Minmax Methods

in Geometric Analysis

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Part 2 : Palais Deformation Theory

in ∞ Dimensional Spaces.

CRITICAL POINT THEORY AND THE MINIMAX PRINCIPLE

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1. **Introduction.** Since the goal of this paper is to present an exposition of a fairly general method of attack on a certain class of problems in analysis, it is perhaps in order to begin with a discussion of the domain of applicability of the concepts and techniques we are going to describe, and to illustrate them in some simple cases.

In a typical problem in analysis, both linear and nonlinear, we are given a space X and a set of "equations" defined on X and are asked to describe the set S of solutions of these equations.

There are really two quite separate types of description, depending on whether one is interested in the properties of the elements of S on the one hand or in describing the nature of the set S on the other.

Typical of the first type of description is classical "complex variable theory." Here we may take for X the set of say C^1 complex valued functions defined in some open set in the complex plane and for S the set of solutions of the Cauchy-Riemann equations. The emphasis is placed on determining the properties that elements of S have as distinguished from the general element of X (e.g. the open mapping property, the maximum modulus property, complex analyticity etc.).

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is a Hausdorff topological space together with a covering by open sets* $(U_i)_{i\in I}$, *a family of Banach vector spaces* $(E_i)_{i\in I}$ *and a family of continuous mappings* $(\varphi_i)_{i\in I}$ *from* U_i *into* E_i *such that*

i) *for every* $i \in I$

$$\varphi_i \; U_i \longrightarrow \varphi_i(U_i) \qquad \textit{is an homeomorphism}$$

ii) *for every pair of indices* $i \neq j$ *in* I

$$\varphi_j \circ \varphi_i^{-1} \; : \; \varphi_i(U_i \cap U_j) \subset E_i \longrightarrow \varphi_j(U_i \cap U_j) \subset E_j$$

is a C^p *diffeomorphism*

□

Example : Let $l\, p > k$

$$\mathcal{M} := W^{l,p}(\Sigma^k, N^n) := \left\{ u \in W^{l,p}(\Sigma^k, \mathbb{R}^m) \; ; \; u(x) \in N^n \; \text{ a.e. } x \in \Sigma^k \right\}$$

Observe : $W^{1,2}(D^2, N^n)$ does not fulfil the conditions.

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact**. □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

Proposition *Every* **Hausdorff paracompact** *space is* **normal**. □

Proof : https://topospaces.subwiki.org/wiki/

Warning ! $\mathcal{M}$ **Banach Paracompact Manifold**, (ϕ, U) a chart s.t.

$$\phi \;:\; U \longrightarrow \phi(U) = (E, \|\cdot\|) \qquad \text{homeomorphism}$$

then $\phi^{-1}(\overline{B_r(x)})$ might not be closed in $\mathcal{M}$.

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$*. Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *, i.e. there exists* $(\phi_\alpha)_{\alpha \in A}$ *where* ϕ_α *is locally lipschitz in* $\mathcal{M}$ *and such that*

i)
$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

ii)
$$\phi_\alpha \geq 0$$

iii)
$$\sum_{\alpha \in A} \phi_\alpha \equiv 1$$

where the sum is locally finite.

□

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$ such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi \searrow & \downarrow \sigma \\ & & U_i \end{array}$$

where σ is the canonical projection from $U_i \times E$ onto U_i. The restriction of τ_i on each fiber $\mathcal{V}_x := \pi^{-1}(\{x\})$ for $x \in U_i$ realizes a continuous isomorphism onto E. Moreover the map

$$x \in U_i \cap U_j \longrightarrow \tau_i \circ \tau_j^{-1}\Big|_{\pi^{-1}(x)} \in \mathcal{L}(E, E)$$

is C^p. □

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\| \cdot \| \ : \ \mathcal{V} \ \longrightarrow \ \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\| \cdot \|_x := \| \cdot \| |_{\pi^{-1}(\{x\})} \qquad \text{is a norm on } \mathcal{V}_x \quad .$$

and the norms are locally uniformly comparable using any trivialization. □

Definition *Let $\mathcal{M}$ be a* **normal** *C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a* **Finsler Manifold**. □

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented $2-$dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. Let $q>2$

$$\mathcal{M} := W^{2,q}_{imm}(\Sigma^2, N^n)$$
$$:= \left\{\Phi \in W^{2,q}(\Sigma^2, N^n)\ ;\ \text{rank}\,(d\Phi_x) = 2 \quad \forall x\in\Sigma^2\right\}$$

The tangent space to $\mathcal{M}$ at a point Φ is

$$T_\Phi\mathcal{M} = \left\{w\in W^{2,q}(\Sigma^2,\mathbb{R}^m)\ ;\ w(x)\in T_{\Phi(x)}N^n \quad \forall\, x\in\,\Sigma^2\right\} \quad .$$

We equip $T_\Phi\mathcal{M}$ with the following norm

$$\|v\|_\Phi := \left[\int_\Sigma \left[|\nabla^2 v|^2_{g_\Phi} + |\nabla v|^2_{g_\Phi} + |v|^2\right]^{q/2}\ dvol_{g_\Phi}\right]^{1/q} + \|\,|\nabla v|_{g_\Phi}\,\|_{L^\infty(\Sigma)}$$

Proposition $\|\cdot\|_\Phi$ *define a* C^2-*Finsler struct. on* $\mathcal{M}$. □

The Palais Distance.

Theorem [Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} \, dt$$

where

$$\Omega_{p,q} := \left\{ \omega \in C^1([0,1], \mathcal{M}) \; ; \; \omega(0) = p \quad \omega(1) = q \right\} \quad .$$

Then d defines a distance on $\mathcal{M}$
and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold.

d is called **Palais distance** of the Finsler manifold $(\mathcal{M}, \|\cdot\|)$.

Corollary *Let* $(\mathcal{M}, \|\cdot\|)$ *be a* **Finsler Manifold** *then* $\mathcal{M}$ *is* **paracompact**. □

Completeness of the Palais Distance.

Proposition *Let* $q > 2$ *and let* $\mathcal{M}$ *be the* **normal**[2] **Banach manifold**

$$W^{2,q}_{imm}(\Sigma^2, N^n) := \left\{\Phi \in W^{2,q}(\Sigma^2, N^n) \; ; \; rank\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

The **Finsler Manifold** *given by*

$$\|v\|_\Phi := \left[\int_\Sigma \left[|\nabla^2 v|^2_{g_\Phi} + |\nabla v|^2_{g_\Phi} + |v|^2\right]^{q/2} \; dvol_{g_\Phi}\right]^{1/q} + \|\, |\nabla v|_{g_\Phi} \,\|_{L^\infty(\Sigma)}$$

is **complete** *for the* **Palais distance**.

[2] Recall that every metric space is normal.

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$.*
Denote

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section $X \ : \ \mathcal{M}^* \to T\mathcal{M}^*$ such that*

i)

$$\forall\, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2\ \|DE_u\|_u$$

ii)

$$\forall\, u \in \mathcal{M}^* \quad \|DE_u\|_u^2 < \langle X(u), DE_u \rangle_{T_u\mathcal{M}^*, T_u^*\mathcal{M}^*}$$

Proposition *Every C^1 function on a Finsler Manifold admits a pseudo-gradient.* □

"Proof" Use that **Finsler Manifolds** are **Paracompact** and "glue together" local pseudo-gradients constructed by local trivializations with an ad-hoc partition of unity.

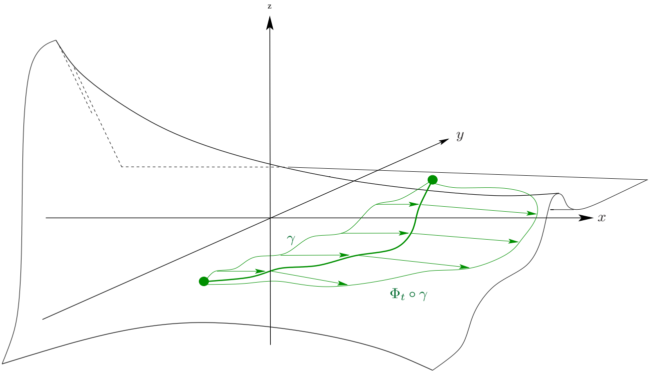

Figure 4: Pull tight going nowhere!

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. One says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d_{\mathbf{P}}(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

Example Let $\mathcal{M}$ be $W^{1,2}(S^1, N^n)$ for the Finsler structure given by

$$\forall\ w \in W^{1,2}(S^1, \mathbb{R}^m) \quad w \cdot u = 0 \qquad \|w\|_u := \|w\|_{W^{1,2}(S^1)}$$

Then the Dirichlet Energy satisfies the Palais Smale condition for every level set. □

Admissible families

Definition *A family of closed subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family** *if for every homeomorphism* Ψ *of* $\mathcal{M}$ *isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Psi(A) \in \mathcal{A}$$

□

Example $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$. Let $c \in \pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \mathbb{Z}_2 \times \mathbb{Z}$

$$\mathcal{A} := \left\{ \Phi \in C^0([0,1], W^{2,q}_{imm}(S^2, \mathbb{R}^3)) \ ; \ \Phi(0,\cdot) = \Phi(1,\cdot) \quad \text{and } [\Phi] = c \right\}$$

is admissible

: for example a sphere eversion is non zero in

$$\pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)/\mathrm{Diff}(S^2)) = \mathbb{Z}$$

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$ and let $E \in C^1(\mathcal{M})$. Let $\mathcal{A}$ admissible. Let

$$\beta := \inf_{A \in \mathcal{A}} \sup_{u \in A} E(u)$$

Assume $(PS)_\beta$ for the level set β. Then there exists $u \in \mathcal{M}$ s.t.

$$\begin{cases} DE_u = 0 \\ \\ E(u) = \beta \end{cases}$$

Proof By **contradiction**. $(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\, \delta > 0 \,, \exists\, \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\, X(\phi_t(u))\, \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\text{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq \; 2\,|t_2 - t_1|^{1/2}\; [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$\lim_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$ Impossible ! $\Rightarrow \forall\, t \in \mathbb{R}_+ \quad \forall\, A \in \mathcal{A}\; \phi_t(A) \in \mathcal{A}$

Take $A \in \mathcal{A}$ s.t. $\max_{u \in A} E(u) < \beta + \varepsilon_0/2$. Apply ϕ_t...cont. ! □

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2 \simeq S^2)$ defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

$\mathcal{A} := \{$ sweep-out$\}$

Palais Theorem $\Rightarrow$

$$W = \inf_{u \in \mathcal{A}} \max_{t \in [0,1]} E(u(t, \cdot)) > 0$$

is achieved by a **closed geodesic**.

This gives a new proof of **Birkhoff existence result**.

Homotopy type of the Loop Space in arbitrary Manifolds.

$$\mathcal{M} := W^{1,2}(S^1, M^m) := \left\{ u \in W^{1,2}(S^1, \mathbb{R}^Q) \ ; \ u(\theta) \in M^m \ , \ \forall \theta \in S^1 \right\}$$

$\mathcal{M} \simeq_{homot} C^0(S^1, M^m)$.
Let $\Omega_p(M^m)$ the *path space* based at p.

Exact sequence of Serre fibration

$$\cdots \pi_n(\Omega_p(M^m)) \longrightarrow \pi_n(C^0(S^1, M^m)) \xrightarrow{ev_*} \pi_n(M^m) \longrightarrow \pi_{n-1}(\Omega_p(M^m)) \cdot$$

It "splits" : $ev_* \circ \iota_* = id_*$ where $\iota_*(q) \equiv q$. Hence

$$\pi_n(C^0(S^1, M^m)) = \pi_n(\Omega_p(M^m)) \oplus \pi_n(M^m)$$

Eckmann-Hilton duality $\pi_n(\Omega_p(M^m)) = \pi_{n+1}(M^m)$. Hence

$$\pi_n(\mathcal{M}) = \pi_{n+1}(M) \oplus \pi_n(M^m)$$

Birkhoff Sweep-outs revisited.

M^m simply connected.
Let $k \in \{2, \cdots, m\}$ s.t.

$$\pi_k(M^m) \neq 0 \quad \text{but } \pi_l(M^m) = 0 \quad \text{for } l \in \{1 \cdots k-1\} \quad .$$

Thus $\pi_{k-1}(\mathcal{M}) = \pi_k(M^m) \neq 0$.

Example : For $M^m = S^2$ we have

$$\pi_1(W^{1,2}(S^1, S^2)) = \pi_2(S^2) = \mathbb{Z}$$

It is generated by Birkhoff Sweep-Out.

Existence of closed Geodesics in arbitrary Manifolds.

Let

$$\mathcal{A} := \left\{ u \in C^0(S^{k-1}, \mathcal{M}) \; ; \; [u] \neq 0 \;\text{ in } \pi_{k-1}(\mathcal{M}) \right\}$$

It is clearly **admissible**.
Introduce the width

$$W_k := \inf_{u \in \mathcal{A}} \max_{s \in S^{k-1}} E(u(s, \cdot))$$

We have

$$W_k > 0$$

Indeed there exists $\delta > 0$ such that

$$\max_{s \in S^{k-1}} E(u(s, \cdot)) < \delta \quad \Rightarrow \quad [u] = 0 \quad (\text{use } \pi_{k-1}(M^m) = 0)$$

The Dirichlet Energy is **Palais Smale** in $W^{1,2}(S^1, M^m)$. Hence

Theorem [Fet-Lyusternik 1951]. *Every closed manifold posses a non trivial closed geodesic.*

More closed Geodesics in arbitrary Manifolds ?

Definition *A geodesic is called* **prime** *if it is not a multiple covering of another one.*

Question *Does there exists* **infinitely many prime geodesics** *in a given closed manifold ?*

This is still open for (S^n, g) when $n \geq 3$.

Question *Which are the manifolds for which we know the existence of* **infinitely many prime geodesics** *?*

Gromov Dimension and non-linear Spectrum

Let

$$\mathcal{M}^\lambda := \left\{ u \in W^{1,2}(S^1, M^m) \quad ; \quad \sqrt{E(u)} \leq \lambda \right\} \quad .$$

Define Gromov dimension for any $\lambda > 0$

$$\mathrm{dm}(\mathcal{M}^\lambda) := \sup\{k \in \mathbb{N} \ ; \ H_l(\mathcal{M}; \mathcal{M}^\lambda; \mathbb{Z}) = 0 \quad \forall l \leq k\}$$

and Gromov Spectrum

$$\lambda_k := \sup \left\{ \lambda \in \mathbb{R}_+ \ ; \ \mathrm{dm}(\mathcal{M}^\lambda) \leq k \right\}$$

Exercise : This formal definition permits to recover the **linear spectrum of the laplacian** for

$$\mathcal{M} := \left\{ u \in W^{1,2}(M^m, \mathbb{R}) \ ; \ \|u\|_{L^2(M^m)} = 1 \right\}$$

A quasi Weyl Law for the Gromov Spectrum

Theorem [Gromov 1978] Assume $\pi_1(M^m)$ is finite then

$$\lambda_k \simeq k$$

□

Morse theory implies that - for a generic metric - at each generator of $H_k(\mathcal{M};\mathbb{R})$ corresponds a geodesic. Combining the two gives

$$\text{Card}\left\{\text{geodesics of length } \leq \lambda\right\} \geq \sum_{k\leq[C\lambda]} \dim(H_k(\mathcal{M};\mathbb{R})) \quad .$$

Which implies

$$\text{Card}\left\{\textbf{prime} \text{ geodesics of length } \leq \lambda\right\} \geq \frac{\sum_{k\leq[C\lambda]} \dim(H_k(\mathcal{M};\mathbb{R}))}{\lambda} \quad .$$

Gromoll Meyer Theorem

Ballman and Ziller improved Gromov lower bound
Theorem [Ballman, Ziller 1982] If $\pi_1(M^m) = 0$ and (M^m, g) generic we have

$$\text{Card}\,\{\textbf{prime} \text{ geodesics of length } \leq \lambda\} \geq \max_{k \leq C\lambda} \dim(H_k(\mathcal{M}; \mathbb{R})) \quad .$$

This permits to deduce in the case of simply connected and generic M^m
Theorem [Gromoll, Meyer 1969] Assume $\pi_1(M^m)$ is finite and

$$\limsup_{k \to +\infty} \dim(H_k(\mathcal{M}; \mathbb{R})) = +\infty \qquad (\star)$$

then (M^m, g) has **infinitely many prime geodesic**

An application of Gromoll Meyer Theorem

The computation of the **minimal model** of M^m (an algebraic procedure introduced by Quillen and Sullivan to compute $\pi_k(M^m) \otimes \mathbb{R}$) implies the following

Theorem [Vigué-Poirrier, Sullivan 1976] If $\pi_1(M^m) = 0$ and $H^k(M, \mathbb{R})$ is not generated by a single element then

$$\limsup_{k \to +\infty} \dim(H_k(\mathcal{M}; \mathbb{R})) = +\infty \qquad (\star)$$

holds and M^m has **infinitely many prime geodesic**.

This does not apply to $M^m := (S^m, g)$. However

Theorem [Franks 1992, Bangert 1993] Let g be an arbitrary metric on S^2 then (S^2, g) has **infinitely many prime geodesic**.