DESCENDENT BOUNDS FOR EFFECTIVE DIVISORS ON $\overline{M}_g$

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ABSTRACT. The slope of $\overline{M}_g$ is bounded from below by $\frac{60}{g+4}$ via a descendent calculation.

1. SLOPE

Let $\overline{M}_g$ be the moduli space of curves for $g \geq 2$. Let $\lambda \in A^1(\overline{M}_g)$ be the first Chern class of the Hodge bundle. Let $\delta_0 \in A^1(\overline{M}_g)$ be the class of the boundary divisor of irreducible nodal curves. For $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, let $\delta_i \in A^1(\overline{M}_g)$ be the class of the corresponding reducible boundary divisor. We will consider *effective* divisors $D \subset \overline{M}_g$ of the form

$$[D] = \alpha\lambda - \beta_0\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \beta_i\delta_i\ , \qquad \alpha, \beta_i \in \mathbb{Q}_{>0}. \tag{1}$$

Our goal is to find lower bounds for the slope $\frac{\alpha}{\beta_0}$ of D.

If $C \subset \overline{M}_g$ is a curve with no components contained in D, then

$$[C] \cdot [D] \geq 0.$$

If also $[C] \cdot \lambda > 0$ and C has no components contained in the boundary divisors, then we conclude

$$\frac{\alpha}{\beta_0} \geq \frac{[C] \cdot \delta_0}{[C] \cdot \lambda}.$$

Hence, such curves C provide lower bounds for the slope.

2. COTANGENT LINES

Let $\psi_i \in A^1(\overline{M}_{g,n})$ be the first Chern class of the cotangent line bundle

$$L_i \rightarrow \overline{M}_{g,n}$$

associated to the i^{th} marked point. The line bundles L_i are well-known to be nef via the Hodge index theorem for surfaces. Hence the class

$$\psi_1^{r_1} \cdots \psi_n^{r_n} \in A_1(\overline{M}_{g,n}), \quad \sum_{i=1}^{n} r_i = 3g - 3 + n - 1 \tag{2}$$

is a limit of effective curve classes which sweep $\overline{M}_{g,n}$. We can push the curve class (2) forward by

$$\epsilon : \overline{M}_{g,n} \rightarrow \overline{M}_g.$$

Date: January 17, 2010.

Therefore,

$$\frac{\int_{\overline{M}_{g,n}} \psi_1^{r_1}\cdots\psi_n^{r_n}\cdot\delta_0}{\int_{\overline{M}_{g,n}} \psi_1^{r_1}\cdots\psi_n^{r_n}\cdot\lambda} \tag{3}$$

is a lower bound for slopes of effective divisors D of the form (1) as long as the denominator is positive.

3. Calculation

We calculate the bound (3) in case $n = 1$ and $r_1 = 3g-3$ by explicit evaluation of the two integrals.

Consider first the numerator. Using the normalization map of the irreducible divisor,

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\delta_0 = \frac{1}{2}\int_{\overline{M}_{g-1,3}} \psi_1^{3g-3} = \frac{1}{2}\int_{\overline{M}_{g-1,1}} \psi_1^{3g-5}.$$

The string equation is used in the last equality. The evaluation $\int_{\overline{M}_{g-1,1}} \psi_1^{3g-5} = \frac{1}{(24)^{g-1}(g-1)!}$ is well-known [1, 2]. Hence

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\delta_0 = \frac{1}{2}\frac{1}{(24)^{g-1}(g-1)!}.$$

The denominator is more complicated to evaluate. The first step is to use the GRR equation for Hodge integrals established in Proposition 2 of [1],

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\lambda = \frac{B_2}{2}\int_{\overline{M}_{g,2}} \psi_1^{3g-3}\psi_2^2 - \frac{B_2}{2}\int_{\overline{M}_{g,1}} \psi_1^{3g-2} + \frac{B_2}{4}\int_{\overline{M}_{g-1,3}} \psi_1^{3g-3}, \tag{4}$$

where $B_2 = 1/6$ is the second Bernoulli number. We have already seen how to evaluate the last two integrals. The first integral on the right side of (4) is evaluated using the L_1 Virasoro constraint [6],

$$\frac{15}{4}\int_{\overline{M}_{g,2}} \psi_1^{3g-3}\psi_2^2 = \frac{(6g-5)(6g-3)}{4}\int_{\overline{M}_{g,1}} \psi_1^{3g-2} + \frac{1}{2}\cdot\frac{1}{4}\int_{\overline{M}_{g-1,3}} \psi_1^{3g-3}. \tag{5}$$

Putting this together, we find

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\lambda = \left(\frac{(6g-5)(6g-3)}{180} - \frac{1}{12}\right)\frac{1}{(24)^g g!} + \frac{2}{45}\frac{1}{(24)^{g-1}(g-1)!}.$$

After taking the ratio, we obtain a simple exact evaluation of the bound,

$$\frac{\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\delta_0}{\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\lambda} = \frac{60}{g+4}.$$

Proposition. *The slope of $D \subset \overline{M}_g$ is always at least $\frac{60}{g+4}$.*

Taking the limit for large g, we find

$$\lim_{g\to\infty}\left(\frac{\int_{\overline{M}_{g,1}}\psi_1^{3g-3}\cdot\delta_0}{\int_{\overline{M}_{g,1}}\psi_1^{3g-3}\cdot\lambda}\right)\sim\frac{60}{g}\,.$$

Hence, we derive the asymptotic $\frac{1}{g}$ bound predicted experimentally in [5].

The method of [5] is the same, but the moving curves there are obtained from Hurwitz covers rather than descendent intersections. The combinatorics of Hurwitz covers makes exact analysis difficult, but computer calculations [5] predict the slope is always at least $\frac{576}{5g}$, which is not very different from the Proposition. The advantage of the descendent approach is the calculational simplicity.

4. HODGE INTEGRALS AND VIRASORO CONSTRAINTS

For readers unfamiliar with Hodge and descendent integrals (and, in particular, unfamiliar with relations (4) and (5)), we provide here a short summary of the methods.

A concise notation for the descendent integrals on the moduli space of curves which exploits the symmetry in the markings is given by

$$\langle\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}\rangle_g=\int_{\overline{M}_{g,n}}\psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}. \tag{6}$$

The integral is well defined when all the k_i are nonnegative integers and the dimension condition $3g-3+n=\sum k_i$ holds. In all other cases, $\langle\prod_{i=1}^n\tau_{k_i}\rangle_g$ is defined to be zero. The empty product $\langle 1\rangle_1$ is also set to zero. The simplest integral is $\langle\tau_0^3\rangle_0=1$.

Let t_i (for $i\geq 0$) be a set of variables. Let $\gamma=\sum_{i=0}^{\infty}t_i\tau_i$ be the formal sum. Consider the formal generating function for the integrals (6),

$$F_g(t_0,t_1,...)=\sum_{n=0}^{\infty}\frac{\langle\gamma^n\rangle_g}{n!}$$

where the expression $\langle\gamma^n\rangle_g$ is defined by monomial expansion and multilinearity in the variables t_i. More concretely,

$$F_g(t_0,t_1,...)=\sum_{\{n_i\}}\prod_{i=1}^{\infty}\frac{t_i^{n_i}}{n_i!}\langle\tau_0^{n_0}\tau_1^{n_1}\tau_2^{n_2}\cdots\rangle_g,$$

where the sum is over all sequences of nonnegative integers $\{n_i\}$ with finitely many nonzero terms. The generating function

$$F=\sum_{g=0}^{\infty}\lambda^{2g-2}F_g$$

captures all the descendent integrals.

The Virasoro constraints of [6] are differential operators which annihilate $\exp(F)$. The L_1 operator is

$$L_1 = -\frac{15}{4}\frac{\partial}{\partial t_2} + \sum_{i=0}^{\infty} \frac{(2i+1)(2i+3)}{4} t_i \frac{\partial}{\partial t_{i+1}} + \frac{\lambda^2}{2}\frac{1}{4}\frac{\partial^2}{\partial t_0^2}. \tag{7}$$

Extraction of the the t_{3g-3} coefficient of the differential equation

$$L_1\Big(\exp(F)\Big) = 0$$

precisely yields (5).

Hodge integrals were defined in [1] to be all integrals of the form

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}\lambda_1^{\ell_1}\ldots\lambda_g^{\ell_g}$$

where λ_i is the i^{th} Chern class of the Hodge bundle. The generating series F is enriched in [1] to include all the Hodge integrals (via the Chern characters of the Hodge bundle instead of the Chern classes). Proposition 2 of [1] provides a set of differential operators D_{2k-1} which annihilate the exponential of the enriched generating series $F_{\mathbb{E}}$. Extraction of the the t_{3g-3} coefficient of the differential equation

$$D_1\Big(\exp(F_{\mathbb{E}})\Big) = 0$$

precisely yields (4). In fact, the operators D_{2k-1} are obtained via an earlier Grothendieck-Riemann-Roch calculation of Mumford.

Several (but certainly not all) series of Hodge integrals admit closed formulas, see [3, 4] for examples of very well-behaved Hodge integrals. Hodge integral techniques are useful for finding exact solutions to intesection calculations on the moduli space of curves.

5. Other bounds

Low genus computations via Faber's program suggest the following property always holds.

Conjecture. For $g \geq 1$ and $\sum_{i=1}^n r_i = 3g-3+n-1$

$$\frac{\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\delta_0}{\int_{\overline{M}_{g,1}} \psi_1^{3g-3}\cdot\lambda} \geq \frac{\int_{\overline{M}_{g,n}} \psi_1^{r_1}\cdots\psi_n^{r_n}\cdot\delta_0}{\int_{\overline{M}_{g,n}} \psi_1^{r_1}\cdots\psi_n^{r_n}\cdot\lambda}.$$

If true, the bound of the Proposition is the best obtainable from descendent integrals on the moduli space of curves.

6. Acknowledgments

The calculation answers a question of D. Chen who asked if any exact bounds for slopes were possible to obtain. G. Farkas has found a related approach using Weil-Petersson volumes. The author was partially supported by NSF grant DMS-0500187.

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