WITTEN'S CONJECTURES/ KONTSEVICH'S THEOREM

RAHUL: FOR JOE & IAN'S BOOK

For each marking i, there exists a canonical line bundle $\mathcal{L}_i$ on the stack $\overline{M}_{g,n}$ determined by the following prescription: the fiber of $\mathcal{L}_i$ at the pointed stable curve C is the cotangent space of C at i. Let $\psi_i \in A^1(\overline{M}_{g,n})$ denote the first Chern class of L_i. Witten's conjectures concern the intersection products of the classes ψ_i. A concise notation for these products which exploits the symmetry in the markings is given by:

$$\langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle_g = \int_{\overline{M}_{g,n}} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n}. \tag{1}$$

Such products are well defined when all the k_i are nonnegative integers and the dimension condition $3g - 3 + n = \sum k_i$ holds. In all other cases, $\langle \prod_{i=1}^n \tau_{k_i} \rangle_g$ is defined to be zero. The empty product $\langle 1 \rangle_1$ is also set to zero. The simplest integral is $\langle \tau_0^3 \rangle_0 = 1$.

Let t_i (for $i \geq 0$) be a set of variables. Let $\gamma = \sum_{i=0}^{\infty} t_i \tau_i$ be the formal sum. Consider the formal generating function for the products (1):

$$F_g(t_0, t_1, ...) = \sum_{n=0}^{\infty} \frac{\langle \gamma^n \rangle_g}{n!}.$$

The expression $\langle \gamma^n \rangle_g$ is defined by monomial expansion and multilinearity in the variables t_i. More concretely,

$$F_g(t_0, t_1, ...) = \sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g,$$

where the sum is over all sequences of nonnegative integers $\{n_i\}$ with finitely many nonzero terms. The generating function

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

arises as a partition function in 2-dimensional quantum gravity. Based on a different physical realization of this function in terms of matrix integrals, Witten [W] conjectured F satisfies two distinct systems of differential equations. Each system determines F uniquely and provides explicit recursions

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which compute all the products (1). These conjectures were proven by Kontsevich [K].

Exercise 1. *Show* $\langle \tau_0^3 \tau_1 \rangle_0 = 1$ *and* $\langle \tau_1 \rangle_1 = 1/24$.

Before describing the full systems, two basic properties are needed. The first is called the *string equation*: for $2g-2+n>0$,

$$\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \rangle_g = \sum_{j=1}^{n} \langle \tau_{k_j-1} \prod_{i \neq j} \tau_{k_i} \rangle_g.$$

Exercise 2. *Show the string equation and* $\langle \tau_0^3 \rangle_0 = 1$ *imply:*

$$\langle \prod_{i=1}^{n} \tau_{k_i} \rangle_0 = \frac{(n-3)!}{\prod_{i=1}^{n} k_i!}.$$

The second property is called the *dilaton equation*: for $2g-2+n>0$,

$$\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \rangle_g = (2g-2+n) \langle \prod_{i=1}^{n} \tau_{k_i} \rangle_g.$$

In genus 1, the string equation, dilaton equation, and $\langle \tau_1 \rangle_1 = 1/24$ determine all the products (1).

Both the string and dilaton equations are derived from a comparision result describing the behavior of the ψ classes under pull-back via the forgetful map $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$. Let $i \in \{1, \ldots, n\}$. The basic formula is:

$$\psi_i = \pi^*(\psi_i) + [D] \tag{2}$$

where D is the boundary divisor on $\overline{M}_{g,n}$ with genus splitting $g+0$ and point splitting $\{1, .., \hat{i}, ..., n\} \cup \{i, n+1\}$.

Exercise 3. *Prove equation (2).*

Exercise 4. *Prove the string and dilaton equations.*

The string and dilaton equations may be written as differential operators annihilating $\exp(F)$ in the following way:

$$L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}, \tag{3}$$

$$L_0 = -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}. \tag{4}$$

Exercise 5. *Prove the string equation and* $\langle \tau_0^3 \rangle_0 = 1$ *imply the equation* $L_{-1} \exp(F) = 0$.

Exercise 6. *Prove the dilaton equation and* $\langle \tau_1 \rangle_1 = 1/24$ *imply the equation* $L_0 \exp(F) = 0$.

The first system of differential equations conjectured by Witten are the KdV equations. These may be written in the following simple form (see [W]). First, define the functions:

$$\langle\langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle\rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_1}} F. \tag{5}$$

Note $\langle\langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle\rangle|_{t_i=0} = \langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle$. Then, the KdV equations are equivalent to the set of equations for $n \geq 1$:

$$(2n+1)\lambda^{-2} \langle\langle \tau_n \tau_0^2 \rangle\rangle = \tag{6}$$

$$\langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle.$$

As an example, consider equation (6) for $n = 3$ evaluated at $t_i = 0$. We obtain:

$$7 \langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + \frac{1}{4} \langle \tau_2 \tau_0^4 \rangle_0.$$

Use of the string equation yields:

$$7 \langle \tau_1 \rangle_1 = \langle \tau_1 \rangle_1 + \frac{1}{4} \langle \tau_0^3 \rangle_0.$$

Hence, we conclude $\langle \tau_1 \rangle = 1/24$. Equations (6) and L_{-1} *together* determine all the products (1) and thus uniquely determine F [W].

The second system of differential equations for F is determined by a representation of a subalgebra of the Virasoro algebra. Consider the Lie algebra L of holomorphic differential operators spanned by

$$L_n = -z^{n+1} \frac{\partial}{\partial z}$$

for $n \geq -1$. The bracket is given by $[L_n, L_m] = (n-m) L_{n+m}$.

Exercise 7. *Prove the differential operators defined in (3) and (4) satisfy:*

$$[L_{-1}, L_0] = -L_{-1}.$$

Equations (3) and (4) may be viewed as the beginning of a representation of L in a Lie algebra of differential operators. In fact, with certain homogeneity restrictions, there is a unique way to extend this assignment of L_{-1} and L_0 to a complete representation of L. For $n \geq 1$, the expression for L_n takes the form:

$$L_n = -\frac{3 \cdot 5 \cdot 7 \cdots (2n+3)}{2^{n+1}} \frac{\partial}{\partial t_{n+1}} \tag{7}$$

$$+\sum_{i=0}^{\infty} \frac{(2i+1)(2i+3)\cdots(2i+2n+1)}{2^{n+1}} t_i \frac{\partial}{\partial t_{i+n}}$$
$$+\frac{\lambda^2}{2}\sum_{i=0}^{n-1}(-1)^{i+1}\frac{(-2i-1)(-2i+1)\cdots(-2i+2n-1)}{2^{n+1}}\frac{\partial^2}{\partial t_i \partial t_{n-1-i}}.$$

Exercise 8. *Prove the formulas define a representation of L.*

The second conjecture is that this representation (7) of L annihilates $\exp(F)$: for all n,

$$L_n \exp(F) = 0. \tag{8}$$

As an example, consider the equation determined by the operator L_3:

$$-\frac{945}{16}\frac{\partial F}{\partial t_4} + \sum_{i=0}^{\infty} t_i \frac{\partial F}{\partial t_{i+3}}$$
$$+\lambda^2 \frac{15}{16}\left(\frac{\partial^2 F}{\partial t_0 \partial t_2} + \frac{\partial F}{\partial t_0}\frac{\partial F}{\partial t_2}\right) + \lambda^2 \frac{9}{32}\left(\frac{\partial^2 F}{\partial t_1 \partial t_1} + \frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_1}\right) = 0.$$

The constant term of the above relation yields:

$$-\frac{945}{16}\langle \tau_4 \rangle_2 + \frac{15}{16}\langle \tau_0 \tau_2 \rangle_1 + \frac{9}{32}(\langle \tau_1 \tau_1 \rangle_1 + \langle \tau_1 \rangle_1^2) = 0.$$

Using the genus 1 numbers, we find $\langle \tau_4 \rangle_2 = 1/1152$. It is quite easy to see that the system of equations (8) also uniquely determines F.

Kontsevich's proof of Witten's conjectures expresses the generating function F in terms of matrix integrals via ribbon graphs. The resulting function may then be shown to satisfy both systems of differential equations [K], [DVV], [Ka], [L]. Direct algebraic arguments in the second conjecture are known only for L_{-1} and L_0.

References

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