Minimax Methods

in Geometric Analysis

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Introduction

Finite Dimension

$$f(x,y) := 1 + x^2 - y^2 \qquad f(0,0) = 1 \qquad \nabla f(0,0) = 0$$

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\mathbb{R}^2 = E_2 \oplus E_{-2} \quad \Longrightarrow \quad \text{Morse Index} = 1$$

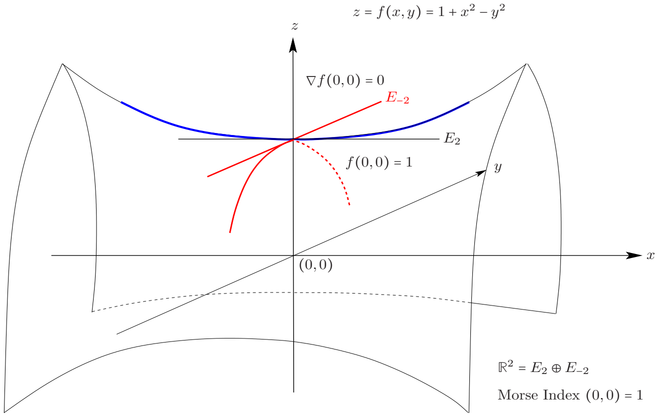

Figure 1: Non-zero Morse Index Critical Point

How to characterize variationally the critical point $(0,0)$?

$$\{(x,y)\ ;\ f(x,y) \leq 0\} \quad \text{has 2 connected components :}$$

$$\Omega_\pm := \{(x,y)\ ;\ f(x,y) \leq 0 \quad \pm y \geq 0\}$$

The notion of admissible families

$$\mathcal{A} := \{\gamma \in C^0([-1,1],\mathbb{R}^2)\ ;\ \gamma(\pm 1) \in \Omega_\pm\}$$

For any homeomorphism Ψ of $\mathbb{R}^2$ s. t.

$$\Psi(x,y) = (x,y) \quad \text{for} \quad f(x,y) \leq 0$$

we have

$$\Psi(\mathcal{A}) = \mathcal{A}$$

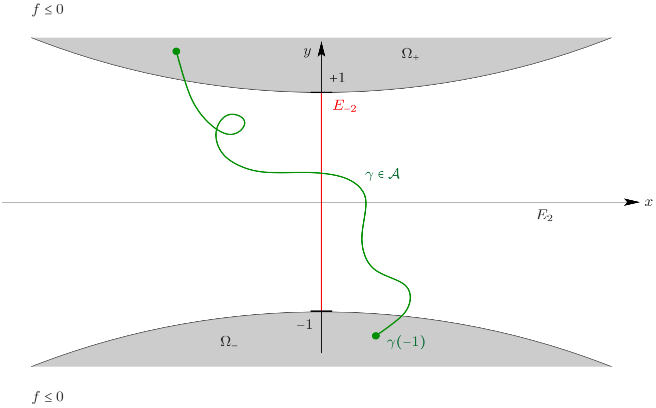

Figure 2: The admissible Family

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$$\Psi(x,y) = (x,y) \quad \text{for} \quad f(x,y) \leq 0$$

we have

$$\Psi(\mathcal{A}) = \mathcal{A}$$

Observe

$$\gamma \in \mathcal{A} \iff [\gamma] \text{ generates } H_1(\mathbb{R}^2, \Omega_+ \cup \Omega_-, \mathbb{Z}) = \mathbb{Z}$$

Homological family of dimension 1.

The width and the pull tight minmax operation.

$$\text{Width} = \inf_{\gamma \in \mathcal{A}} \max_{s \in [0,1]} f(\gamma(s)) = 1 \quad (\text{each } \gamma \in \mathcal{A} \text{ intersects } y = 0)$$

The gradient field

$$X(x,y) := -\max\{f(x,y), 0\}\ \nabla f$$

the gradient flow

$$\begin{cases} \dfrac{\partial \Phi_t}{\partial t}(x,y) = X(\Phi_t(x,y)) \\ \Phi_0(x,y) = (x,y) \end{cases}$$

$$\Phi_t(\mathcal{A}) = \mathcal{A}$$

The pull tight operation :

$$\gamma \longrightarrow \Phi_t \circ \gamma$$

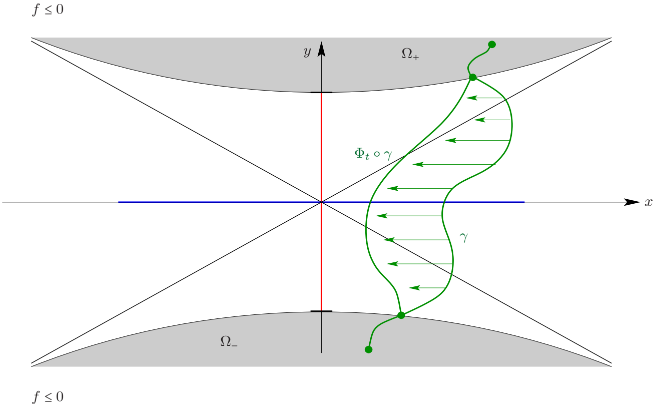


Figure 3: The pull Tight operation

The realization of the Width by a critical point.

Assume

$\exists\, \varepsilon > 0,\ \delta > 0$ s.t.

$$1 - \varepsilon \le f(x,y) \le 1 + \varepsilon \quad \Longrightarrow |\nabla f|(x,y) > \delta$$

Then

$$\exists T > 0 \quad \text{s. t.} \quad \Phi_T(f^{-1}(-\infty, 1+\epsilon)) \subset f^{-1}((-\infty, 1-\epsilon))$$

contradiction. Hence

$$\forall\, \varepsilon \quad \forall\, \delta > 0 \quad \exists\, (x,y) \in f^{-1}((1-\varepsilon, 1+\varepsilon)) \quad \text{and} \quad |\nabla f|(x,y) < \delta$$

and

$$|\nabla f|^{-1}([0,1]) \text{ is compact.}$$

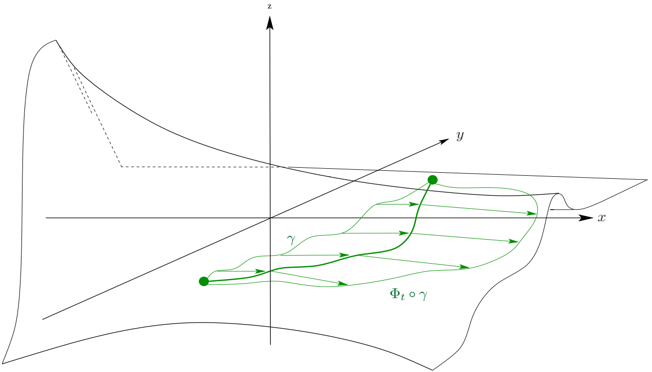


Figure 4: Pull tight going nowhere!

higher dimension : the admissible family

In $\mathbb{R}^{n+m}$

$$f(x_1 \cdots x_m, y_1 \cdots y_n) = 1 + \sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2$$

Let $\Omega := f^{-1}((-\infty, 0])$. Long exact sequence of homology

$$\cdots \quad \underset{H_n(S^{n-1})=0}{\underset{||}{H_n(\Omega)}} \stackrel{i_*}{\to} \underset{0}{\underset{||}{H_n(\mathbb{R}^{m+n})}} \stackrel{j_*}{\to} H_n(\mathbb{R}^{m+n}, \Omega) \stackrel{\partial}{\to} \underset{H_{n-1}(S^{n-1})=\mathbb{Z}}{\underset{||}{H_{n-1}(\Omega)}} \to 0 \cdots$$

The Admissible Family

$$\mathcal{A} := \{\gamma \in C^0(X, \mathbb{R}^{n+m}) \ ; \ X \text{ poly. chain } \ \gamma(X) \neq 0 \text{ in } H_n(\mathbb{R}^{m+n}, \Omega)\}$$

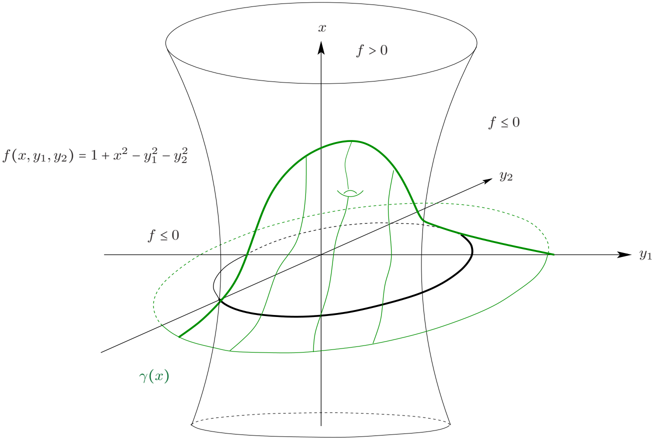


Figure 5: Admissible Family in higher dimension

The Width

Poincaré duality $\implies$

$$\forall\ \gamma \in \mathcal{A} \qquad \gamma \cap \{y = 0\} \neq \emptyset$$

Hence

$$\text{Width} = \min_{\gamma \in \mathcal{A}} \ \max_{t \in X} \ f(\gamma(t)) \geq 1$$

The Width

Poincaré duality $\Longrightarrow$

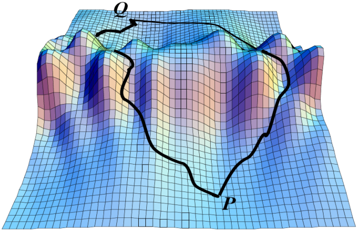
$$\forall\ \gamma \in \mathcal{A} \qquad \gamma \cap \{y = 0\} \neq \emptyset$$

In fact

$$\text{Width} = \min_{\gamma \in \mathcal{A}} \max_{t \in X} f(\gamma(t)) = 1$$

The width is achieved by a critical point of Morse index $= n$

Examples of Minmax Problems in ∞ dimensions



Example 1 : The Origin of Minmax,

The Search of Closed Geodesics

Birkhoff Curve Shortening Process

The Equation of Geodesics

N^n closed sub-manifold of $\mathbb{R}^m$. $u : S^1 \to N^n$

$$L(u) := \int_{S^1} \left| \frac{du}{d\theta} \right| \, d\theta \; .$$

Consider u_s and $w := \partial_s u|_{s=0}$

$$\frac{d}{ds} \int_{S^1} \left| \frac{\partial u_s}{\partial \theta} \right| \, d\theta \Big|_{s=0} = \int_{S^1} \partial_s \partial_\theta u \cdot \frac{\partial_\theta u}{|\partial_\theta u|} \, d\theta = \int_{S^1} \partial_\theta w \cdot \frac{\partial_\theta u}{|\partial_\theta u|} \, d\theta$$

In **normal parametrization** (i.e. $|\partial_\theta u| \equiv Cte$), it gives

$$\forall\, w \in T_u N^n \qquad \int_{S^1} \partial_\theta w \cdot \partial_\theta u \, d\theta = 0 \quad \Longleftrightarrow \quad P_u^T \left(\partial^2_{\theta^2} u \right) = 0$$

$$\Longleftrightarrow \quad \nabla \partial_\theta u = 0 \quad \Longleftrightarrow \quad -\partial^2_{\theta^2} u + \partial_\theta (P_u^T) \partial_\theta u = 0$$

$$\Longleftrightarrow \quad u \text{ is a critical point of } \quad E(u) := \int_{S^1} \left| \frac{du}{d\theta} \right|^2 \, d\theta$$

The Search of Closed Geodesics : $\pi_1(N^n) \neq 0$.

Theorem [Hadamard 1898, Poincaré 1905, Cartan 1927]
Assume $\pi_1(N^n) \neq 0$ *and let* $\alpha \in \pi_1(N^n)$ *with* $\alpha \neq 0$ *then* α *is realized by a closed geodesic.* □

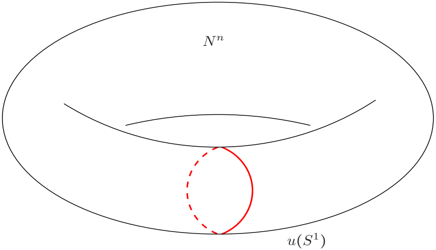


Figure 6: The search of closed geodesics: $\pi_1(N^n) \neq 0$

Proof.

One minimizes

$$E(u) := \int_{S^1} \left| \frac{\partial u}{\partial \theta} \right|^2 \ d\theta$$

Observe

$$L^2(u) \leq 2\pi \, E(u)$$

with equality iff $|\partial_\theta u| \equiv$Cte. Recall

$$W^{1,2}(S^1) \hookrightarrow C^{0,1/2}(S^1)$$

It comes from

$$|u(\theta) - u(\theta')| \leq \int_\theta^{\theta'} \left| \frac{\partial u}{\partial \theta} \right| d\theta \leq |\theta - \theta'|^{1/2} \ E(u)^{1/2}$$

Arzelà Ascoli $\Longrightarrow$

$$u^{k'} \to u^\infty \quad \text{strongly in } C^0$$

Proof being continued.

Observe

$$\exists\ \rho > 0 \quad \text{s.t.} \quad \forall\, z \in N^n \qquad B_\delta^{N^n}(z) \text{ is convex.}$$

Connect $u^{k'}(\theta)$ and $u^\infty(\theta)$ with the constant speed parametrized (between 0 and 1) unique geodesic in $B_\delta^{N^n}(u^\infty(\theta))$

Thus there exists $u_s \in C^0\left([0,1], W^{1,2}(S^1, N^n)\right)$ s.t.

$$u_0 = u^{k'} \quad \text{and} \quad u_1 = u^\infty$$

This realizes an homotopy between $u^{k'}$ and u^∞. Hence $[u^\infty] = \alpha$.

End of the proof

The Euler Lagrange Equation is

$$\forall\, w \in T_{u^\infty} N^n \qquad \int_{S^1} \partial_\theta w \cdot \partial_\theta u^\infty \, d\theta = 0 \quad \Longleftrightarrow \quad P^T_{u^\infty}\left(\partial^2_{\theta^2} u^\infty\right) = 0$$

In particular $\partial_\theta u^\infty \cdot \partial^2_{\theta^2} u^\infty \equiv 0$.

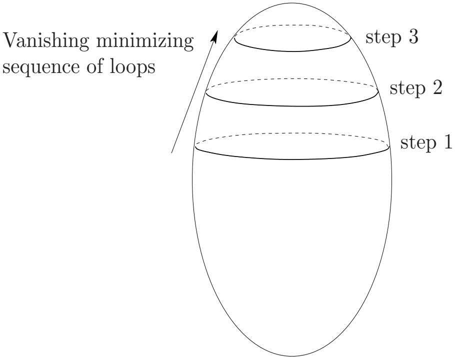
Thus $|\partial_\theta u^\infty| \equiv$Cte and u^∞ is a geodesic.

Moreover

$$L^2(u^\infty) = 2\pi\, E(u^\infty)$$

Thus u^∞ minimizes the length in the class α

The case $\pi_1(N^2) = 0$.



The Notion of Sweep-out.

Birkhoff 1917.

A sweep-out is a map $u : [0,1]\times S^1 \to N^2$ s.t.

i)

$$u \in C^0\left([0,1], W^{1,2}(S^1, N^2)\right)$$

ii)

$$u(0,\cdot) \text{ and } u(1,\cdot) \text{ are constant maps.}$$

iii)

$$u_*\left([0,1]\times S^1\right) \quad \text{generates} \quad H_2(N^2,\mathbb{Z}) = \mathbb{Z}$$

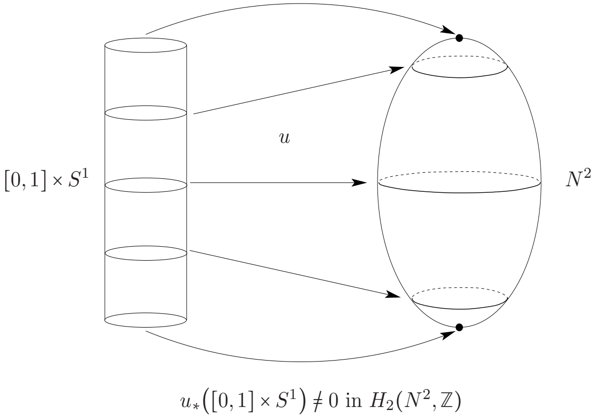


Figure 8: Birkhoff 1917: sweepout of $N^2 = (S^2, g)$

The Width

Birkhoff 1917.

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i)
$$u \in C^0\left([0,1], W^{1,2}(S^1, N^2)\right)$$

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$$u(0,\cdot) \text{ and } u(1,\cdot) \text{ are constant maps.}$$

iii)
$$u_*\left([0,1] \times S^1\right) \quad \text{generates} \quad H_2(N^2, \mathbb{Z}) = \mathbb{Z}$$

Let
$$\mathcal{A} := \{ \text{ sweep-outs } \}$$

Define the width
$$W := \inf_{u \in \mathcal{A}} \max_{t \in [0,1]} E(u(t,\cdot))$$

Non Triviality of the Width

Lemma

$$W > 0$$

□

Proof of the lemma Assume $W = 0$.
Let u_k be a minimizing sequence :

$$\lim_{k \to 0} \max_{t \in [0,1]} E(u_k(t, \cdot)) = 0$$

Use again

$$W^{1,2}(S^1) \hookrightarrow C^{0,1/2}(S^1)$$

Hence

$$\lim_{k \to +\infty} \operatorname{diam}(u_k(t, S^1)) = 0$$

End of the Proof

For k large enough

$$\forall t \in [0,1] \quad u_k(t,S^1) \subset B_\delta^{N^n}(p_k(t)) \quad \text{convex}$$

where

$$p_k(t) := \pi_{N^n}\left(\fint u_k(t,\theta)\ d\theta\right) \in C^0([0,1],N^n)$$

where π_{N^n} normal projection onto N^n. Using shortest geodesic connections

$$\text{homotop } u_k(t,\cdot) \quad \text{to the constant map } p_k(t)$$

Observe

$$p_k([0,1]) \quad \text{is contractible.}$$

Hence $[u_k([0,1]\times S^1)] = 0$ in $H_2(N^2)$. $\square$

Main Question

Does there exists u^∞ such that

$$W = E(u^\infty) = (2\pi)^{-1} L^2(u^\infty)$$

and

$$u^\infty \qquad \text{is a geodesic in constant speed parametrization}$$

that is

$$P^T_{u^\infty}(\partial^2_{\theta^2} u^\infty) = 0 \quad ?$$

Example 2 :

The Search of Minimal Spheres

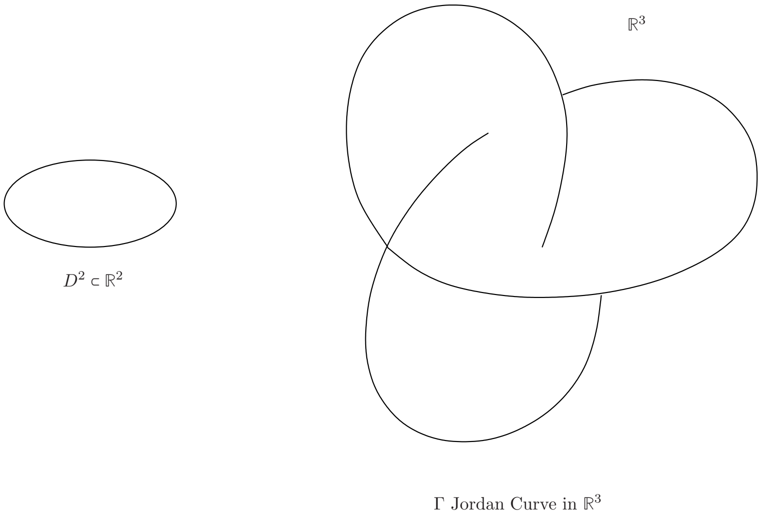


Figure 9 a: The Plateau Problem

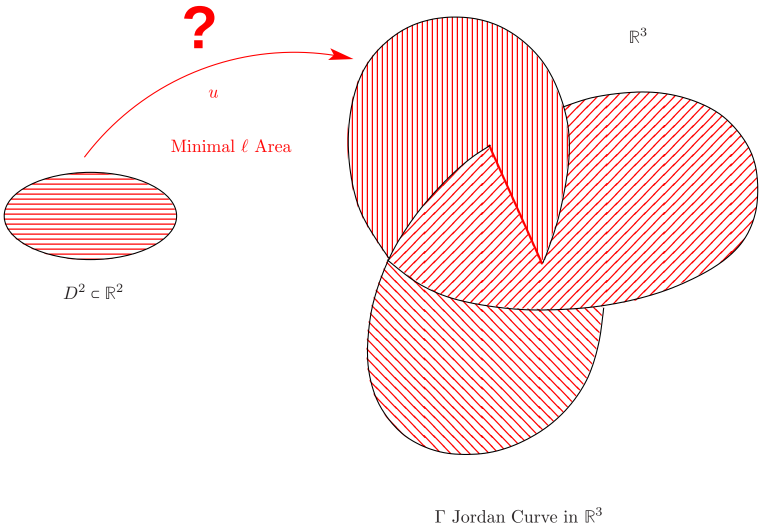


Figure 9 b: The Plateau Problem

The Resolution of the Plateau Problem

Douglas, Rado $\simeq$ 1930 : Instead of considering

$$A(u) := \int_{D^2} |\partial_{x_1} u \wedge \partial_{x_2} u| \; dx^2$$

One takes

$$E(u) := \frac{1}{2} \int_{D^2} |\nabla u|^2 \; dx^2 \quad .$$

One has

$$A(u) \le E(u) \quad \text{with} = \text{iff } u \text{ is conformal : } \left\{ \begin{array}{l} |\partial_{x_1} u|^2 = |\partial_{x_2} u|^2 \\ \partial_{x_1} u \cdot \partial_{x_2} u = 0 \end{array} \right.$$

Uniformization Theorem gives

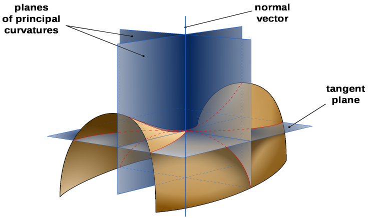
$$\forall u \in \text{Imm}(D^2, \mathbb{R}^3) \quad \exists \Psi \in \text{Diff}(D^2) \quad \text{Area}(u \circ \Psi) = E(u \circ \Psi)$$

Conclusion : Minimizing E should be the same as minimizing A.
E has better coercivity properties.

1st and 2nd Fundamental Forms of $u \in C^2_{imm}(D^2, \mathbb{R}^3)$

$$g_u(X,Y) := u^* g_{\mathbb{R}^3}(X,Y) = \langle u_* X, u_* Y\rangle$$ First fundamental form

$$\vec{n}_u \; D^2 \longrightarrow S^2$$ unit normal : Gauss Map .



Second fundamental form

$$\vec{\mathbb{I}}_u(X,Y) = \langle d\vec{n}_u \cdot X, Y\rangle \;\; \vec{n}_u = (PX)^t \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} PY \;\; \vec{n}_u$$

$P \in SO(2)$. Principal curvatures κ_1, κ_2 Euler 1750.

Recherches sur la Courbure des Surfaces - Euler 1760

Pour donner une conſtruction aiſée de cette formule, qu'on joigne enſemble le plus grand rayon oſculateur & le plus petit en prenant $Of = f$, & $Og = g$, & qu'on décrive ſur la ligne fg, une demi-ellipſe dont un foyer ſoit au point O: alors, pour la ſection MN on n'a qu'à prendre l'angle fOr, le double de l'angle EZM, & la ligne Or ſera égale au rayon oſculateur pour la ſection MN. Ainſi le jugement ſur la courbure des ſurfaces, quelque compliqué qu'il ait paru au commencement, ſe réduit pour chaque élément à la connoiſſance de deux rayons oſculateurs, dont l'un eſt le plus grand & l'autre le plus petit dans cet élément; ces deux choſes déterminent entierement la nature de la courbure en nous découvrant la courbure de toutes les ſections poſſibles, qui ſont perpendiculaires ſur l'élément propoſé. Fig. 5. & 6.

Minimal Immersed Discs

$$\frac{d}{dt}Area(u+t\,w)\Big|_{t=0} = -\,2\int_{D^2}\vec{H}_u\cdot w\; dvol_{g_u}$$ J.-B. Meusnier 1752

where

$$\vec{H}_u = \frac{\kappa_1+\kappa_2}{2}\,\vec{n} = \frac{1}{2}\,\frac{\Delta u}{|\partial_{x_1}u\wedge\partial_{x_2}u|}$$ if u is conformal

$u(D^2)$ is a minimal disc. : $\vec{H}_u = 0 \iff \Delta u = 0$.

Critical points to E satisfy

$\Delta u = 0$ i.e. u is harmonic

If u is harmonic and u is conformal then

$$\vec{H}_u = 0$$

Minimizing the Dirichlet energy under the Boundary Condition

$$\left\{u\in W^{1,2}(D^2,\mathbb{R}^3)\ ;\ u\ :\ \partial D^2\rightarrow\Gamma\ \ \text{monotone continuous}\right\}$$

solves the Plateau Problem : gives a minimal disc of minimal area.

The Hadamard-Poincaré-Cartan 2-D Problem

Let N^n closed in $\mathbb{R}^m$ with $\pi_2(N^n) \neq 0$.

$$A(u) := \int_{S^2} |du \wedge du|_{g_{S^2}} \; dvol_{g_{S^2}}$$

Question : Let $\alpha \in \pi_2(N^n, x_0)$, $\alpha \neq 0$. Does there exist

$$u \; : \; S^2 \; \longrightarrow \; N^n$$

realizing α and minimizing the area ?

The Use of the Dirichlet Energy

$$\text{and} \qquad E(u) := \frac{1}{2} \int_{S^2} |du|^2_{g_{S^2}} \; dvol_{g_{S^2}} \quad .$$

One has

$$A(u) \le E(u) \qquad \text{with } = \text{ iff } u \text{ is conformal : } u^* g_{N^n} = f_u(x)\, g_{S^2}$$

Uniformization Theorem gives

$$\forall u \in \mathrm{Imm}(S^2, N^n) \quad \exists \Psi \in \mathrm{Diff}(S^2) \quad \mathrm{Area}(u \circ \Psi) = E(u \circ \Psi)$$

Critical points to E satisfy

$$P_T(u)(\Delta_{S^2} u) = 0 \qquad \text{i.e. } u \text{ is harmonic into } N^n$$

If u is harmonic and u is conformal then u is minimal.

Preserving the Homotopy Class at the Limit for Minimizing sequences ?

Problem : E is critical in 2 Dim. $W^{1,2} \nrightarrow C^0$

The homotopy class isn't preserved under controlled Energy assumption. Even for minimizers.

$$u_k(x) := \pi^{-1} \circ k\, x \circ \pi \quad \rightharpoonup u_\infty \equiv Cte \quad \text{weakly in } W^{1,2}$$

where $\pi\ S^2 \to \mathbb{C}$ is a stereographic projection.

$$u_k \quad \text{is area minimizing among maps } v \text{ s.t. } deg_{S^2}(v) = 1$$

but

$$deg_{S^2}(u_\infty) = 0$$

Sacks-Uhlenbeck's relaxation of the Dirichlet Energy

Step 1 : Minimize

$$E_\sigma(u) := \int_{S^2} (1+|du|^2_{S^2})^{(1+\sigma)}\, dvol_{S^2} \quad \text{s.t.} \quad [u] = \alpha \in \pi_2(N^n, x_0)\setminus\{0\}$$

Sobolev Embedding

$$W^{1,2+2\sigma}(S^2) \hookrightarrow C^{0,\sigma/(1+\sigma)}(S^2) \hookrightarrow C^0(S^2) \text{ compact Arzela Ascoli}$$

Conclusion :For any $\alpha \in \pi_2(N^n, x_0)$ there exists u_σ minimizing E_σ and realizing α

It solves

$$P_T\left[d^*_{S^2}\left((1+|du|^2)^\sigma \; du\right)\right] = 0 \; .$$

Main question : Can one pass to the limit in the equation ? and get $P_T(u)(\Delta_{S^2} u) = 0$?

Sacks-Uhlenbeck's relaxation of the Dirichlet Energy

Step 2 :

Lemma. Uniform $\epsilon-$regularity $\quad \exists\, \epsilon_N > 0 \quad$ s.t. $\quad \forall\ 0 \leq \sigma \leq 1$

$$\int_{B_r(x)} (1 + |du_\sigma|^2_{S^2})^{(1+\sigma)}\ dvol_{S^2} < \epsilon_N \quad \Longrightarrow \quad |\nabla u_\sigma|(x) \leq C\, r^{-1}$$

Conclusion : There exists u_{σ_k} and $a_1 \cdots a_N \in S^2$ s.t.

$$u_{\sigma_k} \longrightarrow u_0 \quad \text{in } C^l_{loc}(S^2 \setminus \{a_1 \cdots a_N\}, N^n)$$

We have

$$P_T(\Delta_{S^2} u_0) = 0 \quad \text{in } \mathcal{D}'(S^2) \quad \text{and} \quad u_0 \text{ conformal}$$

Moreover $u_0 \in C^\infty(S^2)$ (Point removability).

Problem : $[u_0] = \alpha$? Not necessary : $N^n = S^2$, $\pi_2(S^2) = \mathbb{Z}$,
$u(x) \simeq_{hom} x$

$$u_0^b(x) = (1 - |b|^2)\frac{x - b}{|x - b|^2} - b \qquad b \in B^3 \qquad \text{make} \qquad b \to \partial B^3$$

bubble formation !

Existence of Minimal Spheres. $\pi_2(N^n) \neq 0$.

Concentration compactness :

Uniform $\epsilon-$regularity $\Longrightarrow u_\sigma$ $W^{1,2}-$*bubble tree converges* towards

$$u^1 \cdots u^Q \quad S^2 \ : \longrightarrow \ N^n$$

the u^j are conformal and harmonic hence $u(S^2)$ are minimal S^2 and

$$\alpha \in \pi_1(N^n)[u^1] \oplus \cdots \oplus \pi_1(N^n)[u^Q]$$

where $\pi_1(N^n)[u^j]$ is action of $\pi_1(N^n)$ on the homotopy class of $[u^j]$

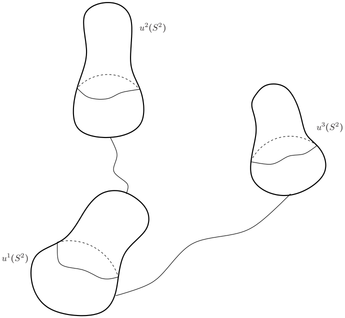


Figure 10: Bubble Tree of Harmonic Spheres in N^n

The case $\pi_2(N) = 0$.

Example : $N^3 \simeq S^3$.

Sweep outs of N^3.

$$\mathcal{A} := \left\{ \begin{array}{c} u \in C^0([0,1], C^1(S^2, N^3)) \quad \text{s. t.} \\ \\ E(u(0,\cdot)) = 0 \quad \& \quad E(u(1,\cdot)) = 0 \\ \\ u_*([0,1] \times S^2) \text{ generates } H_3(N^3, \mathbb{Z}) \end{array} \right\}$$

Let

$$W := \inf_{u \in \text{ Sweep outs}} \max_{t \in [0,1]} \frac{1}{2} \int_{S^2} |du|^2_{S^2} \, dvol_{S^2}$$

Lemma $\mathcal{A} \neq \emptyset$ and

$$W > 0 \, .$$

The space $\mathcal{A}$ moreover is admissible.

Positivity of the Width : Proof of the Lemma

Theorem [R. 1993, Freire 1995, Lin, Longzhi 2013]

$$\exists\, \varepsilon_N > 0 \quad \text{s.t. if} \quad E(u_0) := \frac{1}{2}\int_{S^2} |du_0|^2_{S^2}\, dvol_{S^2} < \varepsilon_N$$

then exists a unique energy decreasing solution to the Harmonic Map Heat Flow

$$\begin{cases} \partial_t u - \Delta_{S^2} u = dP_T(u) \cdot_{S^2} du \\ \\ u(0,\cdot) = u_0 \end{cases}$$

Moreover

$$\lim_{T\to+\infty} u(T,\cdot) \longrightarrow u_\infty \in N^3$$

and u_∞ is a continuous function of u_0

Admissibility of the Family

$\mathcal{A}$ is admissible :

$$\forall\, \Phi_t \in C^0\left([0,1], \mathrm{Homeo}(C^1(S^2, N^3))\right)$$

s.t.

$$\Phi_0 \equiv id \quad \text{on } C^1(S^2, N^3)$$

and

$$\Phi_t \equiv id \quad \text{on } E^{-1}(\{0\})$$

there holds

$$\Phi_1(\mathcal{A}) = \mathcal{A}$$

Example 3 :

Harmonic Mappings of Higher Genus Surfaces

Into Spheres

A Minmax Problem on Riemann Surface Mappings

Let (Σ, g) be a closed oriented Riemannian Surface. Let

$$\mathcal{A} := \left\{ \begin{array}{c} u \in C^0_{L^2}(\overline{B^{n+1}}, W^{1,2}(\Sigma, S^n)) \quad \text{s. t.} \\ \forall\ b \in \partial B^{n+1} \quad L^2 - \lim_{a \to b} u(a, \cdot) = u(b, \cdot) \equiv b \end{array} \right\}$$

Let

$$W := \inf_{u \in \mathcal{A}} \max_{a \in B^{n+1}} \frac{1}{2} \int_\Sigma |du(a, \cdot)|^2_g \ dvol_g$$

Lemma $\mathcal{A} \neq \emptyset$ and

$$W > 0 \ .$$

The space $\mathcal{A}$ moreover is admissible.

Proof of $\mathcal{A} \neq \emptyset$

For $a \in B^{n+1}$ introduce

$$\forall z \in \partial B^{n+1} \quad \Phi_a(z) := (1 - |a|^2) \frac{z - a}{|z - a|^2} - a$$

We have

$$\Phi_a \quad \text{is conformal from } \partial B^{n+1} \quad \text{into itself}$$

and $\forall b \in \partial B^{n+1}$

$$\Phi_a \ \rightarrow -b \quad \in C^0_{loc}(\partial B^{n+1} \setminus \{b\}) \quad \text{as} \quad a \rightarrow b$$

Let $u(0, \cdot)$ be an immersion of (Σ, g) into ∂B^{n+1}

$$u(a, \cdot) := \Phi_{-a} \circ u(0, \cdot) \in \mathcal{A} \ .$$

Proof of $W > 0$

$$\forall u \in \mathcal{A} \quad F_u \ : \ a \in \overline{B^{n+1}} \longrightarrow \fint_\Sigma u(a,\cdot)\ dvol_g\ .$$

We have

$$F_u \in C^0(\overline{B^{n+1}},\overline{B^{n+1}}) \quad \text{and} \quad \forall\, b \in \partial B^{n+1} \quad F(b) = b$$

Hence

$$\exists\, a_0 \in B^{n+1} \qquad F(a_0) = 0$$

This gives

$$\int_\Sigma |du(a_0,\cdot)|_g^2\ dvol_g \ge \lambda_1(\Sigma,g) \int_\Sigma |u(a_0,\cdot)|_g^2\ dvol_g = \lambda_1(\Sigma,g)\,|(\Sigma,g)|\ .$$

Let $\tilde{g} := e^{2\mu}\, g$ there holds

$$\int_\Sigma |du(a_0,\cdot)|_g^2\ dvol_g = \int_\Sigma |du(a_0,\cdot)|_{\tilde{g}}^2\ dvol_{\tilde{g}}$$

Hence

$$2\,W \ge \sup_\mu \lambda_1(\Sigma,\tilde{g})\,|(\Sigma,\tilde{g})| = \Lambda_1(\Sigma,[g])$$ Conformal Spectrum

Example 4 :

Harmonic Maps between Spheres

A Minmax Problem on Maps between Spheres

Let $n > 2$, $n \geq p$ s.t. $\pi_n(S^p) \neq 0$

$$\mathcal{A} := \left\{ \begin{array}{c} u \in C^0(\overline{B^{n+1}}, W^{1,2}(S^n, S^p)) \quad \text{s. t.} \\ \forall \ b \in \partial B^{n+1} \quad \lim_{a \to b} u(a, \cdot) = u(b, \cdot) \equiv v(b) \\ [v] \neq 0 \quad \text{in } \pi_n(S^p) \end{array} \right\}$$

Let

$$W := \inf_{u \in \mathcal{A}} \max_{a \in B^{n+1}} \frac{1}{2} \int_\Sigma |du(a, \cdot)|_g^2 \ dvol_g$$

Lemma $\mathcal{A} \neq \emptyset$ and

$$W > 0 \ .$$

The space $\mathcal{A}$ moreover is admissible.

Proof of $\mathcal{A} \neq \emptyset$

Let $v \in C^1(S^n, S^p)$ s.t. $[v] \neq 0$ in $\pi_n(S^p)$. Let

$$u(a, \cdot) := v \circ \Phi_{-a}$$

where

$$\Phi_a(z) := (1 - |a|^2)\frac{z - a}{|z - a|^2} - a$$

Since $n > 2$ $u \in C^0(\overline{B^{n+1}}, W^{1,2}(S^n, S^p))$. Recall $\forall b \in \partial B^{n+1}$

$$\Phi_a \ \to -b \quad \in C^0_{loc}(\partial B^{n+1} \setminus \{b\}) \quad \text{as} \quad a \to b$$

Hence

$$u \in \mathcal{A} \ .$$

Proof of $W > 0$

Let $u \in \mathcal{A}$. Recall Poincaré inequality

$$\int_{S^n} |u(a,\cdot) - \overline{u(a,\cdot)}|^2 \; dvol_{S^n} \leq C \int_{S^n} |du(a,\cdot)|^2_{S^n} \; dvol_{S^n}$$

Hence

$$\mathrm{dist}(\overline{u(a,\cdot)}, S^p)^2 \leq C \int_{S^n} |du(a,\cdot)|^2_{S^n} \; dvol_{S^n}$$

If

$$\max_{a \in \overline{B^{n+1}}} C \int_{S^n} |du(a,\cdot)|^2_{S^n} \; dvol_{S^n} < \frac{1}{4}$$

then $|\overline{u(a,\cdot)}| > 1/2$. Then

$$a \in \overline{B^{n+1}} \longrightarrow \fint_{S^n} u(a,\cdot) \; dvol_{S^n} \Big/ \left| \fint_{S^n} u(a,\cdot) \; dvol_{S^n} \right|$$

is a continuous extension in $C^0(B^{n+1}, S^p)$ of v. Since $[v] \neq 0$ in $\pi_n(S^p)$ we get a contradiction.

The case $p = n$. $\pi_n(S^n) = \mathbb{Z}$

As before there exists $a_0 \in B^{n+1}$ such that

$$\int_{S^n} u(a_0, \cdot)\ dvol_{S^n} = 0$$

and, since $\lambda_1(S^n) = n$

$$\int_{S^n} |du(a_0, \cdot)|^2_{S^n}\ dvol_{S^n} \geq n \int_{S^n} |u(a_0, \cdot)|^2\ dvol_{S^n} = n\,|S^n|$$

Thus

$$W \geq n\,|S^n|$$

Observe that $|dI_{S^n}|^2 = n$ moreover, due to the conformal invariance of Φ_{-a}, using Hölder

$$W \leq \sup_{a \in B^{n+1}} \int_{S^n} |d\Phi_{-a}|^2\ dvol_{S^n} \quad \leq |S^n|^{\frac{n-2}{n}} \left[\sup_a \int_{S^n} |d\Phi_{-a}|^n\ dvol_{S^n}\right]^{\frac{2}{n}}$$

$$= |S^n|^{\frac{n-2}{n}} \left[\int_{S^n} |dI_{S^n}|^n\ dvol_{S^n}\right]^{\frac{2}{n}} \quad = |S^n|\, n$$

The case $n = 3$ and $p = 2$. $\pi_3(S^2) = \mathbb{Z}$.

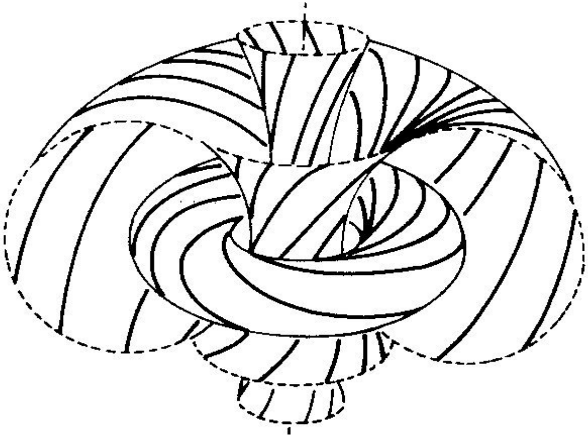
If W is realised we expect to obtain an Index 4 harmonic map.

Theorem [R., JDG 2023] If u is a smooth non constant harmonic map with Morse index ≤ 4 then u is an harmonic morphism : there exists an isometry S of $O(\mathbb{R}^4)$ and an holom. diffeo. φ of $\mathbb{C}P^1$ s. t.

$$u = \varphi \circ \mathfrak{h} \circ S$$

where $\mathfrak{h}$ is the Hopf Fibration.

The Stereographic Projection of the Hopf Fibration



$$\pi \circ \mathfrak{h} \ : \ (z, w) \in S^3 \subset \mathbb{C}^2 \longrightarrow (2\, z\, \overline{w}, |z|^2 - |w|^2) \in S^2 \subset \mathbb{R}^3$$

A Conjecture

$$W = \frac{1}{2} \int_{S^3} |d\mathfrak{h}|^2_{S^3} \; dvol_{S^3} = 8\,\pi^2 \quad ?$$

If we can prove that the *Width* is achieved by a smooth harmonic map of index ≤ 4 then the conjecture is proved.

A "Mapping Version" of the Willmore Conjecture

Conjecture [R., 1995] The Hopf Fibration $\mathfrak{h}$ minimizes the 3-energy

$$E_3(u) := \int_{S^3} |du|^3 \, dvol_{S^3}$$

among smooth maps from S^3 into S^2 non homotopic to a constant.

Let $u \in C^1(S^3, S^2)$ s.t. $[u] \neq 0$ in $\pi_3(S^2)$ and $u(a, \cdot) := u \circ \Phi_{-a}$

$$2\,W \le \sup_a \int_{S^3} |du(a,\cdot)|^2 \, dvol_{S^3} \le [\,2\,\pi^2\,]^{\frac{1}{3}} \left[\sup_a \int_{S^3} |du(a,\cdot)|^3 \, dvol_{S^3}\right]^{\frac{2}{3}}$$

$$= [\,2\,\pi^2\,]^{1/3} \, [\,E_3(u(a,\cdot))\,]^{\frac{2}{3}} = [\,2\,\pi^2\,]^{\frac{1}{3}} \, [\,E_3(u)\,]^{\frac{2}{3}}$$

with $\le$ being an equality for $u = \mathfrak{h}$.

Example 5 :

Sphere Eversions

and the 16π Conjecture

Euler's Elastica

A curve γ in $\mathbb{R}^2$ is called an Euler Elastica if it is an equilibrium of the elastic energy

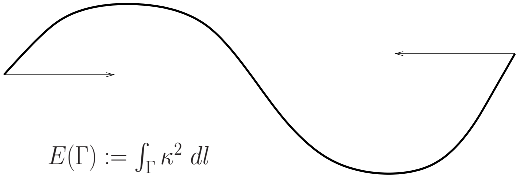


Figure : **A model for Elastic Energy of Rods**

Germain-Poisson's Derivation of the Surface Elastica.

August 1814

(23) Je terminerai ce Mémoire en faisant connaître une propriété curieuse de la surface élastique en équilibre. Celle que je vais considérer est une plaque, également épaisse, pliée par des forces données, qui agissent sur son contour; et, pour simplifier, je fais abstraction de sa pesanteur. Or, je dis que dans l'état d'équilibre, elle est parmi toutes les surfaces de même étendue, la surface dans laquelle l'intégrale

$$\iint \left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^2 k\,dx\,dy$$

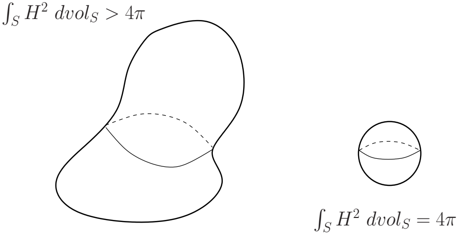
est un *maximum* ou un *minimum* : ρ et ρ' désignent comme plus haut, les deux rayons de courbure principaux qui répondent à un point quelconque; $k\,dx\,dy$ représente l'élément relatif au même point; et cette intégrale double s'étend

Willmore Inequality.

Theorem [Willmore 1965] For any immersion u of a closed oriented surface $u \ : \ S \rightarrow \mathbb{R}^3$

$$\int_S |H_u|^2 \ dvol_u \geq 4\pi$$

with equality if and only if $S = S^2$ and $u(S)$ is a unique covering of a round sphere.



Everting The Sphere ?

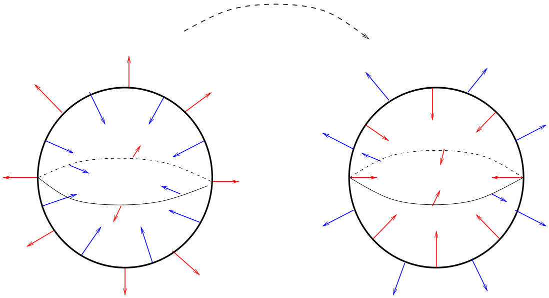


Figure : **Sphere Eversion**

How Much Does It Cost to Evert S^2 in $\mathbb{R}^3$?

Introduce

$$\mathcal{A} := \left\{ \begin{array}{c} u \in C^0([0,1], \mathrm{Imm}(S^2, \mathbb{R}^3)) \\ \\ u(0,x) = x \quad , \quad u(1,x) = -x \end{array} \right\}$$

and

$$W := \inf_{\Phi \in \mathcal{A}} \max_{t \in [0,1]} \int_{S^2} |H_{u(t,\cdot)}|^2 \, dvol_{u(t,\cdot)}$$

Lemma $\mathcal{A} \neq \emptyset$ and

$$W > 4\pi \ .$$

The space $\mathcal{A}$ moreover is admissible.

Remark : $\forall\, u \in \mathcal{A}$

$$t \longrightarrow u(t, S^2)/Diff(S^2)$$

is a non zero element in $\pi_1(Imm(S^2, \mathbb{R}^3)/Diff(S^2))$.

Proof of the Lemma : 1) $\mathcal{A} \neq \emptyset$

Theorem [Smale 1958] $\pi_0(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \{1\}$ i.e. two arbitrary C^2 immersions of S^2 into $\mathbb{R}^3$ are regular homotopic.

Proof of the Lemma : 2) $W > 4\pi$

Theorem [De Lellis, Müller 2005] There exists $\varepsilon_0 > 0$ s.t. $\forall \varepsilon < \varepsilon_0$

$$\int_{S^2} |H_u|^2 \, dvol_u < 4\pi + \varepsilon \quad \text{and} \quad \text{Area}(u) = \int_{S^2} |du \times du| = 4\pi$$

Then $\exists\, \psi \in \text{Diff}(S^2)$ and $v_0 \in \mathbb{R}^3$ s.t.

$$\|u \circ \psi - (Id + v_0)\|_{W^{2,2}(S^2)} \leq C\sqrt{\varepsilon}$$

This implies in particular

$$\left|\left|\int_{S^2} u^* \omega_{S^2}\right| - 4\pi\right| = \left|\int_{S^2} (u \circ \psi)^* \omega_{S^2} - 4\pi\right| \leq C\sqrt{\varepsilon}$$

Renormalise the eversion s.t. $\text{Area}(\Phi(t,\cdot)) = 4\pi$.

$$\int_{S^2} u(t,\cdot)^* \omega_{S^2} \in C^0([0,1]) \quad \int_{S^2} u(0,\cdot)^* \omega_{S^2} = 4\pi = -\int_{S^2} u(1,\cdot)^* \omega_{S^2}$$

$$\Longrightarrow \quad W > 4\pi$$

The 16π Conjecture

Theorem [Bryant 1984, R. 2015, Martino 2023]

$$W \in 4\pi\,\mathbb{N}^* \setminus \{1\}$$

Theorem [Li, Yau 1982] Let u be an immersion of a closed oriented surface Σ. There holds

$$\int_{S^2} |H_u|^2 \; dvol_u \geq 4\pi \max_{p \in \mathbb{R}^3} \mathrm{Card}\left\{u^{-1}(\{p\})\right\}$$

Theorem [Banchoff, Max 1981] Every sphere eversion has a quadruple point.

Corollary

$$W = 4\pi\, N \quad N \in \{4, 5, 6 \cdots\}$$

Conjecture [Kusner 1982]

$$N = 4 \quad \text{i.e.} \quad W = 16\pi$$

The Expected Lowest Energy Saddle of the Eversion

http://www.gang.umass.edu/gallery/willmore/