Quantum cohomology of the Hilbert scheme of points in the plane

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Abstract

We determine the ring structure of the equivariant quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$. The operator of quantum multiplication by the divisor class is a nonstationary deformation of the quantum Calogero-Sutherland many-body system. A relationship between the quantum cohomology of the Hilbert scheme and the Gromov-Witten/Donaldson-Thomas correspondence for local curves is proven.

Contents

1 Introduction **2**
1.1 Overview 2
1.2 Quantum differential equation 4
1.3 Relation to Gromov-Witten and Donaldson-Thomas theories . 4
1.4 Acknowledgments 6

2 The operator M_D **6**
2.1 Fock space formalism 6
2.2 Main theorem 7
2.3 Calogero-Sutherland operator 8
2.4 Eigenvectors 9

3 Proof of Theorem 1 **10**
3.1 3-point functions 10
3.2 Definitions 10
3.3 The Nakajima basis revisited 11

3.4 Reduced virtual classes 12
3.4.1 A fixed domain C 12
3.4.2 Ran's results 14
3.4.3 The reduced absolute theory 14
3.5 Additivity 15
3.6 Induction strategy 17
3.7 Induction step: I 18
3.8 Localization 20
3.8.1 Overview 20
3.8.2 Broken maps 21
3.8.3 Localization contributions 23
3.9 Induction step: II 27
3.9.1 Reduced 3-point function 27
3.9.2 Unbroken maps 27
3.9.3 The contribution of dL 29

4 Properties of the quantum ring **33**
4.1 Proof of Corollary 1 33
4.2 Multipoint invariants 34
4.3 Relation to the Gromov-Witten theory of $\mathbb{C}^2 \times \mathbf{P}^1$ 35
4.4 The orbifold $(\mathbb{C}^2)^n/S_n$ 36
4.5 Higher genus 37

1 Introduction

1.1 Overview

The Hilbert scheme Hilb_n of n points in the plane $\mathbb{C}^2$ parametrizes ideals $\mathcal{I} \subset \mathbb{C}[x, y]$ of colength n,

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/\mathcal{I} = n .$$

An open dense set of Hilb_n parameterizes ideals associated to configurations of n distinct points. The Hilbert scheme is a nonsingular, irreducible, quasi-projective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [13, 28] for an introduction.

The symmetries of $\mathbb{C}^2$ lift to the Hilbert scheme. The algebraic torus

$$T = (\mathbb{C}^*)^2$$

acts on $\mathbb{C}^2$ by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y) .$$

The induced T-action on Hilb_n plays a central role in the subject.

The T-equivariant cohomology of Hilb_n has been recently determined, see [6, 17, 18, 19, 34]. As a ring, $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ is generated by the Chern classes of the tautological rank n bundle

$$\mathcal{O}/\mathcal{I} \rightarrow \mathrm{Hilb}_n,$$

with fiber $\mathbb{C}[x, y]/\mathcal{I}$ over $[\mathcal{I}] \in \mathrm{Hilb}_n$. The operator of classical multiplication by the divisor

$$D = c_1(\mathcal{O}/\mathcal{I})$$

in $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ is naturally identified with the Hamiltonian of the Calogero-Sutherland integrable quantum many-body system. The ring $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ is a module over

$$H_T^*(\mathrm{pt}) = \mathbb{Q}[t_1, t_2],$$

where t_1 and t_2 are the Chern classes of the respective factors of the standard representation. The ratio $-t_2/t_1$ of the equivariant parameters is identified with the coupling constant in the Calogero-Sutherland system.

The goal of our paper is to compute the small quantum product on the T-equivariant cohomology of Hilb_n. The matrix elements of the small quantum product count, in an appropriate sense, rational curves meeting three given subvarieties of Hilb_n. The (non-negative) degree of a curve class $\beta \in H_2(\mathrm{Hilb}_n, \mathbb{Z})$ is defined by

$$d = \int_\beta D.$$

Curves of degree d are counted with weight q^d, where q is the quantum parameter. The ordinary multiplication in T-equivariant cohomology is recovered by setting $q = 0$. See [7, 10] for an introduction to quantum cohomology.

Our main result, Theorem 1, is an explicit formula for the operator M_D of small quantum multiplication by D. As a corollary, D is proven to generate the small quantum cohomology ring over $\mathbb{Q}(q, t_1, t_2)$. The ring structure is therefore determined.

The full T-equivariant quantum cohomology in genus 0 (with arbitrary numbers of insertions) is easily calculated from the 3-point invariants. A procedure is presented in Section 4.2. The higher genus invariants are discussed in Section 4.5.

1.2 Quantum differential equation

Our explicit form for M_D implies, the *quantum differential equation*

$$q\frac{d}{dq}\,\psi = \mathsf{M}_D\,\psi\,, \quad \psi(q) \in H_T^*(\mathrm{Hilb}_n, \mathbb{Q})\,, \tag{1}$$

has regular singularities at $q = 0, \infty$, and certain roots of unity. The monodromy of this linear ODE is remarkable [30]. In particular, we prove the monodromy is invariant under

$$t_1 \mapsto t_1 - 1$$

provided

$$t_1 \neq \frac{r}{s}\,, \quad 0 < r \leq s \leq n\,,$$

and similarly for t_2.

As a corollary, when the sum t_1+t_2 of equivariant parameters is an integer, there is no monodromy around the roots of unity. The full monodromy is then abelian (and, in fact, diagonalizable for generic t_1).

Equation (1) may be viewed as an exactly solvable nonstationary generalization of the Calogero-Sutherland system. Several results and conjectures concerning its solutions, which are deformations of Jack polynomials, are presented in [30].

We expect to find similar integrability in the quantum differential equation for the Hilbert scheme of points of any smooth surface.

1.3 Relation to Gromov-Witten and Donaldson-Thomas theories

Consider the projective line $\mathbf{P}^1$ with three distinguished points

$$0, 1, \infty \in \mathbf{P}^1.$$

The T-equivariant Gromov-Witten theory of $\mathbf{P}^1 \times \mathbb{C}^2$ relative to $\{0, 1, \infty\}$. has been calculated in [5]. Let relative conditions be specified by

$$\lambda, \mu, \nu \in \mathcal{P}(n),$$

where $\mathcal{P}(n)$ is the set of partitions of n. Let

$$\mathsf{Z}'_{GW}(\mathbf{P}^1 \times \mathbb{C}^2)_{n[\mathbf{P}^1],\lambda,\mu,\nu} \in \mathbb{Q}(t_1, t_2)((u))$$

be the reduced Gromov-Witten partition function, see Section 3.2 of [5].

The T-equivariant cohomology of Hilb_n has a canonical Nakajima basis indexed by $\mathcal{P}(n)$. Define the series $\langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}_n}$ of 3-point invariants by a sum over curve degrees:

$$\langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}_n} = \sum_{d \geq 0} q^d \langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}_n}_{0,3,d}.$$

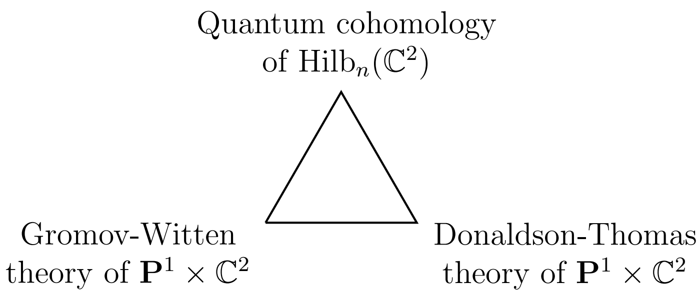
The results of [5] together with our calculation of the 3-point functions of the Hilbert scheme yields a *Gromov-Witten/Hilbert correspondence* discussed in Section 4.3.

Theorem. After the variable change $e^{iu} = -q$,

$$(-iu)^{-n+\ell(\lambda)+\ell(\mu)+\ell(\nu)} \mathsf{Z}'_{GW}(\mathbf{P}^1 \times \mathbb{C}^2)_{n[\mathbf{P}^1],\lambda,\mu,\nu} = (-1)^n \langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}_n}.$$

In fact, our Hilbert scheme calculations were motivated by the correspondence.

The Gromov-Witten and Donaldson-Thomas theories of $\mathbf{P}^1 \times \mathbb{C}^2$ are related by the correspondence conjectured in [24, 25] and refined for the T-equivariant context in [5]. The quantum cohomology of Hilb_n provides a third vertex of equivalence:



The proof of the triangle of equivalences is completed in [29].[1]

The role played by the quantum differential equation (1) in Gromov-Witten and Donaldson-Thomas theories is the following. The fundamental solution of (1) is related, on the one hand, to general triple Hodge integrals on the moduli space of curves and, on the other hand, to the equivariant vertex [24, 25] with one infinite leg, see [29].

[1]The equivalences of the triangle have been extended to the A_n case in [23, 26, 27].

1.4 Acknowledgments

We thank J. Bryan, B. Fantechi, T. Graber, N. Katz, J. Kollár, M. Manetti, D. Maulik, N. Nekrasov, and Z. Ran for many valuable discussions. Both authors were partially supported by the Packard foundation and the NSF.

2 The operator M_D

2.1 Fock space formalism

We review the Fock space description of the T-equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$, see [15, 28]. By definition, the Fock space $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\emptyset$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0\,,$$

and satisfy the commutation relations

$$[\alpha_k, \alpha_l] = k\,\delta_{k+l}\,.$$

A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod \alpha_{-\mu_i}\, v_\emptyset\,. \tag{2}$$

indexed by partitions μ. Here,

$$\mathfrak{z}(\mu) = |\operatorname{Aut}(\mu)| \prod \mu_i$$

is the usual normalization factor. Let the length $\ell(\mu)$ denote the number of parts of the partition μ.

The *Nakajima basis* defines a canonical isomorphism,

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \cong \bigoplus_{n \geq 0} H_T^*(\mathrm{Hilb}_n, \mathbb{Q}). \tag{3}$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\Pi_i \mu_i}[V_\mu]$$

where $[V_\mu]$ is (the cohomological dual of) the class of the subvariety of $\mathrm{Hilb}(\mathbb{C}^2, |\mu|)$ with generic element given by a union of schemes of lengths

$$\mu_1, \dots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of $\mathbb{C}^2$. The vacuum vector $v_\emptyset$ corresponds to the unit in $H_T^*(\mathrm{Hilb}_0, \mathbb{Q})$. As before, t_1, t_2 are the equivariant parameters corresponding to the weights of the T-action on $\mathbb{C}^2$.

The subspace of $\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2]$ corresponding to $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ is spanned by the vectors (2) with $|\mu| = n$. The subspace can also be described as the n-eigenspace of the *energy operator*:

$$|\cdot| = \sum_{k>0} \alpha_{-k}\, \alpha_k \,.$$

The vector $|1^n\rangle$ corresponds by to the identity in $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$. A straightforward calculation shows

$$D = -\left|2, 1^{n-2}\right\rangle \,.$$

The standard inner product on the T-cohomology induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$\langle \mu | \nu \rangle = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)} \,. \tag{4}$$

With respect to the inner product,

$$(\alpha_k)^* = (-1)^{k-1} (t_1 t_2)^{\mathrm{sgn}(k)}\, \alpha_{-k} \,. \tag{5}$$

2.2 Main theorem

The following operator on Fock space plays a central role in the paper:

$$\mathsf{M}(q, t_1, t_2) = (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1}\, \alpha_{-k}\, \alpha_k + \frac{1}{2} \sum_{k,l>0} \Big[t_1 t_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \Big] \,. \tag{6}$$

The q-dependence of M is only in the first sum in (6) which acts diagonally in the basis (2). The two terms in the second sum in (6) are known respectively as the splitting and joining terms. The operator M commutes with the energy operator $|\cdot|$, and

$$\mathsf{M}^* = \mathsf{M} \tag{7}$$

with respect to (5).

Theorem 1. *Under the identification* (3),

$$\mathsf{M}_D = \mathsf{M} - \frac{t_1+t_2}{2} \frac{(-q)+1}{(-q)-1} |\cdot| \tag{8}$$

is the operator of small quantum multiplication by the divisor D in the T-equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$.

Corollary 1. *The divisor class D generates the small quantum ring*

$$QH_T^*(\mathrm{Hilb}_n, \mathbb{Q})$$

over $\mathbb{Q}(q, t_1, t_2)$.

In the basis (2), the matrix elements of M_D are *integral* — the matrix elements lie in $\mathbb{Z}[t_1, t_2][[q]]$.

2.3 Calogero-Sutherland operator

The classical multiplication by the divisor D and the connection to the Calogero-Sutherland operator,

$$\mathsf{H}_{CS} = \frac{1}{2} \sum_i \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \theta(\theta - 1) \sum_{i<j} \frac{1}{|z_i - z_j|^2}, \tag{9}$$

are recovered by setting $q = 0$ in M_D.

The operator H_{CS} describes quantum-mechanical particles on the torus $|z_i| = 1$ interacting via the potentials $|z_i - z_j|^{-2}$. The parameter θ adjusts the strength of the interaction. The function

$$\phi(z) = \prod_{i<j} (z_i - z_j)^\theta$$

is an eigenfunction of H_{CS}, and the operator $\phi \, \mathsf{H}_{CS} \, \phi^{-1}$ preserves the space of symmetric polynomials in the variables z_i. Therefore, via the identification

$$p_\mu(z) = \mathfrak{z}(\mu) \; |\mu\rangle \; ,$$

where

$$p_\mu(z) = \prod_k \sum_i z_i^{\mu_k} \; ,$$

the operator $\phi \, \mathsf{H}_{CS} \, \phi^{-1}$ acts on Fock space.

A direct computation shows the operator $\phi \, \mathsf{H}_{CS} \, \phi^{-1}$ equals

$$\Delta_{CS} = \frac{1-\theta}{2} \sum_k k \, \alpha_{-k} \, \alpha_k + \frac{1}{2} \sum_{k,l>0} \Big[\alpha_{-k-l} \, \alpha_k \, \alpha_l + \theta \, \alpha_{k+l} \, \alpha_{-k} \, \alpha_{-l} \Big] \tag{10}$$

modulo scalars and a multiple of the momentum operator $\sum_i z_i \frac{\partial}{\partial z_i}$, see [33]. We find

$$\mathsf{M}(0) = - \, t_1^{\ell(\,\cdot\,)+1} \, \Delta_{CS} \Big|_{\theta = -t_2/t_1} \, t_1^{-\ell(\,\cdot\,)} \; . \tag{11}$$

The well-known duality $\theta \mapsto 1/\theta$ in the Calogero-Sutherland model corresponds to the permutation of t_1 and t_2.

2.4 Eigenvectors

Let λ be a partition of n. Let $\mathcal{I}_\lambda$ denote the associated T-fixed (monomial) ideal,

$$\mathcal{I}_\lambda = \left(x^{j-1} y^{i-1} \right)_{\square = (i,j) \notin \lambda} \subset \mathbb{C}[x,y] \, . \tag{12}$$

The map $\lambda \mapsto \mathcal{I}_\lambda$ is a bijection between the set of partitions $\mathcal{P}(n)$ and the set of T-fixed points $\mathrm{Hilb}_n^T \subset \mathrm{Hilb}_n$.

The eigenvectors of the classical multiplication by D in $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ are the classes of the T-fixed points of Hilb_n,

$$[\mathcal{I}_\lambda] \in H_T^{2n}(\mathrm{Hilb}_n, \mathbb{Q}) \, , \quad \lambda \in \mathcal{P}(n) \, .$$

The eigenvalues of $\mathsf{M}_D(0)$ are determined by the functions

$$-c(\lambda; t_1, t_2) = - \sum_{(i,j) \in \lambda} \Big[(j-1) t_1 + (i-1) t_2 \Big] \, , \tag{13}$$

for $\lambda \in \mathcal{P}(n)$. The sum in (13) is the trace of the T-action on $\mathbb{C}[x,y]/\mathcal{I}_\lambda$.

We will denote by

$$\mathsf{J}^\lambda \in \mathcal{F} \otimes \mathbb{Q}[t_1, t_2]$$

the image of $[\mathcal{J}_\lambda]$ in Fock space. The corresponding symmetric function is the *integral form* of the Jack polynomial [21].

3 Proof of Theorem 1

3.1 3-point functions

We must prove the matrix elements of M_D yield the T-equivariant 3-point functions of the Hilbert scheme:

$$\left\langle \mu \middle| \mathsf{M}_D \middle| \nu \right\rangle = \sum_{d \geq 0} q^d \left\langle \mu, D, \nu \right\rangle^{\mathrm{Hilb}_n}_{0,3,d} , \tag{14}$$

where $\mu, \nu \in \mathcal{P}(n)$. The matrix elements on the left side of (14) are calculated with the nonstandard inner product (4).

The q^d coefficients of the left and right sides of (14) will be denoted by the respective brackets:

$$\left\langle \mu \middle| \mathsf{M}_D \middle| \nu \right\rangle_d , \quad \left\langle \mu, D, \nu \right\rangle_d .$$

The calculation of the T-equivariant (classical) cohomology of Hilb_n implies the equality (14) in degree 0.

3.2 Definitions

Though Hilb_n is not compact, the T-equivariant Gromov-Witten invariants are well-defined. The T-fixed locus of the moduli space of maps to Hilb_n is a proper Deligne-Mumford stack. The T-equivariant Gromov-Witten theory may be defined by a residue integral on the T-fixed locus via the virtual localization formula [14].

An equivalent geometric definition of the T-equivariant Gromov-Witten theory of Hilb_n is obtained in the fixed point basis. The classes of the T-fixed points of Hilb_n span a basis of the (localized) equivariant cohomology of Hilb_n. The locus of maps to Hilb_n meeting a T-fixed point *is* compact. For example, the 3-point functions in the T-fixed point basis are:

$$\left\langle [\mathcal{J}_\mu], [\mathcal{J}_\nu], [\mathcal{J}_\xi] \right\rangle_d = \int_{[\overline{M}_{0,3}(\mathrm{Hilb}_n, d)]^{vir}} \mathrm{ev}_1^*([\mathcal{J}_\mu]) \cup \mathrm{ev}_2^*([\mathcal{J}_\nu]) \cup \mathrm{ev}_3^*([\mathcal{J}_\xi]),$$

where the integral sign denotes T-equivariant push-forward to a point.

By either definition, the T-equivariant Gromov-Witten invariants of Hilb_n with insertions in $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ have values in the ring $\mathbb{Q}(t_1, t_2)$.

3.3 The Nakajima basis revisited

The Nakajima basis was defined in Section 2.1 with respect to the identity element of $H_T^*(\mathbb{C}^2, \mathbb{Q})$. We may also define a Nakajima basis with respect to the class of the origin

$$[\mathbf{0}] = t_1 t_2 \in H_T^4(\mathbb{C}^2, \mathbb{Q}).$$

The Nakajima basis at the origin is determined by:

$$|\mu([\mathbf{0}])\rangle = (t_1 t_2)^{\ell(\mu)} \, |\mu\rangle \in H_T^{2(|\mu|+\ell(\mu))}(\mathrm{Hilb}_{|\mu|}, \mathbb{Q})\,. \tag{15}$$

By linearity,

$$\begin{aligned} \big\langle \mu, D, \nu \big\rangle_d &= \frac{1}{(t_1 t_2)^{\ell(\mu)}} \big\langle \mu([\mathbf{0}]), D, \nu \big\rangle_d \\ &= \frac{1}{(t_1 t_2)^{\ell(\nu)}} \big\langle \mu, D, \nu([\mathbf{0}]) \big\rangle_d\,. \end{aligned} \tag{16}$$

The Gromov-Witten invariants on the right side of (16) are intersection products in a compact space (because of the occurrences of the class $[\mathbf{0}]$) and, therefore, take values in $\mathbb{Q}[t_1, t_2]$. In order for such Gromov-Witten invariants to be nonzero, the total codimension of the insertions must not be less than the virtual dimension of the moduli space of maps.

The virtual dimension of the moduli space of genus 0 maps to Hilb_n with 3 marked points is:

$$\begin{aligned} \text{vir dim}_{\mathbb{C}}\, \overline{M}_{0,3}(\mathrm{Hilb}_n, \beta) &= \int_\beta c_1(T_{\mathrm{Hilb}_n}) + 2n - 3 + 3 \\ &= 2n. \end{aligned}$$

Since Hilb_n is holomorphic symplectic, the first Chern class of the tangent bundle is trivial.

The (complex) codimensions of the insertions $\mu([\mathbf{0}])$, D, and ν sum to

$$|\mu| + \ell(\mu) + 1 + |\nu| - \ell(\nu) = 2n + 1 + \ell(\mu) - \ell(\nu).$$

Similarly, the codimensions of the insertions μ, D, $\nu([\mathbf{0}])$ sum to

$$2n + 1 + \ell(\nu) - \ell(\mu).$$

The inequality

$$|\ell(\mu) - \ell(\nu)| \leq 1 \tag{17}$$

is therefore a necessary condition for the nonvanishing of the 3-point invariants (14).

If $\ell(\mu) \neq \ell(\nu)$, then one of the invariants on the right in (16) has degree 0 in t_1 and t_2 and, therefore, is purely classical by Lemma 2 below. Hence, for $\ell(\mu) \neq \ell(\nu)$, equation (14) reduces to the known formula for the classical multiplication by D.

3.4 Reduced virtual classes

3.4.1 A fixed domain C

Let C be fixed, pointed, nodal, genus g curve. Let

$$M_C(\mathrm{Hilb}_n, d)$$

denote the moduli space of maps from C to Hilb_n of degree $d > 0$. Let

$$\pi : C \times M_C(\mathrm{Hilb}_n, d) \to M_C(\mathrm{Hilb}_n, d)$$

denote the projection, and let

$$f : C \times M_C(\mathrm{Hilb}_n, d) \to \mathrm{Hilb}_n$$

denote the universal map. The canonical morphism

$$R^\bullet\pi_*(f^* T_{\mathrm{Hilb}_n})^\vee \to L^\bullet_{M_C} \tag{18}$$

determines a perfect obstruction theory on $M_C(\mathrm{Hilb}_n, d)$, see [1, 2, 20]. Here, $L^\bullet_{M_C}$ denotes the cotangent complex of $M_C(\mathrm{Hilb}_n, d)$.

Let $dx \wedge dy$ be the standard holomorphic symplectic form on $\mathbb{C}^2$. Let

$$\mathbb{C}dx \wedge dy$$

be the associated 1-dimensional T-representation of weight $-(t_1 + t_2)$. The form $dx \wedge dy$ induces a canonical holomorphic symplectic form γ on Hilb_n. The T-representation $\mathbb{C}\gamma$ has weight $-n(t_1 + t_2)$.

Let Ω_π and ω_π denote respectively the sheaf of relative differentials and the relative dualizing sheaf. There is a canonical map

$$f^*(\Omega_{\mathrm{Hilb}_n}) \to \Omega_\pi \to \omega_\pi.$$

After dualizing, we obtain

$$\omega_\pi^* \to f^*(T_{\mathrm{Hilb}_n}). \tag{19}$$

The map (19) and the holomorphic symplectic form γ on Hilb_n together yield a map

$$f^*(T_{\mathrm{Hilb}_n}) \to \omega_\pi \otimes (\mathbb{C}\gamma)^\vee.$$

We obtain

$$R^\bullet\pi_*(\omega_\pi)^\vee \otimes \mathbb{C}\gamma \to R^\bullet\pi_*(f^*T_{\mathrm{Hilb}_n})^\vee.$$

Finally, we consider the induced cut-off map

$$\iota : \tau_{\leq -1}R^\bullet\pi_*(\omega)^\vee \otimes \mathbb{C}\gamma \to R^\bullet\pi_*(f^*T_{\mathrm{Hilb}_n})^\vee.$$

The complex $\tau_{\leq -1}R^\bullet\pi_*(\omega)^\vee \otimes \mathbb{C}\gamma$ is represented by a trivial bundle of rank 1 with representation $\mathbb{C}\gamma$ in degree -1. Consider the mapping cone $C(\iota)$ of ι. Certainly $R^\bullet\pi_*(f^*T_{\mathrm{Hilb}_n})^\vee$ is represented by a two term complex. An elementary argument using the positive degree $d > 0$ condition shows the complex $C(\iota)$ is also two term.

By Ran's results[2] on deformation theory and the semiregularity map, there is a canonical map

$$C(\iota) \to L^\bullet_{M_C} \tag{20}$$

induced by (18), see [32]. Ran proves the obstructions to deforming maps from C to a holomorphic symplectic manifold lie in the kernel of the semiregularity map. After dualizing, Ran's result precisely shows (18) factors through the cone $C(\iota)$.

The map (20) defines a *new* perfect obstruction theory on $M_C(\mathrm{Hilb}_n, d)$. The conditions of cohomology isomorphism in degree 0 and the cohomology surjectivity in degree -1 are both induced from the perfect obstruction theory (18).

We view (18) as the *standard* obstruction theory and (20) as the *reduced* obstruction theory. Both obstruction theories are T-equivariant since the morphism of complexes involved are T-equivariant.

[2]The required deformation theory can also be found in a recent paper by M. Manetti [22]. The comments of Sections 3.4.1 and 3.4.2 apply equally to Manetti's results.

3.4.2 Ran's results

Two aspects of the application of Ran's deformation results here warrant further comment.

First, a main technical advance in [32] is the study of obstructions for deformations over Artin local rings: the case of deformations over the curvilinear schemes $\mathbb{C}[\epsilon]/(\epsilon^n)$ was treated earlier in [3, 31]. The Artin local case is needed here.

Second, Ran's proof requires a nonsingular projective target variety with a holomorphic symplectic form. While Hilb_n is *not* complete, Ran's argument can be nevertheless be applied by the following construction. Let

$$f : C \to \mathrm{Hilb}_n$$

be a stable map. The image of f under composition with the Hilbert/Chow morphism,

$$\rho_{HC} : \mathrm{Hilb}_n \to \mathrm{Sym}^n\mathbb{C}^2,$$

must be a point

$$\sum_{i=1}^{\ell(\mu)} \mu_i[p_i].$$

We may view f as a map to the fiber $\rho_{HC}^{-1}(\sum \mu_i[p_i])$ of the Hilbert/Chow morphism.

Let S be a nonsingular, projective, $K3$ surface, and let $q_1, \ldots, q_{\ell(\mu)}$ be distinct points. Consider the fiber

$$\rho_{HC}^{-1}\left(\sum \mu_i[q_i]\right) \subset \mathrm{Hilb}(S, n).$$

Local analytic charts on $\mathbb{C}^2$ and S at the points p_i and q_i induce a local analytic isomorphism of Hilb_n and $\mathrm{Hilb}(S, n)$ in a neighborhood of the two fibers of ρ_{HC}. Hence, the deformation theory of f over Artin local rings can be studied on $\mathrm{Hilb}(S, n)$. Since S and $\mathrm{Hilb}(S, n)$ are holomorphic symplectic, Ran's results imply the obstructions lie in the kernel of the semiregularity map for $\mathrm{Hilb}(S, n)$. The statement implies precisely the required deformation theory statement for Hilb_n.

3.4.3 The reduced absolute theory

The results of Sections 3.4.1-3.4.2 define a T-equivariant reduced obstruction theory of maps to Hilb_n relative to the Artin stack $\mathfrak{M}$ of pointed genus g

curves. A T-equivariant reduced absolute theory is obtained via a distinguished triangle in the usual way, see [1, 2, 20].

Since the new obstruction theory differs from the standard theory by the 1-dimensional obstruction space $(\mathbb{C}\gamma)^\vee$, we find

$$\begin{aligned} [\overline{M}_{g,n}(\mathrm{Hilb}_n, d)]^{vir}_s &= c_1\left((\mathbb{C}\gamma)^\vee\right) \cap [\overline{M}_{g,n}(\mathrm{Hilb}_n, d)]^{vir}_r \\ &= (t_1+t_2)\cdot [\overline{M}_{g,n}(\mathrm{Hilb}_n, d)]^{vir}_r, \end{aligned}$$

for $d > 0$. Here, s and r denote the standard and reduced theories.

Lemma 2. *T-equivariant Gromov-Witten invariants of* Hilb_n *of positive degree with insertions in* $H^*_T(\mathrm{Hilb}_n, \mathbb{Q})$ *are divisible by* $t_1 + t_2$.

Proof. The T-equivariant Gromov-Witten invariants lie in $\mathbb{Q}(t_1, t_2)$. Divisibility is defined by positive valuation at $t_1 + t_2$.

Consider the T-equivariant Gromov-Witten theory of Hilb_n in the Nakajima basis at the origin. By compactness, the invariants lie in $\mathbb{Q}[t_1, t_2]$. By the construction of the reduced virtual class, the invariants are divisible by $t_1 + t_2$.

The Nakajima basis with respect to the identity spans $H^*_T(\mathrm{Hilb}_n, \mathbb{Q})$ as a $\mathbb{Q}$-vector space. The relation (15) concludes the proof. □

3.5 Additivity

Denote the reduced invariants of Hilb_n by curved brackets:

$$\left\langle \mu([\mathbf{0}]), D, \nu \right\rangle_d = (t_1+t_2)\left(\mu([\mathbf{0}]), D, \nu\right)_d, \quad d > 0\,.$$

If $\ell(\mu) = \ell(\nu)$, the integral $\left(\mu([\mathbf{0}]), D, \nu\right)_d$ is a nonequivariant constant. Let

$$\xi = \sum \mu_i [p_i] \ \in \mathrm{Sym}^n\, \mathbb{C}^2,$$

where $\{p_1, \ldots, p_{\ell(\mu)}\} \subset \mathbb{C}^2$ are distinct points. By the definition of the Nakajima basis, we can replace the equivariant class $\mu([\mathbf{0}])$ in the integrand by the nonequivariant class

$$\mu(\xi) = \frac{1}{\mathfrak{z}(\mu)}\, \rho^{-1}_{HC}(\xi) \tag{21}$$

where

$$\rho_{HC} : \text{Hilb}_n \to \text{Sym}^n \mathbb{C}^2$$

is the Hilbert/Chow morphism as before.

Every rational curve in Hilb_n is contracted by the Hilbert/Chow morphism. The moduli space of maps connecting the locus $\mu(\xi)$ and the Nakajima cycle ν is isomorphic to the moduli space of stable maps to the product

$$\prod_{i=1}^{\ell(\mu)} \text{Hilb}_{|\mu_i|, p_i} \tag{22}$$

in case $\mu = \nu$ and empty otherwise. Here, $\text{Hilb}_{m,p} \subset \text{Hilb}_m$ denotes the subspace of schemes supported at p.

The moduli space of maps to the product,

$$\overline{M}_{0,3}\left(\prod_{i=1}^{\ell(\mu)} \text{Hilb}_{|\mu_i|, p_i}, d\right), \tag{23}$$

has components corresponding to the different distributions of the total degree d among the factors. Consider a component

$$\overline{M}[j,k] \subset \overline{M}_{0,3}\left(\prod_{i=1}^{\ell(\mu)} \text{Hilb}_{|\mu_i|, p_i}, d\right),$$

for which the degree splitting has at least two non-zero terms corresponding to the points p_j and p_k.

The moduli space $\overline{M}[j,k]$ has a standard obstruction theory obtained from the standard obstruction theory of $\overline{M}_{0,3}(\text{Hilb}_n, d)$. The standard obstruction theory of $\overline{M}[j,k]$ has a 2-dimensional quotient obtained from the 2-dimensional family of holomorphic symplectic forms (at p_j and p_k). Exactly following the construction of Section 3.4, we obtain a *doubly* reduced obstruction theory by reducing the obstruction space by the 2-dimensional quotient. The nonequivariant integral of the (singly) reduced theory over such a component vanishes since the singly reduced theory contains an additional 1-dimensional trivial factor.

We conclude the only components of (23) which contribute to the integral

$$\Big(\mu(\xi), D, \mu\Big)_d$$

are those for which the degree d is distributed entirely to a single factor of the product.

After unraveling the definitions, we obtain three basic results governing the insertion D:

(i) $\big\langle \mu, D, \nu \big\rangle_{d>0} = 0$ for $\mu \neq \nu$,

(ii) $\big\langle \mu, D, \mu \big\rangle_d = \gamma_{n,d}(t_1t_2)^{-\ell(\mu)}(t_1+t_2)$ where $\gamma_{n,d} \in \mathbb{Q}$,

(iii) the *addition formula*,

$$\frac{\big\langle \mu, D, \mu \big\rangle_{d>0}}{\big\langle \mu \big| \mu \big\rangle} = \sum_i \frac{\big\langle \mu_i, D, \mu_i \big\rangle_{d>0}}{\big\langle \mu_i \big| \mu_i \big\rangle} . \tag{24}$$

The corresponding properties of M_D, including the addition formula,

$$\frac{\big\langle \mu, \mathsf{M}_D, \mu \big\rangle_{d>0}}{\big\langle \mu \big| \mu \big\rangle} = \sum_i \frac{\big\langle \mu_i, \mathsf{M}_D, \mu_i \big\rangle_{d>0}}{\big\langle \mu_i \big| \mu_i \big\rangle} , \tag{25}$$

are directly verified.

3.6 Induction strategy

We will prove Theorem 1 by induction on n. If $n = 0, 1$, the operator M_D vanishes. The insertion D is 0 for $n = 0, 1$, so Theorem 1 is valid.

Let $n > 1$. We proceed by induction on the degree d. The induction step relies upon the addition formulas (24)-(25). For each degree $d \geq 1$, we will compute a 3-point invariant

$$\big\langle \gamma_1, D, \gamma_2 \big\rangle_d$$

for which the expansions of the classes

$$\gamma_1, \gamma_2 \in H_T^{4n}(\mathrm{Hilb}_n, \mathbb{Q}),$$

in the Nakajima basis contain nontrivial multiples *not divisible by* (t_1+t_2) of the class $|n\rangle$. By the addition rules, if

$$\Big\langle \gamma_1 \Big| \mathsf{M}_D \Big| \gamma_2 \Big\rangle_d = \big\langle \gamma_1, D, \gamma_2 \big\rangle_d , \tag{26}$$

then (14) is proven for Hilb_n in degree d.

Both sides of (26) are constant multiples of $t_1^{2n}(t_1+t_2)$ modulo $(t_1+t_2)^2$. Since $\big\langle (n), D, (n) \big\rangle_d$ is determined by the constant $\gamma_{n,d}$ where

$$\big\langle (n), D, (n) \big\rangle_d = -\frac{\gamma_{n,d}}{t_1^2}(t_1+t_2) \mod (t_1+t_2)^2,$$

we need only verify the equality (26) modulo $(t_1+t_2)^2$.

3.7 Induction step: I

Let $n > 1$ and let $d \geq 1$. For the induction step, we will compute the invariant

$$\big\langle \left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right] \big\rangle_d . \tag{27}$$

Following the notation of Section 2.4, $\mathcal{I}_\lambda$ denotes the monomial ideal corresponding to the partition λ, and $[\mathcal{I}_\lambda]$ denotes the T-equivariant class of the associated fixed point in $\mathrm{Hilb}_{|\lambda|}$.

The T-fixed point $[\mathcal{I}_\lambda]$ corresponds to the Jack polynomial

$$\mathsf{J}^\lambda \in \mathcal{F} \otimes \mathbb{Q}[t_1, t_2].$$

For $\theta = -t_2/t_1 = 1$, the Jack polynomials specialize to the Schur functions. Hence,

$$\mathsf{J}^\lambda \equiv \frac{(-1)^{|\lambda|}\,|\lambda|!}{\dim \lambda} \sum_\mu \chi^\lambda_\mu \, t_1^{|\lambda|+\ell(\mu)} \left|\mu\right\rangle \mod t_1+t_2\,,$$

where $\dim \lambda$ is the dimension of the representation λ of the symmetric group and χ^λ_μ is the associated character evaluated on the conjugacy class μ, see [21]. In particular, the coefficient of $\left|n\right\rangle$ in the expansion of both $\mathsf{J}^{(n)}$ and $\mathsf{J}^{(n-1,1)}$ is nonzero.

The operator $\mathsf{M}_D - \mathsf{M}_D(0)$ formed by the terms of positive q degree in the operator M_D acts diagonally in the basis $\left|\mu\right\rangle$. Since $\chi^{(n)}$ is the trivial character and

$$\dim(n-1,1) = n-1\,,$$

we conclude

$$\left\langle \mathsf{J}^{(n)} \middle| \mathsf{M}_D - \mathsf{M}_D(0) \middle| \mathsf{J}^{(n-1,1)} \right\rangle \equiv$$
$$(-1)^n (t_1+t_2)\,\frac{t_1^{2n}\,(n!)^2}{n-1}\left(\chi^{(n-1,1)}, F\right)_{L^2(S(n))} \mod (t_1+t_2)^2\,,$$

where F is the function on the symmetric group $S(n)$ taking the value

$$F(\mu) = -|\mu| \frac{q}{1+q} - \sum_i \mu_i^2 \frac{(-q)^{\mu_i}}{1-(-q)^{\mu_i}} \tag{28}$$

on a permutation with cycle type μ, and $(\,,\,)$ is the standard inner product on $L^2(S(n))$ with respect to which the characters are orthonormal.

The first term in (28) is a constant function and, hence, orthogonal to any nontrivial character. The second term can be written as

$$\sum_i \mu_i^2 \frac{(-q)^{\mu_i}}{1-(-q)^{\mu_i}} = \sum_{k\geq 1} f''(\mu, (-q)^k)\,,$$

where

$$f(\mu, z) = \sum_i z^{\mu_i}$$

and differentiation is taken with respect to the operator $z\frac{d}{dz}$,

$$f''(\mu, z) = \left(z \frac{d}{dz}\right)^2 f(\mu, z)\,.$$

The evaluation of the inner product $\left(\chi^{(n-1,1)}, f''\right)$ is obtained by differentiating the

$$(a, b, c) = (n-1, 1, 0)$$

case of the following result.

Lemma 3. *The Fourier coefficients of f' are:*

$$\left(\chi^\lambda, f'\right)_{L^2(S(n))} = \begin{cases} \sum_{k=1}^a z^k\,, & \lambda = (a)\,, \\ (-1)^{c+1}\, z^{a+c+1} + (-1)^c\, z^{b+c}\,, & \lambda = (a, b, 1^c)\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{29}$$

Proof. The inner product $\left(\chi^\lambda, f\right)$ is the image of the Schur function s_λ under the map on symmetric functions induced by following transformations of the power-sums:

$$p_\mu \mapsto f(\mu, z).$$

We observe

$$p_\mu(1, \underbrace{z, \dots, z}_{N \text{ times}}) = 1 + N\, f(\mu, z) + O(N^2)\,.$$

Hence, the map $p_\mu \mapsto f(\mu, z)$ is the linear coefficient in N of the above expansion. By basic properties of the Schur functions, we find

$$s_\lambda(1, \underbrace{z, \ldots, z}_{N \text{ times}}) = \sum_{\nu \prec \lambda} z^{|\nu|}\, s_\nu(\underbrace{1, \ldots, 1}_{N \text{ times}}) = \sum_{\nu \prec \lambda} z^{|\nu|} \prod_{\square \in \nu} \frac{N + c(\square)}{h(\square)}\,, \tag{30}$$

where $\nu \prec \lambda$ means that ν and λ interlace (or, equivalently, that λ/ν is a horizontal strip). The product is over all squares $\square$ in the diagram of ν, $c(\square)$ denotes the content, and $h(\square)$ denotes the hooklength. The linear coefficient in N vanishes unless ν is a hook, a diagram of the form $(a, 1^c)$, in which case the linear coefficient equals $(-1)^c/(a+c)$.

The summation over ν in (30) telescopes to second case in (29) provided λ has more than one row. The last case in (29) corresponds to diagrams not interlaced by a hook. $\square$

After applying Lemma 3, we conclude

$$\left\langle \mathsf{J}^{(n)} \middle| \mathsf{M}_D - \mathsf{M}_D(0) \middle| \mathsf{J}^{(n-1,1)} \right\rangle \equiv$$
$$(-1)^n (t_1 + t_2) \frac{t_1^{2n}\,(n!)^2}{n-1} \left(\frac{q}{1+q} + n \frac{(-q)^n}{1-(-q)^n} \right) \quad \operatorname{mod}\, (t_1+t_2)^2\,. \tag{31}$$

3.8 Localization

3.8.1 Overview

Our goal now is to reproduce the answer (31) by calculating the 3-point invariant

$$\left\langle \left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right] \right\rangle_d$$

via localization on Hilb_n.

Since the T-fixed locus of the moduli space $\overline{M}_{0,3}(\mathrm{Hilb}_n, d)$ is proper, the virtual localization formula of [14] may be applied. However, since Hilb_n contains positive dimensional families of T-invariant curves, a straightforward application is difficult.

Our strategy for computing the 3-point invariant uses vanishings deduced from the existence of the reduced obstruction theory. Since

$$\left\langle \left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right] \right\rangle_d = (t_1+t_2) \Big(\left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right] \Big)_d, \tag{32}$$

calculation of the reduced T-equivariant integral on the right suffices.

Let $T^{\pm} \subset T$ denote the 1-dimensional anti-diagonal torus determined by the embedding

$$T^{\pm} \ni \xi \mapsto (\xi, \xi^{-1}) \in T.$$

Let t denote the equivariant $T^{\pm}$-weight determined by restriction,

$$t = t_1|_{T^{\pm}} = -t_2|_{T^{\pm}}.$$

Since we need to evaluate the T-equivariant integral (32) modulo $(t_1 + t_2)^2$, calculatation of the $T^{\pm}$-equivariant integral

$$\Big(\left[\mathcal{J}_{(n)}\right], D, \left[\mathcal{J}_{(n-1,1)}\right] \Big)_d$$

suffices.

The $T^{\pm}$-fixed points of Hilb_n coincide with the T-fixed points: monomial ideals indexed by partitions $\mathcal{P}(n)$. However, the $T^{\pm}$-fixed point set of the moduli space $\overline{M}_{0,3}(\mathrm{Hilb}_n, d)$ is much larger than the T-fixed point set.

3.8.2 Broken maps

Consider the moduli space $\overline{M}_{0,k}(\mathrm{Hilb}_n, d)$ for $d > 0$. Let

$$\Big[f : C \to \mathrm{Hilb}_n \Big] \in \overline{M}_{0,k}(\mathrm{Hilb}_n, d)$$

be a $T^{\pm}$-fixed map. If p is a marking of C, a (fractional) $T^{\pm}$-weight w_p is defined by the $T^{\pm}$-representation of the tangent space to C at p. Let $P \subset C$ be a component incident to a node s of C. A (fractional) $T^{\pm}$-weight $w_{P,s}$ is defined by the $T^{\pm}$-representation of the tangent space to P at s.

We define a $T^{\pm}$-fixed map f to be *broken* if either of the following two conditions hold:

(i) C contains a connected, f-contracted subcurve $\tilde{C}$ for which the disconnected curve $C \setminus \tilde{C}$ has at least two connected components which have positive degree under f.

(ii) Two non f-contracted components $P_1, P_2 \subset C$ meet at a node s of C and have tangent weights $w_{P_1,s}$ and $w_{P_2,s}$ satisfying $w_{P_1,s} + w_{P_2,s} \neq 0$.

A maximal connected, f-contracted subcurve satisfying (i) is called a *breaking subcurve.* A node satisfying (ii) is called a *breaking node.* A $T^{\pm}$-fixed map which is not broken is *unbroken.*

A connected component of the $T^{\pm}$-fixed locus of $\overline{M}_{0,n}(\mathrm{Hilb}_n, d)$ is of *broken type* if all the corresponding maps are broken. Similarly, a connected component of the $T^{\pm}$-fixed locus is of *unbroken type* if all the corresponding maps are unbroken. By elementary deformation theory, every connected component is either of broken or unbroken type.

Let $\mu, \nu \in \mathcal{P}(n)$ be two partitions of n. A $T^{\pm}$-fixed map $[f] \in \overline{M}_{0,2}(\mathrm{Hilb}_n, d)$ with markings p_1, p_2 is said to *connect* $\mathcal{I}_\mu$ and $\mathcal{I}_\nu$ if

$$f(p_1) = \mathcal{I}_\mu, \ \ f(p_2) = \mathcal{I}_\nu,$$

and the markings do not lie on f-contracted components. By definition, a connecting map must have positive degree.

Lemma 4. *If f is an unbroken $T^{\pm}$-fixed map of degree d connecting the fixed points I_μ and I_ν, then*

$$w_{p_1} = \frac{-c(\mu; t, -t) + c(\nu; t, -t)}{d}$$

Proof. By our definitions and the stability condition, if f is unbroken then the domain C must be a chain of r rational curves

$$C = P_1 \cup \ldots \cup P_r$$

satisfying the condition

$$w_{P_i, s_i} + w_{P_{i+1}, s_i} = 0$$

at the i^{th} node s_i. Since the tangent $T^{\pm}$-representations at the fixed points of each P_i have opposite weights, we conclude $w_{p_1} = -w_{p_2}$.

A localization calculation of the degree of the map f then proves the Lemma:

$$\begin{aligned} d &= \int_{f_*[C]} D \\ &= \int_C c_1(\mathcal{O}/\mathcal{I}) \\ &= \frac{-c(\mu; t, -t)}{w_{p_1}} + \sum_{i=1}^{r-1} \Big(\frac{-c(f(s_i); t, -t)}{w_{P_i, s_i}} + \frac{-c(f(s_i); t, -t)}{w_{P_{i+1}, s_i}} \Big) + \frac{-c(\nu; t, -t)}{w_{p_2}} \\ &= \frac{-c(\mu; t, -t)}{w_{p_1}} + \frac{c(\nu; t, -t)}{w_{p_1}}. \end{aligned}$$

The trace of $T^{\pm}$-action on $\mathcal{O}/\mathcal{I}_\gamma$ is the function $-c(\gamma; t, -t)$, see Section 2.4. $\square$

Let $w^d_{\mu,\nu}$ denote the tangent weight specified by Lemma 4 at p_1 of an unbroken, $T^{\pm}$-fixed, degree d map connecting $\mathcal{I}_\mu$ to $\mathcal{I}_\nu$. Then,

$$w^d_{\nu,\mu} = -w^d_{\mu,\nu}.$$

The tangent weight $w^d_{\mu,\nu}$ is proportional to a tangent weight of Hilb_n at the fixed point μ. Since the $T^{\pm}$-weights of tangent representation of Hilb_n at the fixed points are *never* 0, we conclude the following result.

Lemma 5. *There are no unbroken maps connecting $\mathcal{I}_\mu$ to $\mathcal{I}_\mu$.*

3.8.3 Localization contributions

We study here 2-pointed, $T^{\pm}$-equivariant invariants of Hilb_n in positive degree,

$$\Big([\mathcal{I}_\mu], [\mathcal{I}_\nu]\Big)_d = \int_{[\overline{M}_{0,2}(\mathrm{Hilb}_n,d)]^{vir}_r} \mathrm{ev}_1^*([\mathcal{I}_\mu]) \cup \mathrm{ev}_2^*([\mathcal{I}_\nu]),$$

where $\mu \neq \nu$. Since the evaluation conditions lead to a proper moduli space, the $T^{\pm}$-equivariant push-forward lies in $\mathbb{Q}[t]$.

The virtual localization formula yields a sum over the connected components of the $T^{\pm}$-fixed loci of the moduli space of maps. Let

$$\Big([\mathcal{I}_\mu], [\mathcal{I}_\nu]\Big)_d \;=\; \Big([\mathcal{I}_\mu], [\mathcal{I}_\nu]\Big)^{broken}_d + \Big([\mathcal{I}_\mu], [\mathcal{I}_\nu]\Big)^{unbroken}_d,$$

denote the separate contributions of the components of broken and unbroken type.

We index the $T^{\pm}$-equivariant localization contributions to the 2-point invariant

$$\Big([\mathcal{I}_\mu], [\mathcal{I}_\nu]\Big)_d$$

by graphs. A 2-pointed tree of degree d is a graph $\Gamma = (V, v_1, v_2, \rho, E, \delta)$,

(i) V is a finite vertex set with distinguished elements $v_1 \neq v_2$,

(ii) $\rho : V \to \mathrm{Hilb}_n^{T^{\pm}}$,

(iii) E is a finite edge set,

(iv) $\delta : E \to \mathbb{Z}_{>0}$ is a degree assignment,

satisfying the following conditions

(a) Γ is a connected tree,

(b) $\rho(v_1) = \mathcal{I}_\mu$, $\rho(v_2) = \mathcal{I}_\nu$,

(c) if $v', v'' \in V$ are connected by an edge, then $\rho(v') \neq \rho(v'')$,

(d) if $v \neq v_1, v_2$ has edge valence 2 with neighbors v', v'', then

$$w^{\delta(e(v,v'))}_{\rho(v),\rho(v')} + w^{\delta(e(v,v''))}_{\rho(v),\rho(v'')} \neq 0,$$

(e) $\sum_{e \in E} \delta(e) = d$.

Let $[f] \in \overline{M}_{0,2}(\mathrm{Hilb}_n, d)$ be a $T^{\pm}$-fixed map. We associate a 2-pointed tree,

$$\Gamma_f = (V, v_1, v_2, \rho, E, \delta),$$

of degree d to f by the following construction. The vertex set V is determined by the connected components of $f^{-1}(\mathrm{Hilb}_n^{T^{\pm}})$ *excluding* the non-breaking nodes. In fact,

$$V = V_1 \cup V_2 \cup V_3,$$

is a union of three disjoint subsets:

(1) V_1 is the set of breaking subcurves,

(2) V_2 is the set of breaking nodes,

(3) V_3 is the set of nonsingular points of C lying on non f-contracted components mapped to $\mathrm{Hilb}_n^{T^{\pm}}$.

The two markings of C are associated to distinct elements of $V_1 \cup V_3$ — the markings determine v_1 and v_2. The function ρ is obtained from f. Chains of non f-contracted curves of C link the vertices of V. The edge set E is determined by such chains. The restriction of f to such a chain is an unbroken connecting map. The degree assignment $\delta(e)$ is obtained from the total f-degree of the unbroken connecting map associated to e.

The 2-pointed tree Γ_f is easily seen to satisfy conditions (a)-(e). Condition (c) holds since there are no self connecting maps. By construction, V_3

is exactly the set of extremal (or edge valence 1) vertices of Γ_f. The tree Γ_f is invariant as $[f]$ varies in a connected component of the $T^{\pm}$-fixed locus of $\overline{M}_{0,2}(\mathrm{Hilb}_n, d)$.

Let G_d denote the finite set of 2-pointed trees of degree d. Let

$$\overline{M}_\Gamma \subset \overline{M}_{0,2}(\mathrm{Hilb}_n, d)$$

denote the substack of $T^{\pm}$-fixed maps corresponding to the tree $\Gamma \in G_d$. Let

$$\Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d^\Gamma$$

denote the localization contribution of $\overline{M}_\Gamma$. There is a unique tree in $\Gamma^* \in G$ with a single edge of degree d corresponding to unbroken maps. Let

$$G_d^* = G_d \setminus \{\Gamma^*\}.$$

By summing contributions,

$$\begin{aligned}\Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d &= \sum_{\Gamma \in G_d} \Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d^\Gamma \\ &= \Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d^{unbroken} + \sum_{\Gamma \in G_d^*} \Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d^\Gamma\end{aligned}$$

Lemma 6. *The $T^{\pm}$-equivariant broken contributions vanish.*

Proof. For each $\Gamma \in G_d^*$, $|E| > 1$. We must show the contribution of each such Γ is 0.

Up to automorphisms, the stack $\overline{M}_\Gamma$ factors as a product:

$$\overline{M}_\Gamma = \Big(\prod_{v \in V_1} \overline{M}_v \times \prod_{e \in E} \overline{M}_e\Big) \Big/ \mathrm{Aut}(\Gamma). \tag{33}$$

Here, $\overline{M}_v$ denotes the f-contracted moduli space of pointed genus 0 curves associated to $v \in V_1$, and $\overline{M}_e$ denotes the moduli space of unbroken $T^{\pm}$-fixed maps of degree $\delta(e)$ connecting the $T^{\pm}$-fixed points associated to the vertices incident to e. By the virtual localization formula [14],

$$\Big([\mathcal{J}_\mu], [\mathcal{J}_\nu]\Big)_d^\Gamma = \int_{[\overline{M}_\Gamma]_r^{vir}} \frac{\mathrm{ev}_1^*([\mathcal{J}_\mu]) \cup \mathrm{ev}_2^*([\mathcal{J}_\nu])}{e(N^{vir})}, \tag{34}$$

where the reduced virtual class on $\overline{M}_\Gamma$ is obtained from the $T^\pm$-fixed part of the reduced obstruction theory and N^{vir} is the virtual normal bundle.

The *standard* obstruction theory of $\overline{M}_\Gamma$ is obtained from the $T^\pm$-fixed part of the complex

$$R^\bullet\pi_*(f^*T_{\mathrm{Hilb}_n})^\vee, \tag{35}$$

and the $T^\pm$-fixed part of the cotangent complex of the Artin stack $L^\bullet(\mathfrak{M})$. Here, we follow the notation of Section 3.4, see also [14]. The *reduced* obstruction of $\overline{M}_\Gamma$ is obtained by removing a trivial 1-dimensional subobject from the standard obstruction theory — see Section 3.4.

The normalization sequence for the universal domain at the breaking curves and breaking nodes relates the complex (35) to the corresponding complexes for each factor in the product (33). The normalization sequence on the universal domain is

$$0 \to \bigoplus_{v\in V_1} \mathcal{O}_{C_v} \oplus \bigoplus_{e\in E} \mathcal{O}_{C_e} \to \mathcal{O}_C \to \bigoplus_{s\in I} \mathcal{O}_s \to 0,$$

where C_v, C_e, are the subcurves associated to $v \in V_1$, $e \in E$, and I is the set of all incidence points of the subcurves. After tensoring with the pull-back of T_{Hilb_n} and taking the derived π-push forward to $\overline{M}_\Gamma$, we find the complex (35) differs from the sum of the corresponding complexes of the factors only by the nodal terms $R^\bullet\pi_*(T_{\mathrm{Hilb}_n} \otimes \mathcal{O}_s)^\vee$.

The cohomology of the complex associated to a node $s \in I$ is concentrated in degree 0 and equals the tangent representations at $f(s) \in \mathrm{Hilb}_n^{T^\pm}$. The tangent representations at the $T^\pm$-fixed points $\mathrm{Hilb}_n^{T^\pm}$ have *no* $T^\pm$-fixed parts. Hence, the $T^\pm$-fixed part of (35) is obtained from the sum of $T^\pm$-fixed parts of corresponding complexes of the factors (33).

The complex $L^\bullet(\mathfrak{M})$ differs from the cotangent complexes of the factors (33) by the deformation spaces at the nodes $s \in I$ and possible automorphism factors at the extremal vertices. The deformation spaces at the nodes s have nontrivial $T^\pm$-weights by definition. The automorphism factors may differ at the extremal vertices since the points of C corresponding to V_3 may not be marked while the ends of $\overline{M}_e$ are taken to be marked. The possible automorphism factors have nontrivial $T^\pm$-weights (proportional to tangent weights at the associated $T^\pm$-fixed points of Hilb_n). The $T^\pm$-fixed part of $L^\bullet(\mathfrak{M})$ is therefore also obtained from the sum of cotangent complexes of the factors (33).

We conclude the standard obstruction theory of $\overline{M}_\Gamma$ is obtained from the sum of the standard obstruction theories of the factors (33). For each edge e, the standard obstruction theory of $\overline{M}_e$ admits a trivial 1-dimensional subobject defining the reduced obstruction theory of $\overline{M}_e$. The standard obstruction theory of $\overline{M}_\Gamma$ therefore admits a trivial $|E|$-dimensional subobject (compatible, by definition, with the trivial 1-dimensional subobject defining by the reduced obstruction theory of $\overline{M}_\Gamma$). Hence, the reduced obstruction theory of $\overline{M}_\Gamma$ admits a trivial $(|E|-1)$-dimensional subobject. If $|E|>1$, the reduced virtual class,

$$[\overline{M}_\Gamma]^{vir}_r,$$

simply vanishes. □

3.9 Induction step: II

3.9.1 Reduced 3-point function

We calculate here the $T^{\pm}$-equivariant, reduced, 3-point function

$$\Big\langle\, \big[\mathcal{I}_{(n)}\big]\,, D,\, \big[\mathcal{I}_{(n-1,1)}\big]\, \Big\rangle_d . \tag{36}$$

Using the divisor equation and Lemma 6, we write the reduced 3-point function (36) as:

$$\begin{aligned}\Big\langle\, \big[\mathcal{I}_{(n)}\big]\,, D,\, \big[\mathcal{I}_{(n-1,1)}\big]\, \Big\rangle_d &= d\Big\langle\, \big[\mathcal{I}_{(n)}\big]\,, \big[\mathcal{I}_{(n-1,1)}\big]\, \Big\rangle_d \\ &= d\Big\langle\, \big[\mathcal{I}_{(n)}\big]\,, \big[\mathcal{I}_{(n-1,1)}\big]\, \Big\rangle_d^{unbroken} .\end{aligned}$$

3.9.2 Unbroken maps

We must now determine the set of unbroken $T^{\pm}$-fixed maps of degree d connecting $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$.

An *unbroken T-fixed map* is an an unbroken $T^{\pm}$-fixed maps which is fixed for the *full* T-action on Hilb_n. If f is an unbroken T-fixed map with an irreducible domain, then by an analogue of Lemma 4, we find,

$$w^d_{(n),(n-1,1)} = \frac{-(n-1)t_1+t_2}{d},$$

where $w^d_{(n),(n-1,1)}$ denotes the full T-representation.

Since the tangent T-weights of $\mathcal{I}_{(n)}$ lie in a half space, the T-action on Hilb_n is isomorphic to a linear T-action on a T-invariant affine neighborhood

$\mathbb{A}$ of $\mathcal{I}_{(n)}$. There is a unique tangent weight of $\mathcal{I}_{(n)}$ proportional to $w^d_{(n),(n-1,1)}$. The line $L \subset \mathrm{Hilb}_n$,

$$L : [w_0 : w_1] \to (w_0 x^{n-1} + w_1 y, x^n, xy, y^2)\,, \tag{37}$$

is the unique irreducible, T-invariant curve meeting $\mathcal{I}_{(n)}$ with tangent weight $-(n-1)t_1 + t_2$. Moreover, L connects $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$.

The d-fold cover of L is therefore the *unique* T-fixed map of degree d with irreducible domain connecting $\mathcal{I}_{(n)}$ and $\mathcal{I}_{(n-1,1)}$.

Lemma 7. *There are no unbroken T-fixed maps with reducible domains connecting $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$.*

Proof. Let $[f] \in \overline{M}_{0,2}(\mathrm{Hilb}_n, d)$ be a unbroken T-fixed map. By definition, f consists of a chain of non-contracted rational curves connecting $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$,

$$f : P_1 \cup \ldots \cup P_r \to \mathrm{Hilb}_n,$$

with every node s_i satisfying

$$(t_1 + t_2) \mid \left(w_{P_i,s_i} + w_{P_{i+1},s_i}\right),$$

where $w_{P_i,si}$ are the T-weights.

We will order partitions by the function $\epsilon : \mathcal{P}(n) \to \mathbb{Z}$,

$$\epsilon(\lambda) = c(\lambda; 1, 0) = \sum_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{2},$$

the t_1 coefficient of $c(\lambda; t_1, t_2)$. By convexity, ϵ achieves a strict maximum at the partition (n). The second largest value of ϵ is achieved uniquely at $(n-1, 1)$.

Consider a pair of T-fixed maps h_1 and h_2 connecting three points

$$\mathcal{I}_\mu \overset{h_1}{\text{---}} \mathcal{I}_\nu \overset{h_2}{\text{---}} \mathcal{I}_\xi$$

of Hilb_n. Assume the maps have irreducible domains P_1 and P_2 respectively and the divisibility condition,

$$(t_1 + t_2) \mid (w_{P_1,\nu} + w_{P_2,\nu})\,,$$

is satisfied in the middle. Let d_1 and d_2 be the respective degrees of h_1 and h_2. By localization,

$$d_1 w_{P_1,\nu} = c(\mu; t_1, t_2) - c(\nu; t_1, t_2),$$

$$d_2 w_{P_2,\nu} = c(\xi; t_1, t_2) - c(\nu; t_1, t_2).$$

The tangent weights of $\mathcal{I}_\nu$ are of the form

$$\alpha t_1 + \beta t_2$$

where either $\alpha \geq 0, \beta \leq 0$ or $\alpha \leq 0, \beta \geq 0$. The T-weights $w_{P_1,\nu}$ and $w_{P_2,\nu}$ are proportional to tangent weights of $\mathcal{I}_\nu$. Hence, by the divisibility condition and the tangent weight inequalities, the t_1 coefficients of $w_{P_1,\nu}$ and $w_{P_2,\nu}$ must have opposite signs. We conclude either the condition

$$\epsilon(\mathcal{I}_\mu) \geq \epsilon(\mathcal{I}_\nu) \geq \epsilon(\mathcal{I}_\xi)$$

or the condition

$$\epsilon(\mathcal{I}_\mu) \leq \epsilon(\mathcal{I}_\nu) \leq \epsilon(\mathcal{I}_\xi)$$

holds.

For the unbroken T-fixed map f, the T-fixed points $f(s_i)$ must have ϵ values lying between $\epsilon(\mathcal{I}_{(n)})$ and $\epsilon(\mathcal{I}_{(n,1)})$. The latter condition is only possible if, for each node,

$$f(s_i) = \mathcal{I}_{(n)} \quad \text{or} \quad \mathcal{I}_{(n-1,1)}.$$

Since f is reducible, there exists at least one node. We reach a contradiction since there are no T-fixed maps with irreducible domains connecting a T-fixed point of Hilb_n to itself. □

The d-fold cover of L is thus the unique unbroken T-fixed map connecting $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$. Since linearized T-actions on positive dimensional varieties must have at least 2 fixed points, we conclude dL is the unique unbroken $T^\pm$-fixed map connecting $\mathcal{I}_{(n)}$ to $\mathcal{I}_{(n-1,1)}$.

3.9.3 The contribution of dL

We have proven the equality:

$$\left\langle \left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right] \right\rangle_d^T \mod (t_1+t_2)^2 = d(t_1+t_2)\Big(\left[\mathcal{I}_{(n)}\right], \left[\mathcal{I}_{(n-1,1)}\right] \Big)_d^{dL,T^\pm},$$

where the respective equivariant groups are made explicit in the notation. The right side is equal to

$$d\left\langle \left[\mathcal{I}_{(n)}\right], \left[\mathcal{I}_{(n-1,1)}\right]\right\rangle_d^{dL,T} \quad \text{mod } (t_1+t_2)^2$$

To match the answer of (31), we will calculate the latter T-equivariant contribution of dL.

The contribution is obtained from the T-weights of the representations

$$H^0(C, f^*(T_{\mathrm{Hilb}_n})), \quad H^1(C, f^*(T_{\mathrm{Hilb}_n})),$$

where

$$f : C \to L$$

is the unique T-fixed unbroken map connecting $\mathcal{I}_{(n)}$ and $\mathcal{I}_{(n-1,1)}$.

For $n > 2$, the restriction of T_{Hilb_n} to L splits into T-equivariant line bundles:

$$T_{\mathrm{Hilb}_n}\Big|_L = \mathcal{O}(2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{2n-4},$$

where the first summand is the tangent bundle of L. The T-weights of the trivial part are:

$$t_1, 2t_1, \ldots, (n-2)t_1, \quad t_2, t_2 - t_1, \ldots, t_2 - (n-3)t_1 .$$

The corresponding flat deformations of the ideal (37) with weight kt_1 can be given explicitly by

$$(w_0 x^{n-1} + w_1 y + \epsilon w_0 x^{n-1-k}, x^n + \epsilon x^{n-k}, xy, y^2), \quad \epsilon^2 = 0,$$

where $k = 1, \ldots, n-2$. Similarly,

$$(w_0 x^{n-1} + w_1 y + \epsilon w_1 x^k, x^n, xy + \epsilon x^{k+1}, y^2 + 2\epsilon \delta_{k,0} y), \quad \epsilon^2 = 0,$$

where $k = 0, \ldots, n-3$, is a flat deformation with weight $t_2 - kt_1$.

The T-weights of the nontrivial summands are recorded in the following table:

	$\mathcal{I}_{(n)}$	$\mathcal{I}_{(n-1,1)}$
$\mathcal{O}(2)$	$t_2-(n-1)\,t_1$	$(n-1)\,t_1-t_2$
$\mathcal{O}(-2)$	nt_1	$2t_2-(n-2)\,t_1$
$\mathcal{O}(1)$	$t_2-(n-2)\,t_1$	t_1
$\mathcal{O}(-1)$	$(n-1)\,t_1$	t_2

For $n = 2$, the T-equivariant splitting of T_{Hilb_2} on L takes a different form,

$$T_{\mathrm{Hilb}_2}\Big|_L = \mathcal{O}(2) \oplus \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}\,.$$

The T-weights are:

	$\mathcal{I}_{(2)}$	$\mathcal{I}_{(1,1)}$
$\mathcal{O}(2)$	$t_2 - t_1$	$t_1 - t_2$
$\mathcal{O}(-2)$	$2t_1$	$2t_2$
$\mathcal{O}$	t_2	t_2
$\mathcal{O}$	t_1	t_1

The weights for the $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ summand for $n > 2$ are switched in the $n = 2$ case for the $\mathcal{O} \oplus \mathcal{O}$ summand.

The T-representation $H^0(C, f^*(T_{\mathrm{Hilb}_n}))$ will be shown below to have a the single 0 weight obtained from reparameterization. As a consequence, $[f] \in \overline{M}_{0,2}(\mathrm{Hilb}_n, d)$ is a nonsingular point of the T-fixed locus. Then, the localization contribution is:

$$d\left\langle \left[\mathcal{I}_{(n)}\right], \left[\mathcal{I}_{(n-1,1)}\right]\right\rangle_d^{dL} =$$

$$\frac{d}{d}\, e\Big(T_{\mathrm{Hilb}_n,\mathcal{I}_{(n)}}\Big) e\Big(T_{\mathrm{Hilb}_n,\mathcal{I}_{(n-1,1)}}\Big) \frac{e\Big(H^1(C, f^*(T_{\mathrm{Hilb}_n}))\Big)}{e\Big(H^0(C, f^*(T_{\mathrm{Hilb}_n})) - 0\Big)}, \quad (38)$$

where e denotes the T-equivariant Euler class. The $1/d$ term in front is obtained from the automorphisms of f.

We start by calculating the weights of $H^0(C, f^*(T_{\mathrm{Hilb}_n}))$. The shorthand

$$\tau = (n-1)t_1 - t_2$$

will be convenient for the formulas.

- The weights of $H^0(C, f^*(\mathcal{O}(2)))$, with the exception of the 0 weight obtained from reparameterization, multiply to

$$\prod_{k=0}^{d-1} \left(t_2 - (n-1)t_1 + \frac{k}{d}\tau\right)^2 (-1)^d \equiv$$

$$(-1)^d \left(\frac{t_1 n}{d}\right)^{2d} (d!)^2 \mod (t_1 + t_2)\,.$$

The calculation of $H^0(C, f^*(\mathcal{O}(1)))$ is separated into two cases:

- If $n \not| d$, the weights of $H^0(C, f^*(\mathcal{O}(1)))$ are

$$\prod_{k=0}^{d} \left(t_2 - (n-2)t_1 + \frac{k}{d}\tau \right) \equiv \left(\frac{t_1 n}{d}\right)^{d+1} \frac{\Gamma\left(\frac{d}{n}+1\right)}{\Gamma\left(\frac{d}{n}-d\right)} \mod (t_1+t_2)\,. \quad (39)$$

- If n divides d, then the factor in (39) corresponding to $k = d - \frac{d}{n}$ equals $\frac{1}{n}(t_1+t_2)$ – reflected by the pole of the Γ-function in the denominator. In case $n|d$, the product (39) equals

$$\frac{1}{n}\left(\frac{t_1 n}{d}\right)^{d} \frac{\Gamma\left(\frac{d}{n}+1\right)}{\Gamma\left(\frac{d}{n}-d+t_1+t_2\right)} \mod (t_1+t_2)^2\,.$$

The trivial summands of $T_{\mathrm{Hilb}_n}|_L$ contribute to $H^0(C, f^*(T_{\mathrm{Hilb}_n}))$.

- The weights of the trivial summands multiply to

$$(-1)^{n-2}(n-2)!^2\, t_1^{2n-4} \mod (t_1+t_2)\,,$$

The computation of the representation $H^0(C, f^*(T_{\mathrm{Hilb}_n}))$ is complete, and the 0 weight assertion is verified.

Next, we calculate the weights of $H^1(C, f^*(T_{\mathrm{Hilb}_n}))$. There are only two summands to consider.

- The weights of $H^1(C, f^*(\mathcal{O}(-2)))$ are

$$\prod_{k=1}^{2d-1} \left(2t_2 - (n-2)t_1 + \frac{k}{d}\tau \right) \equiv (-1)^{d-1}(t_1+t_2)\left(\frac{t_1 n}{d}\right)^{2d-2} (d-1)!^2 \mod (t_1+t_2)^2\,.$$

The calculation of $H^1(C, f^*(\mathcal{O}(-1)))$ is separated into two cases:

- If $n \not| d$, the weights of $H^1(C, f^*(\mathcal{O}(-1)))$ are

$$\prod_{k=1}^{d-1}\left(t_2 + \frac{k}{d}\tau\right) \equiv (-1)^{d-1}\left(\frac{t_1 n}{d}\right)^{d-1} \frac{\Gamma\left(\frac{d}{n}\right)}{\Gamma\left(\frac{d}{n} - d + 1\right)} \quad \bmod (t_1+t_2)\,.$$

- When $n|d$, the weights of $H^1(C, f^*(\mathcal{O}(-1)))$ are

$$(-1)^d\,\frac{n-1}{n}\left(\frac{t_1 n}{d}\right)^{d-2} \frac{\Gamma\left(\frac{d}{n}\right)}{\Gamma\left(\frac{d}{n} - d + 1 + t_1 + t_2\right)} \quad \bmod (t_1+t_2)^2\,.$$

Finally, we require the Euler classes of

$$T_{\mathrm{Hilb}_n, \mathcal{I}_{(n)}}, \quad T_{\mathrm{Hilb}_n, \mathcal{I}_{(n-1,1)}}.$$

The product of the tangent weights at the two points is

$$\frac{(n!)^4}{(n-1)^2}\, t_1^{4n} \quad \bmod (t_1+t_2)\,.$$

The contribution of dL is obtained by substituting the weight calculations in (38). We find, modulo $(t_1+t_2)^2$,

$$\left\langle \left[\mathcal{I}_{(n)}\right], D, \left[\mathcal{I}_{(n-1,1)}\right]\right\rangle_d \equiv \begin{cases} (-1)^{n+d-1}(t_1+t_2)\,\dfrac{t_1^{2n}\,(n!)^2}{n-1}\,, & n \not| d\,, \\ (-1)^{n+d}(t_1+t_2)\,t_1^{2n}\,(n!)^2\,, & n|d\,. \end{cases}$$

The generating function for the numbers on the right is precisely (31). The proof of Theorem 1 is complete. □

4 Properties of the quantum ring

4.1 Proof of Corollary 1

The limiting operator,

$$\lim_{t\to\infty}\frac{1}{t}\,\mathsf{M}_D\left(q, t, t^{-1}\right) = \sum_{k>0}\left(\frac{k}{2}\frac{(-q)^k+1}{(-q)^k-1} - \frac{1}{2}\frac{(-q)+1}{(-q)-1}\right)\alpha_{-k}\,\alpha_k,$$

is diagonal with distinct eigenvalues. Hence, $\mathsf{M}_D(q, t_1, t_2)$ has distinct eigenvalues for generic values of the parameters.

Since the classical ring $H_T^*(\text{Hilb}_n, \mathbb{Q})$ is semisimple after localization, the quantum ring $QH_T^*(\text{Hilb}_n, \mathbb{Q})$ is also semisimple after localization. The idempotents of the quantum ring are eigenvectors of quantum multiplication by D. The vector $|1^n\rangle$ represents the unit in $QH_T^*(\text{Hilb}_n, \mathbb{Q})$. Since the unit is the sum of all idempotents, the action of M_D on $|1^n\rangle$ generates the n-eigenvalue subspace of Fock space. Hence, D generates $QH_T^*(\text{Hilb}_n, \mathbb{Q})$ after extending scalars to the field $\mathbb{Q}(q, t_1, t_2)$. □

4.2 Multipoint invariants

All 3-point, genus 0 T-equivariant Gromov-Witten invariants of Hilb_n in the Nakajima basis,

$$\langle \lambda, \mu, \nu \rangle_{0,3,d}^{\text{Hilb}_n},$$

are determined by Theorem 1 and Corollary 1. The algorithm below can be used to reconstruct multipoint genus 0 invariants from 3-point invariants.

Let $D^{*k} \in QH_T^*(\text{Hilb}_n, \mathbb{Q})$ denote the k^{th} power of D with respect to quantum multiplication. Since the set

$$\{D^{*k}\}_{0 \le k \le |\mathcal{P}(n)|-1},$$

spans $QH_T^*(\text{Hilb}_n, \mathbb{Q})$ after extension of scalars, there is a natural filtration of the quantum ring by degree in D. We will filter the multipoint invariants of Hilb_n by, first, the number of insertions m and, second, the minimal degree k in D among the insertions.

Since all 3-point invariants are known, we assume $m \ge 4$. Since insertions of degree 0 and 1 in D can be removed by the T-equivariant fundamental class and divisor equations, we assume $k \ge 2$.

Let the following bracket denote a series of m-pointed invariants of Hilb_n of minimal degree k,

$$\left\langle D^{*k}, \lambda, \mu, - \right\rangle^{\text{Hilb}_n} = \sum_{d \ge 0} q^d \left\langle D^{*k}, \lambda, \mu, - \right\rangle_{0,m,d}^{\text{Hilb}_n}.$$

The dash stands for $m-3$ other insertions.

Let $\overline{M}_{0,4}$ be the moduli space of 4-pointed, genus 0 curves. Let

$$(12|34),\ (13|24),\ (14|23) \in \overline{M}_{0,4}$$

denote the three boundary divisors. Let

$$\pi : \overline{M}_{0,m+1}(\text{Hilb}_n, d) \to \overline{M}_{0,4}$$

be the T-equivariant map obtained by forgetting all the data except for the first four marking.

Consider the following $(m+1)$-point invariant with domain restriction determined by $\xi \in \overline{M}_{0,4}$,

$$\left\langle D, D^{*(k-1)}, \lambda, \mu, \text{---} \right\rangle_{\xi}^{\mathrm{Hilb}_n} = \sum_{d\geq 0} q^d \int_{[\overline{M}_{0,m+1}(\mathrm{Hilb}_n,d)]^{vir}} \mathrm{ev}_1^*(D)\ \mathrm{ev}_2^*(D^{*(k-1)})\ \mathrm{ev}_3^*(\lambda)\ \mathrm{ev}_4^*(\mu)\ (\text{---})\ \pi^*([\xi]).$$

Since points in $\overline{M}_{0,4}$ are cohomologically equivalent, the equality

$$\left\langle D, D^{*(k-1)}, \lambda, \mu, \text{---} \right\rangle_{(12|34)}^{\mathrm{Hilb}_n} = \left\langle D, D^{*(k-1)}, \lambda, \mu, \text{---} \right\rangle_{(13|24)}^{\mathrm{Hilb}_n}$$

yields the WDVV-equation,

$$\sum_{\nu} \left\langle D, D^{*(k-1)}, \text{---}, \nu \right\rangle \langle \nu^\vee, \lambda, \mu \rangle + \left\langle D, D^{*(k-1)}, \nu \right\rangle \langle \nu^\vee, \lambda, \mu, \text{---} \rangle =$$
$$\sum_{\nu} \langle D, \lambda, \text{---}, \nu \rangle \left\langle \nu^\vee, D^{*(k-1)}, \mu \right\rangle + \langle D, \lambda, \nu \rangle \left\langle \nu^\vee, D^{*(k-1)}, \mu, \text{---} \right\rangle + \ldots.$$

The summation is over partitions $\nu \in \mathcal{P}(n)$. The T-equivariant Poincare dual of ν in the Nakajima basis is denoted by $\nu^\vee$. The dots stand for terms with nontrivial distribution of the insertions (which, therefore, have fewer than m insertions each). The superscript Hilb_n has been dropped from the bracket notation in the WDVV-equation.

By the definition of quantum multiplication,

$$\left\langle D^{*k}, \lambda, \mu, \text{---} \right\rangle^{\mathrm{Hilb}_n} = \sum_{\nu} \left\langle D, D^{*(k-1)}, \nu \right\rangle \langle \nu^\vee, \lambda, \mu, \text{---} \rangle^{\mathrm{Hilb}_n}$$

All the other terms in above WDVV-equation are either 3-point invariants or have minimal degree $k-1$ in D. □

4.3 Relation to the Gromov-Witten theory of $\mathbb{C}^2 \times \mathbf{P}^1$

We follow here the notation of [5] Section 3.2 for the local Gromov-Witten theory of $\mathbb{C}^2 \times \mathbf{P}^1$.

Let $(\mathbf{P}^1, x_1, \ldots, x_r)$ be the sphere with r distinct marked points. Let

$$\overline{M}^{\bullet}_h(\mathbf{P}^1, \lambda^1, \ldots, \lambda^r)$$

denote the moduli space of (possibly disconnected) relative stable maps from genus h curves to $\mathbf{P}^1$ with prescribed ramification λ^i at x_i. The prescribed ramification points on the domain are unmarked, and the maps are required to be nonconstant on all connected components.

The partition function of the local Gromov-Witten theory may be defined by:

$$\mathsf{Z}'_{GW}(\mathbb{C}^2 \times \mathbf{P}^1)_{n[\mathbf{P}^1],\lambda^1,\ldots,\lambda^r} = \sum_{h\in\mathbb{Z}} u^{2h-2} \int_{[\overline{M}^{\bullet}_h(\mathbf{P}^1,\lambda^1,\ldots,\lambda^r)]^{vir}} e(-R^{\bullet}\pi_* f^*(\mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^1})).$$

We will be primarily interested in a shifted generating function,

$$\mathsf{GW}^*_n(\mathbb{C}^2 \times \mathbf{P}^1)_{\lambda^1,\ldots,\lambda^r} = (-iu)^{n(2-r)+\sum_{i=1}^r \ell(\lambda^i)}\ \mathsf{Z}'_{GW}(\mathbb{C}^2 \times \mathbf{P}^1)_{n[\mathbf{P}^1],\lambda^1,\ldots,\lambda^r}.$$

The GW/Hilbert correspondence relates the local theory of $\mathbb{C}^2 \times \mathbf{P}^1$ to the multipoint invariants of Hilb_n with *fixed* complex structure $\xi \in \overline{M}_{0,r}$,

$$\langle \lambda^1, \ldots, \lambda^r \rangle^{\mathrm{Hilb}_n}_{\xi}.$$

Theorem 2. *After the variable change* $e^{iu} = -q$,

$$\mathsf{GW}^*_n(\mathbb{C}^2 \times \mathbf{P}^1)_{\lambda^1,\ldots,\lambda^r} = (-1)^n \langle \lambda^1, \ldots, \lambda^r \rangle^{\mathrm{Hilb}_n}_{\xi}.$$

Proof. A direct comparison of the formulas of Theorem 1 of Section 2.2 and Theorem 6.5 of [5] yields the result in case $r = 3$ *and* λ^1 is the 2-cycle $(1^{n-2}2)$. A verification shows the degeneration formula of local Gromov-Witten theory is compatible via the correspondence with the splitting formula for genus 0 fixed moduli invariants of Hilb_n. By Corollary 1, both sides of the correspondence are canonically determined from the 3-point case with one 2-cycle — see also the reconstruction result of the Appendix of [5] □

4.4 The orbifold $(\mathbb{C}^2)^n/S_n$

Consider the GW/Hilbert correspondence in the 3-point case,

$$\mathsf{GW}^*_n(\mathbb{C}^2 \times \mathbf{P}^1)_{\lambda,\mu,\nu} = (-1)^n \langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}_n}. \tag{40}$$

The 3-pointed, genus 0, T-equivariant Gromov-Witten invariants of the orbifold $(\mathbb{C}^2)^n/S_n$ are easily related to $\mathsf{GW}_n^*(\mathbb{C}^2 \times \mathbf{P}^1)_{\lambda,\mu,\nu}$, see [4]. The Hilbert scheme Hilb_n is a crepant resolution of the (singular) quotient $(\mathbb{C}^2)^n/S_n$. The equivalence (40) may be viewed as relating the T-equivariant quantum cohomology of the quotient *orbifold* $(\mathbb{C}^2)^n/S_n$ to the T-equivariant quantum cohomology of the resolution Hilb_n.

Mathematical conjectures relating the quantum cohomologies of orbifolds and their crepant resolutions in the non-equivariant case have been pursued by Ruan (motivated by the physical predictions of Vafa and Zaslow). Equality (40) suggests the correspondence also holds in the equivariant context.

4.5 Higher genus

Localization may be used to compute the higher genus Gromov-Witten invariants of Hilb_n. Because the T-fixed curves are not isolated, the localization structure is rather complicated. The higher genus invariants are expressed as sums over graphs where the vertex contributions are Hodge integrals over moduli spaces of curves and the edge contributions are integrals over moduli spaces of T-fixed curves in Hilb_n. The latter can be computed recursively from genus 0 descendent invariants of Hilb_n.

We expect the involved localization procedure can be conveniently expressed in Givental's formalism [11, 12, 16] for higher genus potentials for semisimple Frobenius structures. The main issue arising in the application of Givental's ideas is the selection of an R-calibration. We expect the standard Bernoulli R-calibration used in the T-equivariant Gromov-Witten theory of toric varieties is appropriate.

References

[1] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601–617.

[2] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88.

[3] S. Bloch, *Semi-regularity and deRham cohomology,* Invent. Math. **17** (1972), 51–66.

[4] J. Bryan and T. Graber, *The crepant resolution conjecture*, math/0610129.

[5] J. Bryan and R. Pandharipande, *The local Gromov-Witten theory of curves*, math.AG/0411037.

[6] K. Costello and I. Grojnowski, *Hilbert schemes, Hecke algebras and the Calogero-Sutherland system*, math.AG/0310189.

[7] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, American Mathematical Society, Providence, RI, 1999.

[8] D. Edidin, W.-P. Li, Z. Qin, *Gromov-Witten invariants of the Hilbert scheme of 3-points on* $\mathbf{P}^2$, Asian J. Math. **7** (2003), no. 4, 551–574.

[9] G. Ellingsrud and S. Strømme, *Towards the Chow ring of the Hilbert scheme of* $\mathbf{P}^2$, J. Reine Angew. Math. **441** (1993), 33–44.

[10] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, AMS, Providence, RI, 1997.

[11] A. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices (2001), 1265–1286.

[12] A. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Moscow Math. J. **1** (2001), 551–568.

[13] L. Göttsche, *Hilbert schemes of points on surfaces*, ICM Proceedings, Vol. II (Beijing, 2002), 483–494.

[14] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[15] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996), 275–291.

[16] Y.-P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints*, in preparation (Parts I and II available at `www.math.princeton.edu/~rahulp`).

[17] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157–207.

[18] M. Lehn and C. Sorger, *Symmetric groups and the cup product on the cohomology of Hilbert schemes* Duke Math. J. **110** (2001), no. 2, 345–357.

[19] W.-P. Li, Z. Qin, W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, Math. Ann. **324** (2002), no. 1, 105–133.

[20] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, JAMS **11** (1998), 119–174.

[21] I. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press, Oxford University Press, New York, 1995.

[22] M. Manetti, *Lie cylinders and higher obstructions to deforming submanifolds*, math.AG/0507278.

[23] D. Maulik, *Gromov-Witten theory of A_n-resolutions*, math/08022681.

[24] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, math.AG/0312059.

[25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, math.AG/0406092.

[26] D. Maulik and A. Oblomkov, *Quantum cohomology of the Hilbert scheme of points on A_n-resolutions*, math/08022737.

[27] D. Maulik and A. Oblomkov, *Donaldson-Thomas theory of $A_n \times P^1$*, math/08022739.

[28] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, AMS, Providence, RI, 1999.

[29] A. Okounkov and R. Pandharipande, *The local Donaldson-Thomas theory of curves*, math/0512573.

[30] A. Okounkov and R. Pandharipande, *Integrable systems in the quatum cohomology of the Hilbert scheme of points*, in preparation.

[31] Z. Ran, *Hodge theory and the Hilbert scheme*, J. Differential Geom. **37** (1993), 191–198.

[32] Z. Ran, *Semiregularity, obstructions and deformations of Hodge classes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** 28 (1999), 809–820.

[33] R. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), no. 1, 76–115.

[34] E. Vasserot, *Sur l'anneau de cohomologie du schma de Hilbert de* $\mathbf{C}^2$, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 1, 7–12.

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