NEW CALCULATIONS IN GROMOV-WITTEN THEORY

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Dedicated to Fedor Bogomolov on the occasion of his 60th birthday

0. Introduction

0.1. **Overview.** Let X be a nonsingular projective variety over $\mathbb{C}$. The stack $\overline{M}_{g,n}(X,\beta)$ parameterizes stable maps

$$f : C \to X$$

from genus g, n-pointed curves to X representing the class $\beta \in H_2(X,\mathbb{Z})$. The Gromov-Witten invariants of X are defined by integration against the virtual class of the moduli space,

$$\Big\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\Big\rangle^X_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i), \tag{1}$$

where ψ_i is the Chern class of the i^{th} cotangent line,

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to X$$

is the i^{th} evaluation map, and $\gamma_i \in H^*(X,\mathbb{Q})$.

Over the past 10 years, there has been great deal of success in calculating Gromov-Witten invariants if either the domain genus is 0 or the target X carries a strong torus action — a torus action with finitely many 0 and 1 dimensional orbits.

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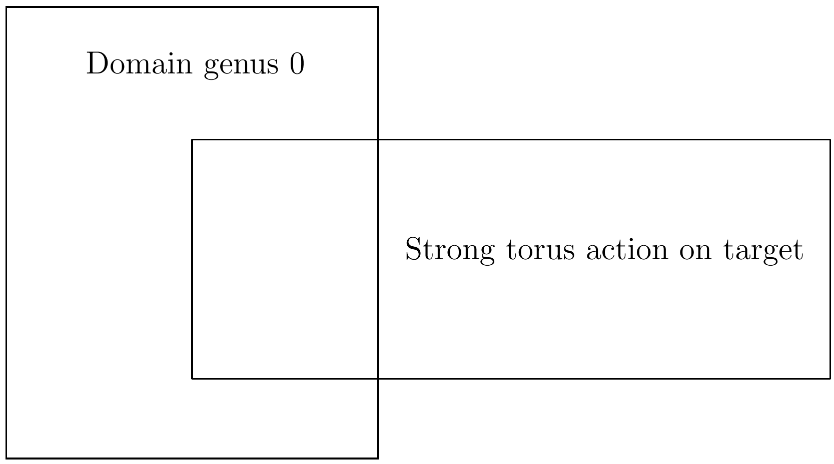

The goal of our paper is to use the methods of [30] for Gromov-Witten calculations outside of the above regions.

We study the higher genus Gromov-Witten invariants of two target geometries: surfaces of general type and compact Calabi-Yau 3-folds. In the surface case, the calculations suggest exact solutions. For Calabi-Yau 3-folds, our results provide the first mathematical encounter with the holomorphic anomaly equation for topological strings.

0.2. **Genus 0.** There are several mathematical approaches to Gromov-Witten theory in genus 0. A wide class of target varieties has been successfully studied via WDVV-equations, Frobenius structures, and quantum Lefschetz formulas. Quantum Schubert calculus is well developed for classical homogeneous spaces. Mirror symmetry relations between genus 0 invariants of Calabi-Yau hypersurfaces and Picard-Fuchs systems have been proven in many cases. The bibliography of [8] is a good source for genus 0 references before 2000.

An example of recent progress in genus 0 is the calculation of the quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ [35].

Genus 0 computation for *orbifold* targets, however, is a largely open subject [2, 3, 6].

0.3. **Strong torus actions.** The situation in higher genus is substantially more difficult. If X carries a strong torus action, then the Gromov-Witten theory of X is reduced to Hodge integrals on $\overline{M}_{g,n}$ via localization of the virtual class [16]. Hodge integrals can be reduced to descendent integrals by Gromov-Witten operators based on a Grothendieck-Riemann-Roch calculation of Mumford [11]. A formalism for expressing the higher genus invariants of X in terms of genus

0 Frobenius structures via [11, 16] has been developed by Givental [15, 22]. An outcome, for example, is Givental's proof of the Virasoro constraints for the Gromov-Witten theory of $\mathbb{P}^n$.

If the dimension of X is 3, calculations in higher genus can be made much more effective. The required Hodge integrals have been treated with increasing sophistication. The culmination has been the topological vertex [1] in the local Calabi-Yau toric case and the equivariant vertex [28, 29] for arbitrary 3-folds with strong torus actions. Neither vertex evaluation of 3-fold Gromov-Witten theory has yet a complete mathematical proof, see [1, 26, 28, 29, 37].

0.4. **New directions.** If we leave the genus 0 or strong torus action realm, very few calculations have been completed.

The Gromov-Witten invariants of all 1-dimensional targets have been fully determined in [32, 33, 34]. The method uses a mix of localization, degeneration, and exact evaluations. Similarly, the local theory of curves in 3-folds has been solved in [5]. The results of [30] present a topological view of Gromov-Witten calculations in all dimensions: the familiar cutting and pasting strategies in the topological category are shown to yield effective Gromov-Witten techniques.

The Virasoro constraints of Eguchi-Hori-Yang [9] and S. Katz are conjectured to hold for the descendent theories of *all* target varieties X. Expositions can be found in [13, 39]. We find the Virasoro constraints to be useful and non-trivial tools. The results here which depend on the Virasoro constraints are clearly indicated.

0.5. **Plan.** The paper starts with a discussion of our topological view of Gromov-Witten theory in Section 1. Applications to surfaces of general type are considered first. The universal framework of Taubes and Lee-Parker [21] is summarized in Section 2.2. Our conjectural formulas based on calculations of the Gromov-Witten theory of branched double covers of $\mathbb{P}^2$ are presented in Section 2.4. Section 3 is devoted to Enriques geometries. The rich Gromov-Witten theory of the Enriques surface is used to study the holomorphic anomaly equation for the Enriques Calabi-Yau 3-fold.

0.6. **Acknowledgments.** Conversations with J. Bryan, T. Parker, and M. Usher played an important role in our surface calculations. Many of these occurred during a workshop on holomorphic curves at the Institute for Advanced Study in Princeton in June of 2005.

Our work on the Enriques Calabi-Yau was largely motivated by a series of lectures by A. Klemm and M. Mariño on heterotic duality and the holomorphic anomaly equation for the Enriques geometry. The lectures (and most of our calculations) took place at a workshop on algebraic geometry and topological strings at the Instituto Superior Técnico in Lisbon in the fall of 2005.

We thank I. Dolgachev for his immediate answers to all of our questions about the classical geometry of the Enriques surface.

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1. Topological view

1.1. **Mayer-Vietoris.** Let X be a nonsingular projective variety. We would like to study Gromov-Witten theory by decomposing X into simpler pieces.

Degeneration may be viewed as an algebraic version of cutting and pasting. Let

$$\epsilon : \mathcal{X} \to \Delta$$

be a flat family over a disk $\Delta \subset \mathbb{C}$ at the origin satisfying:

(i) $\mathcal{X}$ is nonsingular,
(ii) ϵ is smooth over the punctured disk $\Delta^* = \Delta \setminus \{0\}$,
(iii) $\epsilon^{-1}(1) \cong X$,
(iv) $\epsilon^{-1}(0) = X_1 \cup_Y X_2$ is a normal crossings divisor in $\mathcal{X}$.

The family ϵ defines a canonical map

$$H^*(X_1 \cup_Y X_2, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

with image defined to be the *nonvanishing* cohomology of X.

Let $\mathsf{GW}(X)$ denote the descendent Gromov-Witten theory of the target X — the complete set of integrals (1) — and let $\mathsf{GW}(X)_\epsilon$ denote the Gromov-Witten theory of descendents of the nonvanishing cohomology. A Mayer-Vietoris result in Gromov-Witten theory is proven for the family $\mathcal{X}$ in [30].

Theorem 1. $\mathsf{GW}(X)_\epsilon$ *can be uniquely and effectively reconstructed from*

$$\mathsf{GW}(X_1),\ \mathsf{GW}(X_2),\ \mathsf{GW}(Y),$$

and the restriction maps

$$H^*(X_1, \mathbb{Q}) \to H^*(Y, \mathbb{Q}) \leftarrow H^*(X_2, \mathbb{Q}).$$

A degeneration $\epsilon : \mathcal{X} \to \triangle$ satisfying conditions (i)-(iv) above will be called *good*. Theorem 1 requires a good degeneration.

1.2. **Surfaces branched over $\mathbb{P}^2$.** Let S be a nonsingular projective surface which admits a good degeneration to $S_1 \cup_C S_2$. Since $\mathsf{GW}(C)$ has been completely determined in [34], Theorem 1 is particularly applicable.

Surfaces constructed as branched double covers of $\mathbb{P}^2$ provide a basic class of examples. Let

$$S_{2n} \to \mathbb{P}^2$$

be a double cover branched along a nonsingular curve B of degree $2n$. If $n \geq 4$, then S_{2n} is of general type.

Let C be a nonsingular plane curve of degree n generic with respect to B. By degenerating the branch curve B to the square of C, we can construct a 1-parameter family $\mathcal{F}$ of surfaces:

$$\mathcal{F} \to \mathbb{P}^2 \times \mathbb{C}$$

is the double cover along

$$(tB - C^2) \subset \mathbb{P}^2 \times \mathbb{C}$$

where t is the parameter on $\mathbb{C}$. The total space $\mathcal{F}$ has double point singularities above the $2n^2$ points of $B \cap C$. Taking the small resolution, we obtain a nonsingular space

$$\epsilon : \mathcal{S} \to \mathbb{C}$$

which provides a degeneration of S_{2n} to

$$\tilde{\mathbb{P}}^2 \cup_C \mathbb{P}^2$$

where $\tilde{\mathbb{P}}^2$ is the blow-up of $\mathbb{P}^2$ along $B \cap C$.

The cohomology of S_{2n} pulled-back from $\mathbb{P}^2$ is certainly nonvanishing for ϵ. Since the descendent theories $\mathsf{GW}(\tilde{\mathbb{P}}^2)$ and $\mathsf{GW}(\mathbb{P}^2)$ are accessible via various methods (localization, Virasoro, Frobenius structures), Theorem 1 provides an effective approach to $\mathsf{GW}(S_{2n})_\epsilon$.

1.3. **The Enriques surface.** Let σ_{K3} be a fixed point free involution of a K3 surface. By definition, the quotient

$$X = K3/\langle \sigma_{K3} \rangle$$

is an *Enriques surface.* Alternatively, Enriques surfaces arise as elliptic fibrations

$$X \to \mathbb{P}^1 \tag{2}$$

with 12 singular fibers and 2 double fibers.

Let $E \times \mathbb{P}^1$ be the product of an elliptic curve and a projective line. Let t_E denote translation on E by a 2-torsion point, and let $\mathrm{inv}_{\mathbb{P}^1}$ denote an involution on the projective line. Then,

$$\tau : E \times \mathbb{P}^1 \to E \times \mathbb{P}^1$$

acting by $(\mathrm{t}_E, \mathrm{inv}_{\mathbb{P}^1})$ is a fixed point free involution. Let

$$X_1 = \left(E \times \mathbb{P}^1\right) / \langle \tau \rangle$$

denote the quotient. By projecting left,

$$X_1 \to E/\langle \mathrm{t}_E \rangle$$

is a projective bundle over the elliptic curve $E/\langle \mathrm{t}_E \rangle$. Hence $\mathsf{GW}(X_1)$ is determined by localization and [34]. By projecting right,

$$X_1 \to \mathbb{P}^1/\langle \mathrm{inv}_{\mathbb{P}^1} \rangle$$

is an elliptic fibration with no singular fibers and 2 double fibers.

Let $E(1)$ be the *rational elliptic surface* isomorphic to the blow-up of $\mathbb{P}^2$ in 9 points. The surface $E(1)$ admits an elliptic fibration

$$E(1) \to \mathbb{P}^1$$

with 12 singular fibers and no double fibers.

By degenerating the elliptic fibration (2), we find a good degeneration of the Enriques surface X to

$$X_1 \cup_E E(1)$$

where the intersection E is a common elliptic fiber. Since all the cohomology of X is nonvanishing for the degeneration, Theorem 1 provides an effective approach to the Gromov-Witten theory of the Enriques surface.

Since the $K3$ surface is holomorphic symplectic, $\mathsf{GW}(K3)$ is essentially trivial[1] with nonvanishing invariants only for constant maps in genus 0 and 1. The Enriques surface, however, will be seen to have a very rich Gromov-Witten theory.

[1]The $K3$ surface has a rich *modified* Gromov-Witten theory. The investigation of the modified Gromov-Witten theory of the $K3$ surface should also be considered to be outside of the genus 0 and toric realm. Calculations in primitive and twice primitive classes can be found in [4, 20]. At present, the modified virtual class is not amenable to the techniques discussed here.

1.4. **The Enriques Calabi-Yau.** Let σ act freely on the product $K3 \times E$ by an Enriques involution σ_{K3} on the $K3$ and by -1 on the elliptic curve. By definition, the quotient

$$Q = (K3 \times E) / \langle \sigma \rangle$$

is an *Enriques Calabi-Yau 3-fold*. Since $K3 \times E$ carries a holomorphic 3-form invariant under σ,

$$K_Q = 0.$$

By projection on the right,

$$Q \to E/\langle -1 \rangle = \mathbb{P}^1 \tag{3}$$

is a $K3$ fibration with 4 double Enriques fibers.

Let τ act freely on the product $K3 \times \mathbb{P}^1$ by $(\sigma_{K3}, \mathrm{inv}_{\mathbb{P}^1})$. Let

$$R = \left(K3 \times \mathbb{P}^1\right) / \langle \tau \rangle$$

denote the quotient. By projecting left,

$$R \to K3/\langle \sigma_{K3} \rangle = X$$

is a projective bundle over the Enriques surface X. Hence $\mathsf{GW}(R)$ is determined by localization and $\mathsf{GW}(X)$. By projecting right,

$$R \to \mathbb{P}^1/\langle \mathrm{inv}_{\mathbb{P}^1} \rangle$$

is a K3 fibration with 2 double Enriques fibers.

By degenerating the $K3$ fibration (3), we find a good degeneration of the Enriques Calabi-Yau Q to

$$R \cup_{K3} R$$

where the intersection $K3$ is a common fiber. Since all the cohomology of Q is nonvanishing, Theorem 1 provides an effective approach to the Gromov-Witten theory of the Enriques Calabi-Yau.

1.5. **Absolute/Relative.** Relative Gromov-Witten theory plays an essential role in Theorem 1. We review standard notation for relative invariants.

Let (V, W) be a nonsingular projective variety V containing a nonsingular divisor W. Let $\beta \in H_2(V, \mathbb{Z})$ be a curve class satisfying

$$\int_\beta [W] \geq 0.$$

Let $\vec{\mu}$ be an ordered partition,

$$\sum_j \mu_j = \int_\beta [W],$$

with positive parts. The moduli space $\overline{M}_{g,n}(V/W,\beta,\vec{\mu})$ parameterizes stable relative maps from genus g, n-pointed curves to V of class β with multiplicities along W determined by $\vec{\mu}$.

The relative conditions in the theory correspond to partitions *weighted* by the cohomology of W. Let $\delta_1,\ldots,\delta_{m_W}$ be a basis of $H^*(W,\mathbb{Q})$. A cohomology weighted partition ν consists of an *unordered* set of pairs,

$$\Big\{(\nu_1,\delta_{s_1}),\ldots,(\nu_{\ell(\nu)},\delta_{s_{\ell(\nu)}})\Big\},$$

where $\sum_j \nu_j$ is an *unordered* partition of $\int_\beta [W]$. The automorphism group, $\mathrm{Aut}(\nu)$, consists of permutation symmetries of ν.

The *standard* order on the parts of ν is

$$(\nu_i,\delta_{s_i}) > (\nu_{i'},\delta_{s_{i'}})$$

if $\nu_i > \nu_{i'}$ or if $\nu_i = \nu_{i'}$ and $s_i > s_{i'}$. Let $\vec{\nu}$ denote the partition $(\nu_1,\ldots,\nu_{\ell(\nu)})$ obtained from the standard order.

The descendent Gromov-Witten invariants of the pair (V,W) are defined by integration against the virtual class of the moduli of maps. Let $\gamma_1,\ldots,\gamma_{m_V}$ be a basis of $H^*(V,\mathbb{Q})$, and let

$$(4)\quad \Big\langle \tau_{k_1}(\gamma_{l_1})\cdots\tau_{k_n}(\gamma_{l_n}) \ \Big|\ \nu \Big\rangle^{V/W}_{g,\beta} =$$

$$\frac{1}{|\mathrm{Aut}(\nu)|}\int_{[\overline{M}_{g,n}(V/W,\beta,\vec{\nu})]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_{l_i}) \cup \prod_{j=1}^{\ell(\nu)} \mathrm{ev}_j^*(\delta_{s_j}).$$

Here, the second evaluations,

$$\mathrm{ev}_j : \overline{M}_{g,n}(V/W,\beta,\vec{\nu}) \to W.$$

are determined by the relative points. The brackets (4) denote integration over the moduli of maps with *connected* domains.

Gromov-Witten invariants are defined (up to sign) for *unordered* weighted partitions ν. To fix the sign, the integrand on the right side requires an ordering. The ordering is corrected by the automorphism prefactor.

Given a good degeneration of X to $X_1 \cup_Y X_2$, the degeneration formula [10, 18, 23, 24] expresses $\mathsf{GW}(X)_\epsilon$ in terms of the relative Gromov-Witten theories of the pairs (X_1,Y) and (X_2,Y). Hence, Theorem 1 is a consequence of the following result of [30].

Theorem 2. *The relative Gromov-Witten theory of the pair* (V,W) *can be uniquely and effectively reconstructed from* $\mathsf{GW}(V)$*,* $\mathsf{GW}(W)$*, and the restriction map* $H^*(V,\mathbb{Q}) \to H^*(W,\mathbb{Q})$*.*

2. Surfaces of general type

2.1. **Seiberg-Witten theory.** Let S be a nonsingular, projective, minimal surface of general type with $p_g(S) > 0$. Let $K_S \in H_2(S, \mathbb{Z})$ be the canonical class of S, and let

$$g_K = K_S^2 + 1$$

be the adjunction genus in the canonical class. The moduli space $\overline{M}_{g_K}(S, K_S)$ has virtual dimension 0. Taubes has obtained the evaluation

$$\langle 1 \rangle^S_{g_K, K_S} = (-1)^{\chi(\mathcal{O}_S)}. \tag{5}$$

by a connection to Seiberg-Witten theory [17, 31, 40, 41, 42, 43]. Here, $\chi(\mathcal{O}_S)$ denotes the holomorphic Euler characteristic.

We will call a nonsingular, irreducible, canonical divisor $C \subset S$ a *Taubes curve.* If a Taubes curve exists, formula (5) has a simple geometric interpretation. By adjunction, the normal bundle of C in S is a square root of the canonical bundle of C,

$$\begin{aligned} \mathcal{O}_C(2C) &= K_S(C)|_C \\ &= K_C. \end{aligned} \tag{6}$$

The normal bundle is thus a *theta characteristic* of C and has a deformation invariant parity equal to $h^0(C, \mathcal{O}_C(C)) \mod 2$. The *sign* of the Taubes curve is defined by the parity of the normal bundle,

$$\sigma(C) = (-1)^{h^0(C, \mathcal{O}_C(C))}.$$

Lemma 1. *If $C \subset S$ is a Taubes curve, then $\langle 1 \rangle^S_{g_K, K_S}$ equals the sign of C.*

Proof. The cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

starts as

$$\begin{aligned} 0 \ \to \ & H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(C, \mathcal{O}_C(C)) \to \\ & H^1(S, \mathcal{O}_S) \xrightarrow{\phi} H^1(S, \mathcal{O}_S(C)) \to \ldots . \end{aligned}$$

Since C is a Taubes curve,

$$h^0(S, \mathcal{O}_S(C)) = h^0(S, K_S) = h^2(S, \mathcal{O}_S).$$

Hence, the parity of the normal bundle equals the parity of

$$\chi(\mathcal{O}_S) + \dim(\mathrm{Im}(\phi)).$$

The image $\mathrm{Im}(\phi)$ has dimension equal to the rank of the skew-symmetric form on $H^1(S, \mathcal{O}_S)$ defined by

$$H^1(S,\mathcal{O}_S)\times H^1(S,\mathcal{O}_S) \stackrel{(\phi,id)}{\to} H^1(S,K_S)\times H^1(S,\mathcal{O}_S) \stackrel{\cup}{\to} H^2(S,K_S) \stackrel{\sim}{=} \mathbb{C}.$$

The rank of a skew-symmetric form is even. □

Lemma 1 is well-known in various forms. Our purpose is to emphasize the connection to the theta characteristic.

2.2. **Vanishing and universality.** The Gromov-Witten invariants of S of *Severi* type are

$$\Big\langle \tau_0^n([p_S]) \Big\rangle^S_{g,\beta\neq 0},$$

where $[p_S] \in H^4(S,\mathbb{Z})$ is the point class. By results of Taubes, Gromov-Witten invariants of Severi type vanish in the *adjunction genus*

$$2g_\beta - 2 = (K_S + \beta)\cdot\beta$$

if $\beta \neq K_S$. The universal evaluation

$$\Big\langle 1 \Big\rangle^S_{g_K, K_S} = (-1)^{\chi(\mathcal{O}_S)}$$

is an example of nonvanishing.

Let S be a minimal surface of general type. If a Taubes curve $C \subset S$ exists, a stronger vanishing result is proven by J. Lee and T. Parker in [21]. The methods of [21] do not use Seiberg-Witten theory.

I. Vanishing: *The Gromov-Witten invariants*

$$\Big\langle \tau_{\alpha_1}(\gamma_1)\ldots\tau_{\alpha_n}(\gamma_n) \Big\rangle^S_{g,\beta\neq 0}$$

vanish if either $\beta \notin \mathbb{Z}K_S$ *or there exists an insertion satisfying*

$$\gamma_i \in H^{\geq 3}(S,\mathbb{Q}).$$

Constant maps are avoided in the vanishing statement. A complete discussion of $\beta = 0$ invariants of a surface can be found in [14].

II. Universality: $\mathsf{GW}(S)$ *is uniquely determined by the sign of the Taubes curve and the restriction map* $H^*(S,\mathbb{Q}) \to H^*(C,\mathbb{Q})$.

The universality II is slightly stronger than the result of [21] where the precise spin structure (6) is required. Since the sign is the only deformation invariant of a spin structure, we conjecture the stronger universality II.

2.3. **Local theory of surfaces.** We will assume the existence of a Taubes curve $C \subset S$. A sharper universality statement can be made for descendents of the even cohomology

$$H^0(S,\mathbb{Q}) \oplus H^2(S,\mathbb{Q}) \subset H^*(S,\mathbb{Q}).$$

of S. By vanishing, descendents of $H^4(S,\mathbb{Q})$ need not be considered.

II′. Universality: For $d > 0$,

$$\Big\langle \prod_{i=1}^n \tau_{\alpha_i}(D_i) \prod_{j=1}^m \tau_{\tilde{\alpha}_j}(1) \Big\rangle^S_{g,dK_S} =$$

$$d^n \prod_{i=1}^n (K_S \cdot D_i) \ \cdot \ \Big\langle \prod_{i=1}^n \tau_{\alpha_i}([p_C]) \prod_{j=1}^m \tau_{\tilde{\alpha}_j}(1) \Big\rangle^{C,\sigma(C)}_{g,d},$$

where $D_i \in H^2(S,\mathbb{Q})$ and $[p_C] \in H^2(C,\mathbb{Q})$ is the point class.

The brackets $\langle , \rangle^{C,\pm}_{g,d}$ refer to a *local* Gromov-Witten theory of curves in surfaces for which no explicit algebraic construction yet exists. The universality equation II′ may be taken to conjecturally *define* the local theory on the right.

2.4. **Conjectures.** We conjecture evaluations of the local theory of curves in surfaces

$$\Big\langle \prod_{i=1}^n \tau_{\alpha_i}([p_C]) \prod_{j=1}^m \tau_{\tilde{\alpha}_j}(1) \Big\rangle^{C,\pm}_{g,d} \tag{7}$$

for degrees $d = 1, 2$. The dimension constraint for the local theory (7) is

$$g - 1 - d(g_C - 1) + m = \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \tilde{\alpha}_j.$$

By the Virasoro conjecture for the Gromov-Witten theory of surfaces,[2] the insertions $\tau_{\tilde{\alpha}}(1)$ can be removed by universal relations.

We will restrict our attention to the insertions $\tau_\alpha([p_C])$. In degree 1, we conjecture

$$\Big\langle \prod_{i=1}^n \tau_{\alpha_i}([p_C]) \Big\rangle^{\bullet\, C,\pm}_{g,1} = \pm \prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{-\alpha_i}. \tag{8}$$

[2]The Virasoro conjecture is open, but partial results for the local theory of surfaces have been obtained by A. Gholampour.

In degree 2, we conjecture

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}([p_C]) \right\rangle_{g,2}^{\bullet\, C,\pm} = \pm\, 2^{g_C+n-1} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}. \tag{9}$$

The superscripted bullet here denotes Gromov-Witten invariants with possibly disconnected domains *with no degree 0 components*. The heuristic origins of formulas (8) and (9) are based on relationships to exact evaluations in the Gromov-Witten theory of $\mathbb{P}^1$. A discussion will be presented in [27].

Our main evidence for the degree 1 and 2 formulas is obtained from calculations in the Gromov-Witten theory of S_{2n} based on the strategy of Section 1.2. The computations are rather labor intensive — very similar to the quintic surface computations of [30] — and will be omitted here.

For readers seeking worked examples of the method of [30], the 3-fold calculations of Section 3 will be explained in detail.

2.5. **Speculations.** The full evaluation of the local theory

$$\left\langle \ , \ \right\rangle_{g,d}^{C,\pm}$$

of curves in surfaces is not yet known. The Gromov-Witten theory of curves is related to gauge theory for the symmetric group via the Gromov-Witten/Hurwitz correspondence of [32]. The local Gromov-Witten theory of curves in 3-folds is related to $U(1)$-gauge theory via the Gromov-Witten/Donaldson-Thomas correspondence [5, 36]. The local Gromov-Witten theory of curves in surfaces sits in between. We expect the local theory of curves in surfaces to be related to a descendent Donaldson theory of sheaves on surfaces.

3. The Enriques Calabi-Yau 3-fold

3.1. Fiber classes.

3.1.1. *2-torsion.* Let X be the Enriques surface. The middle homology of X has a 2-torsion factor. The torsion-free quotient,

$$H_2(X,\mathbb{Z})' \cong H_2(X,\mathbb{Z})/\mathbb{Z}_2,$$

has rank 10. The intersection pairing $\langle , \rangle$ on $H_2(X,\mathbb{Z})'$ is isomorphic to the quadratic form $U \oplus E_8(-1)$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

is the (negative) Cartan matrix. A good reference for the classical geometry of X is [7].

We will view the curve classes of X as lying in $H_2(X,\mathbb{Z})'$. More precisely, the integral

$$\Big\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\Big\rangle^X_{g,\beta} \tag{10}$$

with $\beta \in H_2(X,\mathbb{Z})'$ is defined to be the *sum* of the integrals associated to the two lifts of β to $H_2(X,\mathbb{Z})$.

Let Q be the Enriques Calabi-Yau 3-fold defined in Section 1.4,

$$Q = (K3 \times E)\,/\langle\sigma\rangle \tag{11}$$

Projection to the first factor,

$$\pi_X : Q \to X,$$

is an elliptic fibration (with all fibers isomorphic to E). The homology mod torsion also projects,

$$\pi_{X*} : H_2(Q,\mathbb{Z})' \to H_2(X,\mathbb{Z})'.$$

Since π_X has a 0 section s_0, there is decomposition

$$H_2(Q,\mathbb{Z})' = H_2(X,\mathbb{Z})' \oplus \mathbb{Z}[E],$$

where $[E]$ is class of the fiber of π_X. The Gromov-Witten invariant

$$N_{g,(\beta,d)} = \langle 1\rangle^Q_{g,(\beta,d)}$$

is defined by summation over all lifts of $(\beta,d) \in H_2(Q,\mathbb{Z})'$ following the convention of (10).

The curve class 0 invariants $N_{g,(0,0)}$ vanish in genus 0 and 1 since the associated moduli spaces of stable maps are empty.

3.1.2. *The projection* $\pi_{\mathbb{P}^1}$. Projection to the second factor of (11),

$$\pi_{\mathbb{P}^1} : Q \to \mathbb{P}^1,$$

is a $K3$ fibration with 4 double Enriques fibers. The homology classes $(\beta, 0)$ of Q project to 0 under $\pi_{\mathbb{P}^1 *}$. The invariants $N_{g,(\beta,0)}$ are the *fiber class* invariants of Q with respect to $\pi_{\mathbb{P}^1}$.

Lemma 2. *The fiber class invariants of Q are Hodge integrals in the Gromov-Witten theory of the Enriques surface X,*

$$N_{g,(\beta,0)} = 4\Big\langle (-1)^{g-1}\lambda_{g-1}\Big\rangle^X_{g,\beta}.$$

Proof. We use the good degeneration of Q to $R \cup_{K3} R$ discussed in Section 1.4. By the degeneration formula,

$$N_{g,(\beta,0)} = N^{R/K3}_{g,(\beta,0)} + N^{R/K3}_{g,(\beta,0)}, \tag{12}$$

where the superscript $R/K3$ denotes the relative invariants of the pair.

By degeneration to the normal cone of $K3 \subset R$,

$$N^{R}_{g,(\beta,0)} = N^{R/K3}_{g,(\beta,0)} + N^{K3\times\mathbb{P}^1/K3}_{g,(\tilde{\beta},0)}, \tag{13}$$

where the relative divisor

$$K3 \subset K3 \times \mathbb{P}^1$$

is a section. Here, $\tilde{\beta}$ stands for all classes pushing forward to β. By the Leray-Hirsch result of [30] and the triviality of $\mathsf{GW}(K3)$,

$$N^{K3\times\mathbb{P}^1/K3}_{g,(\tilde{\beta},0)} = 0.$$

By localization,

$$N^{R}_{g,(\beta,0)} = 2\Big\langle (-1)^{g-1}\lambda_{g-1}\Big\rangle^X_{g,\beta}. \tag{14}$$

The Lemma is obtained by combining (12)-(14). □

Lemma 2 yields a vanishing result for the fiber class invariants of Q,

$$N_{0,(\beta,0)} = 0, \tag{15}$$

in genus 0.

3.1.3. *Heterotic string.* Klemm and Mariño have determined the fiber class invariants $N_{g,(\beta,0)}$ in terms of modular forms by heterotic string calculations [19]. Precise formulas will be discussed in Section 3.2.5. By Lemma 2, the fiber class results of [19] may be viewed as computing λ_{g-1} Hodge integrals in the Gromov-Witten theory of the Enriques surface X.

3.2. **Hodge integrals on the Enriques surface for $g \leq 2$.**

3.2.1. *Overview.* We will study here the fiber class invariants of Q,

$$N_{0,(\beta,0)}, \ N_{1,(\beta,0)}, \ N_{2,(\beta,0)},$$

via Gromov-Witten theory X. In genus 0, vanishing has already been obtained (15). In genus 1,

$$N_{1,(\beta,0)} = 4\langle 1\rangle^X_{1,\beta}$$

by Lemma 2, and Hodge insertions on X do not arise. In genus 2,

$$N_{2,(\beta,0)} = -4\langle \lambda_1\rangle^X_{2,\beta},$$

The required genus 1 and 2 invariants of X will be calculated by a combination of techniques.

3.2.2. *Isotropic classes.* Let X be an Enriques surface presented as an elliptic fibration

$$f : X \to \mathbb{P}^1$$

with 12 singular fibers and 2 double fibers. Let

$$F_f \in H_2(X, \mathbb{Z})'$$

denote half the class of the general fiber of f. A class $\beta \in H_2(X, \mathbb{Z})'$ is *positive* if either

$$\langle F_f, \beta\rangle > 0$$

or β is a positive multiple of F_f.

The classes of $H_2(X, \mathbb{Z})'$ represented by algebraic curves are *effective.* Effective classes must be positive.

A *primitive* class in $H_2(X, \mathbb{Z})'$ is nonzero and not divisible. Let $F \in H_2(X, \mathbb{Z})'$ be a positive, primitive, isotropic class. By the classical theory of Enriques surfaces [7], the class $2F$ is the fiber of an elliptic pencil

$$f_F : X \to \mathbb{P}^1$$

with 12 singular fibers and 2 double fibers.[3] We will compute the invariants $\langle \lambda_{g-1}\rangle^X_{g,nF}$ via the good degeneration of f_F to $X_1 \cup_E E(1)$ discussed in Section 1.3.

There are two cases to consider. If n is odd, the degeneration formula yields

$$\langle \lambda_{g-1}\rangle^X_{g,nF} = \langle \lambda_{g-1}\rangle^{X_1/E}_{g,nF}$$

since nF is not represented on $E(1)$. If n is even, then

$$\langle \lambda_{g-1}\rangle^X_{g,nF} = \langle \lambda_{g-1}\rangle^{X_1/E}_{g,nF} + \langle \lambda_{g-1}\rangle^{E(1)/E}_{g,nF}.$$

[3]We assume here X is generic in the moduli of Enriques surfaces.

There is a good degeneration of an elliptically fibered $K3$ surface to $E(1) \cup_E E(1)$. Hence, for n even,

$$\langle \lambda_{g-1} \rangle^{K3}_{g,nF} = \langle \lambda_{g-1} \rangle^{E(1)/E}_{g,nF} + \langle \lambda_{g-1} \rangle^{E(1)/E}_{g,nF}.$$

By the vanishing of Gromov-Witten invariants of the $K3$ surface,

$$\langle \lambda_{g-1} \rangle^{E(1)/E}_{g,nF} = 0.$$

We conclude

$$\langle \lambda_{g-1} \rangle^{X}_{g,nF} = \langle \lambda_{g-1} \rangle^{X_1/E}_{g,nF}$$

for all n

By degeneration to the normal cone of $E \subset X_1$,

$$\langle \lambda_{g-1} \rangle^{X_1}_{g,nF} = \langle \lambda_{g-1} \rangle^{X_1/E}_{g,nF} + \langle \lambda_{g-1} \rangle^{E\times\mathbb{P}^1/E}_{g,nF}. \tag{16}$$

As in the previous paragraph, the second term on the right of (16) is absent in the n odd case.

Localization may be applied to the Gromov-Witten invariants of X_1,

$$\begin{aligned} \langle \lambda_{g-1} \rangle^{X_1}_{g,nF} &= 2\langle (-1)^{g-1} \lambda_{g-1} \rangle^{E}_{g,n} \\ &= 2\sigma_{-1}(n)\ \delta_{g,1}. \end{aligned}$$

Here,

$$\sigma_{-1}(n) = \sum_{i|n} \frac{1}{i}.$$

Similarly, if n is even,

$$\langle 1 \rangle^{E\times\mathbb{P}^1/E}_{1,nF} = \sigma_{-1}(n/2)\ \delta_{g,1}.$$

Equation (16) then yields the following results.

Lemma 3. *The Gromov-Witten invariants of X in positive isotropic classes are determined by:*

$$\begin{aligned} &\langle 1 \rangle^{X}_{1,nF} = 2\sigma_{-1}(n) && n \text{ odd}, \\ &\langle 1 \rangle^{X}_{1,nF} = 2\sigma_{-1}(n) - \sigma_{-1}(n/2) && n \text{ even}. \end{aligned}$$

Lemma 4. *The fiber class invariants $N_{g,(\beta,0)}$ of Q vanish for nonzero isotropic classes β if $g \geq 2$.*

3.2.3. *Genus 1 invariants of the Enriques surface.* We derive a relation which determines all genus 1 invariants of X using localization equations and the (as yet unproven) Virasoro constraints for X.

Consider the 3-fold $Y = X \times \mathbb{P}^1$. The torus $\mathbb{C}^*$ acts on Y via $\mathbb{P}^1$. Let

$$[X_0], [X_\infty] \in H^*_{\mathbb{C}^*}(Y, \mathbb{Q})$$

denote the classes of the fibers of X over the $\mathbb{C}^*$-fixed points of $\mathbb{P}^1$ with tangent weights 1 and -1 respectively. Let $\beta \in H_2(X, \mathbb{Z})'$ be a nonzero class. Certainly,

$$\Big\langle \tau_1([X_0]^2) \Big\rangle^Y_{2,(\beta,1)} = 0 \tag{17}$$

Calculation of (17) via localization will yield a nontrivial equation in the Gromov-Witten theory of X.

A straightforward application of the virtual localization formula [16] yields

$$\Big\langle \tau_1([X_0]^2) \Big\rangle^Y_{2,(\beta,1)} = 2\Big\langle \tau_2 \Big\rangle^X_{2,\beta} - 4\Big\langle \lambda_1 \Big\rangle^X_{2,\beta} - \sum_{\beta_1+\beta_2=\beta} \Big\langle 1 \Big\rangle^X_{1,\beta_1} \Big\langle 1 \Big\rangle^X_{1,\beta_1} \langle \beta_1, \beta_2 \rangle.$$

The second term on the right side can be evaluated via the Hodge removal equation of [11],

$$\Big\langle \lambda_1 \Big\rangle^X_{2,\beta} = \frac{1}{12}\Big\langle \tau_2 \Big\rangle^X_{2,\beta} + \frac{1}{24}\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \beta \rangle + \frac{1}{24} \sum_{\beta_1+\beta_2=\beta} \Big\langle 1 \Big\rangle^X_{1,\beta_1} \Big\langle 1 \Big\rangle^X_{1,\beta_2} \langle \beta_1, \beta_2 \rangle.$$

The first term on the right side can be evaluated by the Virasoro constraints,

$$\frac{3}{4}\Big\langle \tau_2 \Big\rangle^X_{2,\beta} = \frac{1}{8}\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \beta \rangle + \frac{1}{8} \sum_{\beta_1+\beta_2=\beta} \Big\langle 1 \Big\rangle^X_{1,\beta_1} \Big\langle 1 \Big\rangle^X_{1,\beta_2} \langle \beta_1, \beta_2 \rangle,$$

see [9, 13, 39]. Equation (17) then implies the following result.[4]

Proposition 1. *For all nonzero* $\beta \in H_2(X, \mathbb{Z})'$,

$$\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \beta \rangle = 8 \sum_{\beta_1+\beta_2=\beta} \Big\langle 1 \Big\rangle^X_{1,\beta_1} \Big\langle 1 \Big\rangle^X_{1,\beta_2} \langle \beta_1, \beta_2 \rangle.$$

[4]The derivation depends upon the conjectural Virasoro constraints for X.

The sum on the right in Proposition 1 is taken over all nontrivial decompositions of β into effective classes β_i on X. Effective decomposition defines a partial ordering on the set of effective classes which has no infinite descending chains. Proposition 1 therefore uniquely determines the genus 1 Gromov-Witten theory of X in terms of the isotropic invariants

$$\left\langle 1 \right\rangle_{1,nF}^{X}$$

calculated in Lemma 3.

Lemma 5. *If $\langle \beta, \beta \rangle < 0$, then $\langle 1 \rangle_{1,\beta}^{X} = 0$.*

Proof. Let β be an effective curve class on X satisfying $\langle \beta, \beta \rangle < 0$. Let

$$\beta_1 + \beta_2 = \beta$$

be a decomposition into effective classes. Let H be an ample class on X. Since the intersection form on $H_2(X, \mathbb{R})$ has signature $(1, 9)$, the form is negative definite on $H^{\perp}$. Let

$$\beta_1 = h_1 H + N_1, \quad \beta_2 = h_2 H + N_2$$

where $N_i \in H^{\perp}$. Since the classes β_i are effective, $h_i > 0$. Since β has negative square,

$$(h_1 + h_2)\sqrt{\langle H, H \rangle} < \sqrt{-\langle N_1 + N_2, N_1 + N_2 \rangle}.$$

By the triangle inequality,

$$h_i \sqrt{\langle H, H \rangle} < \sqrt{-\langle N_i, N_i \rangle}$$

must hold for either β_1 or β_2. Hence, either $\langle \beta_1, \beta_1 \rangle < 0$ or $\langle \beta_2, \beta_2 \rangle < 0$. The Lemma is then obtained by induction on the partial ordering. $\square$

By Lemma 5, we may rewrite Proposition 1 purely in terms of the intersection form on $H_2(X, \mathbb{Z})'$ — without regard to effectivity.

Proposition 1′. *For all nonzero $\beta \in H_2(X, \mathbb{Z})'$,*

$$\left\langle 1 \right\rangle_{1,\beta}^{X} \langle \beta, \beta \rangle = 8 \sum_{\beta_1 + \beta_2 = \beta} \left\langle 1 \right\rangle_{1,\beta_1}^{X} \left\langle 1 \right\rangle_{1,\beta_2}^{X} \langle \beta_1, \beta_2 \rangle$$

where the sum is over decompositions into positive classes satisfying $\langle \beta_i, \beta_i \rangle \geq 0$.

3.2.4. *Genus 2 fiber classes.* The Hodge removal and Virasoro equations of Section 3.2.3 yield

$$\left\langle \lambda_1 \right\rangle_{2,\beta}^X = \frac{1}{18}\left\langle 1 \right\rangle_{1,\beta}^X \langle \beta, \beta \rangle + \frac{1}{18} \sum_{\beta_1+\beta_2=\beta} \left\langle 1 \right\rangle_{1,\beta_1}^X \left\langle 1 \right\rangle_{1,\beta_2}^X \langle \beta_1, \beta_2 \rangle.$$

After applying Proposition 1,

$$\left\langle \lambda_1 \right\rangle_{2,\beta}^X = \frac{1}{16}\left\langle 1 \right\rangle_{1,\beta}^X \langle \beta, \beta \rangle.$$

In terms of the invariants of Q,

$$N_{2,(\beta,0)} = -\frac{1}{16} N_{1,(\beta,0)} \langle \beta, \beta \rangle. \tag{18}$$

The Eisenstein series E_{2n} is the modular form defined by the equation

$$-\frac{B_{2n}}{4n} E_{2n}(q) = -\frac{B_{2n}}{4n} + \sum_{k\geq 1} \sigma_{2n-1}(k) q^k,$$

where $\sigma_n(k)$ is the sum of the n^{th} powers of the divisors of k,

$$\sigma_n(k) = \sum_{i|k} i^n.$$

The natural regularization of $\sigma_1(0)$ is

$$\sigma_1(0) = -\frac{B_2}{4} = -\frac{1}{24}.$$

We may rewrite (18) as

$$N_{2,(\beta,0)} = \frac{3}{2}\sigma_1(0) N_{1,(\beta,0)} \langle \beta, \beta \rangle. \tag{19}$$

3.2.5. *Modular forms.* Let $v_1, \dots v_{10} \in H_2(X, \mathbb{Z})'$ be a basis with

$$v_1, v_2 \in U, \quad v_3, \dots, v_{10} \in E_8(-1)$$

with respect to an identification

$$H_2(X, \mathbb{Z})' \stackrel{\sim}{=} U \oplus E_8(-1).$$

Let

$$v(t) = \sum_{i=1}^n t_i v_i$$

be coordinates in the basis. Since v_1 is a primitive isotropic class, positivity can be defined by intersection with v_1. A vector

$$\beta = \sum_{i=1}^n b_i v_i$$

is positive if $b_2 > 0$ *or* if $b_2 = 0$ and β is a positive multiple of v_1.

The fiber class potential function of the Enriques Calabi-Yau 3-fold is defined by

$$F_g(t) = \sum_{\beta > 0} N_{g,(\beta,0)} e^{-\langle \beta, v(t) \rangle}.$$

The heterotic string evaluation of F_g by Klemm and Mariño is

$$F_g(t) = \sum_{\beta>0} c_g\big(\langle \beta, \beta \rangle\big) \Big(2^{3-2g} \mathrm{Li}_{3-2g}(e^{-\langle \beta, v(t) \rangle}) - \mathrm{Li}_{3-2g}(e^{-2\langle \beta, v(t) \rangle}) \Big),$$

for $g > 0$.

The terms on the right side are explained as follows. The coefficients $c_g(n)$ are defined by

$$\sum_n c_g(n) q^n = -\frac{2}{q} \prod_{n=1}^{\infty} (1 - q^{2n})^{-12} \cdot \mathcal{P}_g(q).$$

Here, $\mathcal{P}_g(q)$ is the quasimodular form defined by

$$\sum_{g \geq 0} \mathcal{P}_g(q)\ z^g = \exp\left(\sum_{g=1}^{\infty} \frac{|B_{2g}|}{(2g)!} E_{2g}(q)\ z^g \right).$$

For example,

$$\mathcal{P}_1 = \frac{1}{12} E_2, \quad \mathcal{P}_2 = \frac{1}{1440}(5E_2^2 + E_4).$$

The polylogarithm Li_k is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

Proposition $1'$ should yield a coefficient relation for the various modular series in $F_1(t)$. We have not yet fully checked the compatibility. Many parallel properties hold. For example, equation (19) is valid for the Klemm-Mariño formula, see Section 4.1 of [19].

3.2.6. *Donaldson-Thomas theory.* By the GW/DT correspondence, we may instead study the Donaldson-Thomas theory of Q. The fiber Donaldson-Thomas theory of Q has a reduction to the classical cohomology of the Hilbert scheme of points of X. We expect the resulting vertex algebra calculations to be very closely related to the heterotic string results of [19].

3.3. **Vanishing in genus 0 and 1.** We now consider the Gromov-Witten theory of Q for all classes $(\beta, d) \in H_2(Q, \mathbb{Z})'$.

Proposition 2. $N_{0,(\beta,d)} = 0$.

Proof. If $d = 0$, the vanishing (15) has already been been obtained. We assume $d > 0$.

We use the good degeneration of Q to $R \cup_{K3} R$ discussed in Section 1.4. By the degeneration formula,

$$N_{0,(\beta,d)} = \sum_{\eta} \sum_{\beta_1+\beta_2=\beta} \Big\langle\, 1 \,\Big|\eta\Big\rangle^{\bullet R/K3}_{g_1,(\beta_1,d)} \Big\langle \eta^\vee \Big|\, 1 \,\Big\rangle^{\bullet R/K3}_{g_2,(\beta_2,d)} \tag{20}$$

where $\eta^\vee$ is the partition with cohomology weights Poincaré dual to the cohomology weights of η.

The left side of (20) is a connected invariant. The superscript $\bullet$ on the right side of (20) denotes disconnected invariants. After connecting the disconnected domains via the gluing conditions specified by $(\eta, \eta^\vee)$, a connected domain must be obtained. Also, the genus condition

$$g_1 + g_2 + \ell(\eta) - 1 = 0 \tag{21}$$

must be satisfied.

The relative invariant $\Big\langle\, 1 \,\Big|\eta\Big\rangle^{\bullet R/K3}_{0,(\beta_1,d)}$ can be calculated from the absolute invariants of R and $K3$ by Theorem 2 [30]. Since the Gromov-Witten invariants of R and $K3$ vanish in genus 0 for classes with nonzero push-forwards to X, we conclude

$$\Big\langle\, 1 \,\Big|\eta\Big\rangle^{\bullet R/K3}_{0,(\beta_1,d)} = 0$$

unless $\beta_1 = 0$. Hence, $N_{0,(\beta,d)}$ vanishes if $\beta \neq 0$.

If $\beta = 0$, the degeneration formula (20) yields a vanishing for a more subtle reason. The relative divisor $K3 \subset R$ intersects every fiber of

$$\pi : R \to X$$

in exactly 2 points. Hence, $\ell(\eta)$ must be at least twice the number of connected components of the domain in a nonvanishing invariant $\Big\langle\, 1 \,\Big|\eta\Big\rangle^{\bullet R/K3}_{0,(\beta_1,d)}$. Since each connected domain component must be of genus 0, the genus condition (21) can never be satisfied. □

In genus 1, a similar vanishing result holds for classes which are a proper mix of fiber and base curves.

Proposition 3. *For $\beta \neq 0$ and $d \neq 0$, $N_{1,(\beta,d)} = 0$.*

Proof. Consider the degeneration formula

$$N_{1,(\beta,d)} = \sum_{\eta} \sum_{\beta_1+\beta_2=\beta} \Big\langle\, 1 \,\Big|\eta\Big\rangle^{\bullet R/K3}_{g_1,(\beta_1,d)} \Big\langle \eta^\vee \Big|\, 1 \,\Big\rangle^{\bullet R/K3}_{g_2,(\beta_2,d)}. \tag{22}$$

If the connected components of the relative invariants all have domain genus 0, then $\beta = 0$ as in the proof of Proposition 2. If a connected component of genus 1 occurs, then the genus condition

$$g_1 + g_2 + \ell(\eta) - 1 = 1$$

can not be satisfied unless $d = 0$. □

Lemma 6. *For $d > 0$, $N_{1,(0,d)} = 12\sigma_{-1}(d)$.*

Proof. The Lemma may be obtained by an elementary evaluation of the degeneration formula (22) or from the Gromov-Witten theory of the fibration $\pi_X : Q \to X$. For the latter derivation,

$$\begin{aligned} N_{1,(0,d)} &= \int_{[\overline{M}_1(Q,d[E])]^{vir_\pi}} c_2(\mathbb{E}^\vee \otimes T_X) \\ &= 12 \int_{[\overline{M}_1(E,d[E])]^{vir}} 1 \\ &= 12\sigma_{-1}(d). \end{aligned}$$

Here, vir_π is the relative virtual class of the morphism π, and $\mathbb{E}$ is the Hodge bundle. □

Since the absolute Gromov-Witten theory of a target elliptic curve *E with trivial integrand* vanishing in genus $g \geq 2$, the above proof also yields the following result.

Proposition 4. *$N_{g,(0,d)} = 0$ for $g \geq 2$ and $d \geq 0$.*

3.4. **Holomorphic anomaly in genus 2.**

3.4.1. *Overview.* Our last topic is the Gromov-Witten theory of Q in genus 2. The strategy is to use again the good degeneration

$$Q \to R \cup_{K3} R$$

discussed in Section 1.4. Remarkably, only genus 1 invariants occur on right side of the degeneration formula.

The absolute genus 1 invariants of R are computed in Section 3.4.2. The relative genus 1 invariants of $R/K3$ required for the degeneration formula are then obtained from the absolute invariants in Section 3.4.3. The holomorphic anomaly equation arises naturally in our geometric study in Section 3.4.4.

3.4.2. *Absolute invariants.* We will require the explicit evaluations of several genus 1 invariants of R. Let

$$\iota : K3 \to R$$

denote the inclusion of the relative divisor, and let

$$\pi : K3 \to X$$

denote the induced projection. As in Section 3.2, the small brackets $\langle , \rangle$ denote the intersection pairing on X.

Lemma 7. *For* $\gamma \in H_2(K3, \mathbb{Q})$,

$$\Big\langle \tau_{2d-1}(\iota_*(\gamma)) \Big\rangle^R_{1,(\beta,d)} = \frac{2d}{(d!)^2} \Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma) \rangle.$$

Proof. By parallel localization arguments, we find

$$\Big\langle \tau_{2d-1}(\iota_*(\gamma)) \Big\rangle^R_{1,(\beta,d)} = \Big\langle \tau_{2d-1}(p) \Big\rangle^{E\times\mathbb{P}^1}_{1,(1,d)} \Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma) \rangle$$

where p is a point class on the surface $E \times \mathbb{P}^1$. By degenerating the elliptic factor E of the target to a nodal rational curve,

$$\Big\langle \tau_{2d-1}(p) \Big\rangle^{E\times\mathbb{P}^1}_{1,(1,d)} = 2 \Big\langle (1,p') \ \Big| \ \tau_{2d-1}(p) \ \Big| \ (1,1) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1_0\cup\mathbb{P}^1_\infty}_{0,(1,d)}$$

where p' is a point class of the relative divisor $\mathbb{P}^1_0$. The exchange of relative for absolute insertions takes a simple form here,

$$\Big\langle (1,p') \ \Big| \ \tau_{2d-1}(p) \ \Big| \ (1,1) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1_0\cup\mathbb{P}^1_\infty}_{0,(1,d)} = \Big\langle \tau_0(p')\tau_{2d-1}(p)\tau_0([\mathbb{P}^1_\infty]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(1,d)}.$$

We apply topological recursion relations to the result,

$$\Big\langle \tau_0(p')\tau_{2d-1}(p)\tau_0([\mathbb{P}^1_\infty]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(1,d)} =$$
$$\Big\langle \tau_{2d-2}(p)\tau_0([1,0]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(0,d)} \Big\langle \tau_0([0,1])\tau_0(p')\tau_0([0,1]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(1,0)}$$

where $[1,0], [0,1] \in H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$ denote the classes of the 2 rulings. The evaluations

$$\begin{aligned} \Big\langle \tau_{2d-2}(p)\tau_0([1,0]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(0,d)} &= d \Big\langle \tau_{2d-2}(p) \Big\rangle^{\mathbb{P}^1}_{0,d} \\ &= \frac{d}{(d!)^2} \end{aligned}$$

and

$$\Big\langle \tau_0([0,1]\tau_0(p')\tau_0([0,1]) \Big\rangle^{\mathbb{P}^1\times\mathbb{P}^1}_{0,(1,0)} = 1$$

complete the derivation.

The well-known evaluation of the genus 0 descendent $\langle\tau_{2d-2}(p)\rangle^{\mathbb{P}^1}_{0,d}$ used above can be found in [38]. $\square$

Almost identical arguments yield the evaluations of the following 2-point invariants.

Lemma 8. *Let $a_1 = 2m_1+1$ and $a_2 = 2m_2+1$ be odd integers satisfying $a_1 + a_2 = 2d$. For $\gamma_1, \gamma_2 \in H_2(K3, \mathbb{Q})$,*

$$\Big\langle \tau_{a_1-1}(\iota_*(\gamma_1))\tau_{a_2-1}(\iota_*(\gamma_2)) \Big\rangle^R_{1,(\beta,d)} = \frac{2}{(m_1!)^2(m_2!)^2}\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma_1)\rangle\langle\beta,\pi_*(\gamma_2)\rangle.$$

Lemma 9. *Let $a_1 = 2m_1$ and $a_2 = 2m_2$ be even integers satisfying $a_1 + a_2 = 2d$. For $\gamma_1, \gamma_2 \in H_2(K3, \mathbb{Q})$,*

$$\Big\langle \tau_{a_1-1}(\iota_*(\gamma_1))\tau_{a_2-1}(\iota_*(\gamma_2)) \Big\rangle^R_{1,(\beta,d)} = \frac{2mn}{(m_1!)^2(m_2!)^2}\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma_1)\rangle\langle\beta,\pi_*(\gamma_2)\rangle.$$

3.4.3. *Relative invariants.* The degeneration to the normal cone of $K3 \subset R$ can be applied to determine relative invariants from absolute invariants, see Theorem 2 [30].

Lemma 10. *For $\gamma \in H_2(K3, \mathbb{Q})$,*

$$\Big\langle 1 \ \Big|\ (2d, \gamma) \Big\rangle^{R/K3}_{1,(\beta,d)} = 2\Big\langle 1 \Big\rangle^X_{1,\beta} \langle\beta, \pi_*(\gamma)\rangle.$$

Proof. Let I_d denote the relative invariants to be determined,

$$I_d = \Big\langle 1 \ \Big|\ (2d, \gamma) \Big\rangle^{R/K3}_{1,(\beta,d)}.$$

Degeneration to the normal cone of $K3 \subset R$ yields

$$\Big\langle \tau_{2d-1}(\iota_*(\gamma)) \Big\rangle^R_{1,(\beta,d)} = I_d \ 2d \ \Big\langle (2d) \ \Big|\ \tau_{2d-1}(p) \Big\rangle^{\mathbb{P}^1}_{0,2d} + \sum_{r=1}^{d-1} I_r \ 2r \ \Big\langle (2r), (1)^{d-r} \ \Big|\ \tau_{2d-1}(p) \Big\rangle^{\mathbb{P}^1}_{0,d+r}.$$

The $K3 \times \mathbb{P}^1/K3$ side has been written in terms of the relative Gromov-Witten theory of a vertical $\mathbb{P}^1$ since $\mathsf{GW}(K3)$ is trivial. The coefficients

$$\Big\langle (2d) \ \Big| \ \tau_{2d-1}(p) \Big\rangle^{\mathbb{P}^1}_{0,2d} = \frac{1}{(2d)!},$$

$$\Big\langle (2r), (1)^{d-r} \ \Big| \ \tau_{2d-1}(p) \Big\rangle^{\mathbb{P}^1}_{0,d+r} = \frac{1}{(d+r)!(d-r)!}$$

are easily evaluated by completed cycles [32]. We find

$$I_d = 2\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma) \rangle$$

is the unique solution to the recursion

$$\frac{2d}{(d!)^2}\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma)\rangle = \frac{I_d}{(2d-1)!} + \sum_{r=1}^{d-1} \frac{2rI_r}{(d+r)!(d-r)!}$$

obtained from Lemma 7. □

A parallel (though more complicated) derivation from the evaluations of Lemmas 8 and 9 yields a second result.

Lemma 11. *For $\gamma_1, \gamma_2 \in H_2(K3, \mathbb{Q})$,*

$$\Big\langle 1 \ \Big| \ (d, \gamma_1), (d, \gamma_2) \ \Big\rangle^{R/K3}_{1,(\beta,d)} = 2\Big\langle 1 \Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma_1)\rangle \langle \beta, \pi_*(\gamma_2)\rangle.$$

Lemmas 10 and 11 have a remarkable property — the relative evaluations are independent of d.

3.4.4. *Holomorphic anomaly.* Since the invariant $N_{2,(\beta,d)}$ has been determined by Lemma 4 if $\beta = 0$ and by (19) if $d = 0$, we assume $\beta \neq 0$ and $d > 0$.

We calculate $N_{2,(\beta,d)}$ via the degeneration of Q to $R \cup_{K3} R$,

$$N_{2,(\beta,d)} = \sum_{\eta} \sum_{\beta_1+\beta_2=\beta} \Big\langle \ 1 \ \Big| \eta \Big\rangle^{\bullet R/K3}_{g_1,(\beta_1,d)} \Big\langle \eta^\vee \Big| \ 1 \ \Big\rangle^{\bullet R/K3}_{g_2,(\beta_2,d)}. \tag{23}$$

If a connected component of genus 2 occurs on the right side of degeneration formula, then all other components must be genus 0 vertical classes. Since each genus 0 vertical class must intersect the relative divisor at least twice, the genus condition

$$g_1 + g_2 + \ell(\eta) - 1 = 2$$

can not be satisfied.

Since $\beta \neq 0$, a connected component of genus 1 must occur on the right side of degeneration formula (23). There are exactly two possibilities:

(i) the degeneration graph has a single genus 1 component with a self node,

(ii) the degeneration graph has two genus 1 components.

Genus reduction is the hallmark of the holomorphic anomaly equation.

Consider first the degeneration terms of type (i). The elliptic component may occur on either side of $R \cup_{K3} R$. All other components must be genus 0 vertical curves fully ramified at the intersection points with the relative divisor. Once the side of the elliptic component is specified, the geometric configurations are easily seen to be in bijective correspondence with divisors of d. The term corresponding to the divisor r contributes

$$\begin{aligned} \sum_i \frac{r}{2}\Big\langle 1 \ \Big| \ (r,\gamma_i),(r,\gamma_i^\vee)\ \Big\rangle^{R/K3}_{1,(\beta,d)} &= \sum_i r\Big\langle 1\Big\rangle^X_{1,\beta} \langle \beta, \pi_*(\gamma_i)\rangle\langle\beta,\pi_*(\gamma_i^\vee)\rangle \\ &= 2r\Big\langle 1\Big\rangle^X_{1,\beta}\langle\beta,\beta\rangle, \end{aligned}$$

where the sum is over a basis $\{\gamma_1,\ldots,\gamma_{22}\}$ of $H_2(K3,\mathbb{Q})$. Lemma 11 is used for the first equality.

The full type (i) contribution to the degeneration formula (23) is

$$4\sigma_1(d)\ \Big\langle 1\Big\rangle^X_{1,\beta}\langle\beta,\beta\rangle, \tag{24}$$

counting both sides for the elliptic component.

The degeneration terms of type (ii) have a similar treatment. Again, a bijective correspondence with divisors of d is found. The full type (ii) contribution to (23) is

$$\sum_{\beta_1+\beta_2=\beta} 16\sigma_1(d)\ \Big\langle 1\Big\rangle^X_{1,\beta_1}\Big\langle 1\Big\rangle^X_{1,\beta_2}\langle\beta_1,\beta_2\rangle \tag{25}$$

using the evaluation of Lemma 10.

Summing the contributions and writing the result in terms of the fiber class Gromov-Witten invariants of Q by Lemma 2 yields the following result.

Theorem 3. *For $d > 0$,*

$$N_{2,(\beta,d)} = \sigma_1(d)\Big(N_{1,(\beta,0)}\langle\beta,\beta\rangle + \sum_{\beta_1+\beta_2=\beta} N_{1,(\beta_1,0)}N_{1,(\beta_2,0)}\langle\beta_1,\beta_2\rangle\Big).$$

Our proof of Theorem 3 does not invoke Proposition 1 and is thus *not* dependent upon the Virasoro constraints for the Enriques surface.

Theorem 3 may be interpreted as the holomorphic anomaly equation in genus 2 for the Enriques Calabi-Yau 3-fold. A discussion can be found in Sections 6.2.2 - 6.2.4 of [19]. In fact, Theorem 3 is used in [19] to fix the holomorphic ambiguity.

We may rewrite Theorem 3 using the fiber class results in genus 1 and 2 of Section 3.2. By Proposition 1,

$$N_{2,(\beta,d)} = \frac{3}{2}\sigma_1(d) N_{1,(\beta,0)} \langle \beta, \beta \rangle$$

for $d > 0$. By (19),

$$N_{2,(\beta,0)} = \frac{3}{2}\sigma_1(0) N_{1,(\beta,0)} \langle \beta, \beta \rangle.$$

We obtain the following result.[5]

Corollary 1. *We have*

$$\begin{aligned} \sum_{d \geq 0} N_{2,(\beta,d)} q^d &= -\frac{1}{16} E_2(q) N_{1,(\beta,0)} \langle \beta, \beta \rangle \\ &= E_2(q) N_{2(\beta,0)}. \end{aligned}$$

We have calculated the Gromov-Witten theory of Q in genus $g \leq 2$. We expect the Gromov-Witten theory is exactly solvable in all genera.

The Enriques Calabi-Yau 3-fold may be the most tractable compact Calabi-Yau with nontrivial Gromov-Witten theory. Certainly the higher genus study of the quintic 3-fold in $\mathbb{P}^4$ appears more difficult, see [12, 25, 30].

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