

# NOETHER-LEFSCHETZ THEORY AND THE YAU-ZASLOW CONJECTURE

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ABSTRACT. The Yau-Zaslow conjecture predicts the genus 0 curve counts of  $K3$  surfaces in terms of the Dedekind  $\eta$  function. The classical intersection theory of curves in the moduli of  $K3$  surfaces with Noether-Lefschetz divisors is related to 3-fold Gromov-Witten invariants via the  $K3$  curve counts. Results by Borchers and Kudla-Millson determine these classical intersections in terms of vector-valued modular forms. Proven mirror transformations can often be used to calculate the 3-fold invariants which arise.

Via a detailed study of the STU model (determining special curves in the moduli of  $K3$  surfaces), we prove the Yau-Zaslow conjecture for all curve classes on  $K3$  surfaces. Two modular form identities are required. The first, the Klemm-Lerche-Mayr identity relating hypergeometric series to modular forms after mirror transformation, is proven here. The second, the Harvey-Moore identity, is proven by D. Zagier and presented in the paper.

## CONTENTS

0. Introduction	2
0.1. Yau-Zaslow conjecture	2
0.2. Noether-Lefschetz theory	5
0.3. Three theories	10
0.4. Proof of Theorem 1	11
0.5. Acknowledgments	12
1. The STU model	13
1.1. Overview	13
1.2. Toric varieties	13
1.3. The toric 4-fold $Y$	15
1.4. Fibrations	16
1.5. Divisor restrictions	20
1.6. 1-parameter families	21
1.7. Gromov-Witten invariants	22
2. Noether-Lefschetz numbers and reduced $K3$ invariants	22

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2.1.	Refined Noether-Lefschetz numbers	22
2.2.	STU model	23
2.3.	BPS states	25
2.4.	Invertibility of constraints	26
2.5.	Proof of the Yau-Zaslow conjecture	28
3.	Mirror transform	29
3.1.	Picard-Fuchs	29
3.2.	Solutions	29
3.3.	Mirror transform for $q_3 = 0$	30
3.4.	B-model	33
4.	The Harvey-Moore identity	36
4.1.	Proof of Proposition 4	36
4.2.	Zagier's proof of the Harvey-Moore identity	36
	References	38

## 0. INTRODUCTION

**0.1. Yau-Zaslow conjecture.** Let  $S$  be a nonsingular projective  $K3$  surface, and let

$$\beta \in \text{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C})$$

be a nonzero effective curve class. The moduli space  $\overline{M}_0(S, \beta)$  of genus 0 stable maps (with no marked points) has expected dimension

$$\dim_{\mathbb{C}}^{\text{vir}}(\overline{M}_0(S, \beta)) = \int_{\beta} c_1(S) + \dim_{\mathbb{C}}(S) - 3 = -1.$$

Hence, the virtual class  $[\overline{M}_0(S, \beta)]^{\text{vir}}$  vanishes, and the standard Gromov-Witten theory is trivial.

Curve counting on  $K3$  surfaces is captured instead by the *reduced* Gromov-Witten theory constructed first via the twistor family in [6]. An algebraic construction following [1, 2] is given in [31]. Since the reduced class

$$[\overline{M}_0(S, \beta)]^{\text{red}} \in H_0(\overline{M}_0(S, \beta), \mathbb{Q})$$

has dimension 0, the reduced Gromov-Witten integrals of  $S$ ,

$$(1) \quad R_{0,\beta}(S) = \int_{[\overline{M}_0(S,\beta)]^{\text{red}}} 1 \in \mathbb{Q},$$

are well-defined. For deformations of  $S$  for which  $\beta$  remains a  $(1,1)$ -class, the integrals (1) are invariant.

The second cohomology of  $S$  is a rank 22 lattice with intersection form

$$(2) \quad H^2(S, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (2) is even.

The *divisibility*  $m(\beta)$  is the maximal positive integer dividing the lattice element  $\beta \in H^2(S, \mathbb{Z})$ . If the divisibility is 1,  $\beta$  is *primitive*. Elements with equal divisibility and norm are equivalent up to orthogonal transformation of  $H^2(S, \mathbb{Z})$ . By straightforward deformation arguments using the Torelli theorem for  $K3$  surfaces,  $R_{0,\beta}(S)$  depends, for effective classes, *only* on the divisibility  $m(\beta)$  and the norm  $\langle \beta, \beta \rangle$ . We will omit the argument  $S$  in the notation.

The genus 0 BPS counts associated to  $K3$  surfaces have the following definition. Let  $\alpha \in \text{Pic}(S)$  be a nonzero class which is both effective and primitive. The Gromov-Witten potential  $F_\alpha(v)$  for classes proportional to  $\alpha$  is

$$F_\alpha = \sum_{m>0} R_{0,m\alpha} v^{m\alpha}.$$

The BPS counts  $r_{0,m\alpha}$  are uniquely defined by via the Aspinwall-Morrison formula,

$$(3) \quad F_\alpha = \sum_{m>0} r_{0,m\alpha} \sum_{d>0} \frac{v^{dm\alpha}}{d^3},$$

for both primitive and divisible classes.

The Yau-Zaslow conjecture [36] predicts the values of the genus 0 BPS counts for the reduced Gromov-Witten theory of  $K3$  surfaces. We interpret the conjecture in two parts.

**Conjecture 1.** *The BPS count  $r_{0,\beta}$  depends upon  $\beta$  only through the norm  $\langle\beta, \beta\rangle$ .*

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory since  $R_{0,\beta}$  certainly depends upon the divisibility of  $\beta$ . Let  $r_{0,m,h}$  denote the genus 0 BPS count associated to a class  $\beta$  of divisibility  $m$  satisfying

$$\langle\beta, \beta\rangle = 2h - 2.$$

Assuming Conjecture 1 holds, we define

$$r_{0,h} = r_{0,m,h}$$

independent<sup>1</sup> of  $m$ .

**Conjecture 2.** *The BPS counts  $r_{0,h}$  are uniquely determined by*

$$(4) \quad \sum_{h \geq 0} r_{0,h} q^h = \prod_{n=1}^{\infty} (1 - q^n)^{-24}.$$

Conjecture 2 can be written in terms of the Dedekind  $\eta$  function

$$\sum_{h \geq 0} r_{0,h} q^{h-1} = \eta(\tau)^{-24}$$

where  $q = e^{2\pi i\tau}$ .

The conjectures have been previously proven in very few cases. A proof of the Yau-Zaslow formula (4) for primitive classes  $\beta$  via Euler characteristics of compactified Jacobians following [36] can be found in [3, 7, 11]. The Yau-Zaslow formula (4) was proven via Gromov-Witten theory for primitive classes  $\beta$  by Bryan and Leung [6]. An early calculation by Gathmann [13] for a class  $\beta$  of divisibility 2 was important for the correct formulation of the conjectures. Conjectures 1

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<sup>1</sup>Independence of  $m$  holds when  $2m^2$  divides  $2h - 2$ . Otherwise, no such class  $\beta$  exists and  $r_{0,m,h}$  is defined to vanish.

and 2 have been proven in the divisibility 2 case by Lee and Leung [26] and Wu [35]. The main result of the paper is a proof of Conjectures 1 and 2 in all cases.

**Theorem 1.** *The Yau-Zaslow conjecture holds for all nonzero effective classes  $\beta \in \text{Pic}(S)$  on a K3 surface  $S$ .*

## 0.2. Noether-Lefschetz theory.

0.2.1. *Lattice polarization.* Let  $S$  be a K3 surface. A primitive class  $L \in \text{Pic}(S)$  is a *quasi-polarization* if

$$\langle L, L \rangle > 0 \quad \text{and} \quad \langle L, [C] \rangle \geq 0$$

for every curve  $C \subset S$ . A sufficiently high tensor power  $L^n$  of a quasi-polarization is base point free and determines a birational morphism

$$S \rightarrow \tilde{S}$$

contracting A-D-E configurations of  $(-2)$ -curves on  $S$ . Hence, every quasi-polarized K3 surface is algebraic.

Let  $\Lambda$  be a fixed rank  $r$  primitive<sup>2</sup> embedding

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature  $(1, r-1)$ , and let  $v_1, \dots, v_r \in \Lambda$  be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix}.$$

The sign is chosen so  $\Delta(\Lambda) > 0$ .

A  $\Lambda$ -*polarization* of a K3 surface  $S$  is a primitive embedding

$$j : \Lambda \rightarrow \text{Pic}(S)$$

satisfying two properties:

- (i) the lattice pairs  $\Lambda \subset U^3 \oplus E_8(-1)^2$  and  $\Lambda \subset H^2(S, \mathbb{Z})$  are isomorphic via an isometry which restricts to the identity on  $\Lambda$ ,
- (ii)  $\text{Im}(j)$  contains a quasi-polarization.

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<sup>2</sup>An embedding of lattices is primitive if the quotient is torsion free.

By (ii), every  $\Lambda$ -polarized  $K3$  surface is algebraic.

The period domain  $M$  of Hodge structures of type  $(1, 20, 1)$  on the lattice  $U^3 \oplus E_8(-1)^2$  is an analytic open set of the 20-dimensional nonsingular isotropic quadric  $Q$ ,

$$M \subset Q \subset \mathbb{P}((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}).$$

Let  $M_{\Lambda} \subset M$  be the locus of vectors orthogonal to the entire sublattice  $\Lambda \subset U^3 \oplus E_8(-1)^2$ .

Let  $\Gamma$  be the isometry group of the lattice  $U^3 \oplus E_8(-1)^2$ , and let

$$\Gamma_{\Lambda} \subset \Gamma$$

be the subgroup restricting to the identity on  $\Lambda$ . By global Torelli, the moduli space  $\mathcal{M}_{\Lambda}$  of  $\Lambda$ -polarized  $K3$  surfaces is the quotient

$$\mathcal{M}_{\Lambda} = M_{\Lambda}/\Gamma_{\Lambda}.$$

We refer the reader to [10] for a detailed discussion.

0.2.2. *Families.* Let  $X$  be a compact 3-dimensional complex manifold equipped with holomorphic line bundles

$$L_1, \dots, L_r \rightarrow X$$

and a holomorphic map

$$\pi : X \rightarrow C$$

to a nonsingular complete curve.

The tuple  $(X, L_1, \dots, L_r, \pi)$  is a *1-parameter family of nonsingular  $\Lambda$ -polarized  $K3$  surfaces* if

- (i) the fibers  $(X_{\xi}, L_{1,\xi}, \dots, L_{r,\xi})$  are  $\Lambda$ -polarized  $K3$  surfaces via

$$v_i \mapsto L_{i,\xi}$$

for every  $\xi \in C$ ,

- (ii) there exists a  $\lambda^{\pi} \in \Lambda$  which is a quasi-polarization of all fibers of  $\pi$  simultaneously.

The family  $\pi$  yields a morphism,

$$\iota_{\pi} : C \rightarrow \mathcal{M}_{\Lambda},$$

to the moduli space of  $\Lambda$ -polarized  $K3$  surfaces.

Let  $\lambda^\pi = \lambda_1^\pi v_1 + \cdots + \lambda_r^\pi v_r$ . A vector  $(d_1, \dots, d_r)$  of integers is *positive* if

$$\sum_{i=1}^r \lambda_i^\pi d_i > 0.$$

If  $\beta \in \text{Pic}(X_\xi)$  has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then  $\beta$  has positive degree with respect to the quasi-polarization if and only if  $(d_1, \dots, d_r)$  is positive.

**0.2.3. Noether-Lefschetz divisors.** Noether-Lefschetz numbers are defined in [31] by the intersection of  $\iota_\pi(C)$  with Noether-Lefschetz divisors in  $\mathcal{M}_\Lambda$ . We briefly review the definition of the Noether-Lefschetz divisors.

Let  $(\mathbb{L}, \iota)$  be a rank  $r + 1$  lattice  $\mathbb{L}$  with an even symmetric bilinear form  $\langle, \rangle$  and a primitive embedding

$$\iota : \Lambda \rightarrow \mathbb{L}.$$

Two data sets  $(\mathbb{L}, \iota)$  and  $(\mathbb{L}', \iota')$  are isomorphic if there is an isometry which restricts to identity on  $\Lambda$ . The first invariant of the data  $(\mathbb{L}, \iota)$  is the discriminant  $\Delta \in \mathbb{Z}$  of  $\mathbb{L}$ .

An additional invariant of  $(\mathbb{L}, \iota)$  can be obtained by considering any vector  $v \in \mathbb{L}$  for which

$$(5) \quad \mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v.$$

The pairing

$$\langle v, \cdot \rangle : \Lambda \rightarrow \mathbb{Z}$$

determines an element of  $\delta_v \in \Lambda^*$ . Let  $G = \Lambda^*/\Lambda$  be quotient defined via the injection  $\Lambda \rightarrow \Lambda^*$  obtained from the pairing  $\langle, \rangle$  on  $\Lambda$ . The group  $G$  is abelian of order equal to the discriminant  $\Delta(\Lambda)$ . The image

$$\delta \in G/\pm$$

of  $\delta_v$  is easily seen to be independent of  $v$  satisfying (5). The invariant  $\delta$  is the *coset* of  $(\mathbb{L}, \iota)$

By elementary arguments, two data sets  $(\mathbb{L}, \iota)$  and  $(\mathbb{L}', \iota')$  of rank  $r + 1$  are isomorphic if and only if the discriminants and cosets are equal.

Let  $v_1, \dots, v_r$  be an integral basis of  $\Lambda$  as before. The pairing of  $\mathbb{L}$  with respect to an extended basis  $v_1, \dots, v_r, v$  is encoded in the matrix

$$\mathbb{L}_{h,d_1,\dots,d_r} = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2h - 2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \dots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\dots,d_r}).$$

The coset  $\delta(h, d_1, \dots, d_r)$  is represented by the functional

$$v_i \mapsto d_i.$$

The Noether-Lefschetz divisor  $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$  is the closure of the locus of  $\Lambda$ -polarized  $K3$  surfaces  $S$  for which  $(\text{Pic}(S), j)$  has rank  $r + 1$ , discriminant  $\Delta$ , and coset  $\delta$ . By the Hodge index theorem,  $P_{\Delta,\delta}$  is empty unless  $\Delta > 0$ .

Let  $h, d_1, \dots, d_r$  determine a positive discriminant

$$\Delta(h, d_1, \dots, d_r) > 0.$$

The Noether-Lefschetz divisor  $D_{h,(d_1,\dots,d_r)} \subset \mathcal{M}_\Lambda$  is defined by the weighted sum

$$D_{h,(d_1,\dots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \dots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}]$$

where the multiplicity  $m(h, d_1, \dots, d_r | \Delta, \delta)$  is the number of elements  $\beta$  of the lattice  $(\mathbb{L}, \iota)$  of type  $(\Delta, \delta)$  satisfying

$$(6) \quad \langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, v_i \rangle = d_i.$$

If the multiplicity is nonzero, then  $\Delta | \Delta(h, d_1, \dots, d_r)$  so only finitely many divisors appear in the above sum.

If  $\Delta(h, d_1, \dots, d_r) = 0$ , the divisor  $D_{h,(d_1,\dots,d_r)}$  has an alternate definition. The tautological line bundle  $\mathcal{O}(-1)$  is  $\Gamma$ -equivariant on the



period domain  $M_\Lambda$  and descends to the *Hodge line bundle*

$$\mathcal{K} \rightarrow \mathcal{M}_\Lambda.$$

We define  $D_{h,(d_1,\dots,d_r)} = \mathcal{K}^*$ . See [31] for an alternate view of degenerate intersection.

If  $\Delta(h, d_1, \dots, d_r) < 0$ , the divisor  $D_{h,(d_1,\dots,d_r)}$  on  $\mathcal{M}_\Lambda$  is defined to vanish by the Hodge index theorem.

0.2.4. *Noether-Lefschetz numbers.* Let  $\Lambda$  be a lattice of discriminant  $l = \Delta(\Lambda)$ , and let  $(X, L_1, \dots, L_r, \pi)$  be a 1-parameter family of  $\Lambda$ -polarized  $K3$  surfaces. The Noether-Lefschetz number  $NL_{h,d_1,\dots,d_r}^\pi$  is the classical intersection product

$$(7) \quad NL_{h,(d_1,\dots,d_r)}^\pi = \int_C \iota_\pi^* [D_{h,(d_1,\dots,d_r)}].$$

Let  $\mathrm{Mp}_2(\mathbb{Z})$  be the metaplectic double cover of  $SL_2(\mathbb{Z})$ . There is a canonical representation [4] associated to  $\Lambda$ ,

$$\rho_\Lambda^* : \mathrm{Mp}_2(\mathbb{Z}) \rightarrow \mathrm{End}(\mathbb{C}[G]).$$

The full set of Noether-Lefschetz numbers  $NL_{h,d_1,\dots,d_r}^\pi$  defines a vector valued modular form

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi_\gamma^\pi(q) v_\gamma \in \mathbb{C}[[q^{\frac{1}{2l}}]] \otimes \mathbb{C}[G],$$

of weight  $\frac{22-r}{2}$  and type  $\rho_\Lambda^*$  by results<sup>3</sup> of Borcherds and Kudla-Millson [4, 25]. The Noether-Lefschetz numbers are the coefficients<sup>4</sup> of the components of  $\Phi^\pi$ ,

$$NL_{h,(d_1,\dots,d_r)}^\pi = \Phi_\gamma^\pi \left[ \frac{\Delta(h, d_1, \dots, d_r)}{2l} \right]$$

where  $\delta(h, d_1, \dots, d_r) = \pm\gamma$ . The modular form results significantly constrain the Noether-Lefschetz numbers.

<sup>3</sup>While the results of the papers [4, 25] have considerable overlap, we will follow the point of view of Borcherds.

<sup>4</sup>If  $f$  is a series in  $q$ ,  $f[k]$  denotes the coefficient of  $q^k$ .

0.2.5. *Refinements.* If  $d_1, \dots, d_r$  do not simultaneously vanish, refined Noether-Lefschetz divisors are defined. If  $\Delta(h, d_1, \dots, d_r) > 0$ ,

$$D_{m,h,(d_1,\dots,d_r)} \subset D_{h,(d_1,\dots,d_r)}$$

is defined by requiring the class  $\beta \in \text{Pic}(S)$  to satisfy (6) and have divisibility  $m > 0$ . If  $\Delta(h, d_1, \dots, d_r) = 0$ , then

$$D_{m,h,(d_1,\dots,d_r)} = D_{h,(d_1,\dots,d_r)}$$

if  $m > 0$  is the greatest common divisor of  $d_1, \dots, d_r$  and 0 otherwise.

Refined Noether-Lefschetz numbers are defined by

$$(8) \quad NL_{m,h,(d_1,\dots,d_r)}^\pi = \int_C \iota_\pi^* [D_{m,h,(d_1,\dots,d_r)}].$$

In Section 2.5, the full set of Noether-Lefschetz numbers  $NL_{h,(d_1,\dots,d_r)}^\pi$  is easily shown to determine the refined numbers  $NL_{m,h,(d_1,\dots,d_r)}^\pi$ .

**0.3. Three theories.** The main geometric idea in the proof is the relationship of three theories associated to a 1-parameter family

$$\pi : X \rightarrow C$$

of  $\Lambda$ -polarized  $K3$  surfaces:

- (i) the Noether-Lefschetz numbers of  $\pi$ ,
- (ii) the genus 0 Gromov-Witten invariants of  $X$ ,
- (iii) the genus 0 reduced Gromov-Witten invariants of the  $K3$  fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin. For (ii), we view the theory in terms the Gopakumar-Vafa invariants<sup>5</sup> [16, 17].

Let  $n_{0,(d_1,\dots,d_r)}^X$  denote the Gopakumar-Vafa invariant of  $X$  in genus 0 for  $\pi$ -vertical curve classes of degrees  $d_1, \dots, d_r$  with respect to the line bundles  $L_1, \dots, L_r$ . Let  $r_{0,m,h}$  denote the reduced  $K3$  invariant defined in Section 0.1. The following result is proven<sup>6</sup> in [31] by a comparison

<sup>5</sup>A review of the definitions can be found in Section 2.5.

<sup>6</sup>The result of the [31] is stated in the rank  $r = 1$  case, but the argument is identical for arbitrary  $r$ .

of the reduced and usual deformation theories of maps of curves to the  $K3$  fibers of  $\pi$ .

**Theorem 2.** *For degrees  $(d_1, \dots, d_r)$  positive with respect to the quasi-polarization  $\lambda^\pi$ ,*

$$n_{0,(d_1,\dots,d_r)}^X = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL_{m,h,(d_1,\dots,d_r)}^\pi.$$

**0.4. Proof of Theorem 1.** The STU model described in Section 1 is a special family of rank 2 lattice polarized  $K3$  surfaces

$$\pi^{STU} : X^{STU} \rightarrow \mathbb{P}^1.$$

The fibered  $K3$  surfaces of the STU model are themselves elliptically fibered. The proof of Theorem 1 proceeds in four basic steps:

- (i) The modular form [4, 25] determining the intersections of the base  $\mathbb{P}^1$  with the Noether-Lefschetz divisors is calculated. For the STU model, the modular form has vector dimension 1 and is proportional to the product  $E_4 E_6$  of Eisenstein series.
- (ii) Theorem 2 is used to show the 3-fold BPS counts  $n_{0,(d_1,d_2)}^{X^{STU}}$  then *determine* all the reduced  $K3$  invariants  $r_{0,m,h}$ . Strong use is made of the rank 2 lattice of the STU model.
- (iii) The BPS counts  $n_{0,(d_1,d_2)}^{X^{STU}}$  are calculated via mirror symmetry. Since the STU model is realized as a Calabi-Yau complete intersection in a nonsingular toric variety, the genus 0 Gromov-Witten invariants are obtained after proven mirror transformations from hypergeometric series. The Klemm-Lerche-Mayr identity, proven in Section 3, shows the invariants  $n_{0,(d_1,d_2)}^{X^{STU}}$  are themselves related to modular forms.
- (iv) Theorem 1 then follows from the Harvey-Moore identity which simultaneously relates the modular structures of

$$n_{0,(d_1,d_2)}^{X^{STU}}, \quad r_{0,m,h}, \quad \text{and} \quad NL_{m,h,(d_1,d_2)}^{\pi^{STU}}$$

in the form specified by Theorem 2. D. Zagier's proof of the Harvey-Moore identity is presented in Section 4.

The strategy of proof is special to genus 0. Much less is known in higher genus. The Katz-Klemm-Vafa conjecture [21, 31] for the integral<sup>7</sup>

$$\int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g$$

is a particular generalization of the Yau-Zaslow formula to higher genera. The KKV formula does not yet appear easily approachable in Gromov-Witten theory.<sup>8</sup> However, a proof of the KKV formula for primitive  $K3$  classes in the conjecturally equivalent theory of stable pairs in the derived category is given in [22, 34].

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<sup>7</sup>The integrand  $\lambda_g$  is the top Chern class of the Hodge bundle on  $\overline{M}_g(X, \beta)$ .

<sup>8</sup>For  $g = 1$ , the KKV formula follows for all classes on  $K3$  surfaces from the Yau-Zaslow formula via the boundary relation for  $\lambda_1$ .

## 1. THE STU MODEL

1.1. **Overview.** The STU model<sup>9</sup> is a particular nonsingular projective Calabi-Yau 3-fold  $X$  equipped with a fibration

$$(9) \quad \pi : X \rightarrow \mathbb{P}^1.$$

Except for 528 points  $\xi \in \mathbb{P}^1$ , the fibers

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered  $K3$  surfaces. The 528 singular fibers  $X_\xi$  have exactly 1 ordinary double point singularity each.

The 3-fold  $X$  is constructed as an anticanonical section of a nonsingular projective toric 4-fold  $Y$ . The Picard rank of  $Y$  is 6. The fibration (9) is obtained from a nonsingular toric fibration

$$\pi^Y : Y \rightarrow \mathbb{P}^1.$$

The image of

$$\text{Pic}(Y) \rightarrow \text{Pic}(X_\xi)$$

determines a rank 2 sublattice of each fiber  $\text{Pic}(X_\xi)$  with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The toric data describing the construction of  $X \subset Y$  and the fibration structure are explained here.

1.2. **Toric varieties.** Let  $N$  be a lattice of rank  $d$ ,

$$N \cong \mathbb{Z}^d.$$

A fan  $\Sigma$  in  $N$  is a collection of strongly convex rational polyhedral cones containing all faces and intersections. A toric variety  $V_\Sigma$  is canonically associated to  $\Sigma$ . The variety  $V_\Sigma$  is complete of dimension  $d$  if the support of  $\Sigma$  covers  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . If all cones are simplicial and if all

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<sup>9</sup>The model has been studied in physics since the 80's. The letter  $S$  stands for the dilaton and  $T$  and  $U$  label the torus moduli in the heterotic string. The STU model was an important example for the duality between type IIA and heterotic strings formulated in [20]. The ideas developed in [18, 19, 23, 24, 30] about the STU model play an important role in our paper.

maximal cones are generated by a lattice basis, then  $V_\Sigma$  is nonsingular. See [8, 12, 32] for the basic properties of toric varieties.

Let  $\Sigma$  be a fan corresponding to a nonsingular complete toric variety. A 1-dimensional cone of  $\Sigma$  is a ray with a unique primitive vector. Let  $\Sigma^{(1)}$  denote the set of 1-dimensional cones of  $\Sigma$  indexed by their primitive vectors

$$(10) \quad \{\rho_1, \dots, \rho_n\}.$$

Let  $r^1, \dots, r^\ell$  be a basis over the integers of the module of relations among the vectors (10). We write the  $j^{\text{th}}$  relation as

$$r_1^j \rho_1 + \dots + r_n^j \rho_n = 0.$$

Define a torus

$$(\mathbb{C}^*)^\ell \simeq \prod_{j=1}^{\ell} \mathbb{C}_j^*$$

with factors indexed by the relations.

A simple description of  $V_\Sigma$  is obtained via a quotient construction. Let  $\{z_i\}_{1 \leq i \leq n}$  be coordinates on  $\mathbb{C}^n$  corresponding to the primitives  $\rho_i$  of the rays in  $\Sigma^{(1)}$ . An action of  $\mathbb{C}_j^*$  on  $\mathbb{C}^n$  is defined by

$$(11) \quad \lambda_j \cdot (z_1, \dots, z_n) = (\lambda_j^{r_1^j} z_1, \dots, \lambda_j^{r_n^j} z_n), \quad \lambda_j \in \mathbb{C}_j^*$$

In order to obtain a well-behaved quotient for the induced  $(\mathbb{C}^*)^\ell$ -action on  $\mathbb{C}^n$ , an exceptional set  $Z(\Sigma) \subset \mathbb{C}^n$  consisting of a finite union of linear subspaces is excluded. The linear space defined by  $\{z_i = 0 \mid i \in I\}$  is contained in  $Z(\Sigma)$  if there is no single cone in  $\Sigma$  containing all of the primitives  $\{\rho_i\}_{i \in I}$ . After removing  $Z(\Sigma)$ , the quotient

$$(12) \quad V_\Sigma = \left( \mathbb{C}^n \setminus Z(\Sigma) \right) / (\mathbb{C}^*)^\ell$$

yields the toric variety associated to  $\Sigma$ .

Since  $\ell = n - d$ , the complex dimension of the quotient  $V_\Sigma$  equals the rank  $d$  of the lattice  $N$ . The variety  $V_\Sigma$  is equipped with the action of the quotient torus

$$T = (\mathbb{C}^*)^n / (\mathbb{C}^*)^\ell.$$

The rank of  $\text{Pic}(V_\Sigma)$  is  $\ell$ . The primitives  $\rho_i$  are in 1-to-1 correspondence with the  $T$ -invariant divisors  $D_i$  on  $V_\Sigma$  defined by

$$(13) \quad D_i = \{z_i = 0\} \subset V_\Sigma.$$

Conversely, the homogeneous coordinate  $z_i$  is a section of the line bundle  $\mathcal{O}(D_i)$ . The anticanonical divisor class of  $V_\Sigma$  is determined by

$$(14) \quad -K_{V_\Sigma} = \sum_{i=1}^n D_i.$$

**1.3. The toric 4-fold  $Y$ .** The fan  $\Sigma$  in  $\mathbb{Z}^4$  defining the toric 4-fold  $Y$  has 10 rays with primitive elements

$$\begin{array}{lll} \rho_1 = (1, 0, 2, 3) & \rho_2 = (-1, 0, 2, 3) & \\ \rho_3 = (0, 1, 2, 3) & \rho_4 = (0, -1, 2, 3) & \\ \rho_5 = (0, 0, 2, 3) & \rho_6 = (0, 0, -1, 0) & \rho_7 = (0, 0, 0, -1) \\ \rho_8 = (0, 0, 1, 2) & \rho_9 = (0, 0, 0, 1) & \rho_{10} = (0, 0, 1, 1). \end{array}$$

The full fan  $\Sigma$  is obtained from the convex hull of the 10 primitives. By explicitly checking each of 24 dimension 4 cones,  $Y$  is seen to be a complete nonsingular toric 4-fold.

Generators  $r^1, \dots, r^6$  of the rank 6 module of relations among the primitives can be taken to be

$$\begin{array}{rcccccc} \rho_1 & +\rho_2 & & +4\rho_6 & +6\rho_7 & & = 0 \\ & & \rho_3 & +\rho_4 & +4\rho_6 & +6\rho_7 & = 0 \\ & & & & \rho_5 & +2\rho_6 & +3\rho_7 & = 0 \\ & & & & & \rho_6 & +2\rho_7 & +\rho_8 & = 0 \\ & & & & & & +\rho_7 & & +\rho_9 & = 0 \\ & & & & & & & \rho_6 & +\rho_7 & & +\rho_{10} & = 0 \end{array}$$

By the identification (14) of  $-K_Y$ , the product  $\prod_{i=1}^{10} z_i$  defines an anticanonical section. Hence, every product

$$\prod_{i=1}^{10} z_i^{m_i}, \quad m_i \geq 0$$

which is homogeneous of degree  $\sum_{i=1}^{10} r_i^j$  with respect to the action (11) of  $\mathbb{C}_j^*$  also defines an anticanonical section. Hence,

$$(15) \quad \begin{array}{ll} z_1^{12} z_4^{12} z_5^6 z_8^4 z_9^2 z_{10}^3, & z_1^{12} z_3^{12} z_5^6 z_8^4 z_9^2 z_{10}^3, \\ z_2^{12} z_4^{12} z_5^6 z_8^4 z_9^2 z_{10}^3, & z_2^{12} z_3^{12} z_5^6 z_8^4 z_9^2 z_{10}^3, \\ & z_6^3 z_8 z_9^2, \quad z_7^2 z_{10} \end{array}$$

are all sections of  $-K_Y$ .

From the definitions, we find  $Z(\Sigma)$  consists of the union of the following 11 linear spaces of dimension 2 in  $\mathbb{C}^4$ ,

$$(16) \quad \begin{array}{llll} I_1 = \{1, 2\}, & I_2 = \{3, 4\}, & I_3 = \{5, 6\}, & I_4 = \{5, 7\}, \\ I_5 = \{5, 9\}, & I_6 = \{6, 8\}, & I_7 = \{6, 10\}, & I_8 = \{7, 8\}, \\ I_9 = \{7, 9\}, & I_{10} = \{8, 10\}, & I_{11} = \{9, 10\}. \end{array}$$

Recall,  $I_k$  indexes the coordinates which vanish.

A simple verification show the 6 sections (15) of  $-K_Y$  do not have a common zero on the prequotient  $\mathbb{C}^n \setminus Z(\Sigma)$ . Hence,  $-K_Y$  is generated by global sections on  $Y$ . A hypersurface

$$X \subset Y$$

defined by a generic section of  $-K_Y$  is nonsingular by Bertini's Theorem. By adjunction,  $X$  is Calabi-Yau.

**1.4. Fibrations.** The toric variety  $Y$  admits two obvious fibrations

$$\pi^Y : Y \rightarrow \mathbb{P}^1, \quad \mu^Y : Y \rightarrow \mathbb{P}^1$$

given in homogeneous coordinates by

$$\pi^Y(z_1, \dots, z_{10}) = [z_1, z_2], \quad \mu^Y(z_1, \dots, z_{10}) = [z_3, z_4].$$

Since  $Z(\Sigma)$  contains the linear spaces

$$I_1 = \{1, 2\}, \quad I_2 = \{3, 4\},$$

both  $\pi^Y$  and  $\mu^Y$  are well-defined.



Consider first  $\pi^Y$ . The fibers of  $\pi^Y$  are nonsingular complete toric 3-folds defined by the fan in

$$\mathbb{Z}^3 \subset \mathbb{Z}^4, \quad (c_1, c_2, c_3) \mapsto (0, c_1, c_2, c_3)$$

determined by the primitives  $\rho_3, \dots, \rho_{10}$ .

Let  $X$  be obtained from a generic section of  $-K_Y$ . Let

$$\pi : X \rightarrow \mathbb{P}^1$$

be the restriction  $\pi^Y|_X$ .

**Proposition 1.** *Except for 528 points  $\xi \in \mathbb{P}^1$ , the fibers*

$$X_\xi = \pi^{-1}(\xi)$$

*are nonsingular elliptically fibered K3 surfaces. The 528 singular fibers  $X_\xi$  each have exactly 1 ordinary double point singularity.*

*Proof.* Let  $P_{k,k}(z_1, z_2|z_3, z_4)$  denote a bihomogeneous polynomial of degree  $k$  in  $(z_1, z_2)$  and degree  $k$  in  $(z_3, z_4)$ . Let

$$F = P_{12,12}(z_1, z_2|z_3, z_4), \quad G = P_{8,8}(z_1, z_2|z_3, z_4), \quad H = P_{4,4}(z_1, z_2|z_3, z_4)$$

be bihomogeneous polynomials. Then

$$(17) \quad F z_5^6 z_8^4 z_9^2 z_{10}^3, \quad G z_5^4 z_6 z_8^3 z_9^2 z_{10}^2, \quad H z_5^2 z_6^2 z_8^2 z_9^2 z_{10}, \quad z_6^3 z_8 z_9^2, \quad z_7^2 z_{10}$$

all determine sections of  $-K_Y$ .

Let  $X$  be defined by a generic linear combination of the sections (17). Since the base point free system (15) is contained in (17),  $X$  is nonsingular. We will prove all the fibers  $X_\xi$  are nonsingular, except for finitely many with exactly 1 ordinary double point each, by an explicit study of the equations.

Since  $I_7 = \{6, 10\}$ ,  $I_{10} = \{8, 10\}$ , and  $I_{11} = \{9, 10\}$  are in  $Z(\Sigma)$ , we easily see  $X \cap D_{10} = \emptyset$  if the coefficient of  $z_6^3 z_8 z_9^2$  is nonzero. Similarly

$$X \cap D_8 = \emptyset, \quad X \cap D_9 = \emptyset.$$

Hence, using the last 3 factors of the torus  $(\mathbb{C}^*)^\ell$ , the coordinates  $z_8$ ,  $z_9$ , and  $z_{10}$  can all be set to 1. The equation for  $X$  simplifies to

$$F z_5^6 + G z_5^4 z_6 + H z_5^2 z_6^2 + \alpha z_6^3 + \beta z_7^2.$$

The coordinates  $z_1$  and  $z_2$  do not simultaneously vanish on  $Y$ . There are two charts to consider. By symmetry, the analysis on each is identical, so we assume  $z_1 \neq 0$ . Using the first factor of  $(\mathbb{C}^*)^\ell$ , we set  $z_1 = 1$ . By the same reasoning, we set  $z_3 = 1$  using the second factor of  $(\mathbb{C}^*)^\ell$ . Since  $I_3 = \{5, 6\}$  and  $I_4 = \{5, 7\}$  are in  $Z(\Sigma)$  either  $z_5 \neq 0$  or *both*  $z_6$  and  $z_7$  do not vanish.

**Case**  $z_5 \neq 0$ . Using the third factor of  $(\mathbb{C}^*)^\ell$  to set  $z_5 = 1$ , we obtain the equation

$$(18) \quad F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + \alpha z_6^3 + \beta z_7^2$$

in  $\mathbb{C}^4$  with coordinates  $z_2, z_4, z_6, z_7$ . The map  $\pi$  is given by the  $z_2$  coordinate. The partial derivative of (18) with respect  $z_7$  is  $2\beta z_7$ . Hence, if  $\beta \neq 0$ , all singularities of  $\pi$  occur when  $z_7 = 0$ .

We need only analyze the reduced dimension case

$$(19) \quad F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + z_6^3$$

with coordinates  $z_2, z_4, z_6$ . Here,  $\alpha$  has been set to 1 by scaling the equation. We must show all the fibers of  $\pi$  are nonsingular curves except for finitely many with simple nodes. We view equation (19) as defining a 1-parameter family of paths  $\gamma_{z_2}(z_4)$  in the space

$$\mathcal{C} = \{\gamma_0 + \gamma_1 z_6 + \gamma_2 z_6^2 + z_6^3 \mid \gamma_0, \gamma_1, \gamma_2 \in \mathbb{C}\}$$

of cubic polynomials in the variable  $z_6$ . The coordinate of the path is  $z_4$ . The variable  $z_2$  indexes the family of paths.

Let  $\Delta \subset \mathcal{C}$  be the codimension 1 discriminant locus of cubics with double roots. The discriminant is irreducible with cuspidal singularities in codimension 2 in  $\mathcal{C}$ . The possible singularities of the fiber  $\pi^{-1}(\lambda)$  occur only when the path  $\gamma_\lambda(z_4)$  intersects  $\Delta$ . The fiber  $\pi^{-1}(\lambda)$  is nonsingular over such an intersection point if either

- (i)  $\gamma_\lambda$  is transverse to  $\Delta$  at a nonsingular point of  $\Delta$ ,
- (ii)  $\gamma_\lambda$  is transverse to the codimension 1 tangent cone of a singular point of  $\Delta$ .

The fiber  $\pi^{-1}(\lambda)$  has a simple node over an intersection point of the path  $\gamma_\lambda(z_4)$  with  $\Delta$  if

- (iii)  $\gamma_\lambda$  is tangent to  $\Delta$  at a nonsingular point of  $\Delta$ .

The above are all the possibilities which can occur in a generic 1-parameter family of paths in the space of cubic equations.<sup>10</sup> Possibility (iii) can happen only for finitely many  $\lambda$  and just once for each such  $\lambda$ .

**Case  $z_6 \neq 0$  and  $z_7 \neq 0$ .** Using the third factor of  $(\mathbb{C}^*)^\ell$  to set  $z_6 = 1$ , we obtain the equation

$$(20) \quad F(1, z_2|1, z_4) + H(1, z_2|1, z_4) + G(1, z_2|1, z_4) + \alpha + \beta z_7^2$$

in  $\mathbb{C}^4$  with coordinates  $z_2, z_4, z_5, z_7$ . The partial derivative of (20) with respect  $z_7$  is not 0 for  $z_7 \neq 0$ . Hence, there are no singular fibers of  $\pi$  on the chart.

We have proven all the fibers  $X_\xi$  of  $\pi$  are nonsingular except for finitely many with exactly 1 ordinary double point each. Let  $X_\xi$  be a nonsingular fiber. Let

$$\mu : X \rightarrow \mathbb{P}^1$$

be the restriction  $\mu^Y|_X$ . The fibers of product

$$(\pi, \mu) : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

are easily seen to be anticanonical sections of the nonsingular toric surface<sup>11</sup>  $W$  with fan in  $\mathbb{Z}^2$  determined by the primitives  $\rho_5, \dots, \rho_{10}$ . These anticanonical sections are elliptic curves. Since  $X_\xi$  has trivial canonical bundle by adjunction and the map

$$\mu : X_\xi \rightarrow \mathbb{P}^1$$

is dominant with elliptic fibers, we conclude  $X_\xi$  is an elliptically fibered  $K3$  surface.

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<sup>10</sup>A cusp of  $\pi^{-1}(\lambda)$  occurs, for example, when the path has contact order 3 at a nonsingular point of the discriminant.

<sup>11</sup>Since the product  $(\pi^Y, \mu^Y) : Y \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  has fibers isomorphic to the nonsingular complete (hence projective) toric surface  $W$ , the 4-fold  $Y$  is projective.

The Euler characteristic of  $X$  can be calculated by toric intersection in  $Y$ ,

$$\chi_{top}(X) = -480.$$

The Euler characteristic of a nonsingular  $K3$  fibration over  $\mathbb{P}^1$  is 48. Since each fiber singularity reduces the Euler characteristic by 1, we conclude  $\pi$  has exactly 528 singular fibers.  $\square$

For emphasis, we will sometimes denote the STU model by

$$\pi^{STU} : X^{STU} \rightarrow \mathbb{P}^1.$$

**1.5. Divisor restrictions.** The divisors  $D_1$ ,  $D_2$ ,  $D_8$ ,  $D_9$ , and  $D_{10}$  have already been shown to restrict to the trivial class in  $\text{Pic}(X_\xi)$ . The divisors  $D_3$  and  $D_4$  restrict to the fiber class  $F \in \text{Pic}(X_\xi)$  of the elliptic fibration

$$(21) \quad \mu : X_\xi \rightarrow \mathbb{P}^1.$$

Certainly  $F^2 = 0$ . Let  $S \in \text{Pic}(X_\xi)$  denote the restriction of  $D_5$ . Toric calculations yield the products

$$F \cdot S = 1, \quad S \cdot S = -2.$$

Hence,  $S$  may be viewed as the section class of the elliptic fibration (21). The divisors  $D_6$  and  $D_7$  restrict to classes in the rank 2 lattice generated by  $F$  and  $S$ .

The restriction of  $\text{Pic}(Y)$  to each fiber  $X_\xi$  is a rank 2 lattice generated by  $F$  and  $S$  with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

We may also choose generators  $L_1 = F$  and  $L_2 = F + S$  with intersection form

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**1.6. 1-parameter families.** Let  $X$  be a compact 3-dimensional complex manifold equipped with two holomorphic line bundles

$$L_1, L_2 \rightarrow X$$

and a holomorphic map

$$\pi : X \rightarrow C$$

to a nonsingular complete curve.

The data  $(X, L_1, L_2, \pi)$  determine a *family of  $\Lambda$ -polarized  $K3$  surfaces* if the fibers  $(X_\xi, L_{1,\xi}, L_{2,\xi})$  are  $K3$  surfaces with intersection form

$$\begin{pmatrix} L_{1,\xi} \cdot L_{1,\xi} & L_{2,\xi} \cdot L_{1,\xi} \\ L_{1,\xi} \cdot L_{2,\xi} & L_{2,\xi} \cdot L_{2,\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and there exists a simultaneous quasi-polarization. The 1-parameter family  $(X, L_1, L_2, \pi)$  yields a morphism,

$$\iota_\pi : C \rightarrow \mathcal{M}_\Lambda,$$

to the moduli space of  $\Lambda$ -polarized  $K3$  surfaces.

The construction  $(X^{STU}, L_1, L_2, \pi^{STU})$  of the STU model in Sections 1.3-1.5 is almost a 1-parameter family of  $\Lambda$ -polarized  $K3$  surfaces. The only failing is the 528 singular fibers of  $\pi^{STU}$ . Let

$$\epsilon : C \xrightarrow{2-1} \mathbb{P}^1$$

be a hyperelliptic curve branched over the 528 points of  $\mathbb{P}^1$  corresponding to the singular fibers of  $\pi$ . The family

$$\epsilon^*(X^{STU}) \rightarrow C$$

has 3-fold double point singularities over the 528 nodes of the fibers of the original family. Let

$$\tilde{\pi}^{STU} : \tilde{X}^{STU} \rightarrow C$$

be obtained from a small resolution

$$\tilde{X}^{STU} \rightarrow \epsilon^*(X^{STU}).$$

Let  $\tilde{L}_i \rightarrow \tilde{X}^{STU}$  be the pull-back of  $L_i$  by  $\epsilon$ . The data

$$(\tilde{X}^{STU}, \tilde{L}_1, \tilde{L}_2, \tilde{\pi}^{STU})$$

determine a 1-parameter family of  $\Lambda$ -polarized  $K3$  surfaces, see Section 5.3 of [31]. The simultaneous quasi-polarization is obtained from the projectivity of  $X^{STU}$ .

**1.7. Gromov-Witten invariants.** Since  $X^{STU}$  is defined by an anticanonical section in a semi-positive nonsingular toric variety  $Y$ , the genus 0 Gromov-Witten invariants have been proven by Givental [14, 15, 29, 33] to be related by mirror transformation to hypergeometric solutions of the Picard-Fuchs equations of the Batyrev-Borisov mirror. By Section 5.3 of [31], the Gromov-Witten invariants of  $\tilde{X}^{STU}$  are exactly twice the Gromov-Witten invariants of  $X^{STU}$  for curve classes in the fibers.

## 2. NOETHER-LEFSCHETZ NUMBERS AND REDUCED $K3$ INVARIANTS

**2.1. Refined Noether-Lefschetz numbers.** Following the notation of Section 0.2, let

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

be primitively embedded with signature  $(1, r - 1)$  and integral basis  $v_1, \dots, v_r$ . Let  $(X, L_1, \dots, L_r, \pi)$  be a 1-parameter family of  $\Lambda$ -polarized  $K3$  surfaces. Let  $d_1, \dots, d_r$  be integers which do not all vanish.

**Lemma 1.** *The Noether-Lefschetz numbers  $NL_{h,(d_1,\dots,d_r)}^\pi$  completely determine the refinements  $NL_{m,h,(d_1,\dots,d_r)}^\pi$ .*

*Proof.* By definition, the refined Noether-Lefschetz numbers satisfy two elementary identities. The first is

$$NL_{h,(d_1,\dots,d_r)}^\pi = \sum_{m=1}^{\infty} NL_{m,h,(d_1,\dots,d_r)}^\pi.$$

If  $m$  does not divide all  $d_i$ , then  $NL_{m,h,(d_1,\dots,d_r)}^\pi$  vanishes. If  $m$  divides all  $d_i$ , then a second identity holds:

$$NL_{m,h,(d_1,\dots,d_r)}^\pi = NL_{1,h',(d_1/m,\dots,d_r/m)}^\pi$$

where  $2h - 2 = m^2(2h' - 2)$ .

If  $\Delta(h, d_1, \dots, d_r) = 0$ , the refined number  $NL_{m,h,(d_1,\dots,d_r)}^\pi$  vanishes by definition unless  $m$  is the GCD of  $(d_1, \dots, d_r)$ . In the latter case,

$$NL_{h,(d_1,\dots,d_r)}^\pi = NL_{m,h,(d_1,\dots,d_r)}^\pi.$$

Hence the Lemma is trivial in the  $\Delta(h, d_1, \dots, d_r) = 0$  case.

If  $\Delta(h, d_1, \dots, d_r) > 0$ , we prove the Lemma by induction on  $\Delta$ . The second identity reduces us to the case where  $m = 1$ . The first identity determines the  $m = 1$  case in terms of the Noether-Lefschetz number  $NL_{h,(d_1,\dots,d_r)}$  and refined numbers with

$$\Delta(h', d'_1, \dots, d'_r) < \Delta(h, d_1, \dots, d_r).$$

□

**2.2. STU model.** The resolved version of the STU model

$$\tilde{\pi}^{STU} : \tilde{X}^{STU} \rightarrow C$$

is lattice polarized with respect to

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The application of the results of [4, 25] to the STU model is extremely simple. Since the lattice  $\Lambda$  is unimodular, the corresponding representation  $\rho_\Lambda^*$  is 1-dimensional and, in fact, is the trivial representation of  $\mathrm{Mp}_2(\mathbb{Z})$ . The Noether-Lefschetz degrees are thus encoded by a scalar modular form of weight  $\frac{22-r}{2} = 10$ . The space of such forms is well-known to be of dimension 1 and spanned by the product of Eisenstein series<sup>12</sup>

$$E_{10}(q) = E_4(q)E_6(q) = 1 - 264 \sum_{n \geq 1} \sigma_9(n)q^n.$$

---

<sup>12</sup>The Eisenstein series  $E_{2k}$  is the modular form defined by the equation

$$-\frac{B_{2k}}{4k} E_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n)q^n,$$

where  $B_{2n}$  is the  $2n^{\text{th}}$  Bernoulli number and  $\sigma_n(k)$  is the sum of the  $k^{\text{th}}$  powers of the divisors of  $n$ ,

$$\sigma_k(n) = \sum_{i|n} i^k.$$

Hence, a single Noether-Lefschetz calculation determines the full series.

**Lemma 2.**  $NL_{0,(0,0)}^{\tilde{\pi}} = 1056$ .

*Proof.* By Proposition 1, the STU model

$$\pi^{STU} : X^{STU} \rightarrow \mathbb{P}^1$$

has 528 nodal fibers. Let  $S$  be a fiber of the resolved family  $\tilde{\pi}^{STU}$  lying over a singular fiber of  $\pi$ . The Picard lattice of  $S$  certainly contains

$$(22) \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

spanned by  $L_1$ ,  $L_2$ , and the  $(-2)$ -curve  $E$  of the small resolution. Let

$$\tilde{\tau} : C \rightarrow \mathcal{M}_\Lambda$$

be the map to moduli. Since a class  $\beta$  satisfying

$$\langle \beta, \beta \rangle = -2$$

on a  $K3$  surface is either effective or anti-effective, the set theoretic intersections of  $\tilde{\tau}$  with  $D_{0,(0,0)}$  correspond to fibers of  $\tilde{\pi}$  where  $L_1$  and  $L_2$  do not generate an ample class — precisely the 528 fibers of  $\tilde{\pi}$  lying over the singular fibers of  $\pi$ .

The divisor  $D_{0,(0,0)}$  has multiplicity exactly 2 at the 528 intersections with  $\tilde{\tau}$  since  $E$  and  $-E$  are the only  $-2$  classes orthogonal to  $L_1$  and  $L_2$ . Finally, since  $E$  has normal bundle  $(-1, -1)$  in  $\tilde{X}^{STU}$ , the curve  $\tilde{\tau}$  is transverse to the reduced divisor  $\frac{1}{2}D_{0,(0,0)}$  at the 528 intersections. We conclude  $NL_{0,(0,0)}^{\tilde{\pi}} = 528 \cdot 2 = 1056$ .  $\square$

**Proposition 2.** *The Noether-Lefschetz degrees of the resolved STU model are given by the equation*

$$NL_{h,(d_1,d_2)}^{\tilde{\pi}} = -4E_4(q)E_6(q) \left[ \frac{\Delta(h, d_1, d_2)}{2} \right].$$



**2.3. BPS states.** Let  $(\tilde{X}^{STU}, \tilde{L}_1, \tilde{L}_2, \tilde{\pi}^{STU})$  the  $\Lambda$ -polarized STU model. The vertical classes are the kernel of the push-forward map by  $\tilde{\pi}$ ,

$$0 \rightarrow H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}} \rightarrow H_2(\tilde{X}, \mathbb{Z}) \rightarrow H_2(C, \mathbb{Z}) \rightarrow 0.$$

While  $\tilde{X}$  need not be a projective variety,  $\tilde{X}$  carries a  $(1, 1)$ -form  $\omega_K$  which is Kähler on the  $K3$  fibers of  $\tilde{\pi}$ . The existence of a fiberwise Kähler form is sufficient to define Gromov-Witten theory for vertical classes

$$0 \neq \gamma \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}.$$

The fiberwise Kähler form  $\omega_K$  is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.

Let  $\overline{M}_0(\tilde{X}, \gamma)$  be the moduli space of stable maps from connected genus 0 curves to  $\tilde{X}$ . Gromov-Witten theory is defined by integration against the virtual class,

$$(23) \quad N_{0,\gamma}^{\tilde{X}} = \int_{[\overline{M}_0(\tilde{X}, \gamma)]^{vir}} 1.$$

The expected dimension of the moduli space is 0.

The genus 0 Gromov-Witten potential  $F^{\tilde{X}}(v)$  for nonzero vertical classes is the series

$$F^{\tilde{X}} = \sum_{0 \neq \gamma \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}} N_{0,\gamma}^{\tilde{X}} v^\gamma$$

where  $v$  is the curve class variable. The BPS counts  $n_{0,\gamma}^{\tilde{X}}$  of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^{\tilde{X}} = \sum_{0 \neq \gamma \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}} n_{0,\gamma}^{\tilde{X}} \sum_{d>0} \frac{v^{d\gamma}}{d^3}.$$

Conjecturally, the invariants  $n_{0,\gamma}^{\tilde{X}}$  are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on  $\tilde{X}$ . We do not assume the conjectural properties hold.

Using the  $\Lambda$ -polarization, we define the BPS counts

$$(24) \quad n_{0,(d_1,d_2)}^{\tilde{X}} = \sum_{\gamma \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}, \int_{\gamma} \tilde{L}_i = d_i} n_{0,\gamma}^{\tilde{X}}$$

when  $d_1$  and  $d_2$  are not both 0.

The original STU model,

$$\pi^{STU} : X^{STU} \rightarrow \mathbb{P}^1,$$

with 528 singular fibers is a nonsingular, projective, Calabi-Yau 3-fold. Hence the Gromov-Witten invariants are well-defined. Let  $n_{0,(d_1,d_2)}^X$  denote the fiberwise Gopakumar-Vafa invariant with degrees  $d_i$  measured by  $L_i$ . By the argument of Section 1.7,

$$n_{0,(d_1,d_2)}^{\tilde{X}} = 2n_{0,(d_1,d_2)}^X$$

when  $d_1$  and  $d_2$  are not both 0.

**2.4. Invertibility of constraints.** Let  $\mathcal{P} \subset \mathbb{Z}^2$  be the set of pairs

$$\mathcal{P} = \{ (d_1, d_2) \neq (0, 0) \mid d_1 \geq 0, d_1 \geq -d_2 \}.$$

Pairs  $(d_1, d_2) \in \mathcal{P}$  are certainly positive with respect to any quasi-polarization for  $\tilde{\pi}^{STU}$  since such  $(d_1, d_2)$  can be realized by linear combinations of the effective classes  $F$  and  $S$ .

Theorem 2 applied to the resolved STU model yields the equation

$$(25) \quad n_{0,(d_1,d_2)}^{\tilde{X}} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL_{m,h,(d_1,d_2)}^{\tilde{\pi}}$$

for  $(d_1, d_2) \in \mathcal{P}$ . The BPS states on the left side will be computed by mirror symmetry in Section 3. The refined Noether-Lefschetz degrees are determined by Lemma 1 and Proposition 2. Consequently, equation (25) provides constraints on the reduced  $K3$  invariants  $r_{0,m,h}$

The integrals  $r_{0,m,h}$  are very simple in case  $h \leq 0$ . By Lemma 2 of [31],  $r_{0,m,h} = 0$  for  $h < 0$ ,

$$r_{0,1,0} = 1,$$

and  $r_{0,m,0} = 0$  otherwise.

**Proposition 3.** *The set of integrals  $\{r_{0,m,h}\}_{m \geq 1, h > 0}$  is uniquely determined by the set of constraints (25) for  $(d_1 \geq 0, d_2 > 0)$  and the integrals  $r_{0,m,h \leq 0}$ .*

*Proof.* A certain subset of the linear equations with  $d_2 > 0$  will be shown to be upper triangular in the variables  $r_{0,m,h}$ . Picard rank 2 is crucial for the argument.

Let us fix in advance the values of  $m \geq 1$  and  $h > 0$ . We proceed by induction on  $m$  assuming the reduced invariants  $r_{0,m',h}$  have already been determined for all  $m' < m$ . The assumption is vacuous when  $m = 1$ . We can also assume  $r_{0,m,h'}$  has been determined inductively for  $h' < h$ . If  $2h - 2$  is not divisible by  $2m^2$ , then we have  $r_{0,m,h} = 0$ , so we can further assume

$$2h - 2 = m^2(2s - 2)$$

for an integer  $s > 0$ .

Consider equation (25) for  $(d_1, d_2) = (m(s - 1), m)$ . Certainly

$$NL_{m',h',(m(s-1),m)}^{\tilde{\pi}} = 0$$

unless  $m'$  divides  $m$ . By the Hodge index theorem, we must have

$$(26) \quad \Delta(h', m(s - 1), m) = 2 - 2h' + m^2(2s - 2) \geq 0$$

if  $NL_{m,h',(m(s-1),m)}^{\tilde{\pi}} \neq 0$ . Inequality (26) implies  $h' \leq h$ .

Therefore, the constraint (25) takes the form

$$n_{0,(m(s-1),m)}^{\tilde{X}} = r_{0,m,h} NL_{m,h,(m(s-1),m)}^{\tilde{\pi}} + \dots,$$

where the dots represent terms involving  $r_{0,m',h'}$  with either

$$m' < m \quad \text{or} \quad m' = m, \quad h' < h.$$

The leading coefficient is given by

$$NL_{m,h,(m(s-1),m)}^{\tilde{\pi}} = NL_{h,(m(s-1),m)}^{\tilde{\pi}} = -4.$$

As the system is upper-triangular, we can invert to solve for  $r_{0,m,h}$ .  $\square$

**2.5. Proof of the Yau-Zaslow conjecture.** By Proposition 3, we need only show the answer for  $r_{0,m,h}$  predicted by the Yau-Zaslow conjecture satisfies the constraints (25) for all pairs  $(d_1 \geq 0, d_2 > 0)$ .

Let  $X^{STU}$  be the original Calabi-Yau 3-fold of the STU model. Let

$$(27) \quad D_2^3 F^X = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N_{0, (d_1, d_2)}^X q_1^{d_1} q_2^{d_2}$$

be the third derivative<sup>13</sup> of the genus 0 Gromov-Witten series for  $\pi$ -vertical classes in  $\mathcal{P}$ .

We can calculate  $D_2^3 F^X$  by the constraint (25) *assuming the validity of the Yau-Zaslow conjecture*,

$$(28) \quad D_2^3 F^X = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 c(d_1, d_2) \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}$$

where  $c(k, l)$  is the coefficient of  $q^{kl}$  in

$$-2 \frac{E_4(q) E_6(q)}{\eta^{24}(q)}.$$

**Proposition 4.** *The Yau-Zaslow conjecture is implied by the identity*

$$\sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N_{0, (d_1, d_2)}^X q_1^{d_1} q_2^{d_2} = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 c(d_1, d_2) \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}.$$

*Proof.* The  $q_1^{d_1} q_2^{d_2}$  coefficient of the above identity is simply  $d_2^3$  times the constraint (25). Since we only require the constraints in case

$$(d_1 \geq 0, d_2 > 0) \in \mathcal{P},$$

the identity implies all the constraints we need.  $\square$

The remainder of the paper is devoted to the proof of Proposition 4. The genus 0 Gromov-Witten invariants of  $X$  are related, after mirror transformation, to hypergeometric solutions of the associated Picard-Fuchs system of differential equations. Hence, Proposition 4 amounts to a subtle identity among special functions.

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<sup>13</sup> $D_2 = q_2 \frac{d}{dq_2}$ .

## 3. MIRROR TRANSFORM

3.1. **Picard-Fuchs.** Let  $\pi : X \rightarrow \mathbb{P}^1$  be the STU model. Let

$$\delta_0 \in H^*(X, \mathbb{C})$$

denote the identity class. A basis of  $H^2(X, \mathbb{C})$  is obtained from the restriction of the toric divisors of  $Y$  discussed in Section 1.5,

$$\delta_1 = 2D_1 + 2D_3 + D_5, \quad \delta_2 = D_3, \quad \delta_3 = D_1.$$

Recall,  $\delta_3$  vanishes on the fibers of  $\pi$ . Let  $\{\delta_j\}$  be a full basis of  $H^*(X, \mathbb{C})$  extending the above selections.

Let  $u_1, u_2, u_3$  be the canonical coordinates for the mirror family with respect to the divisor basis  $\delta_1, \delta_2, \delta_3$ . Let

$$\theta_i = u_i \frac{\partial}{\partial u_i}.$$

The Picard-Fuchs system associated to the mirror of  $X^{STU}$  is:

$$\begin{aligned} \mathcal{L}_1 &= \theta_1 (\theta_1 - 2\theta_2 - 2\theta_3) - 12 (6\theta_1 - 5) (6\theta_1 - 1) u_1 \\ (29) \quad \mathcal{L}_2 &= \theta_2^2 - (2\theta_2 + 2\theta_3 - \theta_1 - 2) (2\theta_2 + 2\theta_3 - \theta_1 - 1) u_2, \\ \mathcal{L}_3 &= \theta_3^2 - (2\theta_2 + 2\theta_3 - \theta_1 - 2) (2\theta_2 + 2\theta_3 - \theta_1 - 1) u_3. \end{aligned}$$

The system is obtained canonically from the Batyrev-Borisov construction, see [9] for the formalism.

3.2. **Solutions.** A fundamental solution to the Picard-Fuchs system can be written in terms of GKZ hypergeometric series,

$$(30) \quad \varpi \in H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\log(u_1), \log(u_2), \log(u_3)][[u_1, u_2, u_3]].$$

Let  $\varpi(u, \delta_j)$  be the corresponding coefficient of (30), then

$$\mathcal{L}_i \varpi(u, \delta_j) = 0.$$

The standard normalization of  $\varpi$  satisfies two important properties:

- (i) The  $\delta_0$  coefficient is the unique solution

$$\varpi(u, \delta_0) = 1 + O(u)$$

holomorphic at  $u = 0$ .

(ii) For  $1 \leq i \leq 3$ ,

$$\varpi(u, \delta_i) = \frac{\varpi(u, \delta_0)}{2\pi i} \log(u_i) + O(u)$$

are the logarithmic solutions.

Let  $T_1, T_2, T_3$  be coordinates on  $H^2(X, \mathbb{C})$  with respect to the basis  $\delta$ . The mirror transformation is defined by

$$T_i = \frac{\varpi(u, \delta_i)}{\varpi(u, \delta_0)} = \frac{1}{2\pi i} \log(u_i) + O(u)$$

for  $1 \leq i \leq 3$ .

The mirror transformation relates the genus 0 Gromov-Witten theory of  $X$  to the Picard-Fuchs system for the mirror family. For anti-canonical hypersurfaces in toric varieties, a proof is given in [15].

**3.3. Mirror transform for  $q_3 = 0$ .** We introduce two modular parameters

$$(31) \quad \tau_1 = T_1, \quad \tau_2 = T_1 + T_2.$$

For  $i = 1$  and  $2$ , let

$$\widehat{q}_i = \exp(2\pi i \tau_i),$$

and let  $q_3 = \exp(2\pi i T_3)$ .

Our first step is to find a modular expression for the mirror map and the period  $\varpi(u, \delta_0)$  to leading order in  $q_3$ . We prove two formulas discovered by Klemm, Lerche, and Mayr in [24].

**Lemma 3.** *We have*

$$u_1 = \frac{2(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)}{j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)}} + O(q_3),$$

$$u_2 = \frac{(j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)})^2}{4j(\widehat{q}_1)j(\widehat{q}_2)(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)^2} + O(q_3),$$

where  $\mu = 1728$  and

$$(32) \quad j(q) = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + O(q^2)$$

is the normalized  $j$  function.

**Lemma 4.**  $\lim_{q_3 \rightarrow 0} \varpi(u, \delta_0) = E_4(\widehat{q}_1)^{\frac{1}{4}} E_4(\widehat{q}_2)^{\frac{1}{4}}$ .

*Proof.* We prove Lemmas 3 and 4 together. The first step is to perform the following change of variables

$$u_1 = z_1, \quad u_2 = \frac{z_2}{2} (1 + \sqrt{1 - 4z_3}), \quad u_3 = \frac{z_2}{2} (1 - \sqrt{1 - 4z_3}),$$

with the inverse change

$$z_1 = u_1, \quad z_2 = u_2 + u_3, \quad z_3 = \frac{u_2 u_3}{(u_2 + u_3)^2}.$$

In the new variables, the limit  $u_3 \rightarrow 0$  becomes the limit  $z_3 \rightarrow 0$ .

The statement of Lemma 3 in the variables  $z_i$  remains unchanged to first order in  $q_3$ . We will prove

$$\begin{aligned} z_1 &= \frac{2(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)}{j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)}} + O(q_3), \\ z_2 &= \frac{(j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)})^2}{4j(\widehat{q}_1)j(\widehat{q}_2)(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)^2} + O(q_3). \end{aligned}$$

The Picard-Fuchs differential operators (29) can be rewritten as

$$\begin{aligned} \mathcal{L}'_1(z) &= \mathcal{L}_1(u), \\ z_2 \sqrt{1 - 4z_3} \mathcal{L}'_2(z) &= \mathcal{L}_2(u) - \mathcal{L}_3(u), \\ z_2 \sqrt{1 - 4z_3} \mathcal{L}'_3(z) &= u_3 \mathcal{L}_2(u) - u_2 \mathcal{L}_3(u), \end{aligned}$$

with

$$\begin{aligned} \mathcal{L}'_1 &= \theta_1 (\theta_1 - 2\theta_2) - 12 (6\theta_1 - 5) (6\theta_1 - 1) z_1, \\ \mathcal{L}'_2 &= \theta_2 (\theta_2 - 2\theta_3) - (2\theta_2 - \theta_1 - 2) (2\theta_2 - \theta_1 - 1) z_2, \\ \mathcal{L}'_3 &= \theta_3^2 - (2\theta_3 - \theta_2 - 2) (2\theta_3 - \theta_2 - 1) z_3 \end{aligned}$$

where now  $\theta_i = z_i \frac{d}{dz_i}$ . Since  $\mathcal{L}'_3(z) \rightarrow 0$  in the limit  $z_3 \rightarrow 0$ , we need only focus on  $\mathcal{L}'_1(z)$  and  $\mathcal{L}'_2(z)$ .

Next, we transform  $\mathcal{L}'_1(z)$  and  $\mathcal{L}'_2(z)$  to new variables  $y_1, y_2, y_3$  via the change

$$\begin{aligned} z_1 &= \frac{2(y_1 + y_2 - \mu)}{y_1 y_2 + \sqrt{y_1(y_1 - \mu)}\sqrt{y_2(y_2 - \mu)}}, \\ z_2 &= \frac{(y_1 y_2 + \sqrt{y_1(y_1 - \mu)}\sqrt{y_2(y_2 - \mu)})^2}{4y_1 y_2 (y_1 + y_2 - \mu)^2}, \\ z_3 &= y_3. \end{aligned}$$

We obtain

$$\begin{aligned}\mathcal{L}_1'' &= y_1^2 y_2 (y_1 - \mu) \partial_{y_1}^2 + y_1 y_2 (y_1 - \frac{\mu}{2}) \partial_{y_1} - y_1 y_2^2 (y_2 - \mu) \partial_{y_2}^2 \\ &\quad - y_1 y_2 (y_2 - \frac{\mu}{2}) \partial_{y_2} + 60(y_1 - y_2), \\ \mathcal{L}_2'' &= -y_1^2 (y_1 - \mu) \partial_{y_1}^2 + y_1 (\frac{\mu}{2} - y_1) \partial_{y_1} + y_2^2 (y_2 - \mu) \partial_{y_2}^2 + y_2 (y_2 - \frac{\mu}{2}) \partial_{y_2} \\ &\quad - 2y_1 y_3 (y_1 - \mu) \partial_{y_1} \partial_{y_3} + 2y_2 y_3 (y_2 - \mu) \partial_{y_2} \partial_{y_3}.\end{aligned}$$

In the limit  $y_3 \rightarrow 0$ , the second line on the right for  $\mathcal{L}_2''$  vanishes. We can combine  $\mathcal{L}_1''$  and  $\mathcal{L}_2''$  to obtain the following simple forms:

$$\begin{aligned}\mathcal{L}_1'' + y_1 \lim_{y_3 \rightarrow 0} \mathcal{L}_2'' &= (y_1 - y_2) \left( 60 - \left( y_1 - \frac{\mu}{2} \right) y_1 \partial_{y_1} - (y_1 - \mu) y_1^2 \partial_{y_1}^2 \right), \\ \mathcal{L}_1'' + y_2 \lim_{y_3 \rightarrow 0} \mathcal{L}_2'' &= (y_1 - y_2) \left( 60 - \left( y_2 - \frac{\mu}{2} \right) y_2 \partial_{y_2} - (y_2 - \mu) y_2^2 \partial_{y_2}^2 \right).\end{aligned}$$

The solution  $\varpi(y, \delta_0)_{y_3=0}$  therefore satisfies the differential equation

$$(33) \quad \mathcal{L} = (y - \mu) y^2 \partial_y^2 + \left( y - \frac{\mu}{2} \right) y \partial_y - 60 .$$

in both  $y_1$  and  $y_2$ .

Changing (33) to the variable  $t = \frac{1728}{y}$  yields

$$\mathcal{L} = t(1-t) \partial_t^2 + \left( 1 - \frac{3}{2}t \right) \partial_t - \frac{5}{144} ,$$

which by comparing with the general hypergeometric differential operator

$$\mathcal{L} = t(1-t) \partial_t^2 + (c - (1+a+b)t) \partial_t - ab$$

is identified with the system

$${}_2F_1(a, b; c; t) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t(\tau)\right).$$

According to the results of Klein and Fricke as reviewed in [37], we have a unique (up to scaling) solution  $g_0$  to (33) locally analytic at  $y = \infty$ . The solution can be written as

$$g_0(j(\tau)) = (E_4)^{\frac{1}{4}}(\tau), \quad y(\tau) = j(\tau) .$$

Moreover, the inverse is

$$\tau(y) = \frac{g_1(y)}{2\pi i g_0(y)},$$



where  $g_1$  is a logarithmic solution at  $y = \infty$  of  $\mathcal{L}$ , unique up to normalization and addition of  $g_0$ .

Transformation of the solution  $\varpi(u, \delta_0)$  is seen to be analytic in a neighborhood of  $t_1 = t_2 = 0$ . We conclude

$$\varpi(u, \delta_0)_{u_3=0} = E_4^{\frac{1}{4}}(\tau_1)E_4^{\frac{1}{4}}(\tau_2).$$

By comparing the first few coefficients of the actual solutions  $\varpi(u, \delta_i)$  in the  $u_3 \rightarrow 0$  limit, we can uniquely identify

$$\tau_1(u) = T_1(u), \quad \tau_2(u) = T_1(u) + T_2(u).$$

Hence, Lemma 4 is established. Lemma 3 is proven by transforming back to the  $u_1$  and  $u_2$  variables.  $\square$

Restricted to a  $K3$  fiber of  $\pi : X \rightarrow \mathbb{P}^1$ , we have

$$\delta_1 = 2F + S, \quad \delta_2 = F.$$

The coordinates  $2\pi i\tau_1$  and  $2\pi i\tau_2$  correspond to the divisor basis

$$L_2 = F + S, \quad L_1 = F$$

of the  $K3$  fiber. Since the variables  $q_1$  and  $q_2$  of Section 2 measure degrees against  $L_1$  and  $L_2$ , we see

$$\widehat{q}_1 = q_2 \quad \text{and} \quad \widehat{q}_2 = q_1$$

for the fiber geometry.

**3.4. B-model.** The mirror transformation results of Section 3.3 together with a B-model calculation of the periods will be used to prove the following result discovered by Klemm, Mayr, and Lerche [24].

**Proposition 5.** *We have*

$$2 + \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N_{0, (d_1, d_2)}^X q_1^{d_1} q_2^{d_2} = 2 \frac{E_4(q_1)E_6(q_1)}{\eta^{24}(q_1)} \frac{E_4(q_2)}{j(q_1) - j(q_2)}.$$

The left side of Proposition 5 is the left side of Proposition 4 with an added degree 0 constant 2.

*Proof.* We will use following universal expression for the Gromov-Witten invariants of  $X$  in terms of the periods of the mirror:

$$2 + \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N_{0, (d_1, d_2)}^X q_1^{d_1} q_2^{d_2} = \lim_{q_3 \rightarrow 0} \frac{1}{\varpi(u(T), \delta_0)^2} \sum_{i, j, k=1}^3 \frac{\partial u_i}{\partial \tau_1} \frac{\partial u_j}{\partial \tau_1} \frac{\partial u_k}{\partial \tau_1} Y_{i, j, k}(u(T))$$

where  $Y_{i, j, k}$  are the Yukawa couplings of the mirror family, see [9, 24].

The periods  $Y_{i, j, k}$  can be explicitly computed via Griffith transversality [24] and greatly simplify in the  $q_3 \rightarrow 0$  limit. We tabulate the results below:

$$\begin{aligned} Y_{111} &= \frac{8(1 - \tilde{u}_1)}{\tilde{u}_1^3 \Delta_1}, & Y_{133} &= \frac{2\tilde{u}_1(1 - \tilde{u}_1)}{\tilde{u}_3 \Delta_1}, \\ Y_{112} &= \frac{2(1 - \tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_2 - \tilde{u}_3)}{\tilde{u}_1^2 \tilde{u}_2 \Delta_1}, & Y_{222} &= \frac{(1 - 2\tilde{u}_1) A_2}{2\tilde{u}_2^2 \Delta_1 \Delta_2}, \\ Y_{113} &= \frac{2(1 - \tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_3 - \tilde{u}_2)}{\tilde{u}_1^2 \tilde{u}_3 \Delta_1}, & Y_{223} &= \frac{(1 - 2\tilde{u}_1) A_3}{2\tilde{u}_3 \tilde{u}_2 \Delta_1 \Delta_2}, \\ Y_{122} &= \frac{2\tilde{u}_1(1 - \tilde{u}_1)}{\tilde{u}_2 \Delta_1}, & Y_{233} &= \frac{(1 - 2\tilde{u}_1) A_2}{2\tilde{u}_3 \tilde{u}_2 \Delta_1 \Delta_2}, \\ Y_{123} &= \frac{(1 - \tilde{u}_1) ((1 - \tilde{u}_1)^2 - (\tilde{u}_2 + \tilde{u}_3) \tilde{u}_1^2)}{\tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \Delta_1}, & Y_{333} &= \frac{(1 - 2\tilde{u}_1) A_3}{2\tilde{u}_3^2 \Delta_1 \Delta_2}. \end{aligned}$$

Here, we have introduced the variables

$$\tilde{u}_1 = 432u_1, \quad \tilde{u}_2 = 4u_2, \quad \tilde{u}_3 = 4u_3$$

and the discriminant loci

$$(34) \quad \begin{aligned} \Delta_1 &= (1 - \tilde{u}_1)^4 - 2(\tilde{u}_2 + \tilde{u}_3) \tilde{u}_1^2 (1 - \tilde{u}_1)^2 + (\tilde{u}_2 - \tilde{u}_3)^2 \tilde{u}_1^4, \\ \Delta_2 &= (1 - \tilde{u}_2 - \tilde{u}_3)^2 - 4\tilde{u}_2 \tilde{u}_3. \end{aligned}$$

The quantities  $A_2$  and  $A_3$  are defined by

$$(35) \quad \begin{aligned} A_2 &= (1 + \tilde{u}_2 - \tilde{u}_3) (1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (1 - \tilde{u}_3 - 3\tilde{u}_2) (\tilde{u}_2 - \tilde{u}_3), \\ A_3 &= (1 + \tilde{u}_3 - \tilde{u}_2) (1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (1 - \tilde{u}_2 - 3\tilde{u}_3) (\tilde{u}_3 - \tilde{u}_2). \end{aligned}$$

The normalizations of the Yukawa couplings  $Y_{i, j, k}$  are fixed by the classical intersections.

The leading behavior of the mirror map for  $u_1, u_2$  is obtained by rewriting Lemma 3 in terms of  $E_4(\tau_i)$  and  $E_6(\tau_i)$  as

$$(36) \quad \begin{aligned} u_1 &= \frac{1}{864} \left( 1 - \frac{E_6(\tau_1) E_6(\tau_2)}{E_4(\tau_1)^{\frac{3}{2}} E_4(\tau_2)^{\frac{3}{2}}} \right) + \mathcal{O}(q_3) , \\ u_2 &= \frac{(E_4(\tau_1)^3 - E_6(\tau_1)^2) (E_4(\tau_2)^3 - E_6(\tau_2)^2)}{4 \left( E_4(\tau_1)^{\frac{3}{2}} E_4(\tau_2)^{\frac{3}{2}} - E_6(\tau_1) E_6(\tau_2) \right)^2} + \mathcal{O}(q_3) \end{aligned}$$

Denote the leading behavior of the last mirror map by

$$(37) \quad u_3 = q_3 f_3(\widehat{q}_1, \widehat{q}_2) + \mathcal{O}(q_3^2) .$$

The derivatives of the mirror maps with respect to  $T_2$  are easily evaluated using the standard identities

$$\begin{aligned} q \frac{d}{dq} E_2 &= \frac{1}{12} (E_2^2 - E_4) \\ q \frac{d}{dq} E_4 &= \frac{1}{3} (E_2 E_4 - E_6) \\ q \frac{d}{dq} E_6 &= \frac{1}{2} (E_2 E_6 - E_4^2) \\ q \frac{d}{dq} j &= -j \frac{E_6}{E_4} . \end{aligned}$$

We find, to leading order in  $q_3$ ,

$$\begin{aligned} \frac{\partial u_1}{\partial \tau_1} &= \frac{E_6(\tau_2) (E_4(\tau_1)^3 - E_6(\tau_1)^2)}{1728 E_4(\tau_2)^{\frac{3}{2}} E_4(\tau_1)^{\frac{3}{2}}} \\ \frac{\partial u_2}{\partial \tau_1} &= \frac{\sqrt{E_4(\tau_1)} (E_4(\tau_2)^3 - E_6(\tau_2)^2) \left( - (E_4(\tau_1)^{\frac{3}{2}} E_6(\tau_2)) + E_4(\tau_2)^{\frac{3}{2}} E_6(\tau_1) \right) (E_4(\tau_1)^3 - E_6(\tau_1)^2)}{4 \left( E_4(\tau_2)^{\frac{3}{2}} E_4(\tau_1)^{\frac{3}{2}} - E_6(\tau_2) E_6(\tau_1) \right)^3} \end{aligned}$$

The derivative  $\frac{\partial u_3}{\partial \tau_1}$  can be written to this order as

$$(38) \quad \frac{\partial u_3}{\partial \tau_1} = \frac{u_3}{f_3(\widehat{q}_1, \widehat{q}_2)} \frac{\partial}{\partial \tau_1} f_3(\widehat{q}_1, \widehat{q}_2) + \mathcal{O}(u_3^2) .$$

There are many simplifications in the limit  $u_3 \rightarrow 0$ . First the triple couplings

$$Y_{133}, \quad Y_{233}, \quad Y_{333}$$

do not have enough inverse powers of  $u_3$  and therefore do not contribute by the vanishing (38). Second, the surviving  $Y_{i,j,k}$  simplify in the limit.

We evaluate

$$(39) \quad \lim_{q_3 \rightarrow 0} \frac{1}{\varpi(u(T), \delta_0)^2} \sum_{i,j,k=1}^3 \frac{\partial u_i}{\partial \tau_1} \frac{\partial u_j}{\partial \tau_1} \frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)) = \\ - 2 \frac{E_4(\tau_2) E_4(\tau_1) E_6(\tau_2) (E_4(\tau_1)^3 - E_6(\tau_1)^2)}{E_4(\tau_2)^3 E_6(\tau_1)^2 - E_4(\tau_1)^3 E_6(\tau_2)^2}.$$

The possible linear dependence on  $f_3(\widehat{q}_1, \widehat{q}_2)$  drops out as claimed in [24]! Using the standard identities

$$j = \frac{E_4^3}{\eta^{24}}, \quad \eta^{24} = E_4^3 - E_6^2,$$

we obtain the right side of Proposition 5.  $\square$

#### 4. THE HARVEY-MOORE IDENTITY

**4.1. Proof of Proposition 4.** After evaluating the left side via Proposition 5 and dividing by 2, Proposition 4 amounts to a modular form identity. Let

$$f(\tau) = \frac{E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c(n)q^n$$

where  $q = \exp(2\pi i\tau)$ . Then, we must prove

$$(40) \quad \frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \frac{q_1}{q_1 - q_2} + E_4(\tau_2) - \sum_{d,k,\ell>0} \ell^3 c(k\ell) q_1^{kd} q_2^{\ell d}.$$

Equation (40) is the Harvey-Moore identity conjectured in [18].

**4.2. Zagier's proof of the Harvey-Moore identity.** The Harvey-Moore identity implies Proposition 4 and concludes the proof of the Yau-Zaslow conjecture. We present here Zagier's argument from [38].

Let  $S_k \subset M_k \subset M_k^!$  denote the spaces of cusp forms, modular forms, and weakly holomorphic<sup>14</sup> modular forms for  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ . Certainly

$$f(\tau) \in M_{-2}^!.$$

For each  $n \geq 0$ , there is a unique function  $F_n \in M_4^!$  satisfying

$$F_n(\tau) = q^{-n} + \mathcal{O}(q)$$

<sup>14</sup>Holomorphic except for a possible pole at infinity.

as  $\mathfrak{J}(\tau) \rightarrow \infty$ . Uniqueness follows from the vanishing of  $S_4$ . Existence follows by writing  $F_n(\tau)$  as  $E_4(\tau)$  times a polynomial in  $j(\tau)$ ,

$$F_0 = E_4, \quad F_1 = E_4(j - 984), \quad F_2 = E_4(j^2 - 1728j + 393768) \dots$$

We draw several consequences:

- (i)  $F_1|T_n = n^3 F_n$  for all  $n \geq 1$ , where  $T_n$  is the  $n^{\text{th}}$  Hecke operator in weight 4. Indeed,  $T_n$  sends  $M_4^!$  to itself and, by standard formulas for the action of  $T_n$  on Fourier expansions,  $T_n$  sends  $q^{-1} + \mathcal{O}(q)$  to  $n^3 q^{-n} + \mathcal{O}(q)$ .
- (ii)  $F_1 = -f'''$  where prime denotes differentiation by

$$\frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.$$

We see  $f'''$  lies in  $M_4^!$  by the  $k = 4$  case of Bol's identity

$$\frac{d^{k-1}}{d\tau^{k-1}}(f|_{2-k}\gamma) = \left( \frac{d^{k-1}f}{d\tau^{k-1}} \right) |_{k}\gamma \quad \forall \gamma \in \Gamma.$$

Since, the Fourier expansion of  $f'''$  begins as  $-q^{-1} + \mathcal{O}(q)$ , the claim is proven.

- (iii) For  $\mathfrak{J}(\tau_1) > \max_{\gamma \in \Gamma} \mathfrak{J}(\gamma\tau_2)$ ,

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \sum_{n=0}^{\infty} F_n(\tau_2)q_1^n.$$

Let  $L(\tau_1, \tau_2)$  denote the left side of (4.2). We see  $L(\tau_1, \tau_2)$  is a meromorphic modular form in  $\tau_2$  with a simple pole of residue  $-\frac{1}{2\pi i}$  at  $\tau_2 = \tau_1$  (since  $j' = -E_4^2 E_6 / \eta^{24}$ ) and no poles outside  $\Gamma\tau_1$ . Moreover,  $L(\tau_1, \tau_2)$  tends to 0 as  $\mathfrak{J}(\tau_2) \rightarrow \infty$ . These properties characterize  $L(\tau_1, \tau_2)$  uniquely and show that the  $n^{\text{th}}$  Fourier coefficient with respect to  $\tau_1$  for  $\mathfrak{J}(\tau_1) \rightarrow \infty$  has the properties characterizing  $F_n(\tau_2)$ .

Combining (i) and (ii) with the formula for the action of  $T_n$  on Fourier expansions, we obtain,

$$(41) \quad F_n(\tau) = (-n^{-3} f''')|T_n = n^{-3} \left( q^{-1} - \sum_{m=1}^{\infty} m^3 c(m) q^m \right) |T_n \\ = q^{-n} - \sum_{\substack{\kappa, \ell, d > 0 \\ kd=n}} \ell^3 c(k\ell) q^{\ell d}$$

for  $n > 0$ . The Harvey-Moore identity follows from (41) and (iii) together with the equality  $F_0 = E_4$ .  $\square$

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