NOTES ON LOCAL $\mathbb{P}^1$-ORBIFOLDS

P. JOHNSON, R. PANDHARIPANDE, H.-H. TSENG

1. Local orbifold invariants

Let a and b be positive integers. Denote by $\mathbb{P}^1[a,b]$ be the Deligne-Mumford stack with points of order a at $0 \in \mathbb{P}^1$ and b at $\infty \in \mathbb{P}^1$. More precisely, $\mathbb{P}^1[a,b]$ is the stack obtained by successively applying the a-th root and b-th root constructions along the divisors 0 and ∞ respectively. The constructions and properties of root stacks can be found in [1, 4].

There is a natural map $c : \mathbb{P}^1[a,b] \to \mathbb{P}^1$ to the coarse space. Let

$$L_0 \to \mathbb{P}^1[a,b], \quad L_\infty \to \mathbb{P}^1[a,b]$$

be the stack line bundles determined by

$$L_0 \cong \mathcal{O}_{\mathbb{P}^1[a,b]}([0/\mathbb{Z}_a]), \quad L_\infty \cong \mathcal{O}_{\mathbb{P}^1[a,b]}([\infty/\mathbb{Z}_b]).$$

We have $L_0^{\otimes a} \simeq c^*\mathcal{O}(1)$ and $L_\infty^{\otimes b} \simeq c^*\mathcal{O}(1)$. Therefore,

$$\deg(L_0) = 1/a, \quad \deg(L_\infty) = 1/b.$$

There is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1[a,b]} \to L_0 \oplus L_\infty \to T_{\mathbb{P}^1[a,b]} \to 0. \tag{1}$$

Hence, $\deg(T_{\mathbb{P}^1[a,b]}) = 1/a + 1/b$.

Let $\alpha = (\alpha_1, ..., \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be vectors of integers

$$1 \leq \alpha_i \leq a-1, \quad 1 \leq \beta_j \leq b-1$$

defining nontrivial elements

$$\alpha_i \in \mathbb{Z}/a\mathbb{Z}, \quad \beta_j \in \mathbb{Z}/b\mathbb{Z}.$$

We view α and β as determining tuples of conjugacy classes. Let

$$\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b], d)$$

be the moduli stack of stable maps to $\mathbb{P}^1[a,b]$ of genus g and degree d with $m+n$ domain stack points with structure determined by α and β. We restrict our attention to the case $d > 0$.

Date: April 25, 2008.

The virtual dimension of $\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)$ is computed at a map $[f : C \to \mathbb{P}^1[a,b]]$ by Riemann-Roch,

$$\begin{aligned} \text{vir dim}\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d) &= \chi(f^*T_{\mathbb{P}^1[a,b]}) + 3g - 3 + m + n \\ &= d(\frac{1}{a}+\frac{1}{b}) - \sum_{i=1}^m \frac{\alpha_i}{a} - \sum_{j=1}^n \frac{\beta_j}{b} + (1-3)(1-g) + m + n \\ &= d(\frac{1}{a}+\frac{1}{b}) - \sum_{i=1}^m \frac{\alpha_i}{a} - \sum_{j=1}^n \frac{\beta_j}{b} + 2g - 2 + m + n. \end{aligned} \tag{2}$$

In order for $\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)$ to be non-empty, two parity conditions must hold:

(i) $d - \sum_{i=1}^m \alpha_i = 0 \,(\text{mod } a)$,
(ii) $d - \sum_{j=1}^n \beta_j = 0 \,(\text{mod } b)$.

The integralities are easily deduced from the geometry over $[0/\mathbb{Z}_a], [\infty/\mathbb{Z}_b] \in \mathbb{P}^1[a,b]$.

Let $\mathcal{C}$ denote the universal curve over the moduli space of maps. Consider the universal diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^1[a,b] \\ \pi\big\downarrow & & \\ \overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d). & & \end{array}$$

Let $[f : C \to \mathbb{P}^1[a,b]] \in \overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)$. Since

$$\deg(f^*L_0^\vee) = -d/a < 0, \quad \deg(f^*L_\infty^\vee) = -d/b < 0,$$

we obtain $h^0(\mathcal{C}, f^*(L_0^\vee \oplus L_\infty^\vee)) = 0$. Hence, $R^1\pi_* f^*(L_0^\vee \oplus L_\infty^\vee)$ is a vector bundle over $\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)$ of rank determined by Riemann-Roch,

$$-\Big(d(-1/a - 1/b) - \sum_{i=1}^m (1-\alpha_i/a) - \sum_{j=1}^n (1-\beta_j/b) + 2(1-g)\Big) =$$
$$d(1/a+1/b) - \sum_{i=1}^m \alpha_i/a - \sum_{j=1}^n \beta_j/b + 2g - 2 + m + n,$$

which is equal to the virtual dimension of $\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)$.

We will evaluate the integrals

$$C_{g,\alpha+\beta}(d) = \int_{[\overline{\mathcal{M}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)]^{vir}} e(R^1\pi_* f^*(L_0^\vee \oplus L_\infty^\vee)). \tag{3}$$

by virtual localization. The integrals (3) are the Gromov-Witten invariants of *local* $\mathbb{P}^1[a,b]$.

2. Calculation

The standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$$

lifts canonically to a $\mathbb{C}^*$-action on $\mathbb{P}^1[a,b]$. The tangent weights at $[0/\mathbb{Z}_a]$ and $[\infty/\mathbb{Z}_b]$ are t/a and $-t/b$ respectively. The line bundles L_0, L_∞ are linearized to have weights $[t/a, 0]$ and $[0, -t/b]$ respectively.

2.1. **Vanishing.** The virtual localization formula [6] expresses the integrals (3) sums over graphs of Hurwitz-Hodge integrals. The choice of linearizations yields a vanishing: if a graph contains a vertex of valence greater than 1, then the contribution of the graph is 0.

The argument for the vanishing is as follows. Suppose $f : C \to \mathbb{P}^1[a,b]$ is $\mathbb{C}^*$-fixed, $C_0 \subset C$ is a subcurve contracted by f to $[0/\mathbb{Z}_a]$, and $C_1, C_2 \subset C$ are two subcurves meeting C_0 at b_1, b_2. The normalization sequence for $f^*(L_0^\vee \oplus L_\infty^\vee)$ contains the following part

$$\begin{aligned} &H^0(C_0, f^*(L_0^\vee \oplus L_\infty^\vee)) \oplus H^0(C_1, f^*(L_0^\vee \oplus L_\infty^\vee)) \oplus H^0(C_2, f*(L_0^\vee \oplus L_\infty^\vee)) \\ \to &H^0(b_1, f^*(L_0^\vee \oplus L_\infty^\vee)) \oplus H^0(b_2, f^*(L_0^\vee \oplus L_\infty^\vee)) \\ \to &H^1(C, f^*(L_0^\vee \oplus L_\infty^\vee)). \end{aligned}$$

Since $f(C_0) = [0/\mathbb{Z}_a]$, we see $\mathbb{C}^*$ acts trivially on $L_\infty^\vee|_{b_1}$ and $L_\infty^\vee|_{b_2}$. Also, $\mathbb{Z}_a$ acts trivially on these fibers. By the normalization sequence, one of the weight-0 pieces in $H^0(b_1, f^*(L_0^\vee \oplus L_\infty^\vee)) \oplus H^0(b_2, f^*(L_0^\vee \oplus L_\infty^\vee))$ is cancelled with a weight-0 piece of $H^0(C_0, f^*(L_0^\vee \oplus L_\infty^\vee))$ and the other is mapped injectively into $H^1(C, f^*(L_0^\vee \oplus L_\infty^\vee))$. Thus $H^1(C, f^*(L_0^\vee \oplus L_\infty^\vee))$ contains a weight-0 piece with vanishings equivariant Euler class. The vanishing for vertices of valence greater than 1 over $[\infty/\mathbb{Z}_b]$ is the same.

2.2. **Fixed Loci.** By the above discussion, we need only consider fixed loci of very simple type. A $\mathbb{C}^*$-fixed stable map $f : C \to \mathbb{P}^1[a,b]$ satisfies the following properties:

(i) C is a union of three subcurves,
$$C = C_0 \cup C_1 \cup C_\infty,$$
where C_0 has genus g_1, C_1 has genus 0, and C_∞ has genus g_2.
(ii) C_0 and C_1 meet at a point b_0, C_∞ and C_1 meet at a point b_∞, and $\mathcal{C}_0$ and C_∞ are disjoint.
(iii) $f(C_0) = [0/\mathbb{Z}_a]$ and all marked points of types α lie on C_0.
(iv) $f(C_\infty) = [\infty/\mathbb{Z}_b]$ and all marked points of types β lie on C_∞.
(v) $f|_{C_1}$ is a degree d cover invariant under $\mathbb{C}^*$.

The stack structure at $b_0 \in \mathcal{C}_1$ is given by $d \,(\mathrm{mod}\, a)$ and the stack structure at the corresponding point on $\mathcal{C}_0$ is given by $-d\,(\mathrm{mod}\, a)$. Let r_0 denote the order of this element in $\mathbb{Z}_a$. Similarly the stack structure at $b_\infty \in \mathcal{C}_1$ is given by $d\,(\mathrm{mod}\, b)$ and the stack structure at the corresponding point on $\mathcal{C}_\infty$ is given by $-d\,(\mathrm{mod}\, b)$. Let r_∞ denote the order of this element in $\mathbb{Z}_b$.

If $m \in \{0,1\}$ and C_0 has genus 0, then C_0 is unstable and situation is degenerate. Similiarly if $n \in \{0,1\}$ and C_∞ has genus 0, then C_∞ is unstable. The degenerate cases have to be treated separately.

Let $\mathcal{M}$ denote the moduli space of degree d invariant covers of $\mathbb{P}^1[a,b]$ with stack structures given by $d\,(\mathrm{mod}\, a)$ and $d\,(\mathrm{mod}\, b)$. The fixed locus considered here are of the form

$$\bigcup_{g_1+g_2=g} \overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a) \times_{\bar{I}B\mathbb{Z}_a} \times \mathcal{M} \times_{\bar{I}B\mathbb{Z}_b} \overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b).$$

Since integrands are pulled back from $\bigcup_{g_1+g_2}\overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a)\times\overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b)$, the required integrals over

$$\overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a)\times_{\bar{I}B\mathbb{Z}_a}\times\mathcal{M}\times_{\bar{I}B\mathbb{Z}_b}\overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b)$$

can be evaluated by integrating over $\overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a)\times\overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b)$, provided we include the following factor

$$\frac{1}{d}\frac{a}{r_0}\frac{b}{r_\infty} \tag{4}$$

accounting for the fiber products and automorphisms.

2.3. **Contributions from fixed loci.** Next we need to identify the equivariant Euler classes of $R^1\pi_*f^*(L_0^\vee\oplus L_\infty^\vee)$ and the virtual normal bundle.

Let $f: C\to\mathbb{P}^1[a,b]$ be a $\mathbb{C}^*$-fixed stable map in the $\mathbb{C}^*$-fixed locus indentified in Section 2.2. The normalization sequences yield

$$\begin{aligned} H^1(C,f^*(L_0^\vee\oplus L_\infty^\vee))-H^0(C,f^*(L_0^\vee\oplus L_\infty^\vee)) = \quad & H^1(C_0,f^*(L_0^\vee\oplus L_\infty^\vee))-H^0(C_0,f^*(L_0^\vee\oplus L_\infty^\vee))\\ &+H^1(C_1,f^*(L_0^\vee\oplus L_\infty^\vee))-H^0(C_1,f^*(L_0^\vee\oplus L_\infty^\vee))\\ &+H^1(C_\infty,f^*(L_0^\vee\oplus L_\infty^\vee))-H^0(C_\infty,f^*(L_0^\vee\oplus L_\infty^\vee))\\ &+H^0(b_0,f^*(L_0^\vee\oplus L_\infty^\vee))+H^0(b_\infty,f^*(L_0^\vee\oplus L_\infty^\vee))\end{aligned}$$

for $R^1\pi_*f^*(L_0^\vee\oplus L_\infty^\vee)$ and

$$\begin{aligned} H^1(C,f^*T_{\mathbb{P}^1[a,b]})-H^0(C,f^*T_{\mathbb{P}^1[a,b]}) = \quad & H^1(C_0,f^*T_{\mathbb{P}^1[a,b]})-H^0(C_0,f^*T_{\mathbb{P}^1[a,b]})\\ &+H^1(C_1,f^*T_{\mathbb{P}^1[a,b]})-H^0(C_1,f^*T_{\mathbb{P}^1[a,b]})\\ &+H^1(C_\infty,f^*T_{\mathbb{P}^1[a,b]})-H^0(C_\infty,f^*T_{\mathbb{P}^1[a,b]})\\ &+H^0(b_0,f^*T_{\mathbb{P}^1[a,b]})+H^0(b_\infty,f^*T_{\mathbb{P}^1[a,b]})\end{aligned}$$

for the virtual normal bundle.

2.3.1. *Vertex over* 0. Let U be the 1-dimensional representation of $\mathbb{Z}_a$ given by the character

$$\mathbb{Z}_a\to\mathbb{C}^*,\ \ 1\mapsto e^{\frac{2\pi i}{a}}.$$

The term $H^1(C_0,f^*L_0^\vee)$ yields the vector bundle $(\mathbb{E}^{U^\vee})^\vee$. The term $H^1(\mathcal{C}_0,f^*T_{\mathbb{P}^1[a,b]})$ yields the vector bundle $(\mathbb{E}^U)^\vee$.

Assume $m>1$ to avoid unstable cases. Then, the monodromy representation $\pi_1^{orb}(C_0)\to\mathbb{Z}_a$ is never trivial, and the terms

$$H^0(C_0,f^*L_0^\vee),\ \ H^0(C_0,f^*T_{\mathbb{P}^1[a,b]})$$

vanish. The terms

$$H^0(b_0,f^*L_0^\vee),\ \ H^0(b_0,f^*T_{\mathbb{P}^1[a,b]})$$

vanish unless b_0 is not a stack point in which case a factor of $-t/a$ and t/a are contributed respectively. The term $H^0(b_0,f^*T_{\mathbb{P}^1[a,b]})$ will be disregarded here and considered together with the edge contribution. The terms

$$-H^0(C_0,f^*L_\infty^\vee)+H^0(b_0,f^*L_\infty^\vee)$$

cancel since L_∞ is untwisted at 0.

The Euler class of $R^1\pi_* f^*(L_0^\vee)$ combined with the $H^1 - H^0$ contribution to the virtual normal bundle (disregarding $H^0(b_0, f^*T_{\mathbb{P}^1[a,b]})$) yields

$$(-t/a)^{\epsilon_a^d}\, c_{t/a}((\mathbb{E}^U)^\vee)c_{-t/a}((\mathbb{E}^{U^\vee})^\vee). \tag{5}$$

Here $\epsilon_a^d = 0$ if $\langle d/a\rangle \neq 0$ and $\epsilon_a^d = 1$ if $\langle d/a\rangle = 0$. By Mumford's relation [3], the product (5) is equal to

$$(-t/a)^{\epsilon_a^d}\,(-1)^{\mathrm{rk}\mathbb{E}^{U^\vee}}(t/a)^{\mathrm{rk}\mathbb{E}^U+\mathrm{rk}\mathbb{E}^{U^\vee}}. \tag{6}$$

If $\langle d/a\rangle \neq 0$, then[1]

$$\begin{aligned}\mathrm{rk}\mathbb{E}^U + \mathrm{rk}\mathbb{E}^{U^\vee} &= g_1 - 1 + \sum_{i=1}^m \frac{\alpha_i}{a} + \left\langle \frac{-d}{a}\right\rangle + g_1 - 1 + \sum_{i=1}^m \frac{a-\alpha_i}{a} + \left\langle \frac{d}{a}\right\rangle \\ &= 2g_1 - 1 + m.\end{aligned} \tag{7}$$

If $\langle d/a\rangle = 0$, then

$$\begin{aligned}\mathrm{rk}\mathbb{E}^U + \mathrm{rk}\mathbb{E}^{U^\vee} &= g_1 - 1 + \sum_{i=1}^m \frac{\alpha_i}{a} + g_1 - 1 + \sum_{i=1}^m \frac{a-\alpha_i}{a} \\ &= 2g_1 - 2 + m.\end{aligned} \tag{8}$$

Written together, we find (6) is equal to

$$(-1)^{\epsilon_a^d}\,(-1)^{g_1-1+m-\sum_{i=1}^m \frac{\alpha_i}{a}+\left\langle\frac{d}{a}\right\rangle}(t/a)^{2g_1-1+m}. \tag{9}$$

Since the $\mathbb{C}^*$-weight of L_∞ at $[0/\mathbb{Z}_a]$ is 0, the equivariant Euler class of $R^1\pi_* f^*(L_\infty^\vee)$ is

$$e(\mathbb{E}^\vee) = (-1)^{g_1}\lambda_{g_1}. \tag{10}$$

The node-smoothing contribution is

$$\frac{1}{\frac{1}{r_0}\left(-\bar{\psi}_0 + \frac{t}{d}\right)}. \tag{11}$$

2.3.2. *Vertex over* ∞. Let V be the 1-dimensional representation of $\mathbb{Z}_b$ given by the character

$$\mathbb{Z}_b \to \mathbb{C}^*, \quad 1 \mapsto e^{\frac{2\pi i}{b}}.$$

The term $H^1(C_\infty, f^*L_\infty^\vee)$ yields the vector bundle $(\mathbb{E}^{V^\vee})^\vee$. The term $H^1(\mathcal{C}_\infty, f^*T_{\mathbb{P}^1[a,b]})$ yields the vector bundle $(\mathbb{E}^V)^\vee$.

Assume $n > 1$ to avoid unstable cases. Following the analysis of Section 2.3.1, the Euler class of $R^1\pi_* f^*(L_\infty^\vee)$ combined with the $H^1 - H^0$ contribution to the virtual normal bundle (disregarding $H^0(b_\infty, f^*T_{\mathbb{P}^1[a,b]})$) yields

$$(t/b)^{\epsilon_b^d}\, c_{-t/b}((\mathbb{E}^V)^\vee)c_{t/b}((\mathbb{E}^{V^\vee})^\vee).$$

By Mumford's relation this is equal to

$$(t/b)^{\epsilon_b^d}(-1)^{\mathrm{rk}\mathbb{E}^{V^\vee}}(-t/b)^{\mathrm{rk}\mathbb{E}^V+\mathrm{rk}\mathbb{E}^{V^\vee}}. \tag{12}$$

[1]For $x \in \mathbb{Q}$, we denote the fractional part of x by $\langle x\rangle$. In particular, $< x > + < -x >$ vanishes if $x \in \mathbb{Z}$ and equals 1 is $x \notin \mathbb{Z}$.

If $\langle d/b \rangle \neq 0$, then

$$\begin{aligned}\mathrm{rk}\mathbb{E}^V + \mathrm{rk}\mathbb{E}^{V^\vee} &= g_2 - 1 + \sum_{j=1}^{n} \frac{\beta_j}{b} + \left\langle \frac{-d}{b} \right\rangle + g_2 - 1 + \sum_{j=1}^{n} \frac{b-\beta_j}{b} + \left\langle \frac{d}{b} \right\rangle \\ &= 2g_2 - 1 + n. \end{aligned} \tag{13}$$

If $\langle d/b \rangle = 0$, then

$$\begin{aligned}\mathrm{rk}\mathbb{E}^V + \mathrm{rk}\mathbb{E}^{V^\vee} &= g_2 - 1 + \sum_{j=1}^{n} \frac{\beta_j}{b} + g_2 - 1 + \sum_{j=1}^{n} \frac{b-\beta_j}{b} \\ &= 2g_2 - 2 + n. \end{aligned} \tag{14}$$

Written together, we find (12) is equal to

$$(-1)^{\epsilon_b^d}\,(-1)^{g_2-1+n-\sum_{j=1}^n \frac{\beta_j}{b}+\left\langle\frac{d}{b}\right\rangle}(-t/b)^{2g_2-1+n}. \tag{15}$$

Since the $\mathbb{C}^*$-weight of L_0 at $[\infty/\mathbb{Z}_b]$ is 0, the equivariant Euler class of $R^1\pi_*f^*(L_0^\vee)$ is

$$e(\mathbb{E}^\vee) = (-1)^{g_2}\lambda_{g_2}. \tag{16}$$

The node-smoothing contribution is

$$\frac{1}{\frac{1}{r_\infty}\left(-\bar{\psi}_\infty - \frac{t}{d}\right)}. \tag{17}$$

2.3.3. *Edge contribution.* The edge contribution is independent of the vertex genus splitting. By (1), we have $H^1(C_1, f^*T_{\mathbb{P}^1[a,b]}) = 0$ and

$$\begin{aligned} H^0(C_1, f^*T_{\mathbb{P}^1[a,b]})^{mov} &= H^0(C_1, f^*L_0)^{mov} \oplus H^0(C_1, f^*L_\infty)^{mov} \\ &= H^0(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{d}{a} \rfloor))^{mov} \oplus H^0(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{d}{b} \rfloor))^{mov}. \end{aligned} \tag{18}$$

The weights of the first summand are $t/d, 2t/d, ..., \lfloor \frac{d}{a} \rfloor t/d$, the weights of the second summand are $-t/d, -2t/d, ..., -\lfloor \frac{d}{b} \rfloor t/d$. On the other hand, if $\langle d/a \rangle \neq 0$ and $\langle d/b \rangle \neq 0$, then

$$\begin{aligned} H^1(C_1, f^*(L_0^\vee \oplus L_\infty^\vee)) &= H^1(C_1, f^*L_0^\vee) \oplus H^1(C_1, f^*L_\infty^\vee) \\ &= H^1(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{-d}{a} \rfloor)) \oplus H^1(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{-d}{b} \rfloor)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}(-\lfloor \frac{d}{a} \rfloor - 1)) \oplus H^1(\mathbb{P}^1, \mathcal{O}(-\lfloor \frac{d}{b} \rfloor - 1)). \end{aligned} \tag{19}$$

The weights of the first summand are $-t/d, -2t/d, ..., -\lfloor \frac{d}{a} \rfloor t/d$, the weights of the second summand are $t/d, 2t/d, ..., \lfloor \frac{d}{b} \rfloor t/d$. Therefore, the edge contribution is

$$(-1)^{\lfloor \frac{d}{a} \rfloor + \lfloor \frac{d}{b} \rfloor}.$$

If $\langle d/a \rangle = 0$, the weights of $H^1(\mathcal{C}_1, f^*L_0^\vee)$ are $-t/d, -2t/d, ..., -\lfloor \frac{d-1}{a} \rfloor t/d$. The edge term then contains an uncancelled factor

$$\frac{1}{(-t/a)} \tag{20}$$

from $H^0(C_1, f^*T_{\mathbb{P}^1[a,b]})$. However, under the assumption $m > 0$, the factor (20) is cancelled (up to sign) by the weight t/a of $H^0(b_0, f^*T_{\mathbb{P}^1[a,b]})$ in case $\langle d/a\rangle = 0$. Exactly the same situation holds for if $\langle d/b\rangle = 0$. Hence, the edge term in all cases can be written as

$$(-1)^{\epsilon_a^d+\epsilon_b^d}(-1)^{\lfloor \frac{d}{a}\rfloor + \lfloor \frac{d}{b}\rfloor}.$$

2.3.4. *Vertex contribution.* After combining the vertex factors (4), (9-11), (15-17) with the trivial edge factor, we obtain

$$\begin{aligned}C_g(\alpha+\beta)(d) = {} & \frac{1}{d}\frac{a}{r_0}\frac{b}{r_\infty}(-1)^{\lfloor \frac{d}{a}\rfloor + \lfloor \frac{d}{b}\rfloor} \\ & \cdot \sum_{g_1+g_2=g}\int_{\overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a)} \frac{(-1)^{g_1-1+m-\sum_{i=1}^m \frac{\alpha_i}{a}+\left\langle\frac{d}{a}\right\rangle}\left(\frac{t}{a}\right)^{2g_1-1+m}(-1)^{g_1}\lambda_{g_1}}{\frac{1}{r_0}\left(-\bar\psi_0+\frac{t}{d}\right)} \\ & \cdot \int_{\overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b)} \frac{(-1)^{g_2-1+n-\sum_{j=1}^n \frac{\beta_j}{b}+\left\langle\frac{d}{b}\right\rangle}\left(\frac{-t}{b}\right)^{2g_2-1+n}(-1)^{g_2}\lambda_{g_2}}{\frac{1}{r_\infty}\left(-\bar\psi_\infty-\frac{t}{d}\right)},\end{aligned}$$

which simplifies to

$$\begin{aligned}(-1)^{\lfloor \frac{d}{a}\rfloor + \lfloor \frac{d}{b}\rfloor}\sum_{g_1+g_2=g} & \frac{d(-1)^{m-\sum_{i=1}^m \frac{\alpha_i}{a}+\left\langle\frac{d}{a}\right\rangle+n-\sum_{j=1}^n \frac{\beta_j}{b}+\left\langle\frac{d}{b}\right\rangle}}{a^{2g_1-2+m}b^{2g_2-2+n}} \\ & \cdot \int_{\overline{\mathcal{M}}_{g_1,\alpha-(d)}(B\mathbb{Z}_a)} \frac{t^{2g_1-1+m}\lambda_{g_1}}{-d\bar\psi_0+t} \cdot \int_{\overline{\mathcal{M}}_{g_2,\beta-(d)}(B\mathbb{Z}_b)} \frac{(-t)^{2g_2-1+n}\lambda_{g_2}}{-d\bar\psi_\infty-t}.\end{aligned}$$

After pushing to the moduli of curves, we obtain

$$\begin{aligned}(-1)^{\lfloor \frac{d}{a}\rfloor + \lfloor \frac{d}{b}\rfloor}\sum_{g_1+g_2=g} & \frac{d(-1)^{m-\sum_{i=1}^m \frac{\alpha_i}{a}+\left\langle\frac{d}{a}\right\rangle+n-\sum_{j=1}^n \frac{\beta_j}{b}+\left\langle\frac{d}{b}\right\rangle}}{a^{2g_1-2+m}b^{2g_2-2+n}} \\ & \cdot\ d^{2g_1-2+m}a^{2g_1-1}\int_{\overline{\mathcal{M}}_{g_1,m+1}} \lambda_{g_1}\psi_0^{2g_1-2+m} \ \cdot\ d^{2g_2-2+n}b^{2g_2-1}\int_{\overline{\mathcal{M}}_{g_2,n+1}} \lambda_{g_2}\psi_0^{2g_2-2+n}.\end{aligned}$$

Using the string equation, we have

$$(-1)^{\lfloor \frac{d}{a}\rfloor + \lfloor \frac{d}{b}\rfloor}\sum_{g_1+g_2=g}(-1)^{m-\sum_{i=1}^m \frac{\alpha_i}{a}+\left\langle\frac{d}{a}\right\rangle+n-\sum_{j=1}^n \frac{\beta_j}{b}+\left\langle\frac{d}{b}\right\rangle}\frac{d^{2g-3+m+n}}{a^{m-1}b^{n-1}}\cdot\int_{\overline{\mathcal{M}}_{g_1,1}}\lambda_{g_1}\psi_0^{2g_1-2}\cdot\int_{\overline{\mathcal{M}}_{g_2,1}}\lambda_{g_2}\psi_0^{2g_2-2}.$$

Executing the sum via Lemma 2 of [5], we find

$$\sum_{g\geq 0} C_{g,\alpha+\beta}(d)u^{2g} = (-1)^{m-\sum_{i=1}^m \frac{\alpha_i}{a}+\frac{d}{a}+n-\sum_{j=1}^n \frac{\beta_j}{b}+\frac{d}{b}}\,\frac{d^{m+n-3}}{a^{m-1}b^{n-1}}\left(\frac{ud/2}{\sin(ud/2)}\right)^2 \tag{21}$$

when the two parity conditions

(i) $d-\sum_{i=1}^m \alpha_i = 0 \,(\mathrm{mod}\, a)$,
(ii) $d-\sum_{j=1}^n \beta_j = 0 \,(\mathrm{mod}\, b)$

are satisfied.

Our derivation of formula (21) assumed $m, n > 1$. In fact, a simple analysis of unstable cases shows same formula holds for $m, n \geq 0$.

REFERENCES

[1] D. Abramovich, T. Graber, A. Vistoli, *Gromov-Witten theory for Deligne-Mumford stacks*, math.AG/0603151.
[2] D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), 27–75.
[3] J. Bryan, T. Graber, and R. Pandharipande, *The orbifold quantum cohomology of* $\mathbb{C}^2/\mathbb{Z}_3$ *and Hurwitz-Hodge integrals*, J. Alg. Geom. **17** (2008), 1 – 28.
[4] C. Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), 405–427.
[5] C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.
[6] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.