QUIVERS, CURVES, AND THE TROPICAL VERTEX

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Abstract. Elements of the tropical vertex group are formal families of symplectomorphisms of the 2-dimensional algebraic torus. Commutators in the group are related to Euler characteristics of the moduli spaces of quiver representations and the Gromov-Witten theory of toric surfaces. After a short survey of the subject (based on lectures of Pandharipande at the 2009 *Geometry summer school* in Lisbon), we prove new results about the rays and symmetries of scattering diagrams of commutators (including previous conjectures by Gross-Siebert and Kontsevich). Where possible, we present both the quiver and Gromov-Witten perspectives.

Contents

Introduction

In Sections 1-3 of the paper, we survey the recently discovered relationship of three mathematical structures:

(i) Euler characteristics of the moduli spaces of quiver representations,
(ii) Gromov-Witten counts of rational curves on toric surfaces,
(iii) Ordered product factorizations of commutators in the tropical vertex group.

The tropical vertex group (iii) first arose in the work of Kontsevich and Soibelman [12] and plays a significant role in the program of [9]. A connection of the tropical vertex

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group to (i) has been proven by Reineke [19] using wall-crossing ideas. A connection to (ii) is proven in [8]. Our aim here is to present the shortest path to the simplest cases of the results. Lengthier treatments can be found in the original references.

The definition and basic properties of the tropical vertex group are reviewed in Section 1. Reineke's result is Theorem 1 of Section 2. The formula of [8] relating commutators in the tropical vertex group to rational curve counts is Theorem 2 of Section 3. Put together, Theorems 1 and 2 yield a suprising equivalence between curve counts on toric surfaces and Euler characteristics of moduli spaces of quiver representations. The equivalence is stated in Corollary 3 without any reference to the tropical vertex group.

In Section 4, we address the question of which slopes occur in the ordered product factorizations of commutators (iii). In the language of (i), the question asks which slopes are achieved by semistable representations of particular quivers. In Theorem 5, we find necessary conditions from the perspective of (ii) using the classical geometry of curves on surfaces. The result includes all the previous conjectures on scattering patterns as special cases.

Symmetries of the commutator factorizations are proven in Theorem 7 of Section 5. From the point of view of curve counting, the symmetries are obtained by transformations of blown-up toric surfaces. On the quiver side, the symmetries are a consequence of well-known reflection functors. Further directions in the subject are suggested in Section 6.

1. The tropical vertex group

1.1. **Automorphisms of the torus.** The 2-dimensional complex torus has very few automorphisms

$$\theta : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$$

as an algebraic group. Since θ must take each component $\mathbb{C}^*$ to a 1-dimensional subtorus,

$$\mathrm{Aut}^{\mathrm{Gr}}_{\mathbb{C}}(\mathbb{C}^* \times \mathbb{C}^*) \stackrel{\sim}{=} \mathrm{GL}_2(\mathbb{Z}).$$

As a complex algebraic variety, $\mathbb{C}^* \times \mathbb{C}^*$ has, in addition, only the automorphisms obtained by the translation action on itself,[1]

$$1 \to \mathbb{C}^* \times \mathbb{C}^* \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^* \times \mathbb{C}^*) \to \mathrm{Aut}^{\mathrm{Gr}}_{\mathbb{C}}(\mathbb{C}^* \times \mathbb{C}^*) \to 1.$$

[1] We leave the elementary proof to the reader. An argument can be found by using the characterization

$$\phi(z) = \lambda \cdot z^k \qquad \lambda \in \mathbb{C}^*,\ k \in \mathbb{Z}$$

of all algebraic maps $\phi : \mathbb{C}^* \to \mathbb{C}^*$.

A much richer algebraic structure appears if formal 1-parameter families of automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$ are considered,

$$A = \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}^* \times \mathbb{C}^* \times \mathrm{Spec}(\mathbb{C}[[t]])).$$

Let x and y be the coordinates of the two factors of $\mathbb{C}^* \times \mathbb{C}^*$. Then,

$$\mathbb{C}^* \times \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[x, x^{-1}, y, y^{-1}]).$$

We may alternatively view A as a group of algebra automorphisms,

$$A = \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]).$$

Nontrivial elements of A are easily found. Let $(a, b) \in \mathbb{Z}^2$ be a nonzero vector, and let $f \in \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ be a function of the form

$$f = 1 + tx^a y^b \cdot g(x^a y^b, t), \qquad g(z, t) \in \mathbb{C}[z][[t]].$$

We specify the values of an automorphism on x and y by

$$\theta_{(a,b),f}\ (x) = x \cdot f^{-b}, \quad \theta_{(a,b),f}\ (y) = y \cdot f^a\ . \tag{1.1}$$

The assignment (1.1) extends uniquely to determine an element $\theta_{(a,b),f} \in A$. The inverse is obtained by inverting f,

$$\theta^{-1}_{(a,b),f} = \theta_{(a,b),f^{-1}}\ .$$

1.2. **Tropical vertex group.** The tropical vertex group $H \subset A$ is the completion with respect to the maximal ideal $(t) \subset \mathbb{C}[[t]]$ of the subgroup generated by *all* elements of the form $\theta_{(a,b),f}$. In particular, infinite products are well-defined in H if only finitely many terms are nontrivial mod t^k (for every k). A more natural characterization of H via the associated Lie algebra may be found in Section 1.1 of [8].

The torus $\mathbb{C}^* \times \mathbb{C}^*$ has a standard holomorphic symplectic form given by

$$\omega = \frac{dx}{x} \wedge \frac{dy}{y}.$$

Let $S \subset A$ be the subgroup of automorphisms preserving ω,

$$S = \{\ \theta \in A \mid \theta^*(\omega) = \omega\ \}.$$

Lemma 1.1. *$H \subset S$.*

Proof. The result is obtained from a direct calculation. Let

$$\widetilde{x} = xf^{-b}, \qquad \widetilde{y} = yf^a.$$

From the equations

$$\frac{d\widetilde{x}}{\widetilde{x}} = \frac{dx}{x} - \frac{bf_x}{f}dx - \frac{bf_y}{f}dy, \qquad \frac{d\widetilde{y}}{\widetilde{y}} = \frac{dy}{y} + \frac{af_y}{f}dy + \frac{af_x}{f}dx,$$

we conclude $\theta^*_{(a,b),f}(\omega) = \omega$ if

$$\frac{af_y}{xf} = \frac{bf_x}{yf}.$$

The latter follows from the dependence of f on x and y only through $x^a y^b$. □

A slight variant of the tropical vertex group H first arose in the study of affine structures by Kontsevich and Soibelman in [12]. Further development, related to mirror symmetry and tropical geometry, can be found in [9]. Recently, the tropical vertex group has played a role in wall-crossing formulas for counting invariants in derived categories [13].

1.3. **Commutators.** The first question we can ask about the tropical vertex group is to find a formula for the commutators of the generators. The answer is related to Euler characteristics of moduli spaces of quiver representations and to Gromov-Witten counts of rational curves on toric surfaces. The simplest nontrivial cases to consider are the commutators of the elements

$$S_{\ell_1} = \theta_{(1,0),(1+tx)^{\ell_1}} \quad \text{and} \quad T_{\ell_2} = \theta_{(0,1),(1+ty)^{\ell_2}}$$

where $\ell_1, \ell_2 > 0$. By an elementary result of [12] reviewed in Section 1.3 of [8], there exists a unique factorization

$$T_{\ell_2}^{-1} \circ S_{\ell_1} \circ T_{\ell_2} \circ S_{\ell_1}^{-1} = \overrightarrow{\prod} \theta_{(a,b),f_{a,b}} \tag{1.2}$$

where the product on the right is over *all* primitive vectors $(a,b) \in \mathbb{Z}^2$ lying strictly in the first quadrant.[2,3] The order is determined by increasing slopes of the vectors (a,b). The product (1.2) is very often infinite, but always has only finitely many nontrivial terms mod t^k (for every k). The question is what are the functions $f_{a,b}$ associated to the slopes?

1.4. **Examples.** The easiest example is $\ell_1 = \ell_2 = 1$. The formula

$$T_1^{-1} \circ S_1 \circ T_1 \circ S_1^{-1} = \theta_{(1,1),1+t^2xy}$$

can be directly checked by hand. We will display the information by drawing rays of slope (a,b) in the first quadrant for every term appearing on the right-hand side. Each ray should be thought of as labelled with a function, see Figure 1.1.

[2]A vector (a,b) is primitive if it is not divisible in $\mathbb{Z}^2$. Primitivity implies $(a,b) \neq (0,0)$. Strict inclusion in the first quadrant is equivalent to $a > 0$ and $b > 0$.

[3]Here and throughout the paper, we drop the dependence of $f_{a,b}$ upon (ℓ_1, ℓ_2) for notational convenience.

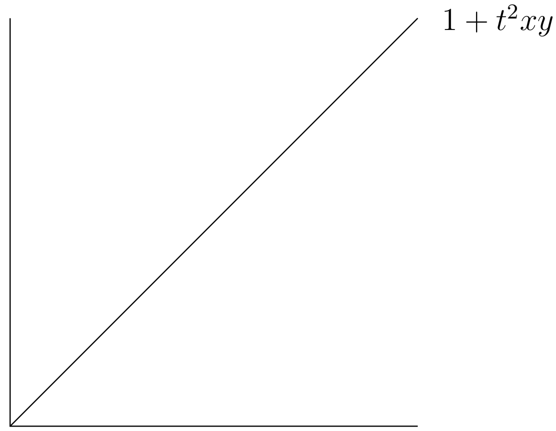


FIGURE 1.1.

For $\ell_1 = \ell_2 = 2$, we already have a much more complicated expansion,

$$\begin{aligned} T_2^{-1} \circ S_2 \circ T_2 \circ S_2^{-1} \quad = \quad & \theta_{(1,2),(1+t^3xy^2)^2} \circ \theta_{(2,3),(1+t^5x^2y^3)^2} \circ \theta_{(3,4),(1+t^7x^3y^4)^2} \circ \cdots \\ & \circ \; \theta_{(1,1),1/(1-t^2xy)^4} \circ \\ & \cdots \circ \theta_{(4,3),(1+t^7x^4y^3)^2} \circ \theta_{(3,2),(1+t^5x^3y^2)^2} \circ \theta_{(2,1),(1+t^3x^2y)^2}. \end{aligned}$$

The values of (a, b) which occur are of the form $(k, k + 1)$ and $(1, 1)$ and $(k + 1, k)$ for all $k \geq 1$. We depict the slopes occuring by rays in the first quadrant as in Figure 1.2. Ideally, we would label each ray $\mathbb{R}_{\geq 0}(a, b)$ with the function $f_{a,b}$, however the diagram would become too difficult to draw. Here

$$\begin{aligned} f_{1,1} &= 1/(1 - t^2xy)^4 \\ f_{k,k+1} &= (1 + t^{2k+1}x^ky^{k+1})^2 \\ f_{k+1,k} &= (1 + t^{2k+1}x^{k+1}y^k)^2 \; . \end{aligned}$$

The case $\ell_1 = \ell_2 = 3$ becomes still more complex, illustrated in Figure 1.3. Extrapolating from calculations, we find rays with primitives

$$(a, b) = (3, 1),\ (8, 3),\ (21, 8),\ \ldots$$

converging to the ray of slope $(3 - \sqrt{5})/2$ and rays with primitives

$$(a, b) = (1, 3),\ (3, 8),\ (8, 21),\ \ldots$$

converging to the ray of slope $(3 + \sqrt{5})/2$. Meanwhile, all rays with rational slope between $(3 - \sqrt{5})/2$ and $(3 + \sqrt{5})/2$ appear to occur.

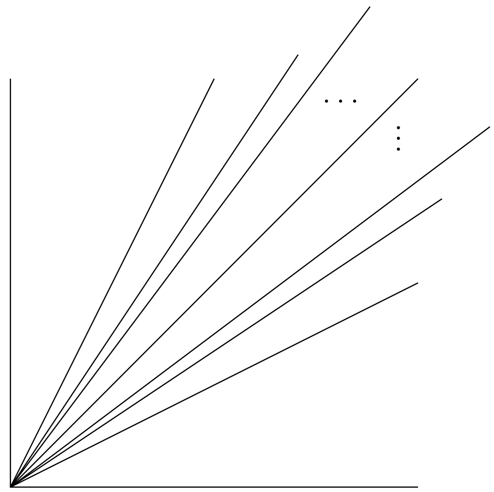


FIGURE 1.2.

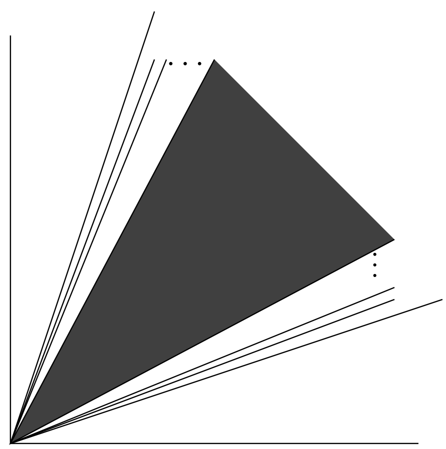


FIGURE 1.3.

We do not know closed forms for the functions associated to each ray. However, Gross conjectured the function attached to the line of slope 1 in Figure 1.3 is

$$\left(\sum_{k=0}^{\infty} \frac{1}{3k+1} \binom{4k}{k} t^{2k} x^k y^k\right)^9. \tag{1.3}$$

Finally, consider the asymmetric case $(\ell_1, \ell_2) = (2, 3)$. We again appear to obtain a discrete series of rays and a cone in which all rays occur. We find rays with primitives

$$(a, b) = (2, 1),\ (5, 2),\ (8, 5),\ (19, 12),\ \ldots$$

converging to a ray of slope $(3 - \sqrt{3})/2$ and rays with primitives

$$(a, b) = (1, 3),\ (2, 5),\ (5, 12),\ (8, 19),\ \ldots$$

converging to a ray of slope $(3+\sqrt{3})/2$. All rays with rational slope in between these two quadratic irrational slopes seem to appear. The function attached to the ray of slope 1 appears to be

$$\left(\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}t^{2k}x^ky^k\right)^6.$$

Inside the exponential is the generating series for Catalan numbers.

Conjecture. *For arbitrary (ℓ_1,ℓ_2), the function attached to the ray of slope 1 is*

$$\left(\sum_{k=0}^{\infty}\frac{1}{(\ell_1\ell_2-\ell_1-\ell_2)k+1}\binom{(\ell_1-1)(\ell_2-1)k}{k}t^{2k}x^ky^k\right)^{\ell_1\ell_2}. \tag{1.4}$$

The above conjecture specializes to the series (1.3) in the $(\ell_1,\ell_2)=(3,3)$ case. The specialization of (1.4) to $\ell_1=\ell_2$ was conjectured by Kontsevich (motivated by (1.3)) and proved by Reineke in [20].

The series (1.4) attached to the ray of slope 1 is not always a rational functional in the variables t,x,y. However, since

$$S_r=\sum_{k=0}^{\infty}\frac{1}{(r-1)k+1}\binom{rk}{k}t^{2k}x^ky^k$$

satisfies the polynomial equation

$$t^2xy(S_r)^r-S_r+1=0,$$

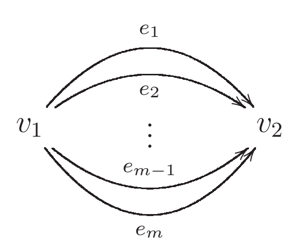
the function (1.4) is algebraic over $\mathbb{Q}(t,x,y)$. Whether the functions attached to other slopes are algebraic over $\mathbb{Q}(t,x,y)$ is an interesting question (asked first by Kontsevich).

2. Moduli of quiver representations

2.1. **Definitions.** A *quiver* is a directed graph. We will consider here only the fundamental m-Kronecker quiver Q_m consisting of two vertices $\{v_1,v_2\}$ and m edges $\{e_1,\dots,e_m\}$ with equal orientations

$$v_1 \xrightarrow{e_j} v_2\ .$$

The m-Kronecker quiver may be depicted with m arrows as:



A representation of $\rho = (V_1, V_2, \tau_1, \ldots, \tau_m)$ of the quiver Q_m consists of the following linear algebraic data

(i) vector spaces V_i associated to the vertices v_i,
(ii) linear transformations $\tau_j : V_1 \to V_2$ associated to the edges e_j.

While representations over any field may be studied, we will restrict our attention to finite dimensional representations over $\mathbb{C}$. Associated to ρ is the *dimension vector*

$$\dim(\rho) = (\dim(V_1), \dim(V_2)) \in \mathbb{Z}^2 \ .$$

A *morphism* $\phi = (\phi_1, \phi_2)$ between two representations ρ and ρ' of Q_m is a pair of linear tranformations

$$\phi_i : V_i \to V_i'$$

satisfying $\tau_j' \circ \phi_1 = \phi_2 \circ \tau_j$ for all j. Two representations are *isomorphic* if there exists a morphism ϕ for which both ϕ_1 and ϕ_2 are isomorphisms of vector spaces. The notions of sub and quotient representations are well-defined. In fact, the representations of Q_m are easily seen to form an abelian category.

There are several accessible references for quiver representations. We refer the reader to papers by King [10] and Reineke [18] where the representation theory of arbitrary quivers is treated. Algebraic background can be found in [1].

2.2. **Moduli.** Consider the moduli space of representations of Q_m with fixed dimension vector (d_1, d_2). Let $\mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$ be the space of $d_1 \times d_2$ matrices. Every element of

$$\mathcal{P}_m(d_1, d_2) = \bigoplus_{j=1}^{m} \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \tag{2.1}$$

determines a representation of Q_m with dimension vector (d_1, d_2). Moreover, the isomorphism class of every representation of Q_m with dimension vector (d_1, d_2) is achieved in the parameter space $\mathcal{P}_m(d_1, d_2)$.

Since $\mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$ carries canonical commuting actions of $\mathbf{GL}_{d_1}$ and $\mathbf{GL}_{d_2}$, we obtain an action of the product $\mathbf{GL}_{d_1} \times \mathbf{GL}_{d_2}$ on the parameter space $\mathcal{P}_m(d_1, d_2)$. In fact, the scalars

$$\mathbb{C}^* \subset \mathbf{GL}_{d_1} \times \mathbf{GL}_{d_2},$$

included diagonally $\xi \mapsto (\xi, \xi)$ are easily seen to act trivially. Hence, we actually have an action of

$$\mathbf{G}_{d_1,d_2} = \Big(\mathbf{GL}_{d_1} \times \mathbf{GL}_{d_2}\Big)\Big/\mathbb{C}^*.$$

To construct an algebraic moduli space of representations of Q_m, we remove the redundancy in the parameter space (2.1) by taking the algebraic quotient

$$\mathcal{P}_m(d_1, d_2)\Big/\mathbf{G}_{d_1,d_2} \ . \tag{2.2}$$

While the quotient (2.2) is well-defined[4], an elementary analysis shows that there are no nontrivial invariants [18]. Indeed, 0 is the only closed $\mathbf{G}_{d_1,d_2}$-orbit in $\mathcal{P}_m(d_1,d_2)$. Hence,

$$\mathcal{P}_m(d_1,d_2)\Big/\mathbf{G}_{d_1,d_2} \;=\; \mathrm{Spec}(\mathbb{C})\ . \tag{2.3}$$

2.3. **Stability conditions.** The trivial quotient (2.3) is hardly a satisfactory answer. Representations of Q_m with dimension vector (d_1,d_2) should typically vary in a

$$\dim\, \mathcal{P}_m(d_1,d_2) - \dim\, \mathbf{G}_{d_1,d_2} = md_1d_2 - d_1^2 - d_2^2 + 1 \tag{2.4}$$

dimensional family. A much richer view of the moduli of quiver representations is obtained by imposing stability conditions.

A *stability condition* ω on Q_m is given by a pair of integers (w_1,w_2). With respect to ω, the slope of a representation ρ of Q_m with dimension vector (d_1,d_2) is

$$\mu(\rho) = \frac{w_1d_1 + w_2d_2}{d_1+d_2}\ .$$

A representation ρ is *(semi)stable* if, for every proper[5] subrepresentation $\widehat{\rho}\subset\rho$,

$$\mu(\widehat{\rho})\ \ (\leq)\ <\ \ \mu(\rho)\ .$$

A central result of [10] is the construction of moduli spaces of semistable representations of quivers. Applied to Q_m, we obtain the moduli space $\mathcal{M}_m^\omega(d_1,d_2)$ of ω-semistable representations with dimension vector (d_1,d_2). We present here a variation of the method of [10].

The two determinants yield two basic characters of the group $\mathbf{GL}_{d_1}\times\mathbf{GL}_{d_2}$,

$$\det{}_1(g_1,g_2) = \det(g_1), \quad \det{}_2(g_1,g_2) = \det(g_2)\ .$$

The stability condition ω defines a character

$$\lambda(g_1,g_2) = \det{}_1^{(w_2-w_1)d_2}\cdot \det{}_2^{(w_1-w_2)d_1}\ .$$

Since λ is trivial on $\mathbb{C}^*\subset\mathbf{GL}_{d_1}\times\mathbf{GL}_{d_2}$, λ descends to a character of $\mathbf{G}_{d_1,d_2}$. Let

$$\mathcal{P}_m^\omega(d_1,d_2) = \lambda\otimes\mathcal{P}_m(d_1,d_2)\oplus\lambda \tag{2.5}$$

be the representation of $\mathbf{G}_{d_1,d_2}$ obtained by tensoring and adding the 1-dimensional character λ to the parameter space (2.1). Let

$$\mathbb{P}\Big(\mathcal{P}_m^\omega(d_1,d_2)\Big)^{ss}\subset\mathbb{P}\Big(\mathcal{P}_m^\omega(d_1,d_2)\Big)$$

denote the semistable locus of the canonically linearized $\mathbf{G}_{d_1,d_2}$-action.

[4]Quotients of reductive groups actions on affine varieties can always be taken.

[5]Both 0 and the entire representation are excluded.

We are not interested in the entire variety $\mathbb{P}\Big(\mathcal{P}_m^\omega(d_1,d_2)\Big)$. There is a canonical open embedding of the parameter space (2.1),

$$\mathcal{P}_m(d_1,d_2) \subset \mathbb{P}\Big(\mathcal{P}_m^\omega(d_1,d_2)\Big),$$

as a $\mathbf{G}_{d_1,d_2}$-equivariant open set defined by the sum structure (2.5). The moduli space of ω-semistable representations of Q_m with dimension vector (d_1,d_2) is the quotient

$$\mathcal{M}_m^\omega(d_1,d_2) = \Big(\mathcal{P}_m(d_1,d_2) \ \cap \ \mathbb{P}\Big(\mathcal{P}_m^\omega(d_1,d_2)\Big)^{ss}\Big) \Big/ \mathbf{G}_{d_1,d_2} \ .$$

Several important properties of the moduli space of ω-semistable representations can be deduced from the construction [10]:

(i) $\mathcal{M}_m^\omega(d_1,d_2)$ is a projective variety.

(ii) An open set $\mathcal{M}_m^\omega(d_1,d_2)^{stable} \subset \mathcal{M}_m^\omega(d_1,d_2)$ parameterizes isomorphism classes of ω-stable representations of Q_m. If nonempty, $\mathcal{M}_m^\omega(d_1,d_2)^{stable}$ is nonsingular of dimension (2.4).

(iii) $\mathcal{M}_m^\omega(d_1,d_2)$ parameterizes isomorphism classes of ω-semistable representations of Q_m modulo Jordan-Holder equivalence (often called S-equivalence).

While properties (ii) and (iii) hold for stability conditions on arbitrary quivers, property (i) is special[6] to Q_m. By the results of [10], $\mathcal{M}_m^\omega(d_1,d_2)$ is projective over the quotient (2.3). Since the quotient (2.3) is Spec($\mathbb{C}$), the moduli space $\mathcal{M}_m^\omega(d_1,d_2)$ is a projective variety.

If $\omega=(0,0)$, all representations are semistable. Then,

$$\mathcal{M}_m^{(0,0)}(d_1,d_2) = \mathcal{P}_m(d_1,d_2)\Big/\mathbf{G}_{d_1,d_2} \ = \ \mathrm{Spec}(\mathbb{C})$$

as before. By the following result of Reineke [18], we will restrict our attention to the stability conditions $(1,0)$ and $(0,1)$.

Lemma 2.1. *ω-(semi)stability is equivalent to (semi)stability with respect to either $(0,0)$, $(1,0)$, or $(0,1)$.*

Proof. Let $\omega=(w_1,w_2)$. By the definition of (semi)stability of representations, we see ω-(semi)stability is equivalent to both

(i) $(w_1+\gamma, w_2+\gamma)$-(semi)stability for $\gamma \in \mathbb{Z}$ and

(ii) $(\lambda w_1, \lambda w_2)$-(semi)stability for $\lambda \in \mathbb{Z} > 0$.

If $w_1 = w_2$, then ω-(semi)stability is equivalent to $(0,0)$-(semi)stability by (i). If $w_1 > w_2$, then ω-(semi)stability is equivalent to $(w_1-w_2,0)$-(semi)stability by (i) and then $(1,0)$-(semi)stability by (ii). Similarly, the $w_1 < w_2$ case leads to $(0,1)$-(semi)stability. □

[6]Projectivity holds for moduli spaces of representations of quivers without oriented cycles.

2.4. **Framing.** Strictly semistable representations of Q_m usually lead to singularities of the moduli space $\mathcal{M}^{\omega}_m(d_1, d_2)$. Following [6], we introduce framing data to improve the moduli behaviour.

We consider two types of framings for representations of Q_m. A *back* framed representation of Q_m is a pair (ρ, L_1) where $\rho = (V_1, V_2, \tau_1, \dots, \tau_m)$ is a standard representation of Q_m and $L_1 \subset V_1$ is a 1-dimensional subspace. A *front* framed representation of Q_m is a pair (ρ, L_2) where $L_2 \subset V_2$ is a 1-dimensional subspace. The subspaces L_i are the framings. Two framed representations are isomorphic if the underlying standard representations admit an isomorphism preserving the framing.

A stability condition ω for Q_m induces a canonical notion of stability for framed representations. A framed representation (ρ, L_i) is *stable* if the following two conditions hold:

(i) ρ is an ω-semistable representation,
(ii) for every proper subrepresentation $\widehat{\rho} \subset \rho$ containing L_i,

$$\mu(\widehat{\rho}) < \mu(\rho).$$

The moduli of stable framed representations admits a GIT quotient construction with no strictly semistables. In fact, stable framed representations can be viewed as stable standard representations for quivers obtained by augmenting Q_m by one vertex (and considering appropriate standard stability conditions). We refer the reader to [6] for a detailed discussion.

Let $\mathcal{M}^{\omega,B}_m(d_1, d_2)$ and $\mathcal{M}^{\omega,F}_m(d_1, d_2)$ denote the moduli spaces of back and front framed representations of Q_m. Both are nonsingular, irreducible, projective varieties.

2.5. **Examples: stability condition** $(0, 1)$**.** Consider first the stability condition $(0, 1)$ on the quiver Q_m. Suppose ρ is a standard representation with dimension vector (d_1, d_2) satisfying $d_1, d_2 > 0$. There exists a proper subrepresentation

$$\widehat{\rho} = (0, \widehat{V}_2, 0, \dots, 0)$$

where $\widehat{V}_2 \subset V_2$ is any 1 dimensional subspace. We see

$$\mu(\widehat{\rho}) = \frac{1}{1} > \frac{d_2}{d_1 + d_2} = \mu(\rho)\ .$$

Hence, ρ can not be $(0, 1)$-semistable.

The dimension vectors of $(0, 1)$-semistable representations of Q_m must be parallel to either $(1, 0)$ or $(0, 1)$. In fact, if framings are placed, only the dimension vectors $(1, 0)$ and $(0, 1)$ are possible. Elementary considerations yield the following result.

Lemma 2.2. *The moduli space of stable framed representations of Q_m with respect to the condition $(0,1)$ is a point in the two cases*

$$\mathcal{M}_m^{(0,1),B}(1,0), \qquad \mathcal{M}_m^{(0,1),F}(0,1),$$

and empty otherwise.

2.6. **Examples: stability condition $(1,0)$.** The stability condition $(1,0)$ on the quiver Q_m leads to much more interesting behavior. Unlike the $(0,1)$ condition, we will here be only able to undertake a case by case analysis.

For the 1-Kronecker quiver Q_1, the moduli spaces of stable framed representations must have dimension vectors equal to $(1,0)$, $(0,1)$, or $(1,1)$. Again, in all four cases (for possible back and front framing), the moduli spaces are points.

For the 2-Kronecker quiver, we find a richer set of possibilities of $(1,0)$-semistable representations.

Lemma 2.3. *If ρ is a $(1,0)$-semistable representation of Q_2, then the dimension vector must be proportional to one of*

$$(k, k+1), \quad (1,1), \quad (k+1, k)$$

for $k \geq 1$.

Proof. Suppose $\rho = (V_1, V_2, \tau_1, \tau_2)$ is a representation of Q_2. We analyze first the case where $d_1 < d_2$. The case $d_1 > d_2$ is obtained by dualizing.[7]

Since the slope of ρ is $\frac{d_1}{d_1+d_2}$, $(1,0)$-semistabiliy is violated if there exists a non-trivial subspace $\widehat{V}_1 \subset V_1$ satisfying

$$\frac{\dim(\widehat{V}_1)}{\dim(\widehat{V}_1) + \dim\Big(\tau_1(\widehat{V}_1) + \tau_2(\widehat{V}_1)\Big)} > \frac{d_1}{d_1+d_2} \ . \tag{2.6}$$

If ρ is $(1,0)$-semistable, the maps τ_1 and τ_2 must be injective (by taking $\widehat{V}_1$ to be $\mathrm{Ker}(\tau_i)$).

We now assume ρ to be $(1,0)$-semistable and construct a candidate for $\widehat{V}_1$ by the following method. Let $S_0 = V_1$, and let

$$S_i = \tau_1^{-1}\left(\tau_2(S_{i-1})\right) \quad \text{for} \ \ i > 0 \ .$$

Since $S_i \subset S_{i-1}$, we obtain a filtration

$$\ldots \subset S_3 \subset S_2 \subset S_1 \subset S_0 \ .$$

[7]The dual of ρ is $\rho^* = (V_2^*, V_1^*, \tau_1^*, \tau_2^*)$, and ρ is $(1,0)$-semistable if and only if ρ^* is $(1,0)$-semistable.

If S_i is nonempty, then the inclusion $S_i \subset S_{i-1}$ must be proper (otherwise $\widehat{V}_1 = S_i$ violates (2.6)). Since the codimension of $S_i \subset V_1$ is at most $i(d_2 - d_1)$, we see

$$S_{\lfloor \frac{d_1-1}{d_2-d_1} \rfloor} \neq 0 \ .$$

We can find a sequence of elements $\epsilon_i \in S_i \setminus S_{i+1}$ for $0 \leq i \leq \lfloor \frac{d_1-1}{d_2-d_1} \rfloor$ such that

$$\tau_2(\epsilon_i) = \tau_1(\epsilon_{i+1}) \ .$$

Let $\widehat{V}_1$ be span of $\epsilon_0, \dots, \epsilon_{\lfloor \frac{d_1-1}{d_2-d_1} \rfloor}$.

Since the ϵ_i are independent, the dimension of $\widehat{V}_1$ is $\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 1$. The dimension of $\tau_1(\widehat{V}_1) + \tau_2(\widehat{V}_1)$ is at most $\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 2$, so

$$\frac{\dim(\widehat{V}_1)}{\dim(\widehat{V}_1) + \dim\Big(\tau_1(\widehat{V}_1) + \tau_2(\widehat{V}_1)\Big)} \geq \frac{\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 1}{2\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 3}$$

Therefore, since ρ is $(1,0)$-semistable, we must have

$$\frac{\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 1}{2\lfloor \frac{d_1-1}{d_2-d_1} \rfloor + 3} \leq \frac{d_1}{d_1 + d_2}$$

or, equivalently,

$$(d_2 - d_1)\lfloor \frac{d_1 - 1}{d_2 - d_1} \rfloor + d_1 + d_2 \leq 3d_1 \ . \tag{2.7}$$

There are now two cases. If $d_2 - d_1$ divides $d_1 - 1$, then the inequality immediately implies $d_2 = d_1 + 1$. If $d_2 - d_1$ does not divide $d_1 - 1$, the inequality implies $d_2 - d_1$ divides d_1. In the second case, the dimension vector is proportional to $(\frac{d_1}{d_2-d_1}, \frac{d_1}{d_2-d_1} + 1)$. □

The construction of $(1,0)$-semistable representations of Q_2 with dimension vectors in the directions permitted by Lemma 2.3 is an easy exercise. We will discuss in more detail the directions $(1,2)$ and $(1,1)$.

The moduli spaces of stable back framed representations of Q_2 of dimension vector $(k, 2k)$ are empty for $k \geq 2$ and $\mathcal{M}_2^{(1,0),B}(1,2)$ is a point. Front framing is slightly more complicated,

$$\mathcal{M}_2^{(1,0),F}(1,2) = \mathbb{P}^1, \quad \mathcal{M}_2^{(1,0),F}(2,4) = \text{point},$$

and $\mathcal{M}_2^{(1,0),F}(k,2k)$ is empty for $k > 2$. These results are obtained by simply unravelling the definitions.

For dimension vector proportional to $(1,1)$, the framed moduli spaces are always nonempty. Their topological Euler characteristics are determined by the following result.

Lemma 2.4. *For $k \geq 1$, we have $\chi\big(\mathcal{M}_2^{(1,0),B}(k,k)\big) = \chi\big(\mathcal{M}_2^{(1,0),F}(k,k)\big) = k + 1$.*

Proof. The simplest approach is to count the fixed points of the $\mathbb{C}^* \times \mathbb{C}^*$-action on the framed moduli spaces obtained by scaling τ_1 and τ_2,

$$(\xi_1, \xi_2) \cdot \Big(\big(\mathbb{C}^k, \mathbb{C}^k, \tau_1, \tau_2 \big), L_i \Big) = \Big(\big(\mathbb{C}^k, \mathbb{C}^k, \xi_1\tau_1, \xi_2\tau_2 \big), L_i \Big) \ .$$

Certainly, $\mathcal{M}_2^{(1,0),B}(1,1)$ and $\mathcal{M}_2^{(1,0),F}(1,1)$ are both $\mathbb{P}^1$ with fixed points given by

$$\tau_1 = 1,\ \tau_2 = 0, \quad \text{and} \quad \tau_1 = 0,\ \tau_2 = 1$$

and unique choice for the framings.

The moduli spaces with dimension vector $(2,2)$ are the first nontrivial cases. Two 2×2 matrices together with a non-zero vector in $\mathbb{C}^2$ specify a back framed representation of Q_2. The three $\mathbb{C}^* \times \mathbb{C}^*$-fixed points of $\mathcal{M}_2^{(1,0),B}(2,2)$ are given by the data

$$\left\{ \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\left\{ \tau_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\left\{ \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ L_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

The analysis for $\mathcal{M}_2^{(1,0),F}(2,2)$ is similar. We leave the higher k examples for the reader to investigate.

A treatment of torus actions on moduli of spaces of representations of quivers can be found in [21]. In fact, $\mathcal{M}_2^{(1,0),B}(k,k) \cong \mathcal{M}_2^{(1,0),F}(k,k) \cong \mathbb{P}^k$. □

2.7. **Reineke's Theorem.** The main result relating commutators in the tropical vertex group to the Euler characteristics of the moduli spaces of representations of Q_m can now be stated. Consider the elements

$$S_m = \theta_{(1,0),(1+tx)^m} \quad \text{and} \quad T_m = \theta_{(0,1),(1+ty)^m}$$

of the tropical vertex group. The unique factorization

$$T_m^{-1} \circ S_m \circ T_m \circ S_m^{-1} = \prod^{\longrightarrow} \theta_{(a,b),f_{a,b}} \tag{2.8}$$

associates a function

$$f_{a,b} \in \mathbb{C}[x^a y^b][[t]]$$

to every primitive vector $(a,b) \in \mathbb{Z}^2$ lying strictly in the first quadrant. Two more functions are obtained from the topological Euler characteristics of the moduli spaces of back and front framed representations of Q_m,

$$B_{a,b} = 1 + \sum_{k\geq 1} \chi\Big(\mathcal{M}_m^{(1,0),B}(ak,bk)\Big) \cdot (tx)^{ak}\ (ty)^{bk}\ ,$$

$$F_{a,b} = 1 + \sum_{k\geq 1} \chi\Big(\mathcal{M}_m^{(1,0),F}(ak,bk)\Big) \cdot (tx)^{ak}\ (ty)^{bk}\ .$$

Theorem 1. (Reineke) *The three functions are related by the equations*

$$f_{a,b} = (B_{a,b})^{\frac{m}{a}} = (F_{a,b})^{\frac{m}{b}}\ .$$

Theorem 1 is proven in [19]. Reineke calculates the Euler characteristics of the framed moduli spaces by counting points over finite fields. The connection to the tropical vertex group is made via a homomorphism from the Hall algebra following the wall-crossing philosophy of [13]. The relevant wall-crossing is from the $(0,1)$ to $(1,0)$ stability condition. The ordered product factorization is then obtained from the Harder-Narasimhan filtration in the abelian category of representations of Q_m.

2.8. **Examples.** For Q_1, the moduli spaces of framed representations are empty for slopes (strictly in the first quadrant) other than 1. Moreover, $\mathcal{M}_1^{(1,0),B}(k,k)$ and $\mathcal{M}_1^{(1,0),F}(k,k)$ are points if $k=1$ and empty otherwise. Theorem 1 then immediately recovers the commutator calculation of Figure 1.1.

For Q_2 and primitive vector $(a,b) = (1,2)$, the results of Section 2.6 yield

$$\begin{aligned} B_{1,2} &= 1 + t^3xy^2\ , \\ F_{1,2} &= 1 + 2t^3xy^2 + t^6x^2y^4\ . \end{aligned}$$

By the commutator results of Section 1.4, we see

$$f_{1,2} = (1 + t^3xy^2)^2$$

verifying Theorem 1. For Q_2 and primitive vector $(a,b) = (1,1)$, we obtain

$$\begin{aligned} B_{1,1} &= (1 - t^2xy)^{-2}\ , \\ F_{1,1} &= (1 - t^2xy)^{-2}\ . \end{aligned}$$

By the commutator results of Section 1.4, we see

$$f_{1,1} = (1 - t^2xy)^{-4}$$

again verifying Theorem 1.

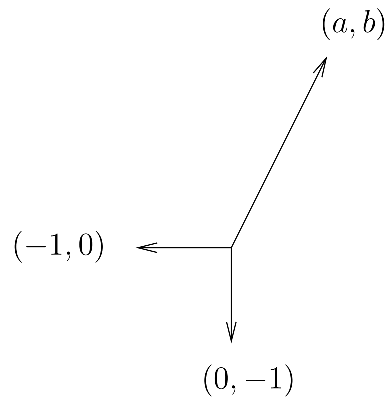


FIGURE 3.1.

3. RATIONAL CURVES ON TORIC SURFACES

3.1. **Toric surfaces.** Let $(a,b) \in \mathbb{Z}^2$ be a primitive vector lying strictly in the first quadrant. The rays generated by $(-1,0)$, $(0,-1)$, and (a,b) determine a complete rational fan[8] in $\mathbb{R}^2$, see Figure 3.1.

Let $X_{a,b}$ be the associated toric surface with toric divisors

$$D_1, D_2, D_{\mathrm{out}} \subset X_{a,b}$$

corresponding to the respective rays. Concretely, $X_{a,b}$ is the weighted projective plane obtained by the quotient

$$X_{a,b} = \big(\mathbb{C}^3 - \{0\}\big)/\mathbb{C}^*$$

where the $\mathbb{C}^*$-action is given by

$$\xi \cdot (z_1, z_2, z_3) = (\xi^a z_1, \xi^b z_2, \xi z_3) \ .$$

The divisors D_1, D_2 and D_{out} correspond respectively to the vanishing loci of z_1, z_2, and z_3.

Let $X^o_{a,b} \subset X_{a,b}$ be the open surface obtained by removing the three toric fixed points

$$[1,0,0], \ [0,1,0], \ [0,0,1] \ .$$

Let $D^o_1, D^o_2, D^o_{\mathrm{out}}$ be the restrictions of the toric divisors to $X^o_{a,b}$.

We denote *ordered partitions* $\mathbf{Q}$ of length ℓ by $q_1 + \ldots + q_\ell$. Ordered partitions differ from usual partitions in two basic ways. First, the ordering of the parts matters. Second, the parts q_i are required only to be non-negative integers (0 is permitted). The size $|\mathbf{Q}|$ is the sum of the parts.

[8]We refer the reader to [7] for background on toric varieties.

Let $k \geq 1$. Let $\mathbf{P}_a = p_1 + \ldots + p_{\ell_1}$ and $\mathbf{P}_b = p'_1 + \ldots + p'_{\ell_2}$ be ordered partitions of size ak and bk respectively. Denote the pair by $\mathbf{P} = (\mathbf{P}_a, \mathbf{P}_b)$. Let

$$\nu : X_{a,b}[\mathbf{P}] \to X_{a,b}$$

be the blow-up of $X_{a,b}$ along ℓ_1 and ℓ_2 distinct points of D_1^o and D_2^o. Let

$$X_{a,b}^o[\mathbf{P}] = \nu^{-1}(X_{a,b}^o).$$

Let $\beta_k \in H_2(X_{a,b}, \mathbb{Z})$ be the unique class with intersection numbers

$$\beta_k \cdot D_1 = ak, \quad \beta_k \cdot D_2 = bk, \quad \beta_k \cdot D_{\text{out}} = k.$$

Let E_i and E'_j be the i^{th} and j^{th} exceptional divisors over D_1^o and D_2^o. Let

$$\beta_k[\mathbf{P}] = \nu^*(\beta_k) - \sum_{i=1}^{\ell_1} p_i[E_i] - \sum_{j=1}^{\ell_2} p'_j[E'_j] \ \in H_2(X_{a,b}[\mathbf{P}], \mathbb{Z}).$$

3.2. **Moduli of maps.** Let $\overline{\mathfrak{M}}(X_{a,b}^o[\mathbf{P}]/D_{\text{out}}^o)$ denote the moduli space of stable relative maps[9] of genus 0 curves representing the class $\beta_k[\mathbf{P}]$ and with full contact order k at an unspecified point of D_{out}^o. By Proposition 4.2 of [8], the moduli space $\overline{\mathfrak{M}}(X_{a,b}^o[\mathbf{P}]/D_{\text{out}}^o)$ is proper (even though the target geometry is open). We can easily calculate the virtual dimension,

$$\begin{aligned}
\dim^{vir} \overline{\mathfrak{M}}(X_{a,b}^o[\mathbf{P}]/D_{\text{out}}^o) &= c_1(X_{a,b}^o[\mathbf{P}]) \cdot \beta_k[\mathbf{P}] - 1 - (k-1) \\
&= \left(\nu^* c_1(X_{a,b}^o) - \sum_{i=1}^{\ell_1} [E_i] - \sum_{j=1}^{\ell_2} [E'_j] \right) \cdot \beta_k[\mathbf{P}] - k \\
&= ak + bk + k - ak - bk - k \\
&= 0 \ ,
\end{aligned}$$

where the formula for the Chern class of a toric variety,

$$c_1(X_{a,b}^o) = D_1 + D_2 + D_{\text{out}},$$

is used in the second line.

Since $\overline{\mathfrak{M}}(X_{a,b}^o[\mathbf{P}]/D_{\text{out}}^o)$ is proper of virtual dimension 0, we may define the associated Gromov-Witten invariant by

$$N_{a,b}[\mathbf{P}] = \int_{[\overline{\mathfrak{M}}(X_{a,b}^o[\mathbf{P}]/D_{\text{out}}^o)]^{vir}} 1 \ \in \mathbb{Q} \ .$$

Proposition 4.2 of [8] shows $N_{a,b}[\mathbf{P}]$ does *not* depend upon the locations of the blow-ups of $X_{a,b}^0$.

[9]We refer the reader to [14] for an introduction to relative stable maps.

Naively, $N_{a,b}[\mathbf{P}]$ counts rational curves on $X^0_{a,b}$ with full contact at a single (unspecified) point of D_{out} and with specified multiple points of orders given by $\mathbf{P}$ on D^0_1 and D^0_2. However, the moduli space $\overline{\mathfrak{M}}(X^o_{a,b}[\mathbf{P}]/D^o_{\mathrm{out}})$ may include multiple covers and components of excess dimension. In particular, $N_{a,b}[\mathbf{P}]$ need not be integral (nor even positive).

3.3. **Formula.** The main result relating commutators in the tropical vertex group to rational curve counts on toric surfaces can now be stated. Consider the elements

$$S_{\ell_1} = \theta_{(1,0),(1+tx)^{\ell_1}} \ \text{ and } \ T_{\ell_2} = \theta_{(0,1),(1+ty)^{\ell_2}}$$

of the tropical vertex group. The unique factorization

$$T^{-1}_{\ell_2} \circ S_{\ell_1} \circ T_{\ell_2} \circ S^{-1}_{\ell_1} = \prod^{\rightarrow} \theta_{(a,b),f_{a,b}} \tag{3.1}$$

associates a function

$$f_{a,b} \in \mathbb{C}[x^a y^b][[t]]$$

to every primitive vector $(a,b) \in \mathbb{Z}^2$ lying strictly in the first quadrant. Since the series $f_{a,b}$ starts with 1, we may take the logarithm. Homogeneity constraints determine the behavior of the variable t. We define the coefficients $c^k_{a,b}(\ell_1,\ell_2) \in \mathbb{Q}$ by

$$\log f_{(a,b)} = \sum_{k\geq 1} k \ c^k_{a,b}(\ell_1,\ell_2) \cdot (tx)^{ak} \ (ty)^{bk}.$$

The function $f_{a,b}$ is linked to Gromov-Witten theory by the following result proven in [8].

Theorem 2. *We have*

$$c^k_{a,b}(\ell_1,\ell_2) = \sum_{|\mathbf{P}_a|=ak} \ \sum_{|\mathbf{P}_b|=bk} N_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)]$$

where the sums are over all ordered partitions $\mathbf{P}_a$ *of size* ak *and length* ℓ_1 *and* $\mathbf{P}_b$ *of size* bk *and length* ℓ_2.

The proof of Theorem 2 starts with the relationship of the tropical vertex group to tropical curve counts on toric surfaces. A transition to holomorphic curve counts with relative constraints is made via [15]. Finally, a degeneration argument is used to separate the virtual and enumerative geometry of the invariant $N_{a,b}[\mathbf{P}]$. The virtual aspects are handled by the multiple cover formulas of [3, 4] and the enumerative aspects by the tropical/holomorphic curve counts.

3.4. **Intuition.** The intuition behind Theorem 2 is as follows. The commutators (3.1) first arose in the work of Kontsevich and Soibelman [12] where rigid analytic K3 surfaces were constructed by gluing together standard charts (akin to $(\mathbb{C}^*)^2$) using elements of the tropical vertex group. The failure of the various automorphisms to commute required corrections which arose naturally from the commutator formulas. Roughly speaking, automorphisms are attached to certain gradient flow lines on an S^2. When the gradient flow lines intersect, new gradient flow lines are added starting at the point of intersection, with new automorphisms attached to these lines as dictated by the commutator expansion. The procedure restores compatibility of the gluing data.

The above description of what is really B-model geometry of K3 surfaces should be mirror to certain A-model geometry. Hence, there should be an enumerative interpretation for the commutator formulas.

The general picture suggested by the B-model is as follows. Consider the big torus orbit $(\mathbb{C}^*)^2 \subset X_{a,b}$ and the log map

$$\log : (\mathbb{C}^*)^2 \to \mathbb{R}^2 , \quad \log(z_1, z_2) = (\log|z_1|, \log|z_2|) .$$

Imagine that we have pieces of holomorphic curves given by ℓ_1 cylinders fibering via log over rays in $\mathbb{R}^2$ generated by $(-1, 0)$, and ℓ_2 cylinders fibering via log over rays in $\mathbb{R}^2$ generated by $(0, -1)$. We imagine trying to glue these cylinders together (perhaps after small perturbation) in some combination in such a way that we end up with a holomorphic curve in $(\mathbb{C}^*)^2$ with one additional unbounded cylinder heading in the direction of the ray generated by (a, b). We allow ourselves to use each of the "incoming" cylinders as many times as we want— the number of times we use the ith cylinder heading in the direction $(-1, 0)$ is p_i, and the number of times we use the jth cylinder headed in the direction $(0, -1)$ is p'_j. The number of ways of gluing the copies of these cylinders, after perturbing, should be $N_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$.

3.5. **Examples.** We consider the examples of §1.4, focusing on the functions attached to the ray of slope 1. For $\ell_1 = \ell_2 = 1$,

$$\log f_{1,1} = \log(1 + t^2xy) = \sum_{k=1}^{\infty} k \cdot \frac{(-1)^{k+1}}{k^2} \cdot (tx)^k(ty)^k.$$

Consider $\mathbb{P}^2$ with the three toric divisors D_1, D_2 and D_{out} making up the toric boundary. There is a unique line passing through a point selected on D_1 and a point selected on D_2. Hence, $N_{1,1}[(1, 1)] = 1$. There are no other rational curves in $\mathbb{P}^2$ passing through these two points and maximally tangent to D_{out}. The result

$$N_{1,1}[(k, k)] = \frac{(-1)^{k+1}}{k^2}$$

comes from multiple covers of the line totally branched over the intersection with D_{out}. The multiple cover contribution is computed in [3].

Next, consider the ray of slope 1 for $\ell_1 = \ell_2 = 2$. We calculate

$$\log f_{1,1} = -4\log(1 - t^2 xy) = 4\sum_{k=1}^{\infty} k \cdot \frac{1}{k^2} \cdot (tx)^k (ty)^k.$$

We now must choose two points each on D_1 and D_2. As above, $N_{(1,1)}[(1+0, 1+0)] = 1$ because there is exactly one line through two points. Similarly

$$N_{1,1}[(1+0, 0+1)] = N_{1,1}[(0+1, 1+0)] = N_{1,1}[(0+1, 0+1)] = 1,$$

giving the desired total for $c^1_{1,1}(2,2) = 4$. The invariant

$$N_{1,1}[(2+0, 2+0)] = -1/4$$

is obtained from the double covers of the line. Hence, double covers of the four lines contribute -1 to $c^2_{1,1}(2,2)$. On the other hand, there is a pencil of conics passing through the four chosen points. Being tangent to D_{out} is a quadratic condition, so

$$N_{1,1}[(1+1, 1+1)] = 2.$$

Putting the calculation together yields

$$c^2_{1,1}(2,2) = (-1) + 2 = 1.$$

All remaining contributions to $c^k_{1,1}(2,2)$ for $k > 2$ come from multiple covers of either one of the lines or one of the conics.

For the ray of slope 1 for $\ell_1 = 2$, $\ell_2 = 3$, we have

$$\log f_{1,1} = 6(tx)(ty) + 2 \cdot \frac{9}{2}(tx)^2(ty)^2 + 3 \cdot \frac{20}{3}(tx)^3(ty)^3 + \cdots .$$

The coefficient $c^1_{1,1}(2,3) = 6$ counts the number of lines passing through one of two points on D_1 and one of three points on D_2. The coefficient

$$c^2_{1,1}(2,3) = 9/2 = 6 - 6/4$$

is obtained as follows. There are six conics passing through the two chosen points on D_1 and two of the three chosen points on D_2 and tangent to D_{out}. The $-6/4$ accounts for double covers of the lines. It is possible to compute

$$N_{1,1}[2+1, 1+1+1] = N_{1,1}[1+2, 1+1+1] = 3.$$

These are the only contributions from non-multiple covers to $c^3_{1,1}(2,3)$ — corresponding to plane cubics with a node at one of the two chosen points on D_1 and passing through all chosen points, with D_{out} being an inflectional tangent. On the other hand, the triple covers of each line contribute 1/9, for a total of

$$c^3_{1,1}(2,3) = 3 + 3 + 6/9 = 20/3.$$

For higher k, there continue to be contributions from curves which are not just multiple covers of curves already found.

3.6. **Correspondence.** Theorems 1 and 2 together yield an interesting correspondence between the moduli space of rational curves on toric sufaces and the moduli spaces of quiver representations.

Corollary 3. *For every $m > 0$ and primitive $(a, b) \in \mathbb{Z}^2$ lying strictly in the first quadrant, we have*

$$\exp\left(\sum_{k\geq 1}\sum_{|\mathbf{P}_a|=ak}\sum_{|\mathbf{P}_b|=bk} kN_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)]\cdot (tx)^{ak}\ (ty)^{bk}\right)$$
$$= \left(1+\sum_{k\geq 1}\chi\Big(\mathcal{M}_m^{(1,0),B}(ak,bk)\Big)\cdot (tx)^{ak}\ (ty)^{bk}\right)^{\frac{m}{a}}$$
$$= \left(1+\sum_{k\geq 1}\chi\Big(\mathcal{M}_m^{(1,0),F}(ak,bk)\Big)\cdot (tx)^{ak}\ (ty)^{bk}\right)^{\frac{m}{b}}$$

where the sums in the first line are over all ordered partitions $\mathbf{P}_a$ of size ak *and length m and $\mathbf{P}_b$ of size* bk *and length m.*

Corollary 3 is a correspondence between rational curve counts for the toric surface $X_{a,b}$ and Euler characteristics of framed moduli spaces of quiver representations of Q_m with dimension vectors proportional to (a, b). At the moment, no direct geometric argument for Corollary 3 is known. Also, while parallels between Corollary 3 and the correspondences of [16] are apparent (both link Gromov-Witten invariants to possibly virtual Euler characteristics of moduli spaces of framed sheaves), again no precise connection is known.

Theorem 2 as stated is more general than Theorem 1 since ℓ_1 and ℓ_2 are not required to be equal. Richer versions of Theorem 1 which capture the $\ell_1 \neq \ell_2$ cases can be obtained from more complicated quiver constructions.[10] Finally, a version of Theorem 2 which casts the commutator calculations in the tropical vertex group (over

[10]M. Reineke has explained to us a method using certain bipartite quivers (up to symmetric group actions). A. King has made a similar proposal.

many variables instead of just t) as equivalent to the determination of the invariants $N_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$ can be found in [8].

4. Scattering patterns

4.1. **Directions.** Consider the basic elements

$$S_{\ell_1} = \theta_{(1,0),(1+tx)^{\ell_1}} \quad \text{and} \quad T_{\ell_2} = \theta_{(0,1),(1+ty)^{\ell_2}}$$

of the tropical vertex group. The unique factorization

$$T_{\ell_2}^{-1} \circ S_{\ell_1} \circ T_{\ell_2} \circ S_{\ell_1}^{-1} = \prod^{\rightarrow} \theta_{(a,b),f_{a,b}} \tag{4.1}$$

associates a function

$$f_{a,b} \in \mathbb{C}[x^a y^b][[t]]$$

to every primitive vector $(a,b) \in \mathbb{Z}^2$ lying strictly in the first quadrant.

Question 4. For which directions is $f_{a,b} \neq 1$?

The *scattering pattern* associated to ℓ_1 and ℓ_2 consists of the directions in the first quadrant for which $f_{a,b} \neq 1$. We have seen several examples of scattering patterns in Section 1.4. Our goal here is to give an answer to Question 4 via Theorem 2 and the the classical geometry of curves on toric surfaces.

4.2. **Curves.** If $f_{a,b} \neq 1$, then there must exist, by Theorem 2, a nonvanishing invariant

$$N_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \neq 0,$$

where $\mathbf{P}_a$ is of size ak and length ℓ_1 and $\mathbf{P}_b$ of size bk and length ℓ_2. The nonvanishing of the invariant implies the nonemptiness of the corresponding moduli space,

$$\overline{\mathfrak{M}}(X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]/D^o_{\text{out}}) \neq \emptyset \ .$$

Recall, following the notation of Section 3.1,

$$\nu : X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \to X^o_{a,b}$$

is the blow-up along ℓ_1 and ℓ_2 distinct points of D^o_1 and D^o_2 respectively.

Let $[\phi] \in \overline{\mathfrak{M}}(X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]/D^o_{\text{out}})$ be a stable relative map,

$$(C,p) \xrightarrow{\phi} \mathfrak{X}^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \xrightarrow{\pi} X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)],$$

satisfying the following properties:

(i) C is a complete connected curve of arithmetic genus 0 with at worst nodal singularities,

(ii) $\mathfrak{X}^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \to X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$ is a destabilization[11] along the relative divisor D^o_{out},

(iii) C has full contact via ϕ with D^o_{out} of order k at p.

For the calculation of intersection numbers, we will often view the composition

$$\pi \circ \phi : C \to X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \subset X_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$$

as having image in the complete surface. Let

$$D_i^{\text{strict}} \subset X_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$$

be the strict transformation under ν of D_i.

Lemma 4.1. *Let $C' \subset C$ be an irreducible component on which $\pi \circ \phi$ is nonconstant. Then,*

$$C' \cdot D_1^{\text{strict}} = C' \cdot D_2^{\text{strict}} = 0\ .$$

Proof. Since $\pi \circ \phi(C') \subset X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$, the component C' can not dominate D_i^{strict}. Hence,

$$C' \cdot D_i^{\text{strict}} \geq 0\ .$$

The intersection number of C with D_1^{strict} is

$$C \cdot D_1^{\text{strict}} = \beta_k \cdot D_1 + \sum_{i=1}^{\ell_1} p_i E_i^2 = 0$$

where $\mathbf{P}_a = p_1 + \ldots + p_{\ell_1}$ and E_i are the exceptional divisors of ν over D_1. Therefore, if $C' \cdot D_1^{\text{strict}} > 0$, then

$$\overline{C \setminus C'} \cdot D_1^{strict} < 0$$

which is impossible since no component of C dominates D_1^{strict}. The argument for D_2^{strict} is identical. □

Lemma 4.2. *Let $C' \subset C$ be an irreducible component on which $\pi \circ \phi$ is nonconstant. The set*

$$C' \ \cap \ (\pi \circ \phi)^{-1}(D^o_{\text{out}})$$

consists of a single point.

[11]A destabilization along a relative divisor is obtained by attaching a finite number of bubbles each of which is a $\mathbb{P}^1$-bundle over the divisor. We refer the reader to Section 1 of [14] for an introduction to the destabilizations required for stable relative maps. Li uses the term *expanded degeneration* for our destabilizations.

Proof. Let $q = \pi \circ \phi(p) \in D^o_{\text{out}}$. Since no components of C dominate D_{out} and $\phi(C)$ has full contact with the extremal $D_{\text{out}} \subset \mathfrak{X}^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$ at a single point, we conclude $\pi \circ \phi(C')$ meets D^o_{out} only at q. Since the dual graph of C has no loops (by the genus 0 condition), the set $C' \cap (\pi \circ \phi)^{-1}(D^o_{\text{out}})$ can not contain more than one point. □

Lemma 4.3. *If $f_{a,b} \neq 1$, then there exists a nonconstant map*

$$\mathbb{P}^1 \to X^o_{a,b}[(\mathbf{P}'_a, \mathbf{P}'_b)]$$

which is both

(i) *a normalization of a subcurve of $X^o_{a,b}[(\mathbf{P}'_a, \mathbf{P}'_b)]$,*
(ii) *an element of $\overline{\mathfrak{M}}(X^o_{a,b}[(\mathbf{P}'_a, \mathbf{P}'_b)]/D^o_{\text{out}})$ where $\mathbf{P}'_a$ is of size ak' and length ℓ_1 and $\mathbf{P}'_b$ of size bk' and length ℓ_2.*

Proof. Let $\mathbb{P}^1 \cong C' \subset C$ be an irreducible component on which $\pi \circ \phi$ is nonconstant. By Lemmas 4.1 and 4.2, the map

$$\pi \circ \phi : C' \to X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] \tag{4.2}$$

lies in the moduli space[12] $\overline{\mathfrak{M}}(X^o_{a,b}[(\mathbf{P}'_a, \mathbf{P}'_b)]/D^o_{\text{out}})$ where $\mathbf{P}'_a$ is of size ak' and length ℓ_1 and $\mathbf{P}'_b$ of size bk' and length ℓ_2 for $k' \leq k$.

If (4.2) is birational onto the image $\pi \circ \phi(C')$, then we have proven the Lemma. If

$$\pi \circ \phi : C' \to \pi \circ \phi(C')$$

is a multiple cover, then, by taking the normalization of $\pi \circ \phi(C')$, we obtain the required map (for $k'' < k'$). □

4.3. **Genus inequalities.** On the surface $X_{a,b}$, the intersection results

$$D_1 \cdot D_2 = 1, \quad D_1 \cdot D_{\text{out}} = \frac{1}{b}, \quad D_2 \cdot D_{\text{out}} = \frac{1}{a}$$

are easily obtained since the divisors intersect transversely (at orbifold points). Since $A_1(X_{a,b})$ is rank 1 over $\mathbb{Q}$, we conclude

$$bD_1 = aD_2 = abD_{\text{out}},$$
$$D_1^2 = \frac{a}{b}, \quad D_2^2 = \frac{b}{a}, \quad D_{\text{out}}^2 = \frac{1}{ab} \ .$$

Since $\beta_k \cdot D_{\text{out}} = k$, we see $\beta_k = abkD_{\text{out}}$.

The arithmetic genus of a complete curve $P \subset X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$ of class

$$\beta_k[(\mathbf{P}_a, \mathbf{P}_b)] = \nu^*(\beta_k) - \sum_{i=1}^{\ell_1} p_i E_i - \sum_{j=1}^{\ell_2} p'_j E'_j$$

[12] Since lengths of the partitions match, the spaces $X^o_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$ and $X^o_{a,b}[(\mathbf{P}'_a, \mathbf{P}'_b)]$ can be taken to be the same.

is given by adjunction,

$$\begin{aligned} 2g_a(P)-2 &= (K_{X^o_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)]}+P)\cdot P \\ &= (-D_1-D_2-D_{\text{out}}+\beta_k)\cdot\beta_k-\sum_{i=1}^{\ell_1}p_i(p_i-1)-\sum_{j=1}^{\ell_2}p'_j(p'_j-1) \\ &= -ak-bk-k+abk^2-\sum_{i=1}^{\ell_1}p_i(p_i-1)-\sum_{j=1}^{\ell_2}p'_j(p'_j-1) \\ &= abk^2-k-\sum_{i=1}^{\ell_1}p_i^2-\sum_{j=1}^{\ell_2}(p'_j)^2\ . \end{aligned}$$

If P is irreducible with normalization of genus 0, then

$$abk^2-k-\sum_{i=1}^{\ell_1}p_i^2-\sum_{j=1}^{\ell_2}(p'_j)^2+2\geq 0$$

since the arithmetic genus is bounded from below by the geometric genus.

Suppose $f_{a,b}\neq 1$. By the existence result of Lemma 4.3, there exists an irreducible curve $P\subset X^o_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)]$ with normalization of genus 0. Hence, there exists an integer $k>0$ and partitions

$$\mathbf{P}_a=p_1+\cdots+p_{\ell_1},\quad |\mathbf{P}_a|=ak,\qquad \mathbf{P}_b=p'_1+\cdots+p'_{\ell_2},\quad |\mathbf{P}_b|=bk \tag{4.3}$$

for which the inequality

$$abk^2-k-\sum_{i=1}^{\ell_1}p_i^2-\sum_{j=1}^{\ell_2}(p'_j)^2+2\geq 0 \tag{4.4}$$

is satisfied.

We define a primitive vector $(a,b)\in\mathbb{Z}^2$ lying strictly in the first quadrant to be *permissible* for the pair (ℓ_1,ℓ_2) if there exist partitions (4.3) with $k>0$ satisfying the inequality (4.4). We have proven the following result.

Proposition 4.4. *If $f_{a,b}\neq 1$ in the order product factorization of $T_{\ell_2}^{-1}\circ S_{\ell_1}\circ T_{\ell_2}\circ S_{\ell_1}^{-1}$, then (a,b) is permissible for the pair (ℓ_1,ℓ_2).*

4.4. **Case I: Continuous range.** Our first result specifies a continuous range of possible slopes of permissible vectors. Consider the quadratic polynomial

$$R_{\ell_1,\ell_2}(z)=\frac{1}{\ell_2}z^2-z+\frac{1}{\ell_1}\ .$$

with discriminant $1-\frac{4}{\ell_1\ell_2}$. For the list of pairs

$$(\ell_1,\ell_2)=(1,1),\ (1,2),\ (2,1),\ (1,3),\ (3,1),$$

$R_{\ell_1,\ell_2}(z) > 0$ for all real z. For all other pairs of positive integers (ℓ_1, ℓ_2), the polynomial R_{ℓ_1,ℓ_2} has two positive real roots

$$\xi_\pm = \frac{\ell_2}{2}\left(1 \pm \sqrt{1 - \frac{4}{\ell_1\ell_2}}\,\right) .$$

For slopes $\xi_- < \frac{b}{a} < \xi_+$ strictly between the roots, $R_{\ell_1,\ell_2}(\frac{b}{a})$ is negative.

Lemma 4.5. *If $R_{\ell_1,\ell_2}(\frac{b}{a}) < 0$, then the vector (a, b) is permissible for (ℓ_1, ℓ_2).*

Proof. If k is chosen to be divisible by both ℓ_1 and ℓ_2, the balanced partitions

$$\mathbf{P}_a = \frac{ak}{\ell_1} + \cdots + \frac{ak}{\ell_1}, \quad \mathbf{P}_b = \frac{bk}{\ell_2} + \cdots + \frac{bk}{\ell_2}$$

can be formed. The inequality (4.4) becomes

$$\left(ab - \frac{a^2}{\ell_1} - \frac{b^2}{\ell_2}\right) k^2 - k + 2 \geq 0 \tag{4.5}$$

Since the coefficient of k^2 is $-a^2 R_{\ell_1,\ell_2}(\frac{b}{a}) > 0$ by the assumed slope condition, the inequality (4.5) can certainly be satisfied for large enough (and divisible) k. □

If $(\ell_1, \ell_2) \in \{(1, 4),\ (4, 1),\ (2, 2)\}$, then the polynomial R_{ℓ_1,ℓ_2} has a double root $\xi_- = \xi_+$. Lemma 4.5 does not permit any slopes in the double root case.

Lemma 4.6. *If $(\ell_1, \ell_2) \notin \{(1, 1),\ (1, 2),\ (2, 1),\ (1, 3),\ (3, 1),\ (1, 4),\ (4, 1), (2, 2)\}$, then the two roots $\xi_\pm$ are real, positive, and irrational.*

Proof. Only the irrational claim is nontrivial. Let 2^s be the largest power of 2 dividing the product $\ell_1\ell_2$,

$$\ell_1\ell_2 = 2^s n$$

where n is odd. There are three cases to consider:

(i) If $s = 0$,

$$\frac{\ell_1\ell_2 - 4}{\ell_1\ell_2} = \frac{n-4}{n}$$

where $n - 4$ and n are relatively prime. But there are no positive pairs of squares separated by 4, so $\sqrt{1 - \frac{4}{\ell_1\ell_2}}$ is irrational.

(ii) If $s = 1$,

$$\frac{\ell_1\ell_2 - 4}{\ell_1\ell_2} = \frac{n-2}{n}$$

and the same argument applies.

(iii) If $s \geq 2$,

$$\frac{\ell_1\ell_2 - 4}{\ell_1\ell_2} = \frac{2^{s-2}n - 1}{2^{s-2}n}$$

and the argument again applies.

The hypotheses in the Lemma are only used to show $\ell_1\ell_2 - 4 > 0$. □

Lemma 4.7. *If $R_{\ell_1,\ell_2}(\frac{b}{a}) = 0$, then we must have $(\ell_1, \ell_2) \in \{(1,4),\ (4,1),\ (2,2)\}$. Moreover, (a,b) is permissible for (ℓ_1,ℓ_2).*

Proof. Since R_{ℓ_1,ℓ_2} has rational roots only in case $(\ell_1, \ell_2) \in \{(1,4),\ (4,1),\ (2,2)\}$, the first claim is clear. For $(\ell_1,\ell_2) = (1,4)$ and $(4,1)$, we have the double roots $(a,b) = (1,2)$ and $(2,1)$ respectively. For $(\ell_1,\ell_2) = (2,2)$, we have the double root $(a,b) = (1,1)$. Permissibility is established in both cases by taking $k = 2$ and balanced partitions. □

4.5. **Case II: Discrete series.**

4.5.1. *Positive values.* Permissibility for $R_{\ell_1,\ell_2}(\frac{b}{a}) \le 0$ has been established by Lemmas 4.5 and 4.7. We now consider the cases where

$$R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) > 0\ . \tag{4.6}$$

Since $\sum_{i=1}^{\ell_1} p_i^2 \ge \frac{a^2}{\ell_1}k^2$ and similarly for the p'_j, we see

$$abk^2 - k - \sum_{i=1}^{\ell_1} p_i^2 - \sum_{j=1}^{\ell_2}(p'_j)^2 + 2 \le -a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right)k^2 - k + 2\ .$$

Certainly for all $k \ge 2$ the right side is negative. Hence, if (a,b) satisfies (4.6) and is permissible for (ℓ_1,ℓ_2), then $k = 1$ and we must have

$$ab - \sum_{i=1}^{\ell_1} p_i^2 - \sum_{j=1}^{\ell_2}(p'_j)^2 + 1 = 0 \tag{4.7}$$

for partitions $p_1 + \cdots + p_{\ell_1} = a$ and $p'_1 + \cdots + p'_{\ell_2} = b$.

There are exactly three possibilities for the solution of (4.7) in the presence of condition (4.6):

(i) $a \equiv 0 \bmod \ell_1$, $b \equiv 0 \bmod \ell_2$, and $a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = 1$.
(ii) $a \equiv \pm 1 \bmod \ell_1$, $b \equiv 0 \bmod \ell_2$, and $a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = \frac{1}{\ell_1}$,
(iii) $a \equiv 0 \bmod \ell_1$, $b \equiv \pm 1 \bmod \ell_2$, and $a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = \frac{1}{\ell_2}$.

A straightforward analysis shows unless one of (i-iii) are satisfied,

$$ab - \sum_{i=1}^{\ell_1} p_i^2 - \sum_{j=1}^{\ell_2}(p'_j)^2 < -a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) - 1 < -1\ .$$

4.5.2. *Analysis of (i).* If ℓ_1 or ℓ_2 equals 1, then (i) is special case of (ii) and (iii). Let $\mathcal{S}_{\ell_1,\ell_2}$ be the set of solutions to (i) with $(a,b) \in \mathbb{Z}^2$ lying in the closed first quadrant. We will show $\mathcal{S}_{\ell_1,\ell_2}$ is empty when $\ell_1, \ell_2 > 1$.

We now assume $\ell_1, \ell_2 > 1$. When specialized to $b = 0$, the equation of (i),

$$a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = 1, \tag{4.8}$$

yields $\frac{a^2}{\ell_1} = 1$ which has *no* solutions satisfying $a \equiv 0 \mod \ell_1$. A similar conclusion holds when $a = 0$. We conclude all elements of $\mathcal{S}_{\ell_1,\ell_2}$ lie strictly in the first quadrant.

Crucial to our analysis are the following two transformations

$$\mathsf{T}_1(a,b) = (\ell_1 b - a, b), \quad \mathsf{T}_2(a,b) = (a, \ell_2 a - b)\ .$$

which leave the expression

$$a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = -ab + \frac{a^2}{\ell_1} + \frac{b^2}{\ell_2}$$

invariant. Both have order two,

$$\mathsf{T}_1^2 = \mathsf{T}_2^2 = \mathrm{Id}\ .$$

If $(a,b) \in \mathcal{S}_{\ell_1,\ell_2}$ is a solution of (i) in the first quadrant, we have seen $a, b > 0$. Let

$$(a_1, b_1) = \mathsf{T}_1(a,b), \quad (a_2, b_2) = \mathsf{T}_2(a,b)\ .$$

By the invariance, we have

$$a_i^2 R_{\ell_1,\ell_2}\left(\frac{b_i}{a_i}\right) = 1$$

for $i = 1, 2$. By the definitions of T_i, the congruence assumptions for a and b hold also for a_i and b_i respectively. Since $b_1 = b > 0$ and

$$\frac{b^2}{\ell_2} > 1,$$

we must have $a_1 > 0$. Hence, $(a_1, b_1) \in \mathcal{S}_{\ell_1,\ell_2}$. Similarly, $(a_2, b_2) \in \mathcal{S}_{\ell_1,\ell_2}$. We have proven the following result.

Lemma 4.8. *Both T_1 and T_2 preserve the set $\mathcal{S}_{\ell_1,\ell_2}$.*

We now apply the transformations twice to obtain two new elements of $\mathcal{S}_{\ell_1,\ell_2}$,

$$(a_{21}, b_{21}) = \mathsf{T}_2(a_1, b_1), \quad (a_{12}, b_{12}) = \mathsf{T}_1(a_2, b_2)\ .$$

Lemma 4.9. *If $(a,b) \in \mathcal{S}_{\ell_1,\ell_2}$ and $\frac{b}{a} > \xi_+$, then*

$$a > a_{12}, \quad b > b_{12}, \quad \frac{b_{12}}{a_{12}} > \frac{b}{a}\ .$$

Proof. Using the formula $a_{12} = \ell_1(\ell_2 a - b) - a$, we find $a > a_{12}$ is equivalent to

$$\frac{b}{a} > \ell_2 - \frac{2}{\ell_1} \,. \tag{4.9}$$

But since $\frac{4}{\ell_1\ell_2} \leq 1$, we see

$$\begin{aligned} \xi_+ &= \frac{\ell_2}{2}\left(1 + \sqrt{1 - \frac{4}{\ell_1\ell_2}}\,\right) \\ &\geq \frac{\ell_2}{2}(1 + 1 - \frac{4}{\ell_1\ell_2}) \\ &\geq \ell_2 - \frac{2}{\ell_1} \,. \end{aligned}$$

Hence, inequality (4.9) follows from the slope assumption $\frac{b}{a} > \xi_+$.

Similarly, using the formula $b_{12} = \ell_2 a - b$, we find $b > b_{12}$ is equivalent to

$$\frac{b}{a} > \frac{\ell_2}{2}$$

which also follows form the slope assumption.

Since $(a_{12}, b_{12}) \in \mathcal{S}_{\ell_1,\ell_2}$, we must have $a_{12} > 0$. Using the ratio of the formulas for b_{12} and a_{12}, we find

$$\frac{b_{12}}{a_{12}} = \frac{\ell_2 - \frac{b}{a}}{\ell_1(\ell_2 - \frac{b}{a}) - 1}.$$

The third claim of the Lemma is

$$\frac{\ell_2 - \frac{b}{a}}{\ell_1(\ell_2 - \frac{b}{a}) - 1} > \frac{b}{a}$$

which is equivalent to

$$0 > -R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = -\frac{1}{a^2}$$

since $(a, b) \in \mathcal{S}_{\ell_1,\ell_2}$. □

Lemma 4.10. *If $(a, b) \in \mathcal{S}_{\ell_1,\ell_2}$ and $\frac{b}{a} < \xi_-$, then*

$$a > a_{21}, \quad b > b_{21}, \quad \frac{b_{21}}{a_{21}} < \frac{b}{a} \,.$$

The proof of Lemma 4.10 is identical to the proof of Lemma 4.9. We are now prepared to prove the emptiness of $\mathcal{S}_{\ell_1,\ell_2}$.

Lemma 4.11. *For $\ell_1, \ell_2 > 1$, we have $\mathcal{S}_{\ell_1,\ell_2} = \emptyset$.*

Proof. Suppose $(a,b) \in \mathcal{S}_{\ell_1,\ell_2}$ exists. Then, since $R_{\ell_1,\ell_2}(\frac{b}{a}) > 0$, we must have either

$$\frac{b}{a} > \xi_+ \ \text{ or } \ \frac{b}{a} < \xi_- \ .$$

In the former case Lemma 4.9 yields a new element $(a_{12}, b_{12}) \in \mathcal{S}_{\ell_1,\ell_2}$ with strictly smaller values $a_{12} < a$ and $b_{12} < b$. In the latter case, we use Lemma 4.10. After finitely many iterations, we must exit the first quadrant contradicting Lemma 4.8. □

4.5.3. *Analysis of (ii).* We assume $\ell_1, \ell_2 > 0$ and $(\ell_1,\ell_2) \neq (1,1)$. Let $\mathcal{A}_{\ell_1,\ell_2}$ be the set of solutions to (ii) with $(a,b) \in \mathbb{Z}^2$ lying in the closed first quadrant. When specialized to $b = 0$, the equation of (ii),

$$a^2 R_{\ell_1,\ell_2}\left(\frac{b}{a}\right) = \frac{1}{\ell_1},$$

yields $\frac{a^2}{\ell_1} = \frac{1}{\ell_1}$ which has a single positive solution $a = 1$. As in Section 4.5.2, no solutions occur when $a = 0$ (using $(\ell_1,\ell_2) \neq (1,1)$). We conclude all elements of $\mathcal{A}_{\ell_1,\ell_2}$ lie strictly in the first quadrant except for $(1,0)$. Let

$$\mathcal{A}^*_{\ell_1,\ell_2} = \mathcal{A}_{\ell_1,\ell_2} - \{(1,0)\} \ .$$

The proof of Lemma 4.8 immediately yields the following result.

Lemma 4.12. *Both* T_1 *and* T_2 *map* $\mathcal{A}^*_{\ell_1,\ell_2}$ *to* $\mathcal{A}_{\ell_1,\ell_2}$ *.*

Assume further $(\ell_1,\ell_2) \notin \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$. The method used in Section 4.5.2 to study the solutions in case (i) yields a complete description of $\mathcal{A}^*_{\ell_1,\ell_2}$.

Proposition 4.13. *The permissible vectors for* (ℓ_1,ℓ_2) *obtained from case (ii) are*

$$\mathcal{A}^*_{\ell_1,\ell_2} = \{\ \mathsf{T}_2(1,0),\ \mathsf{T}_1(\mathsf{T}_2(1,0)),\ \mathsf{T}_2(\mathsf{T}_1(\mathsf{T}_2(1,0))),\ \mathsf{T}_1(\mathsf{T}_2(\mathsf{T}_1(\mathsf{T}_2(1,0)))),\ \ldots\}.$$

Proof. Start with any solution $(a,b) \in \mathcal{A}^*_{\ell_1,\ell_2}$. Depending upon whether $\frac{b}{a}$ is greater than ξ_+ or less than ξ_- apply $\mathsf{T}_1\mathsf{T}_2$ or $\mathsf{T}_2\mathsf{T}_1$. The result is a solution (a',b') with $a' < a$ and $b' < b$. By iterating the process, the solution must eventually leave the strict first quadrant. By Lemma 4.12, we conclude some chain of applications of T_1 and T_2 to (a,b) yields $(1,0)$. □

For the cases $(\ell_1,\ell_2) \in \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$, the group generated by T_1 and T_2 is finite and, in each case, contains elements that move every (a,b) strictly in the first quadrant out of the strict first quadrant. Hence, every element of $\mathcal{A}^*_{\ell_1,\ell_2}$ can be reached from $(1,0)$ by a chain of applications of T_1 and T_2. Since the sets are finite, we can list all the elements:

$$\mathcal{A}^*_{1,1} = \{(1,1)\}, \ \ \mathcal{A}^*_{1,2} = \{(1,2)\}, \ \ \mathcal{A}^*_{2,1} = \{(1,1)\},$$

$$\mathcal{A}^*_{1,3} = \{(1,3),\ (2,3)\},\ \ \mathcal{A}^*_{3,1} = \{(1,1),\ (2,1)\}\ .$$

4.5.4. *Analysis of (iii).* Of course the discussion of (iii) is identical to (ii). Let $\mathcal{B}^*_{\ell_1,\ell_2}$ be the set of solutions to (iii) with $(a,b) \in \mathbb{Z}^2$ lying strictly in the first quadrant. For $(\ell_1,\ell_2) \notin \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$,

$$\mathcal{B}^*_{\ell_1,\ell_2} = \{\ \mathsf{T}_1(0,1),\ \mathsf{T}_2(\mathsf{T}_1(0,1)),\ \mathsf{T}_1(\mathsf{T}_2(\mathsf{T}_1(0,1))),\ \mathsf{T}_2(\mathsf{T}_1(\mathsf{T}_2(\mathsf{T}_1(0,1)))),\ \ldots\}.$$

The special cases are:

$$\mathcal{B}^*_{1,1} = \{(1,1)\},\ \ \mathcal{B}^*_{1,2} = \{(1,1)\},\ \ \mathcal{B}^*_{2,1} = \{(2,1)\},$$

$$\mathcal{B}^*_{1,3} = \{(1,1),\ (1,2)\},\ \ \mathcal{B}^*_{3,1} = \{(3,1),\ (3,2)\}\ .$$

4.6. **Results for scattering patterns.** Let $\ell_1,\ell_2 > 0$. Our main result for scattering patterns determines the set of permissible vectors for (ℓ_1,ℓ_2).

Theorem 5. *If $(\ell_1,\ell_2) \notin \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$, then the set $\mathcal{P}(\ell_1,\ell_2)$ of permissible vectors is the disjoint union*

$$\mathcal{P}_{\ell_1,\ell_2} = \mathcal{A}^*_{\ell_1,\ell_2}\ \cup\ \mathcal{B}^*_{\ell_1,\ell_2}\ \cup\ \{\ (a,b) \in \mathbb{Z}^2 \mid \xi_- \leq \frac{b}{a} \leq \xi_+\ \}\ .$$

Theorem 5 is simply a summary of the result of Sections 4.4-4.5. The sets of permissible vectors for the special pairs (ℓ_1,ℓ_2) excluded in Theorem 5 are:

$$\mathcal{P}_{1,1} = \{(1,1)\},\ \ \mathcal{P}_{1,2} = \{(1,2),(1,1)\},\ \ \mathcal{P}_{2,1} = \{(1,1),(2,1)\},$$

$$\mathcal{P}_{1,3} = \{(1,3),(2,3),(1,1),\ (1,2)\},\ \ \mathcal{P}_{3,1} = \{(1,1),(2,1),(3,1),\ (3,2)\}\ .$$

Returning to Question 4, consider the ordered product factorization (4.1) of the commutator. We have proven in Section 4.3 the implication

$$f_{a,b} \neq 1 \implies (a,b) \in \mathcal{P}_{\ell_1,\ell_2}\ .$$

In other words, the scattering pattern associated to ℓ_1 and ℓ_2 is contained in the directions of $\mathcal{P}_{\ell_1,\ell_2}$. Theorem 5 completely determines $\mathcal{P}_{\ell_1,\ell_2}$. In the nontrivial cases $(\ell_1,\ell_2) = (2,2),(3,3)$ and $(2,3)$ analyzed in §1.4, the behaviour claimed (via calculations) fits precisely with the results predicted by Theorem 5. For $\ell_1 = \ell_2 = m$, the containment of the scattering pattern in $\mathcal{P}_{m,m}$ was conjectured previously by Gross-Siebert and Kontsevich based on computational data.

While very tempting to believe, we have *not* proven the reverse implication

$$(a,b) \in \mathcal{P}_{\ell_1,\ell_2} \implies f_{a,b} \neq 1. \tag{4.10}$$

Certainly (4.10) is consistent with all the gathered data. If $\ell_1 = \ell_2 = m$, the equivalence

$$(a, b) \in \mathcal{P}_{m,m} \iff f_{a,b} \neq 1$$

can be proven via the existence of $(1,0)$-semistable representations of the quiver Q_m discussed in Section 4.7 below.

4.7. **Quivers.** If ℓ_1 and ℓ_2 are both equal to m, then Question 4 is related to the existence of $(1,0)$-semistable representations of Q_m by Theorem 1.

Proposition 4.14. *For $m = \ell_1 = \ell_2$ and primitive $(a, b) \in \mathbb{Z}^2$ lying strictly in the first quadrant, the following are equivalent:*

(i) $f_{a,b} \neq 1$,
(ii) *there exists a nonzero $(1,0)$-semistable representation of Q_m with dimension vector proportional to (a, b),*
(iii) *there exists a nonzero $(1,0)$-stable back framed representation of Q_m with dimension vector proportional to (a, b),*
(iv) *there exists a nonzero $(1,0)$-stable front framed representation of Q_m with dimension vector proportional to (a, b),*

Proof. By Theorem 1, (i) implies (iii) and (iv). The moduli spaces $\mathcal{M}_m^{(1,0),B}(d_1, d_2)$ and $\mathcal{M}_m^{(1,0),F}(d_1, d_2)$ are nonsingular projective varieties with no odd cohomology [11, 17]. For such spaces, nonemptyness implies positive Euler characteristic.[13] Hence, again by Theorem 1, (iii) and (iv) are equivalent and imply (i). By the definition of $(1,0)$-stability for framed represention, the underlying standard representation is $(1,0)$-semistable. So (iii) and (iv) imply (ii).

If (ii) holds, then there exists a $(1,0)$-semistable representation ρ of Q_m with slope

$$\mu(\rho) = \frac{a}{a+b} \ .$$

We will show there exists a subrepresentation $\widehat{\rho} \subset \rho$ of the same slope which is $(1,0)$-stable. If ρ is $(1,0)$-stable, then take $\widehat{\rho} = \rho$. If ρ is strictly $(1,0)$-semistable, then ρ must contain a smaller nonzero $(1,0)$-semistable representation of slope $\frac{a}{a+b}$, and we repeat. By finiteness of chains, we must eventually find a $(1,0)$-stable $\widehat{\rho}$. Since

$$\mu(\widehat{\rho}) = \frac{a}{a+b} \ ,$$

the dimension vector of $\widehat{\rho}$ is proportional to (a, b). For a $(1,0)$-stable standard representation $\widehat{\rho} = (\widehat{V}_1, \widehat{V}_2, \tau_1, \ldots, \tau_m)$, *every* choice of framing data $L_i \subset \widehat{V}_i$ yields a $(1,0)$-stable framed representation. Hence, (ii) implies (iii) and (iv). □

[13] See [21] for better bounds in certain cases.

Reineke has provided us a proof of the following result about representations of Q_m. Given two dimension vectors $\mathbf{d} = (d_1, d_2)$ and $\mathbf{e} = (e_1, e_2)$, let

$$\langle \mathbf{d}, \mathbf{e} \rangle = d_1 e_1 + d_2 e_2 - m d_1 e_2.$$

The form $\langle , \rangle$ is *not* symmetric.

Proposition 4.15. (Reineke) *Let* $\mathbf{d} \in \mathbb{Z}^2$ *be a primitive vector lying in the first quadrant. There exists a* $(1,0)$*-semistable representation of* Q_m *with dimension vector proportional to* $\mathbf{d}$ *if and only if* $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$*.*

Proof. We start by proving the *only if* claim. Let ρ be a $(1,0)$-semistable representation of Q_m with dimension vector proportional to $\mathbf{d}$. We can (as before) assume ρ is $(1,0)$-stable by passing to a subrepresentation if necessary. We have

$$\langle \mathbf{d}, \mathbf{d} \rangle = 1 - \left(\dim \mathcal{P}_m(d_1, d_2) - \dim \mathbf{G}_{d_1,d_2}\right) \; . \tag{4.11}$$

By the stability of ρ, the moduli space $\mathcal{M}_m^{(1,0)}(d_1, d_2)$ is nonempty and of non-negative dimension given by the term in the parentheses on the right side of (4.11). Hence, $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$.

For the claim in the other direction, suppose there does not exist a $(1,0)$-semistable representation with dimension vector $\mathbf{d}$. By Corollary 3.5 of [17], there exists a proper[14] decomposition

$$\mathbf{d} = \mathbf{d}^1 + \cdots + \mathbf{d}^s$$

into nonzero dimension vectors of $(1,0)$-semistable representations of Q_m satisfying

$$\mu(\mathbf{d}^1) > \ldots > \mu(\mathbf{d}^s)$$

and $\langle \mathbf{d}^i, \mathbf{d}^j \rangle = 0$ for all $i < j$. Let $\mathbf{e} = \mathbf{d}^1$ and $\mathbf{f} = \mathbf{d}^2 + \cdots + \mathbf{d}^s$. Then,

$$\mathbf{d} = \mathbf{e} + \mathbf{f}, \quad \mu(\mathbf{e}) > \mu(\mathbf{f}), \quad \langle \mathbf{e}, \mathbf{f} \rangle = 0 \; .$$

After writing the last two inequalities as

$$\frac{e_1}{e_2} > \frac{f_1}{f_2}, \qquad e_1 f_1 + e_2 f_2 - m e_1 f_2 = 0$$

and elementary manipulation, we obtain both $\langle \mathbf{e}, \mathbf{e} \rangle > 0$ and $\langle \mathbf{f}, \mathbf{f} \rangle > 0$. Moreover,

$$\langle \mathbf{f}, \mathbf{e} \rangle = e_1 f_1 + e_2 f_2 - m e_2 f_1 = m(e_1 f_2 - e_2 f_1) > 0.$$

Putting the results together, we conclude

$$\langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{e}, \mathbf{e} \rangle + \langle \mathbf{f}, \mathbf{f} \rangle + \langle \mathbf{e}, \mathbf{f} \rangle + \langle \mathbf{f}, \mathbf{e} \rangle \geq 3$$

[14] By properness, s is at least 2.

since all summands are positive except $\langle \mathbf{e}, \mathbf{f} \rangle = 0$. We have contradicted the assumption $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$. □

For primitive $(a, b) \in \mathbb{Z}^2$ lying strictly in the first quadrant,

$$a^2 R_{m,m}\left(\frac{b}{a}\right) = \frac{1}{m}\langle (a,b),(a,b)\rangle \ .$$

Proposition 4.15 precisely produces $(1,0)$-semistable representations of Q_m in all the permissible directions. The proof of the claim

$$(a,b) \in \mathcal{P}_{m,m} \iff f_{a,b} \neq 1 \tag{4.12}$$

is complete. We do not know a proof of (4.12) via rational curve counting on toric surfaces.

4.8. **Further commutators.** Commutators of more general elements of the tropical vertex group may be similarly considered. Let

$$\begin{aligned} p_1(t,x) &= 1 + c_1(tx)^1 + c_2(tx)^2 + \cdots + c_{\ell_1}(tx)^{\ell_1}, \\ p_2(t,y) &= 1 + c'_1(ty)^1 + c'_2(ty)^2 + \cdots + c'_{\ell_2}(ty)^{\ell_2} \end{aligned}$$

be polynomials of degrees ℓ_1 and ℓ_2 respectively, and let

$$\mathcal{S}_{\ell_1} = \theta_{(1,0),p_1(t,x)}, \qquad \mathcal{T}_{\ell_2} = \theta_{(0,1),p_2(t,y)} \ .$$

Our proof of Theorem 5 yields the following result.

Corollary 6. *The scattering pattern associated to the commutator*

$$\mathcal{T}_{\ell_2}^{-1} \circ \mathcal{S}_{\ell_1} \circ \mathcal{T}_{\ell_2} \circ \mathcal{S}_{\ell_1}^{-1} = \prod^{\rightarrow} \theta_{(a,b),f_{a,b}} \ ,$$

lies in the set $\mathcal{P}_{\ell_1,\ell_2}$.

Proof. By factoring p_1 and p_2 over $\mathbb{C}$, we may instead consider the scattering pattern associated to the commutator of the elements

$$\mathcal{S}_{\ell_1} = \theta_{(1,0),(1+t_1x)(1+t_2x)\cdots(1+t_{\ell_1}x)} \ , \qquad \mathcal{T}_{\ell_2} = \theta_{(0,1),(1+s_1y)(1+s_2y)\cdots(1+s_{\ell_2}y)}$$

in the tropical vertex group over the ring $\mathbb{C}[[t_1, \ldots, t_{\ell_1}, s_1, \ldots, s_{\ell_2}]]$. By using the full strength of Theorem 5.4 of [8], the scattering pattern is constrained by the *same* analysis as in Section 4. □

For $\ell'_1 \leq \ell_1$ and $\ell'_2 \leq \ell_2$, Corollary 6 suggests the inclusion

$$\mathcal{P}_{\ell'_1,\ell'_2} \subset \mathcal{P}_{\ell_1,\ell_2}$$

which can easily be verified directly. Finally, commutators of the elements

$$\theta_{(v_1,v_2),p_1(t,x^{v_1}y^{v_2})} \quad \text{and} \quad \theta_{(w_1,w_2),p_2(t,x^{w_1}y^{w_2})}$$

can be transformed to the case constrained by Corollary 6. We leave the details to the reader.

5. Symmetry of the scattering diagram

5.1. Transformations T_1 and T_2. We return to the basic elements

$$S_{\ell_1} = \theta_{(1,0),(1+tx)^{\ell_1}} \quad \text{and} \quad T_{\ell_2} = \theta_{(0,1),(1+ty)^{\ell_2}}$$

of the tropical vertex group and the unique factorization

$$T_{\ell_2}^{-1} \circ S_{\ell_1} \circ T_{\ell_2} \circ S_{\ell_1}^{-1} = \overrightarrow{\prod} \theta_{(a,b),f_{a,b}} \ . \tag{5.1}$$

We have seen $f_{a,b}$ is a series in the variable $(tx)^a(ty)^b$,

$$f_{a,b}(t,x,y) = \mathsf{f}_{a,b}\Big((tx)^a(ty)^b\Big)$$

where $\mathsf{f}_{a,b}(z) \in \mathbb{Q}[[z]]$. By the following result, the factorization (5.1) is symmetric with respect to the transformations

$$\mathsf{T}_1(a,b) = (\ell_1 b - a, b), \quad \mathsf{T}_2(a,b) = (a, \ell_2 a - b) \ .$$

of Section 4.5.2.

Theorem 7. *Let $(a,b) \in \mathbb{Z}^2$ be a primitive vector lying strictly in the first quadrant. If $\mathsf{T}_1(a,b)$ lies strictly in the first quadrant, then*

$$\mathsf{f}_{a,b} = \mathsf{f}_{\mathsf{T}_1(a,b)} \ .$$

Similarly, if $\mathsf{T}_2(a,b)$ lies strictly in the first quadrant, then $\mathsf{f}_{a,b} = \mathsf{f}_{\mathsf{T}_2(a,b)}$.

We will prove Theorem 7 in Section 5.2 via Theorem 2 and symmetries of Gromov-Witten invariants of toric surfaces.

5.2. Curve counting symmetry. Following the notation of Section 3.1, let $\mathbf{P}_a$ and $\mathbf{P}_b$ be ordered partitions,

$$\begin{aligned} \mathbf{P}_a &= p_1 + \cdots + p_{\ell_1}, \\ \mathbf{P}_b &= p'_1 + \cdots + p'_{\ell_2}, \end{aligned}$$

of size ak and bk respectively. Define partitions $\mathbf{P}'_a$ and $\mathbf{P}'_b$ by

$$\begin{aligned} \mathbf{P}'_a &= (bk - p_1) + \cdots + (bk - p_{\ell_1}), \\ \mathbf{P}'_b &= (ak - p'_1) + \cdots + (ak - p'_{\ell_2}). \end{aligned}$$

The following symmetry of Gromov-Witten invariants is the main step in the proof of Theorem 7.

Proposition 5.1. $N_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)] = N_{\ell_1 b-a,b}[(\mathbf{P}'_a,\mathbf{P}_b)] = N_{a,\ell_2 a-b}[(\mathbf{P}_a,\mathbf{P}'_b)]$.

Proof. We prove the first equality of Proposition 5.1. The argument for

$$N_{a,b}[(\mathbf{P}_a,\mathbf{P}_b)] = N_{a,\ell_2 a-b}[(\mathbf{P}_a,\mathbf{P}'_b)]$$

is, of course, identical.

Consider the surface $Y_{a,b}$ obtained by subdividing the fan for $X_{a,b}$ by adding a ray in the direction $(1,0)$, as depicted in Figure 5.1. Denote by

$$D_1,\ D_2,\ D'_1,\ D_{\text{out}} \subset Y_{a,b}$$

the divisors corresponding to the rays generated by $(-1,0)$, $(0,-1)$, $(1,0)$ and (a,b) respectively. Projection onto the second coordinate induces a map of toric varieties

$$\pi : Y_{a,b} \to \mathbb{P}^1.$$

Both D_2 and D_{out} are fibres of π, but D_{out} occurs with multiplicity b. Away from D_{out}, π is a $\mathbb{P}^1$-bundle. The divisors D_1 and D'_1 are sections of π.

Let $Y^o_{a,b} \subset Y_{a,b}$ be the complement of the four torus fixed points, and let

$$D^o_i = D_i \cap Y^o_{a,b}\ .$$

Choose a set of ℓ_1 points on D^o_1 and a set of ℓ_2 points on D^o_2. Let

$$\nu_Y : Y_{a,b}[\mathbf{P}] \to Y_{a,b}\ , \qquad \mathbf{P} = (\mathbf{P}_a,\mathbf{P}_b)\ ,$$

be the blow-up along all $\ell_1+\ell_2$ chosen points. We use the same notation $D_1, D_2, D'_1, D_{\text{out}}$ for the proper transforms in $Y_{a,b}[\mathbf{P}]$ of the respective divisors. Let $E_1,\ldots,E_{\ell_1}, E'_1,\ldots,E'_{\ell_2}$ be the exceptional divisors of ν_Y.

We can similarly consider $\overline{\mathbf{P}} = (\mathbf{P}'_a,\mathbf{P}_b)$ and perform the same construction for $(\ell_1 b - a, b)$. We obtain

$$\overline{\nu}_Y : Y_{\ell_1 b-a,b}[\overline{\mathbf{P}}] \to Y_{\ell_1 b-a,b}.$$

Let $\overline{D}_1, \overline{D}'_1, \overline{D}_2, \overline{D}_{\text{out}} \subset Y_{\ell_1 b-a,b}$ be the toric divisors. We denote their strict transforms with respect to $\overline{\nu}_Y$ by the same symbols. Let $\overline{E}_1,\ldots,\overline{E}_{\ell_1},\overline{E}'_1,\ldots,\overline{E}'_{\ell_2}$ be the exceptional divisors of $\overline{\nu}_Y$.

Let $x_1,\ldots,x_{\ell_1} \in D^o_1 \subseteq Y_{a,b}$ be the points we have chosen on D^o_1. On $Y_{a,b}[\mathbf{P}]$, the proper transforms of the fibres

$$\pi^{-1}(\pi(x_1)),\ldots,\pi^{-1}(\pi(x_{\ell_1})) \tag{5.2}$$

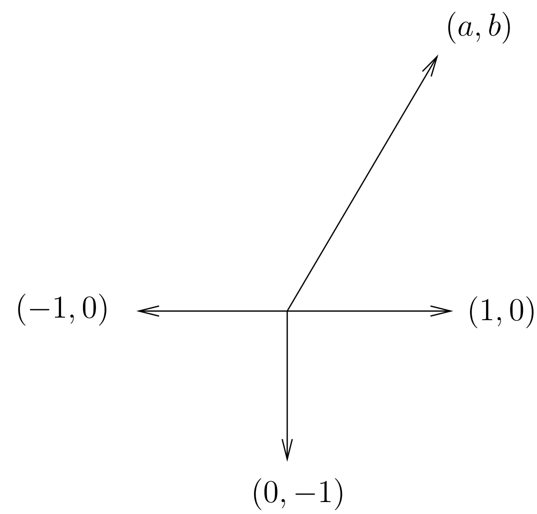


FIGURE 5.1.

are (-1)-curves linearly equivalent to $D_2 - E_1, \ldots, D_2 - E_{\ell_1}$ respectively. Let η be the blow-down of the ℓ_1 curves (5.2) along with $E'_1, \ldots, E'_{\ell_2}$,

$$\eta : Y_{a,b}[\mathbf{P}] \to Z_{a,b}.$$

The rational map $\pi \circ \nu_Y \circ \eta^{-1}$ from $Z_{a,b}$ to $\mathbb{P}^1$ extends to a morphism

$$\pi_Z : Z_{a,b} \to \mathbb{P}^1$$

with all fibres isomorphic to $\mathbb{P}^1$ and reduced (except for the multiple fibre with support $\eta(D_{\text{out}})$).[15] Furthermore, $\eta(D_1)$ and $\eta(D'_1)$ are sections of π_Z. From the above geometry, we easily deduce that $Z_{a,b}$ is a toric variety with toric boundary

$$\eta(D_1) \cup \eta(D'_1) \cup \eta(D_2) \cup \eta(D_{\text{out}}).$$

Which toric variety is $Z_{a,b}$? Because the restriction of π_Z to $Z_{a,b} \setminus \eta(D_{\text{out}})$ is a smooth $\mathbb{P}^1$-bundle over $\mathbb{A}^1$, we see

$$Z_{a,b} \setminus \eta(D_{\text{out}}) \;\cong\; \mathbb{P}^1 \times \mathbb{A}^1$$

as toric varieties. The latter is given, up to lattice isomorphism, by a fan with rays generated by $(\pm 1, 0)$ and $(0, -1)$, so $Z_{a,b}$ must be given by a fan with an additional ray. The fan must look exactly like Figure 5.1, with (a, b) replaced by some (a', b'):

- Since the morphism π_Z is induced by projection onto the second coordinate of the fan and $\eta(D_{\text{out}})$ is still the support of a fibre of π_Z with multiplicity b, we have $b' = b$.

[15] The birational transformation we have described between the $\mathbb{P}^1$-bundles $\pi : Y_{a,b} \to \mathbb{P}^1$ and $\pi_Z : Z_{a,b} \to \mathbb{P}^1$ is known as an *elementary transformation*.

- On $Y_{a,b}$, we have $D_1^2 = \frac{a}{b}$. Hence, $D_1^2 = \frac{a}{b} - \ell_1$ on $Y_{a,b}[\mathbf{P}]$. Then, on $Z_{a,b}$,

$$\eta(D_1)^2 = \frac{a - \ell_1 b}{b}.$$

  Thus, $a' = a - \ell_1 b$.

Using the identification $(a', b') = (a - \ell_1 b, b)$, we conclude

$$Z_{a,b} \;\cong\; Y_{a-\ell_1 b,b} \;\cong\; Y_{\ell_1 b - a,b}$$

where the second isomorphism is obtained by the involution $(m_1, m_2) \mapsto (-m_1, m_2)$ on $\mathbb{Z}^2$ identifying the fans for $Y_{a-\ell_1 b,b}$ and $Y_{\ell_1 b-a,b}$. The composition

$$Y_{a,b}[\mathbf{P}] \xrightarrow{\eta} Z_{a,b} \cong Y_{\ell_1 b-a,b}$$

is the blow-up of ℓ_1 points on $\overline{D}_1^o$ and ℓ_2 points on $\overline{D}_2^o$.

We have shown, if the point sets for the $\ell_1 + \ell_2$ blow-ups are chosen appropriately, there is an isomorphism

$$\varphi : Y_{a,b}[\mathbf{P}] \xrightarrow{\sim} Y_{\ell_1 b-a,b}[\overline{\mathbf{P}}]$$

compatible with boundary geometry

$$\varphi(D_1) = \overline{D}'_1, \quad \varphi(D'_1) = \overline{D}_1, \quad \varphi(D_2) = \overline{D}_2, \quad \varphi(D_{\text{out}}) = \overline{D}_{\text{out}}.$$

Let $\beta_k^Y \in H_2(Y_{a,b}, \mathbb{Z})$ be the unique class with intersection numbers

$$\beta_k^Y \cdot D_1 = ak, \quad \beta_k^Y \cdot D'_1 = 0, \quad \beta_k^Y \cdot D_2 = bk, \quad \beta_k^Y \cdot D_{\text{out}} = k\ ,$$

and let $\beta_k^Y \in H_2(Y_{\ell_1 b-a,b}, \mathbb{Z})$ be the unique class with intersection numbers

$$\beta_k^Y \cdot \overline{D}_1 = (\ell_1 b - a)k, \quad \beta_k^Y \cdot \overline{D}'_1 = 0, \quad \beta_k^Y \cdot \overline{D}_2 = bk, \quad \beta_k^Y \cdot \overline{D}_{\text{out}} = k\ .$$

A straightforward analysis of φ yields the relation

$$\nu_Y^*(\beta_k^Y) - \sum_{i=1}^{\ell_1} p_i[E_i] - \sum_{j=1}^{\ell_2} p'_j[E'_j] = \varphi^*\Big(\overline{\nu}_Y^*(\beta_k^Y) - \sum_{i=1}^{\ell_1}(bk - p_i)[\overline{E}_i] - \sum_{j=1}^{\ell_2} p'_j[\overline{E}'_j]\Big).$$

The equality $N_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)] = N_{\ell_1 b-a,b}[(\mathbf{P}'_a, \mathbf{P}_b)]$ now follows by unravelling the definitions in Section 3.2 of the Gromov-Witten invariants. The isomorphism φ equates the corresponding moduli spaces of relative stable maps

$$\overline{\mathfrak{M}}(X^o_{a,b}[\mathbf{P}]/D^o_{\text{out}}) \;\cong\; \overline{\mathfrak{M}}(X^o_{\ell_1 b-a,b}[\overline{\mathbf{P}}]/D^o_{\text{out}})\ .$$

The extra blow-ups (corresponding to divisors D'_1 and $\overline{D}'_1$) occurring in $Y_{a,b}[\mathbf{P}]$ and $Y_{\ell_1 b-a,b}[\overline{\mathbf{P}}]$ do not affect the relevant moduli spaces. □

The symmetry of Proposition 5.1 applied to Theorem 2 immediately yields the symmetry of Theorem 7. □

If any part of $\mathbf{P}'_a$ is negative, then $N_{\ell_1 b-a,b}[(\mathbf{P}'_a, \mathbf{P}_b)]$ vanishes since

$$\overline{\mathfrak{M}}(X^o_{\ell_1 b-a,b}[\overline{\mathbf{P}}]/D^o_{\text{out}}) = \emptyset \ .$$

Proposition 5.1 then asserts the vanishing of $N_{a,b}[(\mathbf{P}_a, \mathbf{P}_b)]$. Similar logic holds if any part of $\mathbf{P}'_b$ is negative.

Consider the discrete series $\mathcal{B}^*_{\ell_1,\ell_2}$ of the scattering pattern associated to (ℓ_1, ℓ_2). By Theorem 7, all the functions $\mathsf{f}_{a,b}$ for $(a,b) \in \mathcal{B}^*_{\ell_1,\ell_2}$ are *equal* to $\mathsf{f}_{\mathsf{T}_1(0,1)}$. If we apply the transformation T_1 to $\mathsf{T}_1(0,1)$, we leave the strict first quadrant, but Proposition 5.1 still applies. The result is a simple calculation on $\mathbb{P}^1 \times \mathbb{P}^1$ which we leave to the reader.

Lemma 5.2. *In the factorization* (5.1),

$$f_{a,b} = \big(1 + (tx)^a (ty)^b\big)^{\ell_2}$$

for all $(a,b) \in \mathcal{B}^*_{\ell_1,\ell_2}$.

Similarly, by switching the roles of x and y, we obtain the parallel conclusion for the other discrete series.

Lemma 5.3. *In the factorization* (5.1),

$$f_{a,b} = \big(1 + (tx)^a (ty)^b\big)^{\ell_1}$$

for all $(a,b) \in \mathcal{A}^*_{\ell_1,\ell_2}$.

5.3. **Reflection functors for Q_m.** In case $m = \ell_1 = \ell_2$, the symmetry of the factorization (5.1) has a very nice interpretation in terms of the moduli spaces of $(1,0)$-semistable representations of Q_m.

Let $\rho = (V_1, V_2, \tau_1, \ldots, \tau_m)$ be a $(1,0)$-semistable representation of Q_m with dimension vector (d_1, d_2). Consider the canonically associated sequence

$$V_1 \xrightarrow{\tau} \oplus_{i=1}^m V_2 \xrightarrow{\gamma} \mathrm{Coker}(\tau) \to 0 \ , \tag{5.3}$$

where $\tau = (\tau_1, \ldots, \tau_m)$. The $(1,0)$-semistability condition implies τ is injective, hence

$$\dim_{\mathbb{C}} \mathrm{Coker}(\tau) = md_2 - d_1 \ .$$

The *reflection* $R\rho$ is defined to be the representation

$$R\rho = (V_2, \mathrm{Coker}(\tau), \gamma \circ \iota_1, \ldots, \gamma \circ \iota_m) \ ,$$

where ι_i is the inclusion of V_2 as the i^{th} factor of $\oplus_{i=1}^m V_2$, see [2]. The following Lemma is a standard result [21].

Lemma 5.4. *$R\rho$ is $(1,0)$-semistable.*

Proof. The dimension vector of $R\rho$ is $(d_2, md_2 - d_1)$. Suppose

$$U_1 \subset V_2 \ \text{ and } \ U_2 \subset \text{Coker}(\tau)$$

determine a subrepresentation of $R\rho$ with dimension vector (u_1, u_2). If (U_1, U_2) destabilizes $R\rho$, then

$$\frac{u_1}{u_1 + u_2} > \frac{d_2}{(m+1)d_2 - d_1} \ . \tag{5.4}$$

An associated subrepresentation of ρ is obtained from the data

$$\tau^{-1}(\oplus_{i=1}^m U_1) \subset V_1 \ \text{ and } \ U_1 \subset V_2 \ . \tag{5.5}$$

Let u_3 be the dimension of $\tau^{-1}(\oplus_{i=1}^m U_1)$. By sequence (5.3), $u_3 \geq mu_1 - u_2$ and hence

$$\frac{u_3}{u_3 + u_1} \geq \frac{mu_1 - u_2}{(m+1)u_1 - u_2} \ . \tag{5.6}$$

Using (5.4), we conclude the right side of (5.6) is strictly greater than $\frac{d_1}{d_1+d_2}$. Hence, the slope of the subrepresentation (5.5) contradicts the $(1,0)$-semistability of ρ. □

The inverse to R is defined as follows. From ρ, we construct the sequence

$$0 \to \text{Ker}(\tau') \xrightarrow{\gamma'} \oplus_{i=1}^m V_1 \xrightarrow{\tau'} V_2 \ ,$$

where $\tau' = \tau_1 \circ \iota'_1 + \cdots + \tau_m \circ \iota'_m$ and ι'_i is the projection of $\oplus_{i=1}^m V_1$ on the i^{th} factor. The $(1,0)$-semistability of ρ implies the surjectivity of τ'. Hence,

$$\dim_{\mathbb{C}} \text{Ker}(\tau') = md_1 - d_2 \ .$$

Define the representation $R^{-1}\rho = (\text{Ker}(\tau'), V_1, \iota'_1 \circ \gamma', \ldots, \iota'_m \circ \gamma')$. Following the proof of Lemma 5.4, we obtain the parallel result.

Lemma 5.5. *$R^{-1}\rho$ is $(1,0)$-semistable.*

A straightforward verifications shows R and R^{-1} are inverse to each other,

$$R^{-1}R\rho \stackrel{\sim}{=} RR^{-1}\rho \stackrel{\sim}{=} \rho \ , \tag{5.7}$$

for all $(1,0)$-semistable representations of ρ. The transformations R and R^{-1} act on dimension vectors by

$$R(a,b) = (b, mb - a), \quad R^{-1}(a,b) = (ma - b, a) \ .$$

Using (5.7), we find isomorphisms of moduli spaces

$$\mathcal{M}_m^{(1,0)}(d_1, d_2) \stackrel{\sim}{=} \mathcal{M}_m^{(1,0)}(R^{\pm}(d_1, d_2))$$

for (d_1, d_2) and $R^{\pm}(d_1, d_2)$ in the first quadrant.

Next, we consider the role of the framings of Section 2.4. Suppose ρ has a front framing $L_2 \subset V_2$. The subspace $L_2 \subset V_2$ defines a back framing for $R\rho$. The argument of Lemma 5.4 yields a refined result.

Lemma 5.6. *If $(\rho, L_2 \subset V_2)$ is a $(1,0)$-stable front framed representation of Q_m, then $(R\rho, L_2 \subset V_2)$ is a $(1,0)$-stable back framed representation.*

Similarly, the back framing $L_1 \subset V_1$ of ρ determines a front framing of $R^{-1}\rho$.

Lemma 5.7. *If $(\rho, L_1 \subset V_1)$ is a $(1,0)$-stable back framed representation of Q_m, then $(R^{-1}\rho, L_1 \subset V_1)$ is a $(1,0)$-stable front framed representation.*

We conclude the reflections yield isomorphisms of moduli spaces of framed representations as well,[16]

$$\mathcal{M}_m^{(1,0),F}(d_1, d_2) \stackrel{\sim}{=} \mathcal{M}_m^{(1,0),B}(R(d_1, d_2)). \tag{5.8}$$

For primitive (a, b), the generating series of Euler characteristics of Section 2.7 may be written as

$$B_{a,b}(t, x, y) = \mathsf{B}_{a,b}\Big((tx)^a (ty)^b\Big), \quad F_{a,b}(t, x, y) = \mathsf{F}_{a,b}\Big((tx)^a (ty)^b\Big)\ ,$$

where $\mathsf{B}_{a,b}(z)$ and $\mathsf{F}_{a,b}(z) \in \mathbb{Q}[[z]]$.

Proposition 5.8. *Let (a, b) be a primitive vector lying strictly in the first quadrant. If $R(a, b)$ lies in the first quadrant, $\mathsf{f}_{a,b} = \mathsf{f}_{R(a,b)}$.*

Proof. By the isomorphisms (5.8) for all dimension vectors (ak, bk), we conclude

$$\mathsf{F}_{a,b} = \mathsf{B}_{R(a,b)}\ .$$

The result then follows from Theorem 1. □

Since $m = \ell_1 = \ell_2$, there is an additional elementary symmetry given by

$$\mathsf{f}_{a,b} = \mathsf{f}_{b,a}\ . \tag{5.9}$$

In the presence of (5.9), the symmetry generated by R is equivalent to the symmetries generated by T_1 and T_2 of Theorem 7.

In the context of the ordered product factorization (5.1) of the commutator of S_m and T_m, the symmetry R was noticed earlier by Kontsevich.

[16]The spaces $\mathcal{M}_m^{(1,0),B}(d_1, d_2)$ and $\mathcal{M}_m^{(1,0),F}(R(d_1, d_2))$ may fail to be isomorphic.

5.4. **Further commutators.** Symmetries of commutators of more general elements of the tropical vertex group may be similarly considered. Let

$$\begin{aligned} p_1(t,x) &= 1 + c_1(tx)^1 + c_2(tx)^2 + \cdots + c_{\ell_1-1}(tx)^{\ell_1-1} + (tx)^{\ell_1}, \\ p_2(t,y) &= 1 + c'_1(ty)^1 + c'_2(ty)^2 + \cdots + c'_{\ell_2-1}(ty)^{\ell_2-1} + (ty)^{\ell_2} \end{aligned}$$

be polynomials of degrees ℓ_1 and ℓ_2 respectively with highest coefficient equal to 1. Let

$$\begin{aligned} \widehat{p}_1(t,x) &= 1 + c_{\ell_1-1}(tx)^1 + c_{\ell_1-2}(tx)^2 + \cdots + c_1(tx)^{\ell_1-1} + (tx)^{\ell_1}, \\ \widehat{p}_2(t,y) &= 1 + c'_{\ell_2-1}(ty)^1 + c'_{\ell_2-2}(ty)^2 + \cdots + c'_1(ty)^{\ell_2-1} + (ty)^{\ell_2} \ . \end{aligned}$$

Consider the four elements

$$\mathcal{S}_{\ell_1} = \theta_{(1,0),p_1(t,x)}, \qquad \mathcal{T}_{\ell_2} = \theta_{(0,1),p_2(t,y)} \ ,$$

$$\widehat{\mathcal{S}}_{\ell_1} = \theta_{(1,0),\widehat{p}_1(t,x)}, \qquad \widehat{\mathcal{T}}_{\ell_2} = \theta_{(0,1),\widehat{p}_2(t,y)}$$

of the tropical vertex group.

The scattering pattern associated to the commutator

$$\mathcal{T}_{\ell_2}^{-1} \circ \mathcal{S}_{\ell_1} \circ \mathcal{T}_{\ell_2} \circ \mathcal{S}_{\ell_1}^{-1} = \overrightarrow{\prod} \theta_{(a,b),f_{a,b}}$$

is related to the scattering patterns

$$\mathcal{T}_{\ell_2}^{-1} \circ \widehat{\mathcal{S}}_{\ell_1} \circ \mathcal{T}_{\ell_2} \circ \widehat{\mathcal{S}}_{\ell_1}^{-1} = \overrightarrow{\prod} \theta_{(a,b),g_{a,b}} \quad \text{and} \quad \widehat{\mathcal{T}}_{\ell_2}^{-1} \circ \mathcal{S}_{\ell_1} \circ \widehat{\mathcal{T}}_{\ell_2} \circ \mathcal{S}_{\ell_1}^{-1} = \overrightarrow{\prod} \theta_{(a,b),h_{a,b}} \ .$$

As before, let

$$f_{a,b}(t,x,y) = \mathsf{f}_{a,b}\Big((tx)^a(ty)^b\Big), \qquad g_{a,b}(t,x,y) = \mathsf{g}_{a,b}\Big((tx)^a(ty)^b\Big),$$

$$h_{a,b}(t,x,y) = \mathsf{h}_{a,b}\Big((tx)^a(ty)^b\Big).$$

Corollary 8. *Let $(a,b) \in \mathbb{Z}^2$ be a primitive vector lying strictly in the first quadrant. If $\mathsf{T}_1(a,b)$ lies strictly in the first quadrant, then*

$$\mathsf{f}_{a,b} = \mathsf{g}_{\mathsf{T}_1(a,b)} \ .$$

Similarly, if $\mathsf{T}_2(a,b)$ lies strictly in the first quadrant, then $\mathsf{f}_{a,b} = \mathsf{h}_{\mathsf{T}_2(a,b)}$.

Proof. As in the proof of Corollary 6, we start by factoring p_1 and p_2 over $\mathbb{C}$,

$$\mathcal{S}_{\ell_1} = \theta_{(1,0),(1+t_1x)(1+t_2x)\cdots(1+t_{\ell_1}x)} \ , \qquad \mathcal{T}_{\ell_2} = \theta_{(0,1),(1+s_1y)(1+s_2y)\cdots(1+s_{\ell_2}y)} \ .$$

The result then follows from Proposition 5.1 applied to Theorem 5.4 of [8]. □

6. Further directions

There are several interesting questions in the subject which we have not been able to discuss here. We end by stating three:

(i) The functions $f_{a,b}$ associated to the commutator (4.1) should satisfy certain integrality properties. In the $\ell_1 = \ell_2$ case, the relevant integrality is conjectured by Kontsevich and Soibelman in [13] and proven by Reineke in [20]. The integrality of Conjecture 6.2 of [8] constrains all cases (ℓ_1, ℓ_2) and, more generally, genus 0 relative Gromov-Witten invariants of surfaces (where the curves have full contact order at a single point with the relative divisor). Conjecture 6.2 of [8] remains open.

(ii) The curve counting side of Corollary 3 has a very natural higher genus extension discussed in Section 5.8 of [8] involving the top Chern class λ_g of the Hodge bundle on $\overline{M}_g$. The quiver side of Corollary 3 has a natural extension by replacing the Euler characteristic with the Poincaré polynomial. The two extensions do not naively match. What is the meaning of the higher genus Gromov-Witten theory on the quiver side?

(iii) Let m be fixed. M. Douglas has conjectured the function
$$\frac{1}{a}\log\big(\chi\big(\mathcal{M}_m^{(1,0)}(a,b)\big)\big)$$
asymptotically (for large and primitive (a,b)) depends only upon $\frac{b}{a}$. Moreover, the limit function should be continuous. See [21] for a discussion of results toward the conjecture.

A physical context for studying m-Kronecker quivers is explained in Section 4 of [5]. Prediction (iii) fits naturally in the framework of [5].

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