbf{r}$, be two multisections of $E|_{\partial \overline{\mathcal{M}}_{0,k,l}}$ which satisfy

(a) $\mathbf{s}|_C$ and $\mathbf{r}|_C$ are canonical.
(b) The projections of $\mathbf{s}, \mathbf{r}$, to E_1 are identical and transverse to 0.

Then one may find a homotopy H between $\mathbf{s}, \mathbf{r}$, which is transverse to 0 everywhere, does not vanish anywhere on $C \times [0,1]$ and such that its projection to E_1 is constant in time. Moreover, H can be taken to be of the form

$$(23) \qquad H(p,t) = (1-t)\mathbf{s}(p) + t\mathbf{r}(p) + t(1-t)w(p),$$

where $w(p)$ is a canonical multisection.

Proof. Denote by $\mathbf{s}_1$ the projection of $\mathbf{s}$ to E_1. Let

$$w_i, \qquad i \in [m],$$

be a finite set of special canonical multisections which together span the fiber $(E_2)_p$ for all $p \in \partial\overline{\mathcal{M}}_{0,k,l}$. Such multisections exist by Proposition 3.49, case (a). Denote by $\pi : \partial\overline{\mathcal{M}}_{0,k,l} \times [0,1] \to \partial\overline{\mathcal{M}}_{0,k,l}$ the canonical projection. Let $\mathbf{h} \in C_m^\infty(\pi^* E|_{\partial\overline{\mathcal{M}}_{0,k,l}})$ be given by

$$\mathbf{h}\,(p,t) = (1-t)\,\mathbf{s}\,(p) + t\mathbf{r}\,(p) \qquad p \in \partial\overline{\mathcal{M}}_{0,k,l}, \quad t \in (0,1)\,.$$

Apply Theorem 3.52 with

$$N = \partial\overline{\mathcal{M}}_{0,k,l} \times (0,1)\,, \qquad \mathbb{E} = \pi^* E|_{\partial\overline{\mathcal{M}}_{0,k,l}}\,, \qquad V = V_0 = \mathbb{R}^m,$$

and $F = \mathcal{F}$ given by

$$\mathcal{F}_\lambda(p,t) = \mathbf{h}\,(p,t) + t\,(1-t)\sum \lambda_i w_i, \qquad \lambda \in V_0.$$

By assumption (b), the derivatives of $\mathcal{F}^{ev}$ in directions tangent to $\partial\overline{\mathcal{M}}_{0,k,l}$ span the fiber $(E_1)_p$ at each p where $\mathbf{s}_1$ vanishes. Since the multisections w_i span $(E_2)_p$, the derivatives of $\mathcal{F}^{ev}$ in the V directions span the fiber $(E_2)_p$ for all $p \in \partial\overline{\mathcal{M}}_{0,k,l}$. It follows that $\mathcal{F}^{ev} \pitchfork 0$. Thus, Theorem 3.52 implies the set Λ of all $\lambda \in V_0$ such that $\mathcal{F}_\lambda \pitchfork 0$ is residual.

Let $\Gamma \in \mathcal{C}$. Denoting by $E_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}$ the appropriate sum of cotangent line bundles, Observation 3.32 implies that $F_\Gamma^* E_\Gamma = E$. Write

$$N_\Gamma = \mathcal{M}_{\mathcal{B}\Gamma} \times (0,1)$$

and denote by $\pi_\Gamma : N_\Gamma \to \mathcal{M}_{\mathcal{B}\Gamma}$ the canonical projection. Write

$$\mathbb{E}_\Gamma = \pi_\Gamma^* E_\Gamma.$$

It follows from Observation 3.11 and Remark 3.38 that there exists $\mathcal{F}^{\mathcal{B}\Gamma} : V_0 \to C_m^\infty(N_\Gamma, \mathbb{E}_\Gamma)$ such that

$$\mathcal{F}_\lambda|_{\mathcal{M}_\Gamma \times (0,1)} = (F_\Gamma \times \mathrm{Id}_{(0,1)})^* \mathcal{F}_\lambda^{\mathcal{B}\Gamma}, \qquad \lambda \in V_0.$$

Apply Theorem 3.52 with

$$N = N_\Gamma, \qquad \mathbb{E} = \mathbb{E}_\Gamma, \qquad V = V_0, \qquad F = \mathcal{F}^{\mathcal{B}\Gamma}.$$

Since $\mathbf{s}_1 \pitchfork 0$, it follows that $\mathbf{s}_1^{\mathcal{B}\Gamma} \pitchfork 0$. Thus the same argument that shows $\mathcal{F}^{ev} \pitchfork 0$ also shows $(\mathcal{F}^{\mathcal{B}\Gamma})^{ev} \pitchfork 0$. So, the theorem implies the set Λ_Γ of $\lambda \in V_0$ such that $\mathcal{F}_\lambda^{\mathcal{B}\Gamma} \pitchfork 0$ is residual. By Observation 3.15, for $\lambda \in \Lambda_\Gamma$, the homotopy $\mathcal{F}_\lambda^{\mathcal{B}\Gamma}$ does not vanish anywhere. Therefore, the homotopy $\mathcal{F}_\lambda|_{\mathcal{M}_\Gamma \times (0,1)}$ also does not vanish anywhere. We conclude that for

$$\lambda \in \Lambda \cap \bigcap_{\Gamma \in \mathcal{C}} \Lambda_\Gamma,$$

the homotopy $\mathcal{F}_\lambda$ satisfies the requirements of the lemma. $\square$

We will also need the following general lemma on the relative Euler class. For a multisection s that is transverse to zero, we denote by $Z(s)$ its vanishing locus considered as a weighted branched submanifold. For a zero dimensional weighted branched submanifold $Z \subset M$, we denote by $\#Z$ its weighted cardinality. See Appendix A for details.

Lemma 3.55. *Let $E \to M$ be a vector bundle over a manifold with corners with $\operatorname{rk} E = \dim M$, and let $s_0, s_1 \in C_m^\infty(\partial M, E)$ vanish nowhere. Let $p : [0,1] \times M \to M$ denote the projection and let*

$$H \in C_m^\infty([0,1] \times \partial M, p_1^* E)$$

satisfy

$$H|_{\{i\}\times M} = s_i, \qquad i = 0, 1.$$

Moreover, assume H is transverse to zero. Then

$$\int_M e(E; s_1) - \int_M e(E; s_0) = \#Z(H).$$

Proof. For $i = 0, 1$, let $\tilde{s}_i \in C_m^\infty(M, E)$ be an extension of s_i that is transverse to zero. Recall that

$$\partial([0,1] \times M) = \{1\} \times M - \{0\} \times M - [0,1] \times \partial M.$$

So, the multisections $\tilde{s}_0, \tilde{s}_1, H$, fit together to give a multisection

$$r \in C_m^\infty(\partial([0,1] \times M), p_1^* E)$$

that is transverse to zero. Let $\tilde{r} \in C_m^\infty([0,1] \times M, p_1^* E)$ be an extension of r that is transverse to zero. Then $Z(\tilde{r})$ is a weighted branched 1-manifold with boundary. The weighted cardinality of the boundary points of such a weighted branched manifold is zero. Thus

$$\begin{aligned} 0 = \#\partial Z(\tilde{r}) = \#Z(r) &= \#Z(\tilde{s}_1) - \#Z(\tilde{s}_0) - \#Z(H) \\ &= \int_M e(E; s_1) - \int_M e(E; s_0) - \#Z(H). \end{aligned}$$

$\square$

Proof of Theorem 3.3. By Lemma 3.53(a) and Remark 3.38 there exists a nowhere vanishing canonical multisection $\mathbf{s} \in \mathcal{S}$. It remains to show that $e(E, \mathbf{s})$ is independent of the choice of $\mathbf{s}$. By Lemma 3.55 it suffices to construct a nowhere vanishing homotopy between any two canonical multisections $\mathbf{s}, \mathbf{r}$, that each vanish nowhere. But the existence of such a homotopy is a direct consequence of Lemma 3.54, with the bundle $E_1 = 0$, and the collection of boundary strata C being the entire boundary $\partial\overline{\mathcal{M}}_{0,k,l}$. $\square$

We now consider slightly more general bundles, which we shall need later on.

Lemma 3.56. *Let* $1 \le h \le l$. *Let*

$$E \to \overline{\mathcal{M}}_{0,k,l+1}$$

be given by $E = E_1 \oplus E_2 \oplus E_3$ *where*

$$E_1 = \bigoplus_{i=1}^{l+1} \mathbb{L}_i^{\oplus a_i}, \qquad E_2 = \bigoplus_{i=1}^{l} \left(\mathbb{L}_i'\right)^{\oplus a_i'}, \qquad E_3 = \mathcal{O}\left(D_h\right)^{\oplus \varepsilon},$$

and

$$\varepsilon \in \{0,1\}, \qquad (a_1 + \ldots + a_{l+1}) + (a_1' + \ldots a_l') + \varepsilon = \frac{k+2l-1}{2}.$$

One can construct

$$\begin{aligned} s_{ij} &\in \mathcal{S}_i, \qquad i \in [l+1], \quad j \in [a_i], \\ s_{ij}' &\in \mathcal{S}_i', \qquad i \in [l], \quad j \in [a_i'], \end{aligned}$$

such that

$$\mathbf{s} = \bigoplus s_{ij} \oplus \bigoplus s_{ij}' \oplus t_h^{\oplus \varepsilon}$$

does not vanish anywhere. In particular, the relative Euler class $e(E;\mathbf{s})$ *is defined. Moreover, any two choices of such* s_{ij}, s_{ij}', *define the same relative Euler class. Furthermore, the following statements are valid simultaneously:*

(a) Suppose $1 \le i_0 \le l$ *and* $1 \le j_0 \le a_{i_0}$. *If* $\varepsilon = 1$, *suppose that* $i_0 \ne h$. *Then we may assume*

$$s_{i_0 j_0} = s' t_{i_0}, \qquad s' \in \mathcal{S}_{i_0}'.$$

and $s_{i_0 j}$ *does not vanish anywhere on* ∂D_{i_0} *for* $j \ne j_0$.

(b) Suppose $a_{l+1} > 0$. *Then we may assume* $s_{(l+1)1}$ *does not vanish anywhere on* ∂D_i *for all* i.

(c) Suppose $\mathrm{rk}(E_1 \oplus E_3) = 1$. *Then we may assume* $\bigoplus_{i=1}^{l} \bigoplus_{j=1}^{a_i'} s_{ij}'$ *does not vanish anywhere on* $\partial\overline{\mathcal{M}}_{0,k,l+1}$.

Proof. The proof is very similar to that of Lemma 3.53 and Lemma 3.54.

First, we prove cases (a) and (b). Using Proposition 3.49 case (a), choose

$$w_{ijk} \in \mathcal{S}_i, \qquad i \in [l+1], \quad j \in [a_i], \quad k \in [m_{ij}], \quad (i,j) \ne (i_0, j_0),$$

such that for each i, j, the multisections w_{ijk} for $k \in [m_{ij}]$ span the fiber $(\mathbb{L}_i)_p$ for all $p \in \partial\overline{\mathcal{M}}_{0,k,l+1}$. Choose

$$w_k' \in \mathcal{S}_{i_0}', \qquad w_{i_0 j_0 k} = w_k' t_{i_0} \in \mathcal{S}_{i_0}, \qquad k \in [m_{i_0 j_0}],$$

as in Proposition 3.49, case (b), that span the fiber $(\mathbb{L}_{i_0})_p$ for all p not in D_{i_0}, and such that the images of their derivatives at every $q \in D_{i_0}$ span $(\mathbb{L}_{i_0})_q$. Using Proposition 3.49 case (a) over $\overline{\mathcal{M}}_{0,k,l}$ and pulling back by For_{l+1}, choose

$$w'_{ijk} \in \mathcal{S}'_i, \qquad i \in [l+1], \quad j \in [a'_i], \quad k \in [m'_{ij}],$$

such that for each i, j, the multisections w'_{ijk} for $k \in [m'_{ij}]$ span the fiber $(\mathbb{L}'_i)_p$ for all $p \in \partial\overline{\mathcal{M}}_{0,k,l+1}$.

Write

$$J = \{ijk\}_{i \in [l+1], j \in [a_i], k \in [m_{ij}]}, \qquad J' = \{ijk\}_{i \in [l], j \in [a'_i], k \in [m'_{ij}]}.$$

Apply Theorem 3.52 with

$$N = \partial\overline{\mathcal{M}}_{0,k,l+1}, \qquad \mathbb{E} = E, \qquad V = \mathbb{R}^{J \cup J'},$$

and F given by

$$F_\lambda = \sum_{ijk \in J} \lambda_{ijk} w_{ijk} + \sum_{ijk \in J'} \lambda'_{ijk} w'_{ijk} + \delta_{\varepsilon,1} t_h,$$

for

$$\lambda = (\{\lambda_{ijk}\}_{ijk \in J}, \{\lambda'_{ijk}\}_{ijk \in J'}) \in V.$$

We claim that $F^{ev} \pitchfork 0$. Indeed, if $p \in \overline{\mathcal{M}}_{0,k,l+1} \setminus (D_{i_0} \cup D_h)$ then the derivatives of F^{ev} in the directions tangent to V span the fiber E_p. If $p \in D_{i_0}$, then $p \notin D_h$. So, the derivatives of F^{ev} in the directions tangent to $\partial\overline{\mathcal{M}}_{0,k,l+1}$ span the fiber of the j_0^{th} copy of L_{i_0} at p, while the derivatives in the directions tangent to V span the complementary summand of the fiber E_p. If $p \in D_h$, then $p \notin D_{i_0}$. So, the derivatives of F^{ev} in the directions tangent to $\partial\overline{\mathcal{M}}_{0,k,l+1}$ span the fiber $\mathcal{O}(D_h)_p^{\oplus\varepsilon}$, while the derivatives in the directions tangent to V span the complementary summand of the fiber E_p.

Theorem 3.52 implies there exists a residual subset $\Lambda \subset V$ such that if $\lambda \in \Lambda$ then $F_\lambda \pitchfork 0$. By dimension counting, transversality is equivalent to non-vanishing.

Write v_i for the closed abstract vertex with

$$I(v_i) = \{i, l+1, [l] \setminus \{i\}\}.$$

So, $v_i = v_i^*(\Gamma)$ for all $\Gamma \in \mathcal{G}_{D_i}$. Let

$$\Lambda' = \left\{ \lambda \in V \;\middle|\; \begin{array}{ll} \sum_k \lambda_{i_0 jk} w^{v_{i_0}}_{i_0 jk} \neq 0, & j \neq j_0 \\ \sum_k \lambda_{(l+1)jk} w^{v_i}_{l+1jk} \neq 0, & 1 \leq i \leq l \end{array} \right\}.$$

Since $\mathcal{M}_{v_i}$ is a point, Λ' is the complement of a finite union of linear subspaces $U_j, W_i \subseteq V$, one for each inequality. By choice of the sections w_{ijk}, for each $j \geq 2$, there is a $k \in [m_{i_0 j}]$ such that $w^{v_{i_0}}_{i_0 jk} \neq 0$. So U_j

is a proper subspace for $j \geq 2$. Similarly, for each $i \in [l]$, there is a $k \in [m_{(l+1)1}]$ such that $w^{v_i}_{(l+1)jk} \neq 0$. So W_i is a proper subspace for $i \in [l]$. It follows that Λ' is open and dense in V. Thus we may choose $\lambda \in \Lambda \cap \Lambda'$ and set

$$s_{ij} = \sum_k \lambda_{ijk} w_{ijk}, \qquad s'_{ij} = \sum_k \lambda'_{ijk} w'_{ijk}, \qquad s' = \sum_k \lambda_{i_0 j_0 k} w'_k.$$

This proves cases (a) and (b).

Case (c) follows from a similar argument and the fact that the multisections s'_{ij} are pulled back from $\partial\overline{\mathcal{M}}_{0,k,l}$, which has complex dimension one less. So, transversality implies non-vanishing even with one less section.

Using Remark 3.47, the proof of the existence of non-vanishing homotopies in the present case is analogous to the proof of Lemma 3.54. $\square$

4. Geometric recursions

4.1. **Proof of string equation.** Recall Notations 3.40 and 3.44.

Observation 4.1. *$D_i \cap D_j = \emptyset$ for $i \neq j$.* An immediate consequence is the following. Let $E \to \overline{\mathcal{M}}_{0,k,l+1}$ be a bundle containing $\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)$ as a summand. Let $\mathbf{s} \in C^\infty_m(\partial\overline{\mathcal{M}}_{0,k,l+1}, E)$ be a nowhere vanishing multisection that upon projection to $\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)$ agrees with $t_i \oplus t_j$. Then

$$e(E; \mathbf{s}) = 0.$$

Observation 4.2. Let s_i be a special canonical multisection of $\mathbb{L}_i \to \partial\overline{\mathcal{M}}_{0,k,l+1}$ that does not vanish on ∂D_i. Let $E \to \overline{\mathcal{M}}_{0,k,l+1}$ be a bundle that contains $\mathcal{O}(D_i) \oplus \mathbb{L}_i$ as a summand. Let $\mathbf{s} \in C^\infty_m\left(\partial\overline{\mathcal{M}}_{0,k,l+1}, E\right)$ be a nowhere vanishing multisection that upon projection to $\mathcal{O}(D_i) \oplus \mathbb{L}_i$, agrees with $s_i \oplus t_i$. Then

$$e(E; \mathbf{s}) = 0.$$

The same holds if we replace $s_i, \mathbb{L}_i$, everywhere with $s_{l+1}, \mathbb{L}_{l+1}$, respectively.

Proof. Let $\Gamma_i \in \partial\Gamma_{0,k,l+1}$ be the stable graph such that $D_i = \overline{\mathcal{M}}_{\Gamma_i}$, and let v_i be the abstract closed vertex with

$$I(v_i) = \{i, l+1, [l] \setminus \{i\}\}.$$

So $v_i = v_i^*(\Gamma_i) = v_i^*(\Gamma)$ for $\Gamma \in \partial\Gamma_i$. By the definition of a special canonical multisection, there exists a multisection s^{v_i} of $\mathbb{L}_i \to \mathcal{M}_{v_i}$ such that for each $\Gamma \in \mathcal{G}_{D_i}$ we have $s_i^\Gamma = \Phi^*_{\Gamma,i} s^{v_i}$. So, we may extend s_i to a multisection $\tilde{s}_i \in C^\infty_m\left(\overline{\mathcal{M}}_{0,k,l}, \mathbb{L}_i\right)$ such that $\tilde{s}_i|_{\overline{\mathcal{M}}_{\Gamma_i}} = \Phi^*_{\Gamma_i,i} s^v$.

Since s_i does not vanish anywhere on ∂D_i, it follows that $s_i^{v_i}$ does not vanish and thus $\tilde{s}_i$ does not vanish anywhere on D_i. Therefore, $Z(\tilde{s}_i) \cap Z(\tilde{t}_i) = \emptyset$, which implies the Euler class of E vanishes. The same argument works for the case of $s_{l+1}, \mathbb{L}_{l+1}$. $\square$

Lemma 4.3. *Let $E \to X$ be a vector bundle over a manifold with corners with $\operatorname{rk} E = \dim X$. Suppose that $E = L \oplus E'$, where $L \to X$ is a line bundle, and $L = L_1 \otimes L_2$ for line bundles $L_1, L_2 \to X$. Let $\mathbf{s} \in C_m^\infty(\partial X, E)$ vanish nowhere and satisfy $\mathbf{s} = s \oplus \mathbf{s}'$, where $s \in C_m^\infty(\partial X, L)$, and $s = s_1 \otimes s_2$ for $s_1 \in C^\infty(\partial X, L_1)$ and $s_2 \in C_m^\infty(\partial X, L_2)$. Then*

$$e(E; \mathbf{s}) = e(L_1 \oplus E'; s_1 \oplus \mathbf{s}') + e(L_2 \oplus E'; s_2 \oplus \mathbf{s}').$$

Since the multisection $\mathbf{s}$ vanishes nowhere, the multisections $s_i \oplus \mathbf{s}'$ for $i = 1, 2$ also vanish nowhere. Thus the relative Euler classes on the right-hand side are well-defined.

Proof. Let $\tilde{s}_1, \tilde{s}_2$ and $\tilde{\mathbf{s}}'$, be extensions to X of s_1, s_2 and $\mathbf{s}'$ respectively, such that

$$\tilde{s}_i \oplus \mathbf{s}' \pitchfork 0, \quad i = 1, 2, \qquad \tilde{s}_1 \oplus \tilde{s}_2 \oplus \tilde{\mathbf{s}}' \pitchfork 0.$$

By assumption $\operatorname{rk} L_1 \oplus L_2 \oplus E' > \dim X$, so $\tilde{s}_1 \oplus \tilde{s}_2 \oplus \tilde{\mathbf{s}}'$ vanishes nowhere. Therefore,

$$Z(\tilde{s}_1 \oplus \tilde{\mathbf{s}}') \cap Z(\tilde{s}_2 \oplus \tilde{\mathbf{s}}') = \emptyset.$$

Setting $\tilde{s} = \tilde{s}_1 \tilde{s}_2$, it follows that $\tilde{s} \oplus \tilde{\mathbf{s}}'$ is transverse to zero. Thus

$$Z(\tilde{s} \oplus \tilde{\mathbf{s}}') = Z(\tilde{s}_1 \oplus \tilde{\mathbf{s}}') \cup Z(\tilde{s}_2 \oplus \tilde{\mathbf{s}}'),$$

which implies the claim. $\square$

Remark 4.4. In the proof of the preceding Lemma, we cannot make $\tilde{s}$ by itself transverse to zero at any point where both $\tilde{s}_1$ and $\tilde{s}_2$ vanish. Such points are unavoidable in general, but generically they do not intersect $Z(\tilde{\mathbf{s}}')$.

In the following, given pairs

$$\mathcal{E}_i = (E_i, \mathcal{Y}_i), \qquad i = 1, 2,$$

of vector bundles $E_i \to X$ and affine subspaces $\mathcal{Y}_i \subset C_m^\infty(E_i|_{\partial X})$, we write

$$\mathcal{E}_1 \oplus \mathcal{E}_2 = (E_1 \oplus E_2, \mathcal{Y}_1 \oplus \mathcal{Y}_2).$$

For

$$\mathbf{a} = (a_1, \ldots, a_{l+1}) \in \mathbb{Z}_{\geq 0}^{l+1},$$
$$\mathbf{b} = (b_1, \ldots, b_{l+1}) \in \mathbb{Z}_{\geq 0}^{l+1}, \qquad \mathbf{c} = (c_1, \ldots, c_l, 0) \in \mathbb{Z}_{\geq 0}^{l+1},$$
$$\mathbf{b} + \mathbf{c} = \mathbf{a},$$

write

$$E_{\mathbf{b},\mathbf{c}} = \bigoplus_{i=1}^{l+1} \mathbb{L}_i^{\oplus b_i} \oplus \bigoplus_{i=1}^{l} (\mathbb{L}_i')^{\oplus c_i} \to \overline{\mathcal{M}}_{0,k,l+1}.$$

Let

$$\mathcal{S}_{\mathbf{b},\mathbf{c}} = \bigoplus_{i=1}^{l+1} \mathcal{S}_i^{\oplus b_i} \oplus \bigoplus_{i=1}^{l} (\mathcal{S}_i')^{\oplus c_i} \subset C_m^\infty \left(E_{\mathbf{b},\mathbf{c}}|_{\partial \overline{\mathcal{M}}_{0,k,l+1}} \right).$$

and

$$\mathcal{E}_{\mathbf{b},\mathbf{c}} = (E_{\mathbf{b},\mathbf{c}}, \mathcal{S}_{\mathbf{b},\mathbf{c}}).$$

Let $\mathbf{e}_i \in \mathbb{Z}_{\geq 0}^{l+1}$ be the vector with 1 for its i^{th} coordinate and 0 for the others. Let $\mathbf{0} \in \mathbb{Z}^{l+1}$ denote the zero vector and abbreviate

$$E_{\mathbf{a}} = E_{\mathbf{a},\mathbf{0}}, \qquad \mathcal{S}_{\mathbf{a}} = \mathcal{S}_{\mathbf{a},\mathbf{0}}, \qquad \mathcal{E}_{\mathbf{a}} = \mathcal{E}_{\mathbf{a},\mathbf{0}}.$$

Let

$$\hat{\mathbf{a}} = (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l.$$

Thus $\mathcal{E}_{\hat{\mathbf{a}}}$ is a vector bundle over $\overline{\mathcal{M}}_{0,k,l}$. Finally, let

$$\mathcal{O}(\mathcal{D}_i) = (\mathcal{O}(D_i), t_i).$$

Write $|\mathbf{a}| = \sum_i a_i$. For $|\mathbf{a}| = k + 2l - 1$, Lemma 3.56 shows that there exists $\mathbf{s} \in \mathcal{S}_{\mathbf{b},\mathbf{c}}$ that vanishes nowhere, so the relative Euler class $e(E_{\mathbf{b},\mathbf{c}}; \mathbf{s})$ is defined. Furthermore, the same lemma shows that $e(E_{\mathbf{b},\mathbf{c}}; \mathbf{s})$ is independent of the choice of such $\mathbf{s}$. So we define

$$e(\mathcal{E}_{\mathbf{b},\mathbf{c}}) = e(E_{\mathbf{b},\mathbf{c}}; \mathbf{s}).$$

Similarly, for $|\mathbf{a}| = k + 2l - 3$, Lemma 3.56 allows us to define

$$e(\mathcal{E}_{\mathbf{b},\mathbf{c}} \oplus \mathcal{O}(\mathcal{D}_i)) = e(E_{\mathbf{b},\mathbf{c}} \oplus \mathcal{O}(D_i); \mathbf{s} \oplus t_i),$$

for some $\mathbf{s} \in \mathcal{S}_{\mathbf{b},\mathbf{c}}$ such that $\mathbf{s} \oplus t_i$ vanishes nowhere.

Proof of Theorem 1.2, string equation. We start with the open string equation. Consider the intersection number

$$\left\langle \tau_0 \prod_{i=1}^{l} \tau_{a_i} \sigma^k \right\rangle_0^o = 2^{-\frac{k-1}{2}} \int_{\overline{\mathcal{M}}_{0,k,l+1}} e\left(\mathcal{E}_{\mathbf{a}}\right),$$

where $\mathbf{a} = (a_1, \ldots, a_l, 0)$, and $|a| = k + 2l - 3$.

Let $\mathbf{b}, \mathbf{c} \in \mathbb{Z}_{\geq 0}^{l+1}$ satisfy $\mathbf{b} + \mathbf{c} = \mathbf{a}$. For $q \in [l]$ such that $b_q \geq 1$, Lemma 4.3 and case (a) of Lemma 3.56 with $i_0 = q$ and j_0 arbitrary, imply that

$$e(\mathcal{E}_{\mathbf{b},\mathbf{c}}) = e(\mathcal{E}_{\mathbf{b}-\mathbf{e}_q,\mathbf{c}+\mathbf{e}_q}) + e(\mathcal{E}_{\mathbf{b}-\mathbf{e}_q,\mathbf{c}} \oplus \mathcal{O}(\mathcal{D}_q)).$$

For $b_q \geq 2$, Observation 4.2 implies the second summand vanishes. Similarly, Lemma 3.56(a), Lemma 4.3 and Observation 4.1, imply that for $q \neq r \in [l]$,

$$e(\mathcal{E}_{\mathbf{b},\mathbf{c}} \oplus \mathcal{O}(\mathcal{D}_q)) = e(\mathcal{E}_{\mathbf{b}-\mathbf{e}_r,\mathbf{c}+\mathbf{e}_r} \oplus \mathcal{O}(\mathcal{D}_q)).$$

By induction,

$$e(\mathcal{E}_{\mathbf{a}}) = e(\mathcal{E}_{\mathbf{0},\mathbf{a}}) + \sum_{\substack{q\in[l],\\ a_q\geq 1}} e(\mathcal{E}_{\mathbf{0},\mathbf{a}-\mathbf{e}_q} \oplus \mathcal{O}(\mathcal{D}_q)). \tag{24}$$

By definition,

$$e(\mathcal{E}_{\mathbf{0},\mathbf{a}}) = PD[Z(\tilde{\mathbf{s}})],$$

where $\tilde{\mathbf{s}}$ is any transverse extension of a nowhere vanishing section $\mathbf{s} \in \mathcal{S}_{\mathbf{0},\mathbf{a}}$. By definition of $\mathcal{S}_{\mathbf{0},\mathbf{a}}$, there exists $\hat{\mathbf{s}} \in \mathcal{S}_{\hat{\mathbf{a}}}$ such that $\mathbf{s} = For_{l+1}^{*}\hat{\mathbf{s}}$. Since $\operatorname{rk} E_{\hat{\mathbf{a}}} > \dim \overline{\mathcal{M}}_{0,k,l}$, we can choose a nowhere vanishing extension $\bar{\mathbf{s}}$ of $\hat{s}$ by transversality. Taking $\tilde{\mathbf{s}} = For_{l+1}^{*}\bar{\mathbf{s}}$, we obtain

$$e(\mathcal{E}_{\mathbf{0},\mathbf{a}}) = 0. \tag{25}$$

Similarly,

$$e(\mathcal{E}_{\mathbf{0},\mathbf{a}-\mathbf{e}_q} \oplus \mathcal{O}(\mathcal{D}_q)) = PD[Z(\tilde{t}_i) \cap Z(\tilde{\mathbf{s}})],$$

where $\tilde{\mathbf{s}}$ is any transverse extension of a nowhere vanishing section $\mathbf{s} \in \mathcal{S}_{\mathbf{0},\mathbf{a}-\mathbf{e}_q}$. Such $\mathbf{s}$ exists by Lemma 3.56(c). Let $\hat{\mathbf{s}} \in \mathcal{S}_{\hat{\mathbf{a}}-\hat{\mathbf{e}}_q}$ such that $\mathbf{s} = For_{l+1}^{*}\hat{\mathbf{s}}$. Denote by $\bar{\mathbf{s}}$ a transversal extension of $\hat{\mathbf{s}}$ and choose $\tilde{\mathbf{s}} = For_{l+1}^{*}\bar{\mathbf{s}}$. Using Lemma 3.43(b), we obtain

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,k,l+1}} e(\mathcal{E}_{\mathbf{0},\mathbf{a}-\mathbf{e}_q} \oplus \mathcal{O}(\mathcal{D}_q)) &= \#Z(\tilde{t}_i) \cap Z(\tilde{\mathbf{s}}) = \\ = \#Z(\hat{\mathbf{s}}) = \int_{\overline{\mathcal{M}}_{0,k,l}} e\left(\mathcal{E}_{\hat{\mathbf{a}}-\hat{\mathbf{e}}_q}\right) &= 2^{\frac{k-1}{2}} \left\langle \tau_{a_q-1} \prod_{i\neq q} \tau_{a_i}\sigma^k \right\rangle_0^o . \end{aligned} \tag{26}$$

Equations (24), (25), (26) together imply the open string equation. □

4.2. **Proof of dilaton equation.** We continue with the notations of the previous section. The following lemma is the key additional ingredient in the proof of the dilaton equation. In the case $k = 3$ and $l = 0$, the proof of the following lemma calculates the integral $\langle \tau_1 \sigma^3 \rangle$ directly from the definition.

Lemma 4.5. *Let $p \in \mathcal{M}_{0,k,l}$ and $F_p = For_{l+1}^{-1}(p)$ equipped with its complex orientation. Let s be a nowhere vanishing special canonical multisection of $\mathbb{L}_i|_{\partial F_p}$ and let $\tilde{s}$ be an extension of s to F_p that is transverse to zero. Then*

$$\#Z(\tilde{s}) = k + l - 1.$$

Proof. The section s is determined by its value at a single point. Indeed, on each stratum of ∂F_p, the section s is pulled back from a zero dimensional moduli space. In particular, s can only vanish at a given point if it vanishes identically.

It follows that if s' is another section satisfying the same hypotheses as s, then s and s' can be connected by a non-vanishing homotopy. Indeed, the complement of zero in a single fiber of $\mathbb{L}_{l+1}$ is connected. Thus $\#Z(\tilde{s})$ is independent of the choice of s by Lemma 3.55.

To begin, we reduce the calculation to the case $l = 0$. Applying Lemma 3.43(a) with interior labels l and $l+1$ switched, we obtain a canonical map of line bundles

$$\tilde{t} : For_l^*\mathbb{L}_{l+1} \to \mathbb{L}_{l+1},$$

which vanishes transversely exactly at D_l. Write $t = \tilde{t}|_{\partial\overline{\mathcal{M}}_{0,k,l+1}}$.

Let $\hat{p} = For_l(p)$ and let $F_{\hat{p}} = For_{l+1}^{-1}(p) \subset \overline{\mathcal{M}}_{0,k,[l+1]\setminus\{l\}}$. Let $\hat{s}$ be a special canonical multisection of $\mathbb{L}_{l+1} \to \mathcal{M}_{0,k,[l+1]\setminus\{l\}}$ that vanishes nowhere on $\partial F_{\hat{p}}$. By Observation 3.46 with interior labels l and $l+1$ switched, we conclude that $(For_l^*\hat{s})t \in \mathcal{S}_{l+1}$ and vanishes nowhere on ∂F_p. Thus we may take $s = (For_l^*\hat{s})t$. Let $q \in F_p$ be the unique point in $D_l \cap F_p$ and let $\hat{q} = For_l(q) \in F_{\hat{p}}$. Let $\bar{s}$ be a transverse extension of $\hat{s}$ that does not vanish at $\hat{q}$. Then we may take $\tilde{s} = (For_l^*\bar{s})\tilde{t}$ and

$$\#Z(\tilde{s}|_{F_p}) = \#Z(For_l^*\bar{s}|_{F_p}) + \#Z(\tilde{t}|_{F_p}) = \#Z(\bar{s}|_{F_{\hat{p}}}) + 1.$$

In the last equality, we have used the fact that For_l maps F_p diffeomorphically to $F_{\hat{p}}$ as well as Lemma 3.43(b). By induction, appropriately relabelling interior marked points, it suffices to prove the lemma when $l = 0$.

The case $l = 0$, $k = 1$, is exceptional. In this case we take $F_p = \mathcal{M}_{0,1,1}$, which is a point, and the claim is trivial. Below, we assume $l = 0$ and $k \geq 3$.

Let $(\Sigma, \mathbf{x}, \emptyset)$, where $\mathbf{x} = \{x_1, \ldots, x_k\}$, be a marked surface representing $p \in \mathcal{M}_{0,k,0}$. Then F_p is diffeomorphic to the oriented real blowup $\widetilde{\Sigma}$ of Σ at the boundary marked points $x_1, \ldots, x_k$. Indeed, denote by $\pi : \widetilde{\Sigma} \to \Sigma$ the blowup map. Denote by $\mathbb{H} \subset \mathbb{C}$ the upper half-plane. Denote by $\mathrm{HB}_r(s) \subset \mathbb{H}$ the half-disk of radius r centered at $s \in \mathbb{R} = \partial\mathbb{H}$. For $i = 1, \ldots, k$, let $U_i \subset \Sigma$ be an open neighborhood of x_i with a local coordinate

$$\xi_i : U_i \xrightarrow{\sim} \mathrm{HB}_2(0), \qquad \xi_i(x_i) = 0.$$

Possibly shrinking the U_i, we arrange that $U_i \cap U_j = \emptyset$ for $i \neq j$. Write $\widetilde{U}_i = \pi^{-1}(U_i)$. Then we have coordinates

$$r_i : \widetilde{U}_i \to [0,2), \qquad \theta_i : \widetilde{U}_i \to [0,\pi],$$

such that $\xi_i \circ \pi(r_i, \theta_i) = r_i e^{\sqrt{-1}\theta_i}$. For $z \in \operatorname{int}\Sigma$, denote by Σ_z the marked surface $(\Sigma, \mathbf{x}, \{z_1\})$ where $z_1 = z$. For $z \in \partial\Sigma \setminus \mathbf{x}$, denote by Q_z the marked surface $(\Sigma, \{x_0, x_1, \ldots, x_k\}, \emptyset)$, where $x_0 = z$. For $i \in [k]$ denote by P_i the marked surface $(\Sigma, (\mathbf{x} \setminus \{x_i\}) \cup \{x_0\}, \emptyset)$, where $x_0 = x_i$. For $\theta = 0, \pi$, define

$$R_{i,\theta} = (\mathbb{H} \cup \{\infty\}, \{x_{-2}, x_{-3}, x_i\}, \emptyset), \quad x_{-2} = \infty, \quad x_{-3} = \cos\theta, \quad x_i = 0.$$

Furthermore, we define

$$S = (\mathbb{H} \cup \{\infty\}, \{x_{-1}\}, \{z_1\}), \qquad x_{-1} = \infty, \quad z_1 = \sqrt{-1},$$

$$T_{i,\theta} = (\mathbb{H} \cup \{\infty\}, \{x_{-1}, x_i\}, \{z_1\}), \quad x_{-1} = \infty, \quad x_i = 0, \quad z_1 = e^{\sqrt{-1}\theta}.$$

For $z \in \partial\Sigma \setminus \mathbf{x}$, let Σ_z denote the stable surface $(\{Q_z, S\}, \sim)$ where $x_0 \sim x_{-1}$. For $i \in [k]$ and $\theta \in (0,\pi)$, let $\Sigma_{i,\theta}$ denote the stable surface $(\{P_i, T_{i,\theta}\}, \sim)$ where $x_0 \sim x_{-1}$. For $i \in [k]$ and $\theta = 0, \pi$, let $\Sigma_{i,\theta}$ denote the stable surface $(\{P_i, R_{i,\theta}, S\}, \sim)$ where $x_0 \sim x_{-2}$ and $x_{-1} \sim x_{-3}$. We define a diffeomorphism $f : \widetilde{\Sigma} \to F_p$ by

$$f(z) = \begin{cases} [\Sigma_{\pi(z)}], & z \in \pi^{-1}(\Sigma \setminus \mathbf{x}) \\ [\Sigma_{i,\theta_i(z)}] & z \in \pi^{-1}(x_i). \end{cases}$$

Write $g = f^{-1}$. Then there is a tautological isomorphism

$$g^* T^* \widetilde{\Sigma}|_{\operatorname{int}\widetilde{\Sigma}} \simeq \mathbb{L}_i|_{\operatorname{int} F_p}.$$

We aim to construct a section $\bar{s}$ of $T^*\widetilde{\Sigma}|_{\operatorname{int}\widetilde{\Sigma}}$ such that $g^*\bar{s}$ extends to a continuous section $\check{s}$ of $\mathbb{L}_i|_{F_p}$ with $\check{s}|_{\partial F_p}$ special canonical. Indeed, let $\hat{\nu} : \Sigma \to \mathbb{R}$ satisfy

$$\hat{\nu}(z) > 0, \quad z \in \operatorname{int}\Sigma, \qquad \hat{\nu}(z) = 0, \quad z \in \partial\Sigma,$$
$$d\hat{\nu}_z \neq 0, \quad z \in \partial\Sigma,$$

and set $\nu = \hat{\nu} \circ \pi : \widetilde{\Sigma} \to \mathbb{R}$. Let $\{\eta_0, \ldots, \eta_k\}$ be a partition of unity on $\widetilde{\Sigma}$ subordinate to the cover $\{\widetilde{\Sigma} \setminus \pi^{-1}(\mathbf{x}), \widetilde{U}_1, \ldots, \widetilde{U}_k\}$. Let

$$\tau = \eta_0 + \sum_i \eta_i e^{-2\sqrt{-1}\theta_i}, \qquad \bar{s} = \tau \frac{d\nu}{\nu}.$$

For $w \in \partial F_p$, we calculate $\lim_{w' \to w} g^*\bar{s}(w')$ as follows. Write $z = g(w)$ and $z' = g(w')$. Suppose first that $z \in \partial\widetilde{\Sigma} \setminus \pi^{-1}(\mathbf{x})$. Let $U \subset \Sigma$ be an open neighborhood of $\pi(z)$ with a local coordinate $\xi : U \xrightarrow{\sim} \mathrm{HB}_2(0)$

such that $\xi(\pi(z)) = 0$. For $\epsilon > 0$ and $a \in \mathbb{R}$, let $\mu_{\epsilon,a} : \mathbb{H} \to \mathbb{H}$ be given by $\zeta \mapsto \epsilon\zeta + a$. For $\pi(z') \in U$, taking

$$\epsilon = \epsilon(w') = \mathrm{Im}(\xi(\pi(z'))), \qquad a = a(w') = \mathrm{Re}(\xi(\pi(z'))),$$

we have

$$\mu_{\epsilon,a}^{-1}(\xi(z')) = \sqrt{-1} = z_1 \in S.$$

For w' sufficiently close to w, the smooth surface $\Sigma_{\pi(z')}$ is a deformation of the nodal surface $\Sigma_{\pi(z)}$ obtained by removing half-disks around the nodal points $x_0 \in Q_z$ and $x_{-1} \in S$, and identifying half-annuli adjacent to the resulting boundaries. More explicitly, let

$$A_\epsilon = \mathrm{HB}_{\sqrt{2/\epsilon}}(0) \setminus \mathrm{HB}_{1/\sqrt{2\epsilon}}(0).$$

We glue the surfaces

$$Q_z \setminus \xi^{-1}\left(\mathrm{HB}_{\sqrt{\epsilon/2}}(a)\right), \qquad\qquad \mathrm{HB}_{\sqrt{2/\epsilon}}(0) \subset S,$$

along the map $\xi^{-1} \circ \mu_{\epsilon,a}|_{A_\epsilon}$. The identification of $\Sigma_{\pi(z')}$ with the above deformation of $\Sigma_{\pi(z)}$ trivializes $\mathbb{L}_i|_{F_p}$ near w. We use this trivialization to compute

$$\lim_{w' \to w} g^*\bar{s}(w') = \lim_{\substack{\epsilon \to 0 \\ a \to 0}} \left(\pi^{-1} \circ \xi^{-1} \circ \mu_{\epsilon,a}\right)^* \bar{s}\big|_{\sqrt{-1}} \in T^*_{\sqrt{-1}}S = T^*_{z_1}\Sigma_z. \tag{27}$$

Writing $\xi = x + iy$, we have $\hat{\nu}(x,y) = y\chi(x,y)$ where $\chi(x,0) > 0$. So,

$$\begin{aligned}\lim_{\substack{\epsilon \to 0 \\ a \to 0}} \left(\pi^{-1} \circ \xi^{-1} \circ \mu_{\epsilon,a}\right)^* \bar{s}\big|_{\sqrt{-1}} &= \lim_{\substack{\epsilon \to 0 \\ a \to 0}} \left(\xi^{-1} \circ \mu_{\epsilon,a}\right)^* \frac{d\hat{\nu}}{\hat{\nu}}\bigg|_{\sqrt{-1}} = \\ &= \lim_{\substack{\epsilon \to 0 \\ a \to 0}} \frac{dy}{y} + \frac{d(\chi \circ \mu_{\epsilon,a})}{\chi \circ \mu_{\epsilon,a}}\bigg|_{\sqrt{-1}} = dy.\end{aligned} \tag{28}$$

Here, the first equality holds because $\tau|_{\partial\widetilde{\Sigma}\setminus\pi^{-1}(\mathbf{x})} \equiv 1$.

If $z \in \pi^{-1}(x_i)$ and $\theta_i(z) \neq 0, \pi$, we proceed as follows. If $z' \in \widetilde{U}_i$, taking $\epsilon = \epsilon(w') = r_i(z')$, we have

$$\lim_{w' \to w} \mu_{\epsilon,0}^{-1}(\xi_i(\pi(z'))) = e^{\sqrt{-1}\theta_i(z)} = z_1 \in T_{i,\theta_i(z)}.$$

Thus by reasoning similar to the above, we have

$$\begin{aligned}
(29)\qquad \lim_{w'\to w} g^*\bar{s}(w') &= \lim_{\epsilon\to 0}(\pi^{-1}\circ\xi_i^{-1}\circ\mu_{\epsilon,0})^*\bar{s}\Big|_{e^{\sqrt{-1}\theta_i(z)}} \\
&= \tau(z)\lim_{\epsilon\to 0}(\xi_i^{-1}\circ\mu_{\epsilon,a})^*\frac{d\hat{\nu}}{\hat{\nu}}\Bigg|_{e^{\sqrt{-1}\theta_i(z)}} \\
&= e^{-2\sqrt{-1}\theta_i(z)}\ \frac{dy}{y}\Bigg|_{e^{\sqrt{-1}\theta_i(z)}} \\
&= \frac{e^{-2\sqrt{-1}\theta_i(z)}}{\sin\theta_i(z)}dy \in T^*_{e^{\sqrt{-1}\theta_i(z)}}T_{i,\theta_i(z)} = T^*_{z_1}\Sigma_{i,\theta_i(z)}.
\end{aligned}$$

If $\theta_i(z) = 0, \pi$, the situation is slightly more complicated because of the double bubble, but similar reasoning still shows that

$$\lim_{w'\to w} g^*\bar{s}(w') = dy \in T^*_{\sqrt{-1}}S = T^*_{z_1}\Sigma_{i,\theta_i(z)}.$$

Therefore, $g^*\bar{s}$ does indeed extend to a continuous section $\check{s}$ of $\mathbb{L}_i|_{F_p}$.

Moreover, we deduce from the preceding calculations that $\check{s}|_{\partial F_p}$ is special canonical. Indeed, for $z \in \widetilde{\Sigma}\setminus\pi^{-1}(\mathbf{x})$, equations (27) and (28) show that $\check{s}(w) = dy$, independent of w. Thus $\check{s}$ is pulled back from the base on the corresponding components of ∂F_p. For $z \in \pi^{-1}(x_i)$ and $\theta_i(z) \neq 0, \pi$, equation (29) shows that

$$\check{s}(w) = \frac{e^{-2\sqrt{-1}\theta_i(z)}}{\sin\theta_i(z)}dy.$$

The map to the base component forgets $x_{-1} \in T_{i,\theta_i(z)}$. So the remaining marked points x_i and z_1 can be brought to a standard position by a Mobius transformation. Explicitly, let $\beta_\theta : \mathbb{H} \to \mathbb{H}$ be given by

$$\beta_\theta(\zeta) = \frac{\zeta}{\zeta\cos\theta + \sin\theta}.$$

So,

$$\beta_\theta(\sqrt{-1}) = e^{i\theta} = z_1 \in T_{i,\theta}, \qquad \beta_\theta(0) = 0 = x_i \in T_{i,\theta}.$$

Then

$$\beta'_\theta(\zeta) = \frac{\sin\theta}{(\zeta\cos\theta + \sin\theta)^2}.$$

In particular, $\beta'_\theta(\sqrt{-1}) = -e^{2\sqrt{-1}\theta}\sin\theta$. It follows that

$$\beta^*_{\theta_i(z)}\check{s}(w) = -dy,$$

independent of w. So, $\check{s}$ is pulled back from the base on the remaining components of ∂F_p. The case $\theta_i(z) = 0, \pi$, corresponds to a codimension 2 corner of F_p, so it follows by continuity of $\check{s}$.

Finally, choose $\tilde{s}$ to be a transverse perturbation of $\check{s}$ that agrees with $\check{s}$ in a neighborhood V of ∂F_p where $\check{s}$ does not vanish. We calculate $\#Z(\tilde{s})$ by expressing it as a winding number. Let Ξ be a Riemann surface, let $L \to \Xi$ be a complex line bundle, and let $\gamma \subset \Xi$ be a homologically trivial curve. Then $L|_\gamma$ has a distinguished trivialization. Thus if σ is a section of L, the winding number $W(\sigma,\gamma)$ of σ around γ is well defined. Let $\gamma \subset V \cap \operatorname{int} F_p$ be a curve isotopic to ∂F_p and let $\hat{\gamma}$ be the corresponding curve in $\operatorname{int}\widetilde{\Sigma}$. Then

$$\#Z(\tilde{s}) = W(\tilde{s},\gamma) = W(\check{s},\gamma) = W(\bar{s},\hat{\gamma}) = W(d\nu,\hat{\gamma}) + k.$$

But it is well known that $W(d\nu,\hat{\gamma})$ is negative the Euler characteristic of $\widetilde{\Sigma}$, which in our case is -1. The lemma follows. □

Proof of Theorem 1.2, dilaton equation. We have

$$2^{\frac{k-1}{2}}\left\langle \tau_1 \prod_{i=1}^{l} \tau_{a_i}\sigma^k \right\rangle_0^o = \int_{\overline{\mathcal{M}}_{0,k,l+1}} e(\mathcal{E}_{\mathbf{a}}),$$

where $\mathbf{a} = (a_1,\ldots,a_l,1)$. Let $\mathbf{b},\mathbf{c} \in \mathbb{Z}_{\geq 0}^{l+1}$ satisfy $\mathbf{b}+\mathbf{c} = \mathbf{a}$. For all $q \in [l]$ such that $b_q \geq 1$, by Lemma 3.56 cases (a) and (b), Lemma 4.3 and Observation 4.2, we have

$$e(\mathcal{E}_{\mathbf{b},\mathbf{c}}) = e(\mathcal{E}_{\mathbf{b}-\mathbf{e}_q,\mathbf{c}+\mathbf{e}_q}).$$

By induction, we obtain

$$e(\mathcal{E}_{\mathbf{a}}) = e(\mathcal{E}_{\mathbf{e}_{l+1},\mathbf{a}-\mathbf{e}_{l+1}}).$$

Let $\mathbf{s}' \in \mathcal{S}_{\mathbf{0},\mathbf{a}-\mathbf{e}_{l+1}}$ and $s \in \mathcal{S}_{l+1}$ be such that $s \oplus \mathbf{s}'$ vanishes nowhere. By definition of $\mathcal{S}_{\mathbf{0},\mathbf{a}-\mathbf{e}_{l+1}}$, there exists $\hat{\mathbf{s}}' \in \mathcal{S}_{\hat{\mathbf{a}}}$ such that $\mathbf{s}' = For_{l+1}^*\hat{\mathbf{s}}'$. Let $\bar{\mathbf{s}}'$ be a transverse extension of $\hat{\mathbf{s}}'$ to $\overline{\mathcal{M}}_{0,k,l}$ and let $\tilde{\mathbf{s}}' = For_{l+1}^*\bar{\mathbf{s}}'$. Since $Z(\hat{\mathbf{s}}') \subset \mathcal{M}_{0,k,l}$ and $For_{l+1}|_{For_{l+1}^{-1}(\mathcal{M}_{0,k,l})}$ is a submersion, it follows that $\tilde{\mathbf{s}}'$ is a transverse extension of $\mathbf{s}'$. Choose an extension $\tilde{s}$ of s such that $\tilde{s} \oplus \tilde{\mathbf{s}}'$ is transverse. Then,

$$\int_{\overline{\mathcal{M}}_{0,k,l+1}} e(\mathcal{E}_{\mathbf{e}_{l+1},\mathbf{a}-\mathbf{e}_{l+1}}) = \#Z(\tilde{s}) \cap Z(\tilde{\mathbf{s}}') = (k+l-1)\#Z(\hat{\mathbf{s}}') =$$
$$= (k+l-1)\int_{\overline{\mathcal{M}}_{0,k,l}} e(\mathcal{E}_{\hat{\mathbf{a}}}) = (k+l-1)2^{\frac{k-1}{2}}\left\langle \prod_{i=1}^{l} \tau_{a_i}\sigma^k \right\rangle_0^o,$$

where in the second equality, we have used Lemma 4.5 □

4.3. **Proofs of TRR I and II.** Let

$$E = \bigoplus \mathbb{L}_i^{\oplus a_i} \to \overline{\mathcal{M}}_{0,k,l},$$

with $a_1 = n$, and

$$E_1 = \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i=2}^{l} \mathbb{L}_i^{\oplus a_i} \to \overline{\mathcal{M}}_{0,k,l}.$$

Take

$$\mathbf{s} = \bigoplus_{i\in[l] \ \ ,j\in[a_i]} s_{ij}$$

with $s_{ij} \in \mathcal{S}_i$, and

$$\mathbf{s}_1 = \bigoplus_{\substack{i\in[l] \ \ ,j\in[a_i], \\ (i,j)\neq(1,1)}} s_{ij}.$$

The proof hinges on a section $\tilde{\rho} \in C^\infty(\overline{\mathcal{M}}_{0,k,l}, \mathbb{L}_1)$ defined as follows. At a smooth marked disk $D = (D, \mathbf{x}, \mathbf{z})$, which we identify with the upper half plane, set

$$\tilde{\rho}\,(D) = dz \left(\frac{1}{z - x_1} - \frac{1}{z - \bar{z}_1} \right)\bigg|_{z=z_1} \in T^*_{z_1} D. \tag{30}$$

We show the section $\tilde{\rho}$ extends smoothly to the compactified moduli space. Indeed, let $(\Sigma, \mathbf{x}, \mathbf{z})$ be a stable marked disk, and let

$$\Sigma_{\mathbb{C}} = \Sigma \coprod_{\partial\Sigma} \overline{\Sigma}$$

be its complex double, a stable marked sphere. Consider the unique meromorphic differential ω_Σ on the normalization of $\Sigma_{\mathbb{C}}$ with the following properties. At x_1 it has a simple pole with residue 1. At $\bar{z}_1$ it has a simple pole with residue -1. For any node the two preimages have at most simple poles, and the residues at these poles sum to zero. Apart from these points, ω_Σ is holomorphic. Then $\tilde{\rho}\,(\Sigma)$ is the evaluation of ω_Σ at z_1. As z_1 never coincides with a node, $\bar{z}_1$ or x_1, it follows that $\tilde{\rho}$ is smooth. Write $\rho = \tilde{\rho}|_{\partial\overline{\mathcal{M}}_{0,k,l}}$.

Let $\widetilde{T} \subset \partial\Gamma_{0,k,l}$ be the collection of stable graphs Γ with exactly one open vertex v^o_Γ and exactly one closed vertex v^c_Γ, such that $1 \in \ell_I(v^c_\Gamma)$. So for $\Gamma \in \widetilde{T}$, we have $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{v^o_\Gamma} \times \overline{\mathcal{M}}_{v^c_\Gamma}$. Equip $\overline{\mathcal{M}}_\Gamma$ with the orientation o_Γ given by the product of $o_{0,k(v^o_\Gamma),l(v^o_\Gamma)}$ and the complex orientation on $\overline{\mathcal{M}}_{v^c_\Gamma}$.

Lemma 4.6. *The zero locus of $\tilde{\rho}$ is $\bigcup_{\Gamma\in\widetilde{T}} \overline{\mathcal{M}}_\Gamma$. For $\Gamma \in \widetilde{T}$, the subspace $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{0,k,l}$ is cut out transversely by $\tilde{\rho}$ with induced orientation o_Γ.*

Proof. On a component of $\Sigma_{\mathbb{C}}$ containing x_1 or $\bar{z}_1$, the differential ω_Σ vanishes nowhere. Similarly, ω_Σ vanishes nowhere on components whose removal disconnects x_1 from $\bar{z}_1$. On other components it vanishes identically. Thus, $\tilde{\rho}$ vanishes exactly on stable disks Σ such that in $\Sigma_{\mathbb{C}}$ the component containing z_1 is not on the route of components between the components of x_1 and $\bar{z}_1$. This is the case if and only if z_1 belongs to a sphere component of Σ. So the vanishing locus of $\tilde{\rho}$ is as claimed. The proof of transversality is similar to the proof of Lemma 3.43. The equality of orientations follows from induction on dimension by an argument similar to the proof of Lemma 3.16. $\square$

Choose $\mathbf{s}$ satisfying the strong transversality condition of Lemma 3.53 part (b). Put $\mathbf{r} = \rho \oplus \mathbf{s}_1$. We show that $\mathbf{r}$ does not vanish, so $e\left(E;\mathbf{r}\right)$ is defined. Indeed, $Z(\rho)$ consists of boundary strata of codimension at least 3 in $\overline{\mathcal{M}}_{0,k,l}$. So, the transversality requirement of Lemma 3.53 part (b) guarantees that on such boundary strata $\mathbf{s}_1$ does not vanish. Thus, $\mathbf{r}$ does not vanish on $\partial\overline{\mathcal{M}}_{0,k,l}$.

Lemma 4.7. *We have*

$$\int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{r}) = 2^{\frac{k-1}{2}} \sum_{S\coprod R=\{2,\ldots,l\}} \left\langle \tau_0\tau_{n-1}\prod_{i\in S}\tau_{a_i}\right\rangle_0^c \left\langle \tau_0\prod_{i\in R}\tau_{a_i}\sigma^k\right\rangle_0^o .$$

Proof. Choose an extension $\tilde{\mathbf{s}}_1$ of $\mathbf{s}_1$ to $\overline{\mathcal{M}}_{0,k,l}$ that does not vanish on $\overline{\mathcal{M}}_\Gamma \setminus \mathcal{M}_\Gamma$ for $\Gamma \in \tilde{T}$ and such that $\tilde{\mathbf{r}} = \tilde{\rho}\oplus\tilde{\mathbf{s}}_1$ is transverse. Such transversality is generic because $\tilde{\rho}$ is transverse along $\mathcal{M}_\Gamma$ and non-zero outside of $\overline{\mathcal{M}}_\Gamma$ by Lemma 4.6. Again by Lemma 4.6, we obtain

$$\text{(31)} \qquad \int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{r}) = \#Z(\tilde{\rho}\oplus\tilde{\mathbf{s}}_1) = \sum_{\Gamma\in\widetilde{T}} \int_{\overline{\mathcal{M}}_\Gamma} e\left(E_1|_{\overline{\mathcal{M}}_\Gamma};\mathbf{s}_1|_{\partial\overline{\mathcal{M}}_\Gamma}\right).$$

Recall Definitions 2.27 and 2.28. For $\Gamma\in\widetilde{T}$, and $\Lambda\in\partial^!\Gamma$, abbreviate

$$\Lambda^c = \Lambda_{\varsigma^{-1}_{\Lambda,\Gamma}(v^c_\Gamma)}, \qquad \Lambda^o = \Lambda_{\varsigma^{-1}_{\Lambda,\Gamma}(v^o_\Gamma)}.$$

Thus we have a bijection

$$\partial^!\Gamma \xrightarrow{\sim} \partial^!\Gamma^o \times \partial^!\Gamma^c, \qquad \Lambda \mapsto (\Lambda^o,\Lambda^c),$$

and a corresponding diffeomorphism

$$\tilde{b} : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}.$$

given by

$$\tilde{b}|_{\overline{\mathcal{M}}_\Lambda} = For_{\Lambda,\Lambda^o}\times For_{\Lambda,\Lambda^c}, \qquad \Lambda\in\partial^!\Gamma.$$

Additionally, we have a bijection

$$\text{(32)} \qquad \partial^B\Gamma \xrightarrow{\sim} \partial^B\Gamma^o \times \partial^!\Gamma^c, \qquad \Lambda\mapsto(\Lambda^o,\Lambda^c),$$

and a corresponding diffeomorphism

$$b:\partial\overline{\mathcal{M}}_\Gamma\to\partial\overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}$$

given by $b=\tilde{b}|_{\partial\overline{\mathcal{M}}_\Gamma}$. Denote by

$$\tilde{p}_o:\overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}\longrightarrow\overline{\mathcal{M}}_{\Gamma^o},\qquad \tilde{p}_c:\overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}\longrightarrow\overline{\mathcal{M}}_{\Gamma^c},$$
$$p_o:\partial\overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}\longrightarrow\partial\overline{\mathcal{M}}_{\Gamma^o},\qquad p_c:\partial\overline{\mathcal{M}}_{\Gamma^o}\times\overline{\mathcal{M}}_{\Gamma^c}\longrightarrow\overline{\mathcal{M}}_{\Gamma^c},$$

the projection maps. Let

$$E^c_\Gamma=\mathbb{L}_1^{\oplus n-1}\oplus\bigoplus_{i\in\ell_I(v^c_\Gamma)\setminus\{1\}}\mathbb{L}_i^{\oplus a_i}\longrightarrow\overline{\mathcal{M}}_{\Gamma^c},$$
$$E^o_\Gamma=\bigoplus_{i\in\ell_I(v^o_\Gamma)}\mathbb{L}_i^{\oplus a_i}\longrightarrow\overline{\mathcal{M}}_{\Gamma^o}.$$

Thus

$$\tilde{b}^*(\tilde{p}_o^*E^o\oplus\tilde{p}_c^*E^c)=E_1|_{\overline{\mathcal{M}}_\Gamma}.$$

For $\Lambda\in\partial^!\Gamma$, we have

$$v_i^*(\Lambda)=v_i^*(\Lambda^c),\quad i\in\ell_I(\Lambda^c),\qquad v_i^*(\Lambda)=v_i^*(\Lambda^o),\quad i\in\ell_I(\Lambda^o),$$
$$\ell_I(\Lambda)=\ell_I(\Lambda^c)\cup\ell_I(\Lambda^o).$$

So, bijection (32) implies

$$\mathcal{V}^i_{\partial^B\Gamma}=\mathcal{V}^i_{\partial^B\Gamma^o}\cup\mathcal{V}^i_{\partial^!\Gamma^c},\qquad i\in[l]. \tag{33}$$

Since $\partial^B\Gamma\subset\partial^B\Gamma_{0,k,l}$, by the definition of a special canonical multisection, we have

$$s^v_{ij}\in C^\infty_m(\mathcal{M}_v,\mathbb{L}_i),\qquad v\in\mathcal{V}^i_{\partial^B\Gamma},$$

such that $s^\Lambda_{ij}=\Phi^*_{\Lambda,i}s^v_{ij}$ for all $\Lambda\in\partial^B\Gamma$ with $v_i^*(\Lambda)=v$. Let $\mathbf{s}^c_\Gamma\in C^\infty_m(E^c_\Gamma)$ and $\mathbf{s}^o_\Gamma\in C^\infty_m(E^o_\Gamma|_{\partial\overline{\mathcal{M}}_{\Gamma^o}})$ be given by

$$(\mathbf{s}^c_\Gamma)^\Psi=\bigoplus_{\substack{i\in\ell_I(v^c_\Gamma),\,j\in[a_{ij}]\\(i,j)\neq(1,1)}}\Phi^*_{\Psi,i}s_{ij}^{v_i^*(\Psi)},\qquad \Psi\in\partial^!\Gamma^c,$$
$$(\mathbf{s}^o_\Gamma)^\Omega=\bigoplus_{i\in\ell_I(v^o_\Gamma),\,j\in[a_{ij}]}\Phi^*_{\Omega,i}s_{ij}^{v_i^*(\Omega)},\qquad \Omega\in\partial^B\Gamma^o.$$

Here, $s_{ij}^{v_i^*(\Psi)}$ and $s_{ij}^{v_i^*(\Omega)}$ are predetermined because of equation (33). Since we have chosen $\mathbf{s}$ to satisfy the strong transversality condition of Lemma 3.53 part (b), the multisections $\mathbf{s}^c_\Gamma$ and $\mathbf{s}^o_\Gamma$ are transverse to 0. For $\Lambda\in\partial^B\Gamma$, Observation 2.33 and equation (20) imply that

$$\Phi_{\Lambda,i}=\Phi_{\Lambda^o,i}\circ p_o\circ b,\qquad i\in\ell_I(v^o_\Gamma),$$
$$\Phi_{\Lambda,i}=\Phi_{\Lambda^c,i}\circ p_c\circ b,\qquad i\in\ell_I(v^c_\Gamma).$$

It follows that for $\Gamma \in \widetilde{T}$, we have

$$\mathbf{s}_1|_{\partial\overline{\mathcal{M}}_\Gamma} = b^*(p_o^*\mathbf{s}_\Gamma^o \oplus p_c^*\mathbf{s}_\Gamma^c). \tag{34}$$

Choose a transverse extension $\tilde{\mathbf{s}}_\Gamma^o$ of $\mathbf{s}_\Gamma^o$ to $\overline{\mathcal{M}}_{\Gamma^o}$. Then equation (34) implies that $\tilde{b}^*(\tilde{p}_o^*\tilde{\mathbf{s}}_\Gamma^o \oplus \tilde{p}_c^*\mathbf{s}_\Gamma^c)$ is a transverse extension of $\mathbf{s}_1|_{\partial\overline{\mathcal{M}}_\Gamma}$ to $\overline{\mathcal{M}}_\Gamma$. Therefore,

$$\int_{\overline{\mathcal{M}}_\Gamma} e\left(E_1|_{\overline{\mathcal{M}}_\Gamma}; \mathbf{s}_1|_{\partial\overline{\mathcal{M}}_\Gamma}\right) = \#Z(\tilde{p}_o^*\tilde{\mathbf{s}}_\Gamma^o) \cap Z(\tilde{p}_c^*\mathbf{s}_\Gamma^c). \tag{35}$$

Dimension counting and transversality show this number vanishes unless $\operatorname{rk} E_\Gamma^o = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{\Gamma^o}$ and $\operatorname{rk} E_\Gamma^c = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{\Gamma^c}$. In that case, transversality implies that $\mathbf{s}_\Gamma^o$ vanishes nowhere. Thus

$$\#Z(\tilde{p}_o^*\tilde{\mathbf{s}}_\Gamma^o) \cap Z(\tilde{p}_c^*\mathbf{s}_\Gamma^c) = \left(\int_{\overline{\mathcal{M}}_{\Gamma^o}} e(E_\Gamma^o, \mathbf{s}_\Gamma^o)\right)\left(\int_{\overline{\mathcal{M}}_{\Gamma^c}} e(E_\Gamma^c)\right). \tag{36}$$

The graph Γ^o (resp. Γ^c) has a single vertex v_Γ^o (resp. v_Γ^c). By construction, $\mathbf{s}_\Gamma^o$ is a canonical boundary condition. So,

$$\int_{\overline{\mathcal{M}}_{\Gamma^o}} e(E_\Gamma^o, \mathbf{s}_\Gamma^o) = 2^{\frac{k-1}{2}} \left\langle \tau_0 \prod_{i \in \ell_I(v_\Gamma^o)} \tau_{a_i} \sigma^k \right\rangle_0^o, \tag{37}$$

$$\int_{\overline{\mathcal{M}}_{\Gamma^c}} e(E_\Gamma^c) = \left\langle \tau_0 \tau_{n-1} \prod_{\substack{i \in \ell_I(v_\Gamma^c) \\ i \neq 1}} \tau_{a_i} \right\rangle_0^c. \tag{38}$$

Equations (31), (35), (36), (37) and (38), together imply the lemma. □

It remains to analyze the difference between ρ and a canonical multisection. Let $U \subset \partial^B \Gamma_{0,k,l}$ be the collection of graphs Γ with exactly two vertices $v_\Gamma^\pm$, both open, such that

$$1 \in \ell_I(v_\Gamma^-), \qquad 1^\circ \in \ell_B(v_\Gamma^+),$$

and the unique edge e_Γ of Γ is legal for v_Γ^+ and thus illegal for v_Γ^-. Let

$$V = \partial^B \Gamma_{0,k,l} \setminus \partial^! U. \tag{39}$$

Lemma 4.8. *Let $\Gamma \in V$. Then $\rho|_{\mathcal{M}_\Gamma}$ is canonical.*

Proof. If $\Gamma \in V$ and $(\Sigma, \mathbf{x}, \mathbf{z}) \in \mathcal{M}_\Gamma$, then either z_1 and x_1 are in the same component, or the nodal point is legal for the component of z_1. In the first case, ω_Σ has does not depend on the position of the nodal point, so neither does ρ. In the second case, ω_Σ may have a pole at the nodal point and so ρ may depend on the position of the nodal point on the component of z_1. But the nodal point is legal for that component.

In both cases, ρ does not depend on the position of an illegal nodal point, so it is canonical. $\square$

The following observation and lemma quantify the difference between ρ and a canonical multisection on $\overline{\mathcal{M}}_\Gamma$ for $\Gamma \in U$. Let $p \in \mathcal{M}_{\mathcal{B}\Gamma}$. Let F_p be the fiber over p of the map $F_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\mathcal{B}\Gamma}$ equipped with its natural orientation. So F_p is a collection of $a = |\ell_B(v_\Gamma^-)|$ closed intervals corresponding to the a segments between marked points where the illegal nodal point can move. The following observation is a consequence of Observation 3.32.

Observation 4.9. The restriction of the tautological line $\mathbb{L}_i|_{F_p}$ is canonically trivial.

So, we think of sections of $\mathbb{L}_i|_{F_p}$ as complex valued functions well-defined up to multiplication by a constant in $\mathbb{C}^\times$. The following observation is immediate from the definition of a canonical section.

Observation 4.10. A canonical section of $\mathbb{L}_i|_{F_p}$ is constant.

On the other hand, the TRR section ρ twists non-trivially around F_p as follows. For $i \in \ell_B(v_\Gamma^-)$, let Γ_i be the unique stable graph in $\partial^B\Gamma$ with three open vertices $v_i^0, v_i^\pm$ and two boundary edges $e^\pm = \{v_i^\pm, v_i^0\}$ such that $\ell_B(v_i^0) = i$. The boundary ∂F_p consists of two stable disks modelled on each graph Γ_i for $i \in \ell_B(v_\Gamma^-)$, one for each cyclic order of the $3 = k(v_i^0)$ boundary marked points on the component corresponding to v_i^0. Let $\hat{F}_p$ be the quotient space of F_p obtained by identifying the two boundary points corresponding to Γ_i for each $i \in \ell_B(v_\Gamma^-)$. Thus $\hat{F}_p$ is homeomorphic to the circle S^1. The following observation follows from the definition of ρ.

Observation 4.11. The section $\rho|_{F_p}$ descends to a continuous function $\hat{\rho}_p : \hat{F}_p \to \mathbb{C}^\times$.

Lemma 4.12. *The winding number of $\hat{\rho}_p$ is -1.*

Proof. We define a map from a subset $B \subset (0, 2\pi)$, to $\operatorname{int} F_p$ as follows. To each $b \in B$, we assign a surface $\Sigma_b = (\Sigma_b^-, \Sigma^+)$. The component Σ^+ corresponds to the vertex v_Γ^+ and the component Σ_b^- corresponds to the vertex v_Γ^-. As implied by the notation, Σ^+ is independent of b. The exact form of Σ^+ is not important for the present calculation, and its isomorphism class is determined uniquely by p. We fix Σ_b^- as follows. Let $\nu = i_{v_\Gamma^-}(e_\Gamma) \in \mathfrak{L}$ and choose an arbitrary $i_0^\circ \in \ell_B(v_\Gamma^-)$. Identify Σ_b^- with the unit disk $D^2 \subset \mathbb{C}$ in such a way that $z_1 = 0$ and $x_{i_0} = 1$. The position of the remaining boundary marked points in ∂D^2

is then uniquely determined by p. Take B to be the set of arguments of the complement of the marked points in ∂D^2. The parameter $b \in B$ determines the nodal point $x_\nu \in \partial\Sigma_b^-$ by the formula $x_\nu = e^{\sqrt{-1}b}$. The complex double $(\Sigma_b^-)_{\mathbb{C}}$ is naturally identified with the extended complex plane $\mathbb{C} \cup \infty$. The point conjugate to z_1 is ∞ and

$$\omega_{\Sigma_b}|_{\Sigma_b^-} = \frac{dz}{z - x_\nu} = \frac{dz}{z - e^{\sqrt{-1}b}}.$$

So

$$\rho(\Sigma_b) = \omega_{\Sigma_b}|_{z_1} = -e^{-\sqrt{-1}b}. \tag{40}$$

The continuity of $\hat{\rho}_p$ and formula (40) imply that $\hat{\rho}_p$ rotates once in the negative direction around the fiber $\hat{F}_p$. □

Lemma 4.13. *We have*

$$\int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{s}) - \int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{r}) =$$
$$= 2^{\frac{k-1}{2}} \sum_{\substack{S \coprod R = \{2,\ldots,l\} \\ k_1+k_2=k-1}} \binom{k}{k_1} \left\langle \tau_{n-1} \prod_{i\in S} \tau_{a_i} \sigma^{k_1} \right\rangle_0^o \left\langle \prod_{i\in R} \tau_{a_i} \sigma^{k_2+2} \right\rangle_0^o.$$

Proof. Let

$$E_2 = \mathbb{L}_1 \to \overline{\mathcal{M}}_{0,k,l},$$

so $E = E_1 \oplus E_2$. Let $\mathcal{C} = V$. Lemma 4.8 shows that $\mathbf{s}$ and $\mathbf{r}$ satisfy the hypotheses of Lemma 3.54 with the preceding choice of $E_1, E_2, \mathcal{C}$. So, we obtain a homotopy H between $\mathbf{s}$ and $\mathbf{r}$ of the form (23) that is transverse to zero, vanishes nowhere on $\mathcal{M}_\Gamma \times [0,1]$ for $\Gamma \in V$, and such that the projection of H to E_1 equals $\mathbf{s}_1$ independent of time. By Lemma 3.55 and equation (39), we have

$$\int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{s}) - \int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{r}) = \tag{41}$$
$$= -\#Z(H) = -\sum_{\Gamma\in U} \#Z\left(H|_{\overline{\mathcal{M}_\Gamma}\times[0,1]}\right).$$

Denote by $\pi : \partial\overline{\mathcal{M}}_{0,k,l} \times [0,1] \to \partial\overline{\mathcal{M}}_{0,k,l}$ the projection. Decompose $H = H_1 \oplus H_2$, where $H_i \in C_m^\infty(\pi^* E_i)$. Then $H_1 = \pi^*\mathbf{s}_1$. Transversality implies that

$$Z\left(\mathbf{s}_1|_{\overline{\mathcal{M}_\Gamma}}\right) \subset \mathcal{M}_\Gamma.$$

By Remark 3.38, for each $\Gamma \in \partial^B \Gamma_{0,k,l}$ we have $\mathbf{s}_1^\Gamma = F_\Gamma^* \mathbf{s}_1^{\mathcal{B}\Gamma}$. Write

$$\# Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right) = \sum_{p \in Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)} \epsilon(p),$$

where $\epsilon(p)$ is the weight of p as in Definition A.4. It follows from Lemma 3.16 that for $\Gamma \in U$, we have

$$\begin{aligned} \# Z\left(H|_{\overline{\mathcal{M}}_\Gamma \times [0,1]}\right) &= \# Z\left(\pi^* F_\Gamma^* \mathbf{s}_1^{\mathcal{B}\Gamma}\right) \cap Z(H_2) \\ &= \sum_{p \in Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)} \epsilon(p) \cdot \# Z\left(H_2|_{F_p \times [0,1]}\right). \end{aligned} \tag{42}$$

Since H is of the form (23), we have

$$H_2(q,t) = \rho(q)t + s_{11}(q)(1-t) + t(1-t)w_2(q), \tag{43}$$

where w_2 is a canonical multisection. Let $p \in Z(\mathbf{s}_1^{\mathcal{B}\Gamma})$. It follows from equation (43) and Observations 4.9, 4.10 and 4.11, that $H_2|_{F_p \times [0,1]}$ descends to a homotopy $\hat{H}_{2,p}$ on $\hat{F}_p \times [0,1]$, which we may think of as taking values in $\mathbb{C}^\times$. Thus

$$\# Z\left(H_2|_{F_p \times [0,1]}\right) = \# Z\left(\hat{H}_{2,p}\right). \tag{44}$$

Since $\hat{H}_{2,p}|_{\hat{F}_p \times \{0\}}$ is canonical and $\hat{H}_{2,p}|_{\hat{F}_p \times \{1\}} = \hat{\rho}_p$, Observation 4.10 and Lemma 4.12 imply that

$$\# Z\left(\hat{H}_{2,p}\right) = -1. \tag{45}$$

Combining equations (41), (42), (44) and (45), we obtain

$$\int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{s}) - \int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{r}) = \sum_{\Gamma \in U} \# Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right). \tag{46}$$

It remains to analyze $\# Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right)$ for $\Gamma \in U$. Denote by $\hat{v}_\Gamma^\pm \in V(\mathcal{B}\Gamma)$ the vertices corresponding to $v_\Gamma^\pm \in V(\Gamma)$. Recall Definitions 2.27 and 2.28. For $\Lambda \in \partial^! \mathcal{B}\Gamma$, abbreviate

$$\Lambda^\pm = \Lambda_{\varsigma^{-1}_{\Lambda,\mathcal{B}\Gamma}\left(\hat{v}_\Gamma^\pm\right)}.$$

Thus we have a bijection

$$\partial^! \mathcal{B}\Gamma \xrightarrow{\sim} \partial^! \mathcal{B}\Gamma^+ \times \partial^! \mathcal{B}\Gamma^-, \qquad \Lambda \mapsto (\Lambda^+, \Lambda^-), \tag{47}$$

and a corresponding diffeomorphism

$$d : \overline{\mathcal{M}}_{\mathcal{B}\Gamma} \longrightarrow \overline{\mathcal{M}}_{\mathcal{B}\Gamma^+} \times \overline{\mathcal{M}}_{\mathcal{B}\Gamma^-}$$

given by

$$d|_{\overline{\mathcal{M}}_\Lambda} = For_{\Lambda,\Lambda^+} \times For_{\Lambda,\Lambda^-}, \qquad \Lambda \in \partial^! \mathcal{B}\Gamma.$$

Denote by

$$p_\pm : \overline{\mathcal{M}}_{\mathcal{B}\Gamma^+} \times \overline{\mathcal{M}}_{\mathcal{B}\Gamma^-} \longrightarrow \overline{\mathcal{M}}_{\mathcal{B}\Gamma^\pm}$$

the projection maps. Let

$$E^+_{\mathcal{B}\Gamma} = \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i\in\ell_I(v_\Gamma^+)\setminus\{1\}} \mathbb{L}_i^{\oplus a_i} \longrightarrow \overline{\mathcal{M}}_{\mathcal{B}\Gamma^+},$$

$$E^-_{\mathcal{B}\Gamma} = \bigoplus_{i\in\ell_I(v_\Gamma^-)} \mathbb{L}_i^{\oplus a_i} \longrightarrow \overline{\mathcal{M}}_{\mathcal{B}\Gamma^-},$$

$$E_{\mathcal{B}\Gamma} = \mathbb{L}_1^{\oplus n-1} \oplus \bigoplus_{i=2}^{l} \mathbb{L}_i^{\oplus a_i} \longrightarrow \overline{\mathcal{M}}_{\mathcal{B}\Gamma}.$$

Thus

$$d^*\left(p_+^*E^+ \oplus p_-^*E^-\right) = E_{\mathcal{B}\Gamma}.$$

Observation 3.14 and bijection (47) imply that

$$\mathcal{V}^i_{\partial^!\Gamma} = \mathcal{V}^i_{\partial^!\mathcal{B}\Gamma} = \mathcal{V}^i_{\partial^!\mathcal{B}\Gamma^+} \cup \mathcal{V}^i_{\partial^!\mathcal{B}\Gamma^-}. \tag{48}$$

Since $\partial^!\Gamma \subset \partial^B\Gamma_{0,k,l}$, by definition of a special canonical multisection, we have

$$s^v_{ij} \in C^\infty_m(\mathcal{M}_v, \mathbb{L}_i), \qquad v \in \mathcal{V}^i_{\partial^!\Gamma},$$

such that $s^\Lambda_{ij} = \Phi^*_{\Lambda,i} s^v_{ij}$ for all $\Lambda \in \partial^B\Gamma$ with $v_i^*(\Lambda) = v$. Let

$$\mathbf{s}^\pm_{\mathcal{B}\Gamma} \in C^\infty_m(E^\pm_{\mathcal{B}\Gamma})$$

be given by

$$\left(\mathbf{s}^+_{\mathcal{B}\Gamma}\right)^\Omega = \bigoplus_{\substack{i\in\ell_I(v_\Gamma^+),\, j\in[a_{ij}],\\ (i,j)\neq(1,1)}} \Phi^*_{\Omega,i} s_{ij}^{v_i^*(\Omega)}, \qquad \Omega \in \partial^!\mathcal{B}\Gamma^+,$$

$$\left(\mathbf{s}^-_{\mathcal{B}\Gamma}\right)^\Omega = \bigoplus_{i\in\ell_I(v_\Gamma^-),\, j\in[a_{ij}]} \Phi^*_{\Omega,i} s_{ij}^{v_i^*(\Omega)}, \qquad \Omega \in \partial^!\mathcal{B}\Gamma^-.$$

Here, $s_{ij}^{v_i^*(\Omega)}$ are predetermined because of equation (48). Since we have chosen $\mathbf{s}$ to satisfy the strong transversality condition of Lemma 3.53 part (b), the multisections $\mathbf{s}^\pm_{\mathcal{B}\Gamma}$ are transverse to 0. For $\Omega \in \partial^!\mathcal{B}\Gamma$, Observation 2.33 and equation (20) imply that

$$\Phi_{\Omega,i} = \Phi_{\Omega^\pm,i} \circ p_\pm \circ d, \qquad i \in \ell_I(v_\Gamma^\pm).$$

It follows that

$$\mathbf{s}_1^{\mathcal{B}\Gamma} = d^*(p_+^*\mathbf{s}^+_{\mathcal{B}\Gamma} \oplus p_-^*\mathbf{s}^-_{\mathcal{B}\Gamma}).$$

Thus

$$\#Z\left(\mathbf{s}_1^{\mathcal{B}\Gamma}\right) = \#Z\left(p_+^*\mathbf{s}^+_{\mathcal{B}\Gamma}\right) \cap Z\left(p_-^*\mathbf{s}^-_{\mathcal{B}\Gamma}\right). \tag{49}$$

Dimension counting and transversality show this number vanishes unless $\operatorname{rk} E^{\pm}_{\mathcal{B}\Gamma} = \dim_{\mathbb{C}} \overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}$. In that case, transversality implies that $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}|_{\partial\overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}}$ vanishes nowhere. Thus

$$\#Z\left(p_+^*\mathbf{s}^+_{\mathcal{B}\Gamma}\right) \cap Z\left(p_-^*\mathbf{s}^-_{\mathcal{B}\Gamma}\right) = \tag{50}$$
$$= \left(\int_{\overline{\mathcal{M}}_{\mathcal{B}\Gamma^+}} e\left(E^+_{\mathcal{B}\Gamma}, \mathbf{s}_{\mathcal{B}\Gamma^+}|_{\partial\overline{\mathcal{M}}^+_{\mathcal{B}\Gamma}}\right)\right)\left(\int_{\overline{\mathcal{M}}_{\mathcal{B}\Gamma^-}} e\left(E^-_{\mathcal{B}\Gamma}, \mathbf{s}_{\mathcal{B}\Gamma^-}|_{\partial\overline{\mathcal{M}}^-_{\mathcal{B}\Gamma}}\right)\right).$$

The graph $\mathcal{B}\Gamma^{\pm}$ has a single vertex $v^{\pm}_{\mathcal{B}\Gamma}$. By construction, $\mathbf{s}^{\pm}_{\mathcal{B}\Gamma}|_{\partial\overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}}$ is a canonical boundary condition. So,

$$\int_{\overline{\mathcal{M}}_{\mathcal{B}\Gamma^{\pm}}} e\left(E^{\pm}_{\mathcal{B}\Gamma}, \mathbf{s}_{\mathcal{B}\Gamma^{\pm}}|_{\partial\overline{\mathcal{M}}^{\pm}_{\mathcal{B}\Gamma}}\right) = 2^{\frac{k\left(\hat{v}^{\pm}_{\Gamma}\right)-1}{2}} \left\langle \prod_{i\in\ell_I(\hat{v}^{\pm}_{\Gamma})} \tau_{a_i}\sigma^{k\left(\hat{v}^{\pm}_{\Gamma}\right)} \right\rangle^o_0. \tag{51}$$

For each $\Gamma \in U$, we have $1^\circ \in \ell_B(v^+_\Gamma)$ and e_Γ is legal for v^+_Γ. It follows that

$$k\left(\hat{v}^+_\Gamma\right) \geq 2, \qquad \Gamma \in U.$$

Keeping in mind that

$$k\left(\hat{v}^+_\Gamma\right) + k\left(\hat{v}^-_\Gamma\right) = k+1, \qquad \ell_I(\hat{v}^+_\Gamma) \cup \ell_I(\hat{v}^-_\Gamma) = \{2,\ldots,l\},$$

equations (46), (49), (50) and (51), imply the lemma. □

Proof of Theorem 1.5. The differential equation TRR I is equivalent to the following recursion relation:

$$\left\langle \tau_n \prod_{i=2}^{l} \tau_{a_i}\sigma^k \right\rangle^o_0 = \tag{52}$$
$$= \sum_{S\coprod R=\{2,\ldots,l\}} \left\langle \tau_0\tau_{n-1}\prod_{i\in S}\tau_{a_i}\right\rangle^c_0 \left\langle \tau_0 \prod_{i\in R}\tau_{a_i}\sigma^k\right\rangle^o_0 +$$
$$+ \sum_{\substack{S\coprod R=\{2,\ldots,l\}\\ k_1+k_2=k-1}} \binom{k}{k_1} \left\langle \tau_{n-1}\prod_{i\in S}\tau_{a_i}\sigma^{k_1}\right\rangle^o_0 \left\langle \prod_{i\in R}\tau_{a_i}\sigma^{k_2+2}\right\rangle^o_0.$$

By definition

$$\left\langle \tau_n \prod_{i=2}^{l} \tau_{a_i}\sigma^k \right\rangle^o_0 = \int_{\overline{\mathcal{M}}_{0,k,l}} e(E;\mathbf{s}).$$

So, recursion (52) follows immediately from Lemmas 4.7 and 4.13. The proof of TRR II is similar, except that we define ω_Σ to be the unique meromorphic differential on the normalization of Σ with the following properties. At $\bar{z}_1$ it has a simple pole with residue -1, and at z_2 it has a simple pole with residue 1. For any node the two preimages have at most simple poles and the residues at these poles sum to zero. Apart from these points, ω_Σ is holomorphic. As in the proof of TRR I, the section $\tilde{\rho}(\Sigma)$ is the evaluation of ω_Σ at z_1. □

5. Proof of Theorem 1.1

5.1. **Virasoro in genus** 0**.** The open Virasoro operators $\mathcal{L}_n$ and the partitions functions F_0^c and F_0^o were defined in Section 1.6. Define

$$G_r = \mathcal{L}_r \ \exp(u^{-2}F_0^c + u^{-1}F_0^o)$$

for $r \geq -1$. The genus 0 term of G_r is defined by

$$\text{Coeff}_{\ u^{-1}}\Big(G_r \exp(-u^{-2}F_0^c - u^{-1}F_0^o)\Big) \ .$$

The claim of Theorem 1.1 is:

$$\forall r \geq -1, \quad \text{Coeff}_{\ u^{-1}}\Big(G_r \exp(-u^{-2}F_0^c - u^{-1}F_0^o)\Big) = 0 \ .$$

By the open string and dilaton equations, genus 0 terms of G_{-1} and G_0 vanish. Using the Virasoro bracket, Theorem 1.1 follows from the vanishing

$$\text{Coeff}_{\ u^{-1}}\Big(G_2 \exp(-u^{-2}F_0^c - u^{-1}F_0^o)\Big) = 0 \ . \tag{53}$$

However, for the proof of (53), we will require the vanishing

$$\text{Coeff}_{\ u^{-1}}\Big(G_1 \exp(-u^{-2}F_0^c - u^{-1}F_0^o)\Big) = 0 \ . \tag{54}$$

5.2. **Vanishing for** $r = 1$**.** By unravelling the definition of $\mathcal{L}_1$, we can write the vanishing (54) explicitly for G_1. Using the Virasoro bracket

$$[\mathcal{L}_{-1}, \mathcal{L}_1] = -2\mathcal{L}_0,$$

we need only check the vanishing of G_1 at coefficients independent of t_0. The resulting equation is

$$\begin{aligned} -\frac{15}{4}\Big\langle \tau_2 \prod_{i=1}^{l} \tau_{a_i} \ \sigma^k \Big\rangle_0^o + \sum_{i=1}^{l} \frac{(2a_i+1)(2a_i+3)}{4} \Big\langle \tau_{a_i+1} \prod_{j\neq i} \tau_{a_j} \ \sigma^k \Big\rangle_0^o \\ + \sum_{S\cup T=\{1,\ldots,l\}} \Big\langle \prod_{i\in S} \tau_{a_i} \ \sigma^{k_S} \Big\rangle_0^o \, k \binom{k-1}{k_S-1} \Big\langle \prod_{i\in T} \tau_{a_i} \ \sigma^{k_T} \Big\rangle_0^o \ = 0. \end{aligned} \tag{55}$$

for $a_i \geq 1$ for all i.

Following the notation (71), the number of boundary markings in (55), is set by the dimension constraint

$$\begin{aligned} k &= 5 + 2A - 2l \\ k_S &= 3 + 2A_S - 2l_S \ , \end{aligned}$$

where the conventions

$$A = \sum_{i=1}^{l} a_i, \qquad \forall i \ a_i \geq 1 \ ,$$

$$A_S = \sum_{i \in S} a_i, l_S = |S|, \forall S \subseteq \{1, 2, \ldots, l\}$$

are used.

After substituting the evaluation of Theorem 1.4 and cancelling factors and simplifying, we reduce (55) to the identity:

$$(56) \quad \frac{20 + 8A - 8l}{4}(3 + 2A - l)! =$$
$$\sum_{S \cup T = \{1,\ldots,l\}} (5+2A-2l)\binom{4 + 2A - 2l}{2 + 2A_S - 2l_S}(1+2A_S-l_S)!(1+2A_T-l_T)! \ .$$

5.3. **Closed TRR.** In order to prove (56), we use the closed TRR in genus 0 to derive combinatorial identities. The following identity is obtained from closed TRR:

$$\Big\langle \tau_2 \tau_2 \tau_0 \prod_{i=1}^{l} \tau_{2a_i - 1} \ \tau_0^{4+2A-2l} \Big\rangle_0^c =$$
$$\sum_{S \cup T = \{1,\ldots,l\}} \Big\langle \tau_1 \tau_0 \prod_{i \in S} \tau_{2a_i - 1} \ \tau_0^{2+2A_S - 2l_S} \Big\rangle_0^c \binom{4 + 2A - 2l}{2 + 2A_S - 2l_S}$$
$$\cdot \Big\langle \tau_2 \tau_0^2 \prod_{i \in T} \tau_{2a_i - 1} \ \tau_0^{2+2A_T - 2l_T} \Big\rangle_0^c$$

After substituting the closed genus 0 evaluation, cancelling factors, and simplifying, we find:

$$(57) \quad \frac{4 + 2A - l}{4}(3 + 2A - l)! =$$
$$\sum_{S \cup T = \{1,\ldots,l\}} \frac{4 + 2A - l}{4}\binom{4 + 2A - 2l}{2 + 2A_S - 2l_S}(1+2A_s-l_s)!(1+2A_T-l_T)! \ .$$

And Equation 57 is clearly equivalent to Equation 56, as needed.

The proof of the vanishing (54) for $r = 1$ is complete. Hence, the open Virasoro constraint $\mathcal{L}_1$ is established in genus 0.

5.4. **Vanishing for $r = 2$.** By the definition of $\mathcal{L}_2$, we can write the vanishing (53) explicitly for G_2. Using the Virasoro bracket

$$[\mathcal{L}_{-1}, \mathcal{L}_2] = -3\mathcal{L}_1$$

and the validity of the constraint $\mathcal{L}_1$ in genus 0, we need only check the vanishing of G_2 at coefficients independent of t_0.

After unravelling the definition of $\mathcal{L}_2$ (just as we did for $\mathcal{L}_1$), we must prove the following identity:

$$\frac{42 + 12A - 12l}{8}(5 + 2A - l)! = \sum_{S \cup T \cup U = \{1,\ldots,l\}} (7+2A-2l) \binom{6 + 2A - 2l}{2 + 2A_S - 2l_S, 2 + 2A_T - 2l_T, 2 + 2A_U - 2L_U} \cdot (1 + 2A_S - l_S)!(1 + 2A_T - l_T)!(1 + 2A_U - l_U)! \ . \tag{58}$$

By applying the closed TRR twice, we obtain the following relation among closed descendent invariants:

$$\Big\langle \tau_2\tau_2\tau_2 \prod_{i=1}^{l} \tau_{2a_i-1}\ \tau_0^{6+2A-2l} \Big\rangle_0^c = \sum_{S \cup T \cup U = \{1,\ldots,l\}} \binom{6 + 2A - 2l}{2 + 2A_S - 2l_S, 2 + 2A_T - 2l_T, 2 + 2A_U - 2l_U} \cdot \Big\langle \tau_1\tau_0 \prod_{i\in S} \tau_{2a_i-1}\ \tau_0^{2+2A_S-2l_S} \Big\rangle_0^c \cdot \Big\langle \tau_2\tau_0^2 \prod_{i\in T} \tau_{2a_i-1}\ \tau_0^{2+2A_T-2l_T} \Big\rangle_0^c \cdot \Big\langle \tau_1\tau_0 \prod_{i\in U} \tau_{2a_i-1}\ \tau_0^{2+2A_U-2l_U} \Big\rangle_0^c \ .$$

After substituting the closed genus 0 evaluation, we find the identity

$$\frac{6 + 2A - l}{8}(5 + 2A - l)! = \sum_{S \cup T \cup U = \{1,\ldots,l\}} \frac{6 + 2A - l}{6} \binom{6 + 2A - 2l}{2 + 2A_S - 2l_S, 2 + 2A_T - 2l_T, 2 + 2A_U - 2L_U} \cdot (1 + 2A_S - l_S)!(1 + 2A_T - l_T)!(1 + 2A_U - l_U)! \ . \tag{59}$$

Identity 59 is clearly equivalent to Identity 58.

The proof vanishing (53) for $r = 2$ is complete. Hence, the open Virasoro constraint $\mathcal{L}_2$ is established in genus 0, and the proof of Theorem 1.1 is complete. $\square$

6. Proof of Theorem 1.3

6.1. **KdV.** Our goal is to prove the open KdV relation in genus 0:

$$(2n+1)\big\langle\big\langle\tau_n\big\rangle\big\rangle_0^o = \big\langle\big\langle\tau_{n-1}\tau_0\big\rangle\big\rangle_0^c\big\langle\big\langle\tau_0\big\rangle\big\rangle_0^o + 2\big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma\big\rangle\big\rangle_0^o \tag{60}$$

for $n \geq 1$. After differentiating both sides by s, we obtain

$$\begin{aligned}(2n+1)\big\langle\big\langle\tau_n\sigma\big\rangle\big\rangle_0^o = \\ \big\langle\big\langle\tau_{n-1}\tau_0\big\rangle\big\rangle_0^c\big\langle\big\langle\tau_0\sigma\big\rangle\big\rangle_0^o + 2\big\langle\big\langle\tau_{n-1}\sigma\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma\big\rangle\big\rangle_0^o + 2\big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma^2\big\rangle\big\rangle_0^o\end{aligned} \tag{61}$$

for $n \geq 1$. Since all nonvanishing genus 0 open invariants have at least a single σ insertion, equation (61) implies the open KdV (60) in genus 0.

Since we already have proven the TRR relation

$$\big\langle\big\langle\tau_n\sigma\big\rangle\big\rangle_0^o = \big\langle\big\langle\tau_{n-1}\tau_0\big\rangle\big\rangle_0^c\big\langle\big\langle\tau_0\sigma\big\rangle\big\rangle_0^o + \big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma^2\big\rangle\big\rangle_0^o ,$$

equation (61) follows from the differential equation

$$2n\big\langle\big\langle\tau_n\sigma\big\rangle\big\rangle_0^o = 2\big\langle\big\langle\tau_{n-1}\sigma\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma\big\rangle\big\rangle_0^o + \big\langle\big\langle\tau_{n-1}\big\rangle\big\rangle_0^o\big\langle\big\langle\sigma^2\big\rangle\big\rangle_0^o \tag{62}$$

for $n \geq 1$.

We observe equation (62) holds trivially for $n = 0$. The compatibility of (62) with the open string equation is easily checked. Hence, to prove equation (62), we need only consider additional insertions τ_{a_i} with $a_i \geq 1$. Using (69) for such insertions, we reduce (62) to the relation

$$\begin{aligned}(2n-1)\big\langle\tau_n\tau_{a_1}\ldots\tau_{a_l}\sigma^k\big\rangle_0^o = \\ 2\sum_{S\cup T=\{1,\ldots,l\}}\big\langle\tau_{n-1}\prod_{i\in S}\tau_{a_i}\ \sigma^{k_S}\big\rangle_0^o\binom{k-1}{k_S-1}\big\langle\prod_{i\in T}\tau_{a_i}\ \sigma^{k-k_S+1}\big\rangle_0^o .\end{aligned} \tag{63}$$

The sum is over all disjoint unions $S \cup T$ of the index set $\{1,\ldots,l\}$. The number of boundary markings in (63),

$$\begin{aligned} k &= 2n + 2A - 2l + 1 \\ k_S &= 2n + 2A_S - 2l_S - 1 ,\end{aligned}$$

is as in (69). As before, we use the notation (71).

6.2. **Binomial identities.** Recall the evaluation of Theorem 3,

$$\big\langle\tau_n\tau_{a_1}\ldots\tau_{a_l}\sigma^k\big\rangle_0^o = \frac{(2n+2A-l)!}{(2n-1)!!\prod_{i=1}^l(2a_i-1)!!} \tag{64}$$

in case $n \geq 1$ and $a_i \geq 1$ for all i. After substituting evaluation (64), relation (63) reduces to the following binomial identity (after cancelling all the equal factors on both sides):

$$(65) \qquad 2n + 2A - l = 2 \sum_{S \cup T = \{1,\ldots,l\}} \frac{\binom{2n+2A-2l}{2n+2A_S-2l_S-2}}{\binom{2n+2A-l-1}{2n+2A_S-l_S-2}} .$$

The sum is over all disjoint unions $S \cup T$ of the index set $\{1, \ldots, l\}$.

6.3. **Closed TRR.** As before, instead of a combinatorial proof of (65), we present a geometric argument using the following closed genus 0 topological recursion relation in genus 0,

$$(66) \qquad \big\langle\big\langle \tau_{2n-2}\tau_0\tau_2 \big\rangle\big\rangle_0^c = \big\langle\big\langle \tau_{2n-2}\tau_0^2 \big\rangle\big\rangle_0^c \big\langle\big\langle \tau_0\tau_1 \big\rangle\big\rangle_0^c .$$

Expanding (66) explicitly, we find

$$(67) \quad \Big\langle \tau_{2n-2}\tau_0\tau_2 \prod_{i=1}^{l} \tau_{2a_i-1} \cdot \tau_0^{2n+2A-2l} \Big\rangle_0^c = \sum_{S \cup T = \{1,\ldots,l\}} \Big\langle \tau_{2n-2}\tau_0^2 \prod_{i \in S} \tau_{2a_i-1} \cdot \tau_0^{2n+2A_S-2l_S-2} \Big\rangle_0^c \cdot \binom{2n+2A-2l}{2n+2A_S-2l_S-2} \cdot \Big\langle \tau_0\tau_1 \prod_{i \in T} \tau_{2a_i-1} \cdot \tau_0^{2A_T-2l_T+2} \Big\rangle_0^c .$$

We substitute the closed genus 0 formula

$$\big\langle \tau_{b_1} \ldots \tau_{b_m} \big\rangle_0^c = \binom{m-3}{b_1, \ldots, b_m}$$

in (67). After cancelling equal factors on both sides, we arrive exactly at the desired binomial identity (65). □

7. Proof of Theorem 1.4

7.1. **TRR.** Our goal is to prove the evaluation

$$(68) \qquad \big\langle \tau_{a_1} \ldots \tau_{a_l} \sigma^k \big\rangle_0^o = \frac{(\sum_{i=1}^{l} 2a_i - l + 1)!}{\prod_{i=1}^{l} (2a_i - 1)!!}$$

in case $a_i \geq 1$ for all i. The dimension constraint for the bracket (68) yields

$$-3 + k + 2l = \sum_{i=1}^{l} 2a_i .$$

Hence, k must be odd (and at least 1). The dilaton equation,

$$\left\langle \tau_1 \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o = (-1 + k + l) \left\langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o,$$

is easily seen to be compatible with the evaluation (68).

Writing the TRR relation

$$\left\langle\left\langle \tau_n \sigma \right\rangle\right\rangle_0^o = \left\langle\left\langle \tau_{n-1} \tau_0 \right\rangle\right\rangle_0^c \left\langle\left\langle \tau_0 \sigma \right\rangle\right\rangle_0^o + \left\langle\left\langle \tau_{n-1} \right\rangle\right\rangle_0^o \left\langle\left\langle \sigma^2 \right\rangle\right\rangle_0^o$$

of Theorem 4 explicitly, we find

$$(69) \quad \left\langle \tau_n \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o = \sum_{S \cup T = \{1,\dots,l\}} \left\langle \tau_{n-1} \prod_{i \in S} \tau_{a_i} \ \sigma^{k_S} \right\rangle_0^o \binom{k-1}{k_S} \left\langle \prod_{i \in T} \tau_{a_i} \ \sigma^{k-k_S+1} \right\rangle_0^o .$$

The sum is over all disjoint unions $S \cup T$ of the index set $\{1, \dots, l\}$. The number of boundary markings in (69),

$$\begin{aligned} k &= 2n + 2 \sum_{i=1}^{l} a_i - 2l + 1 \\ k_S &= 2n + 2 \sum_{i \in S} a_i - 2|S| - 1 \ , \end{aligned}$$

is set by the dimension constraint. The condition $a_i \geq 1$ forces the term

$$\left\langle\left\langle \tau_{n-1} \tau_0 \right\rangle\right\rangle_0^c \left\langle\left\langle \tau_0 \sigma \right\rangle\right\rangle_0^o$$

of the TRR to vanish. The right side of (69) is obtained from the second term of the TRR.

7.2. **Induction.** We prove the evaluation (68) by descending induction on the a_i. The base of the induction is when $a_i = 1$ for all i. By the compatibility of the evaluation (68) and the dilation equation, the base case is easily established.

By further use of the compatibility with the dilaton equation, we need only consider invariants

$$\left\langle \tau_n \tau_{a_1} \dots \tau_{a_l} \sigma^k \right\rangle_0^o$$

where $n \geq 2$ and $a_i \geq 1$. We will prove the induction step by applying the TRR relation (69). We observe the right side of (69) contains *no* disk invariants with τ_0 insertions. To complete the induction step, we need only prove the evaluation (68) satisfies the TRR relation (69). We are left with a combinatorial formula to verify.

7.3. **Binomial identities.** The combinatorial formula which arises in the induction step can be written as the following binomial identity (after cancelling all the equal factors on both sides):

$$\frac{2n+2A-l}{2n-1} = \sum_{S\cup T=\{1,\ldots,l\}} \frac{\binom{2n+2A-2l}{2n+2A_S-2l_S-1}}{\binom{2n+2A-l-1}{2n+2A_S-l_S-2}} \ . \tag{70}$$

The sum is over all disjoint unions $S \cup T$ of the index set $\{1,\ldots,l\}$, and

$$A = \sum_{i=1}^{l} a_i, \quad A_S = \sum_{i\in S} a_i, \quad A_T = \sum_{i\in T} a_i, \quad l_S = |S|, \quad l_T = |T| \ . \tag{71}$$

Instead of a direct combinatorial proof of (70), we present a geometric argument using the closed topological recursion relations in genus 0,

$$\big\langle\big\langle \tau_{2n-1}\tau_0\tau_1 \big\rangle\big\rangle_0^c = \big\langle\big\langle \tau_{2n-2}\tau_0 \big\rangle\big\rangle_0^c \big\langle\big\langle \tau_0^2\tau_1 \big\rangle\big\rangle_0^c \ . \tag{72}$$

First, we write (72) explicitly in the following specially chosen case:

$$\begin{aligned} \Big\langle \tau_{2n-1}\tau_0\tau_1 \prod_{i=1}^{l} \tau_{2a_i-1} \cdot \tau_0^{2n+2A-2l} \Big\rangle_0^c = \\ \sum_{S\cup T=\{1,\ldots,l\}} \Big\langle \tau_{2n-2}\tau_0 \prod_{i\in S} \tau_{2a_i-1} \cdot \tau_0^{2n+2A_S-2l_S-1} \Big\rangle_0^c \\ \cdot \binom{2n+2A-2l}{2n+2A_S-2l_S-1} \\ \cdot \Big\langle \tau_0^2\tau_1 \prod_{i\in T} \tau_{2a_i-1} \cdot \tau_0^{2A_T-2l_T+1} \Big\rangle_0^c \ . \end{aligned} \tag{73}$$

Second, we substitute the closed genus 0 formula

$$\big\langle \tau_{b_1} \ldots \tau_{b_m} \big\rangle_0^c = \binom{m-3}{b_1,\ldots,b_m}$$

in (73). After cancelling equal factors on both sides, we arrive precisely at the binomial identity (70). □

Appendix A. Multisections and the relative Euler class

We summarise relevant definitions concerning multisections and their zero sets. For the most part, we follow [3]. As usual, all manifolds may have corners.

Definition A.1. Let M be a $n-$dimensional manifold. A *weighted branched submanifold* N of dimension k is a function

$$\mu : M \to \mathbb{Q} \cap [0, \infty), \qquad \operatorname{supp}(\mu) = N,$$

which satisfies the following condition. For each $x \in M$ there exists an open neighborhood U of x, a finite collection of $k-$dimensional submanifolds, $N_1, \ldots, N_m$, of M which are relatively closed in U and positive rational numbers $\mu_1, \ldots, \mu_m$, such that

$$\forall y \in U, \qquad \mu|_U = \sum_{i=1}^{m} \mu_i \chi_{N_i}.$$

Here, χ_{N_i} is the characteristic function of N_i.

We call the submanifolds N_i *branches* of N in U and the numbers μ_i their *weights*.

A weighted branched submanifold is *compact* if the support of μ is compact.

Throughout the article we refer to branched weighted manifolds by their support, N. In this appendix, however, it is more convenient to work with the representing function μ, and this is indeed what we do. We say that N is represented by μ and we use both notations for the same notion.

Remark A.2. A usual submanifold $N \hookrightarrow M$ is a special case of a weighted branched submanifold. Indeed, take

$$\mu = \chi_N, \qquad m = 1, \qquad N_1 = N, \qquad \mu_1 = 1.$$

Notation A.3. For a vector space V, denote by $Gr_k(V)$ the Grassmannian of k-dimensional vector subspaces of V, and by $Gr_k^+(V)$ the Grassmannian of oriented k-dimensional vector subspaces of V. The oriented Grassmannian of zero dimensional subspaces Gr_0^+ consists of two points labeled $+$ and $-$. Given a vector bundle $E \to M$, we denote the associated (oriented) Grassmannian bundle by

$$Gr_k^{(+)}(E) = \left\{ (x, W) \middle| x \in M, W \in Gr_k^{(+)}(E_x) \right\}.$$

Definition A.4. Let M be a manifold of dimension n, and μ a weighted branched submanifold of dimension k.

(a) The *tangent bundle* of μ is the unique k-dimensional weighted branched submanifold $T\mu$ of $Gr_k(TM)$, such that

$$T\mu(x, W) = \sum_{T_x N_i = W} \mu_i,$$

where μ_i, N_i are weights and branches at x respectively.

(b) An *orientation* of μ is a function

$$\mu^+ : Gr_k^+(TM) \to \mathbb{Q},$$

which satisfies the following condition. For all

$$(x, W) \in Gr_k^+(TM),$$

there exists an open neighborhood U of x in which there are branches N_i of μ each with a given orientation, and weights μ_i of μ, such that

$$\mu^+(x, W) = \sum_{T_x N_i = W} \mu_i - \sum_{T_x N_i = -W} \mu_i.$$

Here, vector spaces are oriented and $-W$ stands for the vector space W with orientation reversed.

(c) If μ is compact, oriented, of dimension 0 and $(\operatorname{supp}\mu) \cap \partial M = \emptyset$, the *weighted cardinality* of μ is given by

$$\#\mu = \sum_{x \in M} \mu^+(x, +).$$

The existence of the tangent bundle was established in [3] .

Remark A.5. Again, the definitions generalize the standard ones for submanifolds. Indeed, let μ be as in Remark A.2. We take $T\mu(x, W)$ to be 1 if and only if $\mu(x) = 1$ and $W = T_x N$. Otherwise, it is 0. Similarly, if N is oriented, we define

$$\mu^+(x, W) = \begin{cases} 1, & \mu(x) = 1 \text{ and } W = T_x N, \\ -1, & \mu(x) = 1 \text{ and } W = -T_x N, \\ 0, & \text{otherwise.} \end{cases}$$

We can now define weighted versions of unions and intersections.

Definition A.6. Let μ, λ, be two branched weighted submanifolds of M of dimensions k, l, respectively. We say that μ is *transverse* to λ and write $\mu \pitchfork \lambda$ if for all $x \in M$, $W \in Gr_k(TM), V \in Gr_l(TM)$, with

$$T\mu(x, W), T\lambda(x, V) > 0,$$

W and V intersect transversally.

If $\mu \pitchfork \lambda$, we define the *intersection* $\mu \cap \lambda$ of μ and λ by

$$\mu \cap \lambda(x) = \mu(x)\lambda(x).$$

Given orientations μ^+, λ^+ of μ, λ, respectively, and given an orientation on M, we define the *induced orientation* of $\mu \cap \lambda$,

$$\mu^+ \cap \lambda^+ : Gr_{k+l-\dim M}^+ \to \mathbb{Q},$$

by

$$\mu^+ \cap \lambda^+ (x, U) = \sum_{U = V \cap W} \mu^+ (W) \lambda^+ (V) .$$

Here, we need the orientation on M in order to induce the orientation on $V \cap W$. If $k + l = \dim M$, we define the *intersection number* by

$$\left(\mu, \mu^+\right) \cdot \left(\lambda, \lambda^+\right) = \#(\mu \cap \lambda, \mu^+ \cap \lambda^+).$$

If $k = l$, we define the *union* of μ and λ by

$$\mu \cup \lambda(x) = \mu(x) + \lambda(x).$$

The transverse intersection of branched weighted manifolds of dimensions k and l has dimension $k + l - \dim M$.

Remark A.7. It is easy to see that both intersection and union are commutative and associative. In addition, we have the distributive property. That is, any three branched weighted submanifolds λ, μ, ν, satisfy

$$(\mu \cup \lambda) \cap \nu = (\mu \cap \nu) \cup (\lambda \cap \nu) .$$

We now move to multisections and operations between them.

Definition A.8. Let $p : E \to M$ be a rank k vector bundle over an n-dimensional manifold. A *multisection* s of E, is a weighted branched submanifold

$$\sigma : E \to \mathbb{Q} \cap [0, \infty) ,$$

of the following special form. For all $x \in M$ there exists a neighborhood U, smooth sections $s_1, \ldots, s_m : U \to E$ called *branches*, and rational numbers $\sigma_1 \ldots, \sigma_m$, called *weights*, with sum 1, such that

$$\sigma (x, v) = \sum_{s_i(x) = v} \sigma_i, \qquad \forall (x, v) \in E|_U .$$

That is, the total weight of the fiber is 1. We say that s is represented by σ and we use both notations for the same notion.

Given a submanifold $N \subseteq M$ and a multisection s of $E \to M$, we define the *restriction* of s to N by

$$s|_N = \sigma|_{p^{-1}(N)} .$$

Let $f : M \to N$ be a map of smooth manifolds with corners and let $E \to N$ be a vector bundle. Denote by $\tilde{f} : f^*E \to E$ the canonical map covering f. Let σ be a multisection of E. Then the *pull-back* $f^*\sigma$ is the multisection of f^*E given by

$$(f^*\sigma)(x, v) = \sigma(\tilde{f}(x, v)).$$

A multisection is said to be *transverse* if it and the zero section are transverse as weighted branched manifolds.

Definition A.9. Let χ_0 denote the indicator function of the zero section. Given a scalar a in the base field and a multisection σ, we define the product multisection $a\sigma$ by

$$(a\sigma)(x,v) = \begin{cases} \sigma(x, a^{-1}v), & a \neq 0, \\ \chi_0, & a = 0. \end{cases}$$

Given several multisections $\sigma_1, \ldots, \sigma_m$, we define their *sum*

$$\sigma = \sigma_1 + \ldots + \sigma_m$$

by

$$\sigma\left(x, v\right) = \sum_{v_1 + \ldots + v_m = v} \prod_{i=1}^{m} \sigma_i\left(x, v_i\right).$$

The sum of multisections is commutative and associative.

Let $pr : [0,1] \times M \to M$ denote the projection. A *homotopy* between two multisections σ_1, σ_2, of $E \to M$ is a multisection σ of

$$pr^* E \to M \times [0,1],$$

such that

$$\sigma|_{E \times \{0\}} = \sigma_1, \qquad \sigma|_{E \times \{1\}} = \sigma_2.$$

We say that a multisection *vanishes* at a point if one of its branches vanishes there.

Given multisections σ_i of $E_i \to M$ for $i = 1, 2$, we define the multisection $\sigma_1 \oplus \sigma_2$ of $E_1 \oplus E_2$ by

$$\sigma\left(\left(x, v_1 \oplus v_2\right)\right) = \sigma_1\left(x, v_1\right)\sigma_2\left(x, v_2\right).$$

Given a multisection σ of $E \to M$, and a section t of a line bundle $L \to M$, we define the multisection σt of $E \otimes L$, by

$$\left(\sigma t\right)\left(x, v \otimes w\right) = \sigma\left(x, v\right)\delta_{t(x) - w},$$

where $\delta_{t(x)-w} = 1$ if $t\left(x\right) = w$, and otherwise it is 0.

Given a multisection s of $E \to \partial M$, an extension of s to all M is a multisection s' whose restriction to ∂M is s.

Let G be a discrete group, and let $E \to M$ be a G-equivariant vector bundle. Given a multisection σ of E, we define the multisection $g \cdot \sigma$ by

$$\left(g \cdot \sigma\right)\left(x, v\right) = \sigma\left(g^{-1} \cdot \left(x, v\right)\right).$$

We say that σ is *G-equivariant* if

$$\sigma = g \cdot \sigma, \qquad \forall g \in G.$$

Definition A.10. In case G is finite we define the $G-$*symmetrization* of σ by

$$\sigma^G(x,v) = \frac{1}{|G|}\sum_{g\in G} g\cdot\sigma(x,v).$$

The symmetrization is G invariant.

Notation A.11. We denote by $C_m^\infty(E)$, the space of multisections of E. If a group G acts on E, we use the notation $C_m^\infty(E)^G$ for the $G-$invariant multisections.

In case M is oriented of dimension n, the image of a section s of a vector bundle $E \to M$ inherits a canonical orientation through the diffeomorphism

$$s : M \to s(M).$$

In a similar manner, every multisection $s \in C_m^\infty(E)$, carries a natural orientation described as follows. Assume s is represented by σ, take $x \in M, W \in Gr_n^+(T_xM)$, and let U, σ_i, s_i be as in the definition of a multisection. We define

$$\sigma^+(x,W) = \sum\nolimits^+ \sigma_i - \sum\nolimits^- \sigma_i,$$

where $\sum^\pm$ is taken over indices i such that

$$W = \pm(ds_i(T_xU)).$$

This definition agrees with the usual orientation for sections. With these definitions in hand we define the zero set of a multisection as follows.

Definition A.12. Let $s \in C_m^\infty(E)$ be a transverse multisection. We define its *unoriented zero set* $\tilde{Z}(s)$, as the intersection of the multisections s and 0 as branched weighted submanifolds.

In case M and $E \to M$ are oriented we define the zero set $Z(s)$, to be $\tilde{Z}(s)$ with the orientation induced from the canonical orientations of s and 0.

Remark A.13. Let $E \to M$ be a vector bundle with $\operatorname{rk} E = \dim M$ and let $s \in C_m^\infty(E)$ be transverse. Suppose that at a point x several branches s_{i_j} vanish. Then the weight of x in the zero set of s is the signed sum of σ_{i_j}. The sign is the sign of the intersection of s_{i_j} and the zero section at x.

We will use the following theorem. In [3], a proof of this theorem is given in the case that M has no boundary. The proof for a manifold with corners is similar and will be omitted.

Theorem A.14. *Let $E \to M$ be a rank n bundle over a manifold of dimension n. Let $s \in C_m^\infty(E|_{\partial M})$ vanish nowhere and let $\tilde{s} \in C_m^\infty(E)$ be a transverse extension. Then $\#Z(\tilde{s})$ depends only on s and not on the choice of $\tilde{s}$.*

In other words, the homology class $[Z(\tilde{s})] \in H_0(M)$ depends only on E and s. It is Poincaré dual to a relative cohomology class in $H^n(M, \partial M)$, which we call the *relative Euler class of E* with respect to s.

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