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The Germain–Poisson Problem and Variational Aspects of the Willmore Lagrangian

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presented by

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Contents

Abstract

The present dissertation is concerned with two topics, eminently of analytic nature, on Willmore surfaces in Euclidean space. The first one is the *Germain–Poisson problem*, which is a geometric boundary value problem consisting in finding a Willmore disk–type surface spanning a given boundary curve, boundary Gauss map and having prescribed surface area. The second one is the *parametric approach to the Willmore flow*, which merges the study of the Willmore gradient flow with the framework developed by T. Rivière for the calculus of variation of Willmore surfaces from a parametric point of view.

The study of such problems leads to two connected results that are of independent interest. The first one is the validity – and failure – of *Wente-type estimates for Neumann problems involving Jacobians* under various boundary conditions. The second one is an *elliptic regularity result for the inhomogeneous Willmore equation with L^p datum, where* $1 < p < \infty$.

The contents of this thesis are as follows.

Chapter 1 is of introductory nature. The concepts of Willmore energies and Willmore surfaces are introduced and few historical remarks and classical references on the subject are given; afterwards the Germain–Poisson problem and the notion of gradient flow for the Willmore energy are introduced, motivated and pertaining references are given. It is also explained why one naturally comes to consider Wente–type estimates for problems with Neumann boundary conditions while considering the Germain–Poisson problem. Finally, a general presentation of the results of the thesis is given.

Chapter 2 is concerned with Wente–type estimates for boundary value problems with boundary conditions of Neumann type. We show that such estimates do not in general hold under the same hypotheses on the data for Dirichlet boundary conditions and also not under boundary conditions that are natural from a variational perspective. Finally some positive results that hold under specific assumptions are given, some of which will be used in the study of the Germain–Poisson problem. The results in this chapter are obtained in collaboration with F. Da Lio.

Chapter 3 serves as an introduction to the differential geometry present in the following chapters. No original result is present in the chapter but the statements are presented in a way that is useful for the following analysis. Some basic notions about curvature are recalled, then some first variations formulas are computed – among these the derivation of the Willmore equation – and finally the conservation laws issuing form the conformal invariance of the Willmore energy are deduced.

Chapter 4 is concerned in establishing elliptic regularity results for the inhomogeneous Willmore equation. For a conformal, Lipschitz $W^{2,2}$ immersion with distributional Willmore operator in L^p, $1 < p < \infty$, it is proven that such immersion is locally $W^{4,p}$, just as is expected for an elliptic problem. A quantitative estimate is also given for the case $p = 2$. This theorem is a generalization of the classical result of Rivière for the regularity of weak Willmore surfaces, and the proof follows, essentially, similar ideas. This result will be used while studying the Willmore flow.

Chapter 5 is concerned with the parametric approach to the Willmore gradient flow. The introduction of such parametric theory allows to consider a general class of weak, energy–level solutions and opens the possibility to study, in the future, energy quantization and finite–time singularities. We restrict to a small–energy regime and prove that, for small-energy weak immersions, the Cauchy problem in this class admits a unique solution. The results in this chapter are obtained in collaboration with T. Rivière.

Chapter 6 is concerned with the Germain–Poisson problem. We find a disk–type surface $\mathcal{D} \subset \mathbb{R}^n$ of least Willmore energy among all immersed surfaces having the same boundary, boundary Gauss map and area. We present a solution in the case of boundary data of class $C^{1,1}$ and when the boundary curve is simple and closed. The minimum is realised by an immersed disk, possibly with a finite number of branch points in its interior, which is of class $C^{1,\alpha}$ up to the boundary for some $0 < \alpha < 1$, and whose Gauss map extends to a map of class $C^{0,\alpha}$ up to the boundary. The results in this chapter are obtained in collaboration with F. Da Lio & T. Rivière.

Sunto

La presente tesi tratta di due argomenti, di natura eminentemente analitica, circa le superfici di Willmore nello spazio euclideo. Il primo di questi è il *problema di Germain–Poisson*, un problema al bordo geometrico che consiste nel trovare una superficie di Willmore di tipo disco avente bordo, mappa di Gauss al bordo e area prescritti. Il secondo è *l'approccio parametrico al flusso di Willmore*, che congiunge lo studio del flusso gradiente di Willmore con l'approccio sviluppato da T. Rivière per il calcolo delle variazioni di superfici di Willmore da un punto di vista parametrico.

Lo studio di tali problemi porta in modo naturale ad ottenere due risultati di interesse autonomo. Il primo riguarda la validità – e la non–validità – di *stime alla Wente per problemi con condizione di Neumann che riguardano giacobiani*, con vari tipi di condizioni al bordo. Il secondo è un *risultato di regolarità ellittica per l'equazione inomogenea di Willmore con dato in* L^p*, ove* $1 < p < \infty$.

Di seguito sono riportati i contenuti dei capitoli della presente tesi.

Il Capitolo 1 è di natura introduttiva. Si richiamano i concetti di energia di Willmore e di superfici di Willmore, vengono date alcune note storiche e referenze calssiche sul tema; dopodiché vengono introdotti il problema di Germain–Poisson e il flusso gradiente per l'energia di Willmore, motivandoli e dando pertinenti referenze. Si chiarisce perché si giunge in modo spontaneo a considerare stime di tipo Wente per problemi al bordo con condizione di Neumann mentre si studia il problema di Germain–Poisson.

Il Capitolo 2 riguarda stime di tipo Wente per problemi al bordo con condizioni di Neumann. Si mostra che tali stime non sono in generale valide se si assumono le medesime ipotesi del caso con condizione al bordo di Dirichlet e nemmeno sotto ipotesi che sono naturali per problemi variazionali. Infinte si danno alcuni risultati positivi validi sotto specifiche ipotesi, alcune delle quali saranno poi usate nello studio del problema di Germain–Poisson. I risultati in questo capitolo sono ottenuti in collaboraione con F. Da Lio.

Il Capitolo 3 serve da introduzione alla geometria differenziale presente nei capitoli che seguono. Non vi sono risultati originali ma gli enunciati sono presentati in modo funzionale all'analisi dei capitoli successivi. Vengono richiamate alcune nozioni di base sulla curvatura, si calcolano alcune variazioni prime – tra queste l'equazione di Willmore – e infine vengono dedotte le leggi di convervazione dovute all'invarianza conforma dell'energia di Willmore.

Nel Capitolo 4 si dimostrano dei risultati di regolarità ellittica per l'equazione di Willmore inomogenea. Si dimostra che, per una immersione Lipschitz e $W^{2,2}$, se il suo operatore (distribuzionale) di Willmore è in L^p con $1 < p < \infty$, allora tale immersione è localmente di classe $W^{4,p}$, come atteso qualora si studi un problema ellittico. Si dà intolre una stima quantitativa per il caso $p = 2$. Tale teorema generalizza il risultato classico di Rivière sulla regolarità delle immersioni di Willmore e la dimostrazione è simile nelle idee essenziali. Tale risultato verrà utilizzato nello studio del flusso di Willmore.

Il Capitolo 5 presenta l'approccio parametrico allo studio del flusso gradiente di Willmore. Tale approccio permette di considerare uan classe generale di soluzioni deboli a "livello energia" e apre

la possibilità ad un futuro studio studio della quantizzazione dell'energia e delle singolarità che si formano in tempo finito. Si lavora in regime di energia bassa e si mostra che, per immersioni deboli, il problema di Cauchy ammette un'unica soluzione nella classe sopra menzionata. I risultati in questo capitolo sono ottenuti in collaborazione con T. Rivière.

Il Capitolo 6 presenta lo studio del problema di Germain–Poisson. Si ottiene l'esistenza di un disco topologico $\mathcal{D} \subset \mathbb{R}^n$ avente energia di Willmore minima tra tutte le superfici aventi stesso bordo, stessa mappa di Gauss al bordo e stessa area. I dati al bordo devono essere di classe $C^{1,1}$ e la curva di bordo semplice e chiusa. Il minimo consiste in un disco immerso, possibilmente con un numero finito di punti di ramificazione interni, di regolarità $C^{1,\alpha}$ fino al boardo per qualche $0 < \alpha < 1$ e con mappa di Gauss di classe $C^{0,\alpha}$ fino al bordo. I risultati in questo capitolo sono ottenuti in collaborazione con F. Da Lio & T. Rivière.

Acknowledgements

First and foremost, I thank my advisors Francesca Da Lio and Tristan Rivière who made this work possible. I am deeply grateful for the invaluable mathematical training they provided me, as well as for the time, patience, resources and overall energy they decided to invest on me.

Next, I thank colleagues, other professors and friends for precious discussions, both about math and non–math subjects, and for their overall support and warmth they transmitted me over the years.

Finally I wish to express my gratitude to professor Michael Struwe for discussions concerning parabolic PDE and geometric analysis, for his answers to my questions and his insight on these subjects, particularly during the last period while the work concerning the Willmore flow presented in this thesis was conducted.

1 Introduction

Worum geht es?

The present thesis is concerned with topics, eminently of analytic nature, on *Willmore surfaces* in Euclidean space.

These are surfaces whose peculiar shape is determined by the fact that they are critical points of the so–called *Willmore energy*, a concept quite known also outside mathematics, appearing, for instance, in nonlinear models for elasticity theory (Germain–Poisson model for elastic plates), cell biology (shaping of red blood cells after Canham and of lipid bilayers of membranes after Helfrich), general relativity (the Willmore energy is the main term appearing in Hawking quasi–local mass).

We shall study

- the *Germain–Poisson problem*, which is a geometric boundary value problem consisting in finding a Willmore disk–type surface spanning a given boundary curve, boundary Gauss map and having prescribed surface area,
- the *parametric approach to the Willmore flow*, which merges the study of the Willmore gradient flow with the framework developed by T. Rivière for the calculus of variation of Willmore surfaces from a parametric point of view.

The study of such problems led to two connected results that are of independent interest:

- The validity – and failure – of *Wente–type estimates for Neumann problems* involving Jacobians,
- An *elliptic regularity result for the inhomogeneous Willmore equation with L^p datum, where $1 < p < \infty$.*

Such results have been obtained by the author together with his advisors Francesca Da Lio and Tristan Rivière during his graduate studies and are contained in the papers [DP17, DPR20, PR]. The material in this thesis will essentially consist in an expanded, adapted and slightly revised version of these works.

In this introduction we shall recall and elaborate on the concept of Willmore surfaces, recall fundamental concepts from Rivière's parametric approach and give an overall description of the results.

1.1 Willmore Surfaces and Their Gradient Flow

The *Willmore energy* of a surface was initially considered in the works of Poisson [Poi16] and Germain [Ger21] on elastic plates. It was reconsidered in a purely geometric perspective in the works of Thomsen [Tho23] and Blaschke [Bla29], in attempt to merge the study of minimal surfaces with conformal invariance. It was reintroduced in recent times in the works of Willmore [Wil65] in pure mathematics and by Canham [Can70] and Helfrich [Hel73] in theoretical biology while modeling, respectively, the shape of blood cells and of lipid bilayers of membranes (in fact, it is sometimes referred to as Canham–Helfrich energy). It is since then subject of constant research; some of the best-known examples are the works by Li and Yau [LY82], Bryant [Bry84], Marques and Neves [MN14].

1.1.1 Let us recall the fundamental concepts. The *Willmore energy* of a closed surface $\mathcal{S} \subset \mathbb{R}^n$ is commonly known in the following three variants:

$$\mathcal{W}_0(\mathcal{S}) = \frac{1}{2}\int_{\mathcal{S}} |A^\circ|^2 \, d\sigma_g, \qquad \mathcal{W}_1(\mathcal{S}) = \int_{\mathcal{S}} |H|^2 \, d\sigma_g, \qquad \mathcal{W}_2(\mathcal{S}) = \frac{1}{4}\int_{\mathcal{S}} |A|^2 \, d\sigma_g, \tag{1.1.1}$$

were H the mean curvature, A is the 2nd fundamental form, A° it tracefree part and $d\sigma_g$ the area element for the induced metric g. If K denotes the Gauss curvature of $\mathcal{S}$, there holds

$$\frac{1}{2}|A^\circ|^2 = |H|^2 - K = \frac{1}{4}|A|^2 - \frac{1}{2}K, \tag{1.1.2}$$

hence, by Gauss–Bonnet theorem, if the topology of $\mathcal{S}$ is fixed, at least in a smooth setting such energies are all variationally equivalent and, in particular, they have the same Euler–Lagrange operator. Depending on the context, it may however be more favorable to work with one rather than another. Consequently, in what follows, when there is no need to choose one over another we shall simply denote them collectively by $\mathcal{W}(\mathcal{S})$.

The *Willmore operator* is the associated Euler–Lagrange operator:

$$\delta\mathcal{W} = \Delta^\perp H + Q(A^\circ)H,$$

where $\Delta^\perp$ is the Laplace operator on the normal bundle and

$$Q(A^\circ)H = \Big\langle A^\circ, \langle H, A^\circ\rangle \Big\rangle = g^{\mu\sigma} g^{\nu\tau} \Big\langle A^\circ_{\mu\nu}, \langle A^\circ_{\sigma\tau}, H\rangle \Big\rangle.$$

Similarly as for the mean curvature, $\delta\mathcal{W}$ is a normal–valued vector field along $\mathcal{S}$. When $n = 3$, the expression simplifies somewhat:

$$\delta\mathcal{W} = \Delta^\perp H + |A^\circ|^2 H = (\Delta H_{\mathrm{sc}} + 2(H_{\mathrm{sc}}^2 - K)H_{\mathrm{sc}})N, \tag{1.1.3}$$

where N is the Gauss map of $\mathcal{S}$ and $H_{\mathrm{sc}} = \langle H, N\rangle$ is the scalar mean curvature.

Willmore surfaces are those surfaces with vanishing Willmore operator. Equivalently, (closed) Willmore surfaces are the stationary points of the Willmore energy, namely if $\{\mathcal{S}_t\}_{t\in(-\varepsilon,\varepsilon)}$ is any 1–parameter family of surfaces with $\mathcal{S} = \mathcal{S}_0$, then $\mathcal{S}$ is Willmore if and only if

$$\mathcal{W}(\mathcal{S}_t) = \mathcal{W}(\mathcal{S}) + o(t) \quad \text{as } t \to 0.$$

One may extend these concepts, with the needed care, also for surfaces that are not closed, e.g. with boundary or noncompact – and in such cases, the equivalence of (1.1.1) may not hold any more.

$$\frac{1+q^2}{k^3}\cdot\frac{d^2R}{dx^2}-\frac{2pq}{k^3}\cdot\frac{d^2R}{dx\,dy}+\frac{1+p^2}{k^3}\cdot\frac{d^2R}{dy^2}$$
$$-\frac{pR}{k}\cdot\frac{dR}{dx}-\frac{qR}{k}\cdot\frac{dR}{dy}+\frac{R}{2}(R^2-c)-\frac{R}{2k^4}\cdot\left(\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2\right)=0;$$

Figure 1.1: Euler-Lagrange Equation with area constraint (for graphs) derived by Poisson.

1.1.2 Any of the Willmore energies (1.1.1) is invariant under conformal transformations of $\mathbb{R}^n$ and, in fact, the Lagrangian density $|A^\circ|^2 d\sigma_g$ is a pointwise conformal invariant, see Chen [Che74]. As a consequence, the Willmore operator and the notion of Willmore surface are also conformal invariants. This was already observed by Blaschke and Thomsen; in fact in [Tho23] they were referred to as "conformal minimal surfaces" (*Konformminimalflächen*). Since minimal surfaces, namely surfaces with vanishing mean curvature: $H = 0$, are clearly Willmore, one may think, broadly speaking, that the equivalence class of minimal surfaces with respect to conformal transformations consists of Willmore surfaces.

In fact, at least in codimension one, it is possible to show that, essentially, the Willmore energy is the only Lagrangian which is conformally invariant, see Bryant [Bry88] and Mondino and Nguyen [MN18].

1.1.3 A different perspecive on Willmore surfaces comes historically from the seminal works of Poisson [Poi16] and Germain [Ger21] on elastic plates.

Germain, building on earlier one–dimensional models concerning beams by Euler and J. Bernoulli, formulated the hypotesis that the elastic energy density stored in a thin, elastic plate is proportional to the mean curvature squared.

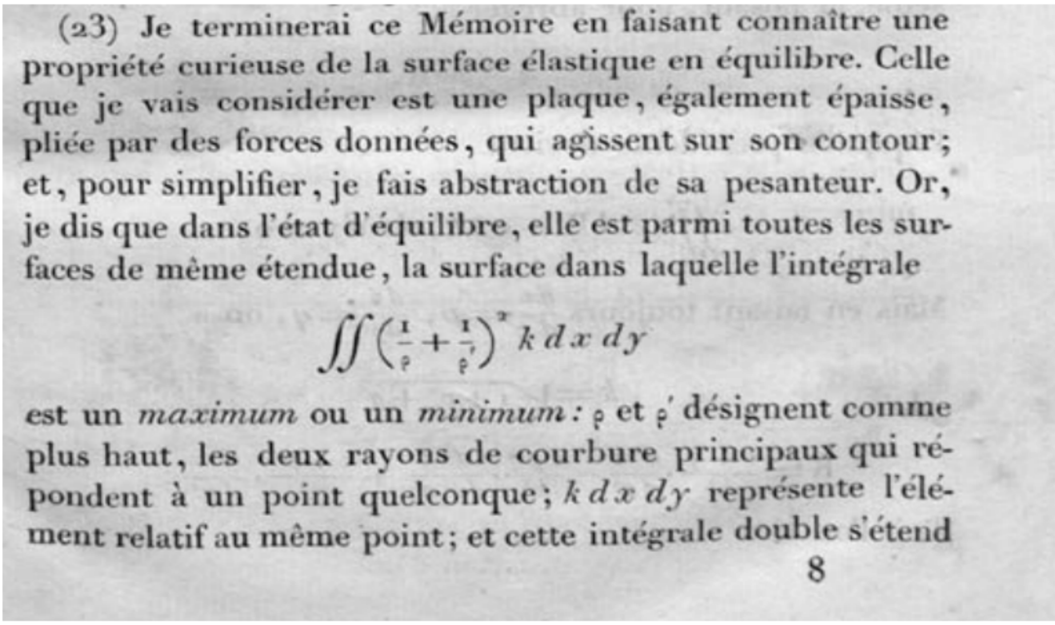
Poisson, considering thin, clamped elastic plates of given surface area, found that in a state of equilibrium they should satisfy the equation corresponding to (1.1.3) complemented with a term corresponding to a fixed-area constraint (see Fig. 1.1).

In light of this, we name *Germain–Poisson problem* the following:

> Given a simple, closed curve $\Gamma \subset \mathbb{R}^n$, and a unit normal $(n-2)$-vector field N_0 along Γ and a value $a > 0$, find an immersed disk $\mathcal{D} \subset \mathbb{R}^n$ bounding Γ, having boundary Gauss map N_0, area $a > 0$ and minimizing the Willmore energy.

It should be said that in Poisson's memoir the variational nature of the equation is just referred to, in a concluding remark, as a "curious property" (see Fig. 1.2). The equation is instead deduced with a different argument concerning the state of equilibrium of every particle of the surface.

The understanding of elasticity has advanced since then, although we remark that a linearized version of the Willmore energy (i.e. the biharmonic energy) is still in use today in models concerning small deformations of thin elastic plates. We refer to [LL86, Vil97, FJM06, GGS10] for more on modern theories of elasticity and to [BD80, DD87, Sza01] for a historical perspective on the development of the subject.



(23) Je terminerai ce Mémoire en faisant connaître une propriété curieuse de la surface élastique en équilibre. Celle que je vais considérer est une plaque, également épaisse, pliée par des forces données, qui agissent sur son contour; et, pour simplifier, je fais abstraction de sa pesanteur. Or, je dis que dans l'état d'équilibre, elle est parmi toutes les surfaces de même étendue, la surface dans laquelle l'intégrale

$$\iint \left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^2 k\,dx\,dy$$

est un *maximum* ou un *minimum*: ρ et ρ' désignent comme plus haut, les deux rayons de courbure principaux qui répondent à un point quelconque; $k\,dx\,dy$ représente l'élément relatif au même point; et cette intégrale double s'étend

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Figure 1.2: Another excerpt from Poisson's memoir, where the Willmore lagrangian appears

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1.1.4 We want to give another reason, this time purely analytic, on why one may come to study the Poisson–Germain problem, and thus to Willmore surfaces. Probably the simplest and most important problem in Calculus of Variations consists in finding minimizers of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx,$$

over functions $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ subject to the the prescribed boundary condition $u|_{\partial\Omega} = u_0$ for a given u_0. This yields to the study of the classical Dirichlet bounday value problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \quad\; u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.4}$$

The study of existence, uniqueness and regularity of weak solutions for such problem are among the basic notions of the subject. Necessity and research yields to consider three fundamental variants of this problem, where:

1. the *domain is curved*, i.e. Ω is replaced with a Riemannian manifold $(\mathcal{M}, g)$;
2. the *target is curved*, i.e. $\mathbb{R}^n$ is replaced with a Riemannian manifold $(\mathcal{N}, h)$;
3. the maps are *immersions* of Ω into $\mathbb{R}^n$ and the energy is with respect to the induced metric $g = u^\star g_{\mathbb{R}^n}$. In this case $E(u)$ is the *area* of $u(\Omega)$.

The first variant is rather mild: classical theory for elliptic PDE yields an existence, uniqueness and regularity theory for the associated Dirichlet problem similar to the basic case. Quite different is the scenario for the other two variants: the second one leads to the theory of *harmonic maps* and the third one to that of *minimal surfaces*, whose study offers formidable challenges.

For minimal surfaces, in the simplest case $m = 2$, $n = 3$ and $\Omega = B_1$, the minimization problem becomes the *Plateau problem*: given a closed, simple curve $\Gamma \subset \mathbb{R}^3$, find an immersed disk $\mathcal{D} = u(B_1) \subset \mathbb{R}^3$ bounding Γ and with least possible area. The boundary value problem (1.1.4) translates into

$$\begin{cases} H = 0, \\ \partial\Sigma = \Gamma, \end{cases}$$

where H is the mean curvature vector of $\mathcal{D}$. The analogous boundary value problem associated to the Germain-Poisson problem is then

$$\begin{cases} \delta\mathcal{W} = cH, \\ \partial\mathcal{D} = \Gamma, \\ N|_{\partial\mathcal{D}} = N_0, \end{cases}$$

where $c \in \mathbb{R}$ is a constant due to the area constraint. From an analysis perspective, up to the area constraint this is, as for the Plateau problem, the variant nr. 3 of the Dirichlet problem above described but for the biharmonic energy $B(u) = \int_\Omega |\Delta u|^2 \, dx$. It is then also clear to see that that the two boundary constraints are natural since the problem is of fourth order.

1.1.5 We briefly mention that the Willmore energy appears also in theoretical biology while modeling the shapes of blood cells and lipid bilayers; in fact, it is sometimes referred to as Canham–Helfrich energy. We refer to the introduction in [KMR14] for more references on the subject.

1.1.6 The analogue of the Poisson problem in the closed case, namely the minimization of the Willmore energy among closed surfaces of a given genus, is already widely studied and solved in the works, among others, of Willmore [Wil65], Simon [Sim93], Bauer and Kuwert [BK03] and Rivière [Riv14].

For boundary value problems, the scenario is far from complete and the subject is in constant expansion. What follows is a selection of the works on the subject.

Nitsche [Nit93] discussed various boundary conditions for the Willmore and related type of functionals, and proved existence and uniqueness results for a class of such problems, also considering a volume constraint, when the surfaces are graphs in $\mathbb{R}^3$ and the boundary data are sufficiently small in $C^{4,\alpha}$-norm.

Deckelnick, Grunau amd Rögers [DGR17] also consider the minimisation over graphs in $\mathbb{R}^3$ of the Willmore functional (also plus a constant times integral of the Gauss curvature) subject to various boundary conditions and deduced compactness results in the L^1-topology, and from this, also a lower-semicontinuity for a suitably defined relaxation of the Willmore functional.

A considerable series of results [EK17, DGR17, DDW13, BDF13, DFGS11, BDF10, DG09, DDG08] is available when considering boundary value problems for the Willmore functional under the hypothesis that the surfaces in consideration are surfaces of revolution around an axis in $\mathbb{R}^3$ (hence the boundary consist of two circles).

Schätzle [Sch10], by working on the sphere $S^n \subset \mathbb{R}^{n+1}$, has proved the existence, for arbitrary smooth boundary data Γ and N_0 and without area constraint, of an immersion (even for surfaces with rich topology) which is smooth and Willmore away from the finitely many points; the resulting surface however may not be solution of the Germain–Poisson problem and may also be noncompact.

ALEXAKIS and MAZZEO [AM15] consider smooth, properly embedded and complete Willmore surfaces in the hyperbolic space $\mathbb{H}^3$ and relate the regularity of their asymptotic boundary with the smallness of a suitable version of the Willmore energy.

ALESSANDRONI and KUWERT [AK16] consider a free–boundary problem for the Willmore functional and proved the existence (and non–uniqueness) of smooth Willmore disk-type surfaces in $\mathbb{R}^3$ with prescribed but small value of the area whose boundary lays on the boundary of a smooth, bounded domain.

KUWERT and LAMM [KL20] proved a first boundary regularity result for free–boundary Willmore surfaces when the free boundary lies in a plane or a line.

Finally, we also mention the very recent works of EICHMANN [Eic19] and POZZETTA [Poz20] that also deal with minimization problems for Willmore–type energy similar to ours.

1.1.7 A *Willmore L^2–gradient flow* in $\mathbb{R}^n$ (*Willmore flow* for short) of a closed, abstract surface Σ is a 1–parameter family of immersions $\Phi(t,\cdot) : \Sigma \to \mathbb{R}^n$, $t \in [0,T)$ evolving according to the law

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U \quad \text{in } [0,T) \times \Sigma, \tag{1.1.5}$$

where, for each t, $\delta\mathcal{W}$ is the Willmore operator of $\mathcal{S}_t = \Phi(t,\Sigma)$ and $U = U^\mu \partial_\mu \Phi$ is a tangent tangent vector field, possibly time–dependent.

A Willmore flow can be regarded as a continuous deformation of the initial surface $\mathcal{S}_0 = \mathcal{S} = \Phi(0,\Sigma)$ constructed so that the Willmore energy (in any of the forms given in (1.1.1)) decreases most rapidly in time, and the deformation stops as soon as the surface becomes Willmore. Thus, at least in principle, Willmore flows have the potential to converge efficiently to Willmore immersions as $t \to +\infty$. This is a feature common to gradient flows for other type of Lagrangians (for instance, the area or the Dirichlet energy) that makes them so useful. The energy decrease is expressed quantitatively by the energy identity:

$$\mathcal{W}(\Phi(t,\cdot)) - \mathcal{W}(\Phi_0) = -\int_0^t \int_\Sigma |\delta\mathcal{W}|^2 \, d\sigma_g \, d\tau, \quad \text{for } 0 \le t < T, \tag{1.1.6}$$

which may be rephrased by saying that, among all families of immersions whose velocity vector has normal part with L^2-norm equal to $\|\delta\mathcal{W}\|_{L^2(S^2)}$, Willmore flows are those with most rapidly decreasing Willmore energy.

The first to consider L^2–gradient flows in geometric analysis were EELLS and SAMPSON [ES64] in the context of harmonic maps. Since then, the study of parabolic geometric flows has widened to the extent that some of them constitute research areas on their own right, the mean curvature flow and Hamilton's Ricci flow being two of the best–known examples.

It should be noted right away that what is typically called a Willmore flow is a family solving (1.1.5) with $U = 0$, which we will call a *normal Willmore flow*. Since Σ is closed, and $\delta\mathcal{W}$ is a tensor, it is classical fact that, at least in a smooth situation, there is a bijective correspondence between tangential components and family of reparametrizations of Σ, see e.g. MANTEGAZZA [Man11, Proposition 1.3.4] for the case, entirely analogous in this regard, of the mean curvature flow. Consequently, for every family solving (1.1.5) there is a unique family of diffeomorphisms $\varphi : [0,T) \times \Sigma \to \Sigma$ with $\varphi(0,\cdot) = \mathrm{id}_\Sigma$ so that the reparametrized family $\Phi(t, \varphi(t,\cdot))$, $t \in [0,T)$ is a normal Willmore flow, and on the other hand, every reparametrization of a normal Willmore flow will be a Willmore flow (1.1.5) for some U.

Thus, in this sense, similarly as for immersions of surfaces, flows can be regarded as equivalence classes of solutions to (1.1.5), two of them being equivalent if one can be reparametrized into another. As for surfaces, depending on the situation one may choose one parametrization over another, and in this case this may be done through the choice of the tangential component.

The study of Willmore flows was introduced by Kuwert and Schätzle [KS01, KS02] and Simonett [Sim01] and is since then subject of a growing number of works.

Essentially, the central result that was there proved is a small–energy existence theorem: there exists $\varepsilon_0(n) > 0$ so thatm for any smooth immersion $\Phi_0 : \Sigma \to \mathbb{R}^n$ so that $\mathcal{W}_0(\Phi_0) = \mathcal{W}_0(\Phi_0(\Sigma)) < \varepsilon_0$, then the normal Willmore flow has a unique solution in the smooth category, which furthermore exists for all times and converges to a round sphere.

Particularly close in spirit to the two original works is the paper Kuwert and Scheuer [KS20] providing asymptotic estimates on the area and barycenter along the flow.

We also mention that Lamm and Koch in [KL12] obtained (among other results of geometric interest) an existence and uniqueness result for the Willmore flow for entire graphs in a weak framework with Lipschitz initial datum. Such datum needs to be small in the Lipschitz norm.

1.1.8 We limited ourselves,in the above exposition, to aspects on Willmore surfaces that pertain the work of the present thesis. Many more topics regarding (or strictly connected to) Willmore surfaces are subject of current research, for instance: Willmore immersions in curved manifolds [MR13, MR14], refined singularity and bubbling analysis [KS04, BR13, BR14], Morse–theoretic and classification aspects [Mic20, RM, Mic], Willmore surfaces of saddle–type (minmax constructions) [Riv21].

1.2 The Parametric Approach

We now recall some basic ideas from the so–called *parametric approach to the study of Willmore surfaces* that will be of major importance in this thesis. Such approach has been introduced by Rivière [Riv08, Riv14, Riv16] building on works by Toro [Tor94, Tor95], Müller–Šverák [MŠ95], Hélein [Hél02], and others.

It is similar in spirit with the parametric approach to the Plateau problem developed, to mention few names, by Douglas, Radò, Courant and Morrey (see for instance [Str88, CI11, DHS10, DHT10]) and the analysis of 2–dimensional harmonic maps (see for instance [Hél02]). It consists, briefly speaking, in introducing an appropriately weak notion immersion of a given abstract surface, namely that of *Lipschitz (conformal) $W^{2,2}$ immersion*, where calculus of variations of the Willmore energy can be performed.

Such framework differs from the one introduced by Simon [Sim93], and later developed extensively by Kuwert, Schätzle [KS04, KS07] and other authors. This is more an *geometric measure theory, or ambient, approach* in that, instead of looking at the immersions, one studies the immersed surface itself with techniques mainly from geometric measure theory.

1.2.1 The central definition is the following.

Definition (Weak Immersions)**.** *Let Σ be a surface and let g_0 be a fixed reference metric on Σ. The set of* Lipschitz $W^{2,2}$ immersions*, or* weak immersions *for short, is*

$$\mathscr{E}(\Sigma,\mathbb{R}^n) = (W^{2,2}\cap W^{1,\infty}_{imm})(\Sigma,\mathbb{R}^n),$$

namely, Φ belongs to $\mathscr{E}(\Sigma,\mathbb{R}^n)$ if and only if it is $W^{2,2}$ and there exists $C>0$ so that a.e. in Σ there holds, in the sense of metrics,

$$\frac{1}{C}g_0 \leq g \leq Cg_0,$$

where $g_{\mu\nu} = \langle\partial_\mu\Phi,\partial_\nu\Phi\rangle = (\Phi^\star g_{R^3})_{\mu\nu}$ is the metric induced by Φ.

Note that, if Σ is closed, any smooth metric g_0 above yields the same set of maps; in the other cases (e.g. noncompact, noncomplete) one should clarify which choice is made. In any case, such definition is sensible for two reasons.

First, any element in $\mathscr{E}(\Sigma,\mathbb{R}^n)$ admits a distributional Willmore operator. Indeed, since the Willmore operator can be written in divergence form[1]:

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H - 2(\nabla H)^\top - |H|^2 d\Phi\Big) = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big), \tag{1.2.1}$$

then the (distributional) Willmore operator of $\Phi\in\mathscr{E}(S^2,\mathbb{R}^3)$ is defined as the vector–valued, distribution–valued two form given by

$$\Big(\delta\mathcal{W}d\sigma_g,\varphi\Big)_{\mathcal{D}'} = \int_\Sigma \Big(\Big\langle H,\Delta_g\varphi\Big\rangle + \Big\langle\langle A^\circ,H\rangle^{\sharp_g} + \langle A,H\rangle^{\sharp_g},\nabla\varphi\Big\rangle_g\Big)d\sigma_g,$$

for every $\varphi\in C^\infty(\Sigma,\mathbb{R}^3)$.

Second, the theory asserts that, at least when Σ is closed, any map in $\mathscr{E}(\Sigma,\mathbb{R}^n)$ admits a (bi–Lipschitz) reparametrization which makes it conformal. For a conformal $\Phi\in\mathscr{E}(\Sigma,\mathbb{R}^n)$ with induced metric $g = \mathrm{e}^{2\lambda}g_0$, one can see that its classical Willmore energy is

$$\mathcal{W}_1(\Phi) = \frac{1}{4}\int_\Sigma |\Delta\Phi|^2\mathrm{e}^{-2\lambda}\,d\sigma,$$

where Δ and $d\sigma$ refer to the metric g_0. Thus, for a conformal map in $\mathscr{E}(\Sigma,\mathbb{R}^n)$, its Willmore energy is equivalent to the standard bi–harmonic energy, such equivalence depending however on a uniform control, above and below, of the conformal factor. The reader familiar with calculus of variation will then recognize that, if such uniform control can be achieved, tools from the standard machinery of calculus of variation and functional analysis can be brought into play.

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[1] We denote here

- $\nabla^{*_g}(Z\otimes\omega) = \frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}\,g^{\mu\nu}\omega_\nu Z\big)$ the (negative) formal L^2-adjoint of the covariant derivative induced on the pull-back bundle $\Phi^\star(T\mathbb{R}^n)$ acting on sections of $\Phi^\star(T\mathbb{R}^3)\otimes T^*\Sigma$, and
- $\langle A,H\rangle^{\sharp_g} = g^{\mu\xi}\langle A_{\xi\nu},H\rangle\partial_\mu\otimes dx^\nu \simeq g^{\mu\xi}\langle A_{\xi\nu},H\rangle\partial_\mu\Phi\otimes dx^\nu$ the the 1st-index raising of $\langle A,H\rangle$, and similarly for $\langle A^\circ,H\rangle$.

1.2.2 To achieve the fore–mentioned uniform control, one argues as follows. Given a conformal immersion $\Phi : \Sigma \to \mathbb{R}^n$, the logarithm of its conformal factor λ, satisfies the classical Liouville equation:

$$-\Delta\lambda = K_g e^{2\lambda} - K_0 \quad \text{on } \Sigma, \tag{1.2.2}$$

where K_0 is the Gauss curvature of the reference metric g_0. This yields the estimate

$$\|\Delta\lambda\|_{L^1(\Sigma,g_0)} \leq C\Big(\mathcal{W}_2(\Phi) + 1\Big),$$

for a constant $C > 0$ independent of Φ. Such estimate is however not sufficient to establish a uniform (i.e. in L^∞) control for λ. This is a critical situation: standard elliptic thery would have sufficed for the purpose if, on the left–hand side, we had the norm $\|\Delta\lambda\|_{L^p(\Sigma,g_0)}$ for some $p > 1$.

It follows however from Gauss' equation for the curvature of immersed submanifolds that $K_g e^{2\lambda}$ can be written locally as a Jacobian. Namely, if (e_1, e_2) is any local orhonormal frame for the immersion Φ, one has

$$K_g e^{2\lambda} = \langle *\overline{\nabla} e_1, \overline{\nabla} e_2\rangle = \langle \overline{\nabla}^\perp e_1, \overline{\nabla} e_2\rangle = \langle \partial_1 e_1, \partial_2 e_2\rangle - \langle \partial_2 e_1, \partial_2 e_1\rangle.$$

where $\overline{\nabla}$ denotes the standard Euclidean covariant derivative. So if $\Phi \in \mathscr{E}(B_1, \mathbb{R}^n)$ is conformal (for instance a local parametrization of some conformal immersion), we can recast (1.2.2) on B_1 as

$$-\Delta\lambda = \langle \overline{\nabla}^\perp e_1, \overline{\nabla} e_2\rangle \quad \text{in } B_1,$$

with all the operators being the one for the flat metric $g_{\mathbb{R}^2}$.

Thus, by the classical estimate of Wente (or, equivalently, by integrability by compensation), this special Jacobian structure allows, in particular, to have a uniform estimate for λ:

$$\|\lambda - \ell\|_{L^\infty(B_{1/2})} + \|d\lambda\|_{L^2(B_{1/2})} \leq C\Big(\|\overline{\nabla} e_1\|_{L^2(B_1)} \|\overline{\nabla} e_2\|_{L^2(B_1)} + \mathcal{W}_2(\Phi)\Big).$$

for a constant $\ell \in \mathbb{R}$.

To complete the argument, one needs to finally show that there a special choice of the frame (e_1, e_2) whose energy is controlled by the Willmore energy, namely

$$\|\overline{\nabla} e_1\|^2_{L^2(B_1)} + \|\overline{\nabla} e_2\|^2_{L^2(B_1)} \leq C\mathcal{W}_2(\Phi),$$

and *this is possible if $\mathcal{W}_2(\Phi)$ is sufficiently small*, namely when $\mathcal{W}_2(\Phi) \leq \varepsilon_0$ for some $\varepsilon_0 = \varepsilon_0(n)$.

1.2.3 The reader familiar in calculus of variation will now recognize that this leads to a *concentration–compactness* scheme, consisting roughly speaking in the following. Let Σ be closed and let $(\Phi_k)_{k\in\mathbb{N}} \subset \mathscr{E}(\Sigma, \mathbb{R}^n)$ be a sequence of weak, conformal immersions, each conformal with respect to a reference background metric $g_{0,k}$ and with uniformly bounded Willmore energy

$$\limsup_{k\to\infty} \mathcal{W}_2(\Phi_k) < \infty.$$

If $(g_{0,k})_{k\in\mathbb{N}}$ is compact in the moduli space of Σ (for instance if it can be taken constant), then, up to the choice of a subsequence, there is a suitable sequence of Moebius transformations of $\mathbb{R}^n$ $(\Xi_k)_{k\in\mathbb{N}}$, so that the normalized sequence $\Phi'_k = \Xi_k \circ \Phi_k$ converges, in a suitable topology, to a conformal map $\mathscr{E}(\Sigma \setminus \{p_1, \ldots, p_\ell\}, \mathbb{R}^n)$ with respect to the limiting the metric $g_0 = \lim_{k\to\infty} g_{0,k}$.

Note that the p_j's are points where the Willmore energy concentrates along the sequence above a certain level, and the composition with the Moebius transformations is needed to control the conformal invariance of the Willmore energy.

1.2.4 The theory clarifies that the nature of the limiting map above around each of the singular points is that of a *branched immersion*, and the p_j's as above are *branch points* as in the following definition.

Definition (Branch Point)**.** *A map in* $\Phi \in \mathscr{E}(B_1 \setminus \{0\}, \mathbb{R}^n)$ *conformal with respect to the Euclidean metric* $g_{\mathbb{R}^2}$ *has a* branch point *at 0 if* Φ *extends as a map in* $W^{1,2}(B_1)$ *and if*

$$\lim_{\delta \to 0} \int_{B_1 \setminus B_\delta} |A|^2 d\sigma_g \quad \textit{is finite.}$$

Note that the last condition may be equivalently rephrased by saying the that Gauss map of Φ extends to a map in $W^{1,2}(B_1)$.

The theory yields that, if Φ is as above, then it extends to a Lipschitz map also at 0 and there exists an non–negative integer ϑ, called *the order or the branch point*, so that, for some $C > 0$, there holds

$$C(|z|^\vartheta - o(1)) \leq |d\Phi(z)| \leq C(|z|^\vartheta + o(1)) \quad \text{as } z \to 0.$$

So, branch points are in a way similar to branch points of holomorphic functions, such as $z \mapsto z^2$. As already the study of minimal surfaces indicates, they are generally unavoidable and, after all, quite natural in many situations.

Thus, one extends the definition of weak immersion as follows.

Definition (Weak Branched Immersions)**.** *Let* Σ *be a surface and let* g_0 *be a fixed reference metric on* Σ*. The set of* weak conformal branched immersions *is the set* $\mathscr{F}(\Sigma, \mathbb{R}^n)$ *of maps* $\Phi \in \mathscr{E}(\Sigma \setminus S, \mathbb{R}^n)$*, where* $S = S(\Phi) \subset \Sigma$ *is a finite set consisting of branch points of* Φ*.*

Such definition completes, at least in this brief description, the variational set–up for the parametric theory of Willmore surfaces.

The study of Willmore surfaces near branch points in the parametric framework is subject to an increasingly fine number of works, see for instance [BR13] and [RM]. We shall only need them to a limited extent in the present thesis.

1.2.5 With the appropriate notion of weak immersions comes the necessity of proving the regularity of weak Willmore immersions. The problem is not trivial, since, even using the divergence form of the equation (1.2.1):

$$\delta\mathcal{W} = \Delta_g H + \nabla^{*_g}\Big(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big),$$

the quantity on the left–hand side inside the divergence–type operator ∇^{*_g} is merely in L^1, thus the equation is critical and, written as it is, does not bootstrap.

We do not give here more detail about this part, since in this thesis we shall present a elliptic regularity results for the inhomogeneous Willmore equation which revisits all the ideas of such regularity theory, in more general context which however does not obscure the original ideas. We thus refer the reader to Chapter 4 (see also §1.3.3 below).

1.3 Contents and Results of This Thesis

We now describe the results of this thesis. Detailed information, pertaining references, useful but more technical consequences, extensive commentary and possible future developments are found at the introduction each pertaining chapter.

1.3.1 Neumann Problem With Jacobian Data (Chapter 2) Integrability by compensation plays an important role in the geometric analysis of many problems, for instance in the study of harmonic maps, CMC surfaces and, of course, in the fore–mentioned analysis of Willmore surfaces.

In particular, a central role is played by Wente's inequality, in the following form: for $a, b \in W^{1,2}(B_1)$, then the solution to the boundary value problem problem on the unit disk $B_1 \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } B_1, \\ \quad u = 0 & \text{on } \partial B_1, \end{cases}$$

is continuous in $\overline{B}_1$, in $W^{1,2}(B_1)$ and in $W^{2,1}(B_1)$ and exists $C > 0$ independent of u, a, b such that

$$\|u\|_{L^\infty(B_1)} + \|\nabla u\|_{L^2(B_1)} + \|\nabla^2 u\|_{L^1(B_1)} \leq C \|\nabla a\|_{L^2(B_1)} \|\nabla b\|_{L^2(B_1)}. \tag{1.3.1}$$

Here $\langle \nabla^\perp a, \nabla b \rangle = \partial_1 a \, \partial_2 b - \partial_2 a \, \partial_1 b$ denotes the Jacobian of the map $(a, b) : B_1 \to \mathbb{R}^2$.

Such inequality is remarkable since it does not follow from stardard elliptic theory, as, on a first glance, the Jacobian is just in L^1.

The study of certain kind of geometric problems (in our case, the Germain–Poisson problem) leads naturally to ask whether Wente's estimate is valid in the case where Dirichlet boundary conditions are replaced by Neumann ones.

More specifically, we consider the homogeneous problem:

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } B_1, \\ \ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.3.2}$$

where the obvious compatibility condition $\int_{B_1} \langle \nabla^\perp a, \nabla b \rangle \, dx = 0$ is assumed, and the following inhomogeneous problem:

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } B_1, \\ \ \partial_\nu u = (\partial_\tau a) b & \text{on } \partial B_1, \end{cases} \tag{1.3.3}$$

where for $(x^1, x^2) \in \partial B_1$, $\tau(x^1, x^2) = (-x^2, x^1)$ is the unit tangent vector to ∂B_1.

The first main result is a negative answer for (1.3.2) for general a and b.

Theorem. *There exist $a, b \in (L^\infty \cap W^{1,2})(B_1)$ with $\int_{B_1} \langle \nabla^\perp a, \nabla b \rangle \, dx = 0$ such that every solution of (1.3.2) is neither in $W^{1,2}(B_1)$ nor in $L^\infty(B_1)$; in particular (1.3.1) cannot hold for this problem.*

One guesses that such result holds after realizing that elementary boundary reflection arguments never preserve the Jacobian structure for the extended problem, unless additional assumptions are made on a, b. In this regard, one has the following.

Theorem. *Let $a \in W_0^{1,2}(B_1)$ and $b \in W^{1,2}(B_1)$ and let u be a solution of (1.3.2). Then for the solution with zero average, (1.3.1) holds for this problem.*

Also in the case of (1.3.3) (which is of variational nature, and hence more natural, one may think) one can produce explicit counterexamples:

Theorem. *There exist $a, b \in L^\infty(B_1) \cap W^{1,2}(B_1)$ such that every solution of (1.3.3) is not in $L^\infty(B_1)$ and in particular the estimate (1.3.1) cannot hold for this problem.*

Additional assumptions to deduce a positive result can however be given arguing in terms of Lorentz spaces:

Theorem. *Let $a, b \in W^{1,2}(B_1)$ be so that $\nabla a, \nabla b$ are in the Lorentz space $L^{(2,1)}(B_1)$ and with $\fint_{B_1} a\,dx = 0$ and let $u \in W^{1,1}(B_1)$ be the solution with zero mean to (1.3.3). Then $\nabla u \in L^{(2,1)}(B_1)$ and there exists an absolute constant $C_1 > 0$ so that:*

$$\|\nabla u\|_{L^{(2,1)}(B_1)} \le C_1 \|\nabla a\|_{L^{(2,1)}(B_1)} \|\nabla b\|_{L^{(2,1)}(B_1)}.$$

As a consequence, there exists an absolute constant $C_2 > 0$ so that:

$$\|u\|_{L^\infty(B_1)} \le C_2 \|\nabla a\|_{L^{(2,1)}(B_1)} \|\nabla b\|_{L^{(2,1)}(B_1)}.$$

Finally, a positive result that will be used in the solution of the Germain–Poisson problem is the following.

Theorem. *Let $a, b \in W^{1,2}(B_1)$ be so that their traces $\operatorname{tr}(a), \operatorname{tr}(b)$ belong to $W^{1,p}(\partial B_1)$ for some $p > 1$. Then there exist a constant $C > 0$ independent of u, a, b and constant $C(p) > 0$ depending only on p so that every solution $u \in W^{1,2}(B_1)$ of the problem (1.3.3) belongs to $C^0(\overline{B_1})$ and the following estimate holds:*

$$\begin{aligned}\|\nabla u\|_{L^2(B_1)} + \inf_{c \in \mathbb{R}} \|u - c\|_{L^\infty(B_1)} &\le C \|\nabla a\|_{L^2(B_1)} \|b\|_{W^{1,2}(B_1)} \\ &\quad + C(p)\Big(\|\partial_\tau \operatorname{tr}(a)\|_{L^p(\partial B_1)} \|\operatorname{tr}(b)\|_{W^{1,p}(\partial B_1)}\Big).\end{aligned}$$

Such results are obtained by deducing a suitable representation formula for the trace of the solution u at the boundary.

1.3.2 Differential Geometry of Willmore Immersions (Chapter 3) This chapter serves as an introduction to the differential geometry present in the following chapters.

First some basic notions about curvature are recalled, then some first variations formulas are computed (among these is the derivation of the Willmore equation) and finally the conservation laws issuing form the conformal invariance of the Willmore energy are deduced.

No original result is present in the chapter but the statements are presented in a way that is useful for the following analysis.

1.3.3 Inhomogeneous Willmore Equation (Chapter 4) It is often the case that, when dealing with PDE of variational nature, one is led to study the associated inhomogeneous equation. If the PDE in question is elliptic, then one expects to gain as many degrees of information from the inhomogeneous datum as is the order of the equation, similarly as in the classical study of the Laplace operator and the Poisson equation.

For nonlinear elliptic equations, success or failure of this gain depends on the particular nature of the nonlinearities in relation to the background initial function space and, if one also needs quantitative estimates, on suitable "smallness" of the energy associated with the PDE (or a related one).

The study of the Willmore flow leads to the following elliptic regularity question for Willmore surfaces: for a conformal immersion $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ with distributional Willmore operator $\delta\mathcal{W} \in L^p(B_1)$, $1 < p < \infty$, does it follow that $\Phi \in W^{4,p}_{\mathrm{loc}}(B_1)$, and, when Willmore energy is small, does it also imply an estimate of elliptic type? The answer is positive:

Theorem. *Let $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ be conformal with conformal factor e^λ and Willmore operator $\delta\mathcal{W}$ in $L^p(B_1)$ for some $1 < p < \infty$. Then $\Phi \in W^{4,p}_{loc}(B_1)$ and furthermore, for the case $p = 2$, if $C_{(2,\infty)} > 0$ is a constant bounding the Lorentz (quasi)norm:*

$$\|d\lambda\|_{L^{(2,\infty)}(B_1)} \leq C_{(2,\infty)},$$

there exists an $\varepsilon_0 > 0$ depending only on $C_{(2,\infty)}$ so that if

$$\mathcal{W}_2(\Phi) = \frac{1}{4}\int_{B_1} |A|^2 \, d\sigma_g \leq \varepsilon_0,$$

then the following estimate holds:

$$\|d\Phi\|_{W^{3,2}(B_{1/2})} \leq C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|\mathrm{e}^\lambda\|_{L^2(B_1)}\Big), \tag{1.3.4}$$

where $C = C(C_{(2,\infty)}) > 0$.

Note that the fact that the estimate (1.3.4) does not include $\|\Phi\|_{L^2(B_{1/2})}$ on the left hand-side is motivated by the translation invariance of all the quantities on the right-hand side.

This theorem is a generalization of the classical result of Rivière for the regularity of weak Willmore surfaces, and the proof follows, essentially, similar ideas. As for that result, the distributional Willmore equation is critical with respect to the space $\mathscr{E}(B_1, \mathbb{R}^3)$ and does not bootstrap. To gain an initial amount of information, additional equations following from the conformal invariance of the Willmore Lagrangian and suitable Hodge decompositions performed on these equations have to be used.

1.3.4 Parametric Approach to the Willmore Flow (Chapter 5) Seminal works by Kuwert and Schätzle on the Willmore flow proved long–time existence, uniqueness and convergence of the solution to the corresponding Cauchy problem in the smooth category when the Willmore energy of the initial datum is small.

As is the case for other geometric flows, an effective study of singularities and bubbling analysis requires eventually to develop an energy–level theory, namely to consider appropriate notions of weak solution. We have in mind as a particular example the classical work on the harmonic map flow done by Struwe and complemented by Rivière, Freire and other authors.

There is good evidence for the parametric framework introduced by Rivière to be, when suitably adapted, the appropriate one. We substantiate this claim by introducing, under particularly favourable hypotheses, an energy–level class of weak Willmore flow and prove a uniqueness statement for the corresponding Cauchy problem in this class for a broad set of weak initial data, which we believe to be sufficiently close to the largest possible one, at least among unbranched surfaces.

We shall work in low energy regime, namely we shall arrange things so that the Willmore energy of the surfaces in consideration $\mathcal{W}_0(\mathcal{S})$ is as small as needed; furthermore, we shall also work in codimension one, namely $n = 3$. The first major consequence of this is that (as it is easy to see) small energy implies that the underlying topology is that of the standard sphere S^2. The second one is that we can take advantange of results from the work of De Lellis and Müller which provides for surfaces with small Willmore energy the existence of a conformal parametrization satisfying favourable estimates.

Central in the theory developed in by Rivière is the idea of working with conformal immersions and exploit conservation laws issuing from the conformal invariance of the Willmore operator, turning the Willmore equation (a 4th order quasilinear elliptic system) in a 2nd order semilinear system involving Jacobian–type nonlinearities, which allows regularity bootstrap by means of integrability by compensation.

We exploit this theory by considering Willmore flows in conformal gauge and then use a slice–wise in time (elliptic) integrability by compensation arguments to bootstrap the regularity of the equation, which – as is often the case when working with parabolic PDE in small energy regime – will suffice to get the regularity also in the time variable.

We are going to consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U, & \text{in } (0,T)\times S^2, \\ \Phi(0,\cdot) = \Phi_0 & \text{on } S^2. \end{cases}$$

where the tangential vector field U is chosen so that $\Phi(t,\cdot)$ is conformal for every t. It is a fairly simple matter to find an explicit characterization for U, and one verifies that, while working on the sphere, it defines an uniformly elliptic, zero–cokernel operator.

To control the gauge action of the $\mathrm{Aut}(S^2)$ (the conformal self–maps of S^2) an additional, finite–dimensional constraint is needed and is defined by the following.

Definition. *An immersion $\Phi : S^2 \to \mathbb{R}^3$ is called* well–balanced *if there holds*

$$\int_{S^2} I d\sigma_g = 0 \quad \textit{and} \quad \int_{S^2} \Phi \times I \, d\sigma = 0,$$

where I denotes standard embedding of S^2, $d\sigma$ its area element and $d\sigma_g$ the area element for the induced metric $g = \Phi^\star g_{\mathbb{R}^3}$.

It is also necessary to fix the gauge invariance relative to the choice of the conformal parametrization for the initial datum. To this aim, we adapt the special conformal parametrization constructed in the work of De Lellis and Müller.

The set of initial data for the conformal Willmore flow will consist of weak $W^{2,2}$–closure of the set of immersed surfaces $\mathcal{S} \subset \mathbb{R}^3$ with Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area 4π and vanishing barycenter $\mathcal{C}(\mathcal{S}) = 0$. Parametrically we shall choose a parametrization provided by De Lellis and Müller, which is in addition well–balanced. More precisely:

Definition. *For $\varepsilon > 0$, the set of weak initial data $\mathscr{W}^{\varepsilon}(S^2, \mathbb{R}^3)$ is the weak $W^{2,2}$–closure of the set $\mathscr{D}^{\varepsilon}(S^2, \mathbb{R}^3)$ consisting of smooth conformal immersions $\Phi : S^2 \to \mathbb{R}^3$ so that the surface $\mathcal{S} = \Phi(S^2)$ has Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area $\mathcal{A}(\mathcal{S}) = 4\pi$, barycenter $\mathcal{C}(\mathcal{S}) = 0$, is well–balanced and so that the estimate of De Lellis and Müller holds:*

$$\|\Phi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^{\lambda} - 1\|_{L^{\infty}(S^2)} \leq C\mathcal{W}_0(\mathcal{S}),$$

where $I : S^2 \to \mathbb{R}^3$ denotes the standard immersion of S^2, $c = \fint_{S^2} \Phi \, d\sigma$ and e^{λ} is the conformal factor.

The only essential requirement in this definition is the control (smallness) of the Willmore energy. All the others can be seen as normalizations.

We will consider $\mathscr{W}^{\varepsilon}(S^2, \mathbb{R}^3)$ only for $\varepsilon > 0$ sufficiently small. As a consequence of the theory of Rivière, $\mathscr{W}^{\varepsilon}(S^2, \mathbb{R}^3)$ is a subset of the space $\mathscr{E}(S^2, \mathbb{R}^3)$.

We now define an energy-level class of maps where one can consider weak conformal Willmore flows.

Central to the definitions we shall give shortly is the validity of the energy identity (1.1.6) for the Willmore flow, always true in smooth settings but taken as an assumption in weak ones (in fact, a slightly weaker version will suffice, see condition (iii) in the definition below). This should be, broadly speaking, an obvious requirement to avoid the presence of pathological solutions that invalidate the uniqueness of the solution to the Cauchy problem, as is the case for the case of the harmonic map flow.

Definition (Well–Balanced Energy Class)**.** *For ε, δ, $T > 0$, $\mathscr{W}^{\varepsilon,\delta}_{[0,T]}(S^2, \mathbb{R}^3)$ is set of locally integrable maps $\Phi : (0, T) \times S^2 \to \mathbb{R}^3$ so that*

(i) *For almost every t, $\Phi(t, \cdot)$ is in $\mathscr{E}(S^2, \mathbb{R}^3)$ and conformal,*

(ii) *There holds*

$$\|\Phi - I - c\|_{L^{\infty}((0,T),W^{2,2}(S^2))} + \|\mathrm{e}^{\lambda} - 1\|_{L^{\infty}((0,T)\times S^2)} \leq \delta,$$

where I denotes standard embedding of S^2, $\mathrm{e}^{\lambda} = \mathrm{e}^{\lambda(t,\cdot)}$ is the conformal factor of $\Phi(t, \cdot)$ and $c(t) = \fint_{S^2} \Phi(t, \cdot) \, d\sigma$,

(iii) *There holds*

$$\delta\mathcal{W} \in L^2((0, T) \times S^2) \quad \text{and} \quad \mathcal{W}_0(\Phi(t, \cdot)) \leq \varepsilon \ \text{ for a.e. } t,$$

(iv) *Φ is well-balanced for a.e. t.*

Finally we let, also for $T = +\infty$,

$$\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2, \mathbb{R}^3) = \bigcap_{\tau\in(0,T)} \mathscr{W}^{\varepsilon,\delta}_{[0,\tau]}(S^2, \mathbb{R}^3).$$

In the energy class it is possible to define the notion of weak (distributional) Willmore flows.

Our main result is the following.

Theorem. *There exists $\varepsilon_0 > 0$ so that the Cauchy problem for the conformal Willmore flow with initial datum in $\mathscr{W}^{\varepsilon_0}(S^2, \mathbb{R}^3)$ has a weak solution in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,\infty)}(S^2, \mathbb{R}^3)$ for some $\delta > 0$, assuming the initial datum in the sense of traces. Such solution is smooth, exists for all times and smoothly converges to the standard embedding I of S^2 in $\mathbb{R}^3$. Furthermore, if the initial datum is smooth, such weak solution is also unique.*

We can compare this result with the classical one by Kuwert and Schätzle. They obtain, in the smooth class, long–time existence, uniqueness and convergence to a round sphere for the Cauchy problem of the (normal) Willmore flow. A central feature our result is that the uniqueness of this smooth solution is in the broad class of finite energy solutions, and the fact that it converges exactly to the standard embedding.

We expect the solution to be unique also if the initial datum is nonsmooth.

The proof of the regularity part of the above Theorem shares evident similarities with the corresponding one for the harmonic map flow or Rivière and Freire. In those works, the core estimate that was obtained for weak solutions of the harmonic map flow was of the form

$$\|u(t,\cdot)\|_{W^{2,2}} \leq C\Big(\|\partial_t u(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

which could then be squared and integrated in time to yield higher regularity, and eventually smoothness by the classical theory by Struwe.

We shall obtain a similar result, namely an inequality of the form

$$\|\Phi(t,\cdot)\|_{W^{4,2}} \leq C\Big(\|e^{\lambda}\delta\mathcal{W}(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

for weak solutions of the conformal Willmore flow, and likewise obtain higher regularity from it.

1.3.5 Germain–Poisson Problem (Chapter 6) A solution to the Germain–Poisson problem is presented. We recall it once more:

> Given a simple, closed curve $\Gamma \subset \mathbb{R}^n$, and a unit normal $(n-2)$-vector field N_0 along Γ and a value $a > 0$, find an immersed disk $\mathcal{D} \subset \mathbb{R}^n$ bounding Γ, having boundary Gauss map N_0 and area $a > 0$ minimising the Willmore energy.

We minimise here the version of the Willmore energy

$$\mathcal{W}_2(\mathcal{D}) = \int_{\mathcal{D}} |A|^2 \, d\sigma_g,$$

which has good coercivity properties and (differently from the other variants in (1.1.1)) controls the number of branch points.

The class of "admissible" data (Γ, N_0, a) for which we can solve the problem is the following.

Definition. *A triple (Γ, N_0, a) curve $\Gamma \subset \mathbb{R}^n$, a unit–normal $(n-2)$-vector field N_0 and a real number $a > 0$ is called* admissible for the Germain–Poisson problem *if Γ and N_0 are of class $C^{1,1}$, Γ is simple and closed, and if there is at least one weak, branched conformal immersion $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n)$ so that*

(i) its branch points are only on the interior of B_1,

(ii) *it assumes geometrically the boundary data, namely if* $\gamma : ([0, \mathcal{H}^1(\Gamma)]/\sim) \to \mathbb{R}^m$ *is a chosen arc-length parametrisation of* Γ*, there exist a homeomorphism* $\sigma_\Phi : S^1 \to [0, \mathcal{H}^1(\Gamma)]/\sim$ *so that, for every* $x \in \partial B_1 = S^1$ *there holds*

$$\Phi(x) = \gamma(\sigma_\Phi(x)) \quad \textit{and} \quad N = N_0(\gamma(\sigma_\Phi(x))),$$

(iii) *it has area equal to* a*, namely,* $\mathrm{Area}(\Phi) = \frac{1}{2}\int_{B_1} |\nabla\Phi|^2\, dx = a$.

An elementary application of the $h-$*principle* allows us, for any given Γ and N_0 ad in the above definition to prove the existence of some $a_0 > 0$ so that, for every $a \geq a_0$ the triple (Γ, N_0, a) is admissible. One notices that when $n = 3$, if one requires the map Φ not to have any branch points, (Γ, N_0) need to satisfy a topological constraint, namely, if $\mathbf{t}$ denotes the tangent vector of Γ, the map $x \mapsto (\mathbf{t} \times N_0, \mathbf{t}, N_0)(x)$, $x \in S^1$, has to define a non-nullhomotopic loop in the space of special orthogonal matrices $SO(3)$.

The existence result is the following.

Theorem. *Let* (Γ, N_0, a) *be an admissible triple for the Germain–Poisson problem. Then, there exists conformal weak, branched immersion* $\Phi : B_1 \to \mathbb{R}^n$ *(whose branch points lie of the interior of* B_1*) assuming this data which minimizes the Willmore energy* $\mathcal{W}_2$ *in this class.*

The (partial) regularity result is the following.

Theorem. *Let* (Γ, N_0, a) *be an admissible triple for the Germain–Poisson problem. Every minimizing map* Φ *as in the previous theorem satisfies distributional Willmore equation with area constraint:*

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big) = cH, \quad \textit{in } \mathcal{D}'(B_1),$$

where $c \in \mathbb{R}$*, and such equation is in particular satisfied* at the branch points*.*

Such map Φ *is smooth in* B_1 *away from the branch points and for every* $0 < \beta < 1$*,* Φ *is of class* $C^{2,\beta}$ *at the branch points and its Gauss map* N *extends to a map of class* $C^{1,\beta}$ *at the branch points.*

Finally, there exists $0 < \alpha < 1$ *so that* Φ *is of class* $C^{1,\alpha}$ *up to the boundary and its Gauss map* N *extends to a map of class* $C^{0,\alpha}$ *up to the boundary.*

In the above result, the interior regularity part (both away and at the branch points) comes from the works of Bernard, Rivière and Michelat. The boundary part is instead obtained through suitable biharmonic comparison.

Central in the proofs of these theorems is the use of the results about the Neumann problem with Jacobian data discussed in Chapter 2. Indeed, for a conformal immersion $\Phi : B_1 \to \mathbb{R}^n$, the logarithm of the conformal factor solves the Liouville equations with a Neumann boundary condition:

$$\begin{cases} -\Delta\lambda = K\mathrm{e}^{2\lambda} & \text{in } B_1, \\ \partial_\nu\lambda = k_g\mathrm{e}^{\lambda} - 1 & \text{on } \partial B_1, \end{cases}$$

where k_g is the geodesic curvature of the boundary curve. If (e_1, e_2) is a tangent ortho–normal frame for Φ, such boundary value problem is recast as

$$\begin{cases} -\Delta\lambda = \langle \overline{\nabla}^\perp e_1, \overline{\nabla} e_2\rangle & \text{in } B_1, \\ \partial_\nu\lambda = \langle \partial_\tau e_1, e_2\rangle & \text{on } \partial B_1, \end{cases}$$

so the connection with the content of Chapter 2 is evident.

2 Notes on the Poisson Equation with Jacobian Data

Summary: In this chapter we study the validity of Wente–type estimates for boundary value problems with Neumann boundary conditions. We show in particular that such estimates do not in general hold under the same hypotheses on the data as for Dirichlet boundary conditions and also not under boundary conditions that are natural from a variational perspective. Finally we give some positive results that hold under specific assumptions.

2.1 Introduction

Integrability by compensation has played a central role in the last decades in the geometric analysis of conformally invariant problems. At the beginning of this theory there is WENTE's discovery [Wen69, Wen81] that the distribution

$$\varphi = \log|\cdot| * \langle \nabla^{\perp} a, \nabla b\rangle = \log|\cdot| * (\partial_{x^1} a\, \partial_{x^2} b - \partial_{x^2} a\, \partial_{x^1} b)$$

with $\nabla a, \nabla b \in L^2(\mathbb{R}^2)$ is in $(L^\infty \cap W^{1,2})(\mathbb{R}^2)$ and the following estimate holds:

$$\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq C \|\nabla a\|_{L^2(\mathbb{R}^2)} \|\nabla b\|_{L^2(\mathbb{R}^2)},$$

for $C > 0$ independent of a, b. Subsequent works by BREZIS and CORON [BC84], TARTAR [Tar85], COIFMAN, LIONS, MEYER and SEMMES in [CLMS93], produced, together with various generalizations, the following.

Theorem 2.1.1 (Wente's Inequality)**.** *Let $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk and $a, b \in W^{1,2}(B_1)$. Then the solution $u \in W_0^{1,1}(\Omega)$ to the problem:*

$$\begin{cases} -\Delta u = \langle \nabla^{\perp} a, \nabla b\rangle & \text{in } B_1, \\ \quad u = 0 & \text{on } \partial B_1, \end{cases}$$

is continuous in $\overline{B}_1$ and in $(W^{1,2} \cap W^{2,1})(B_1)$ with $C > 0$ independent of u, a, b such that

$$\|u\|_{L^\infty(B_1)} + \|\nabla u\|_{L^2(B_1)} + \|\nabla^2 u\|_{L^1(B_1)} \leq C \|\nabla a\|_{L^2(B_1)} \|\nabla b\|_{L^2(B_1)}. \tag{2.1.1}$$

The reader familiar with classical elliptic theory will recognize that such result is remarkable since, on a first glance, the Jacobian $\langle \nabla^{\perp} a, \nabla b\rangle$ is just in L^1.

Proofs, detailed accounts and geometric applications can be found in HÉLEIN's book [Hél02]. We also explicitly mention the following generalization which shall be used in next chapters.

Theorem 2.1.2. *([Bet92], [Riv93], [Ge99]) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain, let $a \in W^{1,2}(\Omega)$ and $f \in L^p(\Omega)$ for $1 < p < 2$. Let $u \in W^{1,(2,\infty)}(\Omega)$*[1] *be a solution to*

$$-\Delta u = \langle \nabla^\perp a, \nabla u \rangle + f \quad \textit{in } \Omega.$$

Then $u \in W^{2,p}_{loc}(\Omega)$.

The study of the Germain–Poisson problem (Chapter 6) leads to ask whether Wente's estimate is valid in the case where Dirichlet boundary condition are replaced by Neumann boundary conditions. More specifically, we shall look at the homogeneous problem (assuming of course the compatibility condition $\int_{B_1} \langle \nabla^\perp a, \nabla b \rangle \, dx = 0$):

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \tag{2.1.2}$$

and at the following inhomogeneous one:

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } B_1, \\ \partial_\nu u = (\partial_\tau a) b & \text{on } \partial B_1, \end{cases} \tag{2.1.3}$$

where for every $(x^1, x^2) \in \partial B_1$, $\tau(x^1, x^2) = (-x^2, x^1)$ is the unit tangent vector to ∂B_1. Solutions to (2.1.3) are more natural since they are critical points in $W^{1,2}(B_1)$ of the Lagrangian:

$$\mathcal{L}(u) = \frac{1}{2} \int_{B_1} |\nabla u + (\nabla^\perp a) b|^2 dx.$$

The first main result is a negative answer for (2.1.2) for general a and b.

Theorem 2.1.3. *There exist $a, b \in (L^\infty \cap W^{1,2})(B_1)$ with $\int_{B_1} \langle \nabla^\perp a, \nabla b \rangle \, dx = 0$ such that every solution of (2.1.2) is neither in $W^{1,2}(B_1)$ nor in $L^\infty(B_1)$; in particular (2.1.1) cannot hold for this problem.*

Two preliminary considerations motivate this result.

The first one is that the elementary proof of Theorem 2.1.1 that is illustrated e.g. in [Hél02, Theorem 3.1.2] cannot be adapted to the Neumann case: indeed, if ui solves (2.1.2) and $\alpha : B_1 \to B_1$ is a conformal self-map (Moebius transformation) of the disk, $u \circ \alpha$ still solves (2.1.2) but differs from u by a constant that depends on α. This poses an obstacle in obtaining an uniform L^∞-estimate up to the boundary.

The second one is that, if we consider (for better clarity) the analogous problem in the upper half-plane $\mathbb{R}^2_+ := \{(x^1, x^2) : x^2 > 0\}$, to which (2.1.2) is equivalent by conformal invariance:

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b \rangle & \text{in } \mathbb{R}^2_+, \\ \partial_\nu u = 0 & \text{on } \partial \mathbb{R}^2_+, \end{cases}$$

and call $\tilde{u}$ the even reflection of u with respect to $\partial \mathbb{R}^2_+$, then one sees that

$$-\Delta \tilde{u}(x) = \langle \nabla^\perp a, \nabla b \rangle (x^1, x^2) \chi_{\mathbb{R}^2_+}(y) + \langle \nabla^\perp a, \nabla b \rangle (x^1, -x^2) \chi_{\mathbb{R}^2_-}(x);$$

for general a, b, the right-hand-side can no longer be interpreted as a Jacobian. The same holds for the case of the disk, where the reflection is replaced by the inversion with respect to the unit circle.

However, if either a or b have vanishing trace, say a, then, extending a oddly and u and b evenly, one can see that the Jacobian structure is preserved and that the following holds.

[1] That is, $u \in W^{1,1}(\Omega)$ and ∇u is in the Lorentz space $L^{(2,\infty)}(\Omega)$.

Theorem 2.1.4. *Let $a \in W_0^{1,2}(B_1)$ and $b \in W^{1,2}(B_1)$ and let u be a solution of (2.1.2). Then for the solution with zero average, (2.1.1) holds for this problem.*

Theorem 2.1.4 has been used by Rivière in [Riv07]. We also refer also to the work of Schikorra [Sch18] for related results.

Also in the case of (2.1.3) the assumption $a, b \in (L^\infty \cap W^{1,2})(B_1)$ is not enough to guarantee the boundedness of the solution.

Theorem 2.1.5. *There exist $a, b \in (L^\infty \cap W^{1,2})(B_1)$ such that every solution of (2.1.3) is not in $L^\infty(B_1)$ and in particular the estimate (2.1.1) cannot hold for this problem.*

A similar result was proved by Hirsch [DP17]. The boundedness of the solution is however obtained if we assume a bit more on a, b, namely that their gradient lies in the Lorentz space $L^{(2,1)}(B_1)$ (see e.g. [Ste70] for definition and fundamental properties) rather than just in $L^2(B_1)$.

Theorem 2.1.6. *Let $a, b \in W^{1,2}(B_1)$ be so that $\nabla a, \nabla b \in L^{(2,1)}(B_1)$ and with $\fint_{B_1} a\, dx = 0$ and let $u \in W^{1,1}(B_1)$ be the solution with zero mean to (2.1.3). Then $\nabla u \in L^{(2,1)}(B_1)$ and there exists an absolute constant $C_1 > 0$ so that:*

$$\|\nabla u\|_{L^{(2,1)}(B_1)} \leq C_1 \|\nabla a\|_{L^{(2,1)}(B_1)} \|\nabla b\|_{L^{(2,1)}(B_1)}.$$

As a consequence, there exists an absolute constant $C_2 > 0$ so that:

$$\|u\|_{L^\infty(B_1)} \leq C_2 \|\nabla a\|_{L^{(2,1)}(B_1)} \|\nabla b\|_{L^{(2,1)}(B_1)}.$$

We observe that the assumption $\nabla b \in L^{(2,1)}(B_1)$ is in particular satisfied if $b \in W^{2,1}(B_1)$, see e.g. [Hél02]. We also remark that the assumptions $\nabla a \in L^{(2,1)}(B_1)$ and $\fint_{B_1} a\, dx = 0$ imply $a \in L^\infty(D^2)$ with $\|a\|_{L^\infty} \leq C \|\nabla a\|_{L^{(2,1)}}$.

Finally, a positive result that shall be used later for the Germain–Poisson problem is the following.

Theorem 2.1.7. *Let $a, b \in W^{1,2}(B_1)$ be so that their traces $\operatorname{tr}(a), \operatorname{tr}(b)$ belong to $W^{1,p}(\partial B_1)$ for some $p > 1$. Then there exist a constant $C > 0$ independent of u, a, b and constant $C(p) > 0$ depending only on p so that every solution $u \in W^{1,2}(B_1)$ of the problem (2.1.3) belongs to $C^0(\overline{B_1})$ and the following estimate holds:*

$$\|\nabla u\|_{L^2(B_1)} + \inf_{c \in \mathbb{R}} \|u - c\|_{L^\infty(B_1)} \leq C \|\nabla a\|_{L^2(B_1)} \|b\|_{W^{1,2}(B_1)} + C(p) \Big(\|\partial_\tau \operatorname{tr}(a)\|_{L^p(\partial B_1)} \|\operatorname{tr}(b)\|_{W^{1,p}(\partial B_1)} \Big).$$

For future convenience, we also state the following "localized" version. Its proof follows from Theorem 2.1.7 and standard cut-off–type arguments.

Lemma 2.1.8. *Let $f \in L^1(B_1)$, $g \in L^1(\partial B_1)$, $a_1, \dots a_\ell$ be points in B_1 and $\alpha_1, \dots, \alpha_\ell$ be real numbers satisfying*

$$\int_{B_1} f\, dx + \sum_i \alpha_i = -\int_{\partial B_1} g\, d\sigma.$$

Let $u \in W^{1,1}(B_1)$ be a weak solution to the problem

$$\begin{cases} -\Delta u = f + \displaystyle\sum_{i=1}^{\ell} \alpha_i \delta_{a_i} & \text{in } B_1, \\ \partial_\nu u = g & \text{on } \partial B_1. \end{cases}$$

Assume that, for a given $x_0 \in \partial B_1$ and $0 < r < 1$, $B_1 \cap B_r(x_0)$ contains none of the a_i's and there holds

$$\begin{aligned} & f = \langle \nabla^{\perp} a, \nabla b \rangle \text{ in } B_1 \cap B_r(x_0), \\ & g = \partial_\tau a\, b \text{ on } \partial B_1 \cap B_r(x_0), \end{aligned}$$

for some $a, b \in W^{1,2}(B_1 \cap B_r(x_0))$ so that the traces on ∂B_1 $\operatorname{tr}(a), \operatorname{tr}(b)$ belong to $W^{1,p}(\partial B_1 \cap B_r(x_0))$ for some $p > 1$. Then $u \in C^0(\overline{B_1 \cap B_{r/2}(x_0)}) \cap W^{1,2}(B_1 \cap B_{r/2}(x_0))$ and there exists constants $C(r) > 0$ $C(r,p) > 0$ so that

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \|u - c\|_{L^\infty(B_1 \cap B_{r/2}(x_0))} \\ & \leq C(r) \left(\|f\|_{L^1(B_1)} + \|g\|_{L^1(\partial B_1)} + \sum_{i=1}^{\ell} |\alpha_i| \right) + C(r) \|\nabla a\|_{L^2(B_1 \cap B_r(x_0))} \|b\|_{W^{1,2}(B_1 \cap B_r(x_0))} \\ & \quad + C(r,p) \|\partial_\tau \operatorname{tr}(a)\|_{L^p(\partial B_1 \cap B_r(x_0))} \|\operatorname{tr}(b)\|_{W^{1,p}(\partial B_1 \cap B_r(x_0))}. \end{aligned}$$

2.2 Proof of Theorems 2.1.4 and 2.1.6

In the following, we denote by $\iota : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$, the inversion with respect to the unit circle, namely $\iota(z) = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$. This is a conformal map mapping B_1 to its complement in the plane with conformal factor $\mathrm{e}^{\mu} = \frac{1}{\sqrt{2}} |\nabla \iota| = \frac{1}{|z|^2}$.

Proof of Theorem 2.1.4. Extend a by odd reflection by means of ι:

$$\tilde{a}(z) = \begin{cases} a(z) & \text{if } z \in B_1, \\ -a(\iota(z)) & \text{if } z \in \mathbb{R}^2 \setminus B_1, \end{cases}$$

and b and u by even reflection:

$$\tilde{b}(z) = \begin{cases} b(z) & \text{if } z \in B_1, \\ b(\iota(z)) & \text{if } z \in \mathbb{R}^2 \setminus B_1, \end{cases} \qquad \tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in B_1, \\ u(\iota(z)) & \text{if } z \in \mathbb{R}^2 \setminus B_1. \end{cases}$$

Since a has vanishing trace, both $\tilde{a}$ and $\tilde{b}$ belong to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and by the conformal invariance of the Dirichlet energy, there holds

$$\int_{\mathbb{R}^2} |\nabla \tilde{a}|^2 \, dx = 2 \int_{B_1} |\nabla a|^2 \, dx \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla \tilde{b}|^2 \, dx = 2 \int_{B_1} |\nabla b|^2 \, dx.$$

Again by conformality, we have

$$\Delta \tilde{u} = \mathrm{e}^{2\mu} (\Delta u) \circ \iota \quad \text{in } \mathbb{R}^2 \setminus B_1,$$

on the other hand, by the chain rule we have

$$\langle \nabla^\perp \widetilde{a}, \nabla \widetilde{b} \rangle = \det[D(\widetilde{a}, \widetilde{b})] = (\det[D(a,b)] \circ \iota) \det[D\iota] = \mathrm{e}^{2\mu} \langle \nabla^\perp a, \nabla b \rangle \circ \iota \quad \text{in } \mathbb{R}^2 \setminus B_1;$$

consequently, we have

$$\Delta \widetilde{u} = \langle \nabla^\perp \widetilde{a}, \nabla \widetilde{b} \rangle \quad \text{in } \mathbb{R}^2 \setminus B_1.$$

Since u has vanishing normal derivative, its distributional Laplace operator does not have jump terms across ∂B_1 and hence

$$\Delta \widetilde{u} = \Delta u \chi_{B_1} + \Delta \widetilde{u} \chi_{\mathbb{R}^2 \setminus B_1} = \langle \nabla^\perp a, \nabla b \rangle \chi_{B_1} + \langle \nabla^\perp \widetilde{a}, \nabla \widetilde{b} \rangle \chi_{\mathbb{R}^2 \setminus B_1} = \langle \nabla^\perp \widetilde{a}, \nabla \widetilde{b} \rangle \quad \text{in } \mathbb{R}^2.$$

From here, it is a standard matter to deduce the conclusion using Theorem 2.1.1. □

Proof of Theorem 2.1.6. *Step 1.* We start by observing that we can recast (2.1.3) in divergence form:

$$\begin{cases} \operatorname{div}(\nabla u + (\nabla^\perp a) b) = 0 & \text{in } B_1, \\ \qquad\qquad \partial_\nu u = (\partial_\tau a) b & \text{on } \partial B_1. \end{cases}$$

Therefore, there exists $C \in W^{1,2}(B_1)$ such that:

$$\nabla^\perp C = \nabla u + (\nabla^\perp a) b.$$

which then solves:

$$\begin{cases} -\Delta C = -\operatorname{div}((\nabla a) b) & \text{in } B_1, \\ \ \partial_\tau C = 0 & \text{on } \partial B_1. \end{cases}$$

Since C is determined up to a constant, we can reduce ourselves to study the following Dirichlet problem:

$$\begin{cases} -\Delta C = -\operatorname{div}((\nabla a) b) & \text{in } B_1, \\ \quad C = 0 & \text{on } \partial B_1. \end{cases}$$

Step 2. In this step and in the following we use basic facts about the theory of Calderón-Zygmund operators and interpolation theory, for which we refer to [Hél02, Ste70]. We first assume $b \in W^{1,p}(B_1)$. For a fixed but arbitrary $1 < p < \infty$. Letting

$$f = -(\nabla a) b \in L^p(B_1),$$

we have:

$$\|f\|_{L^p(B_1)} \leq C \|\nabla a\|_{L^p(B_1)} \|b\|_{L^\infty(B_1)}.$$

We denote by $\tilde{f} = f \chi_{B_1}$ its extension by 0 to $\mathbb{R}^2$. We write $C = C_1 + C_2$ where:

$$C_1(x) = \left(-\frac{1}{2\pi} \log |\cdot| * \operatorname{div} \tilde{f} \right)(x), \quad x \in \mathbb{R}^2,$$

and $C_2 = C - C_1$ is the harmonic rest, namely the solution to

$$\begin{cases} -\Delta C_2 = 0 & \text{in } B_1, \\ \quad C_2 = -C_1 & \text{on } B_1. \end{cases}$$

Since

$$\nabla C_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde{f}(y)\Big(\frac{y-x}{|y-x|^3}\Big)dy,$$

and the function $\mathcal{K}(x,y) = \frac{y-x}{|y-x|^3}$ is a CZ-operator, using that $p > 1$ we may estimate

$$\|\nabla C_1\|_{L^p(\mathbb{R}^2)} \leq C\|\tilde{f}\|_{L^p(\mathbb{R}^2)} = C\|f\|_{L^p(B_1)}.$$

As for C_2, since its trace is in $W^{1-1/p,p}(\partial B_1)$, elliptic estimates imply $C_2 \in W^{1,p}(B_1)$ with

$$\|\nabla C_2\|_{L^p(B_1)} \leq C_p\|\operatorname{tr}(C_1)\|_{W^{1-1/p,p}(\partial B_1)} \leq C_p\|f\|_{L^p(B_1)}.$$

We then deduce that

$$\|\nabla C\|_{L^p(B_1)} \leq C_p\|f\|_{L^p(B_1)},$$

and therefore:

$$\|\nabla u\|_{L^p(B_1)} \leq C_p\|f\|_{L^p(B_1)} \leq C_p\|\nabla a\|_{L^p(B_1)}\|b\|_{L^\infty(B_1)}.$$

We shall keep in mind that, as long as p belongs to a compact interval $I \subset (0,\infty)$, the constant C_p is uniformly bounded.

Now we define:

$$\mathfrak{G}_p(B_1) := \{X = (X_1, X_2) \in L^p(D^2, \mathbb{R}^2) : \operatorname{curl}(X) = -\partial_{x^2}X^1 + \partial_{x^1}X^2 = 0\}.$$

Note that since B_1 is simply connected, Poincaré's lemma ensures that every $X \in \mathfrak{G}_p(B_1)$ is of the form $X = \nabla f$ for some $f \in W^{1,p}(B_1)$. By Step 1, if we fix $b \in L^\infty(D^2)$, the linear operator $T\colon \mathfrak{G}_p(D^2) \to L^p(D^2)$, which maps $X = \nabla a$ to ∇u, where u is the zero-mean solution to (2.1.3), is continuous for each $p > 1$.

Step 3. Note now that, since $b \in L^\infty(B_1)$ and $\nabla a \in L^{(2,1)}(B_1)$ $f \in L^{(2,1)}(D^2)$ with

$$\|f\|_{L^{(2,1)}(B_1)} \leq C\|\nabla a\|_{L^{(2,1)}(B_1)}\|b\|_{L^\infty(B_1)}.$$

By interpolation and Step 2, we get that $\nabla u \in L^{(2,1)}(B_1)$ with:

$$\begin{aligned}\|\nabla u\|_{L^{(2,1)}(B_1)} &\leq C\|f\|_{L^{(2,1)}(B_1)} \\ &\leq C\|\nabla a\|_{L^{(2,1)}(B_1)}\|b\|_{L^\infty(B_1)} \\ &\leq C\|\nabla a\|_{L^{(2,1)}(B_1)}\|\nabla b\|_{L^{(2,1)}(B_1)}.\end{aligned}$$

for some $C = C_1 > 0$, which concludes the proof. □

2.3 Proof of Theorems 2.1.3, 2.1.5 and 2.1.7

Because of conformal invariance, we now consider the problems on the half-plane $\mathbb{R}^2_+ = \{x^2 > 0\}$, the homogeneous one being

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b\rangle & \text{in } \mathbb{R}^2_+, \\ \partial_\nu u = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.3.1}$$

and the inhomogeneous one being

$$\begin{cases} -\Delta u = \langle \nabla^\perp a, \nabla b\rangle & \text{in } \mathbb{R}^2_+, \\ \partial_\nu u = (\partial_\tau a) b & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.3.2}$$

For a, b in the homogeneous space $\dot{W}^{1,2}(\mathbb{R}^2)$ namely so that $\nabla a, \nabla b \in L^2(B_1)$. In this case $\nu = (0, -1)$ and $\tau = (1, 0)$. Therefore $\partial_\nu u = -\partial_{x^2} u$ and $\partial_\tau u = \partial_{x^1} u$.

We recall that Green's function for the Neumann problem in the half-plane $\mathcal{G} : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}$ is that solution, for every $x \in \mathbb{R}^2_+$ of the problem:

$$\begin{cases} -\Delta_y \mathcal{G}(x, \cdot) = \delta_x & \text{in } \mathbb{R}^2_+, \\ \partial_{\nu_y} \mathcal{G}(x, \cdot) = 0 & \text{in } \partial\mathbb{R}^2_+, \end{cases}$$

given by:

$$\mathcal{G}(x, y) = -\frac{1}{2\pi}\left\{\log(|x - y|) + \log(|y - \tilde{x}|)\right\},$$

where $x = (x^1, x^2)$, $y = (y^1, y^2)$, $\tilde{x} = (x^1, -x^2)$. We are going to consider the solutions to the problems obtained through the representation formula:

$$u(x) = \int_{\mathbb{R}^2_+} \mathcal{G}(x, \cdot)(-\Delta u)\, dy + \int_{\partial\mathbb{R}^2_+} \mathcal{G}(x, \cdot)\partial_\nu u \, d\sigma \tag{2.3.3}$$

and deduce representation formulas for its trace.

Lemma 2.3.1. *The trace of the solution u given by the formula (2.3.3) is given, for the inhomogeneous problem (2.3.2), by*

$$\mathrm{tr}(u)(x^1) = A(x^1) + B(x^1),$$

and, for the homogenoeus problem (2.3.1), by

$$\mathrm{tr}(u)(x^1) = A(x^1) + B(x^1) + \frac{1}{\pi}\Big(\log|\cdot| * (\partial_\tau a) b\Big)(x^1),$$

where $A = A(a, b)$ anb $B = B(a, b)$ are two functions so that B is estimated uniformly as

$$\operatorname*{ess\,sup}_{\mathbb{R}} |B| \leq C\|\nabla a\|_{L^2(\mathbb{R}^2_+)}\|\nabla b\|_{L^2(\mathbb{R}^2_+)}, \tag{2.3.4}$$

and A can be written as

$$A(x^1) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{1}{t} a(x^1 - t, 0) b(x^1 - t, 0) dt + \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{1}{t} a(x^1 + t, 0) b(x^1 - t, 0) dt. \tag{2.3.5}$$

Proof. Let us start with the inhomogeneous case. By translation invariance it suffices to prove that (2.3.4) and (2.3.5) hold at the origin. We may also assume that a, b are in $C_c^\infty(\mathbb{R}^2)$, and deduce the general case by density.

Step 1. Writing $\langle \nabla^\perp a, \nabla b\rangle = \operatorname{div}((\nabla^\perp a)b)$, we integrate by parts (2.3.3) and get:

$$u(x) = \int_{\mathbb{R}^2_+} \mathcal{G}(x,\cdot)\langle \nabla^\perp a, \nabla b\rangle\, dy + \int_{\partial\mathbb{R}^2_+} \mathcal{G}(x,\cdot)(\partial_\tau a) b\, d\sigma = -\int_{\mathbb{R}^2_+} \langle \nabla_y \mathcal{G}(x,\cdot), \nabla^\perp a\rangle b\, dy,$$

hence, since $\mathcal{G}((x^1,0),y) = -\frac{1}{\pi}\log|(x^1,0)-y|$, at the origin we obtain the expression

$$u(0) = \frac{1}{\pi}\int_{\mathbb{R}^2_+} \langle \nabla(\log|y|), \nabla^\perp a(y)\rangle b(y)\, dy,$$

which, in polar coordinates, reads as

$$u(0) = \frac{1}{\pi}\int_{\mathbb{R}^2_+} \frac{1}{r}(\partial_\theta a) b\, dr d\theta = \int_0^\infty \int_0^\pi \frac{1}{r}(\partial_\theta a) b\, dr d\theta.$$

We now split such expression as follows: we add and subtract $\bar{b}_r = \frac{1}{2}(b(r,\pi) + b(r,0))$ and let

$$u(0) = \frac{1}{\pi}\int_0^\infty \int_0^\pi \frac{1}{r}(\partial_\theta a)\bar{b}_r\, dr d\theta + \frac{1}{\pi}\int_0^\infty \int_0^\pi \frac{1}{r}(\partial_\theta a)(b - \bar{b}_r)\, dr d\theta := A + B,$$

so we see that

$$\begin{aligned}
A &= \frac{1}{\pi}\int_0^\infty \int_0^\pi \frac{1}{r}(\partial_\theta a)\bar{b}_r\, dr d\theta \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{r}\int_0^\pi (a(r,\pi) - a(r,0))\bar{b}_r\, dr d\theta \\
&= \frac{1}{2\pi}\int_0^\infty \frac{1}{r}(a(r,\pi) - a(r,0))(b(r,\pi) + b(r,0))\, dr,
\end{aligned}$$

which is then equivalent to

$$\begin{aligned}
A &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{t}(a(-t,0) - a(t,0))(b(-t,0) + b(t,0))\, dt \\
&= \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \frac{1}{t} a(-t,0) b(-t,0) dt + \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \frac{1}{t} a(t,0) b(-t,0) dt.
\end{aligned}$$

This yields the expression for (2.3.4). For the term B, it can be estimated with Hölder and Poincaré's inequality:

$$\begin{aligned}
|B| &\le \left| \int_0^\infty \int_0^\pi \frac{1}{r}(\partial_\theta a)(b - \bar{b}_r)\, dr d\theta \right| \\
&\le C\Big(\int_0^\infty \int_0^\pi \frac{1}{r}|\partial_\theta a|^2\, dr d\theta\Big)^{1/2}\Big(\int_0^\infty \int_0^\pi \frac{1}{r}|\partial_\theta b|^2\, dr d\theta\Big)^{1/2} \\
&\le C\|\nabla a\|_{L^2(\mathbb{R}^2_+)}\|\nabla b\|_{L^2(\mathbb{R}^2_+)},
\end{aligned}$$

thus yielding (2.3.5).

The formula for the homogeneous problem is proved similarly: integrating by parts we get

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^2_+} \mathcal{G}(x,\cdot)\langle \nabla^\perp a, \nabla b\rangle\, dy \\
&= -\int_{\mathbb{R}^2_+} \langle \nabla_y \mathcal{G}(x,\cdot), \nabla^\perp a\rangle b\, dy - \int_{\partial\mathbb{R}^2_+} \mathcal{G}(x,\cdot)(\partial_\tau a) b\, d\sigma,
\end{aligned}$$

and the conclusion is reached as before. □

The following lemma is elementary but provides the core material to produce the counterexamples.

Lemma 2.3.2. *Let $\alpha \in \mathbb{R}$ and consider the function $f : [-1/2, 1/2] \to \mathbb{R}$ given by*

$$f(t) = \left(\frac{1}{-\log|t|}\right)^{\alpha}.$$

Then:

(i) if $\alpha \geq 0$, then f belongs to $(L^\infty \cap W^{\frac{1}{2},2})((-1/2, 1/2))$;

(ii) If $\alpha > \frac{1}{2}$, then fH belongs to $(L^\infty \cap W^{\frac{1}{2},2})((-1/2, 1/2))$, where H is the Heavyside function: $H(t) = \chi_{[0,+\infty)}(t)$.

Proof. *Part (i).* Boundedness is clear. It is sufficient to prove that f is the trace on$(-1/2, 1/2) \times \{0\}$ of a function belonging to $W^{1,2}(B_{1/2}(0))$. Such function can be taken as $F(y) = \left(\frac{1}{-\log|y|}\right)^{\alpha}$, since through polar coordinates we see that:

$$\begin{aligned} |\nabla F(y)|^2\, dy &= \left|\alpha \left(\frac{1}{-\log|y|}\right)^{\alpha+1} \frac{y}{|y|^2}\right|^2 dy \\ &= \alpha^2 \left(\frac{1}{-\log\rho}\right)^{2(\alpha+1)} \frac{1}{\rho} d\theta d\rho, \end{aligned}$$

so if $\alpha > -\frac{1}{2}$, $\int_{B_{1/2}(0)} |\nabla F(y)|^2\, dy$ is convergent.

Part (ii). $g = H\, f$ is bounded, so it is enough to check that the $W^{\frac{1}{2},2}$- seminorm:

$$[g]_{W^{\frac{1}{2},2}((-1/2,1/2))} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, dsdt$$

is finite. Since the support of g lies in $\left[0, \frac{1}{2}\right]$, we see that:

$$\begin{aligned} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds, dt &= \int_{-1/2}^{1/2} \left(\int_{-1/2}^{0} \frac{|g(t)|^2}{|s-t|^2}\, ds\right) + \left(\int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\right) dt \\ &= \int_0^{1/2} \left(\int_{-1/2}^{0} \frac{|g(t)|^2}{|s-t|^2}\, ds\right) dt + \int_{-1/2}^{0} \left(\int_0^{1/2} \frac{|g(s)|^2}{|s-t|^2}\, ds\right) dt + \int_0^{1/2} \int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\, dt \\ &= 2\int_0^{1/2} \int_{-1/2}^{0} \frac{|g(t)|^2}{|s-t|^2}\, ds\, dt + \int_0^{1/2} \int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\, dt \\ &= 2\int_0^{1/2} |g(t)|^2 \left(\frac{1}{t} - \frac{1}{t+\frac{1}{2}}\right) dt + \int_0^{1/2} \int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\, dt \\ &= \int_0^{1/2} |g(t)|^2 \frac{1}{t\left(t+\frac{1}{2}\right)}\, dt + \int_0^{1/2} \int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\, dt \\ &= \int_0^{1/2} \left(\frac{1}{-\log|t|}\right)^{2\alpha} \frac{1}{t\left(t+\frac{1}{2}\right)}\, dt + \int_0^{1/2} \int_0^{1/2} \frac{|g(s) - g(t)|^2}{|s-t|^2}\, ds\, dt. \end{aligned}$$

The second integral is bounded by $[f]_{W^{\frac{1}{2},2}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)}$, and is consequently convergent for (i). The first integral is convergent if and only if $\alpha > \frac{1}{2}$. □

Proof of Theorem 2.1.5. Because of conformal invariance, it suffices to prove that the thesis holds for the problem in the upper half-plane (2.3.1), where we can use Lemma 2.3.1. Fix $\psi : \mathbb{R} \to \mathbb{R}$ a radial, symmetric smooth cut-off function $\psi : \mathbb{R} \to \mathbb{R}$:

$$\psi(t) = \begin{cases} 1 & \text{in } \left[-\frac{1}{2}, \frac{1}{2}\right], \\ 0 & \text{in } \mathbb{R} \setminus \left[-\frac{3}{4}, \frac{3}{4}\right], \end{cases}$$

Consider any bounded, compactly supported extension to $a, b \in W^{1,2}(\mathbb{R}^2_+, \mathbb{R})$ (that exist thanks to Lemma 2.3.2) of:

$$a(t) = \left(\frac{1}{-\log|t|}\right)^{\alpha} H(t)\psi(t), \quad \text{and} \quad b(t) = \psi(t), \quad t \in \mathbb{R} \simeq \partial\mathbb{R}^2_+,$$

where again $H(t) = \chi_{[0,+\infty)}(t)$, and $\frac{1}{2} < \alpha < 1$. By Lemma 2.3.1,

$$u(x^1, 0) = A(a, b)(x^1) + B(a, b)(x^1),$$

and $B(a, b)$ is bounded. By the symmetry of ψ, $b(y^1, 0) = b(-y^1, 0)$ so we deduce:

$$A(a, b)(0) = \frac{1}{\pi} \int_0^{3/4} \frac{1}{y^1} \left(\frac{1}{-\log|y^1|}\right)^{\alpha} \psi(y^1)^2 \, dy^1, \tag{2.3.6}$$

and this integral is divergent since $\alpha < 1$. □

Proof of Theorem 2.1.3. Because of conformal invariance, it suffices to prove that the thesis holds for the problem in the upper half-plane (2.3.2), where we can use Lemma 2.3.1. With the same functions a, b and exponent $\frac{1}{2} < \alpha < 1$ of the previous proof, and again the representation formula of Lemma 2.3.1,

$$u(x_1, 0) = A(a, b)(x^1) + B(a, b)(x^1) + \frac{1}{\pi}\Big(\log|\cdot| * (\partial_\tau a) b\Big)(x^1),$$

we have that $A(a, b) + B(a, b)$ belong to $W^{1/2,2}(\partial\mathbb{R}^2_+)$, because it represents the trace for the compatible solution. To see that the last term does not belong to $W^{1/2,2}(\partial\mathbb{R}^2_+)$, we note that we can write it as:

$$\left(\frac{1}{\pi} \log|\cdot| * (\partial_\tau a)\, b\right)(x^1) = (-\Delta)^{-1/2}((\partial_\tau a)\, b)(x^1)$$

From the mapping properties of the Riesz Potential (see [Ste70]) it is sufficient that $\partial_\tau a\, b$ does not belong to $W^{-1/2,2}(\partial\mathbb{R}^2_+)$, and this verified since, being

$$\partial_\tau a\, b(t) = \psi(x)\left(\frac{\alpha}{t}\left(\frac{1}{-\log|t|}\right)^{\alpha-1} \psi(t)H(t) + \left(\frac{1}{-\log|t|}\right)^{\alpha} \psi'(t)\right),$$

if we test it against $t \mapsto (-\log|t|)^{\alpha}\psi(t) \in W^{1/2,2}(\partial\mathbb{R}^2_+)$, the resulting integral is divergent.

Finally, combining such expression with the one above (2.3.6) we see once more that $\mathrm{tr}(u)(x^1, 0)$ is unbounded as $x^1 \to 0$. □

Proof of Lemma 2.1.8. We let χ be a function in $C^\infty(\overline{B_1})$ so that $\chi = 1$ in $B_1 \cap B_{3r/4}(x_0)$ and whose support is contained in $B_1 \cap B_{7r/8}(x_0)$, and we let $\tilde{a}$ be an extension of a to B_1, obtained through a suitable Moebius tranformation of B_1 so that $\|\nabla \tilde{a}\|_{L^2(B_1)} \leq C \|\nabla a\|_{L^2(B_1 \cap B_r(x_0))}$ and $\|\partial_\tau \tilde{a}\|_{L^p(\partial B_1)} \leq C(p) \|\partial_\tau a\|_{L^p(\partial B_1 \cap B_r(x_0))}$ for constants $C, C(p) > 0$. Up to a constant, we may write $u = u_1 + u_2$, with

$$\begin{cases} -\Delta u_1 = \langle \nabla^\perp \tilde{a}, \nabla(\chi b) \rangle & \text{in } B_1, \\ \partial_\nu u_1 = [\partial_\tau \tilde{a}]\,(\chi b) & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} -\Delta u_2 = \langle \nabla^\perp a, \nabla((1-\chi) b) \rangle + \sum_i \alpha_i \delta_{a_i} & \text{in } B_1, \\ \partial_\nu u_2 = [\partial_\tau a]\,((1-\chi) b) & \text{on } \partial B_1, \end{cases}$$

with the convention that

$$\langle \nabla^\perp a, \nabla((1-\chi) b) \rangle = f \text{ in } B_1 \setminus B_{7r/8}(x_0) \quad \text{and}$$
$$[\partial_\tau a]\,((1-\chi) b) = g \text{ on } \partial B_1 \setminus B_{7r/8}(x_0).$$

From Theorem 2.1.7, we deduce that for some constant $c_1 \in \mathbb{R}$ there holds

$$\begin{aligned} &\|u_1 - c_1\|_{L^\infty(B_1)} + \|\nabla u_1\|_{L^2(B_1)} \\ &\leq C \|\nabla \tilde{a}\|_{W^{1,2}(B_1)} \|\chi b\|_{W^{1,2}(B_1)} \\ &\quad + C(p) \left(\|\partial_\tau \operatorname{tr}(\tilde{a})\|_{W^{1,p}(\partial B_1)} \|\operatorname{tr}(\chi b)\|_{W^{1,p}(\partial B_1)} \right). \end{aligned} \tag{2.3.7}$$

To estimate u_2, we use the representation formula:

$$\begin{aligned} u_2(x) - \overline{u}_2 = &\int_{B_1} \mathcal{G}(x,y) \langle \nabla^\perp a, \nabla((1-\chi) b) \rangle (y)\, dy \\ &+ \int_{\partial B_1} \mathcal{G}(x,y) \langle \partial_\tau a, (1-\chi) b \rangle\, d\sigma(y) + \sum_{i=1}^{\ell} \alpha_i \mathcal{G}(x, a_i), \end{aligned}$$

since none of the a_i's is in $B_1 \cap B_r(x_0)$ and $1 - \chi$ vanishes $B_1 \cap B_{3r/4}(x_0)$, we may estimate on $B_1 \cap B_{r/2}(x_0)$:

$$\begin{aligned} &\|u_2 - \overline{u}_2\|_{L^\infty(B_1 \cap B_{r/2}(x_0))} \\ &\leq C(r) \left(\|\langle \nabla^\perp a, \nabla((1-\chi) b) \rangle\|_{L^1(B_1)} + \|\operatorname{tr}((\partial_\tau a)(1-\chi) b)\|_{L^1(\partial B_1)} + \sum_{i=1}^{\ell} |\alpha_i| \right). \end{aligned} \tag{2.3.8}$$

Since χ can always be chosen so that $\|\nabla \chi\|_{L^\infty(B_1)} \leq C/r$, by joining estimates (2.3.7) and (2.3.8) we reach the conclusion. □

3 Differential Geometry of Willmore Immersions

Summary: This chapter serves as an introduction to the differential geometry present in the following chapters. No original results are present but we establish some notation and recall a few formulas for curvature operators; also some first variation formulas (and among these, the derivation of the Willmore Euler–Langrange equation) and conservation laws issuing from the conformal invariance of the Willmore Lagrangian are deduced with self–contained proofs.

3.1 Curvature Operators

We recall some classical notions and facts concerning curvature operators. Let $(\mathcal{M}, g)$ be a Riemannian manifold, where we also denote $g = \langle \cdot, \cdot \rangle$ and let ∇ be the associated Levi-Civita connection.

3.1.1 The *Riemann curvature* operator of M, $R = R^g$ is the $\binom{1}{3}$-tensor:

$$R(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad R = R^{\tau}_{\mu\nu\sigma}.$$

It may equivalently expressed as a $\binom{0}{4}$-tensor by lowering one index:

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle \quad \text{i.e.} \quad R_{\mu\nu\sigma\tau} = g_{\tau\zeta} R^{\zeta}_{\mu\nu\sigma}.$$

The *Ricci curvature* is defined as the trace of the operator $Z \mapsto R(Z, X)(Y)$:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(R(\cdot, X)Y), \quad \text{i.e.} \quad \mathrm{Ric}_{\mu\nu} = R^{\sigma}_{\sigma\mu\nu} = g^{\sigma\tau} R_{\sigma\mu\nu\tau} = g^{\sigma\tau} R_{\mu\sigma\tau\nu}.$$

From the basic properties of R it follows that Ric is a symmetric $\binom{0}{2}$-tensor. We may identify with a $\binom{1}{1}$-tensor by raising and index:

$$\mathrm{Ric}^{\nu}_{\mu} = g^{\nu\alpha} \mathrm{Ric}_{\mu\alpha} = g^{\nu\alpha} g^{\sigma\tau} R_{\sigma\mu\alpha\tau}.$$

The *scalar curvature* is the trace of the Ricci operator with respect to g:

$$S = \mathrm{tr}_g(\mathrm{Ric}), \quad \text{i.e.} \quad S = g^{\mu\nu} \mathrm{Ric}_{\mu\nu} = g^{\mu\nu} R^{\sigma}_{\sigma\mu\nu} = g^{\mu\nu} g^{\sigma\tau} R_{\sigma\mu\nu\tau}.$$

The *sectional curvature* $\mathrm{Sec}(\pi)$ *of a plane* $\pi \subset T_x\mathcal{M}$, $x \in \mathcal{M}$, is the Gauss curvature at x of the surface which is image of π through the exponential map in $\mathcal{M}$ which is (by definition on abstract surfaces) the sectional curvature of such surface divided by two. If $\pi = \mathrm{span}\{v_1, v_2\}$ denoting $g_{ij} = \langle v_i, v_j \rangle$, there holds:

$$\mathrm{Sec}(\pi) = \frac{1}{2} g^{ij} g^{kl} R(v_k, v_i, v_j, v_l),$$

and in particular if v_1, v_2 are orthonormal it is

$$\mathrm{Sec}(\pi) = R(v_1, v_2, v_2, v_1).$$

One may equivalently write

$$\mathrm{Sec}(\pi) = \frac{R(v,w,w,v)}{|v|_g^2|w|_g^2 - \langle v,w\rangle^2} = 2\frac{R(v,w,w,v)}{|v\wedge w|_g^2},$$

where the last equality follows form the definition of wedge product and of extension of g to tensor products:

$$\begin{aligned} |v\wedge w|_g^2 &= |v\otimes w - w\otimes v|_g^2 \\ &= |v\otimes w|_g^2 + |w\otimes v|_g^2 - 2\langle v\otimes w, w\otimes v\rangle \\ &= 2\left(|v|_g^2|w|_g^2 - \langle v,w\rangle^2\right). \end{aligned}$$

3.1.2 Let us now assume that $\mathcal{M}$ is a sub-manifold of another manifold $(\mathcal{N}, h)$, with the induced metric, which we denote by g. If no risk of confusion arises we denote both h and g by $\langle\cdot,\cdot\rangle$. We identify the tangent plane at $x \in \mathcal{M}$ as a subspace of the tangent space $T_x\mathcal{N}$, we denote with A the second fundamental form of $\mathcal{M}$, and we use superscripts "$\mathcal{M}$" or "$\mathcal{N}$", or "g" and "h", to mark the belonging of the different objects below. When $\mathcal{N} = \mathbb{R}^n$, the preferred notation for the Euclidean covariant derivative is $\overline{\nabla}$.

The *Gauss equation* relates the curvature tensors with the second fundamental form: for every $U_1, U_2, V_1, V_2 \in \mathfrak{X}(\mathcal{M})$ there holds

$$\begin{aligned} &\Big\langle A(U_1,U_2), A(V_1,V_2)\Big\rangle - \Big\langle A(U_1,V_2), A(V_1,U_2)\Big\rangle \\ &\qquad = \Big\langle R^{\mathcal{M}}(U_1,V_1)V_2, U_2\Big\rangle - \Big\langle R^{\mathcal{N}}(U_1,V_1)V_2, U_2\Big\rangle \end{aligned} \tag{3.1.1}$$

i.e.

$$R^{\mathcal{M}}_{\mu\nu\sigma\tau} = R^{\mathcal{N}}_{\mu\nu\sigma\tau} + \langle A_{\mu\tau}, A_{\nu\sigma}\rangle - \langle A_{\mu\sigma}, A_{\nu\tau}\rangle,$$

where the Greek indices refer to local coordinates (or local frames) on $\mathcal{M}$. We deduce the following expression for the scalar curvature of $\mathcal{M}$:

$$S^{\mathcal{M}} = g^{\mu\nu}g^{\sigma\tau}\left(R^{\mathcal{N}}_{\mu\nu\sigma\tau} + \langle A_{\mu\tau}, A_{\nu\sigma}\rangle - \langle A_{\mu\sigma}, A_{\nu\tau}\rangle\right).$$

When $\mathcal{M} = \Sigma$ is a surface, we have

$$\begin{aligned} S^{\Sigma} &= 2\,\mathrm{Sec}^{\mathcal{N}}(TM) + g^{\mu\nu}g^{\sigma\tau}\left(\langle A_{\mu\tau}, A_{\nu\sigma}\rangle - \langle A_{\mu\sigma}, A_{\nu\tau}\rangle\right) \\ &= 2\,\mathrm{Sec}^{\mathcal{N}}(TM) + 2(g^{11}g^{22} - (g^{12})^2)\left(\langle A_{11}, A_{22}\rangle - \langle A_{12}, A_{12}\rangle\right) \\ &= 2\left(\mathrm{Sec}^{\mathcal{N}}(T\mathcal{N}) + \frac{\langle A_{11}, A_{22}\rangle - |A_{12}|^2}{\det g}\right). \end{aligned}$$

From this expression we recall in particular that when Σ is immersed in $\mathbb{R}^n$ and K is the Gauss curvature, it is

$$K = \frac{1}{2}S^{\Sigma}.$$

We take this equality as *definition of Gauss curvature* when Σ is immersed in a general $\mathcal{N}$; in this way the Theorema Egregium and the Gauss-Bonnet theorem still hold. Note that when $\dim \mathcal{N} = 3$, then, in terms of principal curvatures it is

$$K = k_1 k_2 + \mathrm{Sec}^{\mathcal{N}}(T\Sigma).$$

The *Codazzi equation* complements the Gauss equation, in the sense that, if (3.1.1) specified the tangential component of $R^{\mathcal{N}}$ with respect to $\mathcal{M}$, then Codazzi specifies the normal one. In the following, $\nabla^{\perp}$ refers to the induced covariant derivative on the normal bundle $(T\mathcal{M})^{\perp}$. For instance:

$$\begin{aligned}
\nabla^{\perp} H &= (\nabla^h H)^{\perp}, \\
\nabla^{\perp} A(X,Y) &= \left(\nabla^h A(X,Y)\right)^{\perp} = \left(\nabla^h (A(X,Y))\right)^{\perp} - A(\nabla^g X, Y) - A(X, \nabla^g Y), \\
\Delta^{\perp} H &= (\nabla^{\perp})^* (\nabla^{\perp} H) = ((\nabla^h)^* (\nabla^{\perp} H))^{\perp}.
\end{aligned}$$

The equation of Codazzi is then the following: for every $X, Y, Z \in \mathfrak{X}(\mathcal{M})$ there holds

$$(R^{\mathcal{N}}(X,Y)(Z))^{\perp} = \nabla^{\perp}_X A(Y,Z) - \nabla^{\perp}_Y A(X,Z). \tag{3.1.2}$$

If we contract this identity with respect to X and Y and use the fact that metric contraction commute with covariant derivative, see that (expressing the contraction by means of a local orthonormal frame $e_1, \ldots, e_m$ on $\mathcal{M}$)

$$\begin{aligned}
(\nabla^{\perp})^* A &= \sum_j \nabla^{\perp}_{e_j} A(e_j, \cdot) = \sum_j \nabla^{\perp}_{e_j} A(\cdot, e_j) \\
&= \sum_j \nabla^{\perp}_{(\cdot)} A(e_j, e_j) + (R^{\mathcal{N}}(e_j, e_j)(\cdot))^{\perp} \\
&= m(\nabla^{\perp} H) + \sum_j (R^{\mathcal{N}}(e_j, e_j)(\cdot))^{\perp} = m(\nabla^{\perp} H) + \mathfrak{R},
\end{aligned}$$

where by definition we set

$$\mathfrak{R}(X) = (\mathrm{tr}_g R^{\mathcal{N}}(\cdot, \cdot)(X))^{\perp} \quad \text{for } X \in \mathfrak{X}(\mathcal{M}).$$

Thus, (3.1.2) implies the following:

$$\begin{aligned}
\nabla^{\perp} A &= m(\nabla^{\perp} H) + \mathfrak{R}, \\
(\nabla^{\perp})^* A^{\circ} &= (m-1)(\nabla^{\perp} H) + \mathfrak{R},
\end{aligned}$$

where the last identity follows from the first and the fact that $(\nabla^{\perp})^*(Hh) = \nabla^{\perp} H$.

When $\mathcal{M} = \Sigma$ is a surface, if e_1, e_2 is any local ortho-normal frame on Σ, we have:

$$\begin{aligned}
S^{\Sigma} &= 2\Big(\mathrm{Sec}^{\mathcal{N}}(T\Sigma) + h(A(e_1,e_1), A(e_2,e_2)) - |A(e_1,e_2)|^2\Big), \\
|A|^2 &= |A(e_1,e_1)|^2 + |A(e_2,e_2)|^2 + 2|A(e_1,e_2)|^2, \\
|H|^2 &= \frac{1}{4}\left(|A(e_1,e_1)|^2 + |A(e_2,e_2)|^2 + 2h(A(e_1,e_1), A(e_2,e_2)\right), \\
|A^{\circ}|^2 &= \frac{1}{2}\left(|A(e_1,e_1)|^2 + |A(e_2,e_2)|^2\right) - h\left(A(e_1,e_1), A(e_2,e_2)\right) + 2|A(e_1,e_2)|^2,
\end{aligned}$$

and thus we have the relations

$$|A|^2 = 4|H|^2 - S^\Sigma + 2\operatorname{Sec}^{\mathcal{N}}(T\Sigma) = 2|A^\circ|^2 + S^\Sigma - 2\operatorname{Sec}^{\mathcal{N}}(T\Sigma),$$
$$|A^\circ|^2 = 2|H|^2 - S^\Sigma + 2\operatorname{Sec}^{\mathcal{N}}(T\Sigma) = |A|^2 - 2|H|^2.$$

In a 3-manifold $(\mathcal{N}^3, h)$ we may talk about principal curvatures and from the above formulas we see that

$$|A|^2 = k_1^2 + k_2^2, \quad |H|^2 = \frac{(k_1+k_2)^2}{4}, \quad |A^\circ|^2 = \frac{(k_1-k_2)^2}{2}, \quad S^\Sigma - 2\operatorname{Sec}^N(T\Sigma) = 2k_1k_2.$$

3.1.3 Let us recall some formulas for *conformal changes* in the ambient Space. Let $\widetilde{h} = e^{2u}h$ be a metric conformal to h, where $u \in C^\infty(\mathcal{N}, \mathbb{R})$.

The Levi-Civita connection of $\widetilde{h}$ satisfies

$$\nabla^{\widetilde{h}}_X Y = \nabla^h_X Y + X(u)Y + Y(u)X - h(X,Y)\operatorname{grad}_h u. \tag{3.1.3}$$

If $\mathcal{M}$ is a m-sub-manifold of $\mathcal{N}$ and g and $\widetilde{g}$ denote the metrics induced on $\mathcal{M}$ by h and $\widetilde{h}$ respectively, then its second fundamental form with respect to $\widetilde{h}$ is given by

$$\widetilde{A}(X,Y) = \sum_a \widetilde{h}\left(\nabla^{\widetilde{h}}_X Y, \widetilde{n}_a\right)\widetilde{n}_a = \sum_a h\left(\nabla^{\widetilde{h}}_X Y, n_a\right)n_a,$$

consequently thanks to formula (3.1.3) we deduce that

$$\widetilde{A}(X,Y) = A(X,Y) - h(X,Y)(\operatorname{grad}_h u)^\perp$$

and thus

$$\widetilde{H} = e^{-2u}(H - (\operatorname{grad}_h u))^\perp$$

where the orthogonal projection "$\perp$" is indifferently that of h or, since they coincide, that of $\widetilde{h}$. These relations imply that the trace-free second fundamental form

$$A^\circ = A - \frac{\operatorname{tr}_g A}{m}h = A - Hh,$$

is point-wise conformally invariant:

$$\widetilde{A}^\circ = A^\circ.$$

Finally, since the Riemanninan volume form can be locally expressed as

$$d\operatorname{vol}_h = e_1^* \wedge \cdots \wedge e_n^*,$$

where $e_1, \ldots, e_n$ is any local ortho-normal frame for h and $e_1^*, \ldots, e_n^*$ is its dual co-frame, from the relation $e_i^* = e^u e_i^*$ we deduce that the change is given by

$$d\operatorname{vol}_{\widetilde{h}} = e^{nu} d\operatorname{vol}_h,$$

and similarly for M there holds

$$d\operatorname{vol}_{\widetilde{g}} = e^{mu} d\operatorname{vol}_g.$$

Now we assume that $\mathcal{M} = \Sigma$ is a surface. By definition of the norm on the tensor products and the point-wise conformal invariance of the trace-free second fundamental form, we have

$$|\widetilde{A}^\circ|^2_{\widetilde{h}\otimes\widetilde{g}} = e^{-2u}|A^\circ|^2_{h\otimes g},$$

and so the 2-form $|A^\circ|^2 d\operatorname{vol}_g = |A^\circ|^2\, d\sigma_g$ is also pointwise conformally invariant.

3.1.4 Recall that for a conformal metric g (with respect some fixed background metric g_0) on a surface Σ, if $g = \mathrm{e}^{2\lambda} g_0$, *Liouville's equation* holds:

$$-\Delta_{g_0}\lambda = \mathrm{e}^{2\lambda}K - K_0,$$

where K and K_0 are the Gauss curvatures of g and g_0 respectively.

In the particular case where $\Phi : B_1 \to \mathbb{R}^n$ is a conformal immersion, i.e. $g = \Phi^\star g_{\mathbb{R}^n}$ (such as the local parametrization of a conformally immersed surface), then from Gauss equation we see that

$$K = \langle A(e_1, e_1), A(e_2, e_2)\rangle - |A(e_1, e_2)|^2,$$

where (e_1, e_2) is any local ortho-normal frame for $\Phi_\star(TB_1)$. A simple computation reveals that the right-hand side has a Jacobian structure:

$$\langle A(e_1, e_1), A(e_2, e_2)\rangle - |A(e_1, e_2)|^2 = \mathrm{e}^{-2\lambda}\Big(\langle \partial_1 e, \partial_2 e\rangle - \langle \partial_2 e, \partial_1 e\rangle\Big) = \langle *\overline{\nabla} e_1, \overline{\nabla} e_2\rangle_g$$

where, since the metric is conformal, the Hodge operator acts exactly as the Euclidean one, namely

$$*\overline{\nabla} e_1 = *\Big(\partial_1 e_1 \otimes dx^1 + \partial_2 e_2 \otimes dx^2\Big) = -\partial_1 e_1 \otimes dx^2 + \partial_2 e_2 \otimes dx^1 =: \overline{\nabla}^\perp e_1.$$

With this fact, we finally deduce that, for any ortho-normal frame (e_1, e_2), there holds:

$$-\Delta\lambda = \langle *\overline{\nabla} e_1, \overline{\nabla} e_2\rangle = \langle \overline{\nabla}^\perp e_1, \overline{\nabla} e_2\rangle.$$

3.2 Families of Immersions

3.2.1 Recall that if $\mathcal{M}$ is any manifold and $f : \mathcal{M} \to \mathcal{N}$ is any map, a vector field along f is a section of the pullback bundle $f^\star(T\mathcal{N})$, that is a map $X : M \to T\mathcal{N}$ so that $X(x) \in T_{f(x)}\mathcal{N}$ for every $x \in M$. We may locally write

$$X(x) = X^i(x) e_i(f(x)),$$

where $(e_1, \ldots, e_n)$ is any local frame for $T\mathcal{N}$ near $f(x)$. One may covariantly differentiate vector fields along f as follows: if $x \in \mathcal{M}$ and $V \in T_x\mathcal{M}$, for X as above,

$$\nabla^h_V X(x) = dX^i(x)(V)\, e_i(f(x)) + X^i(x)\nabla^h_{df(x)(V)} e_i\,(f(x)).$$

The following holds.

Lemma 3.2.1. *Let $f : \mathcal{M} \to \mathcal{N}$ be a differentiable map between manifolds and let $h = \langle\cdot,\cdot\rangle$ be a Riemannian metric on N. Denote by $x^1, \ldots, x^m$ local coordinates on M. Then:*

- *If X, Y are vector fields along f, then*

$$\frac{\partial}{\partial x^\mu}\langle X, Y\rangle = \left\langle \nabla^h_{\partial_\mu} X, Y\right\rangle + \left\langle X, \nabla^h_{\partial_\mu} Y\right\rangle; \tag{3.2.1}$$

- *There holds*

$$\nabla^h_{\partial_\mu}\partial_\nu f = \nabla^h_{\partial_\nu}\partial_\mu f. \tag{3.2.2}$$

- *If Y is a vector field along F, then*

$$\nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu} Y = \nabla^h_{\partial_\nu}\nabla^h_{\partial_\mu} Y + R^h(\partial_\mu f, \partial_\nu f)Y, \tag{3.2.3}$$

where R^h is the Riemann curvature tensor for the Levi-Civita covariant derivative of h.

Proof. *Formula* (3.2.1). If $e_1, \ldots, e_n$ is a local frame for $T\mathcal{N}$ around $f(x)$ (such as $e_i = \partial_i$), we may write $X(x) = X^i(x)e_i(f(x))$ and $Y(x) = Y^j(x)e_j(f(x))$ for some locally defined smooth function X^i, Y^j. By the metric properties of the covariant derivative there holds

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\langle e_i(f), e_j(f)\rangle(x) &= d(\langle e_i, e_j\rangle)(f(x))[\partial_\mu f(x)] \\ &= \left\langle \nabla^h_{\partial_\mu f(x)} e_i(f(x)), e_j(f(x))\right\rangle + \left\langle e_i(f(x)), \nabla^h_{\partial_\mu f(x)} e_j(f(x))\right\rangle,\end{aligned}$$

hence we may compute

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\langle X, Y\rangle(x) &= \frac{\partial}{\partial x^\mu}\left(X^i Y^j\langle e_i(f), e_j(f)\rangle\right)(x) \\ &= \left\langle \partial_\mu X^i(x)e_i(f(x)), Y(x)\right\rangle + \left\langle X^i(x)\nabla^h_{\partial_\mu f(x)} e_i(f(x)), Y(x)\right\rangle \\ &\quad + \left\langle X(x), \partial_\mu Y^j(x)e_j(f(x))\right\rangle + \left\langle X(x), Y^j(x)\nabla^h_{\partial_\mu f(x)} e_j(f(x))\right\rangle,\end{aligned}$$

which, by definition of covariant derivative along a map, yields (3.2.1).

Formula (3.2.2). We may write locally $\partial_\mu f(x) = \partial_\mu f^i(x)\partial_i|_{f(x)}$ and $\partial_\nu f(x) = \partial_\nu f^j(x)\partial_j|_{f(x)}$, so we see that, if Γ^k_{ij} are the Christoffel symbols of ∇^h,

$$\begin{aligned}\nabla^h_{\partial_\nu}\partial_\mu f(x) &= \partial^2_{\mu\nu} f^i(x)\partial_i|_{f(x)} + \partial_\mu f^i(x)\nabla^h_{\partial_\nu f(x)}\partial_i(f(x)) \\ &= \partial^2_{\mu\nu} f^i(x)\partial_i|_{f(x)} + \partial_\mu f^i(x)\partial_\nu f^j(x)\nabla^h_{\partial_j}\partial_i(f(x)) \\ &= \partial^2_{\mu\nu} f^i(x)\partial_i|_{f(x)} + \partial_\mu f^i(x)\partial_\nu f^j(x)\Gamma^k_{ji}(f(x))\partial_k|_{f(x)} \\ &= \partial^2_{\mu\nu} f^i(x)\partial_i|_{f(x)} + \partial_\mu f^i(x)\partial_\nu f^j(x)\Gamma^k_{ij}(f(x))\partial_k|_{f(x)} \\ &= \nabla^h_{\partial_\mu}\partial_\nu f(x),\end{aligned}$$

by the symmetry of Γ^k_{ij} in i and j. This proves (3.2.2).

Formula (3.2.3). We compute in local coordinates:

$$\begin{aligned}\nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu} Y(x) &= \nabla^h_{\partial_\mu}\left(\partial_\nu Y^i(x)\partial_i|_{f(x)} + Y^i(x)\nabla^h_{\partial_\nu f(x))}\partial_i(f(x))\right) \\ &= \partial^2_{\mu\nu} Y^i(x)\partial_i|_{f(x)} + \partial_\nu Y^i(x)\nabla^h_{\partial_\mu f(x)}\partial_i|_{f(x)} \\ &\quad + \partial_\mu Y^i(x)\nabla^h_{\partial_\nu f(x))}\partial_i(f(x)) + Y^i(x)\nabla^h_{\partial_\mu f(x)}\nabla^h_{\partial_\nu f(x)}\partial_i(f(x));\end{aligned}$$

we now note that the first line in the above expression is symmetric in μ and ν, while for the second one we have:

$$
\begin{aligned}
&\nabla^h_{\partial_\mu f(x)} \nabla^h_{\partial_\nu f(x)} \partial_i(f(x)) \\
&= \nabla^h_{\partial_\mu f(x)} \left(\partial_\nu f^j(x) \nabla^h_{\partial_j} \partial_i(f(x)) \right) \\
&= \partial^2_{\mu\nu} f^j(x) \nabla^h_{\partial_j} \partial_i(f(x)) + \partial_\mu f^k(x) \partial_\nu f^j(x) \nabla^h_{\partial_k} \nabla^h_{\partial_j} \partial_i(f(x)) \\
&= \partial^2_{\mu\nu} f^j(x) \nabla^h_{\partial_j} \partial_i(f(x)) + \partial_\mu f^k(x) \partial_\nu f^j(x) \left(\nabla^h_{\partial_j} \nabla^h_{\partial_k} \partial_i(f(x)) + \mathbf{R}^h_{f(x)}(\partial_k, \partial_j) \partial_i(f(x)) \right) \\
&= \nabla^h_{\partial_\nu f(x)} \nabla^h_{\partial_\mu f(x)} \partial_i(f(x)) + \mathbf{R}^h_{f(x)}(\partial_\mu f, \partial_\nu f) \partial_i(f(x)),
\end{aligned}
$$

which then leads to (3.2.3). □

3.2.2 A preliminary remark on differentiation of tensors:

- Let $\mathcal{M}$ be a manifold let $I \subseteq \mathbb{R}$ be some open interval and let $S = S(t, \cdot)$ be a family tensor fields of $\mathcal{M}$ which is also differentiable as a function $I \times \mathcal{M} \to \mathcal{T}_l^k(\mathcal{M})$. It is then possible to consider the "time" derivative of S:

  $$\frac{\partial S}{\partial t}(t, x) = \lim_{h \to 0} \frac{S(t + h, x) - S(t, x)}{h}, \quad x \in \mathcal{M}.$$

  First of all, the limit exists and is a smooth function since we may interpret $\partial_t S(t, x) = dS(x, t)[\partial_t]$. Since $S(\cdot, x)$ is an element of $T_l^k(T_x\mathcal{M})$ for every x, the incremental ratio is well-defines and is a tensor; and thus $\partial_t S(t, \cdot)$ also defines a family of (k, l)-tensors.

- Let now $\mathcal{N}$ be another manifold and let $\Phi : I \times \mathcal{M} \to \mathcal{N}$ be a family of smooth maps. If $S : I \times \mathcal{M} \to T\mathcal{N}$ is a family of vector fields along Φ, that is $S(t, x) \in T_{\Phi(t,x)}\mathcal{N}$ for every (t, x). In general, $S(t, x)$ and $S(t + h, x)$ will belong to two different spaces. To be able to differentiate with repsect to t we introduce a covariant derivative ∇^h on $\mathcal{N}$ and so the analogue of $\partial_t S(t, x)$ will be $\nabla^h_{\partial_t} S(t, x)$.

  A similar procedure must be considered when S is a more general tensor along Φ. In this case the one has to consider the differentiation with respect to tensor connections.

For an immersion $\Phi : \mathcal{M} \to (\mathcal{N}, h)$ inducing on $\mathcal{M}$ the metric $g = \Phi^\star h$ we introduce the following "Ricci–type" curvature operator:

$$\mathscr{R}(X) := \operatorname{tr}_g R^h(X, \cdot)(\cdot) = g^{\mu\nu} R^h(X, \partial_\nu \Phi)(\partial_\mu \Phi),$$

for vector fields along Φ $X \in \Gamma(\Phi^\star(T\mathcal{N}))$. By the symmetry properties of R^h, we have the symmetry property

$$\left\langle \mathscr{R}(X), Y \right\rangle = \left\langle X, \mathscr{R}(Y) \right\rangle,$$

for vector fields along Φ $X, Y \in \Gamma(\Phi^\star(T\mathcal{N}))$.

We identify abstract tangent vectors in $\mathcal{M}$ with their counterpart in $\mathcal{N}$, i.e.

$$V = V^\mu \partial_\mu \simeq V^\mu \partial_\mu \Phi,$$

and we denote with $(\cdot)^\sharp$ the index-raising musical isomorphism in M:

$$\omega^\sharp = (\omega_\mu dx^\mu)^\sharp = g^{\mu\nu}\omega_\nu \partial_\mu \simeq g^{\mu\nu}\omega_\nu \partial_\mu \Phi.$$

The raising for a tensor with more than one covariant component is the one for the first (or left-most) argument.

Proposition 3.2.2 (Some First Variation Formulas). *Let $\Phi : I \times \mathcal{M} \to \mathcal{N}$, with $I \subseteq \mathbb{R}$ be a family of immersions between manifolds with velocity*

$$X(t,x) = \frac{\partial}{\partial t}\Phi(t,x).$$

We assume $\mathcal{M}$ to be orientable and we let $h = \langle \cdot, \cdot \rangle$ be Riemannian metric with associated Levi-Civita covariant derivative ∇^h and Riemann curvature tensor R^h. On $\mathcal{M}$ we let $g = \Phi^\star h$ be the induced metric. The following formulas hold:

(i) First variation of the metric:

$$\frac{\partial}{\partial t} g(U,V) = \left\langle \nabla^h_U X, V \right\rangle + \left\langle U, \nabla^h_V X \right\rangle \tag{3.2.4}$$

for U, V vector fields over $\mathcal{M}$, and

$$\frac{\partial}{\partial t} g(\omega,\vartheta) = -\left\langle \nabla^h_{\omega^\sharp} X, \vartheta^\sharp \right\rangle - \left\langle \omega^\sharp, \nabla^h_{\vartheta^\sharp} X \right\rangle, \tag{3.2.5}$$

for ω, ϑ differential forms over $\mathcal{M}$.

(ii) First variation of the volume element[1]*:*

$$\frac{\partial}{\partial t} d\sigma = \left\langle d\Phi, \nabla^h X \right\rangle d\sigma. \tag{3.2.6}$$

(iii) First variation of the Christoffel symbols of ∇^g:

$$\frac{\partial}{\partial t}\Gamma^\sigma_{\mu\nu} = g^{\sigma\tau}\Big(\left\langle \nabla^2_{\mu\nu} X, \partial_\tau \Phi \right\rangle + \left\langle A_{\mu\nu}, \nabla^h_{\partial_\tau} X \right\rangle + \left\langle R^h(\partial_\tau \Phi, \partial_\nu \Phi)(\partial_\mu \Phi), X \right\rangle\Big). \tag{3.2.7}$$

(iv) First variation of the second fundamental form:

$$\nabla^h_{\partial_t} A(U,V) = \left(\nabla^2_{U,V} X + R^h(X,U)(V)\right)^\perp - \langle A(U,V), \nabla^h X\rangle^\sharp, \tag{3.2.8}$$

and of its norm:

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 = {} & 2\langle A, \nabla^2 X\rangle - 2 g^{\mu\nu} g^{\sigma\tau}\left\langle R^h(X, \partial_\sigma \Phi)(\partial_\mu \Phi), A_{\nu\tau}\right\rangle \\ & + 4m\left\langle (\nabla^h H)^\top, \nabla^h X\right\rangle - 4\left\langle (\mathscr{R})^\top, \nabla^h X\right\rangle + 4\left\langle \mathrm{Ric}, d\Phi \dot{\otimes} \nabla^h X\right\rangle\end{aligned} \tag{3.2.9}$$

where Ric *is the Ricci tensor of the metric g.*

[1] Useful: since $\langle X, d\Phi\rangle^\sharp \simeq g^{\mu\nu}\langle X, \partial_\nu \Phi\rangle_\Phi \partial_\mu \Phi = X^\top$, it is possible to write, by the Leibniz rule:

$$\frac{\partial}{\partial t}\sigma = -\langle \Delta\Phi, X\rangle\, d\sigma + \mathrm{div}_g(X^\top)\, d\sigma = -\langle 2H, X\rangle\, d\sigma + \mathrm{div}_g(X^\top)\, d\sigma.$$

(v) First Variation of the mean curvature:

$$\nabla^h_{\partial_t} H = \frac{1}{m}(\Delta X + \mathscr{R}(X))^\perp - \frac{2}{m}\Big\langle A, d\Phi \dot{\otimes} \nabla^h X \Big\rangle - \langle H, \nabla^h X \rangle^\sharp, \tag{3.2.10}$$

and of its norm:

$$\frac{\partial}{\partial t}|H|^2 = \frac{2}{m}\Big(\Big\langle H, \Delta X \Big\rangle + 2\Big\langle (\nabla^h H)^\top, \nabla^h X \Big\rangle + \Big\langle \mathscr{R}(H), X \Big\rangle\Big). \tag{3.2.11}$$

Proof. We will repeatedly use Lemma 3.2.1 with the domain manifold $M = I \times \mathcal{M}$ map $f = \Phi$, and X vector field along Φ.

Variation of the metric. Writing locally $g_{\mu\nu} = \langle \partial_\mu \Phi, \partial_\nu \Phi \rangle$, thanks to (3.2.1) and (3.2.2) we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{\mu\nu} &= \frac{\partial}{\partial t}\left(\langle \partial_\mu \Phi, \partial_\nu \Phi \rangle\right) \\
&= \Big\langle \nabla^h_{\partial_t} \partial_\mu \Phi, \partial_\nu \Phi \Big\rangle + \Big\langle \partial_\mu \Phi, \nabla^h_{\partial_t} \partial_\nu \Phi \Big\rangle \\
&= \Big\langle \nabla^h_{\partial_\mu} \partial_t \Phi, \partial_\nu \Phi \Big\rangle + \Big\langle \partial_\mu \Phi, \nabla^h_{\partial_\nu} \partial_t \Phi \Big\rangle \\
&= \Big\langle \nabla^h_{\partial_\mu} X, \partial_\nu \Phi \Big\rangle + \Big\langle \partial_\mu \Phi, \nabla^h_{\partial_\nu} X \Big\rangle,
\end{aligned}$$

which yields (3.2.4). If $g^{\mu\nu}$ denotes the induced metric on the cotangent bundle, since $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$, it follows that

$$\partial_t g^{\mu\nu}\, g_{\nu\sigma} + g^{\mu\nu} \partial_t g_{\nu\sigma} = 0,$$

and consequently we deduce that

$$\frac{\partial}{\partial t} g^{\mu\nu} = -g^{\mu\sigma} g^{\nu\tau} \frac{\partial}{\partial t} g_{\sigma\tau} = -g^{\mu\sigma} g^{\nu\tau}\left(\Big\langle \nabla^h_{\partial_\sigma} X, \partial_\tau \Phi \Big\rangle + \Big\langle \partial_\sigma \Phi, \nabla^h_{\partial_\tau} X \Big\rangle\right),$$

which yields (3.2.5).

Variation of the volume element. With Jacobi's formula:

$$\partial_t(A(t)) = \det A(t)\, \mathrm{tr}\left(A^{-1}(t) \partial_t A(t)\right),$$

we see that

$$\begin{aligned}
\frac{\partial}{\partial t} \det g &= \det g\, \mathrm{tr}\left(g^{\mu\sigma} \frac{\partial}{\partial t} g_{\sigma\tau}\right) \\
&= \det g\ g^{\mu\sigma} \frac{\partial}{\partial t} g_{\sigma\mu} \\
&= \det g\ g^{\mu\sigma}\left(\Big\langle \nabla^h_{\partial_\sigma} X, \partial_\mu \Phi \Big\rangle + \Big\langle \partial_\sigma \Phi, \nabla^h_{\partial_\mu} X \Big\rangle\right) \\
&= 2 \det g\, g^{\mu\sigma} \Big\langle \partial_\mu \Phi, \nabla^h_{\partial_\sigma} X \Big\rangle \\
&= 2h \otimes g\left(d\Phi, \nabla^h X\right) \det g.
\end{aligned}$$

Consequently

$$\frac{\partial}{\partial t}\sqrt{\det g} = \frac{1}{2\sqrt{\det g}}\frac{\partial}{\partial t}\det g = h \otimes g\left(d\Phi, \nabla^h X\right)\sqrt{\det g},$$

and so the derivative of the volume element $d\sigma = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^m$ is given by (3.2.6).

Variation of the Christoffel Symbols. By definition it is

$$\Gamma^\sigma_{\mu\nu} = \left(\nabla^h_{\partial_\mu}\partial_\nu\Phi\right)^\top = g^{\sigma\tau}\left\langle \nabla^h_{\partial_\mu}\partial_\nu\Phi, \partial_\tau\Phi\right\rangle.$$

We compute first with (3.2.2), (3.2.3):

$$\nabla^h_{\partial_t}\nabla^h_{\partial_\mu}\partial_\nu\Phi = \nabla^h_{\partial_\mu}\nabla^h_{\partial_t}\partial_\nu\Phi + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi) = \nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi),$$

so with (3.2.5), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^\sigma_{\mu\nu} &= -g^{\sigma\xi}g^{\tau\zeta}\left(\left\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\right\rangle + \left\langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\right\rangle\right)\left\langle\nabla^h_{\partial_\mu}\partial_\nu\Phi, \partial_\tau\Phi\right\rangle \\
&\quad + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi), \partial_\tau\Phi\right\rangle + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\partial_\nu\Phi, \nabla^h_{\partial_\tau}\Phi\right\rangle \\
&= -g^{\sigma\xi}\Gamma^\zeta_{\mu\nu}\left(\left\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\right\rangle + \left\langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\right\rangle\right) \\
&\quad + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi), \partial_\tau\Phi\right\rangle + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\partial_\nu\Phi, \nabla^h_{\partial_\tau}\Phi\right\rangle \\
&= -g^{\sigma\xi}\Gamma^\zeta_{\mu\nu}\left\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\right\rangle - g^{\sigma\xi}\Gamma^\zeta_{\mu\nu}\left\langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\right\rangle + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X, \partial_\tau\Phi\right\rangle \\
&\quad + g^{\sigma\tau}\left\langle R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi), \partial_\tau\Phi\right\rangle + g^{\sigma\tau}\left\langle\nabla^h_{\partial_\mu}\partial_\nu\Phi, \nabla^h_{\partial_\tau}X\right\rangle.
\end{aligned}$$

Now recalling that

$$\begin{aligned}
\nabla^{(2)}_{\partial_\mu,\partial_\nu}X &= \nabla^h_{\partial_\mu}\nabla^h_{\partial_\mu}X - \nabla^h_{\nabla^g_{\partial_\mu}\partial_\nu}X = \nabla^h_{\partial_\mu}\nabla^h_{\partial_\mu}X - \Gamma^\sigma_{\mu\nu}\nabla^h_{\partial_\sigma}X, \\
A_{\mu\nu} &= \nabla^{(2)}_{\partial_\mu,\partial_\nu}\Phi = \nabla^h_{\partial_\mu}\partial_\mu\Phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma\Phi,
\end{aligned}$$

we may collect, respectively, the 2nd and 3rd and the 1st and 4th term in the above expression to get:

$$\frac{\partial}{\partial t}\Gamma^\sigma_{\mu\nu} = g^{\sigma\tau}\left(\left\langle\nabla^2_{\partial_\mu,\partial_\nu}X, \partial_\tau\Phi\right\rangle + \left\langle A_{\mu\nu}, \nabla^h_{\partial_\tau}X\right\rangle + \left\langle R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi), \partial_\tau\Phi\right\rangle\right).$$

Since by the symmetry properties of the curvature tensor there holds

$$\left\langle R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi), \partial_\tau\Phi\right\rangle = \left\langle R^h(\partial_\tau,\partial_\mu\Phi)(\partial_\nu\Phi), X\right\rangle,$$

we arrive at (3.2.7).

Variation of the Second Fundamental Form. We have

$$\nabla^h_{\partial_t}A_{\mu\nu} = \nabla^h_{\partial_t}\left(\nabla^{(2)}_{\partial_\mu,\partial_\nu}\Phi\right) = \nabla^h_{\partial_t}\left(\nabla^h_{\partial_\mu}\partial_\nu\Phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma\Phi\right);$$

on the one hand, by (3.2.1), (3.2.2) we have

$$\nabla^h_{\partial_t}\nabla^h_{\partial_\mu}\partial_\nu\Phi = \nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi),$$

on the other hand with (3.2.7) we have

$$\begin{aligned}\nabla^h_{\partial_t}\left(\Gamma^\sigma_{\mu\nu}\partial_\sigma\Phi\right) &= \left(\partial_t\Gamma^\sigma_{\mu\nu}\right)\partial_\sigma\Phi + \Gamma^\sigma_{\mu\nu}\nabla^n_{\partial_t}\partial_\sigma\Phi \\ &= g^{\sigma\tau}\left(\left\langle\nabla^2_{\partial_\mu,\partial_\nu}X,\partial_\tau\Phi\right\rangle + \left\langle A_{\mu\nu},\nabla^h_{\partial_\tau}X\right\rangle + \left\langle R^h(\partial_\tau\Phi,\partial_\nu\Phi)(\partial_\mu\Phi),X\right\rangle\right)\partial_\sigma\Phi \\ &\quad + \Gamma^\sigma_{\mu\nu}\nabla^h_{\partial_\sigma}X,\end{aligned}$$

hence

$$\begin{aligned}\nabla^h_{\partial_t}A_{\mu\nu} = &\quad \nabla^h_{\partial_\mu}\nabla^h_{\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi) \\ &- g^{\sigma\tau}\left(\left\langle\nabla^2_{\partial_\mu,\partial_\nu}X,\partial_\tau\Phi\right\rangle + \left\langle A_{\mu\nu},\nabla^h_{\partial_\tau}X\right\rangle + \left\langle R^h(\partial_\tau\Phi,\partial_\nu\Phi)(\partial_\mu\Phi),X\right\rangle\right)\partial_\sigma\Phi \\ &- \Gamma^\sigma_{\mu\nu}\nabla^h_{\partial_\sigma}X \\ = &\quad \nabla^2_{\partial_\mu,\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi) \\ &- g^{\sigma\tau}\left(\left\langle\nabla^2_{\partial_\mu,\partial_\nu}X,\partial_\tau\Phi\right\rangle + \left\langle A_{\mu\nu},\nabla^h_{\partial_\tau}X\right\rangle + \left\langle R^h(\partial_\tau\Phi,\partial_\nu\Phi)(\partial_\mu\Phi),X\right\rangle\right)\partial_\sigma\Phi \\ = &\quad \nabla^2_{\partial_\mu,\partial_\nu}X + R^h(X,\partial_\mu\Phi)(\partial_\nu\Phi) \\ &- g^{\sigma\tau}\left(\left\langle\nabla^2_{\partial_\mu,\partial_\nu}X,\partial_\tau\Phi\right\rangle + \left\langle R^h(X,\partial_\nu\Phi)(\partial_\mu\Phi),\partial_\tau\Phi\right\rangle\right)\partial_\sigma\Phi \\ &- g^{\sigma\tau}\left\langle A_{\mu\nu},\nabla^h_{\partial_\tau}X\right\rangle\partial_\sigma\Phi \\ = &\left(\nabla^2_{\partial_\mu,\partial_\nu}X + R^h(X,\partial_\nu\Phi)(\partial_\mu\Phi)\right)^\perp - g^{\sigma\tau}\left\langle A_{\mu\nu},\nabla^h_{\partial_\tau}X\right\rangle\partial_\sigma\Phi,\end{aligned}$$

which yields (3.2.8). Recall next that by definition it is

$$|A|^2 = g^{\mu\nu}g^{\sigma\tau}\langle A_{\mu\sigma},A_{\nu\tau}\rangle,$$

and hence

$$\partial_t|A|^2 = \partial_t(g^{\mu\nu}g^{\sigma\tau})\langle A_{\mu\sigma},A_{\nu\tau}\rangle + g^{\mu\nu}g^{\sigma\tau}\partial_t\langle A_{\mu\sigma},A_{\nu\tau}\rangle.$$

On the one hand, with (3.2.8) we have

$$\begin{aligned}\partial_t\langle A_{\mu\sigma},A_{\nu\tau}\rangle &= \langle\nabla^h_{\partial_t}A_{\mu\sigma},A_{\nu\tau}\rangle + \langle A_{\mu\sigma},\nabla^h_{\partial_t}A_{\nu\tau}\rangle \\ &= \left\langle\nabla^2_{\mu\sigma}X + R^h(X,\partial_\sigma\Phi)(\partial_\mu\Phi),A_{\nu\tau}\right\rangle + \left\langle A_{\mu\sigma},\nabla^2_{\nu\tau}X + R^h(X,\partial_\tau\Phi)(\partial_\nu\Phi)\right\rangle,\end{aligned}$$

and so using the (anti)symmetry properties of the curvature tensor we have

$$\begin{aligned}g^{\mu\nu}g^{\sigma\tau}\partial_t\langle A_{\mu\sigma},A_{\nu\tau}\rangle &= 2\langle A,\nabla^2X\rangle + 2g^{\mu\nu}g^{\sigma\tau}\left\langle R^h(X,\partial_\sigma\Phi)(\partial_\mu\Phi),A_{\nu\tau}\right\rangle \\ &= 2\langle A,\nabla^2X\rangle - 2\left\langle R^h(\partial_\mu\Phi,A_{\nu\tau})(\partial_\sigma\Phi),X\right\rangle.\end{aligned}$$

On the other hand, with Gauss equation

$$\langle A_{\mu\sigma},A_{\nu\tau}\rangle = \langle A_{\mu\nu},A_{\sigma\tau}\rangle - R^g_{\mu\tau\sigma\nu} + R^h_{\mu\tau\sigma\nu}$$

(where we have denoted $R^h_{\mu\tau\sigma\nu} = \Big\langle R^h(\partial_\mu\Phi, \partial_\tau\Phi)(\partial_\sigma\Phi), \partial_\nu\Phi\Big\rangle$) and (3.2.5) we have

$$\begin{aligned}
&\partial_t(g^{\mu\nu}g^{\sigma\tau})\langle A_{\mu\sigma}, A_{\nu\tau}\rangle\\
&= -(g^{\mu\xi}g^{\nu\zeta}g^{\sigma\tau} + g^{\mu\nu}g^{\sigma\xi}g^{\tau\zeta})\langle A_{\mu\sigma}, A_{\nu\tau}\rangle\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big)\\
&= -g^{\mu\xi}g^{\nu\zeta}g^{\sigma\tau}\Big(\langle A_{\mu\nu}, A_{\sigma\tau}\rangle - R^g_{\mu\tau\sigma\nu} + R^h_{\mu\tau\sigma\nu}\Big)\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big)\\
&\quad - g^{\mu\nu}g^{\sigma\xi}g^{\tau\zeta}\Big(\langle A_{\mu\nu}, A_{\sigma\tau}\rangle - R^g_{\mu\tau\sigma\nu} + R^h_{\mu\tau\sigma\nu}\Big)\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big)\\
&= -g^{\mu\xi}g^{\nu\zeta}\Big(m\langle A_{\mu\nu}, H\rangle - \mathrm{Ric}_{\mu\nu} + g^{\sigma\tau}R^h_{\mu\tau\sigma\nu}\Big)\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big)\\
&\quad - g^{\sigma\xi}g^{\tau\zeta}\Big(m\langle H, A_{\sigma\tau}\rangle - \mathrm{Ric}_{\tau\sigma} + g^{\mu\nu}R^h_{\mu\tau\sigma\nu}\Big)\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big)\\
&= -2g^{\mu\xi}g^{\nu\zeta}\Big(m\langle A_{\mu\nu}, H\rangle - \mathrm{Ric}_{\mu\nu} + g^{\sigma\tau}R^h_{\mu\tau\sigma\nu}\Big)\Big(\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle + \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle\Big).
\end{aligned}$$

We now notice that, since H is a normal vector, we have

$$g^{\mu\xi}g^{\nu\zeta}\langle A_{\mu\nu}, H\rangle\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\rangle = -g^{\mu\xi}g^{\nu\zeta}\Big\langle\langle\nabla^h_{\partial_\mu}H, \partial_\nu\Phi\rangle, \nabla^h_{\partial_\xi}X\Big\rangle = -\Big\langle(\nabla^h H)^\top, \nabla^h X\Big\rangle,$$

and similarly

$$g^{\mu\xi}g^{\nu\zeta} = \langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\rangle = -\Big\langle(\nabla^h H)^\top, \nabla^h X\Big\rangle;$$

also note that

$$\begin{aligned}
g^{\mu\xi}g^{\nu\zeta}g^{\sigma\tau}R^h_{\mu\tau\sigma\nu}\Big\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\Big\rangle &= g^{\mu\xi}g^{\nu\zeta}g^{\sigma\tau}\Big\langle R^h(\partial_\mu\Phi, \partial_\tau\Phi)(\partial_\sigma\Phi), \partial_\nu\Phi\Big\rangle\Big\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\Big\rangle\\
&= g^{\mu\xi}g^{\nu\zeta}\Big\langle\mathscr{R}(\partial_\mu\Phi), \partial_\nu\Phi\Big\rangle\Big\langle\nabla^h_{\partial_\xi}X, \partial_\zeta\Phi\Big\rangle\\
&= g^{\mu\xi}\Big\langle\nabla^h_{\partial_\xi}X, g^{\nu\zeta}\langle\mathscr{R}(\partial_\mu\Phi), \partial_\nu\Phi\rangle\partial_\zeta\Phi\Big\rangle\\
&= \Big\langle\nabla^h X, (\mathscr{R})^\top\Big\rangle,
\end{aligned}$$

and similarly

$$g^{\mu\xi}g^{\nu\zeta}g^{\sigma\tau}R^h_{\mu\tau\sigma\nu}\Big\langle\partial_\xi\Phi, \nabla^h_{\partial_\zeta}X\Big\rangle = \Big\langle\nabla^h X, (\mathscr{R})^\top\Big\rangle,$$

and so we have (3.2.9).

Variation of the Mean Curvature. Recall that by definition it is $H = \frac{1}{m}\Delta\Phi = \frac{1}{m}g^{\mu\nu}A_{\mu\nu}$; with (3.2.4) and (3.2.8) we have

$$\begin{aligned}
\nabla^h_{\partial_t}H &= \frac{1}{m}\nabla^h_{\partial_t}(g^{\mu\nu}A_{\mu\nu})\\
&= -\frac{1}{m}\Big(g^{\mu\sigma}g^{\nu\tau}\Big(\langle\nabla^h_{\partial_\sigma}X, \partial_\tau\Phi\rangle + \langle\partial_\sigma\Phi, \nabla^h_{\partial_\tau}X\rangle\Big)\Big)A_{\mu\nu}\\
&\quad + \frac{1}{m}g^{\mu\nu}\Big(\Big(\nabla^2_{\mu\nu}X + R^h(X, \partial_\nu\Phi)(\partial_\mu\Phi)\Big)^\perp - g^{\sigma\tau}\langle A_{\mu\nu}, \nabla^h_{\partial_\tau}X\rangle\partial_\sigma\Phi\Big)\\
&= -\frac{2}{m}\Big\langle A, d\Phi\,\dot{\otimes}\nabla^h X\Big\rangle + \frac{1}{m}\Big(\Delta X + g^{\mu\nu}R^h(X, \partial_\nu\Phi)(\partial_\mu\Phi)\Big)^\perp - g^{\sigma\tau}\langle H, \nabla^h_{\partial_\tau}X\rangle\partial_\sigma\Phi,
\end{aligned}$$

which yields (3.2.10). Using this formula we compute

$$\frac{\partial}{\partial t}|H|^2 = -2\Big\langle\nabla^h_{\partial_t}H, H\Big\rangle = -\frac{4}{m}\Big\langle\langle A, H\rangle, \langle d\Phi, \nabla^h X\rangle\Big\rangle + \frac{2}{m}\Big\langle\Delta X + \mathscr{R}(X), H\Big\rangle.$$

We now notice that, since H is a normal vector and $A_{\mu\nu} = \nabla^h_\mu \partial_\nu \Phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma \Phi$ we have

$$\begin{aligned}\Big\langle \langle A, H\rangle, d\Phi \dot{\otimes} \nabla^h X \Big\rangle &= g^{\mu\sigma} g^{\nu\tau} \langle A_{\mu\nu}, H\rangle \langle \partial_\sigma \Phi, \nabla^h_{\partial_\tau} X\rangle \\ &= -g^{\mu\sigma} g^{\nu\tau} \langle \partial_\nu \Phi, \nabla^h_{\partial_\mu} H\rangle \langle \partial_\sigma \Phi, \nabla^h_{\partial_\tau} X\rangle \\ &= -g^{\nu\tau} \Big\langle g^{\mu\sigma} \langle \partial_\nu \Phi, \nabla^h_{\partial_\mu} H\rangle \partial_\sigma \Phi, \nabla^h_{\partial_\tau} X \Big\rangle \\ &= -\Big\langle (\nabla^h H)^\top, \nabla^h X \Big\rangle,\end{aligned}$$

and, by the symmetry properties of $\mathscr{R}$, there holds

$$\langle \mathscr{R}(X), H\rangle = \langle \mathscr{R}(H), X\rangle,$$

whence (3.2.11) follows. $\square$

3.2.3 We also point out the following.

Proposition 3.2.3 (First Variation of the Gauss Curvature Density)**.** *In the setting as Proposition 3.2.2 when $\mathcal{M} = \Sigma$ is a surface and when $\mathcal{N} = \mathbb{R}^n$, the first variation of the Gauss curvature density of the immersion $\Phi : \Sigma \to \mathbb{R}^N$ is*

$$\frac{\partial}{\partial t}(K\, d\sigma) = d^*\Big(2\langle H, \overline{\nabla} X\rangle - \langle A, \overline{\nabla} X\rangle\Big) d\sigma = d^*\Big(\langle H, \overline{\nabla} X\rangle - \langle A^\circ, \overline{\nabla} X\rangle\Big) d\sigma.$$

Remark 3.2.4 In light of the Gauss-Bonnet theorem, $K d\sigma$ is null Lagrangian and thus the fact that its first variation has a divergence structure is expected. If moreover we recall that

$$A = \nabla^2 \Phi, \qquad H = \frac{1}{2}\Delta\Phi, \qquad 2K = 4|H|^2 - |A|^2 = |\Delta\Phi|^2 - |\nabla^2\Phi|^2,$$

some Euclidean heuristics can be helpful. Indeed for any $\varphi, \psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$, on the one hand we have

$$\begin{aligned}\Big\langle \nabla^2\varphi, \nabla^2\psi \Big\rangle &= \sum_{\mu,\nu} \partial^2_{\mu\nu}\varphi\, \partial^2_{\mu\nu}\psi = \sum_{\mu,\nu} \partial_\mu \Big(\partial^2_{\mu\nu}\varphi\, \partial_\nu \psi\Big) - \partial^3_{\mu\mu\nu}\varphi\, \partial_\nu\psi \\ &= \operatorname{div}\Big(\nabla^2\varphi \cdot \nabla\psi\Big) - \Big\langle \Delta\nabla\varphi, \nabla\psi\Big\rangle,\end{aligned}$$

on the other hand

$$\begin{aligned}\Big\langle \Delta\varphi, \Delta\psi \Big\rangle &= \sum_{\mu,\nu} \partial^2_{\mu\mu}\varphi\, \partial^2_{\nu\nu}\psi = \sum_{\mu,\nu} \partial_\nu \Big(\partial^2_{\mu\mu}\varphi\, \partial_\nu \psi) - \partial^3_{\nu\mu\mu}\varphi\, \partial_\nu\psi \\ &= \operatorname{div}(\Delta\varphi\, \nabla\psi) - \Big\langle \nabla\Delta\varphi, \nabla\psi\Big\rangle,\end{aligned}$$

and since in this case it is $\nabla\Delta\varphi = \Delta\nabla\varphi$, we see that the two expressions differ by a divergence term:

$$\Big\langle \Delta\varphi, \Delta\psi \Big\rangle - \Big\langle \nabla^2\varphi, \nabla^2\psi \Big\rangle = \operatorname{div}\Big(\Delta\varphi\nabla\psi - \nabla^2\varphi \cdot \nabla\psi\Big).$$

In our case however the gradient and the Laplacian will commute up to a curvature term (a consequence of Weizenböck formula) but this will be compensated by the fact that the variation of the area elemtent is nonzero.

Proof of Proposition 3.2.3. Recall first that, for surfaces, the Ricci tensor is simply given by $\mathrm{Ric} = \frac{S}{2} g = Kg$. Thus, from computations of Lemma 3.2.2, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= 2\Big\langle A, \nabla^2 X\Big\rangle + 8\Big\langle (\overline{\nabla} H)^\top, \overline{\nabla} X\Big\rangle + 4K\Big\langle d\Phi, \overline{\nabla} X\Big\rangle,\\
\frac{\partial}{\partial t}|H|^2 &= \Big\langle H, \Delta X\Big\rangle + 2\Big\langle (\overline{\nabla} H)^\top, \overline{\nabla} X\Big\rangle,\\
\frac{\partial}{\partial t} d\sigma &= \Big\langle d\Phi, \overline{\nabla} X\Big\rangle d\sigma,
\end{aligned}$$

and thus we have

$$\frac{\partial}{\partial t}(2K) = \frac{\partial}{\partial t}\Big(4|H|^2 - |A|^2\Big) = 4\Big\langle H, \Delta X\Big\rangle - 2\Big\langle A, \nabla^2 X\Big\rangle - 4K\Big\langle d\Phi, \overline{\nabla} X\Big\rangle,$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(2K d\sigma\Big) &= \Big(4\Big\langle H, \Delta X\Big\rangle - 2\Big\langle A, \nabla^2 X\Big\rangle - 2K\Big\langle d\Phi, \overline{\nabla} X\Big\rangle\Big) d\sigma\\
&= 2\Big(\Big\langle \Delta\Phi, \Delta X\Big\rangle - \Big\langle \nabla^2\Phi, \nabla^2 X\Big\rangle - K\Big\langle d\Phi, \overline{\nabla} X\Big\rangle\Big) d\sigma.
\end{aligned}$$

Now, from the definition of adjoint we have

$$\Big\langle \Delta\Phi, \Delta X\Big\rangle = \Big\langle \Delta\Phi, \overline{\nabla}^*\overline{\nabla} X\Big\rangle = d^*\Big\langle \Delta\Phi, \overline{\nabla} X\Big\rangle - \Big\langle \overline{\nabla}\Delta\Phi, \overline{\nabla} X\Big\rangle$$

and similarly, by working on the bundle $\Phi^\star(T\mathbb{R}^3) \otimes T^*\Sigma$ with induced connection and Laplace operator,

$$\begin{aligned}
\Big\langle \nabla^2\Phi, \nabla^2 X\Big\rangle = \Big\langle \nabla(d\Phi), \nabla(\overline{\nabla} X)\Big\rangle &= d^*\Big\langle \nabla(d\Phi), \overline{\nabla} X\Big\rangle - \Big\langle \nabla^*\nabla(d\Phi), \overline{\nabla} X\Big\rangle\\
&= d^*\Big\langle \nabla^2\Phi, \overline{\nabla} X\Big\rangle - \Big\langle \Delta(d\Phi), \overline{\nabla} X\Big\rangle.
\end{aligned}$$

(recall that here it is $\langle \nabla(d\Phi), \overline{\nabla} X\Big\rangle = g^{\mu\nu}\langle \nabla_{\partial_\mu}(d\Phi), \overline{\nabla}_{\partial_\nu} X\rangle = g^{\mu\nu}\langle \nabla^2_{\mu,\cdot}\Phi, \overline{\nabla}_{\partial_\nu} X\rangle$).

To understand how the two vector-valued 1–forms $\overline{\nabla}\Delta\Phi$ and $\Delta(d\Phi)$ are related, it it suffices to consider $\Phi = (\Phi^1, \ldots, \Phi^n)$ as n–tuple of functions, on which the differential operators in question act component-wise. We can consequently reduce to the familiar case of differential forms on Σ, where, with the usual small abuse of notation,

$$\begin{aligned}
\Delta(d\Phi) &= \nabla^*\nabla(d\Phi) = \Delta_C(d\Phi),\\
\overline{\nabla}\Delta\Phi &= d(\nabla^* d\Phi) = d(d^* d\Phi) = \Delta_H d\Phi,
\end{aligned}$$

where Δ_C and Δ_H denote the connection and Hodge Laplacian respectively and we also used the fact that $\nabla^* = d^*$ when restricted to forms. Hence, by the Weizenböck formula, with our sign conventions we have

$$\overline{\nabla}\Delta\Phi = \Delta(d\Phi) - \mathrm{Ric}(d\Phi) = \Delta(d\Phi) - K d\Phi.$$

We conclude that

$$\Big\langle \Delta\Phi, \Delta X\Big\rangle - \Big\langle \nabla^2\Phi, \nabla^2 X\Big\rangle = d^*\Big(\Big\langle \Delta\Phi, \overline{\nabla} X\Big\rangle - \Big\langle \nabla^2\Phi, \overline{\nabla} X\Big\rangle\Big) + K\Big\langle d\Phi, \overline{\nabla} X\Big\rangle,$$

and consequently that

$$\frac{\partial}{\partial t}(K d\sigma) = d^*\Big(\Big\langle \Delta\Phi, \overline{\nabla} X\Big\rangle - \Big\langle \nabla^2\Phi, \overline{\nabla} X\Big\rangle\Big) d\sigma = d^*\Big(2\Big\langle H, \overline{\nabla} X\Big\rangle - \Big\langle A, \overline{\nabla} X\Big\rangle\Big) d\sigma.$$

Finally if we replace A with $A = A^\circ + Hg$, since $\langle Hg, \overline{\nabla} X\rangle = \langle H, \overline{\nabla} X\rangle$, we deduce the validity of the claimed formulas. □

3.3 The Willmore Operator in Divergence Form

We present some details about the Willmore operator.

3.3.1 First, a remark about differential geometry. If $\mathcal{M}$ is a manifold immersed in $\mathbb{R}^n$, there is a fundamental difference in the covariant differentiation, in the ambient space, of vector fields and differential forms, namely if $X, V \in \mathfrak{X}(\mathcal{M})$ and $\omega \in \Omega^1(\mathcal{M})$, then

$$\begin{aligned}
&\overline{\nabla}_X V = \nabla_X V + A(X, V),\\
&\overline{\nabla}_X \omega = \nabla_X \omega,
\end{aligned}$$

where A is the second fundamental form of $\mathcal{M}$ and $\overline{\nabla}$ the Euclidean covariant derivative in $\mathbb{R}^n$. Consequently, the musical isomorphisms on $\mathcal{M}$ do *not* commute with the ambient covariant derivative.

An obvious example is the following. Denote by Φ the map defining the immersion of $\mathcal{M}$ into $\mathbb{R}^n$. If we take the (flat) metric $g = g_{\mathbb{R}^n}$ restricted to $\mathcal{M}$, for which it holds $\overline{\nabla} g = \nabla g \equiv 0$, and we raise the first index we obtain the identity map over $\mathcal{M}$:

$$g^\sharp = \partial_\mu \otimes dx^\mu \simeq \partial_\mu \Phi \otimes dx^\mu = d\Phi,$$

we have

$$\overline{\nabla}(g^\sharp) = \overline{\nabla} d\Phi = \nabla^2 \Phi = A.$$

More generally, in the same setting as above, we will use the following.

Lemma 3.3.1. *Let $B = B_{\mu\nu} dx^\mu \otimes dx^\nu \in \mathcal{T}^0_2(\mathcal{M})$ be a 2-covariant tensor and let*

$$B^\sharp = g^{\mu\alpha} B_{\alpha\nu}\, \partial_\mu \otimes dx^\nu \simeq g^{\mu\alpha} B_{\alpha\nu}\, \partial_\mu \Phi \otimes dx^\nu$$

be the tensor obtained raising the first index. Then the ambient covariant derivative of $B^\sharp$ is as follows:

$$\overline{\nabla}_X (B^\sharp)(V) = (\nabla_X B)^\sharp(V) + A(X, B^\sharp(V)),$$

for every $X, V \in \mathfrak{X}(M)$.

Proof. Write $B^\sharp = B^\mu_\nu\, \partial_\mu \otimes dx^\nu \simeq B^\mu_\nu\, \partial_\mu \Phi \otimes dx^\nu$ and compute in local coordinates:

$$\begin{aligned}
\overline{\nabla}_X (B^\sharp) &= X(B^\mu_\nu) \partial_\mu \Phi \otimes dx^\nu + B^\mu_\nu \Big(\nabla_X \partial_\mu + A(X, \partial_\mu) \Big) \otimes dx^\nu + B^\mu_\nu \partial_\mu \Phi \otimes \nabla_X dx^\nu \\
&= \nabla_X (B^\sharp) + B^\mu_\nu A(X, \partial_\mu) \otimes dx^\nu \\
&= \nabla_X (B^\sharp) + A(X, B^\mu_\nu \partial_\mu) \otimes dx^\nu,
\end{aligned}$$

and conclude using the fact that index raising commutes with the covariant derivative on $\mathcal{M}$. $\square$

3.3.2

Proposition 3.3.2. *With the same hypotesis as in Proposition 3.2.2, when $\mathcal{M} = \Sigma$ is a surface and $\mathcal{N} = \mathbb{R}^n$, we have*

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(|A^\circ|^2\, d\sigma\Big) = (\overline{\nabla}^* w)\, d\sigma + d^*\Big(\langle A^\circ, \overline{\nabla} X\rangle - \langle w, X\rangle\Big)\, d\sigma, \tag{3.3.1}$$

w is the $T\mathbb{R}^n$-valued 1-form written in any of these equivalent ways:

$$\begin{aligned} w &= \overline{\nabla} H - 2(\overline{\nabla} H)^\top - |H|^2 d\Phi \\ &= \overline{\nabla} H + \langle H, A\rangle^\sharp + \langle H, A^\circ\rangle^\sharp && (3.3.2)\\ &= \nabla^\perp H + \langle H, A^\circ\rangle^\sharp && (3.3.3)\\ &= (\nabla^\perp)^* A^\circ + \langle H, A^\circ\rangle^\sharp. && (3.3.4)\end{aligned}$$

Proof. From Proposition 3.2.2, integration by parts yields

$$\begin{aligned}\frac{\partial}{\partial t}\Big(|H|^2 d\sigma\Big) &= \Big(\Big\langle H, \Delta X\Big\rangle + 2\Big\langle (\overline{\nabla} H)^\top, \overline{\nabla} X\Big\rangle + |H|^2\Big\langle d\varphi, \overline{\nabla} X\rangle\Big) d\sigma \\ &= \Big\langle \overline{\nabla}^*\Big(\overline{\nabla} H - 2(\overline{\nabla} H)^\top - |H|^2 d\varphi\Big), X\Big\rangle d\sigma \\ &\quad + d^*\Big(\Big\langle H, \overline{\nabla} X\Big\rangle - \Big\langle \overline{\nabla} H - 2(\overline{\nabla} H)^\top - |H|^2 d\varphi, X\Big\rangle\Big) d\sigma,\end{aligned}$$

and since $\frac{1}{2}|A^\circ|^2 = |H|^2 - K$, with Proposition 3.2.3 we deduce (3.3.1). To deduce the other equivalent expression for w, we claim that

$$\begin{aligned} (\overline{\nabla} H)^\top &= -\langle H, A\rangle^\sharp, && (3.3.5)\\ |H|^2 d\Phi &= \langle H, Hg\rangle^\sharp, && (3.3.6)\\ (\overline{\nabla} H)^\top + |H|^2 d\Phi &= -\langle H, A^\circ\rangle^\sharp. && (3.3.7)\end{aligned}$$

Indeed, since H is a normal vector, we have

$$\begin{aligned}(\overline{\nabla} H)^\top &= g^{\mu\nu}\langle \overline{\nabla} H, \partial_\nu \Phi\rangle \partial_\mu \Phi \\ &= -g^{\mu\nu}\langle H, \overline{\nabla}\partial_\nu\Phi\rangle\partial_\mu\Phi \\ &= -g^{\mu\nu}\langle H, A(\cdot, \partial_\nu)\rangle\partial_\mu\Phi \\ &= -g^{\mu\nu}\langle H, A_{\alpha\nu}\rangle\partial_\mu\Phi \otimes dx^\alpha,\end{aligned}$$

which is (3.3.5). Similarly for (3.3.6) since

$$|H|^2 d\Phi = \langle H, H\rangle\partial_\mu\Phi \otimes \partial x^\mu = g^{\mu\nu}\langle H, Hg_{\alpha\nu}\rangle\partial_\mu\Phi \otimes dx^\alpha,$$

and (3.3.7) is obtained adding (3.3.5) to (3.3.6). With these identites, decomposing $\overline{\nabla} H = (\overline{\nabla} H)^\top + \nabla^\perp H$ gives (3.3.2) and (3.3.3), while (3.3.4) is obtained in turn since Codazzi's equation implies

$$\nabla^\perp H = (\nabla^\perp)^* A^\circ = \frac{1}{2}(\nabla^\perp)^* A.$$

□

Remark 3.3.3 We may equivalently have used $|A^\circ| = |A|^2 - 2|H|^2$ and

$$\begin{aligned}\frac{1}{4}\frac{\partial}{\partial t}\Big(|A|^2 d\sigma\Big) &= \frac{1}{2}\bigg(\Big\langle A, \nabla^2 X\Big\rangle + 4\Big\langle(\overline{\nabla}H)^\top, \overline{\nabla}X\Big\rangle + \bigg(2K + \frac{|A|^2}{2}\bigg)\Big\langle d\varphi, \overline{\nabla}X\Big\rangle\bigg)d\sigma \\ &= \Big\langle \overline{\nabla}^*\Big(\overline{\nabla}H - 2(\overline{\nabla}H)^\top - |H|^2 d\varphi\Big), X\Big\rangle d\sigma \\ &\quad + d^*\bigg(\frac{1}{2}\Big\langle A, \overline{\nabla}X\Big\rangle - \Big\langle \overline{\nabla}H - 2(\overline{\nabla}H)^\top - |H|^2 d\varphi, X\Big\rangle\bigg)d\sigma,\end{aligned}$$

to deduce, since $K = 2|H|^2 - \frac{1}{2}|A|^2$,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\Big(|A^\circ|^2 d\sigma\Big) &= \Big(\Big\langle H, \Delta X\Big\rangle + 2\Big\langle(\overline{\nabla}H)^\top, \overline{\nabla}X\Big\rangle + |H|^2\Big\langle d\varphi, \overline{\nabla}X\rangle\Big)d\sigma \\ &\quad - d^*\Big(\Big\langle H, \overline{\nabla}X\Big\rangle - \Big\langle A^\circ, \overline{\nabla}X\Big\rangle\Big)d\sigma \\ &= \Big\langle \overline{\nabla}^*\Big(\overline{\nabla}H - 2(\overline{\nabla}H)^\top - |H|^2 d\varphi\Big), X\Big\rangle d\sigma \\ &\quad + d^*\Big(\Big\langle A^\circ, \overline{\nabla}X\Big\rangle - \Big\langle \overline{\nabla}H - 2(\overline{\nabla}H)^\top - |H|^2 d\varphi, X\Big\rangle\Big)d\sigma.\end{aligned}$$

and deduce in this way (3.3.1).

From Proposition 3.3.2 we deduce an expression, in divergence form, of the Willmore operator; the following shows its equivalence with the classical one.

Proposition 3.3.4. *For an immersion $\Phi : \Sigma \to \mathbb{R}^n$, its Willmore operator may be expressed equivalently as*

$$\delta\mathcal{W} = \overline{\nabla}^* w = \Delta^\perp H + Q(A^\circ)H, \tag{3.3.8}$$

where $\Delta^\perp$ denotes the Laplace operator on the normal bundle of $\Phi(\Sigma)$ and

$$Q(A^\circ)H = \Big\langle A^\circ, \langle H, A^\circ\rangle\Big\rangle = g^{\mu\sigma}g^{\nu\tau}\Big\langle A^\circ_{\mu\nu}, \langle A^\circ_{\sigma\tau}, H\rangle\Big\rangle.$$

Proof. We see that

$$\begin{aligned}\overline{\nabla}^*(\nabla^\perp H) &= \Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\top + \Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\perp \\ &= \Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\top + (\nabla^\perp)^*(\nabla^\perp H) \\ &= \Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\top + \Delta^\perp H.\end{aligned}$$

We claim that

$$\Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\top = -\langle \overline{\nabla}H, A\rangle^\sharp. \tag{3.3.9}$$

Indeed, since $\nabla^\perp H$ is normal-valued, we have

$$\begin{aligned}\Big(\overline{\nabla}^*(\nabla^\perp H)\Big)^\top &= g^{\mu\nu}\langle \overline{\nabla}^*\nabla^\perp H, \partial_\nu\Phi\rangle\partial_\mu\Phi \\ &= -g^{\mu\nu}\langle \nabla^\perp H, \overline{\nabla}\partial_\nu\Phi\rangle\partial_\mu\Phi \\ &= -g^{\mu\nu}\langle \nabla^\perp H, A(\cdot, \partial_\nu)\rangle\partial_\mu\Phi,\end{aligned}$$

that gives (3.3.9). So we deduced:

$$\overline{\nabla}^*(\overline{\nabla}^\perp H) = \Delta^\perp H - \langle \overline{\nabla} H, A\rangle^\sharp. \tag{3.3.10}$$

Next, from Lemma 3.3.1 and Codazzi's equation we have

$$\begin{aligned}
\overline{\nabla}^*(\langle H, A^\circ\rangle^\sharp) &= (\overline{\nabla}^*\langle H, A^\circ\rangle)^\sharp + \langle A, \langle H, A^\circ\rangle\rangle \\
&= \Big(\langle \overline{\nabla} H, A^\circ\rangle + \langle H, \overline{\nabla}^* A^\circ\rangle\Big)^\sharp + \langle A, \langle H, A^\circ\rangle\rangle \\
&= \Big(\langle \overline{\nabla} H, A^\circ\rangle + \langle H, \overline{\nabla} H\rangle\Big)^\sharp + \langle A, \langle H, A^\circ\rangle\rangle \\
&= \langle \overline{\nabla} H, A\rangle^\sharp + \langle A, \langle H, A^\circ\rangle\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\overline{\nabla}^* w &= \overline{\nabla}^*\Big(\nabla^\perp H + \langle H, A^\circ\rangle^\sharp\Big) \\
&= \Delta^\perp H - \langle \overline{\nabla} H, A\rangle^\sharp + \langle \overline{\nabla} H, A\rangle^\sharp + \langle A, \langle H, A^\circ\rangle\rangle \\
&= \Delta^\perp H + \langle A, \langle H, A^\circ\rangle\rangle;
\end{aligned}$$

however, it is

$$\langle A, \langle H, A^\circ\rangle\rangle = \langle A^\circ, \langle H, A^\circ\rangle\rangle = Q(A^\circ)H, \tag{3.3.11}$$

since, being A° trace-free, there holds

$$\langle Hg, \langle H, A^\circ\rangle\rangle = Hg^{\mu\sigma}g^{\nu\tau}g_{\mu\nu}\langle H, A^\circ_{\sigma\tau}\rangle = Hg^{\sigma\tau}\langle H, A^\circ_{\sigma\tau}\rangle = 0.$$

So $\overline{\nabla}^* w$ is equivalent to the classical form of the Willmore operator. □

3.3.3 For future use, we also compute explicitly how the Willmore operator compares to the Bilaplacian $\Delta^2\Phi = 2\Delta_g H$.

Lemma 3.3.5. *The Bilaplacian operator of an immersion $\Phi : \Sigma \to \mathbb{R}^n$ decomposes orthogonally as follows:*

$$\begin{aligned}
\frac{1}{2}\Delta_g^2\Phi = \Delta_g H &= (\Delta_g H)^\perp + (\Delta_g H)^\top \\
&= \Delta^\perp H - \langle A, \langle H, A\rangle\rangle - 2\langle \overline{\nabla} H, A\rangle^\sharp - \langle \overline{\nabla} H, H\rangle^\sharp.
\end{aligned} \tag{3.3.12}$$

Proof. Orthogonal decomposition gives

$$\Delta_g H = \overline{\nabla}^*\overline{\nabla} H = \overline{\nabla}^*(\overline{\nabla} H)^\top + \overline{\nabla}^*(\nabla^\perp H);$$

with (3.3.5), Lemma 3.3.1 and Codazzi we deduce

$$\begin{aligned}
\overline{\nabla}^*(\overline{\nabla} H)^\top &= -\overline{\nabla}^*(\langle H, A\rangle^\sharp) \\
&= -\Big(\langle \overline{\nabla} H, A\rangle + \langle H, \overline{\nabla}^* A\rangle\Big)^\sharp - \langle A, \langle H, A\rangle\rangle \\
&= -\Big(\langle \overline{\nabla} H, A\rangle + \langle H, \overline{\nabla} H\rangle\Big)^\sharp - \langle A, \langle H, A\rangle\rangle
\end{aligned}$$

so together with (3.3.10) we get (3.3.12). □

From (3.3.12) it is also clear that the highest-order term in the expression is $\Delta^\perp H$. In a similar fashion, if we want to write $\delta\mathcal{W}$ as full bilaplacian $\Delta_g^2\Phi$ plus lower order terms, we have the following.

Lemma 3.3.6. *For an immersion $\Phi : \Sigma \to \mathbb{R}^n$ there holds*

$$\delta\mathcal{W} = \frac{1}{2}\Delta_g^2\Phi + \overline{\nabla}^*\Big(\langle H, A^\circ\rangle^\sharp + \langle H, A\rangle^\sharp\Big) \tag{3.3.13}$$

$$= \frac{1}{2}\Delta_g^2\Phi + \langle\overline{\nabla}H, H\rangle^\sharp + 2\langle\overline{\nabla}H, A\rangle^\sharp + \langle A^\circ, \langle H, A^\circ\rangle\rangle + \langle A, \langle H, A\rangle\rangle. \tag{3.3.14}$$

Proof. Identity (3.3.13) line is immediate from (3.3.2). Using (3.3.5), (3.3.6), (3.3.7), (3.3.11), Lemma 3.3.1 and Codazzi, we get

$$\begin{aligned}\delta\mathcal{W} &= \frac{1}{2}\Delta_g^2\Phi + \overline{\nabla}^*\Big(\langle H, A^\circ\rangle^\sharp + \langle H, A\rangle^\sharp\Big)\\ &= \frac{1}{2}\Delta_g^2\Phi + \Big(\langle\overline{\nabla}H, A^\circ\rangle + \langle H, \overline{\nabla}^*A^\circ\rangle\Big)^\sharp + \langle A, \langle H, A^\circ\rangle\rangle\\ &\qquad + \Big(\langle\overline{\nabla}H, A\rangle + \langle H, \overline{\nabla}^*A\rangle\Big)^\sharp + \langle A, \langle H, A\rangle\rangle,\\ &= \frac{1}{2}\Delta_g^2\Phi + \Big(\langle\overline{\nabla}H, A^\circ\rangle + 2\langle H, \overline{\nabla}H\rangle + \langle\overline{\nabla}H, A\rangle\Big)^\sharp + \langle A^\circ, \langle H, A^\circ\rangle\rangle + \langle A, \langle H, A\rangle\rangle.\end{aligned}$$

Since

$$\langle\overline{\nabla}H, A^\circ\rangle + 2\langle H, \overline{\nabla}H\rangle + \langle\overline{\nabla}H, A\rangle = \langle\overline{\nabla}H, H\rangle + 2\langle\overline{\nabla}H, A\rangle,$$

we get (3.3.14). □

Finally we also point out the following.

Lemma 3.3.7. *Let g_0 be a fixed reference metric on Σ and let $\Phi : \Sigma \to \mathbb{R}^n$ be a conformal immersion with conformal factor e^λ. Then*

$$\Delta_g^2\Phi = \mathrm{e}^{-4\lambda}\Big(\Delta_{S^2}^2\Phi - 4\langle d\lambda, \overline{\nabla}\Delta_{g_0}\Phi\rangle_{g_0} + (4|d\lambda|_{g_0}^2 - 2\Delta_{g_0}\lambda)\Delta_{g_0}\Phi\Big), \tag{3.3.15}$$

and likewise

$$\Delta_{g_0}^2\Phi = \mathrm{e}^{4\lambda}\Big(\Delta_g^2\Phi + 4\langle d\lambda, \overline{\nabla}\Delta_g\Phi\rangle_g + (4|d\lambda|_g^2 + 2\Delta_g\lambda)\Delta_g\Phi\Big).$$

The proof is elementary: it suffices to expand the identities $\Delta_g^2\Phi = \mathrm{e}^{-2\lambda}\Delta_{g_0}(\mathrm{e}^{-2\lambda}\Delta_{g_0}\Phi))$ and $\Delta_{g_0}^2\Phi = \mathrm{e}^{2\lambda}\Delta_{g_0}(\mathrm{e}^{2\lambda}\Delta_{g_0}\Phi))$.

3.4 Willmore Energy and Conservation Laws

In this section and the following, we revisit some of the work done in [Riv08], [Riv16], [Ber16] with particular regard to surfaces that that are not necessarily Willmore.

We shall work henceforth in codimension 1, i.e. $n = 3$, The assumption greatly simplifies the calculations, although is it not essential.

Proposition 3.4.1. *Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion with induced metric $g = \Phi^\star g_{\mathbb{R}^3} = \Phi^\star\langle\cdot,\cdot\rangle$ and let $\delta\mathcal{W}$ be the Willmore Euler-Lagrange operator applied to Φ. Let $w \in \Gamma(\Phi^\star(T\mathbb{R}^3) \otimes T^*\Sigma)$ be the vector-valued form along Φ defined in Proposition 3.3.2, namely*

$$w = \overline{\nabla}H - 2(\overline{\nabla}H)^\top - |H|^2 d\Phi = \overline{\nabla}H + \langle H, A^\circ\rangle^\sharp + \langle H, A\rangle^\sharp.$$

Then the following formulas hold true:

$$\delta\mathcal{W} = \overline{\nabla}^* w, \tag{3.4.1}$$

$$\Phi \times \delta\mathcal{W} = \overline{\nabla}^*\Big(- d\Phi \times A^\circ + \Phi \times w\Big), \tag{3.4.2}$$

$$\Big\langle \Phi, \delta\mathcal{W}\Big\rangle = d^*\Big\langle \Phi, w\Big\rangle, \tag{3.4.3}$$

$$\Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\Big\langle \Phi, \delta\mathcal{W}\Big\rangle = \overline{\nabla}^*\Big(- 2\Phi \times (d\Phi \times A^\circ) + \Phi \times (\Phi \times w) + \Phi\langle \Phi, w\rangle\Big). \tag{3.4.4}$$

Proof. Since the Lagrangian

$$\frac{1}{2}|A^\circ|^2 d\sigma_g$$

is point-wise invariant under conformal transformation on the target (see Chen [Che74](, if we consider family of diffeomorphisms (onto their image) $F : I \times \Omega \to \mathbb{R}^3$, with $I \times \Omega \subseteq \mathbb{R} \times \mathbb{R}^3$ that are conformal at every time and so that $F(0,\cdot) = \mathrm{id}_{\mathbb{R}^3}$, the Willmore Lagrangian of the family

$$\Phi(t,x) = F(t,\Phi(x))$$

will be constant in time. Thus letting

$$\mathcal{X}(y) = \frac{\partial}{\partial t}F(t,y)\bigg|_{t=0}$$

be the associated vector field and

$$X(t,x) = \partial_t\Phi(t,x) = \mathcal{X}(\Phi(t,x))$$

be the velocity of the family, from Proposition 3.3.2 it will follow that

$$\Big\langle \delta\mathcal{W}, X\Big\rangle + d^*\Big(\Big\langle A^\circ, \overline{\nabla}X\Big\rangle - \Big\langle w, X\Big\rangle\Big) = 0. \tag{3.4.5}$$

Invariance by Translations. For every fixed vector $v \in \mathbb{R}^3$, we set $F(t,y) = y + sv$, so $\mathcal{X}(y) \equiv v$ and $X(x) \equiv v$. Thus (3.4.5) gives immediately (3.4.1). As expected this conservation law is equivalent to the Euler-Lagrange equation.

Invariance by Rotations. If v is a unit vector in $\mathbb{R}^3$ and $R(v,\vartheta)$ is the counter-clockwise rotation of angle ϑ in direction v, we set $F(t,y) = R(v,t)(y)$, so $\mathcal{X}(y) = v \times y$ and $X = v \times \Phi$. Thus (3.4.5) gives

$$\Big\langle \delta\mathcal{W}, v \times \Phi\Big\rangle + d^*\Big(\Big\langle A^\circ, v \times d\Phi\Big\rangle - \Big\langle w, v \times \Phi\Big\rangle\Big) = 0,$$

and hence, with the triple product rule and since v is arbitrary, (3.4.2) follows.

Invariance by Dilations. We set $F(s,y) = \mathrm{e}^s y$, so $\mathcal{X} = \mathrm{id}_{\mathbb{R}^n}$ and $X = \Phi$. Thus, since A° is a normal-valued, (3.4.5) gives (3.4.3).

Invariance by Inversions. Let $\mathcal{I}(y) = y/|y|^2$ be the inversion with respect to the unit sphere and let $\tau_v(y) = y + v$ be the translation of vector $v \in \mathbb{R}^n$. Then the composition

$$f_v(y) = \mathcal{I} \circ \tau_v \circ \mathcal{I} = \frac{y + v|y|^2}{|y + |y|v|^2}|x|^2 = \frac{y + v|y|^2}{1 + |y|^2|v|^2 + 2\langle y, v\rangle}$$

is smooth away from $-v/|v|^2$ and satisfies $f_v \circ f_w = f_{v+w}$. Then for each unit vector in $v \in \mathbb{R}^3$ the map $F : (-s_0, s_0) \times B_{1/s_0}(0) \to \mathbb{R}^n$ given by

$$F(s,y) = f_{sv}(y) = \frac{y + sv|y|^2}{1 + s^2|y|^2 + 2s\langle y, v\rangle}.$$

is a family of diffeomorphisms, and is in fact the local flow associated to the vector field $\mathcal{X}(y) = v|y|^2 - 2y\langle y, v\rangle$. Then

$$X = v|\Phi|^2 - 2\Phi\langle \Phi, v\rangle = \Phi \times (v \times \Phi) - \Phi\langle \Phi, v\rangle.$$

Thus computing with the triple product rule,

$$\begin{aligned}
\Big\langle \delta\mathcal{W}, X \Big\rangle &= \Big\langle \delta\mathcal{W}, \Phi \times (v \times \Phi)\Big\rangle - \Big\langle \delta\mathcal{W}, \Phi\langle \Phi, v\rangle\Big\rangle \\
&= \Big\langle v \times \Phi, \delta\mathcal{W} \times \Phi\Big\rangle - \Big\langle \delta\mathcal{W}, \Phi\langle \Phi, v\rangle\Big\rangle \\
&= \Big\langle v, \Phi \times (\delta\mathcal{W} \times \Phi)\Big\rangle - \Big\langle \delta\mathcal{W}, \Phi\langle \Phi, v\rangle\Big\rangle \\
&= \Big\langle -\Phi \times (\Phi \times \delta\mathcal{W}) - \Phi\langle \Phi, \delta\mathcal{W}\rangle, v\Big\rangle,
\end{aligned}$$

similarly we have

$$\Big\langle w, X \Big\rangle = \Big\langle -\Phi \times (\Phi \times w) - \Phi\langle \Phi, w\rangle, v\Big\rangle,$$

and finally since

$$\overline{\nabla} X = 2\Big(\langle d\Phi, \Phi\rangle v - d\Phi\langle \Phi, v\rangle - \Phi\langle d\Phi, v\rangle\Big) = 2\Big(\Phi \times (v \times d\Phi) - \Phi\langle d\Phi, v\rangle\Big)$$

with a similar computation we have

$$\begin{aligned}
\Big\langle A^\circ, \overline{\nabla} X\Big\rangle &= 2\Big\langle A^\circ, \Phi \times (v \times d\Phi) - \Phi\langle d\Phi, v\rangle\Big\rangle \\
&= 2\Big\langle -d\Phi \times (\Phi \times A^\circ) - \Big\langle d\Phi, \langle \Phi, A^\circ\rangle\Big\rangle, v\Big\rangle \\
&= 2\Big\langle -d\Phi \times (\Phi \times A^\circ) - A^\circ \times (d\Phi \times \Phi), v\Big\rangle \\
&= 2\Big\langle \Phi \times (A^\circ \times d\Phi), v\Big\rangle = -2\Big\langle \Phi \times (d\Phi \times A^\circ), v\Big\rangle,
\end{aligned}$$

where we used that $\Big\langle d\Phi, \langle \Phi, A^\circ\rangle\Big\rangle = A^\circ \times (d\Phi \times \Phi)$ since A° is normal valued and Jacobi identity for the vector product. Hence (3.4.5) gives

$$\begin{aligned}
&-\Big\langle \Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\langle \Phi, \delta\mathcal{W}\rangle, v\Big\rangle \\
&+ d^*\Big(-2\Big\langle \Phi \times (d\Phi \times A^\circ), v\Big\rangle + \Big\langle \Phi \times (\Phi \times w) + \Phi\langle \Phi, w\rangle, v\Big\rangle = 0,
\end{aligned}$$

and since v is an arbitrary unit vector this is equivalent to (3.4.4). □

Remark 3.4.2 Note that (3.4.4) can be deduced from (3.4.2) and (3.4.3), but for instance (3.4.3) does not seem directly deducible directly using only (3.4.2) and (3.4.4). Thus there seems to be some sort of "hierarchy" between these equations.

The equations in Proposition 3.4.1 involve the tracefree second fundamental form. We obtain an equivalent set of equations, analogous to the one in [Riv08], [Ber16], involving the mean curvature instead.

Corollary 3.4.3. *In the same hypothesis of Proposition 3.4.1, the following formulas hold true:*

$$\begin{aligned}
\delta\mathcal{W} &= \overline{\nabla}^* w, \\
\Phi \times \delta\mathcal{W} &= \overline{\nabla}^*\Big(- d\Phi \times H + \Phi \times w\Big), \\
\Big\langle \Phi, \delta\mathcal{W}\Big\rangle &= d^*\Big\langle \Phi, w\Big\rangle, \\
\Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\Big\langle \Phi, \delta\mathcal{W}\Big\rangle + 4H &= \overline{\nabla}^*\Big(- 2\Phi \times (d\Phi \times H) + \Phi \times (\Phi \times w) + \Phi\langle \Phi, w\rangle\Big).
\end{aligned}$$

Proof. Since the Gauss curvature density is point-wise invariant by rotations, applying the same procedure as in the proposition, from Proposition 3.2.3 we get the formula

$$\overline{\nabla}^*\Big(d\Phi \times H - d\Phi \times A^\circ\Big) = 0, \tag{3.4.8}$$

from which one computes that

$$-2H = \overline{\nabla}^*\Big(\Phi \times (d\Phi \times H) - \Phi \times (d\Phi \times A^\circ)\Big). \tag{3.4.9}$$

By subtracting (3.4.8) from (3.4.2) and (3.4.9) from (3.4.4) we get the result. □

4 Elliptic Estimates for the Inhomogeneous Willmore Equation

Summary: This chapter presentes an ellpitic regularity result for the inhomogeneous Willmore equation. For a conformal, Lipschitz $W^{2,2}$ immersion with distributional Willmore operator in L^p, $1 < p < \infty$, it is proven that such immersion is locally $W^{4,p}$, just as is expected for an elliptic problem. A quantitative estimate is also given for the case $p = 2$. This theorem is a generalization of the classical result of Rivière for the regularity of weak Willmore surfaces, and the proof follows, essentially, similar ideas.

4.1 Introduction

It is often the case that, when dealing with PDE related to a variational problem, one is led to study the associated inhomogeneous equation. If the PDE in question is elliptic, then one expects to gain as many degrees of information from the inhomogeneous datum as is the order of the equation, similarly as in the classical study of the Laplace operator and the Poisson equation.

For instance, considering

$$-\Delta u = f \quad \text{in } B_1,$$

one wants to deduce information on u from f. By the classical elliptic estimates, if for instance $f \in L^2(B_1)$ then, regardless of the initial space where u is (e.g. $W^{1,2}$, or even only a distribution), there holds $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ with a quantitative estimate

$$\|u\|_{W^{2,2}(B_{1/2})} \leq C\Big(\|f\|_{L^2(B_1)} + \|u\|_{L^2(B_1)}\Big),$$

holding for a constant C independent of u and f.

For nonlinear elliptic equations, success or failure of this gain depend on the particular nature of the nonlinearities in relation to the background initial function space and, if one also needs quantitative estimates, on suitable "smallness" of the energy associated with the PDE (or a related one).

In the present situation, we are concerned with the Willmore equation, a nonlinear, 4th order elliptic PDE, and motivated by the study of the Willmore flow in the next chapter, we are interested in deducing local regularity properties of a weak conformal immersion from that of its Willmore operator.

To this aim we recall the basic definitions.

Definition 4.1.1. *Let B_1 be the unit disk in $\mathbb{R}^2$ with background Euclidean metric $g_{\mathbb{R}^2}$. The class of* Lipschitz $W^{2,2}$ immersions, *or* weak immersions *for short, is*

$$\mathscr{E}(B_1, \mathbb{R}^n) = (W^{2,2} \cap W^{1,\infty}_{imm})(B_1, \mathbb{R}^n),$$

namely, Φ belongs to $\mathscr{E}(B_1, \mathbb{R}^n)$ if and only if it is $W^{2,2}$ and there exists $C > 0$ so that a.e. in B_1 there holds, in the sense of metrics,

$$\frac{1}{C} g_{\mathbb{R}^2} \leq g \leq C g_{\mathbb{R}^2}.$$

The Willmore operator *of $\Phi \in \mathscr{E}(B_1, \mathbb{R}^n)$ is defined as the vector-valued, distribution-valued two form on B_1 given by*

$$\Big(\delta\mathcal{W}(\Phi) d\sigma_g, \varphi\Big)_{\mathcal{D}'} = \Big(\overline{\nabla}^{*_g}\Big(\overline{\nabla} H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A^\circ, H\rangle^{\sharp_g}\Big) d\sigma_g, \varphi\Big)_{\mathcal{D}'},$$

for every $\varphi \in C^\infty(B_1, \mathbb{R}^n)$ or, equivalently, as

$$\Big(\delta\mathcal{W} d\sigma_g, \varphi\Big)_{\mathcal{D}'} = \int_{B_1} \Big(\langle H, \Delta_g \varphi\rangle - \Big\langle \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}, \nabla\varphi\Big\rangle_g\Big) d\sigma_g.$$

This chapter is devoted to prove the following regularity result.

Theorem 4.1.2. *Let $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ be conformal with conformal factor e^λ and Willmore operator $\delta\mathcal{W}$ in $L^p(B_1)$ for some $1 < p < \infty$. Then $\Phi \in W^{4,p}_{loc}(B_1)$, and furthermore when $p = 2$ if $C_{(2,\infty)} > 0$ is constant so that*

$$\|d\lambda\|_{L^{(2,\infty)}(B_1)} \leq C_{(2,\infty)},$$

there exists an $\varepsilon_0 > 0$ depending only on $C_{(2,\infty)}$ so that if

$$\mathcal{W}_2(\Phi) = \frac{1}{4}\int_{B_1} |A|^2_g \, d\sigma_g \leq \varepsilon_0,$$

then the following estimate holds:

$$\|d\Phi\|_{W^{3,2}(B_{1/2})} \leq C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|\mathrm{e}^\lambda\|_{L^2(B_1)}\Big), \tag{4.1.1}$$

where $C = C(C_{(2,\infty)}) > 0$.

Remark 4.1.3 The fact that the estimate (4.1.1) does not include $\|\Phi\|_{L^2(B_{1/2})}$ on the left hand-side is motivated by the translation invariance of all the quantities on the right-hand side.

Such result generalizes the classical proof of the regularity of weak Willmore immersions by Rivière [Riv08, Riv14, Riv16], and the proof is similar in spirit and methodology. We now briefly summarize the essential ideas.

The first thing one notices is that the Willmore equation is critical in $\mathscr{E}(B_1, \mathbb{R}^3)$: indeed if we write

$$\delta\mathcal{W} = \Delta_g H + \overline{\nabla}^{*_g}\Big(\langle A^\circ, H\rangle^{\sharp_g} + \langle A^\circ, H\rangle^{\sharp_g}\Big), \tag{4.1.2}$$

even assuming conformality and, say, $\delta\mathcal{W} \in L^2$ the last term on the right hand side represent, written explicitly, divergence of L^1–quantities, a fact thats prevents standard elliptic regularity theory from being applied to H. Only a slightly better information on H namely $H \in L^p_{\mathrm{loc}}(B_1)$ for some $p > 2$ rather than just $p = 2$ would suffice to start the bootstrapping process.

To deduce such information, a new substantial idea is needed, namely, one exploits the fact that Willmore surfaces and Willmore energies are conformal invariant. More precisely, in our version of the proof we shall use the fact that the Lagrangian density $|A^\circ|^2 d\sigma_g$ is pointwise invariant under conformal transformations of $\mathbb{R}^n$ ($n = 3$ in this case) see Chen [Che74]. This yields a set of new equations (see Proposition 3.4.1 and Corollary 3.4.3 of Chapter 3), true for any immersion Φ, and performing suitably a Hodge decomposition on such equations, one is led to a system that comprises linear combination of Jacobians, plus other therms due to the fact of higher regularity.

This in the end is yields the required extra information on H and, through integrability by compensation estimates (see Chapter 2) allows for the regularity bootstrap.

4.2 The Inhomogeneous Willmore System

We first point out the following fact.

Lemma 4.2.1. *The vector-valued form w defined in Proposition 3.3.2 satisfies $\langle w, d\Phi\rangle_g = 0$.*

Proof. Since $|d\Phi|^2 = 2$ and H is a normal vector field, we have

$$\begin{aligned}
\langle w, d\Phi\rangle &= \left\langle \overline{\nabla} H, d\Phi\right\rangle - 2\left\langle \pi_\tau(\overline{\nabla} H), d\Phi\right\rangle - |H|^2|d\Phi|^2 \\
&= -\left\langle \overline{\nabla} H, d\Phi\right\rangle - 2|H|^2 \\
&= -\frac{1}{2}\left\langle \overline{\nabla}\Delta_g\Phi, d\Phi\right\rangle - 2|H|^2 \\
&= \frac{1}{2}\left\langle \Delta_g\Phi, \overline{\nabla}^* d\Phi\right\rangle - 2|H|^2 \\
&= \frac{1}{2}|\Delta_g\Phi|^2 - \frac{1}{2}|\Delta_g\Phi|^2 = 0.
\end{aligned}$$

□

Since we are working on the disk, every closed 1–form is also exact. For the same reason, if $\omega \in \Omega^1(B_1)$ is a 1–form which is co–closed: $d^*\omega = 0$, then there exist a function $f \in C^\infty(B_1)$, unique up to an additive constant, so that $d^*(*f) = \omega$ i.e. so that $*df = \omega$. The same holds for vector-valued forms, replacing "d" with "$\overline{\nabla}$".

For the following proposition, the starting point are Proposition 3.4.1 and Corollary 3.4.3 of Chapter 3.

Proposition 4.2.2. *$\Phi : B_1 \to \mathbb{R}^3$ be an immersion. Consider the Hodge decomposition*

$$w = \overline{\nabla}\mathscr{L} + *\overline{\nabla} L, \tag{4.2.1}$$

and consider further the Hodge decompositions

$$-d\Phi \times H - (*d\Phi) \times L = \overline{\nabla}\mathscr{R} + *\overline{\nabla} R, \tag{4.2.2}$$

$$-\langle *d\Phi, L\rangle = d\mathscr{S} + *dS. \tag{4.2.3}$$

Then the following relations hold:

$$\Delta_g \Phi = d\Phi \times \left(\overline{\nabla}\mathscr{R} + *\overline{\nabla}R\right) + \left\langle d\Phi, d\mathscr{S} + *dS\right\rangle \tag{4.2.4}$$

$$\Delta_g R = \overline{\nabla}N \times (\overline{\nabla}\mathscr{R} + *\overline{\nabla}R) - \langle \overline{\nabla}N, d\mathscr{S} + *dS\rangle + (*d\Phi) \times \overline{\nabla}\mathscr{L}, \tag{4.2.5}$$

$$\Delta_g S = \langle \overline{\nabla}N, \overline{\nabla}\mathscr{R} + *dR\rangle + \langle *d\Phi, \overline{\nabla}\mathscr{L}\rangle, \tag{4.2.6}$$

where N is the Gauss map of Φ.

Proof. Note first that, if the Hodge decompositions (4.2.1), (4.2.2), (4.2.3) hold, we necessarily have

$$\Delta_g \mathscr{L} = \overline{\nabla}^* w = \delta \mathcal{W},$$

$$\Delta_g \mathscr{R} = -d\Phi \times \overline{\nabla}\mathscr{L}, \tag{4.2.7}$$

$$\Delta_g \mathscr{S} = -\langle d\Phi, \overline{\nabla}\mathscr{L}\rangle. \tag{4.2.8}$$

With (3.4.4) and the identity

$$\overline{\nabla}^*(-\Phi \times d\Phi \times H) = 2H - \Phi \times d\Phi \times \overline{\nabla}H = 2H - \Phi \times d\Phi \times w,$$

we see that there holds

$$\begin{aligned}
&\Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\langle \Phi, \delta\mathcal{W}\rangle + 4H \\
&= \overline{\nabla}^*\Big(-2\Phi \times (d\Phi \times H) + \Phi \times (\Phi \times w) + \Phi\langle \Phi, w\rangle\Big) \\
&= \overline{\nabla}^*\Big(-\Phi \times (d\Phi \times H)\Big) \\
&\quad + \overline{\nabla}^*\bigg(\Phi \times \Big(-d\Phi \times H + \Phi \times (\overline{\nabla}\mathscr{L} + *\overline{\nabla}L)\Big) + \Phi\Big\langle \Phi, \overline{\nabla}\mathscr{L} + *\overline{\nabla}L\Big\rangle\bigg) \\
&= 2H - \Phi \times (d\Phi \times w) \\
&\quad + \overline{\nabla}^*\bigg(\Phi \times \Big(-d\Phi \times H + \Phi \times \overline{\nabla}\mathscr{L} + *\overline{\nabla}(\Phi \times L) - (*d\Phi) \times L\Big)\bigg) \\
&\quad + \overline{\nabla}^*\bigg(\Phi\Big(\langle \Phi, \overline{\nabla}\mathscr{L}\rangle + *d\langle \Phi, L\rangle - \langle *d\Phi, L\rangle\Big)\bigg) \\
&= 2H - \Phi \times (d\Phi \times w) \\
&\quad + \overline{\nabla}^*\bigg(\Phi \times \Big(\Phi \times \overline{\nabla}\mathscr{L} + \overline{\nabla}\mathscr{R} + *\overline{\nabla}R + *\overline{\nabla}(\Phi \times L)\Big)\bigg) \\
&\quad + \overline{\nabla}^*\bigg(\Phi\Big(\langle \Phi, \overline{\nabla}\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle \Phi, L\rangle\Big)\bigg).
\end{aligned}$$

Now, on the one hand we have, from (4.2.7),

$$\begin{aligned}
&\overline{\nabla}^*\bigg(\Phi \times \Big(\Phi \times \overline{\nabla}\mathscr{L} + \overline{\nabla}\mathscr{R} + *\overline{\nabla}R + *\overline{\nabla}(\Phi \times L)\Big)\bigg) \\
&= d\Phi \times \Big(\Phi \times \overline{\nabla}\mathscr{L} + \overline{\nabla}\mathscr{R} + *\overline{\nabla}R + *\overline{\nabla}(\Phi \times L)\Big) + \Phi \times \Big(d\Phi \times \overline{\nabla}\mathscr{L} + \Phi \times \Delta_g\mathscr{L} + \Delta_g\mathscr{R}\Big) \\
&= d\Phi \times \Big(\Phi \times d\mathscr{L} + \overline{\nabla}\mathscr{R} + *\overline{\nabla}R + *\overline{\nabla}(\Phi \times L)\Big) + \Phi \times (\Phi \times \delta\mathcal{W})
\end{aligned}$$

on the other hand from (4.2.8) it follows that

$$
\begin{aligned}
&\overline{\nabla}^*\Big(\Phi\big(\langle\Phi,\overline{\nabla}\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\big)\Big)\\
&= \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle + \Phi\big(\langle d\Phi,\overline{\nabla}\mathscr{L}\rangle + \langle\Phi,\Delta_g\mathscr{L}\rangle + \Delta_g\mathscr{S}\big)\\
&= \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle + \Phi\langle\Phi,\delta\mathcal{W}\rangle,
\end{aligned}
$$

thus we deduce

$$
\begin{aligned}
2H &= -\Phi\times(d\Phi\times w)\\
&\quad + d\Phi\times\big(\Phi\times\overline{\nabla}\mathscr{L} + \overline{\nabla}\mathscr{R} + *\overline{\nabla}R + *\overline{\nabla}(\Phi\times L)\big)\\
&\quad + \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle\\
&= -\Phi\times(d\Phi\times w) + d\Phi\times\big(\overline{\nabla}\mathscr{R} + *\overline{\nabla}R\big) + \Big\langle d\Phi, d\mathscr{S} + *dS\Big\rangle\\
&\quad + d\Phi\times\big(\Phi\times\overline{\nabla}\mathscr{L} + *\overline{\nabla}(\Phi\times L)\big) + \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L}\rangle + *\langle\Phi, L\rangle\Big\rangle.
\end{aligned}
$$

By definition of $\mathscr{L}$ and L, with Lemma 4.2.1 the last line in the above expression is

$$
\begin{aligned}
&d\Phi\times\big(\Phi\times\overline{\nabla}\mathscr{L} + *\overline{\nabla}(\Phi\times L)\big) + \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L}\rangle + *\langle\Phi, L\rangle\Big\rangle\\
&= d\Phi\times\big(\Phi\times(\overline{\nabla}\mathscr{L} + *\overline{\nabla}L) + (*d\Phi)\times L\big) + \Big\langle d\Phi, \langle\Phi,\overline{\nabla}\mathscr{L} + *\overline{\nabla}L\rangle + \langle *d\Phi, L\rangle\Big\rangle\\
&= d\Phi\times\big(\Phi\times w + (*d\Phi)\times L\big) + \Big\langle d\Phi, \langle\Phi, w\rangle + \langle *d\Phi, L\rangle\Big\rangle\\
&= d\Phi\times\big(\Phi\times w\big) + \Big\langle d\Phi, \langle\Phi, w\rangle\Big\rangle + d\Phi\times\big((*d\Phi)\times L\big) + \Big\langle d\Phi, \langle *d\Phi, L\rangle\Big\rangle\\
&= \Phi\times(d\Phi\times w),
\end{aligned}
$$

and this yields (4.2.4). Next, since $\langle d\Phi\times H, N\rangle = 0$, we see that

$$
\begin{aligned}
\Big\langle\overline{\nabla}\mathscr{R} + *\overline{\nabla}R, N\Big\rangle &= \Big\langle -d\Phi\times H - (*d\Phi)\times L, N\Big\rangle\\
&= -\Big\langle N\times(*d\Phi), L\Big\rangle\\
&= -\Big\langle d\Phi, L\Big\rangle\\
&= - * d\mathscr{S} + dS
\end{aligned}
$$

and similarly, using the rules of the vector product, we have

$$
\begin{aligned}
\big(\overline{\nabla}\mathscr{R} + *\overline{\nabla}R\big)\times N &= \big(-d\Phi\times H - (*d\Phi)\times L\big)\times N\\
&= N\times\big(d\Phi\times H\big) + N\times\big((*d\Phi)\times L\big)\\
&= -H\times(N\times d\Phi) - d\Phi\times(H\times N)\\
&\quad - L\times\big(N\times(*d\Phi)\big) - (*d\Phi)\times(L\times N)\\
&= H\times(*d\Phi) - L\times d\Phi - (*d\Phi)\times(L\times N)\\
&= -(*d\Phi)\times H + d\Phi\times L + N\langle *d\Phi, L\rangle\\
&= *\overline{\nabla}\mathscr{R} - \overline{\nabla}R - N(d\mathscr{S} + *dS),
\end{aligned}
$$

and thus, codifferentiating these identities we have

$$\begin{aligned}
&\Delta_g \mathscr{R} \times N + \left(\overline{\nabla}\mathscr{R} + *\overline{\nabla}R\right) \times \overline{\nabla}N = -\Delta_g R - \left\langle \overline{\nabla}N, d\mathscr{S} + *dS\right\rangle - N\Delta_g \mathscr{S},\\
&\left\langle \Delta_g \mathscr{R}, N\right\rangle + \left\langle \overline{\nabla}\mathscr{R} + *\overline{\nabla}R, \overline{\nabla}N\right\rangle = \Delta_g S.
\end{aligned}$$

It now suffices to notice that, by definition of $\mathscr{R}$ and $\mathscr{S}$ it is

$$\begin{aligned}
N\Delta_g \mathscr{S} &= -N\langle d\Phi, \overline{\nabla}\mathscr{L}\rangle\\
\Delta_g \mathscr{R} \times N &= -(d\Phi \times \overline{\nabla}\mathscr{L}) \times N = N \times (d\Phi \times \overline{\nabla}\mathscr{L}) = \left\langle d\Phi, \langle N, \overline{\nabla}\mathscr{L}\rangle\right\rangle,\\
\left\langle \Delta_g \mathscr{R}, N\right\rangle &= -\langle d\Phi \times \overline{\nabla}\mathscr{L}, N\rangle = -\langle N \times d\Phi, \overline{\nabla}\mathscr{L}\rangle = \langle *d\Phi, \overline{\nabla}\mathscr{L}\rangle
\end{aligned}$$

and in particular

$$\begin{aligned}
\Delta_g \mathscr{R} \times N + N\Delta_g \mathscr{S} &= \left\langle d\Phi, \langle N, \overline{\nabla}\mathscr{L}\rangle\right\rangle - N\langle d\Phi, \overline{\nabla}\mathscr{L}\rangle\\
&= \overline{\nabla}\mathscr{L} \times (d\Phi \times N)\\
&= \overline{\nabla}\mathscr{L} \times (*d\Phi).
\end{aligned}$$

Substituting these relations in the ones above then gives (4.2.5) and (4.2.6). □

4.3 Qualitative Estimates

Proposition 4.3.1. *Let $1 < q < \infty$ and let $\Phi \in \mathscr{E}(B_1, \mathbb{R}^n)$ be a conformal with Willmore operator $\delta\mathcal{W}$ in $L^q(B_1)$. It suffices to know that $H \in L^p(B_1)$ for some $p > 2$ to deduce that $\Phi \in W^{4,q}_{loc}(B_1)$.*

Proof. We may certainly assume $2 < p < 4$. Since

$$\Delta\Phi = 2e^{2\lambda}H \in L^p,$$

elliptic regularity theory gives that $\Phi \in W^{2,p}_{\text{loc}}$, and this in turn implies

$$A = (\nabla^2\Phi)^\perp \in L^p_{\text{loc}}.$$

Looking at the Willmore equation (4.1.2):

$$-\Delta H = \nabla^*(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}) - e^{2\lambda}\delta\mathcal{W},$$

we see that

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} = e^{-2\lambda}\left(\langle A^\circ, H\rangle^{\sharp} + \langle A, H\rangle^{\sharp}\right) \in L^{\frac{p}{2}}_{\text{loc}},$$

and, since $q > 1$, we have in particular that $\delta\mathcal{W} \in W^{-1,2}$; thus $\Delta H \in W^{-1,\frac{p}{2}}_{\text{loc}}$, whence elliptic regularity and Sobolev embedding give

$$H \in W^{1,\frac{p}{2}}_{\text{loc}} \hookrightarrow L^{\left(\frac{p}{2}\right)^*}_{\text{loc}},$$

and from this, it follows that $A \in L_{\mathrm{loc}}^{\left(\frac{p}{2}\right)^*}$. Then

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} \in L_{\mathrm{loc}}^{\frac{1}{2}\left(\frac{p}{2}\right)^*},$$

whence $\Delta H \in W_{\mathrm{loc}}^{-1,\frac{1}{2}\left(\frac{p}{2}\right)^*}$, so by elliptic regularity

$$H \in W_{\mathrm{loc}}^{1,\frac{1}{2}\left(\frac{p}{2}\right)^*}.$$

This process can be iterated, and since the sequence p, $\left(\frac{p}{2}\right)^*$, $\left(\frac{1}{2}\left(\frac{p}{2}\right)^*\right)^*$, $\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{p}{2}\right)^*\right)^*\right)^*$,... is strictly monotone increasing and unbounded, after a finite number (depending on p) of steps we get that

$$\left(\frac{1}{2}\left(\frac{1}{2}\left(\cdots\left(\frac{p}{2}\right)^*\cdots\right)^*\right)^*\right)^* \geq 2.$$

We then deduce that $-\Delta H \in W_{\mathrm{loc}}^{-1,2}$, and thus that $H \in W_{\mathrm{loc}}^{1,2}$. By Sobolev embedding, this yields $H \in L_{\mathrm{loc}}^r$ for every $r < \infty$, and in turn elliptic estimates give

$$\Phi \in W_{\mathrm{loc}}^{2,r} \quad \forall\, r < \infty,$$

hence also $A \in L_{\mathrm{loc}}^r$ for every $r < \infty$. From Liouville equation

$$-\Delta\lambda = \mathrm{e}^{2\lambda} K,$$

since $|K| \leq C|A|^2$, we have $\Delta\lambda \in L_{\mathrm{loc}}^r$ and hence $\lambda \in W_{\mathrm{loc}}^{2,r}$ for every $r < \infty$. With this we infer that in fact $A, H \in W_{\mathrm{loc}}^{1,r}$ and so

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} \in W_{\mathrm{loc}}^{1,r},$$

Thus $-\Delta H \in L_{\mathrm{loc}}^q$, which implies $H \in W_{\mathrm{loc}}^{2,q}$ and hence (again since $\lambda \in W_{\mathrm{loc}}^{2,r}$) that $\Phi \in W_{\mathrm{loc}}^{4,q}$. □

Proposition 4.3.2. *Let $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ be conformal with conformal factor e^λ and so that $\delta\mathcal{W} \in L^p(B_1)$ for some $p > 1$. Then $H \in L_{loc}^r(B_1)$ for every $r < \infty$.*

Proof. We may certainly assume $1 < p < 2$.

Step 1: there exists $\mathscr{L}$ and L realizing the Hodge decomposition (4.2.1) with $\mathscr{L} \in W^{2,p}(B_1)$ and $L \in L_{loc}^{(2,\infty)}(B_1)$. Indeed, we let $\mathscr{L}$ solve

$$\begin{cases} \Delta\mathscr{L} = \mathrm{e}^{2\lambda}\delta\mathcal{W} & \text{in } B_1, \\ \quad u = 0 & \text{on } \partial B_1. \end{cases}$$

Elliptic regularity theory gives then that $\mathscr{L} \in W^{2,p}$. By construction we have

$$d^*(w - d\mathscr{L}) = 0 \quad \text{in } B_1,$$

thus L exists as a distribution in B_1 thanks to Poincaré's lemma and it is determined up to an additive constant. Note now that

$$\begin{aligned} \Delta L &= d^*(*d\mathscr{L} - *w) \\ &= d^*\Big(*\, d\mathscr{L} - *(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big) \\ &= -d^*\Big(*\,(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big), \end{aligned}$$

so if we let

$$L_0(x) = -\int_{B_1} \Big\langle dK(x-y), *(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big\rangle \, dy,$$

where $K(x) = -\frac{1}{2\pi}\log|x|$ is the fundamental solution of the Laplace operator, given that

$$|dK(x-y)| \le \frac{C}{|x-y|} \quad \Longrightarrow \quad \sup_{x\in B_1} \|dK(x-\cdot)\|_{L^{2,\infty}(B_1)} \le C,$$

we see that $L_0 \in L^{(2,\infty)}$, and hence, since $L - L_0$ is harmonic, that $L \in L^{(2,\infty)}_{\mathrm{loc}}$.

Step 2: There exist $\mathscr{R}$, R, $\mathscr{S}$ and S realizing the Hodge decompositions (4.2.2), (4.2.3) *with $\mathscr{R}, \mathscr{S} \in W^{2,p^*}(B_1)$ and $R, S \in W^{1,(2,\infty)}_{loc}(B_1)$.* Indeed, define $\mathscr{R}$ and $\mathscr{S}$ by

$$\begin{cases} \Delta\mathscr{R} = -d\Phi \times d\mathscr{L} & \text{in } B_1, \\ \mathscr{R} = 0 & \text{on } \partial B_1, \end{cases} \qquad \begin{cases} \Delta\mathscr{S} = -\langle d\Phi, d\mathscr{L}\rangle & \text{in } B_1, \\ \mathscr{S} = 0 & \text{on } \partial B_1. \end{cases}$$

Since by Sobolev embedding $d\mathscr{L} \in W^{1,p} \hookrightarrow L^{p^*}$, so elliptic regularity gives $\mathscr{R}, \mathscr{S} \in W^{2,p^*}$. By construction we then have

$$\begin{aligned} d^*\Big(- d\Phi \times H - (*d\Phi) \times L - d\mathscr{R}\Big) &= 0, \\ d^*\Big(- \langle *d\Phi, L\rangle - d\mathscr{S}\Big) &= 0, \end{aligned}$$

and thus R and S exist as distributions by Poincaré lemma and are determined up to additive constants, and, since L is in $L^{(2,\infty)}_{\mathrm{loc}}$, so are dR and dS.

Step 3: conclusion. From relations (4.2.5), (4.2.6), we see that R and S satisfy a system Jacobians plus some extra terms, namely

$$\begin{cases} \Delta R = dN \times (*dR) - \langle dN, *dS\rangle + f_R, \\ \Delta S = \langle dN, *dR\rangle + f_S, \end{cases}$$

where, since $d\mathscr{L} \in L^{p^*}$ and $d\mathscr{R}, d\mathscr{S} \in W^{1,p^*} \hookrightarrow L^\infty$, we have

$$\begin{aligned} f_R &= dN \times d\mathscr{R} - \langle dN, d\mathscr{S}\rangle + (*d\Phi)\times d\mathscr{L} && \in L^2, \\ f_S &= \langle dN, d\mathscr{R}\rangle + \langle *d\Phi, d\mathscr{L}\rangle && \in L^2. \end{aligned}$$

Thanks to Theorem 2.1.2 of Chapter 2, we get that $R, S \in W^{2,q}_{\mathrm{loc}}$ for every $q < 2$ and hence that

$$dR, dS \in W^{1,r}_{\mathrm{loc}} \quad \text{for every } r < \infty.$$

Inserting thin information in (4.2.4) gives that $\Delta\Phi$, and so H, is in L^r_{loc} for every $r < \infty$. $\square$

4.4 Quantitative Estimates

4.4.1 Control of the Conformal Factor We firs recall the following basic fact about weak immersions, providing uniform control of the conformal factor with sufficiently small Willmore energy.

Theorem 4.4.1 ([Hél02], [Riv16]). *There exists $\varepsilon_0 > 0$ so that, if $\Phi \in \mathscr{E}(B_1, \mathbb{R}^n)$ is conformal with conformal factor e^λ and so that*

$$\mathcal{W}_2(\Phi) = \frac{1}{4}\int_{B_1} |A|_g^2 \, d\sigma_g \leq \varepsilon_0,$$

then, if $C_{(2,\infty)} > 0$ is constant so that

$$\|d\lambda\|_{L^{(2,\infty)}(B_1)} \leq C_{(2,\infty)},$$

for any domain $0 < r < 1$ there holds

$$\|d\lambda\|_{L^2(B_r)} + \|\lambda - \ell\|_{L^\infty(B_r)} \leq C \int_{B_1} |A|_g^2 \, d\sigma_g,$$

for some constants $\ell \in \mathbb{R}$ and $C = C(r, C_{(2,\infty)}) > 0$.

Remark 4.4.2 By the triangle inequalily, without loss of generality we can take $\ell = \lambda(0)$ in the above estimate.

4.4.2 In the computations that follows, we shall make use of various Gagliardo-Nirenberg inequalities, namely, of multiplicative Sobolev inequalities such as

$$\|u\|_{L^4(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W^{1,2}(\Omega)}^{1/2},$$

for $u \in W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$ bounded, regular domain. We refer for instance to [Nir59].

Proposition 4.4.3. *Let $\Omega \subset \mathbb{R}^2$ is a bounded, regular domain with $0 \in \Omega$ and $\Phi : \Omega \to \mathbb{R}^n$ is a conformal immersion of class $W^{4,2}$ with conformal factor e^λ. Let $E > 0$ and $C_\infty > 0$ be constants so that*

$$\|e^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)} \leq E \quad \text{and} \quad \|\lambda - \Lambda\|_{L^\infty(\Omega)} \leq C_\infty,$$

where $\Lambda = \lambda(0)$. Then, the following estimate holds:

$$\|\Delta^2\Phi\|_{L^2(\Omega)} \leq 2\|e^{4\lambda}\delta\mathcal{W}\|_{L^2(\Omega)} + C\|e^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)}\|\nabla^2\Phi\|_{W^{2,2}(\Omega)}, \tag{4.4.1}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Lemma 4.4.4. *The following pointwise estimates hold for an absolute constant $C > 0$ and $k = 0, 1, 2$:*

$$\frac{1}{C}|e^{-\lambda}\nabla^2\Phi| \leq |e^{-\lambda}A| + |d\lambda| \leq C\, e^{-\lambda}|\nabla^2\Phi|, \tag{4.4.2}$$

$$|\nabla^k(e^{2\lambda}H)| \leq C|\nabla^{k+2}\Phi|, \tag{4.4.3}$$

$$|\nabla^k(e^{2\lambda}H)| \leq C|\nabla^k A|, \tag{4.4.4}$$

and the following estimate holds:

$$\|A\|_{W^{k,2}(\Omega)} \leq C\|\nabla^2\Phi\|_{W^{k,2}(\Omega)}, \tag{4.4.5}$$

for $C = C(\Omega, E, C_\infty) > 0$ and $k = 0, 1, 2$.

Proof. Since

$$A_{\mu\nu} = \partial^2_{\mu\nu}\Phi - \Gamma^\sigma_{\mu\nu}\,\partial_\sigma\Phi \quad \text{and} \quad e^{2\lambda} = \frac{1}{2}|d\Phi|^2,$$

and

$$\begin{aligned}
&\Gamma^1_{11} = \partial_1\lambda, && \Gamma^1_{12} = \Gamma^1_{21} = \partial_2\lambda, && \Gamma^1_{22} = -\partial_1\lambda,\\
&\Gamma^2_{11} = -\partial_2\lambda, && \Gamma^2_{12} = \Gamma^1_{21} = \partial_1\lambda, && \Gamma^2_{22} = \partial_2\lambda,
\end{aligned}$$

estimates (4.4.2) and consequently (4.4.5) for $k = 0$ follow. Next, since

$$\mathrm{e}^{2\lambda}H = \frac{1}{2}\Delta\Phi = \frac{1}{2}(\Delta\Phi)^\perp = \frac{1}{2}(A_{11} + A_{22}),$$

estimates (4.4.3) and (4.4.4) follow. Now differentiating of the above identities:

$$\begin{aligned}
\partial_\xi A_{\mu\nu} &= \partial^3_{\xi\mu\nu}\Phi - \partial_\xi\Gamma^\sigma_{\mu\nu}\,\partial_\sigma\Phi - \Gamma^\sigma_{\mu\nu}\,\partial^2_{\xi\sigma}\Phi,\\
\partial^2_{\zeta\xi} A_{\mu\nu} &= \partial^4_{\zeta\xi\mu\nu}\Phi - \partial^2_{\zeta\xi}\Gamma^\sigma_{\mu\nu}\,\partial_\sigma\Phi - \partial_\xi\Gamma^\sigma_{\mu\nu}\,\partial^2_{\zeta\sigma}\Phi - \partial_\zeta\Gamma^\sigma_{\mu\nu}\,\partial^2_{\xi\sigma}\Phi - \Gamma^\sigma_{\mu\nu}\,\partial^3_{\zeta\xi\sigma}\Phi,
\end{aligned}$$

and

$$\begin{aligned}
2d\lambda\mathrm{e}^{2\lambda} &= \langle\nabla^2\Phi, d\Phi\rangle,\\
(2\nabla^2\lambda + 4d\lambda\otimes d\lambda)\mathrm{e}^{2\lambda} &= \langle\nabla^3\Phi, d\Phi\rangle + \nabla^2\Phi\langle\dot{\otimes}\rangle\nabla^2\Phi,\\
(2\nabla^3\lambda + 8\nabla^2\lambda\otimes d\lambda + 8d\lambda\otimes d\lambda\otimes d\lambda)\mathrm{e}^{2\lambda} &= \langle\nabla^4\Phi, d\Phi\rangle + 2\nabla^3\Phi\langle\dot{\otimes}\rangle\nabla^2\Phi,
\end{aligned}$$

(where $\langle\dot{\otimes}\rangle$ means scalar product in the vector part, inner product in one of the covariant entries and tensor product in the remaining ones) yields the estimates

$$|\Gamma^\sigma_{\mu\nu}| \le C|d\lambda|, \qquad |\partial_\xi\Gamma^\sigma_{\mu\nu}| \le C|\nabla^2\lambda|, \qquad |\partial^2_{\zeta\xi}\Gamma^\sigma_{\mu\nu}| \le C|\nabla^3\lambda|,$$

and

$$\begin{aligned}
|d\lambda| &\le C\mathrm{e}^{-\lambda}|\nabla^2\Phi|,\\
|\nabla^2\lambda| &\le C\Big(|d\lambda|^2 + \mathrm{e}^{-\lambda}|\nabla^3\Phi| + \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\Big)\\
&\le C\Big(\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2 + \mathrm{e}^{-\lambda}|\nabla^3\Phi|\Big),\\
|\nabla^3\lambda| &\le C\Big(|\nabla^2\lambda||d\lambda| + |d\lambda|^3 + \mathrm{e}^{-\lambda}|\nabla^4\Phi| + \mathrm{e}^{-2\lambda}|\nabla^3\Phi||\nabla^2\Phi|\Big)\\
&\le C\Big(\mathrm{e}^{-2\lambda}|\nabla^3\Phi||\nabla^2\Phi| + \mathrm{e}^{-3\lambda}|\nabla^2\Phi|^3 + \mathrm{e}^{-\lambda}|\nabla^4\Phi|\Big),
\end{aligned}$$

thanks to which we estimate in turn

$$\begin{aligned}
|\nabla A| &\le C\Big(|\nabla^3\Phi| + \mathrm{e}^{\lambda}|\nabla^2\lambda| + |d\lambda||\nabla^2\Phi|\Big)\\
&\le C\Big(|\nabla^3\Phi| + \mathrm{e}^{-\lambda}|\nabla^2\Phi|^2\Big),\\
|\nabla^2 A| &\le C\Big(|\nabla^4\Phi| + \mathrm{e}^{\lambda}|\nabla^3\lambda| + |\nabla\lambda||\nabla^2\Phi| + |d\lambda||\nabla^3\Phi|\Big)\\
&\le C\Big(|\nabla^4\Phi| + \mathrm{e}^{-\lambda}|\nabla^3\Phi||\nabla^2\Phi| + \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^3\Big).
\end{aligned}$$

Thus, with the help of Gagliardo-Nirenberg we can estimate

$$\begin{aligned}\|\nabla A\|_{L^2} &\le C\Big(\|\nabla^3\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}|\nabla^2\Phi|^2\|_{L^2}\Big)\\ &\le C\Big(\|\nabla^3\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|^2_{L^4}\Big)\\ &\le C\Big(\|\nabla^3\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{1,2}}\Big)\\ &\le C\Big(\|\nabla^3\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{1,2}}\Big),\end{aligned}$$

yielding (4.4.5) for $k=1$. Similarly, since

$$\begin{aligned}\|\nabla^2 A\|_{L^2} &\le C\Big(\|\nabla^4\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}|\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} + \|\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^3\|_{L^2}\Big)\\ &\le C\Big(\|\nabla^4\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\||\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|^3_{L^6}\Big),\end{aligned}$$

with Hölder and Gagliardo-Nirenberg we estimate

$$\begin{aligned}\||\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} &\le \|\nabla^3\Phi\|_{L^4}\|\nabla^2\Phi\|_{L^2}\\ &\le C\|\nabla^2\Phi\|^{\frac14}_{L^2}\|\nabla^2\Phi\|^{\frac34}_{W^{2,2}}\|\nabla^2\Phi\|^{\frac34}_{L^2}\|\nabla^2\Phi\|^{\frac14}_{W^{2,2}}\\ &\le C\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}},\end{aligned}$$

and

$$\|\nabla^2\Phi\|^3_{L^6} \le C\|\nabla^2\Phi\|^2_{L^2}\|\nabla^2\Phi\|_{W^{2,2}},$$

so to obtain

$$\|\nabla^2 A\|_{L^2} \le C\Big(\|\nabla^4\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|^2_{L^2}\|\nabla^2\Phi\|_{W^{2,2}}\Big),$$

yielding (4.4.5) also for $k=2$. □

Proof of Proposition 4.4.3. It can be deduced from the following three lemmas and (4.4.5).

Lemma 4.4.5. *There holds*

$$\|\mathrm{e}^{4\lambda}\Delta_g^\perp H\|_{L^2(\Omega)} \le C\|\mathrm{e}^{-\lambda}A\|_{L^2(\Omega)}\|A\|_{W^{2,2}(\Omega)} + \|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(\Omega)}, \tag{4.4.6}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Lemma 4.4.6. *There holds*

$$\|\mathrm{e}^{4\lambda}\Delta_g H\|_{L^2(\Omega)} \le \|\mathrm{e}^{4\lambda}\Delta_g^\perp H\|_{L^2(\Omega)} + C\|\mathrm{e}^{-\lambda}A\|_{L^2(\Omega)}\|A\|_{W^{2,2}(\Omega)}, \tag{4.4.7}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Lemma 4.4.7. *There holds*

$$\|\Delta^2\Phi\|_{L^2(\Omega)} \le 2\|\mathrm{e}^{4\lambda}\Delta H\|_{L^2(\Omega)} + C\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)}\|\nabla^2\Phi\|_{W^{2,2}(\Omega)}, \tag{4.4.8}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Proof of Lemma 4.4.5. With the classical form of the Willmore operator (see (3.3.8) of Chapter 3) and the pointwise estimate (4.4.4), by means of Hölder's and Gagliardo-Nirenberg inequalities we estimate

$$\begin{aligned}
\|\langle A^\circ, \langle H, A^\circ\rangle\rangle_g\|_{L^2} &\le C\|\mathrm{e}^{-4\lambda}\langle A^\circ, \langle H, A^\circ\rangle\rangle\|_{L^2} \\
&\le C\|\mathrm{e}^{-4\lambda}|A^\circ|^2 H\|_{L^2} \\
&\le C\|\mathrm{e}^{-6\lambda}|A|^3\|_{L^2} \\
&\le C\mathrm{e}^{-6\Lambda}\|A\|_{L^6}^3 \\
&\le C\mathrm{e}^{-6\Lambda}\|A\|_{L^2}^2\|A\|_{W^{2,2}} \\
&\le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}},
\end{aligned}$$

which yields (4.4.6). □

Proof of Lemma 4.4.6. From formula (3.3.12), we have

$$\|\Delta_g H\|_{L^2} \le \|\Delta_g^\perp H\|_{L^2} + C\Big(\|\langle A, \langle H, A\rangle\rangle_g\|_{L^2} + \|\langle \nabla H, A\rangle_g^{\sharp_g}\|_{L^2} + \|\langle \nabla H, H\rangle^{\sharp_g}\|_{L^2}\Big).$$

Similarly as in the proof of Lemma (4.4.5), we have

$$\|\langle A, \langle H, A\rangle\rangle_g\|_{L^2} \le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}}.$$

Next, since

$$\nabla H = \mathrm{e}^{-2\lambda}\nabla(\mathrm{e}^{2\lambda}H) - 2H \otimes d\lambda,$$

with (4.4.4) we may pointwise estimate

$$|\nabla H| \le C\mathrm{e}^{-2\lambda}\Big(|\nabla A| + |A||d\lambda|\Big),$$

thus allowing to deduce

$$\begin{aligned}
\|\langle \nabla H, H\rangle^{\sharp_g}\|_{L^2} &\le C\|\mathrm{e}^{-\lambda}|\nabla H||H|\|_{L^2} \\
&\le C\|\mathrm{e}^{-3\lambda}(|\nabla A| + |A||d\lambda|)|H|\|_{L^2} \\
&\le C\|\mathrm{e}^{-5\lambda}(|\nabla A| + |A||d\lambda|)|A|\|_{L^2};
\end{aligned}$$

now with Hölder and Gagliardo-Nirenberg we see that, on the one hand,

$$\begin{aligned}
\|\mathrm{e}^{-5\lambda}|\nabla A||A|\|_{L^2} &\le C\mathrm{e}^{-5\Lambda}\||\nabla A||A|\|_{L^2} \\
&\le C\mathrm{e}^{-5\Lambda}\|\nabla A\|_{L^4}\|A\|_{L^4} \\
&\le C\mathrm{e}^{-5\Lambda}\|A\|_{L^2}\|A\|_{W^{2,2}} \\
&\le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\|\mathrm{e}^{-5\lambda}|A|^2|d\lambda|\|_{L^2} &\le C\mathrm{e}^{-5\Lambda}\|d\lambda\|_{L^2}\|A\|_{L^\infty}^2 \\
&\le C\mathrm{e}^{-5\Lambda}\|d\lambda\|_{L^2}\|A\|_{L^2}\|A\|_{W^{2,2}} \\
&\le C\mathrm{e}^{-4\Lambda}\|d\lambda\|_{L^2}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},
\end{aligned}$$

so we deduce

$$\|\langle \nabla H, H\rangle^{\sharp_g}\|_{L^2} \le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}}.$$

Similarly, we see that

$$\begin{aligned}\|\langle \nabla H, A\rangle_g^{\sharp_g}\|_{L^2} &\le C\|\mathrm{e}^{-3\lambda}\langle \nabla H, A\rangle^{\sharp}\|_{L^2}\\ &\le C\|\mathrm{e}^{-3\lambda}|\nabla H||A|\|_{L^2}\\ &\le C\|\mathrm{e}^{-5\lambda}(|\nabla A| + |A||d\lambda|)|A|\|_{L^2},\end{aligned}$$

and so similarly as before we deduce

$$\|\langle \nabla H, A\rangle_g^{\sharp_g}\|_{L^2} \le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},$$

yielding (4.4.7). □

Proof of Lemma 4.4.7. From formula for the bilaplace operator in conformal coordinates (formula (3.3.15) of Chapter 3), it follows that

$$\|\Delta\Phi\|_{L^2} \le 2\|\mathrm{e}^{4\lambda}\Delta_g H\|_{L^2} + C\Big(\|\mathrm{e}^{4\lambda}\langle d\lambda, \nabla H\rangle_g\|_{L^2} + \|\mathrm{e}^{4\lambda}|d\lambda|_g^2 H\|_{L^2} + \|\mathrm{e}^{4\lambda}\Delta_g\lambda\, H\|_{L^2}\Big).$$

Now with (4.4.2), (4.4.3), and (again) the identity

$$\nabla H = \mathrm{e}^{-2\lambda}\nabla(\mathrm{e}^{2\lambda}H) - 2H\otimes d\lambda,$$

we estimate with Hölder and Galgliardo-Nirenberg

$$\begin{aligned}\|\mathrm{e}^{4\lambda}\langle d\lambda, \nabla H\rangle_g\|_{L^2} &\le C\|\mathrm{e}^{2\lambda}\langle d\lambda, \nabla H\rangle\|_{L^2}\\ &\le C\|\mathrm{e}^{2\lambda}|d\lambda||\nabla H|\|_{L^2}\\ &\le C\Big\|\mathrm{e}^{2\lambda}\Big(\mathrm{e}^{-\lambda}|\nabla^2\Phi|\Big)\Big(\mathrm{e}^{-2\lambda}|\nabla^3\Phi| + \mathrm{e}^{-3\lambda}|\nabla^2\Phi|^2\Big)\Big\|_{L^2}\\ &\le C\Big(\mathrm{e}^{-\Lambda}\||\nabla^3\Phi|\nabla^2\Phi|\|_{L^2} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^6}^3\Big)\\ &\le C\Big(\mathrm{e}^{-\Lambda}\|\nabla^3\Phi\|_{L^4}\|\nabla^2\Phi\|_{L^4} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^6}^3\Big)\\ &\le C\Big(\mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^2}^2\|\nabla^2\Phi\|_{W^{2,2}}\Big)\\ &\le C\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}}.\end{aligned}$$

Similarly, again with (4.4.3) and Gagliardo-Nirenberg we estimate

$$\begin{aligned}\|\mathrm{e}^{4\lambda}|d\lambda|_g^2 H\|_{L^2} &\le C\|\mathrm{e}^{2\lambda}|d\lambda|^2 H\|_{L^2}\\ &\le C\|\mathrm{e}^{2\lambda}(\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2)(\mathrm{e}^{-2\lambda}|\nabla^2\Phi|)\|_{L^2}\\ &\le C\|\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\|_{L^2}\\ &\le C\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}.\end{aligned}$$

Finally, with Liouville's equation

$$-\Delta\lambda = \mathrm{e}^{2\lambda}K,$$

the pointwise estimate

$$|K| \le |A|_g^2 \le \mathrm{e}^{-4\lambda}|A|^2,$$

we can estimate with Gagliardo-Nirenberg and (4.4.5):

$$\begin{aligned}
\|\mathrm{e}^{4\lambda}\Delta_g \lambda H\|_{L^2} &\le C\||A|^2 H\|_{L^2}\\
&\le C\|\mathrm{e}^{-2\lambda}|A|^3\|_{L^2}\\
&\le C\mathrm{e}^{-2\Lambda}\|A\|_{L^2}^2\|A\|_{W^{2,2}}\\
&\le C\|\mathrm{e}^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}}\\
&\le C\|\mathrm{e}^{-\lambda}A\|_{L^2}^2\|\nabla^2\Phi\|_{W^{2,2}}.
\end{aligned}$$

All these estimates together yield (4.4.8). □

The combination of lemmas 4.4.5, 4.4.6 and 4.4.7 immediately gives (4.4.1), and concludes the proof of Proposition 4.4.3. □

Proof of Theorem 4.1.2. The qualitative statement, namely that $\Phi \in W^{4,p}_{\mathrm{loc}}$, is an immediate consequence of Propositions 4.3.1 and 4.3.2, so we now seek to establish the estimate for the case $p = 2$. By Proposition 4.4.3, (4.4.1) jointly with elliptic estimates for the bilaplacian give

$$\|d\Phi\|_{W^{3,2}(B_{1/2})} \le C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(B_1)}\|\nabla^2\Phi\|_{W^{2,2}(B_1)} + \|d\Phi\|_{L^2(B_1)}\Big) \tag{4.4.9}$$

for $C = C(E, C_\infty) > 0$. Now we consider rescalings. For $0 < r < 1$ we let

$$\widetilde{\Phi}(x) = \Phi(rx), \quad x \in B_1,$$

and we denote with a tilde all the quantities pertaining to $\widetilde{\Phi}$. From for $k \in \mathbb{N}$ it follows in particular that for $\Omega \subseteq B_1$ we have

$$\begin{aligned}
\nabla^k\widetilde{\Phi}(x) &= r^k\nabla^k\Phi(rx),\\
\|\nabla^k\widetilde{\Phi}\|_{L^2(\Omega)} &= r^{k-1}\|\nabla^k\Phi\|_{L^2(r\Omega)},\\
\mathrm{e}^{\widetilde{\lambda}(x)} &= r\mathrm{e}^{\lambda(rx)},\\
\widetilde{\lambda}(x) - \widetilde{\lambda}(0) &= \lambda(rx) - \lambda(0),\\
\widetilde{\delta\mathcal{W}}(x) &= \delta\mathcal{W}(rx),
\end{aligned}$$

(the last equality follows either by direct inspection or at once recalling that $\delta\mathcal{W}$ is a vector field) and from these relations, we deduce in particular that

$$\begin{aligned}
\|\mathrm{e}^{4\widetilde{\lambda}}\widetilde{\delta\mathcal{W}}\|_{L^2(B_1)} &= r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)},\\
\|\mathrm{e}^{-2\widetilde{\lambda}}\nabla^2\widetilde{\Phi}\|_{L^2(B_1)} &= \|\mathrm{e}^{-2\lambda}\nabla^2\Phi\|_{L^2(B_r)},\\
\|\widetilde{\lambda} - \widetilde{\Lambda}\|_{L^\infty(B_1)} &= \|\lambda - \Lambda\|_{L^\infty(B_r)},
\end{aligned}$$

and the last two relations in particular give that $\widetilde{E} \le E$ and $\widetilde{C_\infty} \le C_\infty$. Consequently, applying (4.4.9) to $\widetilde{\Phi}$ gives

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \le C\big(r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(B_r)}\|\nabla^2\Phi\|'_{W^{2,2}(B_r)} + \|d\Phi\|_{L^2(B_r)}\big), \tag{4.4.10}$$

where for $k = 1, 2$ we denoted $\|\Phi\|'_{W^{k,2}(B_\rho)} = \left(\sum_{h=1}^{k} \rho^{k-1}\|\Phi\|^2_{W^{h,2}(B_\rho)}\right)^{1/2}$ the scale-invariant version of the Sobolev norms where, as we said abefore $C = C(E, C_\infty) > 0$.

Recall now that from (4.4.2) it is

$$\|e^{-\lambda}\nabla^2\Phi\|_{L^2(B_{1/2})} \leq C\Big(\|e^{-\lambda}A\|_{L^2(B_{1/2})} + \|d\lambda\|_{L^2(B_{1/2})}\Big),$$

so by Theorem 4.4.1 we let ε_0 be sufficiently small so to have

$$\|d\lambda\|_{L^2(B_{1/2})} + \|\lambda - \lambda(0)\|_{L^\infty(B_{1/2})} \leq C\|e^{-\lambda}A\|_{L^2(B_1)} \leq C\varepsilon_0,$$

for $C = C(C_{(2,\infty)}) > 0$, whence (4.4.10) can be improved to

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \leq C\Big(r^3\|e^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)} + \varepsilon_0\|\nabla^2\Phi\|'_{W^{2,2}(B_r)} + \|d\Phi\|_{L^2(B_r)}\Big),$$

for $C = C(C_{(2,\infty)}) > 0$.

Choose finally $\varepsilon_0 > 0$ be sufficiently small so that $C\varepsilon_0 \leq \frac{1}{2}$ to obtain

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \leq \frac{1}{2}\|d\Phi\|'_{W^{3,2}(B_r)} + C\Big(r^3\|e^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|d\Phi\|_{L^2(B_1)}\Big),$$

for every $0 < r \leq 1/2$. A classical iteration/interpolation argument applied to $\phi(r) = \|d\Phi\|'_{W^{3,2}(B_r)}$ and a covering argument yield to (4.1.1). $\square$

5 The Willmore Flow in Conformal Gauge

Summary: In this chapter we introduce a parametric framework for the study of Willmore gradient flows which enables to consider a general class of weak, energy-level solutions and opens the possibility to study energy quantization and finite–time singularities. We restrict to a small–energy regime and prove that, for small–energy weak immersions, the Cauchy problem in this class admits a unique solution.

5.1 Introduction

In the present chapter we move the first steps towards a parametric theory for the Willmore flow that, we believe, will lead to an effective study of singularities, bubbling analysis and energy quantization.

5.1.1 Willmore Gradient Flows A *Willmore L^2–gradient flow* in $\mathbb{R}^n$ (*Willmore flow* for short) of a closed, abstract surface Σ is a 1-parameter family of immersions $\Phi(t,\cdot) : \Sigma \to \mathbb{R}^n$, $t \in I \subseteq \mathbb{R}$ evolving according to the law

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U \quad \text{in } I \times \Sigma, \tag{5.1.1}$$

where, for each t, $\delta\mathcal{W}$ is the Willmore operator of $\mathcal{S}_t = \Phi(t,\Sigma)$ and $U = U^\mu \partial_\mu \Phi$ is a tangent tangent vector field, possibly time-dependent.

One good reason to consider Willmore flows is that they satisfy the energy identity, namely if $I = (0,T)$, then

$$\mathcal{W}_0(\Phi(t,\cdot)) - \mathcal{W}_0(\Phi_0) = -\int_0^t \int_\Sigma |\delta\mathcal{W}|^2 \, d\sigma_g \, d\tau, \quad \text{for } 0 \le t < T. \tag{5.1.2}$$

Willmore flows can be regarded as a continuous deformation of the initial surface $\mathcal{S}_0 = \mathcal{S} = \Phi(0,\Sigma)$ constructed so that the Willmore energy (in any of the forms given in (1.1.1)) decreases most rapidly in time, and the deformation stops as as soon as the deformed surface becomes Willmore. Thus, at least in principle, these flows have the potential to converge efficiently to Willmore immersions as $t \to +\infty$.

This is a feature common to gradient flows that makes them particularly worth studying. The first to consider L^2–gradient flows in a geometric context were Eells and Sampson [ES64] in the context of harmonic maps. Since then, the study of parabolic geometric flows has widened to the extent that some of them constitute research areas on their own right, the mean curvature flow and Hamilton's Ricci flow being two of the best-known examples.

It should be noted right away that what is typically called a Willmore flow is a family solving (5.1.1) with $U = 0$, which we will call here a *normal Willmore flow*. Since Σ is closed, and

$\delta\mathcal{W}$ is a tensor, it is classical fact that there is a bijective correspondence between tangential components and family of reparametrizations of Σ, see e.g. MANTEGAZZA [Man11, Proposition 1.3.4] for the case, entirely analogous in this regard, of the mean curvature flow. Consequently, if, say, I is a connected interval containing 0, for every family solving (5.1.1) there is a unique family of diffeomorphisms $\varphi : I \times \Sigma \to \Sigma$ with $\varphi(0, \cdot) = \mathrm{id}_\Sigma$ so that the reparametrized family $\Phi(t, \varphi(t, \cdot))$, $t \in I$ is a normal Willmore flow, and on the other hand, every reparametrization of a normal Willmore flow will be a Willmore flow (5.1.1) for some U.

Thus, in this sense, similarly as for immersions of surfaces, flows can be regarded as equivalence classes of solutions to (5.1.1), two of them being equivalent if one can be reparametrized into another. As for surfaces, depending on the situation one may choose one parametrization over another, and in this case this may be done through the choice of the tangential component. This will be a crucial fact in the present discussion.

The study of Willmore flows was introduced by KUWERT and SCHÄTZLE [KS01, KS02] and SIMONETT [Sim01] and is since then subject of a growing number of works. Particularly useful for us will be the one by KUWERT and SCHEUER [KS20] providing asymptotic estimates on the area and barycenter along the flow.

Our attention here focuses on the following foundational result. Consider the Cauchy problem for the normal Willmore flow:

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi = -\delta\mathcal{W}, & \text{in } (0,T)\times\Sigma, \\ \Phi(0,\cdot) = \Phi_0 & \text{on } \Sigma. \end{cases} \tag{5.1.3}$$

Theorem ([KS01, KS02])**.** *There exists $\varepsilon_0(n) > 0$ so that, for a smooth immersion $\Phi_0 : \Sigma \to \mathbb{R}^n$ satisfying $\mathcal{W}_0(\Phi_0) = \mathcal{W}_0(\Phi_0(\Sigma)) < \varepsilon_0$, then (5.1.3) has a unique solution in the smooth category, which furthermore exists for all times and converges to a round sphere.*

It should be said immediately that if $\mathcal{W}_0(\Phi)$ is sufficiently small, Σ must be a sphere. Indeed, as already noticed in [Wil65], it is always $\mathcal{W}_1(\Phi) \geq 4\pi$ and from (1.1.2), using Gauss-Bonnet one sees that

$$\chi(\Sigma) = \frac{1}{2\pi}\int_\Sigma K\, d\sigma = \frac{1}{2\pi}\Big(\mathcal{W}_1(\Phi) - \mathcal{W}_0(\Phi)\Big) \geq \frac{1}{2\pi}\Big(4\pi - \varepsilon_0\Big) > 1, \tag{5.1.4}$$

where $\chi(\Sigma)$ is the Euler-Poincaré characteristic of Σ.

Such theorem was concerned with *smooth* solutions. As for other geometric flows however, an effective study of singularities and bubbling analysis requires eventually to work at the *energy level*, namely, to consider appropriate notions of weak solution.

We have in mind as a particular example the classical work on the harmonic map flow done by STRUWE [Str85, Str08] and complemented by the works of others, the 2nd author [Riv93], FREIRE [Fre95b, Fre95a], CHANG, DING and YE [CDY92], TOPPING [Top02] BERTSCH, DAL PASSO and VAN DER HOUT [BDvdH02] just to mention a few.

We believe that the framework introduced by Rivière in a series of works [Riv08, Riv14, Riv16], which led for instance to an effective energy quantization analysis of Willmore surfaces by BERNARD and the 2nd author [BR14] to be, when suitably adapted, the appropriate one. We want to give in the present chapter an idea of why this should be true by introducing, under particularly favourable hypotheses, an energy-level class of weak Willmore flow and prove a uniqueness statement for the corresponding Cauchy problem in this class for a broad set of weak initial data, which we believe to be sufficiently close to the largest possible one (among unbranched surfaces).

Let us mention that Lamm and Koch in [KL12] obtained (among other results of geometric interest) an existence and uniqueness result for the Willmore flow for entire graphs in a weak framework with Lipschitz initial datum. Such datum needs to be small in the Lipschitz norm.

5.1.2 Well-Balanced Conformal Willmore Flows We shall work, in the present situation, always in a low energy regime, namely we shall arrange things so that the Willmore energy of the surfaces in consideration $\mathcal{W}_0(\mathcal{S})$ is as small as needed; furthermore, we shall also work in codimension one, namely $n = 3$. The first major consequence of this is that, as already said above, with (5.1.4) we may directly assume that the underlying topology is that of the standard sphere S^2. The second one is that we can take advantange of results from the work of De Lellis and Müller [DM05, DM06].

So, from now, it is $\Sigma = S^2$, and the underling reference metric and complex structure are the standard ones.

Central in the theory developed in [Riv08, Riv14, Riv16] and in the present one is the idea of working with *conformal immersions*. The first advantage of doing so is that the Willmore operator becomes uniformly elliptic, with ellipticity constants depending on the conformal factor, and this permits eventually the regularity bootstrap. The second one is that, exploiting conservation laws issuing from the conformal invariance (as explained in Bernard [Ber16] see also Chapters 3 and 4), the Willmore operator of a conformal immersion, which is a 4th order quasilinear elliptic system, can be recast as a 2nd order semilinear system involving Jacobian-type nonlinearities, which allows regularity bootstrap by means of integrability by compensation, similarly as in the work of Hélein [Hél02] on weakly harmonic maps in two dimensions.

The idea is then to consider *Willmore flows in conformal gauge*, where the equation becomes uniformly parabolic, if the conformal factor is uniformly bounded away from zero, and then use a slice–wise in time (elliptic) integrability by compensation arguments to bootstrap the regularity of the equation, which – as is often the case when working with parabolic PDEs in small energy regime – will suffice to get the regularity also in the time variable. This approach was successfully used by the 2nd author in [Riv93] for the case of the harmonic map flow.

Indeed, (5.1.1) is invariant under reparametrizations and thus it is degenerate parabolic, as is the case for others geometric flows such as the mean curvature flow or the Ricci flow - and this can be a serious source of troubles. The celebrated trick of DeTurck [DeT83], originally devised for the Ricci flow but easily adapted to the present situation, is one way of overcoming this problem. For the sake of compleneteness, we outlined it in §5.4. Such method has the advantage of working regardless of the topology of Σ, but, as an inspection of the proof reveals, does not seem to be suitable when working with low degrees of smoothness and moreover does not give explicitly a control on the parametrization that is chosen by such gauge.

We are instead going to consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U, & \text{in } (0,T)\times S^2, \\ \Phi(0,\cdot) = \Phi_0 & \text{on } S^2. \end{cases} \tag{5.1.5}$$

where the tangential vector field U is chosen so that $\Phi(t,\cdot)$ is conformal for every t.

It is a fairly simple matter to find an explicit characterization for U, which – unsurprisingly, given the relationship between complex and conformal structures on surfaces – is best expressed in complex notation. To this aim, we recall that if $\Phi : S^2 \to \mathbb{R}^3$ is a conformal immersion with metric $g = \mathrm{e}^{2\lambda} g_{S^2}$, the second fundamental form may be written in complex notation as

$$A = h_0 + \overline{h}_0 + H \otimes g,$$

where H is the mean curvature and

$$h_0 = A_{zz}\, dz \otimes dz = \frac{1}{4}(A_{11} - A_{22} - 2iA_{12}) dz \otimes dz.$$

Similarly, the tracefree second fundamental form is written as

$$A^\circ = A - Hg = h_0 + \overline{h}_0.$$

With this formalism, we have the following.

Lemma 5.1.1. *The tangential component of a conformal Willmore flow satisfies*

$$\bar{\partial} U^{(1,0)} = -\langle \delta\mathcal{W}, \overline{h}_0 \rangle^{\sharp_g}, \tag{5.1.6}$$

that is

$$\partial_{\bar{z}}(U^1 + iU^2)\partial_z \otimes d\bar{z} = 2\mathrm{e}^{-2\lambda}\Big\langle -\delta\mathcal{W}, A_{\bar{z}\bar{z}}\Big\rangle \partial_z \otimes d\bar{z}.$$

Basic facts for the $\overline{\partial}$-operator on vector fields is recalled in §5.5. It is certainly good news that, at least on the sphere, it defines an uniformly elliptic, zero-cokernel operator.

However, (5.1.6) do not suffices *per se* to guarantee that (5.1.5) is uniformly parabolic, not even for short time, since the control on the conformal factor in time still needs to be addressed. Geometrically this is evident: any conformal Willmore flow can be composed with any 1-parameter family of conformal self-maps of S^2 and remain conformal, and $\mathrm{Aut}(S^2)$, the set conformal (i.e. biholomorphic) self-map of S^2 is not compact. One can clearly see this also by noting that (5.1.6) has a nontrivial kernel given by the conformal Killing (holomorphic, with complex identification) vector fields on S^2 which is 6 dimensional, as $\mathrm{Aut}(S^2)$.

Moreover, (5.1.6) is an equation satisfied for every fixed t, and does not give any information on the regularity in time of U. Geometrically, this means that the 1-parameter family of maps in $\mathrm{Aut}(S^2)$ which we may compose a conformal Willmore flow may be taken nonsmooth with respect to t.

Precisely because $\mathrm{Aut}(S^2)$ is 6-dimensional however, a final choice of a 6-dimensional constraint will be enough to tame the action of such gauge group. This will be defined by the following.

Definition 5.1.2. *An immersion $\Phi : S^2 \to \mathbb{R}^3$ is called* well-balanced *if there holds*

$$\int_{S^2} I d\sigma_g = 0 \quad \textit{and} \quad \int_{S^2} \Phi \times I\, d\sigma = 0, \tag{5.1.7}$$

where I denotes standard embedding of S^2, $d\sigma$ its area element and $d\sigma_g$ the area element for the induced metric $g = \Phi^\star g_{\mathbb{R}^3}$.

Remark 5.1.3 Note that being well-balanced is a translation-invariant condition, namely if Φ is well-balanced, so is $\Phi + k$ for every $k \in \mathbb{R}^3$.

Conditions (5.1.6) and (5.1.7) together with a good choice for the parametrization of the initial datum, which we shall now discuss, will be sufficient to control the behaviour of the tangential component U in on (5.1.5). Moreover, they are meaningful also for the notion of weak conformal Willmore flow that we are going to define.

5.1.3 Chosing an ad-hoc Parametrization for initial Data with Small Energy. From a geometric perspective, both the Cauchy problems (5.1.3) and (5.1.5)–(5.1.6) possess an obvious "gauge invariance" for the initial datum, namely if $\Phi_0(S^2) = \mathcal{S}$ is the immersed sphere representing the initial datum, and φ is any diffeomorphism of S^2, then $\Phi_0 \circ \varphi$ is again a parametrization for the same surface $\mathcal{S}$, and there is no a priori preferred choice – or possibility to distinguish – between Φ_0 and $\Phi_0 \circ \varphi$. This a relevant issue for a parametric theory.

The conformal gauge choice helps to reduce this invariance (Φ_0 has to be conformal, and so φ must belong to $\mathrm{Aut}(S^2)$), but does not break it entirely. To this aim, we shall use the following result contained in the work of De Lellis and Müller [DM05, DM06].

Theorem ([DM05, DM06])**.** *There exist ε_0, $C > 0$ so that, if $\mathcal{S} \subset \mathbb{R}^3$ is an immersed surface with area $\mathcal{A}(\mathcal{S}) = 4\pi$ and Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon_0$, there exists a conformal parametrization $\Phi : S^2 \to \mathcal{S}$ satisfying*

$$\|\Phi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\lambda - 1\|_{L^\infty(S^2)} \leq C\mathcal{W}_0(\mathcal{S}), \tag{5.1.8}$$

where $I : S^2 \to \mathbb{R}^3$ denotes the standard immersion of S^2 and $c = ⨍_{S^2} \Phi \, d\sigma$.

In this theorem the fact that the area of the surface is 4π can be seen to a normalization achievable by scaling. We shall need another one, achievable by translations; to this aim recall that the *barycenter* of an immersed surface $\mathcal{S} \subset \mathbb{R}^3$ is defined as

$$\mathcal{C}(\mathcal{S}) = \int_{\mathcal{S}} \mathrm{id}_{\mathbb{R}^3} \, d\mathcal{H}^2 = \int_\Sigma \Phi \, d\sigma_g,$$

where $\Phi : \Sigma \to \mathcal{S}$ is any parametrization of $\mathcal{S}$.

The set of initial data for the conformal Willmore flow will consist *geometrically* of the set of immersed surfaces $\mathcal{S} \subset \mathbb{R}^3$ with Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area 4π and vanishing barycenter $\mathcal{C}(\mathcal{S}) = 0$. *Parametrically* we shall choose a parametrization provided by the above theorem, which is in addition well-balanced as in Definition 5.1.2. More precisely:

Definition 5.1.4. *For $\varepsilon > 0$, $\mathscr{D}^\varepsilon(S^2, \mathbb{R}^3)$ is the set of smooth conformal immersions $\Phi : S^2 \to \mathbb{R}^3$ so that the surface $\mathcal{S} = \Phi(S^2)$ has Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area $\mathcal{A}(\mathcal{S}) = 4\pi$, barycenter $\mathcal{C}(\mathcal{S}) = 0$, is well-balanced and so that* (5.1.8) *holds for $C > 0$ given by that estimate.*

This is, when restricted to the smooth category, the suitable class of initial data that shall be considered in this work, for sufficiently small $\varepsilon > 0$. It will be enlarged to its weak $W^{2,2}$-closure when considering the extension of the theory to the weak framework, which we discuss below.

We want to stress that the only essential requirement in Definition 5.1.4 is the control (smallness) of the Willmore energy. All the others can be seen as normalizations. More precisely, the first result of this work, which will be used to prove the main one, is the following extension of the theorem above:

Proposition 5.1.5. *There are ε_0, δ, $C > 0$ with the following properties:*

(i) Any immersed surface $\mathcal{S} \subset \mathbb{R}^3$ with area 4π and Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon_0$ admits a conformal parametrization satisfying (5.1.8) which is also well-balanced.

(ii) For any well-balanced conformal parametrization $\Psi : S^2 \to \mathcal{S}$ with conformal factor e^{ν} and any vector $c \in \mathbb{R}^3$ so that

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^{\nu} - 1\|_{L^\infty(S^2)} \leq \delta, \tag{5.1.9}$$

there holds

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^{\nu} - 1\|_{L^\infty(S^2)} \leq C\mathcal{W}_0(\mathcal{S}).$$

Furthermore, the following local uniqueness *property holds: if Ψ' is another well-balanced conformal immersion satisfying (5.1.9), and $\psi \in \mathrm{Aut}(S^2)$ is the conformal diffeomorphism so that $\Psi' = \Psi \circ \psi$, there is a neighborhood $\mathcal{O} \subset \mathrm{Aut}(S^2)$ of the identity e (depending only on δ) so that, if $\psi \in U$, then $\psi = e$.*

5.1.4 Conformal Weak Flows

We now define an energy– class of maps where one can consider weak conformal Willmore flows. We believe it to the a prototype for future works concerned with Willmore flows at energy level.

Central to the definitions we shall give shortly is that the validity of the energy identity (5.1.2) (in fact, a slightly weaker version will suffice). This should be, broadly speaking, a requirement to avoid the presence of pathological solutions that invalidate the uniqueness of the solution to the Cauchy problem, as the examples of Topping [Top02] and Bertsch, Dal Passo and van der Hout [BDvdH02] show in the case of the harmonic map flow.

From [Riv08, Riv14, Riv16] we recall the notion of weak $W^{2,2}$-Lipschitz immersion. If $W^{1,\infty}_{\mathrm{imm}}(S^2, \mathbb{R}^3)$ denotes the set of Lipschitz immersions, namely those Lipschitz maps $\Phi : S^2 \to \mathbb{R}^3$ so that there exists $C = C(\Phi) > 0$ with

$$\frac{1}{C} g_{S^2} \leq g = \Phi^\star g_{\mathbb{R}^3} \leq C g_{S^2}$$

almost everywhere in the sense of metrics, we let

$$\mathscr{E}(S^2, \mathbb{R}^3) = W^{1,\infty}_{\mathrm{imm}}(S^2, \mathbb{R}^3) \cap W^{2,2}(S^2).$$

Every map in such set admits a conformal reparametrization and moreover, it is possible to define its Willmore operator in the sense of distributions. Starting from the divergence form of the Willmore operator introduced in [Riv08]:[1]

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H - 2(\nabla H)^{\top_\Phi} - |H|^2 d\Phi\Big) = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big), \tag{5.1.10}$$

[1] We denote, here and in the sequel:

- $\nabla^{*_g}(Z \otimes \omega) = \frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}\, g^{\mu\nu}\omega_\nu Z\big)$, *minus* the formal L^2-adjoint of the covariant derivative induced on the pull-back bundle $\Phi^\star(T\mathbb{R}^3)$ acting on sections of $\Phi^\star(T\mathbb{R}^3) \otimes T^*S^2$,
- $\langle A, H\rangle^{\sharp_g} = g^{\mu\xi}\langle A_{\xi\nu}, H\rangle \partial_\mu \otimes dx^\nu \simeq g^{\mu\xi}\langle A_{\xi\nu}, H\rangle \partial_\mu \Phi \otimes dx^\nu$ the the 1st-index raising of $\langle A, H\rangle$, and similarly for $\langle A^\circ, H\rangle$.

the Willmore operator of $\Phi \in \mathscr{E}(S^2,\mathbb{R}^3)$ is defined as the distribution-valued two form given by

$$\Big(\delta\mathcal{W} d\sigma_g, \varphi\Big)_{\mathcal{D}'} = \int_{S^2} \Big(\Big\langle H, \Delta_g \varphi\Big\rangle + \Big\langle \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}, \nabla\varphi\Big\rangle_g\Big) d\sigma_g,$$

for every $\varphi \in C^\infty(S^2,\mathbb{R}^3)$.

We can now give the following central definitions.

Definition 5.1.6 (Weak Initial Data). *For $\varepsilon > 0$, $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ is the closure of $\mathscr{D}^\varepsilon(S^2,\mathbb{R}^3)$ (from Definition 5.1.4) with respect to the weak $W^{2,2}(S^2)$-topology.*

We will consider $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ only for $\varepsilon > 0$ sufficiently small. As a consequence of the works [Riv08, Riv14, Riv16], $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ is a subset of the space $\mathscr{E}(S^2,\mathbb{R}^3)$, which would be the broadest possible choice for the theory (among nonbranched surfaces, at least). We do not know at present whether $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ coincides, or strictly contained in, $\mathscr{E}(S^2,\mathbb{R}^3)$.

Definition 5.1.7 (Well-Balanced Energy Class). *For ε, δ, $T > 0$, $\mathscr{W}^{\varepsilon,\delta}_{[0,T]}(S^2,\mathbb{R}^3)$ is set of locally integrable maps $\Phi : (0,T) \times S^2 \to \mathbb{R}^3$ so that*

(i) For almost every t, $\Phi(t,\cdot)$ is in $\mathscr{E}(S^2,\mathbb{R}^3)$ and conformal,

(ii) There holds

$$\|\Phi - I - c\|_{L^\infty((0,T),W^{2,2}(S^2))} + \|e^\lambda - 1\|_{L^\infty((0,T)\times S^2)} \leq \delta, \tag{5.1.11}$$

where I denotes standard embedding of S^2, $e^\lambda = e^{\lambda(t,\cdot)}$ is the conformal factor of $\Phi(t,\cdot)$ and $c(t) = \fint_{S^2} \Phi(t,\cdot)\, d\sigma$,

(iii) There holds

$$\delta\mathcal{W} \in L^2((0,T)\times S^2) \quad \text{and} \quad \mathcal{W}_0(\Phi(t,\cdot)) \leq \varepsilon \;\; \text{for a.e. } t, \tag{5.1.12}$$

(iv) Φ is well-balanced for a.e. t.

Finally we let, also for $T = +\infty$,

$$\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3) = \bigcap_{\tau\in(0,T)} \mathscr{W}^{\varepsilon,\delta}_{[0,\tau]}(S^2,\mathbb{R}^3).$$

Assumption (5.1.11) is quite natural if we look at Proposition 5.1.5. In the energy class, a weak Willmore flow is defined as follows.

Definition 5.1.8 (Weak Willmore Flow). *$\Phi \in \mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ is a weak solution of the Willmore flow with tangential component $U = U^\mu \partial_\mu \Phi$:*

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U \quad \text{in } (0,T)\times S^2,$$

if for every $\varphi \in C^\infty_c((0,T)\times S^2,\mathbb{R}^3)$ there holds

$$-\int_0^T \int_{S^2} \Big\langle \Phi, \frac{\partial}{\partial t}\varphi\Big\rangle\, d\sigma_g\, dt = -\int_0^T \Big(\delta\mathcal{W} d\sigma_g, \varphi(t,\cdot)\Big)_{\mathcal{D}'} dt + \int_{S^2} \Big\langle U, \varphi\Big\rangle\, d\sigma_g\, dt.$$

Our main result is the following.

Theorem 5.1.9. *There exists $\varepsilon_0 > 0$ so that the Cauchy problem for the conformal Willmore flow (5.1.5)-(5.1.6) with initial datum in $\mathscr{W}^{\varepsilon_0}(S^2, \mathbb{R}^3)$ has a weak solution in $\mathscr{W}^{\varepsilon_0;\delta}_{[0,T)}(S^2, \mathbb{R}^3)$ for some $\delta > 0$, assuming the initial datum in the sense of traces. Such solution is smooth, exists for all times and smoothly converges to the standard embedding I of S^2 in $\mathbb{R}^3$. Furthermore, if the initial datum is smooth, such weak solution is also unique.*

We can compare this result with the above metioned one of Kuwert and Schätzle [KS01, KS02]. They obtain, in the smooth class, long-time existence, uniqueness and convergence to a round sphere for the Cauchy problem of the normal flow (5.1.3). A central feature our result is that the uniqueness of this smooth solution is in the broad class of finite energy solutions, and the fact that it converges exactly to the standard embedding.

We expect the solution to be unique also if the initial datum is nonsmooth; we plan to address this question in the future.

The proof of the regularity part of Theorem 5.1.9 shares evident similarities with the corresponding one for the harmonic map flow obtained in [Riv93]. In that work, the core estimate that was obtained for weak solutions of the harmonic map flow was of the form

$$\|u(t,\cdot)\|_{W^{2,2}} \leq C\Big(\|\partial_t u(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

which could then be squared and integrated in time to yield higher regularity, and eventually smoothness by the classical theory by Struwe [Str85, Str08]. We shall obtain a similar result, namely an inequality of the form

$$\|\Phi(t,\cdot)\|_{W^{4,2}} \leq C\Big(\|\mathrm{e}^{\lambda}\delta\mathcal{W}(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

for weak solutions of the conformal Willmore flow, and likewise obtain higher regularity from it. The overall procedure shall be however more technical.

5.1.5 Final Comments We have intentionally decided to work in a small-energy regime in this first contribution on the subject. We plan to consider more general scenarios in future works, where more technical, localization/energy-concentration arguments will be dealt with.

One has also to take into account that, when the underlying surface is not a sphere, there is more than one conformal class, so to properly work with a conformal Willmore flow, one has to take into account the nontriviality of the corresponding Teichmüller space. The work of Rupflin and Topping [RT16] on the Teichmüller harmonic map flow also faces the difficulty of "following" the conformal class along the flow.

In the future we plan to determine whether the class $\mathscr{W}^{\varepsilon}(S^2, \mathbb{R}^3)$ coincides or not with $\mathscr{E}(S^2, \mathbb{R}^3)$, and whether the solution given by Theorem 5.1.9 is unique also in the case of weak initial data. We shall also seek to extend the argument for branched weak initial data.

Finally, we plan to carry an accurate study of singularities (blow-up points, degeneration of conformal factor or conformal class...) in forthcoming works. For this we will likely build upon some of the work already done on the subject, Mayer and Simonett [MS02], Blatt [Bla09] and Chill, Fašangová and Schätzle [CFS09] just to mention a few.

One of the questions relative to the parametric approach of the Willmore flow is the following: can the conformal class of a conformal Willmore flow – suitably normalized to remove any obvious gauge invariance – degenerate in finite time?

5.2 Consequences of De Lellis-Müller Theorem

First we recall:

Theorem 5.2.1 ([DM05, DM06]). *Let $\mathcal{S} \subset \mathbb{R}^3$ be an immersed surface with area $\mathcal{A}(\mathcal{S}) = 4\pi$ and let g be the induced metric. Then there holds*

$$\int_{\mathcal{S}} |A_{sc} - g|_g^2 \, d\sigma_g \leq C \int_{\mathcal{S}} |A^\circ|_g^2 \, d\sigma_g, \tag{5.2.1}$$

where $A_{sc}(\cdot\,,\cdot) = \langle A(\cdot\,,\cdot), N\rangle$ is the scalar second fundamental form of $\mathcal{S}$. Furthermore, there is $\varepsilon_0 > 0$ so that if

$$\mathcal{W}_0(\mathcal{S}) = \int_{\mathcal{S}} |A^\circ|_g^2 \, d\sigma_g \leq \varepsilon_0,$$

there exists a conformal parametrization $\Phi : S^2 \to \mathbb{R}^3$ satisfying

$$\|\Phi - (c_{\mathcal{S}} + I)\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\lambda - 1\|_{L^\infty(S^2)} \leq C\mathcal{W}_0(\mathcal{S}), \tag{5.2.2}$$

for some vector $c_{\mathcal{S}} \in \mathbb{R}^3$ and for an absolute constant $C > 0$, where e^λ is the conformal factor of the induced metric and I is the standard immersion of S^2 into $\mathbb{R}^3$.

Remark 5.2.2 From the minimality property of the average:

$$\left\| f - ⨍_{S^2} f \, d\sigma \right\|_{L^2(S^2)} = \inf_{c\in\mathbb{R}} \|f - c\|_{L^2(S^2)},$$

and since $\int_{S^2} I \, d\sigma$, we may suppose $c_{\mathcal{S}} = ⨍_{S^2} \Phi \, d\sigma_{S^2}$ in (5.2.2).

The proof of Proposition 5.1.5 will follow from two lemmas.

Lemma 5.2.3. *The function $F : \mathrm{Aut}(S^2) \to \mathbb{R}^6$ given by*

$$F(\psi) = (F_1(\psi), F_2(\psi)) = \Big(\int_{S^2} I \, \psi^\star d\sigma, \int_{S^2} (I \circ \psi) \times I \, d\sigma \Big), \tag{5.2.3}$$

where $\psi^\star d\sigma$ denotes the pullback of area element $d\sigma$ via ψ, is differentiable and $dF(e)$ is an isomorphism.

Proof. Differentiability follows since F is composition of smooth functions and operations. We shall now use the language of differential forms and so along this proof it is convenient to temporarily change our notation for the area element from $d\sigma$ to ω_{S^2}.

Recall that $\omega_{S^2} = \omega_{\mathbb{R}^3} \llcorner N = \omega_{\mathbb{R}^3}(N, \cdot, \cdot)$, where N Gauss map of S^2. More explicitly, since $N(y) = y$, we have the formula

$$\omega_{S^2} = (dy^1 \wedge dy^2 \wedge dy^3)(y, \cdot, \cdot) = y^1 dy^2 \wedge dy^3 - y^2 dy^1 \wedge dy^3 + y^3 dy^1 \wedge dy^2.$$

Now $\mathrm{Aut}(S^2)$ has dimenion 6 as a manifold and we consider the basis for the tangent space $T_e \mathrm{Aut}(S^2)$ given by the vector fields generating, respectively, rotations and "spherical dilations" about the coordinate axes:

$$Z_1(y) = (0, -y^3, y^2), \qquad Z_2(y) = (y^3, 0, -y^1), \qquad Z_3(y) = (-y^2, y^1, 0),$$
$$Z_4(y) = e_1 - y^1 y, \qquad Z_5(y) = e_2 - y^2 y, \qquad Z_6(y) = e_3 - y^3 y.$$

We shall prove that

$$\left(\frac{\partial F^j}{\partial Z_a}(e)\right)_{1\le a,j\le 6} = -\frac{8\pi}{3}\mathbf{1}_{6\times 6}. \tag{5.2.4}$$

To compute $\partial_X F(e)$, if Φ^X denotes the local flow of the vector field X, we have to evaluate

$$\frac{\partial}{\partial X}F(e) = \frac{d}{dt}F(\Phi^X(t,\cdot))\bigg|_{t=0}.$$

Let us look at F_1. By Cartan's formula, since $d\omega_{S^2} = 0$, it is

$$\frac{\partial}{\partial t}\Phi^X(t,\cdot)\omega_{S^2}\bigg|_{t=0} = \mathcal{L}_X\omega_{S^2} = d(X \llcorner \omega_{S^2}),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Since Z_a's for $a = 1,2,3$ generate isometries, $\mathcal{L}_{Z_a}\omega_{S^2} = 0$ and hence *a fortiori*

$$\frac{d}{dt}F_1\Big(\Phi^{Z_a}(t,\cdot)\Big)\bigg|_{t=0} = \int_{S^2} I\,\mathcal{L}_{Z_a}\omega_{S^2} = 0 \qquad \text{for } a = 1,2,3.$$

As for the Z_a's for $a = 4,5,6$, we see that

$$\begin{aligned} Z_4 \llcorner \omega_{S^2} &= -y^2dy^3 + y^3dy^2, & Z_5 \llcorner \omega_{S^2} &= y^1dy^3 - y^3dy^1, & Z_6 \llcorner \omega_{S^2} &= -y^1dy^2 + y^2dy^1, \\ d(Z_4 \llcorner \omega_{S^2}) &= -2dy^2\wedge dy^3, & d(Z_5 \llcorner \omega_{S^2}) &= 2dy^1\wedge dy^3, & d(Z_6 \llcorner \omega_{S^2}) &= -2dy^1\wedge dy^2, \end{aligned}$$

and hence with Stokes' theorem we get

$$\begin{aligned} \frac{d}{dt}F_1(\Phi^{Z_4}(t,\cdot))\bigg|_{t=0} &= \int_{S^2} I\mathcal{L}_{Z_4}\omega_{S^2} \\ &= \Big(\int_{S^2} y^1(-2dy^2\wedge dy^3), \int_{S^2} y^2(2dy^1\wedge dy^3), \int_{S^2} y^3(-2dy^1\wedge dy^2)\Big) \\ &= -\frac{8\pi}{3}(1,0,0), \end{aligned}$$

and similarly

$$\frac{d}{dt}F_1\Big(\Phi^{Z_5}(t,\cdot)\Big)\bigg|_{t=0} = -\frac{8\pi}{3}(0,1,0), \qquad \frac{d}{dt}F_1\Big(\Phi^{Z_6}(t,\cdot)\Big)\bigg|_{t=0} = -\frac{8\pi}{3}(0,0,1).$$

Now we consider F_2. Since we can write

$$F_2\Big(\Phi^X(t,\cdot)\Big) = \int_{S^2}\Phi^X(t,\cdot)\times I\,d\sigma,$$

it is

$$\frac{d}{dt}F_2\Big(\Phi^X(t,\cdot)\Big)\bigg|_{t=0} = \int_{S^2} X\times I\,d\sigma,$$

and thus one directly computes that for $a = 1,2,3$ it is

$$\int_{S^2} Z_a\times I\,d\sigma = -\frac{8\pi}{3}e_a, \qquad \int_{S^2} Z_{a+3}\times I\,d\sigma = 0.$$

Putting together all these computations yields (5.2.4) and hence the thesis. $\square$

Lemma 5.2.4. *There exists $E > 0$ with the following property. For any $\eta_1 \le E$ there exist $\eta_2 > 0$ so that, if $\mathcal{S} \subset \mathbb{R}^3$ is an immersed surface with area $\mathcal{A}(\mathcal{S}) = 4\pi$ and $\Phi : S^2 \to \mathcal{S}$ is a conformal immersion with conformal factor e^λ and $c \in \mathbb{R}^3$ is a vector so that*

$$\|\Phi - I - c\|_{W^{2,2}(S^2)} + \|e^\lambda - 1\|_{L^\infty(S^2)} \le \eta_1, \tag{5.2.5}$$

then there exists a conformal self-map $\psi \in \mathrm{Aut}(S^2)$ so that $\Psi = \Phi \circ \psi$ is well-balanced and ψ is the unique self-map with such property in the Riemannian ball $B_{\eta_2}(e) \subset \mathrm{Aut}(S^2)$. In addition, if e^ν denotes the conformal factor of Ψ, there holds

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)} \le 2\eta_1. \tag{5.2.6}$$

Proof. Let $\mathcal{F} : W^{2,2}(S^2, \mathbb{R}^3) \times \mathrm{Aut}(S^2) \to \mathbb{R}^6$ be given by

$$\mathcal{F}(f, \psi) = \Big(\int_{S^2} I \, \frac{1}{2} |d(f \circ \psi)|^2 d\sigma, \int_{S^2} (f \circ \psi) \times I \, d\sigma \Big).$$

Note that this definition makes sense for every $f \in W^{2,2}(S^2)$ and, if Φ is a conformal immersion $\mathcal{F}(\Phi, e) = 0$ means that Φ is well-balanced as in Definition 5.1.2. Moreover $\mathcal{F}(I, \cdot) = 0$ coincides with F given in (5.2.3). Finally $\mathcal{F}$ is invariant by translations in its first component: $\mathcal{F}(f, \cdot) = \mathcal{F}(f + k, \cdot)$ for every $k \in \mathbb{R}^3$.

As a consequence of Lemma 5.2.3, $d_\psi \mathcal{F}(I, e) = d_\psi \mathcal{F}(I, \cdot)(e)$ is an isomorphism, and hence by the implicit function theorem, there exists $E, \eta_2 > 0$ so that if (5.2.5) holds for $\eta_1 \le E$, there is a unique $\psi = \psi_\Phi$ in the Riemannian ball $B_{\eta_2}(e) \subset \mathrm{Aut}(S^2)$ so that $\mathcal{F}(\Phi - c, \psi) = \mathcal{F}(\Phi, \psi) = 0$, i.e. so that $\Psi = \Phi \circ \psi$ is well-balanced (recall also Remark 5.1.3).

Finally, since ψ is biholorphic, $\forall N \in \mathbb{N}$ we can estimate $\sum_{k=1}^N \mathrm{dist}(\nabla^k \psi, \nabla^k e) \le C_N \,\mathrm{dist}(\psi, e)$ for some $C_N > 0$ independent of ψ. So (5.2.5) holds, by the triangle inequality and the continuity of the Lebesgue integral,

$$\begin{aligned}
&\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)} \\
&= \|\Phi \circ \psi - I - c\|_{W^{2,2}(S^2)} + \|\tfrac{1}{\sqrt{2}} |d\psi| \mathrm{e}^{\lambda \circ \psi} - 1\|_{L^\infty(S^2)} \\
&= \|(\Phi - c) \circ \psi - I\|_{W^{2,2}(S^2)} + \|\tfrac{1}{\sqrt{2}} |d\psi| \mathrm{e}^{\lambda \circ \psi} - 1\|_{L^\infty(S^2)} \\
&= \eta_1 + o(1) \quad \text{as } \mathrm{dist}(\psi, e) \to 0,
\end{aligned}$$

and, since $\|\Psi - c\|_{L^2(S^2)}$ and $\|e^\lambda\|_{L^\infty(S^2)}$ are uniformly bounded, the remainder $o(1)$ can be taken uniform in Φ, Ψ, ψ, c and hence, choosing η_2 sufficiently small we obtain to (5.2.6). $\square$

Proof of Proposition 5.1.5. It suffices to prove the thesis for $\mathcal{W}_0(\mathcal{S}) \le \varepsilon_0$ sufficiently small. For part (i), combine Theorem 5.2.1 and Lemma 5.2.4.

For part (ii), Let $\varepsilon_0 > 0$ be sufficiently small so that

$$C\varepsilon_0 \le \frac{1}{2} E,$$

where H is as in Lemma 5.2.4 and C is the constant of Theorem 5.2.1, and let $\Phi : S^2 \to \mathcal{S}$ be the conformal parametrization given by that theorem. By Lemma 5.2.4 there exists a unique choice of $\alpha = \alpha_\Phi$ in $B_{\eta_2}(e) \subset \mathrm{Aut}(S^2)$ with $\mathcal{F}(\Phi, \alpha) = 0$ i.e. $\Phi' = \Phi \circ \alpha$ is well-balanced and

$$\|\Phi' - I - c\|_{W^{2,2}(S^2)} + \|e^{\lambda'} - 1\|_{L^\infty(S^2)} \le 2C\mathcal{W}_0(\Phi') \le E.$$

So now if δ is taken so that

$$\delta \leq \frac{1}{2}E,$$

since Ψ is already well-balanced, by uniqueness it must be $\Phi' = \Psi$, and the thesis follows also for the local uniqueness part, with $\mathcal{O} = B_{\eta_2}(e)$. $\square$

The following simple consequence of Theorem 5.2.1 will also be needed later.

Lemma 5.2.5. *If $\Phi : S^2 \to \mathbb{R}^3$ is a conformal immersion with conformal factor e^λ and $B_r(x_0) \subset S^2$ is a disk of radius r (in the standard metric of S^2), there holds*

$$\int_{B_r(x_0)} |A|_g^2 \, d\sigma_g \leq C\Big(\int_{S^2} |A^\circ|_g^2 \, d\sigma_g + \mathrm{e}^{4C_0} r^2 \Big), \tag{5.2.7}$$

where $C_0 = \|\lambda\|_{L^\infty(S^2)}$ and $C > 0$ is an absolute constant.

Proof. It is a consequence of (5.2.1) applied to the immersed surface $\mathcal{S} = a\,\Phi(S^2)$, where $a = \sqrt{\frac{4\pi}{\mathcal{A}(\Phi)}}$ and $\mathcal{A}(\Phi) = \int_{S^2} \mathrm{e}^{2\lambda}\, d\sigma$ is the area of $\Phi(S^2)$. Indeed, since

$$4\pi \mathrm{e}^{-2C_0} \leq \mathcal{A}(\Phi) \leq 4\pi \mathrm{e}^{2C_0},$$

it follows that $e^{-C_0} \leq a \leq \mathrm{e}^{C_0}$ and we can estimate

$$\int_{B_r(x_0)} |a\, g|_g^2 d\sigma_g = a \int_{B_r(x_0)} 2\mathrm{e}^{2\lambda} d\sigma \leq C\mathrm{e}^{4C_0} r^2.$$

and thus

$$\begin{aligned}\int_{B_r(x_0)} |A|_g^2 \, d\sigma_g &\leq 2\int_{B_r(x_0)} |A_{\mathrm{sc}} - a_\Phi g|_g^2 \, d\sigma_g + 2\int_{B_r(x_0)} |a\, g|_g^2 \, d\sigma_g \\ &\leq C\int_{B_r(x_0)} |A^\circ|_g^2 \, d\sigma_g + C\mathrm{e}^{4C_0} r^2,\end{aligned}$$

which proves (5.2.7). $\square$

5.3 Conformal Willmore Flows

Proof of Lemma 5.1.1. A metric g is conformal if and only if its Hopf differential (computed with respect to the background complex structure of S^2) vanishes identically. In our case it is

$$\mathrm{Hopf}(g) = g_{zz} dz \otimes dz = \langle \partial_z \Phi, \partial_z \Phi \rangle dz \otimes dz.$$

Since $\delta\mathcal{W}$ is a normal vector field, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \partial_z\Phi, \partial_z\Phi\rangle = \langle \partial_z\partial_t\Phi, \partial_z\Phi\rangle = \langle -\partial_z\delta\mathcal{W} + \partial_z U, \partial_z\Phi\rangle = \langle \delta\mathcal{W}, \partial^2_{zz}\Phi\rangle + \langle \partial_z U, \partial_z\Phi\rangle.$$

Since $U = U^z \partial_z\Phi + U^{\bar z}\partial_{\bar z}\Phi$ with $U^z = U^1 + iU^2$ and $U^{\bar z} = U^1 - iU^2$ we have

$$\begin{aligned}\langle \partial_z U, \partial_z \Phi\rangle &= \Big\langle \partial_z U^z \partial_z\Phi + U^z\partial^2_{zz}\Phi + \partial_z U^{\bar z}\partial_{\bar z}\Phi + U^{\bar z}\partial^2_{z\bar z}\Phi, \partial_z\Phi\Big\rangle \\ &= \partial_z U^z g_{zz} + U^z\Big\langle \partial^2_{zz}\Phi, \partial_z\Phi\Big\rangle + \partial_z U^{\bar z} g_{\bar z z} + U^{\bar z}\Big\langle \partial^2_{z\bar z}\Phi, \partial_z\Phi\Big\rangle \\ &= \partial_z U^z g_{zz} + \partial_z U^{\bar z} g_{\bar z z} + \frac{1}{2}\Big(U^z\partial_z g_{zz} + U^{\bar z}\partial_{\bar z} g_{zz}\Big),\end{aligned}$$

thus we have

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \partial_z \Phi, \partial_z \Phi\rangle = \partial_z U^z g_{zz} + \partial_z U^{\bar{z}} g_{\bar{z}z} + \frac{1}{2}\Big(U^z \partial_z g_{zz} + U^{\bar{z}} \partial_{\bar{z}} g_{zz}\Big) + \Big\langle \delta\mathcal{W}, \partial^2_{zz}\Phi\Big\rangle.$$

If the flow Φ is conformal, then $\partial_t \operatorname{Hopf}(g) \equiv 0$ and $g_{zz}, g_{\bar{z}\bar{z}}$ vanish identically. Since moreover $\delta\mathcal{W}$ is a normal vector we may replace $\partial^2_{zz}\Phi$ with A_{zz} and thus obtain, after conjugation,

$$g_{\bar{z}z}\partial_z U^{\bar{z}} = -\langle \delta\mathcal{W}, A_{zz}\rangle,$$

and so since $\frac{1}{2}e^{2\lambda} = g_{\bar{z}z}$, this yields (5.1.6). □

Next, we recall the following.

Theorem 5.3.1 ([KS01, KS02, KS20]). *There exists an $\varepsilon_0 = \varepsilon_0(n) > 0$ so that, if $\Phi : [0,T) \times S^2 \to \mathbb{R}^n$ is a smooth normal Willmore flow*

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W},$$

with smooth initial datum $\Phi(0,\cdot) = \Phi_0$ and $\mathcal{W}_0(\Phi_0) \le \varepsilon_0$, then:

(i) its area satisfies

$$|\mathcal{A}(\Phi(t,\cdot)) - \mathcal{A}(\Phi_0)| \le C\mathcal{A}_0(\Phi_0)\mathcal{W}_0(\mathcal{S}),$$

for a constant $C = C(n) > 0$.

(ii) Its barycenter $\mathcal{C}(\Phi) = \fint_{S^2} \Phi \, d\sigma_g$ satisfies

$$|\mathcal{C}(\Phi(t,\cdot)) - \mathcal{C}(\Phi_0)| \le C\mathcal{W}_0(\mathcal{S})$$

for a constant $C = C(n) > 0$.

(iii) Φ exists for all times and smoothly converges to a round sphere $t \to \infty$.

The proof of the following proposition follows directly by direct inspection of the proof of the original theorems; for the proof of (ii) one uses additionally that conformal Willmore flows are $W^{4,2}$ for almost every time, as proved in Proposition 5.3.3 below.

Proposition 5.3.2. *(i) If the surface $\mathcal{S}$ is immersed through a $W^{4,2}$-map, then the same conclusion of Theorem 5.2.1 and all its consequences obtained in Section 5.2 still hold.*

(ii) For conformal Willmore flows in $\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2, \mathbb{R}^3)$ the area and barycenter bounds in Theorem 5.3.1 still hold if ε is chosen sufficiently small.

A first regularity improvement for conformal Willmore flows is the following.

Proposition 5.3.3. *For any ε_0, $\delta > 0$, any conformal Willmore flow $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2, \mathbb{R}^3)$ is in $W^{4,2}(S^2)$ for a.e. $t \in (0,T)$. Futhermore there exist ε_0, δ, $C > 0$ independent of Φ so that $\Phi \in L^2((0,T), W^{4,2}(S^2))$ with*

$$\|d\Phi\|_{L^2((0,T),W^{3,2}(S^2))} \le C(\sqrt{T} + \|e^{\lambda}\delta\mathcal{W}\|_{L^2((0,T)\times S^2)}), \tag{5.3.1}$$

for a constant $C > 0$.

Proof. Since $\delta\mathcal{W} \in L^2((0,T) \times S^2)$, Fubini's theorem implies $\delta\mathcal{W}(t,\cdot) \in L^2(S^2)$ for a.e. $t \in (0,T)$. Consequently, by Propositions 4.3.1 and 4.3.2 of Chapter 4 it follows that $\Phi(t,\cdot)$ is in $W^{4,p}(S^2)$. This yields that $A(t,\cdot) \in L^\infty(S^2)$ an information that inserted back in (5.1.6) yields that $U(t,\cdot) \in L^2(S^2)$, whence that $\Phi(t,\cdot) \in W^{4,2}(S^2)$ by Proposition 4.3.2.

From Liouville's equation

$$\Delta_{S^2}\lambda = \mathrm{e}^{2\lambda}K - 1,$$

the pointwise inequality $|K| \leq |A|_g^2 = \mathrm{e}^{-4\lambda}|A|^2$ and elliptic estimates

$$\|d\lambda\|_{L^\infty((0,T),L^{(2,\infty)}(S^2))} \leq C. \tag{5.3.2}$$

Choose now $\varepsilon_1 > 0$ sufficiently small so that, for a.e. $t \in (0,T)$ so that $\Phi(t,\cdot)$ is $W^{4,2}$, we can fix a value $r > 0$ which satisfies, according to Lemma 5.2.5,

$$\int_{B_r(x_0)} |A|^2 \, d\sigma \leq C\Big(\varepsilon_1 + \mathrm{e}^C r^2\Big) \leq \varepsilon_0, \tag{5.3.3}$$

for every $x_0 \in S^2$, where ε_0 is as in Theorem 4.1.2 of Chapter 4. With the estimates (5.3.2) and (5.3.3), an application of Theorem 4.1.2 to a covering S^2 with balls of radius $r/2$ gives, for a.e. $t \in (0,T)$,

$$\|d\Phi\|_{W^{3,2}(S^2)} \leq C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(S^2)} + \|\mathrm{e}^{\lambda}\|_{L^2(S^2)}\Big) \leq C\Big(\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)} + 1\Big).$$

Now if we square and integrate in t such inequality, recalling that, since $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}$, (5.1.12) holds by assumption, we obtain (5.3.1). $\square$

One can see that, in fact, along the proof of Proposition 5.3.3, well-balanced condition (iv) in Definition 5.1.7 was not needed, an in fact the proof is completely independent of the tangential component U. Such condition is however essential to prove the following.

Proposition 5.3.4. *For every $p < 2$ there exist $\delta > 0$ with the following property. Let $\Phi \in \mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ be a weak Willmore flow. Then its tangential component is in $L^2((0,T),L^p(S^2))$, with*

$$\|U\|_{L^2((0,T),L^p(S^2))} \leq C\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2((0,T),L^2(S^2))}, \tag{5.3.4}$$

for a constant $C = C(p) > 0$.

Proof. We may certainly assume $p > 1$. In this proof we find it convenient to clearly distinguish between the non-immersed (pulled-back) tangential component $U = U^\mu\partial_\mu$, the immersed one $d\Phi(U) = U^\mu\partial_\mu\Phi$, and the associated tangent vector field on S^2, $dI(U) = U^\mu\partial_\mu I$, where $I : S^2 \to \mathbb{R}^3$ denotes is the standard immersion. With this notation Φ satisfies weakly

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + d\Phi(U) \quad \text{in } (0,T) \times S^2.$$

In what follows, it will be implicitly understood that all the slice-wise operations are valid for a.e. fixed t. Finally we shall make use of the notation and concepts recalled in Appendix 5.5.

From Lemma 5.1.1, we deduce that U is given by

$$U^{(1,0)} = -\overline{\partial}^{-1}(\langle \delta\mathcal{W}, \overline{h}_0\rangle^{\sharp_g}) + \Omega$$

for some time-dependent holomorphic vector field $\Omega = \Omega(t,\cdot) \in \mathfrak{X}^{\omega}(S^2)$. Classical elliptic estimates and the simple inequality

$$1 = \mathrm{e}^{-\lambda}\mathrm{e}^{\lambda} \leq \sup_{S^2}(\mathrm{e}^{-\lambda})\mathrm{e}^{\lambda} \leq C(1+\delta)\mathrm{e}^{\lambda}$$

issuing from property (5.1.11) and Hölder permit to estimate

$$\begin{aligned}
\|\overline{\partial}^{-1}(\langle \delta\mathcal{W}, \overline{h}_0\rangle^{\sharp_g})\|_{L^p(S^2)} &\leq C_p\|\langle \delta\mathcal{W}, \overline{h}_0\rangle^{\sharp}_g\|_{L^1(S^2)} \\
&\leq C_p\||\delta\mathcal{W}|\mathrm{e}^{-2\lambda}|A^\circ|\|_{L^1(S^2)} \\
&\leq C_p\|\mathrm{e}^{-\lambda}|A^\circ|\|_{L^2(S^2)}\|\delta\mathcal{W}\|_{L^2(S^2)} \\
&\leq C_p\mathcal{W}_0(\Phi)\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)} \\
&\leq C_p\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)}.
\end{aligned} \tag{5.3.5}$$

We now examine Ω. It will be more practical to look at the associated (time-dependent) conformal Killing vector field i.e. generating conformal transformations:

$$V = \Omega + \overline{\Omega},$$

Similarly as in the proof of Lemma 5.2.3, a basis for the vector space of conformal Killing fields $T_e\,\mathrm{Aut}(S^2)$ is given, in its immersed representative, is given by

$$\begin{aligned}
&dI(Z_1)(y) = (0,-y^3,y^2), \qquad &&dI(Z_2)(y) = (y^3,0,-y^1), \qquad &&dI(Z_3)(y) = (-y^2,y^1,0), \\
&dI(Z_4)(y) = e_1 - y^1 y, &&dI(Z_5)(y) = e_2 - y^2 y, &&dI(Z_6)(y) = e_3 - y^3 y.
\end{aligned}$$

One checks that this basis is orthogonal with respect the L^2–scalar product and each element has the same length. Thus, we may write

$$V = \sum_{a=1}^{6} V^a Z_a = C\sum_{a=1}^{6}(V,Z_a)_{L^2(S^2)}Z_a = C\sum_{a=1}^{6}(U,Z_a)_{L^2(S^2)}Z_a, \tag{5.3.6}$$

where the last inequality is a consequence of the fact that, by construction, the normal solution of the $\overline{\partial}$-operator is L^2-orthogonal to the space of holomorphic vector fields. So, to estimate V it suffices to estimate the (time-dependent) coefficients

$$(V,Z_a)_{L^2} = \int_{S^2}\langle U, Z_a\rangle\, d\sigma \quad \text{for } a = 1,\dots,6.$$

Now, we may write the integrand as

$$\langle U, Z_a\rangle = \langle dI(U), dI(Z_a)\rangle = \langle d\Phi(U), dI(Z_a)\rangle + \langle dI(U) - d\Phi(U), dI(Z_a)\rangle$$

where the second term can be estimated as

$$\Big|\langle dI(U) - d\Phi(U), dI(Z_a)\rangle\Big| \leq C|dI - d\Phi||U|$$

and thus, upon integration, Hölder's inequality and property (5.1.11) of the set $\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$, we see that the estimate

$$\begin{aligned}\left|\int_{S^2}\langle dI(U)-d\Phi(U),dI(Z_a)\rangle d\sigma\right| &\leq C\int_{S^2}|dI-d\Phi||U|\,d\sigma \\ &\leq C\|dI-d\Phi\|_{L^{p'}(S^2)}\|U\|_{L^p(S^2)} \\ &\leq C\|\Phi-I-c\|_{W^{2,2}(S^2)}\|U\|_{L^p(S^2)} \\ &\leq C_p\,\delta\,\|U\|_{L^p(S^2)},\end{aligned} \tag{5.3.7}$$

holds, which will suit our purposes. We are left to estimate the terms

$$\int_{S^2}\langle d\Phi(U),dI(Z_a)\rangle\,d\sigma \quad \text{for } a=1,\dots,6.$$

Differentiating the well-balanced conditions (5.1.7) gives,

$$\begin{aligned}0&=\frac{d}{dt}\int_{S^2}I\times\Phi\,d\sigma=\int_{S^2}I\times(-\delta\mathcal{W}+d\Phi(U))\,d\sigma,\\ 0&=\frac{d}{dt}\int_{S^2}Id\sigma_g=\int_{S^2}I\Big(\langle 2H,\delta\mathcal{W}\rangle+\operatorname{div}_g(U)\Big)d\sigma_g,\end{aligned}$$

and thus that

$$\int_{S^2}I\times d\Phi(U)\,d\sigma=\int_{S^2}I\times\delta\mathcal{W}\,d\sigma, \tag{5.3.8}$$

and, upon integration by parts, that

$$\int_{S^2}dI(U)\,d\sigma_g=\int_{S^2}I\langle 2H,\delta\mathcal{W}\rangle\,d\sigma_g.$$

On the other hand, a direct calculation shows that

$$\begin{pmatrix}\langle d\Phi(U),dI(Z_1)\rangle\\ \langle d\Phi(U),dI(Z_2)\rangle\\ \langle d\Phi(U),dI(Z_3)\rangle\end{pmatrix}=I\times d\Phi(U), \tag{5.3.9}$$

and similarly that

$$\begin{pmatrix}\langle d\Phi(U),dI(Z_4)\rangle\\ \langle d\Phi(U),dI(Z_5)\rangle\\ \langle d\Phi(U),dI(Z_6)\rangle\end{pmatrix}=d\Phi(U)-\langle d\Phi(U),I\rangle I=(d\Phi(U))^\top, \tag{5.3.10}$$

where $(\cdot)^\top$ denotes the orthogonal projection onto the tangent space of S^2 in the standard immersion. Integrating (5.3.9) and using (5.3.8) we can estimate for $a=1,2,3$

$$\Big|(d\Phi(U),dI(X_a))_{L^2}\Big|\leq C\left|\int_{S^2}I\times\delta\mathcal{W}\,d\sigma\right|\leq C\|e^\lambda\delta\mathcal{W}\|_{L^2(S^2)}. \tag{5.3.11}$$

Integrating (5.3.10) we get instead

$$\sum_{a=1}^{3}C(d\Phi(U),dI(Y_a))_{L^2}e_a=\int_{S^2}(d\Phi(U))^\top d\sigma,$$

and if we write the integrand as

$$(d\Phi(U))^\top = (dI(U))^\top + (d\Phi(U) - dI(U))^\top$$

and notice that, similarly as for (5.3.7) we can estimate

$$\begin{aligned}\left|\int_{S^2}(d\Phi(U) - dI(U))^\top d\sigma\right| &\leq C\int_{S^2}|d\Phi(U) - dI(U)|\,d\sigma\\ &\leq C\|d\Phi - dI\|_{L^{p'}(S^2)}\|U\|_{L^p(S^2)}\\ &\leq C_p\,\delta\,\|U\|_{L^p(S^2)},\end{aligned}$$

using (5.3.8) (since $dI(U) = dI(U)^\top$), we have for $a = 4, 5, 6$

$$\begin{aligned}|(d\Phi(U), dI(Y_a))_{L^2}| &\leq C\left|\int_{S^2} dI(U)d\sigma\right| + C_p\,\delta\,\|U\|_{L^p(S^2)}\\ &\leq C\left|\int_{S^2} I\langle 2H, \delta\mathcal{W}\rangle\, d\sigma_g\right| + C_p\,\delta\,\|U\|_{L^p(S^2)}\\ &\leq C\|H\mathrm{e}^\lambda\|_{L^2(S^2)}\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + C_p\,\delta\,\|U\|_{L^p(S^2)}\\ &\leq C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big),\end{aligned} \tag{5.3.12}$$

where we also used that $\|H\mathrm{e}^\lambda\|_{L^2(S^2)} = C\mathcal{W}_1(\Phi)$ is bounded uniformly in t.

Estimates (5.3.7), (5.3.11) and (5.3.12) inserted in (5.3.6) yield

$$\|V\|_{L^\infty(S^2)} \leq C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big).$$

and so, in conjunction with (5.3.5), we get

$$\|U\|_{L^p(S^2)} \leq C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big).$$

Taking the L^2-norm in time of such inequality gives

$$\|U\|_{L^2((0,T),L^p(S^2))} \leq C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2((0,T),L^2(S^2))} + \delta\,\|U\|_{L^2((0,T),L^p(S^2))}\Big),$$

and so, if δ is chosen sufficiently small, we reach (5.3.4). □

Remark 5.3.5 The δ of Proposition 5.3.4 may be smaller than that given by Proposition 5.1.5. An inspection of the proof however shows however that we may equivalently have taken the same δ at the price of choosing $\varepsilon > 0$ sufficiently small, because we may apply instead Propositions 5.3.2 and 5.1.5 for a.e. t so that $\Phi(t,\cdot)$ is $W^{4,2}$, and argue in the end similarly as above. This variant would have been equally fine for our purposes.

Corollary 5.3.6. *There exists $\varepsilon_0, \delta > 0$ so that any Willmore flow in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ is in $C^\infty((0,T]\times S^2)$, and, if it is a solution to the Cauchy problem (5.1.5) for smooth initial datum Φ_0, then it is in $C^\infty([0,T]\times S^2)$.*

Proof. By definition and by Proposition 5.3.3, $d\Phi$ is in $L^\infty((0,T),W^{1,2}(S^2))\cap L^2((0,T),W^{3,2}(S^2))$, we have sufficient regularity to expand the Willmore operator in the flow equation by means of formulas (5.1.10) and (3.3.12) of Chapter 3:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi + \frac{1}{2}\mathrm{e}^{-4\lambda}\Delta^2\Phi = \frac{1}{2}\mathrm{e}^{-4\lambda}\Big(&2\langle d\lambda, \nabla H\rangle_g + (2|d\lambda|_g^2 + \Delta_g\lambda)H\Big)\\ &- \nabla^{*_g}\Big(\langle A, H\rangle^{\sharp_g} + \langle A^\circ, H\rangle^{\sharp_g}\Big) + U;\end{aligned}$$

since e^{λ} is by assumption uniformly bounded, the equation is uniformly parabolic, and by Proposition 5.3.4, $U \in L^2(0,T), L^p(S^2))$ for any $p < 2$.

This is enough to start a boostrapping procedure using first L^p-L^q and then Schauder parabolic estimates in a fashion similar to the elliptic case discussed in Proposition 4.3.1 of Chapter 4. To bootstrap the regularity of the tangential component U, one uses higher-regularity variants of Proposition 5.3.4, whose proofs are similar to the basic case. □

Proof of Theorem 5.1.9. *Case of smooth initial datum.* By Corollary 5.3.6, it suffices to prove that there exists a unique smooth solution in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ with the required properties.

An application DeTurck's trick (see Appendix 5.4) yields existence and uniqueness of a smooth solution to the Cauchy problem for the normal Willmore flow:

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi^0 = -\delta\mathcal{W} & \text{in } (0,T)\times S^2, \\ \Phi^0(0,\cdot) = \Phi_0 & \text{on } S^2, \end{cases}$$

and if $\varepsilon_0 > 0$ is small enough, by Theorem 5.3.1 Φ^0 exists for all and smoothly converges to a round sphere.[2] We conformalize such flow composing it with the family $(\phi(t,\cdot))_{t\in[0,+\infty)}$ of canonical quasi-conformal mappings associated to the family of metrics $g^0(t,\cdot) = \Phi^0(t,\cdot)^\star g_{\mathbb{R}^3}$, see [AB60]. The fact that it is $\phi(0,\cdot) = e$ that such family is smooth both in the space and in time follows from the theory of quasi-conformal mappings. Then $\Phi^1(t,\cdot) = \Phi^0(t,\phi(t,\cdot))$ is a conformal Willmore flow defined for all times and converging to a conformal parametrization of a round sphere.

Let

$$a(t) = \sqrt{\frac{4\pi}{\mathcal{A}(\mathcal{S}_t)}}$$

the normalizing function of time so that $a(t)\mathcal{S}_t$ has always area 4π and let $\varepsilon_0 > 0$ be sufficiently small as in Theorem 5.2.1 and also so that $C\varepsilon_0 \le \delta$ where C is as in (5.2.2) and δ is sufficiently small as in Propositions 5.1.5 and 5.3.4.[3] In this way, there exists a family of conformal diffeomorphisms $(\psi(t,\cdot))_{t\in[0,+\infty)} \subset \mathrm{Aut}(S^2)$ so that $\Phi(t,\cdot) = \Phi^1(t,\psi(t,\cdot))$ is conformal, well-balanced and

$$\|a(t)\Phi(t,\cdot) - I - c(t)\|_{W^{2,2}(S^2)} + \|a(t)e^{\lambda(t,\cdot)} - 1\|_{L^\infty(S^2)} \le C\mathcal{W}_0(\mathcal{S}_t), \tag{5.3.13}$$

where $c(t) = ⨍_{S^2}\Phi(t,\cdot)\,d\sigma$ (see Remark 5.2.2) and moreover there exist a neighborhood of the identity $\mathcal{O} \subset \mathrm{Aut}(S^2)$ where such choice is unique.

As for $a(t)$, thanks the area control of Theorem 5.3.1 (recall that $\mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathcal{S}_0) = 4\pi$) it is

$$|\mathcal{A}(\mathcal{S}_t) - 4\pi| \le C\mathcal{W}_0(\mathcal{S}_t) = o(1) \quad \text{as } t\to+\infty,$$

and hence

$$|a(t) - 1| \le C\mathcal{W}_0(\mathcal{S}_t) = o(1) \quad \text{as } t\to+\infty,$$

[2] Note carefully: the convergence is to *some* smooth parametrization, of a round sphere of *some* center and radius, not to I modulo dilation and translation. For instance, if Φ_0 is *any* smooth parametrization of a round sphere, then Φ^0 trivially converges to Φ_0.

[3] See in this regard Remark 5.3.5.

which means that we may remove $a(t)$ from the estimate (5.3.13).

As for $c(t)$, we may write

$$c(t) = \fint_{S^2} \Phi(t,\cdot)\, d\sigma = \fint_{S^2} \Phi(t,\cdot) e^{2\lambda(t,\cdot)}\, d\sigma + \fint_{S^2} \Phi(t,\cdot)(1 - e^{2\lambda(t,\cdot)}) d\sigma$$
$$= \mathcal{C}(\mathcal{S}_t) + \fint_{S^2} \Phi(t,\cdot)(1 - e^{2\lambda(t,\cdot)}) d\sigma,$$

where $\mathcal{C}(\mathcal{S}_t)$ denotes the barycenter as in Theorem 5.3.1. Thus, from: the barycenter control of Theorem 5.3.1, the control on the conformal factor issuing from (5.3.13), the fact that $\psi(t,\cdot) \in \mathcal{O}$, smooth convergence and $\mathcal{C}(\mathcal{S}) = 0$, we can estimate

$$|c(t)| \leq C\mathcal{W}_0(\mathcal{S}_t),$$

and hence also $c(t)$ can be removed from estimate (5.3.13) as well and deduce that

$$\|\Phi(t,\cdot) - I\|_{W^{2,2}(S^2)} + \|e^{\lambda(t,\cdot)} - 1\|_{L^\infty(S^2)} \leq C\mathcal{W}_0(\mathcal{S}_t). \tag{5.3.14}$$

Since $\mathcal{W}_0(\mathcal{S}_t) = o(1)$ as $t \to +\infty$, we obtain that $\Phi(t,\cdot)$ converges to I in $W^{2,2}$ and that its conformal factor converges uniformly to 1. Convergence of higher order derivatives to I then follow from this and the smooth convergence of Φ^1.

Case of weak initial datum. Let $\Phi_{0,j} \in \mathscr{D}^{\varepsilon_0}(S^2, \mathbb{R}^3)$ be a sequence appoximating Φ_0 in the weak $W^{2,2}$-topology, i.e.

$$\Phi_{0,j} \rightharpoonup \Phi_0 \quad \text{in } W^{2,2}(S^2).$$

If ε_0 is taken sufficiently small, it follows from the analysis in [Riv08, Riv14, Riv16] (see for instance the proof of Theorem 3.36 in [Riv16]) that also

$$e^{\lambda_j} \overset{*}{\rightharpoonup} e^{\lambda} \quad \text{in } L^\infty(S^2).$$

For each j, we let $\Phi_j \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,+\infty)}(S^2, \mathbb{R}^3)$ be the well-balanced conformal Willmore flow given by Theorem 5.1.9 with initial datum $\Phi_{0,j}$, which we know to be smooth. By estimate (5.3.14) we have that for every $t > 0$ there holds

$$\|\Phi_j(t,\cdot) - I\|_{W^{2,2}(S^2)} + \|e^{\lambda_j(t,\cdot)} - 1\|_{L^\infty(S^2)} \leq C\mathcal{W}_0(\mathcal{S}_t) \leq \delta,$$

and by Proposition 5.3.3 it follows that for every fixed choice of $T > 0$ there holds

$$\|\Phi_j\|_{L^2((0,T),W^{4,2}(S^2))} \leq C,$$

and finally by Proposition 5.3.4 we also have, for a fixed choice of $1 < p < 2$,

$$\|\partial_t \Phi_j\|_{L^2((0,T),L^p(S^2))} = \| - \delta\mathcal{W}_j + U_j\|_{L^2((0,1),L^p(S^2))} \leq C,$$

where C does not depend on j.

Weak sequential compactness properties of Sobolev spaces then imply that, up to the extraction of a subsequence, there exists measurable functions $\Phi : (0,+\infty) \times S^2 \to \mathbb{R}^3$ and

$\lambda : (0, +\infty) \times S^2 \to \mathbb{R}$ so that, for every fixed $T > 0$, as $j \to \infty$,

$$\begin{aligned}
&\Phi_j \to \Phi \quad \text{a.e. in } (0, +\infty) \times S^2, && (5.3.15)\\
&\Phi_j \rightharpoonup \Phi \quad \text{in } W^{1,p}((0,T) \times S^2), && (5.3.16)\\
&\Phi_j \rightharpoonup \Phi \quad \text{in } L^2((0,T), W^{4,2}(S^2)),\\
&\Phi_j \overset{*}{\rightharpoonup} \Phi \quad \text{in } L^\infty((0,T), W^{2,2}(S^2)),\\
&\mathrm{e}^{\lambda_j} \overset{*}{\rightharpoonup} \mathrm{e}^{\lambda} \quad \text{in } L^\infty((0,T) \times S^2). && (5.3.17)
\end{aligned}$$

Thus, for every $T > 0$, Φ satisfies the conditions (i), (ii), (iv) of Definition 5.1.2 (and e^λ is its conformal factor), its Willmore operator is in $L^2((0,T) \times S^2)$ and finally since also

$$\mathcal{W}_0(\Phi(t,\cdot)) \le \liminf_{j \to \infty} \mathcal{W}_0(\Phi_j) \le \varepsilon_0,$$

which means that (ii) is satisfied as well and so $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,+\infty)}(S^2, \mathbb{R}^3)$. Moreover, since $p > 1$ by (5.3.16) Φ has a trace at initial time, which, by uniqueness and continuity of the trace operator, must coincide with Φ_0.

Finally, the convergence properties (5.3.15)-(5.3.17) are enough to pass to the limit as $j \to \infty$ in Definition 5.1.8 and thus deduce that Φ is also a weak Willmore flow. By Corollary 5.3.6 Φ is smooth on $(0, +\infty) \times S^2$ as well. □

5.4 Appendix: DeTurck's Trick for the Willmore Flow

We outline here one way to obtain, in the smooth category, short-time existence for the Cauchy's problem (5.1.3) adapting an idea originally devised by DeTurck [DeT83] in the context of the Ricci flow. There are other possibilities, such as the graph Ansatz [HP99] (in codimension 1) or through the Nash-Moser Implicit Function Theorem [Ham82a, Ham82b].

The key idea is as follows. From the divergence form of the Willmore operator (5.1.10), we write

$$\delta\mathcal{W} = \Delta_g H + \nabla^{*_g}\left(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\right) = \frac{1}{2}\Delta_g^2\Phi + F\left(d\Phi, \nabla^2\Phi, \nabla^3\Phi\right),$$

where F is a smooth function and the covariant derivatives are with respect to a fixed smooth reference metric on Σ. Now in local coordinates we may write

$$\Delta_g\Phi = g^{\mu\nu}\partial^2_{\mu\nu}\Phi - g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma\Phi,$$

and so

$$\begin{aligned}
\Delta_g^2\Phi &= \Delta_g(\Delta_g\Phi)\\
&= g^{\mu\nu}\partial^2_{\mu\nu}(\Delta_g\Phi) - g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma(\Delta_g\Phi)\\
&= g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}\partial^2_{\alpha\beta}\Phi - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma\Phi\right) + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right),
\end{aligned}$$

for some smooth function f. Now, because also the metric depends on Φ: $g_{\mu\nu} = \langle\partial_\mu\Phi, \partial_\nu\Phi\rangle$, the term $g^{\alpha\beta}\left(\partial^2_{\mu\nu}\Gamma^\gamma_{\alpha\beta}\right)\partial_\gamma\Phi$ also contains derivative of order 4 in Φ. This causes the Willmore operator to be degenerate elliptic, and the corresponding flow to be degenerate parabolic.

However, writing:

$$\Delta_g^2 \Phi = g^{\mu\nu} g^{\alpha\beta} \partial^4_{\mu\nu\alpha\beta} \Phi - g^{\mu\nu} \partial^2_{\mu\nu} \left(g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \right) \partial_\gamma \Phi + f\left(\partial_x \Phi, \partial^2_{xx} \Phi, \partial^3_{xxx} \Phi \right),$$

one guesses that it may be possible to add a tangent vector field to the Willmore operator which for which the corresponding flow is a uniformly parabolic one. This motivates the following.

Definition 5.4.1. *Let Σ be a closed, orientable surface and let $\Phi_0 : \Sigma \to \mathbb{R}^3$ be a smooth immersion. If $\Phi : \Sigma \to \mathbb{R}^3$ is another smooth immersion,* DeTurck's vector field for the Willmore flow *(for Φ relative to Φ_0) is the vector field tangent to Φ given by*

$$V = V(\Phi_0, \Phi) = -\frac{1}{2} \Delta_g W = -\frac{1}{2} (\Delta_g W)^\gamma \partial_\gamma \Phi,$$

where $W = W^\gamma \partial_\gamma$ is the vector field on Σ given by

$$W^\gamma = g^{\alpha\beta} \left(\Gamma^\gamma_{\alpha\beta} - \breve{\Gamma}^\gamma_{\alpha\beta} \right),$$

where $\Gamma^\gamma_{\alpha\beta}$ and $\breve{\Gamma}^\gamma_{\alpha\beta}$ denote respectively the Christoffel symbols of Φ and Φ_0.

Note that this definition makes sense, since it is a well-known fact in differential geometry that, although the Christoffel symbols are themselves not tensor, the expression $\Gamma^\gamma_{\alpha\beta} - \breve{\Gamma}^\gamma_{\alpha\beta}$ (and consequently its trace W) is.

Proposition 5.4.2 (Short-Time Existence for the Smooth DeTurck-Willmore Flow)**.** *Let Σ be a closed, orientable surface and let $\Phi_0 : \Sigma \to \mathbb{R}^3$ be a smooth immersion. There exists some $T = T(\Phi_0) > 0$ so that the Cauchy problem*

$$\begin{cases} \partial_t \Phi = -\delta \mathcal{W} + V & \text{in } (0,T) \times \Sigma, \\ \Phi(0,\cdot) = \Phi_0, & \text{on } \Sigma, \end{cases} \tag{5.4.1}$$

has a unique solution in the class $C^\infty([0,T] \times \Sigma, \mathbb{R}^3)$, where $V = V(\Phi_0, \Phi)$ is DeTurck's vector field for the Willmore flow.

Proof. It is sufficent to prove that (5.4.1) defines a uniformly parabolic system for Φ over Σ. The existence, uniqueness and smoothness of a solution follows then from the general theory for such systems in Hölder spaces [Sol65] (transl. English [MR067]), [LSU68]. We have

$$\Delta_g W = \operatorname{tr}_g \left(\nabla^{(2)} W \right) = g^{\mu\nu} \left(\nabla^g_{\partial_\mu} \nabla^g_{\partial_\nu} W - \nabla^g_{\nabla^g_{\partial_c}} W \right) = g^{\mu\nu} \left(\nabla^g_{\partial_\mu} \nabla^g_{\partial_\nu} W - \Gamma^\sigma_{\mu\nu} \nabla^g_{\partial_\sigma} W \right),$$

so computing directly we see that

$$\begin{aligned} \Delta_g W = g^{\mu\nu} \Big(& \partial^2_{\mu\nu} W^\xi + \partial_\nu W^\sigma \Gamma^\xi_{\mu\sigma} + \partial_\mu W^\sigma \Gamma^\xi_{\nu\sigma} \\ & - \Gamma^\tau_{\mu\nu} \partial_\tau W^\xi + W^\sigma \partial_\mu \Gamma^\xi_{\nu\sigma} + W^\sigma \Gamma^\tau_{\nu\sigma} \Gamma^\xi_{\mu\tau} - \Gamma^\sigma_{\mu\nu} \Gamma^\xi_{\tau\sigma} W^\sigma \Big) \partial_\xi \\ = g^{\mu\nu} & \left(\partial^2_{\mu\nu} W^\xi + f(W, \partial_x W, \partial_x \Phi, \partial^2_{xx} \Phi) \right) \partial_\xi \\ = g^{\mu\nu} & \left(\partial^2_{\mu\nu} W^\xi + f\left(\partial_x \Phi, \partial^2_{xx} \Phi, \partial^3_{xxx} \Phi \right) \right) \partial_\xi, \end{aligned}$$

thus we have, in *every* choice of local coordinates, that

$$\begin{aligned}
-\delta\mathcal{W} + V &= -\frac{1}{2}\Delta_g^2\Phi + V + f\left(\partial_x\Phi, \partial_{xx}^2\Phi, \partial_{xxx}^3\Phi\right) \\
&= -\frac{1}{2}\Big(g^{\mu\nu}g^{\alpha\beta}\partial_{\mu\nu\alpha\beta}^4\Phi - g^{\mu\nu}\partial_{\mu\nu}^2\left(g^{\alpha\beta}\Gamma_{\alpha\beta}^{\gamma}\right)\partial_\gamma\Phi \\
&\qquad\qquad + g^{\mu\nu}\partial_{\mu\nu}^2\left(g^{\alpha\beta}(\Gamma_{\alpha\beta}^{\gamma} - \breve{\Gamma}_{\alpha\beta}^{\gamma})\right)\partial_\gamma\vec{\Phi}\Big) + f\left(\partial_x\Phi, \partial_{xx}^2\Phi, \partial_{xxx}^3\Phi\right) \\
&= -\frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta}\partial_{\mu\nu\alpha\beta}^4\Phi - g^{\mu\nu}\partial_{\mu\nu}^2\left(g^{\alpha\beta}(\breve{\Gamma}_{\alpha\beta}^{\gamma})\right)\partial_\gamma\vec{\Phi}\right) + f\left(\partial_x\Phi, \partial_{xx}^2\Phi, \partial_{xxx}^3\Phi\right) \\
&= -\frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta}\partial_{\mu\nu\alpha\beta}^4\Phi\right) + f\left(\partial_x\Phi, \partial_{xx}^2\Phi, \partial_{xxx}^3\Phi\right),
\end{aligned}$$

so that (5.4.1) defines, for T sufficiently small and in the smooth category, a uniformly parabolic system of fourth order in Φ. □

One now obtains the analogue short-time existence and uniqueness statement for the Cauchy problem (5.1.3) combining Proposition 5.4.2 and the the fact that there is a bijective correspondence between tangential components an reparametrizations (see the analogous discussion in [Man11]).

5.5 Appendix: The $\overline{\partial}$-operator on Vector Fields

Let Σ be a Riemann surface. the complexified tangent bundle $T^{\mathbb{C}}\Sigma = T\Sigma + iT\Sigma$ splits in two sub-bundles:

$$T^{\mathbb{C}}\Sigma = T\Sigma^{(1,0)} \oplus T\Sigma^{(0,1)},$$

whose sections are respectively $(1,0)$ and $(0,1)$-vector fields:

$$\begin{aligned}
\mathfrak{X}^{(1,0)}(\Sigma) = \Gamma(T\Sigma^{(1,0)}) \quad &\ni \quad V = (V^1 + iV^2)\partial_z, \\
\mathfrak{X}^{(0,1)}(\Sigma) = \Gamma(T\Sigma^{(0,1)}) \quad &\ni \quad W = (W^1 + iW^2)\partial_{\bar{z}}.
\end{aligned}$$

One can identify $(1,0)$- and real vector fields by means of conjugation and $(1,0)$-projection:

$$\mathfrak{X}^{(1,0)}(\Sigma) \simeq \mathfrak{X}(\Sigma) \quad : \quad V \to V + \overline{V} \quad \text{and} \quad W^{(1,0)} \leftarrow W.$$

With this identification, we can consider holomorphic vector fields as real vector fields, whose flows consist precisely of families of $\mathrm{Aut}(\Sigma)$, the space of conformal self-maps (Moiebius transformations) of Σ. Calling $B = \Gamma(T\Sigma^{(1,0)} \otimes T^*\Sigma^{(0,1)})$, the $\overline{\partial}$-operator over $(1,0)$-vector fields is

$$\overline{\partial} : \mathfrak{X}^{(1,0)}(\Sigma) \to B, \quad \overline{\partial}V = \frac{1}{2}\partial_{\bar{z}}(V^1 + iV^2)\partial_z \otimes d\bar{z},$$

whose kernel is the space of holomorphic vector fields $\ker(\overline{\partial}) = \mathfrak{X}^{\omega}(\Sigma)$. Fix now a conformal metric over Σ, $g = \mathrm{e}^{2\lambda}|dz|^2$. The formal L^2-adjoint of $\overline{\partial}$, defined through the formula $(\overline{\partial}V, F)_{L^2} = (V, \overline{\partial}^*F)_{L^2}$ is, if $F = f\partial_z \otimes d\bar{z}$,

$$\overline{\partial}^* : B \to \mathfrak{X}^{(1,0)}(\Sigma), \quad \overline{\partial}^*F = -2\mathrm{e}^{-4\lambda}\partial_z(\mathrm{e}^{2\lambda}f)\partial_z.$$

We deduce that $\ker(\overline{\partial}^*)$ consists of those tensors $F = f\partial_z \otimes d\bar{z}$ so that $e^{2\lambda} f$ is antiholomorphic. If we lower the fist index of F:

$$F^\flat = f g_{\bar{z}z}\, d\bar{z} \otimes \bar{z} = \frac{1}{2} e^{2\lambda} f\, d\bar{z} \otimes d\bar{z},$$

then $F^\flat$ is an antiholomorphic quadratic differential, and so its conjugate is a holomorphic quadratic differential. With these identifications, we have

$$\ker(\overline{\partial}^*) \simeq Q^\omega(\Sigma).$$

In particular, we note that even though $\overline{\partial}^*$ does depend on the chosen metric, $Q^\omega(\Sigma)$ does not.

For given $F \in B$, we consider the equation

$$\overline{\partial} V = F \quad \text{on } \Sigma. \tag{5.5.1}$$

Then, (5.5.1) has a solution if and only if

$$F \in \overline{\partial}(\mathfrak{X}^{(1,0)}(\Sigma)) = \ker(\overline{\partial}^*)^\perp$$

In such case, if V_0 is one such solution, every other one is of the form $V = V_0 + v$ for $v \in \mathfrak{X}^\omega(\Sigma)$.

The *normal solution* to (5.5.1) is the only one in $\ker(\overline{\partial})^\perp$, and we denote it by $\overline{\partial}^{-1} F$. Normal solutions satisfy the typical elliptic estimates, such as for instance

$$\|\overline{\partial}^{-1} F\|_{W^{1,2}(\Sigma)} \leq C \|F\|_{L^2(\Sigma)} \quad \forall\, F \in \ker(\overline{\partial}^*)^\perp.$$

for a constant $C = C(\Sigma, g) > 0$.

As a consequence of the Riemann-Roch formula, if γ is the genus of Σ, we have

$$\dim_{\mathbb{C}} Q^\omega(\Sigma) = \begin{cases} 0 & \text{if } \gamma = 0 \\ 1 & \text{if } \gamma = 1, \\ 3\gamma - 3 & \text{if } \gamma \geq 2, \end{cases} \quad \text{and} \quad \dim_{\mathbb{C}} \mathfrak{X}^\omega(\Sigma) = \begin{cases} 3 & \text{if } \gamma = 0, \\ 1 & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma \geq 2. \end{cases}$$

In particular, (5.5.1) can be solved for any F when $\Sigma = S^2$.

6 The Germain–Poisson Problem

Summary: This chapter is dedicated to the solve the Germain–Poisson problem. We find a disk–type surface $\mathcal{D} \subset \mathbb{R}^n$ of least Willmore energy $\mathcal{W}_2(\mathcal{D})$ among all immersed surfaces having the same boundary, boundary Gauss map and area. We present a solution in the case of boundary data of class $C^{1,1}$ and when the boundary curve is simple and closed. The minimum is realised by an immersed disk, possibly with a finite number of branch points in its interior, which is of class $C^{1,\alpha}$ up to the boundary for some $0 < \alpha < 1$, and whose Gauss map extends to a map of class $C^{0,\alpha}$ up to the boundary.

6.1 Introduction

From the introduction to this thesis we recall that the Germain–Poisson problem (for disk–type surfaces) is the following:

> Given a simple, closed curve $\Gamma \subset \mathbb{R}^n$, and a unit normal $(n-2)$-vector field N_0 along Γ and a value $a > 0$, find an immersed disk $\mathcal{D} \subset \mathbb{R}^n$ bounding Γ, having boundary Gauss map N_0 and area $a > 0$ minimising the Willmore energy.

The name is after Germain's [Ger21] and Poisson's [Poi16], seminal work on elasticity theory of plates. The hypothesis that the elastic energy density be proportional to the mean curvature is due to Germain's seminal research on elastic plates, who in turn built on earlier one-dimensional models concerning the vibration of elastic beams investigated by Euler and Jacques Bernoulli. Poisson found they should satisfy the corresponding Euler–Lagrange equation complemented with a term corresponding to a fixed–area constraint.

Of course, the understanding of elasticity has advanced since then; however, a linearized version of such energy, namely where the Willmore energy is replaced by the simpler biharmonic energy of a graph, is still in use today in models concerning small deformations of thin elastic plates. In such models however the non–stretchability of the plate is imposed by requiring that the immersion is an isometry with respect to a reference metric on the surface instead of just requiring the area to be fixed, as we are doing here following Poisson's memoir. Such a constraint, certainly more physical, would however greatly change the way the problem should be treated. We refer for instance to [LL86, Vil97, FJM06, GGS10] for more on modern theories of elasticity and to [BD80, DD87, Sza01] for a historical perspective on the development of the subject. We briefly mention that considering Willmore–type energies with area constraints is natural according to models for cell membranes in cell biology: we mention the celebrated paper by Helfrich [Hel73] and we refer to the introduction in [KMR14] for more references on the subject.

We are here interested in studying the Germain–Poisson problem from a geometric perspective; we will consequently not dwell on the treatment of potentially undesired features of the solution (such as self-intersections) that may not be desirable when dealing with physical elastic plates.

Finally, we remark that the Germain–Poisson problem may be seen as a generalization of the Plateau's problem (see for instance [Str88, CI11, DHS10, DHT10]), since, for certain special choices of the field N_0 and of a, there will be minimal surfaces Σ (i.e. satisfying $H = 0$) bounding Γ which are then absolute minimisers for the Germain–Poisson problem.

Among the possible variants for the Willmore energy (see (1.1.1) in the introduction of this thesis) we shall work here with

$$\mathcal{W}_2(\mathcal{D}) = \frac{1}{4} \int_{\mathcal{D}} |A|^2 \, d\sigma_g, \tag{6.1.1}$$

(sometimes called *total curveture energy* of $\mathcal{D}$), since it has the advantage of having better coercivity properties and is capable of controlling the number of branch points (recalled in §1.2.4 of the introduction of this thesis, see also [MR14, Riv16]), so the variational problem will be well–posed with $\mathcal{W}_2$.

Note also that, if $\Phi : B_1 \to \mathcal{D}$ is a conformal parametrization and $N : B_1 \to \mathrm{Gr}_{n-2}(\mathbb{R}^n)$ is its Gauss map (see §6.1.1 below for more information), then

$$\mathcal{W}_2(\mathcal{D}) = \frac{1}{4} \int_{B_1} |\nabla N|^2 \, dx.$$

With respect to the classical Willmore energy $\mathcal{W}_2(\mathcal{D}) = \int_{\mathcal{D}} |H|^2 \, d\sigma_g$, we recall once more that, since there holds

$$|A|^2 = 4|H|^2 - 2K,$$

where K denotes the Gauss curvature of $\mathcal{D}$, we have that

$$4\mathcal{W}_1(\mathcal{D}) - \mathcal{W}_2(\mathcal{D}) = 2 \int_{\mathcal{D}} K \, d\sigma_g,$$

so by virtue of the Gauss–Bonnet theorem (see for instance [AT12, Chapter 6]) there holds

$$\int_{\mathcal{D}} K \, d\sigma_g = 2\pi\chi(\mathcal{D}) - \int_{\partial\mathcal{D}} k_g \, d\ell_g = 2\pi - \int_{\partial\mathcal{D}} k_g \, d\ell_g, \tag{6.1.2}$$

since $\chi(\mathcal{D}) = 1$ is the the Euler-Poincaré characteristic of $\mathcal{D}$ and $\int_{\partial\mathcal{D}} k_g \, d\ell_g$ is the integral of the geodesic curvature of $\partial\mathcal{D}$ as a positively oriented curve in $\mathcal{D}$.

We observe that there could be differences between minimizers of the two Lagrangians $\mathcal{W}_1$ and $\mathcal{W}_2$ in the case of interior branch points since the identity (6.1.2) does not hold anymore.

We present in this paper a solution to the Germain–Poisson problem when Γ is a connected, simple, closed curve and $\mathcal{D}$ is (the image of) a parametrised, possibly branched, immersed disk $\Phi : B_1 \to \mathbb{R}^n$, where $B_1 = \{z \in \mathbb{R}^2 : |z| < 1\}$. Let us define the class of "admissible" data (Γ, N_0, a) for which we can solve the problem.

In accordance with the terminology given in the introduction of this thesis, in what follows a (conformal) "weak, branched immersion" is understood to be a conformal Lipschitz, branched immersion with second fundamental form in L^2.

Definition 6.1.1. *A triple* (Γ, N_0, a) *curve* $\Gamma \subset \mathbb{R}^n$*, a unit-normal* $(n-2)$*-vector field* N_0 *and a real number* $a > 0$ *is called* admissible for the Germain–Poisson problem *if* Γ *and* N_0 *are of class* $C^{1,1}$*,* Γ *is simple and closed, and if there is at least one weak, branched conformal immersion* $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n)$ *so that* *(i) its branch points are only on the interior of B_1,*

(ii) it assumes geometrically the boundary data, namely if $\gamma : ([0, \mathcal{H}^1(\Gamma)]/\sim) \to \mathbb{R}^m$ is a chosen arc–length parametrization of Γ, there exist a homeomorphism $\sigma_\Phi : S^1 \to [0, \mathcal{H}^1(\Gamma)]/\sim$ so that, for every $x \in \partial B_1 = S^1$ there holds

$$\Phi(x) = \gamma(\sigma_\Phi(x)) \quad \textit{and} \quad N = N_0(\gamma(\sigma_\Phi(x))),$$

(iii) it has area equal to a, namely, $\mathrm{Area}(\Phi) = \mathcal{A}(\Phi) = \frac{1}{2}\int_{B_1} |\nabla\Phi|^2 \, dx = a$.

Note that, for a given triple (Γ, N_0, a) it is not so obvious to determine directly whether is it admissible of not. However, an elementary application of the $h-$*principle* (see [EM02, Gro86]) allows us, for any given Γ and N_0 as in Definition 6.1.1, to prove the existence of some $a_0 > 0$ so that, for every $a \geq a_0$ the triple (Γ, N_0, a) is admissible. As the proof of this fact shows, when $n = 3$, if one requires the map Φ as in the definition 6.1.1 not to have any branch points, (Γ, N_0) need to satisfy a topological constraint, namely, if $\mathbf{t}$ denotes the tangent vector of Γ, the map

$$x \mapsto (\mathbf{t} \times N_0, \mathbf{t}, N_0)(x), \quad x \in S^1,$$

has to define a non–nullhomotopic loop in the space of special orthogonal matrices $SO(3)$.

The first main result of this paper is the following.

Theorem 6.1.2. *Let (Γ, N_0, a) be an admissible triple for the Germain–Poisson problem. Then, there exists conformal weak, branched immersion $\Phi : B_1 \to \mathbb{R}^n$ (whose branch points lie of the interior of B_1) assuming this data which minimizes the Willmore energy $\mathcal{W}_2$ in this class.*

The second main result is the following.

Theorem 6.1.3. *Let (Γ, N_0, a) be an admissible triple for the Germain–Poisson problem. Every minimizing map Φ as in Theorem 6.1.2 satisfies the distributional Willmore equation with area constraint:*

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big) = cH, \quad \textit{in } \mathcal{D}'(B_1),$$

where $c \in \mathbb{R}$, and such equation is in particular satisfied at the branch points.

Such map Φ is smooth in B_1 away from the branch points and for every $0 < \beta < 1$, Φ is of class $C^{2,\beta}$ at the branch points and its Gauss map N extends to a map of class $C^{1,\beta}$ at the branch points.

Finally, there exists $0 < \alpha < 1$ so that Φ is of class $C^{1,\alpha}$ up to the boundary and its Gauss map N extends to a map of class $C^{0,\alpha}$ up to the boundary.

In the above result, the interior regularity part, away and at the branch points follows adapting the results in [Riv08], similarly as done in [MR13, MR14], while the regularity at the branch points follows from the study of singularities of Willmore surfaces (see [Riv08, BR13] and [KS04, KS07]).

Remark 6.1.4 Notice that the assumptions $\Gamma, N_0 \in C^{1,1} = W^{2,\infty}$ are far from being optimal. We expect that the conclusions of theorems 6.1.2 6.1.3 to hold under weaker assumptions on the admissible boundary data.

Let us put our results in a broader context. Nitsche [Nit93] discussed various boundary conditions for the Willmore and related type of functionals, and proved existence and uniqueness results for a class of such problems, also considering a volume constraint, when the surfaces are graphs in $\mathbb{R}^3$ and the boundary data are sufficiently small in $C^{4,\alpha}$-norm. Recently Deckelnick-Grunau-Rögers [DGR17] also consider the minimisation over graphs in $\mathbb{R}^3$ of the Willmore functional (also plus a constant times integral of the Gauss curvature) subject to various boundary conditions and deduced compactness results in the L^1-topology, and from this, also a lower-semicontinuity for a suitably defined relaxation of the Willmore functional. A considerable series of results (we refer to [EK17, DGR17, DDW13, BDF13, DFGS11, BDF10, DG09, DDG08] and the references therein) is available when considering boundary value problems for the Willmore functional under the hypothesis that the surfaces in consideration are surfaces of revolution around an axis in $\mathbb{R}^3$ (hence the boundary consist of two circles). Schätzle [Sch10], by working on the sphere $S^n \subset \mathbb{R}^{n+1}$, has proved the existence, for arbitrary smooth boundary data Γ and N_0 and without area constraint, of a branched immersion, smooth away from the finitely many branch points, satisfying the classical form of the Willmore equation in S^n, namely

$$\Delta^\perp H + Q(A^\circ)H = 0,$$

away from the branch points However this equation has no meaning at the branch points and it is not proved that, starting from arbitrary data Γ, N_0 in $\mathbb{R}^n$, projecting them stereographically into S^n, and then considering the projected-back Willmore surface obtained in S^n (recall that the Willmore equation is conformally invariant), one gets a surface which is "Germain–Poisson–minimal", i.e. that it solves the Poisson problem or also whether it passes through ∞ or not.

Alexakis and Mazzeo [AM15] consider smooth, properly embedded and complete Willmore surfaces in the hyperbolic space $\mathbb{H}^3$ and relate the regularity of their asymptotic boundary with the smallness of a suitable version of the Willmore energy. In a recent paper Alessandroni and Kuwert [AK16], considering a free-boundary problem for the Willmore functional, have proved the existence (and non-uniqueness) of smooth Willmore disk-type surfaces in $\mathbb{R}^3$ with prescribed but small value of the area whose boundary lays on the boundary of a smooth, bounded domain. Finally, we also mention the very recent works of Eichmann [Eic19] and Pozzetta [Poz20] that also deal with minimization problems for Willmore–type energy similar to ours.

We would like to mention some interesting questions and open problems related to the the Germain–Poisson problem, some of which are the aim of future investigation. Beside our expectations sharpening the boundary regularity one may for instance consider:

- existence and properties of non–minimizing i.e. saddle-type Willmore surfaces having prescribed boundary data and area (a partial answer is in [Sch10] described above);
- the solution to the Germain–Poisson problem in the case of a boundary curve consisting of multiple connected components (in the spirit of its Plateau-counterpart as done in the classical work by Meeks and Yau [MY82]) or among manifold with a non-trivial topology (an important tool for this should be the work by Bauer and Kuwert [BK03]), or both;
- an investigation on a version of the Germain–Poisson problem where no area bound is prescribed;

- free-boundary versions of the Germain–Poisson problem in the case of surfaces with arbitrary prescribed area and boundary laying in some submanifold of $\mathbb{R}^m$ such as the sphere (as done in the forementioned work [AK16] but with no restriction on the prescribed value of the area).

Central ingredient for our solution to the Germain–Poisson problem, are the estimate for Neumann problems involving Jacobians treated in Chapter 2.

The regularity *up to the boundary* of a minimising map in Theorem 6.1.3 follows from a boundary ε-regularity Lemma which is in turn obtained by suitable biharmonic comparison arguments. It is expected such an ε-regularity Lemma up to the boundary to hold for general critical points of the Energy (6.1.1) (an interior ε-regularity for such critical points has been already proved by RIVIÈRE in [Riv08]).

6.1.1 Some Notation, Definitions and a Lemma Beside the common notation, in this paper we denote:

$\mathbb{R}^2_+ = \{(x^1,x^2) : x^2 > 0\}$	upper half-space,
$B_r^+(0) = B_r(0) \cap \mathbb{R}^2_+ \quad \text{with } r > 0$	upper half-ball of radius r,
$\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$	canonical basis of $\mathbb{R}^n$.

We will write $\partial B_r^+(0) = rI + rS$, where

$rI = \{(x^1, 0) : -r < x^1 < r\} \simeq (-r, r)$	base diameter,
$rS = \{(x^1, \sqrt{1-(x^1)^2}) : -r < x^1 < r\}$	upper semi-circle.

The Gauss map of an immersion $\Phi : B_1 \to \mathbb{R}^n$ is seen here as the map $N : B_1 \to \mathrm{Gr}_{n-2}(\mathbb{R}^n)$ given by

$$N(x) = \star \frac{\partial_1\Phi(x) \wedge \partial_2\Phi(x)}{|\partial_1\Phi(x) \wedge \partial_2\Phi(x)|},$$

where $\star$ is the Hodge operator in $\mathbb{R}^n$ (here $\mathrm{Gr}_{n-2}(\mathbb{R}^n)$ denotes the $(n-2)$-dimensional Grassmannian of $\mathbb{R}^n$ and $\star$ is the Hodge operator in $\mathbb{R}^n$ see e.g. [BG80, Chapter 2]). When when $n = 3$, we have the canonical identification $\star(V \wedge W) = V \times W$, where $\times$ is the vector product, so in this case N can be considered as a S^2-valued map.

It can be represented as $N = N_1 \wedge \cdots \wedge N_{n-2}$, where the N_i's are normal vector fields for Φ so that the n-ple $(\partial_1\Phi, \partial_2\Phi, N_1, \dots, N_{n-2})$ defines a positively oriented basis of $\mathbb{R}^n$.

If $\Gamma \subset \mathbb{R}^n$ is a simple, closed curve with a chosen arc-length parametrization $\gamma : [0, \mathcal{H}^1(\Gamma)]/\sim \longrightarrow \Gamma$ and $N_0 = N_0(\gamma(\cdot))$ is a unit-normal $(n-2)$-vector field along Γ, the geodesic curvature k_g of Γ (with respect to N_0) is defined as follows: if $\mathbf{t} = \dot\gamma$ denotes the unit-tangent vector of Γ we set (see [AT12, Chapter 5])

$$k_g = \left\langle \dot{\mathbf{t}}, \star(N_0 \wedge \mathbf{t}) \right\rangle = \langle \dot\gamma, \star(N_0 \wedge \dot\gamma) \rangle \,. \tag{6.1.3}$$

Definition 6.1.5 (Weak Branched Immersions)**.**

(i) $\mathscr{F}(B_1, \mathbb{R}^n)$ *denotes the set of weak branched immersions whose branch points lie only in the interior of* B_1*.*

(ii) *If* $\Gamma \subset \mathbb{R}^n$ *is a simple, closed curve and* N_0 *is a unit-normal* $(n-2)$*-vector field along* Γ, $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0)$ *is the set of maps* $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n)$ *so that for some homeomorphism* $\sigma_\Phi : S^1 \to [0, \mathcal{H}^1(\Gamma)]/\sim$ *there holds*

$$\Phi(x) = \gamma(\sigma_\Phi(x)) \quad \textit{and} \quad N(x) = N_0(\sigma_\Phi(x)), \quad \textit{for } x \textit{ in } \partial B_1 = S^1,$$

where $\gamma : [0, \mathcal{H}^1(\Gamma)]/\sim \to \Gamma$ *is a fixed arc-length parametrization of* Γ*.*

(iii) *For* $a > 0$, $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ *is the set of maps* $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0)$ *with area* a*, namely* $\mathcal{A}(\Phi) = \mathrm{Area}(\Phi) = \frac{1}{2}\int_{B_1} |\nabla\Phi|^2\, dx = a$*.*

If $\Phi : B_1 \to \mathbb{R}^n$ is a weak branched conformal immersion, the logarithm of its conformal factor $\lambda = \log(|d\Phi|/\sqrt{2})$ will be a weak solution of the so-called Liouville equation:

$$\begin{cases} -\Delta\lambda = K e^{2\lambda} - 2\pi \sum_{i=1}^{\ell} n_i \delta_{a_i} & \text{in } B_1, \\ \partial_\nu \lambda = k_g(\sigma_\Phi) e^{\lambda} - 1 & \text{on } \partial B_1, \end{cases} \tag{6.1.4}$$

where $a_1, \dots, a_\ell$ are the branch points of Φ and $n_i \in \mathbb{N}$ are their respective multiplicities.

The existence of a conformal re–parametrizations for branched immersions that are not necessarily conformal is implied by the following result which makes use Müller-Šverák theory of weak isothermic charts [MŠ95] and Hélein's moving frame technique (for more deails we refer to [Hél02] and [Riv12]):

Lemma 6.1.6. *Let* $\Phi \in W^{1,\infty}(B_1, \mathbb{R}^n)$ *satisfy the following conditions:*

i) *there exists some* $C > 0$ *such that*

$$|\nabla\Phi|^2 \le C\sqrt{\det g_\Phi} \quad \textit{a.e. in } B_1,$$

ii) *there exists a finite (possibly empty) set* $\{a_1, \dots, a_\ell\} \subset B_1$ *such that for every compact set* $K \subset \overline{B_1} \setminus \{a_1, \dots, a_\ell\}$, Φ *defines a Lipschitz,* $W^{2,2}$ *immersion,*

iii) *Its Gauss maps extends to a* $W^{1,2}$*–map also at the* a_k*'s,*

Then there is a bi–Lipschitz diffeomorphism $\psi : B_1 \to B_1$ *so that* $\Phi \circ \psi$ *is conformal.*

6.2 Existence of Immersed Disks with Given Boundary Data

In this section we prove, along with some other facts and comments, the following result.

Lemma 6.2.1. *Let* $\Gamma \subset \mathbb{R}^n$ *be a simple, closed curve of class* $C^{k,\alpha}$ *for* $k \in \mathbb{N}_{\ge 1}$ *and* $\alpha \in (0,1]$ *whose unit tangent vector we denote by* $\mathbf{t}$ *and let* N_0 *be a unit-normal* $(n-2)$*-vector field along* Γ *of class* $C^{k,\alpha}$*. There exists a possibly branched conformal immersion* $\Phi : B_1 \to \mathbb{R}^n$ *of class* $C^{k,\alpha}$ *and boundary* Γ *and whose Gauss map along* Γ *is* N_0*. In particular, a branch-point-free immersion* Φ *can be produced when either* $n > 3$ *or when* $n = 3$ *and the map*

$$x \mapsto (\mathbf{t} \times N_0, \mathbf{t}, N_0)(x), \quad x \in S^1,$$

defines a non-nullhomotopic loop in $SO(3)$*.*

We treat the case $n = 3$; when $n \geq 3$ see the final Remark 6.2.4. For the elementary concepts of algebraic topology here mentioned we refer the reader to [Hat02, DFN85, DFN92].

Any (non-branched) immersion $\Phi : \overline{B_1} \to \mathbb{R}^3$ naturally defines a map into the space invertible of matrices with positive determinant, $E = E_\Phi : \overline{B_1} \to GL^+(3, \mathbb{R})$, by

$$E(x) = (\partial_1\Phi(x), \partial_2\Phi(x), N(x)), \quad x \in \overline{B_1}^2,$$

where N denotes the Gauss map of Φ. The classical Gram-Schmidt algorithm gives the existence of a deformation retraction of $GL^+(3, \mathbb{R})$ to the 3-dimensional special orthogonal group $SO(3)$, and in particular the map E is homotopic in $GL^+(3, \mathbb{R})$ to the coordinate frame map

$$e(x) = e_\Phi(x) = (e_1(x), e_2(x), e_3(x)), \quad x \in \overline{B_1},$$

where

$$\begin{aligned}
e_1(x) &= \frac{\partial_1\Phi(x)}{|\partial_1\Phi(x)|}, \\
e_2(x) &= \frac{\partial_2\Phi(x)}{|\partial_2\Phi(x)|} - \left\langle \frac{\partial_2\Phi(x)}{|\partial_2\Phi(x)|}, e_1(x) \right\rangle e_1(x), \\
e_3(x) &= e_1 \times e_2(x) = N(x).
\end{aligned}$$

We can similarly define the *polar frame map* defined by means of polar coordinates $x = re^{i\theta}$ in $\overline{B_1} \setminus \{0\}$ as $p(x) = (p_1(x), p_2(x), p_3(x))$, where

$$p(re^{i\theta}) = (e_1, e_2, e_3)(re^{i\theta}) \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We recall that the fundamental group of $SO(3)$ consists precisely of two components:

$$\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z},$$

the non-trivial class being represented for instance by the family realising a complete rotation around the z-axis:

$$R(\theta, \hat{z}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta \in [0, 2\pi].$$

Recall moreover that, being $SO(3)$ a topological group, the matrix product operation is compatible with the one of $\pi_1(SO(3))$. Since the restriction of the coordinate frame map e to $S^1 = \partial B_1$ defines a nullohomotopic loop in $SO(3)$, the homotopy being induced by the immersion:

$$e_t(x) = e\,(tx)\,, \quad x \in \partial B_1,\, t \in [0, 1].$$

the polar frame defines then a non-contractible loop in $SO(3)$. This argument implies that, given an immersed curve $\gamma : S^1 \to \mathbb{R}^3$ and a unit-normal vector field $N_0 : S^1 \to S^2$ along γ, a necessary condition for the existence of an immersion $\Phi : \overline{B_1} \to \mathbb{R}^3$ bounding γ and so that $N_\Phi = N_0$ on ∂B_1 is that

$$x \mapsto (\mathbf{t} \times N_0, \mathbf{t}, N_0)(x), \quad x \in S^1,$$

is *not* a nullhomotopic loop in $SO(3)$. Examples of couples (γ, n_0) that do not satisfy this condition are easy to produce.

Example 6.2.2 (Dirac Belt) Let $\gamma : [0, 2\pi] \to \mathbb{R}$ be the unit circle:

$$\gamma(\theta) = (\cos\theta, \sin\theta, 0).$$

We consider a rotation of angle θ around the tangent vector of γ, namely $\mathbf{t}(\theta) = (-\sin\theta, \cos\theta, 0)$. Its matrix is given by

$$R(\theta, \mathbf{t}(\theta)) = B^T(\theta) R(\theta, \hat{z}) B(\theta),$$

where

$$B(\theta) = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix},$$

hence we consider the polar frame map given by

$$p(\theta) = \Big(R(\theta, \mathbf{t}(\theta))\hat{z} \times \mathbf{t}(\theta), \mathbf{t}(\theta), R(\theta, \mathbf{t}(\theta))\hat{z}\Big),$$

where $\hat{z} = (0, 0, 1)$ is the z-versor, which corresponds to rotating the polar frame map of the standard unit disk:

$$p(\theta) = R(\theta, \mathbf{t}(\theta)) \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the compatibility of the product operations between the $SO(3)$ and $\pi_1(SO(3))$, we see that

$$\begin{aligned} [p(\theta)] &= [R(\theta, \mathbf{t}(\theta))] + [R(\theta, \hat{z})] \\ &= [R(\theta, \mathbf{t}(\theta))] + 1. \end{aligned}$$

To prove that also $R(\theta, \mathbf{t}(\theta))$ belongs to the non-trivial class of $\pi_1(SO(3))$, we can use its quaternion representation:

$$\begin{aligned} R(\theta, \mathbf{t}(\theta)) &= (\cos(\theta/2), \sin(\theta/2)\,\mathbf{t}(\theta)) \\ &= \cos(\theta/2) - \sin(\theta/2)(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}). \end{aligned}$$

With this representation, since there holds

$$R(0, \mathbf{t}(0)) = 1 \quad \text{and} \quad R(2\pi, \mathbf{t}(2\pi)) = -1,$$

the lift of $R(\theta, \mathbf{t}(\theta))$ to the universal cover S^3 is not a closed loop, and this means that the base loop is not nullhomotopic. We then conclude that

$$[p(\theta)] = 1 + 1 = 0 \quad \text{in } \mathbb{Z}/2\mathbb{Z}.$$

This example demonstrates also that a couple (γ, N_0) needs not to bound an immersion of $\Phi : \overline{B_1} \to \mathbb{R}^3$ not even if γ is planar and injective. We now want to prove that the only additional requirement for Φ to exist is to have a branch point. The key step to prove it, obtained through an elementary application of the so-called *h-principle* [EM02, Gro86], is the following lemma. In what follows, let us denote, for $0 \leq r \leq R < \infty$,

$$A[R, r] = \overline{B_R} \setminus B_r$$

the annulus of radii R and r centered at 0.

Lemma 6.2.3. *Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be regular, closed curves of class $C^{k,\alpha}$ for $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $\alpha \in (0,1]$, whose unit tangent vectors we denote by $\mathbf{t}_1$ and $\mathbf{t}_2$, and let $N_0, N_1 : S^1 \to S^2$ be unit normal vector fields along γ_1 and γ_2 respectively of class $C^{k,\alpha}$. There exists a regular, immersed strip of class $C^{k,\alpha}$*

$$\Phi : A[2,1] \to \mathbb{R}^3,$$

satisfying

$$\Phi|_{\partial B_1} = \gamma_1, \quad N|_{\partial B_1} = N_1 \quad \textit{and} \quad \Phi|_{\partial B_1} = \gamma_2, \quad N|_{\partial B_1} = N_2 \tag{6.2.1}$$

if and only if the maps

$$p_1(x) = (\mathbf{t}_1 \times N_1, \mathbf{t}_1, N_1)(x) \quad \textit{and} \quad p_2(x) = (\mathbf{t}_2 \times N_2, \mathbf{t}_2, N_2)(x), \quad x \in S^1,$$

are homotopic in $SO(3)$.

Proof of Lemma 6.2.3. The necessity of the condition is clear, we prove the sufficiency.

Step 1. Set, for $\delta > 0$, $K_\delta = S^1 \times (-\delta, \delta)^2$ and define the following maps for $i = 1, 2$:

$$\phi_i(\xi, u, v) = \gamma_i(\xi) + u\,(\mathbf{t}_i \times N_i)(\xi) + v\,N_i(\xi), (\xi, u, v) \in \overline{K_\delta}.$$

If δ is chosen small enough, ϕ_1 and ϕ_2 define regular immersions (i.e. the Jacobian matrix $D\phi(x)$ has rank 3 for every $x \in \overline{K}_\delta$) of class $C^{k,\alpha}$. Since S^1 and $SO(3)$ are strong deformation retracts of $\overline{K_\delta}$ and $GL^+(3,\mathbb{R})$ respectively, an homotopy between p_1 and p_2 in $SO(4)$ induce an homotopy in $GL^+(3,\mathbb{R})$ between $D\phi_1$ and $D\phi_2$. Let $(x,t) \mapsto m(x,t)$, $(x,t) \in \overline{K_\delta} \times [0,1]$ be such an homotopy.

Step 2. Let $J^1(\mathbb{R}^3, \mathbb{R}^3)$ be the 1-jet space of maps from $\mathbb{R}^3$ to itself (see [EM02, Chapter 1]) and let us consider the (local) section

$$F : S^1 \times \left[-\delta, \frac{5}{2}\delta\right] \times [-\delta, \delta] \to J^1(\mathbb{R}^3, \mathbb{R}^3), \quad x \mapsto (x, \phi(x), M(x)),$$

where

$$\begin{aligned}
&\phi(x) = \phi(\xi, u, v) \\
&= \begin{cases} \phi_1(\xi, u, v) & \text{if } u \in \left[-\delta, \frac{\delta}{2}\right], \\ \frac{2}{\delta}(\delta - u)\phi_1(\xi, u, v) + \frac{2}{\delta}\left(\frac{\delta}{2} - u\right)\phi_2\left(\xi, u - \frac{3}{2}\delta, v\right) & \text{if } u \in \left[-\frac{\delta}{2}, \delta\right], \\ \phi_2\left(\xi, u - \frac{3}{2}\delta, v\right) & \text{if } u \in \left[\delta, \frac{5}{2}\delta\right], \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&M(x) = M(\xi, u, v) \\
&= \begin{cases} D\phi_1(\xi, u, v) & \text{if } u \in \left[-\delta, \frac{\delta}{2}\right], \\ m\left(\xi, u - 3\left(u - \frac{\delta}{2}\right), v, \frac{2}{\delta}\left(1 - \frac{\delta}{2}\right)\right) & \text{if } u \in \left[-\frac{\delta}{2}, \delta\right], \\ D\phi_2\left(\xi, u - \frac{3}{2}\delta, v\right) & \text{if } u \in \left[\delta, \frac{5}{2}\delta\right], \end{cases}
\end{aligned}$$

Performing a normalisation of the parameters, we obtain a section $G : \overline{K_{1/2}} = S^1 \times [-1,1] \times [-1,1] \to J^1(\mathbb{R}^3, \mathbb{R}^3)$ which is holonomic in the set $S^1 \times [-1, -1/4] \times [-1,1] \cup S^1 \times [1/4, 1] \times [-1,1]$.

Step 3. By the relative version of the Holonomic Approximation theorem ([EM02, Theorems 3.1.1, 3.2.1]) with

$$A = S^1 \times \left[-\frac{3}{4}, \frac{3}{4}\right] \times \{0\}, \quad B = S^1 \times \left\{-\frac{1}{2}, \frac{1}{2}\right\} \times \{0\},$$

we may obtain, for every $\varepsilon_1 > 0$, a diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $\|h - \mathrm{Id}_{\mathbb{R}^3}\|_{C^0(\mathbb{R}^3)} \leq \varepsilon_1$ and satisfying $h \equiv \mathrm{Id}_{\mathbb{R}^3}$ on a open neighbourhood U of B, a holonomic section $\tilde{G} : V \to J^1(\mathbb{R}^3, \mathbb{R}^3)$, where $V \supseteq U$ is an open neighbourhood of $h(A)$, satisfying $\tilde{G} \equiv G$ on U and $\|\tilde{F} - F\|_{C^0(V)} \leq \varepsilon_1$. By choosing ε_1 small enough, $\tilde{G}$ is then the 1-jet extension of an immersion coinciding with the one of ϕ_0 in a open neighbourhood of $S^1 \times \{-1/2\} \times \{0\}$ and with the one of ϕ_1 in a open neighbourhood of $S^1 \times \{1/2\} \times \{0\}$; in particular it is of class $C^{k,\alpha}$ in such neighbourhoods.

Possibly reducing U and V, the existence of a diffeomorphism $g : K_{1/2} \to V$ that shrinks $K_{1/2}$ into V wile keeping U fixed is ensured. If we consider the restriction of holonomic section $H = G \circ g : \overline{K_{1/2}} \to J^1(\mathbb{R}^3, \mathbb{R}^3)$ to $S^1 \times [-1/2, 1/2] \times \{0\}$, and denote by $\Psi : S^1 \times [-1/2, 1/2] \simeq A[2,1] \to \mathbb{R}^3$ the base map, we have that Ψ is realises a regular immersion of class C^1 which, in a small neighdourhood of the boundary, containing, say, $S^1 \times [1/2 + \varepsilon_2, 1/2 - \varepsilon_2]$ for some small $\varepsilon_2 > 0$, and of class $C^{k,\alpha}$ and satisfies the desired boundary prescriptions (6.2.1).

Step 4. To ensure the global $C^{k,\alpha}$ regularity, we us use a localised mollification for Ψ:

$$\Phi(x) = \Psi * \rho_{\varepsilon(x)}(x) \quad x \in S^1 \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

where ρ is the standard mollification kernel and $\varepsilon(x) = \varepsilon_3 \chi(x)$, where χ is smooth cut-off function identically 1 on $S^1 \times [-1/2 + \varepsilon_3, 1/2 - \varepsilon_3]$ and compactly supported on $S^1 \times (-1/2, 1/2)$ and $\varepsilon_3 > 0$ has be chosen so small that Φ has maximum rank. We conclude that the map Φ thus defined is the one we have been looking for. □

From this lemma we can prove lemma 6.2.1, in the case $n = 3$, that is we can construct, given any couple curve-normal vector field (γ, n_0) of class $C^{k,\alpha}$, a possibly branched immersion $\Phi : \overline{B_1} \to \mathbb{R}^3$ of class $C^{k,\alpha}$ assuming such data at the boundary.

Proof of Lemma 6.2.1. We consider the the loop $p(x) = (\mathbf{t} \times n_0, \mathbf{t}, n_0)(x)$ in $SO(3)$ induced by Γ and N_0. If it is not nullhomotopic, we can connect, in a C^1 way, the couple (γ, N_0) and the flat immersion of the disk $z \mapsto (z, 0)$ by means of a regular strip of class $C^{k,\alpha}$. If it is nullhomotopic instead we can do the same with the branched immersion $z \mapsto (z^2, 0)$. If necessary, we smooth out the immersion near the junction as done in Step 4 of the proof of Lemma 6.2.3; a final reparametrization with Lemma 6.1.6 gives then the immersion $\Phi : \overline{B_1}^2 \to \mathbb{R}^3$ we have been looking for (since, when $k = \alpha = 1$, or when $k \geq 2$, Φ satisfies the assumptions of lemma). □

Remark 6.2.4 (Higher codimension case) For general $n \geq 3$, a regular curve $\gamma : S^1 \to \mathbb{R}^n$ and $(n-2)$-unit normal vector field $N_0 : S^1 \to \mathrm{Gr}_{n-2}(\mathbb{R}^n)$ along γ uniquely determine a loop into the set of couples of ortho-normal vectors or $\mathbb{R}^n$:

$$x \mapsto (\star(\mathbf{t} \wedge N_0), \mathbf{t})(x) \quad x \in S^1,$$

that is to say, into the Stiefel manifold $V_2(\mathbb{R}^n)$ (see e.g. [Hat02]), which for the case $n = 3$ we could identify with $SO(3)$. As it is well-known, $\pi_1(V_2(\mathbb{R}^n)) = 0$ for $n > 3$ and hence, the higher-dimensional version of lemma 6.2.3 basically says that a regular strip bounding any two couples (γ_1, N_1) and (γ_2, N_2) can always be constructed. As a consequence, with the aid of the Holonomic approximation, in a similar fashion as the one just described, we may always find a regular immersion bounding γ and N_0. This is perhaps not so surprising, as higher codimension gives us more freedom.

6.3 An Estimate for the Neumann Problem

For given $f \in L^1(B_1)$, $g \in L^1(\partial B_1)$ we recall that a function $u \in W^{1,1}(B_1)$ is said to weakly solve the Neumann problem for the Poisson equation in B_1:

$$\begin{cases} -\Delta u = f & \text{in } B_1, \\ \partial_\nu u = g & \text{on } \partial B_1, \end{cases} \tag{6.3.1}$$

if, for every $\psi \in C^\infty(\overline{B_1})$, there holds

$$\int_{B_1} f\,\psi\,dx + \int_{\partial B_1} g\,\psi\,d\mathcal{H}^1 = \int_{B_1} \langle \nabla u, \nabla \psi \rangle\,dx.$$

From this expression, it is immediate to see that a necessary condition for the existence of a weak solution is that $\int_{B_1} f = -\int_{\partial B_1} g$ (see e.g. [Ken94] for more on weak formulations of Neumann problems). Such condition is also sufficient and we have the following representation formula:

$$u(x) - \fint_{\partial B_1} u\,d\mathcal{H}^1 = \int_{B_1} \mathcal{G}(x,y)\,f(y)\,dy + \int_{\partial B_1} \mathcal{G}(x,y) g(y)\,d\mathcal{H}^1(y) \quad x \in B_1,$$

where $\mathcal{G}$ is the Green function for the Neumann problem (with zero average on ∂B_1), that is the function:

$$\mathcal{G}(x,y) = -\frac{1}{2\pi}(\log|x-y| + \log|\tilde{x} - y||x|) + \frac{|y|^2}{4\pi} - \frac{1}{4\pi}, \tag{6.3.2}$$

which satisfies, for every $x \in B_1$,

$$\begin{cases} -\Delta_y \mathcal{G}(x,\cdot) = \delta_x - \dfrac{1}{|B_1|}, & \text{in } B_1, \\ \partial_{\nu_y} \mathcal{G}(x,\cdot) = 0 & \text{on } \partial B_1, \\ \displaystyle\int_{\partial B_1} \mathcal{G}(x,y)\,d\mathcal{H}^1(y) = 0. \end{cases} \tag{6.3.3}$$

Note that the presence of the constant $-1/|B_1| = -1/\pi$, computed directly from expression (6.3.2), is indeed the correct one for the Neumann problem (6.3.3) to admit a solution. Two such solutions to (6.3.1) differ by a constant.

In what follows, we denote $\frac{1}{2\pi}\int_{\partial B_1} \phi$ by $\overline{\phi}$. The results we now present are in the spirit of Brezis-Merle [BM91, Theorem 1] and Da Lio-Martinazzi-Rivière [DMR15, Theorem 3.2].

Theorem 6.3.1. *Let $f \in L^1(B_1)$, $g \in L^1(\partial B_1)$ satisfy $\int_{B_1} f = -\int_{\partial B_1} g$ and let $u \in W^{1,1}(B_1)$ be a weak solution to the problem (6.3.1). Then, for every $\varepsilon > 0$ verifying $\|g\|_{L^1(\partial B_1)} + 2\|f\|_{L^1(B_1)} < \pi - \varepsilon$ we have*

$$\|e^{u-\overline{u}}\|_{L^p(\partial B_1)} \leq C_\varepsilon,$$

for some $C_\varepsilon > 0$ depending on ε and

$$p = \frac{\pi - \varepsilon}{\|g\|_{L^1(\partial B_1)} + 2\|f\|_{L^1(B_1)}}.$$

Proof of Theorem 6.3.1. Set $k = \frac{1}{2\pi}\int_{B_1} f = -\frac{1}{2\pi}\int_{\partial B_1} g$. We write the solution of (6.3.1) as:

$$u(x) - \overline{u} = u_1(x) + u_2(x) + \frac{k}{2}(1 - |x|^2), \quad x \in B_1, \tag{6.3.4}$$

where:

$$\begin{cases} -\Delta u_1 = f - 2k, & \text{in } B_1, \\ \partial_\nu u_1 = 0 & \text{on } \partial B_1, \\ \overline{u_1} = 0, & \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = 0, & \text{in } B_1, \\ \partial_\nu u_2 = g + k & \text{on } \partial B_1, \\ \overline{u_2} = 0. & \end{cases}$$

Step 1: study of u_1. We set $F = f - 2k = f - ⨍_{B_1} f$; note that there holds $\int_{B_1} F = 0$ and $\|F\|_{L^1(B_1)} \le 2\|f\|_{L^1(B_1)}$. The Green function for the Neumann problem can be written as

$$\mathcal{G}(x,y) = \frac{1}{2\pi}\left(\log\left(\frac{2}{|x-y|}\right) + \log\left(\frac{2}{|\tilde{x}-y||x|}\right)\right) + \frac{|y|^2}{4\pi} - \frac{1}{4\pi} - \frac{1}{\pi}\log 2,$$

where, since $|x-y| \le 2$ and $|\tilde{x}-y||x| \le 2$, the term in brackets is non negative. Then we may write u_1 as:

$$u_1(x) = \frac{1}{2\pi}\int_{B_1}\left(\log\left(\frac{2}{|x-y|}\right) + \log\left(\frac{2}{|\tilde{x}-y||x|}\right)\right) F(y)\,dy + \int_{B_1}\frac{|y|^2}{4\pi}F(y)\,dy,$$

and in particular, for $x \in \partial B_1$, we have the formula:

$$u_1(x) = \frac{1}{\pi}\int_{B_1}\log\left(\frac{2}{|x-y|}\right) F(y)\,dy + \int_{B_1}\frac{|y|^2}{4\pi}F(y)\,dy, \quad x \in \partial B_1.$$

For $\gamma > 0$ then there holds:

$$\frac{\gamma|u_1(x)|}{\|F\|_{L^1(B_1)}} \le \frac{\gamma}{\pi}\int_{B_1}\log\left(\frac{2}{|x-y|}\right)\frac{|F(y)|}{\|F\|_{L^1(B_1)}}\,dy + \frac{\gamma}{4\pi},$$

so by Jensen's inequality (see e.g. [Eva10, Appendix B]):

$$\exp\left(\frac{\gamma|u_1(x)|}{\|F\|_{L^1(B_1)}}\right) \le \int_{B_1}\left(\frac{2}{|x-y|}\right)^{\frac{\gamma}{\pi}}\frac{|F(y)|}{\|F\|_{L^1(B_1)}}\,dy \cdot e^{\frac{\gamma}{4\pi}}.$$

Integrating on ∂B_1 and using Tonelli's theorem yields:

$$\int_{\partial B_1}\exp\left(\frac{\gamma|u_1(x)|}{\|F\|_{L^1(B_1)}}\right) d\mathcal{H}^1(x) \le \int_{B_1}\left\{\int_{\partial B_1}\left(\frac{2}{|x-y|}\right)^{\frac{\gamma}{\pi}} d\mathcal{H}^1(x)\right\}\frac{|F(y)|}{\|F\|_{L^1(B_1)}}\,dy \cdot e^{\frac{\gamma}{4\pi}}.$$

Then one sees that, for $\gamma < \pi$, the integral:

$$\int_{\partial B_1}\left(\frac{2}{|x-y|}\right)^{\frac{\gamma}{\pi}} d\mathcal{H}^1(x)$$

is convergent and its value uniformly bounded in $y \in B_1$. We conclude that:

$$\int_{\partial B_1} \exp\left(\frac{\gamma |u_1(x)|}{\|F\|_{L^1(D)}}\right) d\mathcal{H}^1(x) \leq C_\gamma, \quad \text{for } \gamma < \pi, \tag{6.3.5}$$

for some constant $C_\gamma > 0$ depending on γ.

Step 2: study of u_2. Note that for $x = \mathrm{e}^{i\phi}$, $y = \mathrm{e}^{i\theta}$, $\mathcal{G}(x,y)$ takes the form:

$$\mathcal{G}(\mathrm{e}^{i\phi}, \mathrm{e}^{i\theta}) = -\frac{1}{\pi}\log|x-y| = -\frac{1}{2\pi}\log(2(1-\cos(\phi-\theta))) =: G(\phi-\theta), \tag{6.3.6}$$

hence the boundary value of u_2 can be written as:

$$u_2(\phi) = u_2(\mathrm{e}^{i\phi}) = G * (g+k) = G * g, \quad \text{on } \partial B_1,$$

where the last equality follows since G has zero average. Using again this property, we may write:

$$\begin{aligned} u_2(\mathrm{e}^{i\phi}) &= -\frac{1}{2\pi}\int_{\partial B_1} \log(2(1-\cos(\phi-\theta)))g(\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_{\partial B_1} \log\left(\frac{1-\cos(\phi-\theta)}{2}\right) g(\theta)\, d\theta, \end{aligned}$$

where now the argument in the logarithm is always bigger than 1. As in step 1, for $\gamma > 0$ and Jensen's inequality one deduces:

$$\exp\left(\frac{\gamma u_1(\mathrm{e}^{i\theta})}{\|g\|_{L^1(\partial B_1)}}\right) \leq \int_{\partial B_1}\left(\frac{2}{1-\cos(\phi-\theta)}\right)^{\frac{\gamma}{2\pi}} \frac{|g(\theta)|}{\|g\|_{L^1(\partial B_1)}}\, d\theta,$$

hence with Tonelli's theorem:

$$\int_{\partial B_1} \exp\left(\frac{\gamma u_1(\mathrm{e}^{i\theta})}{\|g\|_{L^1(\partial B_1)}}\right) d\theta \leq \int_{\partial B_1}\left\{\int_{\partial_{B_1}}\left(\frac{2}{1-\cos(\phi-\theta)}\right)^{\frac{\gamma}{2\pi}} d\phi\right\}\frac{|g(\theta)|}{\|g\|_{L^1(\partial B_1)}}\, d\theta.$$

Provided $\gamma < \pi$ the integral:

$$\int_{\partial B_1}\left(\frac{2}{1-\cos(\phi-\theta)}\right)^{\frac{\gamma}{2\pi}} d\phi = \int_{\partial B_1}\left(\frac{2}{1-\cos(\phi)}\right)^{\frac{\gamma}{2\pi}} d\phi$$

is convergent, hence we conclude that:

$$\int_{\partial B_1} \exp\left(\frac{\gamma |u_2(x)|}{\|g\|_{L^1(\partial B_1)}}\right) dx \leq C_\gamma, \quad \text{for } \gamma < \pi, \tag{6.3.7}$$

and for some $C_\gamma > 0$.

Step 3. Finally from (6.3.4), we may write

$$\mathrm{e}^{u-\overline{u}} = \mathrm{e}^{u_1}\mathrm{e}^{u_2} \quad \text{on } \partial_{B_1}.$$

In particular, if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, by Hölder's inequality there holds

$$\|\mathrm{e}^{u-\overline{u}}\|_{L^p(\partial(B_1))} \leq \|\mathrm{e}^{u_1}\|_{L^{p_1}(\partial B_1)}\|\mathrm{e}^{u_2}\|_{L^{p_2}(\partial B_1)}.$$

Choosing

$$p_1 = \frac{\pi - \varepsilon}{2\|f\|_{L^1(B_1)}} \quad \text{and} \quad p_2 = \frac{\pi - \varepsilon}{\|g\|_{L^1(\partial B_1)}},$$

we reach the conclusion by using estimates (6.3.5) and (6.3.7) with $\gamma = \pi - \varepsilon$. This proves theorem 6.3.1. $\square$

Remark 6.3.2 In step 2, G defined in (6.3.6) has zero average, the computation is invariant by translations of g. Consequently, the assumption on f and g on theorem 6.3.1 may be replaced by

$$\|g - \alpha\|_{L^1(\partial B_1)} + 2\|f\|_{L^1(B_1)} < \pi - \varepsilon \quad \text{for some } \alpha \in \mathbb{R}.$$

The following is a localised version of theorem 6.3.1.

Lemma 6.3.3. *Let $f \in L^1(B_1)$, $g \in L^1(\partial B_1)$, $a_1, \dots a_\ell$ be points in B_1 and $\alpha_1, \dots, \alpha_\ell$ be real numbers satisfying $\int_{B_1} f + \sum_i \alpha_i + \int_{\partial B_1} g = 0$. Let $u \in W^{1,1}(B_1)$ be a weak solution to the problem*

$$\begin{cases} -\Delta u = f + \displaystyle\sum_{i=1}^{\ell} \alpha_i \delta_{a_i} & \text{in } B_1, \\ \partial_\nu u = g & \text{on } \partial B_1. \end{cases} \tag{6.3.8}$$

Assume that, for a given $x_0 \in \partial B_1$ and $0 < r < 1$, $B_r(x_0) \cap \{a_1, \dots a_\ell\} = \emptyset$ and $\|g\|_{L^1(\partial B_1 \cap B_r(x_0))} + 2\|f\|_{L^1(B_1 \cap B_r(x_0))} < \pi - \varepsilon$, for some $0 < \varepsilon < \pi$. Then

$$\|e^{u - \overline{u}}\|_{L^p(\partial B_1 \cap B_{r/2}(x_0))} \le C_\varepsilon\, C_1 \left(\frac{1}{r}\right)^{C_2}, \tag{6.3.9}$$

where

$$p = \frac{\pi - \varepsilon}{\|g\|_{L^1(\partial B_1 \cap B_r(x_0))} + 2\|f\|_{L^1(B_1 \cap B_r(x_0))}},$$

where $C_\varepsilon > 0$ is a constant depending on ε and C_1, C_2 depend on $\|f\|_{L^1(B_1)}$, $\|g\|_{L^1(\partial B_1)}$) and the α_i's.

Proof of Lemma 6.3.3. Let $\chi : B_1 \to \mathbb{R}$ be a function in $C^\infty(\overline{B_1})$ so that $\chi = 1$ in $B_1 \cap B_{3r/4}(x_0)$ and with support in $B_1 \cap B_r(x_0)$. We write the solution of (6.3.8) as: $u - \overline{u} = u_1 + u_2$, where:

$$\begin{cases} -\Delta u_1 = f\chi & \text{in } B_1, \\ \partial_\nu u_1 = g\chi - c & \text{on } \partial B_1, \\ \overline{u}_1 = 0, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = f(1-\chi) + \displaystyle\sum_{i-1}^{\ell} \alpha_i \delta_{a_i} & \text{in } B_1, \\ \partial_\nu u_2 = g(1-\chi) + c & \text{on } \partial B_1, \\ \overline{u}_2 = 0, \end{cases}$$

with $c = \frac{1}{\pi} \int_{B_1} f\chi + \frac{1}{2\pi} \int_{\partial B_1} g\chi$. Applying theorem 6.3.1 together with remark 6.3.2, we deduce the existence of $C_\varepsilon > 0$ such that:

$$\|e^{u_1}\|_{L^p(\partial B_1)} \le C_\varepsilon. \tag{6.3.10}$$

To estimate u_2 we use the representation formula

$$u_2(x) = \int_{B_1} \mathcal{G}(x,y) f(y)(1-\chi(y))\, dy + \int_{\partial B_1} \mathcal{G}(x,y) g(y)(1-\chi(y))\, d\mathcal{H}^1(y) + \sum_{i=1}^{\ell} \alpha_i \mathcal{G}(x,a_i).$$

Notice that for $x \in \partial B_1$ we have $\mathrm{e}^{\alpha_i \mathcal{G}(x,a_i)} = |x-a_i|^{-\alpha_i/\pi} \mathrm{e}^{\alpha_i(|a_i|^2-1)/4\pi}$. Since none of the a_i's is in $B_1 \cap B_r(x_0)$ we have $r/2 \le |x-a_i| \le 2$ for $x \in \partial B_1 \cap B_{r/2}(x_0)$ and $|x-a_i|^{-\alpha_i/\pi} \lesssim r^{-|\alpha_i|/\pi}$. Observe also that and $1-\chi$ vanishes in $B_1 \cap B_{3r/4}(x_0)$, therefore for $x \in \partial B_1 \cap B_{r/2}(x_0)$ we have the estimate

$$\mathrm{e}^{|u_2(x)|} \le C_1 \left(\frac{1}{r}\right)^{C_2} \tag{6.3.11}$$

where C_1 and C_2 depend on $\|f\|_{L^1(B_1)}$, $\|g\|_{L^1(\partial B_1)}$) and $\sum_{i=1}^{\ell} |\alpha_i|$. Hence joining estimates (6.3.10) and (6.3.11), we then deduce the validity of (6.3.9). This proves the lemma. □

6.4 Facts About Moving Frames

A *moving frame*, or simply a *frame*, from a domain $\Omega \subseteq \mathbb{R}^2$ into $\mathbb{R}^n$ is a map $f = (f_1, f_2) : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ so that $\langle f_i, f_j \rangle = \delta_{ij}$. If f and g are two frames from Ω into $\mathbb{R}^n$, a function $\phi : \Omega \to \mathbb{R}$ defines a way to pass from f to g, i.e. a *gauge transformation*:

$$\begin{cases} g_1(x) = f_1(x)\cos\phi(x) + f_2(x)\sin\phi(x), \\ g_2(x) = -f_1(x)\sin\phi(x) + f_2(x)\cos\phi(x), \end{cases}$$

which can be written using the complex notation as

$$g_1 + i g_2 = \mathrm{e}^{-i\phi}(f_1 + i f_2) \quad \text{in } \Omega.$$

Differentiating this relation, we deduce the following *Change of gauge formula*:

$$\langle \nabla g_1, g_2 \rangle = \nabla\phi + \langle \nabla f_1, f_2 \rangle \quad \text{in } \Omega. \tag{6.4.1}$$

For a map $N : B_1 \to \mathrm{Gr}_{n-2}(\mathbb{R}^n)$ expressed as $N = N_1 \wedge \ldots \wedge N_{n-2}$ for some $(n-2)$-tuple of ortho-normal sections: $\langle N_i, N_j \rangle = \delta_{ij}$, we say that the frame (f_1, f_2) *is a lift for* N if:

$$N = \star(f_1 \wedge f_2) \quad \text{in } B_1,$$

where $\star$ denotes the Euclidean Hodge operator in $\mathbb{R}^n$ transforming 2 vectors into $n-2$ vectors and vice-versa (see e.g. [BG80]). By orthonormality, there holds:

$$\begin{aligned}
\nabla f_1 &= \langle \nabla f_1, f_2 \rangle f_2 + \sum_{i=1}^{n-2} \langle \nabla f_1, N_i \rangle N_i = \langle \nabla f_1, f_2 \rangle f_2 - \sum_{i=1}^{n-2} \langle f_1, \nabla N_i \rangle N_i, \\
\nabla f_2 &= \langle \nabla f_2, f_1 \rangle f_1 + \sum_{i=1}^{n-2} \langle \nabla f_2, N_i \rangle N_i = \langle \nabla f_2, f_1 \rangle f_1 - \sum_{i=1}^{n-2} \langle f_2, \nabla N_i \rangle N_i, \\
\nabla N_i &= \langle \nabla N_i, f_1 \rangle f_1 + \langle \nabla N_i, f_2 \rangle f_2 \quad (i = 1, \ldots, n-2),
\end{aligned}$$

in particular:

$$|\nabla f_1|^2 + |\nabla f_2|^2 = 2|\langle \nabla f_1, f_2\rangle|^2 + |\nabla N|^2. \tag{6.4.2}$$

When $N = N_\Phi$ is the Gauss map of an immersion $\Phi : B_1 \to \mathbb{R}^n$ and (f_1, f_2) is a positively oriented ortho-normal basis of the tangent space. When f and N correspond to a orthonormal frame and Gauss map of an immersion Φ, an elementary computation reveals that

$$K\, d\sigma_g = \langle \nabla^\perp f_1, N\rangle\langle \nabla f_2, N\rangle\, dx = \langle \nabla^\perp f_1, \nabla f_2\rangle\, dx, \tag{6.4.3}$$

where K is the Gauss curvature of Φ and $d\sigma_g$ its area element. This equation has two important consequences: first, it reveals that the the left-hand-side is a sum of Jacobians. Second, if N is sufficiently regular (namely, that it admits a lifting frame f: see lemmas 6.4.1 and 6.4.2 below) then it has a meaning also when Φ is singular, for example at a branch point, when a tangent plane is not defined. Finally, it is useful to note that

$$|\langle \nabla^\perp f_1, \nabla f_2\rangle| \le \frac{1}{2}\left(|\langle \nabla^\perp f_1, N\rangle|^2 + |\langle \nabla f_2, N\rangle|^2\right) = \frac{|\nabla N|^2}{2}.$$

Recall Hélein's lifting lemma ([Hél02, Lemma 5.1.4], see also [LLT13]).

Lemma 6.4.1. *There is an $\varepsilon_0 > 0$ so that, for every $0 < \varepsilon < \varepsilon_0$, there exists a constant $C > 0$ independent of ε with the following property. If a map $N \in W^{1,2}(B_1, \mathrm{Gr}_2(\mathbb{R}^n))$ satisfies*

$$\|\nabla N\|^2_{L^2(B_1)} \le \varepsilon$$

for some $0 < \varepsilon < \varepsilon_0$, then there exist an orthonormal frame $f = (f_1, f_2) \in W^{1,2}(B_1, \mathbb{R}^n \times \mathbb{R}^n)$ which is a positive orthonormal basis for N, i.e.:

$$N = \star(f_1 \wedge f_2) \quad \textit{in } B_1,$$

that satisfies the following Coulomb condition:

$$\begin{cases} \mathrm{div}(\langle \nabla f_1, f_2\rangle) & = 0 \ \textit{in } B_1, \\ \langle \partial_\nu f_1, f_2\rangle & = 0 \ \textit{on } \partial B_1, \end{cases}$$

and whose energy is controlled as follows:

$$\|\nabla f\|^2_{L^2(B_1)} \le C\|\nabla N\|^2_{L^2(B_1)}.$$

The proof consists on a minimizing procedure and the L^2-bound relies on Wente's lemma. The following variant concerns the existence of a energy-controlled lift with a prescribed boundary value.

Lemma 6.4.2. *There is an $\varepsilon_0 > 0$ so that, for every $0 < \varepsilon < \varepsilon_0$, there exists a constant $C > 0$ independent of ε with the following property. For any map $N \in W^{1,2}(B_1, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$ and any ortho-normal frame $e = (e_1, e_2) \in H^{1/2}(\partial B_1, \mathbb{R}^n \times \mathbb{R}^n)$ lifting N:*

$$N = \star(e_1 \wedge e_2) \quad \textit{on } \partial B_1,$$

and satisfying the estimate

$$\|\nabla N\|_{L^2(B_1)} + [e]_{W^{1/2,2}(\partial B_1)} \leq \varepsilon, \tag{6.4.4}$$

there exists an ortho-normal frame $g = (g_1, g_2) \in W^{1,2}(B_1, \mathbb{R}^n \times \mathbb{R}^n)$ *lifting* N*:*

$$N = \star(g_1 \wedge g_2) \quad \text{in } B_1, \tag{6.4.5}$$

whose trace on ∂_{B_1} *coincides with* e*, satisfying the Coulomb condition*

$$\operatorname{div}(\langle \nabla g_1, g_2 \rangle) = 0 \quad \text{in } B_1, \tag{6.4.6}$$

and the estimate

$$\|\nabla g\|_{L^2(B_1)} \leq C\left(\|\nabla N\|_{L^2(B_1)} + [e]_{W^{1/2,2}(\partial B_1)}\right).$$

Proof of Lemma 6.4.2. Let us start by fixing $\varepsilon_0 < 2\pi$, so that by lemma (6.4.1) we deduce the existence of a Coulomb frame f on B_1 satisfying

$$\|\nabla f\|_{L^2(B_1)} \leq \sqrt{2}\|\nabla N\|_{L^2(B_1)}. \tag{6.4.7}$$

We now want to identify the angle $\alpha_0 : \partial B_1 \to \mathbb{R}$ which rotates f to e, implicitly defined in complex notation by $e_1 + ie_2 = \mathrm{e}^{-i\alpha}(f_1 + if_2)$. To this aim, let us define the S^1-valued function:

$$u = \langle e_1, f_1 \rangle - i\langle e_2, f_1 \rangle \quad \text{on } \partial B_1,$$

and note that it belongs to $W^{1/2,2}(\partial B_1, S^1)$, satisfying [1] because of (6.4.4) and (6.4.7) the estimate

$$[u]_{W^{1/2,2}(\partial_{B_1})} \leq 2\left([e]_{W^{1/2,2}(\partial B_1)} + [f]_{W^{1/2,2}(\partial B_1)}\right) \leq C\varepsilon, \tag{6.4.8}$$

for some constant $C > 0$. By choosing $\varepsilon_0 > 0$ sufficiently small, we may then invoke theorem [PV17, Theorem 1] and deduce the existence of a function $U \in W^{1,2}(B_1, S^1)$ whose trace on ∂B_1 coincides with u and satisfying the estimate

$$\|\nabla U\|_{L^2(B_1)} \leq C[u]_{W^{1/2,2}(\partial B_1)}, \tag{6.4.9}$$

for some $C > 0$. For such U, we may now deduce the existence of a lift $\alpha \in W^{1,2}(B_1)$ (see e.g. [BBM00, Theorem 3]), that is,

$$U(x) = \mathrm{e}^{i\alpha(x)} \quad x \in B_1,$$

satisfying the point-wise almost everywhere estimate $|\nabla U| = |\nabla \alpha|$. if $\widetilde{\alpha}$ denotes the harmonic extension of the trace of α on ∂B_1, the Dirichlet principle together with the inequalities (6.4.4) and (6.4.8) imply

$$\|\nabla \widetilde{\alpha}\|_{L^2(B_1)} \leq C\left(\|\nabla N\|_{L^2(B_1)} + [e]_{W^{1/2,2}(\partial B_1)}\right). \tag{6.4.10}$$

[1] Recall that if Ω is a domain of $\mathbb{R}$ or of S^1, the space $(W^{1/2,2} \cap L^\infty)(\Omega)$ is an algebra with:

$$[ab]^2_{W^{1/2,2}(\Omega)} \leq 2(\|a\|^2_{L^\infty(\Omega)}[b]^2_{W^{1/2,2}(\Omega)} + \|b\|^2_{L^\infty(\Omega)}[a]^2_{W^{1/2,2}(\Omega)}).$$

Finally, we set:

$$g_1 + ig_2 = e^{-i\widetilde{\alpha}}(f_1 + if_2) \quad \text{in } B_1.$$

By construction, the frame g has trace equal to e on ∂B_1, and satisfies conditions (6.4.5), (6.4.6) and from formula (6.4.2), we see that almost everywhere in B_1 the relation

$$\begin{aligned}|\nabla g|^2 &= 2|\langle \nabla g_1, g_2\rangle|^2 + |\nabla N|^2 \\ &= 2|\langle \nabla f_1, f_2\rangle + \nabla\widetilde{\alpha}|^2 + |\nabla N|^2 \\ &\le 4|\nabla f|^2 + |\nabla\widetilde{\alpha}|^2\end{aligned}$$

holds, from which we deduce, thanks to inequalities (6.4.7) and (6.4.10), the validity of (6.4.9). This concludes the proof of the lemma. □

A localised version of the above lemma is as follows.

Lemma 6.4.3. *There is an $\varepsilon_0 > 0$ so that, for every $0 < \varepsilon < \varepsilon_0$, a constant $C > 0$ independent of ε with the following property exists. Let $x_0 \in \partial B_1$ and $0 < r < 1$ be fixed. For any map $N \in W^{1,2}(B_1, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$ and any ortho-normal frame $e = (e_1, e_2) \in W^{1/2,2}(\partial B_1 \cap B_r(x_0), \mathbb{R}^n \times \mathbb{R}^n)$ lifting N:*

$$N = \star(e_1 \wedge e_2) \quad \text{on } \partial B_1 \cap B_r(x_0),$$

and satisfying the estimate

$$\|\nabla N\|_{L^2(B_1\cap B_r(x_0))} + [e]_{W^{1/2,2}(\partial B_1\cap B_r(x_0))} \le \varepsilon,$$

there exists an ortho-normal frame $g = (g_1, g_2) \in W^{1,2}(B_1 \cap B_r(x_0), \mathbb{R}^n \times \mathbb{R}^n)$ lifting N:

$$N = \star(g_1 \wedge g_2) \quad \text{in } B_1 \cap B_r(x_0),$$

whose trace on $\partial B_1 \cap B_r(x_0)$ coincides with e, satisfying the Coulomb condition

$$\operatorname{div}(\langle \nabla g_1, g_2\rangle) = 0 \quad \text{in } B_1 \cap B_r(x_0),$$

and the estimate

$$\|\nabla g\|_{L^2(B_1\cap B_r(x_0))} \le C\left(\|\nabla N\|_{L^2(B_1\cap B_r(x_0))} + [e]_{W^{1/2,2}(\partial B_1\cap B_r(x_0))}\right).$$

The proof makes use of the following elementary result.

Lemma 6.4.4. *Let $0 < \theta_0 < \pi$ be fixed and $f : (-\theta_0, \theta_0) \to \mathbb{C}$ be a $W^{1/2,2}$-function. Let F be its extension to $S^1 \simeq (-\pi, \pi]/\sim$ by even reflection defined by:*

$$F(x) = \begin{cases} f(x) & \text{if } x \in (-\theta_0, \theta_0), \\ f(m(x - \operatorname{sign}(x)\pi)) & \text{if } x \in (-\pi, \pi] \setminus (-\theta_0, \theta_0),\end{cases}$$

where $m = \frac{\theta_0}{\theta_0 - \pi}$. Then $F \in W^{1/2,2}(S^1)$ and there holds:

$$[F]_{W^{1/2,2}(S^1)} \le 2[f]_{W^{1/2,2}((-\theta_0,\theta_0))}.$$

Proof of Lemma 6.4.4. Note first of all that we may equivalently write $F = f \circ j$, where $j : [-\pi, \pi] \to [-\theta_0, \theta_0]$ is defined as:

$$j(x) = \begin{cases} x & \text{if } x \in (-\theta_0, \theta_0), \\ m(x - \operatorname{sign}(x)\pi) & \text{if } x \in (-\pi, \pi] \setminus (-\theta_0, \theta_0). \end{cases}$$

Using the invariance by rescaling of the $W^{1/2,2}$-seminorm and Tonelli's theorem, we see that:

$$[F]^2_{W^{1/2,2}(S^1)} = 2[f]^2_{W^{1/2,2}((-\theta_0,\theta_0))} + 2\int_{S^1\setminus(-\theta_0,\theta_0)}\int_{(-\theta_0,\theta_0)} \frac{|F(x)-F(y)|^2}{|e^{ix}-e^{iy}|^2}\,dx\,dy.$$

Thinking of j as a diffeomorphism from $S^1 \setminus [-\theta_0, \theta_0]$ to $[-\theta_0, \theta_0]$ with $j' = m$, we perform a change variable in the above integral as follows:

$$\begin{aligned} &\int_{S^1\setminus(-\theta_0,\theta_0)}\int_{(-\theta_0,\theta_0)} \frac{|F(x)-F(y)|^2}{|e^{ix}-e^{iy}|^2}\,dx\,dy \\ &= \int_{S^1\setminus(-\theta_0,\theta_0)}\int_{(-\theta_0,\theta_0)} \frac{|f(x)-f(j(y))|^2}{|e^{ix}-e^{iy}|^2}\,dx\,dy \\ &= \frac{1}{|m|}\int_{(-\theta_0,\theta_0)}\int_{(-\theta_0,\theta_0)} \frac{|f(x)-f(\eta)|^2}{|e^{ix}-e^{i(j^{-1}(\eta))}|^2}\,dx\,d\eta. \end{aligned}$$

Suppose now that $\theta_0 = \pi/2$: we have $j^{-1}(\eta) = -\eta + \operatorname{sign}(\eta)\pi$ and since $x, \eta \in (-\pi/2, \pi/2)$, there holds $|e^{ix} - e^{i(j^{-1}(\eta))}| \geq |e^{ix} - e^{i\eta}|$, hence:

$$\begin{aligned} &\int_{S^1\setminus(-\pi/2,\pi/2)}\int_{(-\pi/2,\pi/2)} \frac{|F(x)-F(y)|^2}{|e^{ix}-e^{iy}|^2}\,dx\,dy \\ &\leq \int_{(-\pi/2,\pi/2)}\int_{(-\pi/2,\pi/2)} \frac{|f(x)-f(\eta)|^2}{|e^{ix}-e^{i\eta}|^2}\,dx\,d\eta = [f]^2_{W^{1/2,2}((-\pi/2,\pi/2))}, \end{aligned}$$

So we conclude that:

$$[F]^2_{W^{1/2,2}(S^1)} \leq 4[f]^2_{W^{1/2,2}((-\pi/2,\pi/2))}.$$

For a general $0 < \theta_0 < \pi$, we may reduce to the case $\theta_0 = \pi/2$ by using the fact that the $H^{1/2}$-seminorm is invariant with respect to the restriction to S^1 of Moebius transformation of D. In our particular case, the transformation we need is:

$$M_a(z) = \frac{z+a}{\bar{a}z+1} \quad \text{with } a = \frac{\pi/2-\theta_0}{1-(\pi/2)\theta_0}, \quad \text{for } z \in S^1.$$

In other words:

$$[F]_{W^{1/2,2}(S^1)} = [F \circ M_a]_{W^{1/2,2}(S^1)},$$

so we may apply the previous inequality and reach the conclusion. This concludes the proof of the lemma. □

Remark 6.4.5 Lemma 6.4.4 holds also for domains which are conformally equivalent to B_1.

Proof of Lemma 6.4.3. Without loss of generality we may assume $x_0 = 1$, so that we have the identification $\partial_{B_1} \cap B_r(x_0) \simeq (-\theta_0, \theta_0)$ for some $0 < \theta_0 < \pi$.
We define a map $N \in W^{1,2}(B_1, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$ which coincides with the given N in $B_1 \cap B_r(x_0)$ and has globally controlled energy, as follows. First, define $N' \in \dot{W}^{1,2}(\mathbb{C}, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$ as the extension of N to $\mathbb{C}$ through even reflection:

$$N'(z) = \begin{cases} N(z) & \text{if } z \in B_1, \\ N\left(\frac{z}{|z|^2}\right) & \text{if } z \in \mathbb{C} \setminus B_1. \end{cases}$$

By the conformal invariance of the Dirichlet energy, there holds $\|\nabla N'\|^2_{L^2(\mathbb{C})} = 2\|\nabla N\|^2_{L^2(B_1)}$ and [2]

$$\|\nabla N'\|^2_{L^2(B_r(x_0))} \leq 2\|\nabla N\|^2_{L^2(D \cap B_r(x_0))}. \tag{6.4.11}$$

Consider now N' as a map from $B_r(x_0)$ and define $N \in \dot{W}^{1,2}(\mathbb{C}, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$ to be its extension through even reflection:

$$N(z) = \begin{cases} N'(z) & \text{if } z \in B_r(x_0), \\ N'\left(\frac{r^2}{|z-x_0|^2}(z - x_0)\right) & \text{if } z \in \mathbb{C} \setminus B_r(x_0). \end{cases}$$

By the conformal invariance of the Dirichlet energy and (6.4.11), there holds $\|\nabla N\|^2_{L^2(\mathbb{C})} \leq 4\|\nabla N\|^2_{L^2(B_1 \cap B_r(x_0))}$, hence a fortiori:

$$\|\nabla N\|^2_{L^2(B_1)} \leq 4\|\nabla N\|^2_{L^2(B_1 \cap B_r(x_0))}.$$

Consequently, by assuming $4\varepsilon_0 < 2\pi$, we may invoke lemma 6.4.1 and find a Coulomb orthonormal frame $f = (f_1, f_2) \in W^{1,2}(B_1, \mathbb{R}^n \times \mathbb{R}^n)$ lifting N in B_1 and satisfying:

$$\|\nabla f\|_{L^2(B_1)} \leq 2\sqrt{2}\|\nabla N\|^2_{L^2(B_1 \cap B_r(x_0))}.$$

As in the proof of lemma 6.4.2, the angle $\alpha_0 : \partial B_1 \cap B_r(x_0) \to \mathbb{R}$ which rotates f to e is implicitly defined trough the S^1-valued function $u = \langle e_1, f_1 \rangle - i\langle e_2, f_1 \rangle$, which belongs to $W^{1/2,2}((-\theta_0, \theta_0), S^1)$ and satisfies the estimate

$$[u]_{W^{1/2,2}((-\theta_0,\theta_0))} \leq 2\left([e]_{W^{1/2,2}((-\theta_0,\theta_0))} + [f]_{W^{1/2,2}((-\theta_0,\theta_0))}\right) \leq C\varepsilon$$

for some constant $C > 0$. By means of lemma 6.4.4, we may extend u to $S^1 = \partial B_1$, thus obtaining a function $v \in W^{1/2,2}(\partial B_1, S^1)$ satisfying $[v]_{W^{1/2,2}(\partial B_1)} \leq 2[u]_{W^{1/2,2}((-\theta_0,\theta_0))}$. The rest of the argument is now similar to that of the proof of lemma 6.4.2, with v in place of u. This concludes the proof of the lemma. □

6.5 Existence of a Minimizer

This section is devoted to prove the main part of theorem 6.1.2, namely the existence for an admissible triple (Γ, N_0, a) of a minimiser for Willmore energy (6.1.1) in the class $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ introduced in definition 6.1.5. In the section 6.6 it is shown that such a minimiser is actually C^1 and not only Lipschitz.

[2] If $I(z) = z/|z|^2$ denotes the inversion with respect to the unit circle, then:

$$(\mathbb{C}^2 \setminus D) \cap B_r(x_0)) \subset I(D \cap B_r(x_0)).$$

6.5.1 Singular Point of Lipschitz Immersions

We recall the following.

Lemma 6.5.1 ([Riv14], Lemma A.5). *Let $\Phi \in \mathscr{E}(B_1 \setminus \{0\}, \mathbb{R}^n)$ be a conformal weak immersion which extends to a map in $W^{1,2}(B_1, \mathbb{R}^n)$ and so that*

$$\lim_{\delta\to 0}\int_{B_1\setminus B_\delta} |A|^2 d\sigma_g = \lim_{\delta\to 0}\int_{B_1\setminus B_\delta} |\nabla N|^2\, dx \quad \textit{is finite.}$$

Then Φ extends to a map in $W^{1,\infty}(B_1)$ and there exists an non–negative integer ϑ so that, for some $C>0$, there holds

$$C(|z|^\vartheta - o(1)) \le |d\Phi(z)| \le C(|z|^\vartheta + o(1)) \quad \textit{as } z\to 0.$$

We now prove the following fact for boundary points.

Lemma 6.5.2. *Let $\{b_1,\dots,b_M\}$ be points on ∂B_1 and let $\Phi : B_1 \to \mathbb{R}^n$ be a measurable map so that, for every $\delta>0$, $\Phi : B_1\setminus \cup_{i=1}^M B_\delta(b_i) \to \mathbb{R}^n$ defines a conformal Lipschitz $W^{2,2}$ immersion, possibly branched at finite number of points $\{a_1,\dots,a_\ell\}\subset D$, with ℓ independent of δ. Assume that*

1. *Φ extends to a map in $W^{1,2}(B_1,\mathbb{R}^n)$ and N extends to a map in $W^{1,2}(B_1, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$*
2. *$\log|d\Phi|$ extends to a map in $W^{1,p}(B_1)$ for some $p>1$,*
3. *$\Phi|_{\partial B_1} = \gamma\circ\sigma$ and $N|_{\partial B_1} = N_0\circ\sigma$ for some homeomorphism σ,*

where γ is an arc-length parametrization of a closed, simple curve Γ in $\mathbb{R}^n$ of class $C^{1,1}$ and N_0 is a unit-normal $n-2$ vector field along Γ of class $C^{1,1}$. Then Φ extends to a weak Lipschitz immersion at every point b_i, $i=1,\dots,M$.

Proof of Lemma 6.5.2. We call $\lambda = \log(|d\Phi|/\sqrt{2})$. It is enough to prove that, for every $i=1,\dots,M$, there exists some $0<s<1$ so that

$$\|\lambda\|_{L^\infty(B_1\cap B_s(b_j))} < +\infty.$$

Claim 1: For every $\varepsilon>0$, the coordinate ortho-normal frame of Φ denoted by $e=(e_1,e_2)$ extends to a map in $W^{1,2}(B_1\setminus\cup_{j=1}^\ell B_\varepsilon(a_j), \mathbb{R}^n\times\mathbb{R}^n)$.
Proof of claim 1. We need to prove that e extends to a a $W^{1,2}$-map in a neighbourhood of each b_i. From the relation (6.4.2) we have

$$|\nabla e|^2 = 2|\langle\nabla e_1, e_2\rangle|^2 + |\nabla N|^2 = 2|\nabla\lambda|^2 + |\nabla N|^2 \quad \text{in } \mathcal{D}'(B_1\setminus\{a_1,\dots,a_\ell\})$$

consequently, since $\nabla\lambda$ belongs to $L^p(B_1)$ and $|\nabla N|$ belongs to $L^2(B_1)$, we deduce that $|\nabla e|$ belongs to $L^p(B_1\setminus\cup_{j=1}^M B_\varepsilon(a_j))$ for every $\varepsilon>0$. Hence e belongs to $W^{1,p}(B_1\setminus\cup_{j=1}^M B_\varepsilon(a_j),\mathbb{R}^n\times\mathbb{R}^n)$ and the trace of e on ∂B_1 is well-defined and belongs to $W^{1/2,1-1/p}(\partial B_1,\mathbb{R}^n\times\mathbb{R}^n)$. Moreover, if $\mathbf{t}$ is the unit tangent vector of Γ, from the boundary conditions this trace is given in complex notation by

$$e_1 + ie_2 = \mathrm{e}^{-i\theta}(\star(\mathbf{t}\wedge N_0) + i\mathbf{t})(\sigma) \quad \text{on } \partial B_1,$$

so we see that it actually lies in $(W^{1/2,2}\cap C^0)(\partial B_1,\mathbb{R}^n\times\mathbb{R}^n)$. Fix now $i=1,\dots,M$ and choose a sufficiently small $0<r<1$ so that no branch point a_j, $j=1,\dots M$, lies in $B_r(b_i)\cap D$ and so that, thanks to lemma 6.4.3, we find an ortho-normal Coulomb frame $g=(g_1,g_2)$ belonging to $W^{1,2}(B_1\cap B_r(b_i),\mathbb{R}^n\times\mathbb{R}^n)$, which lifts N and whose trace on $\partial B_1\cap B_r(b_i)$ coincides with that of e. If φ denotes the angle which rotates g to e , from the change of Gauge formula (6.4.1) we deduce that φ is harmonic in $B_1\cap B_r(b_i)$, moreover

$$|\nabla\varphi|\leq|\langle\nabla g_1,g_2\rangle|+|\langle\nabla e_1,e_2\rangle|\quad\text{in. }\mathcal{D}'(B_1\cap B_r(b_i)).$$

Hence $\varphi\in W^{1,p}(B_1\cap B_r(b_j))$ and thus has a well-defined trace in $\partial B_1\cap B_r(b_j)$ which is is zero by construction. Hence φ is smooth on $B_1\cap B_{r/2}(b_i)$ and so we deduce that $e\in W^{1,2}(B_1\cap B_{r/2}(b_j),\mathbb{R}^n\times\mathbb{R}^n)$. Since $i=1,\dots,N$ was arbitrary, claim 1 follows.

Claim 2. σ' extends to a map in $L^1(\partial B_1)$.

Proof of Claim 2. From the boundary conditions on Φ we have that, uniformly on $\delta>0$,

$$\int_{\partial B_1\setminus\cup_i B_\delta(b_i)}\sigma'=\int_{\partial B_1\setminus\cup_i B_\delta(b_i)}\mathrm{e}^{\lambda}|_{\partial B_1}\leq\mathcal{H}^1(\Gamma),$$

hence, since σ is continuous, the classical Schwartz lemma for distributions implies $\sigma'=\mathrm{e}^{\lambda}|_{\partial B_1}$ extends to a map in $L^1(\partial B_1)$. This proves claim 2.

Combining claims 1 and 2, we deduce that λ is a weak solution of Liouville's equation 6.1.4. From claim 1, $k_g(\sigma)\mathrm{e}^{\lambda}-1$ belongs to $L^1(\partial B_1)$ hence we may find a sufficiently small $0<r<1$ so that, from lemma 6.3.3 there holds $\|e^{\lambda-\overline{\lambda}}\|_{L^p(\partial B_1\cap B_{r/2}(b_i)}$ for some $p>1$. Thanks to claim 2 and possibly reducing r so that no branch point a_j lies in $D\cap B_r(b_i)$, we can invoke theorem 2.1.7 of Chapter 2 and conclude that $\|\lambda-\overline{\lambda}\|_{L^\infty(\partial B_1\cap B_{r/4}(b_i))}$ is finite, which gives the desired estimate 6.5.1. This concludes the proof of the lemma. □

6.5.2 We now need several preliminary lemmas. Along this section, Γ and N_0 will be fixed as in the statement of Theorem 6.1.2, $\gamma:[0,\mathcal{H}^1(\Gamma)]/\sim\to\Gamma$ will denote a fixed arc-length parametrization of Γ and k_g its geodesic curvature defined in (6.1.3). When dealing with a sequence of immersions, we denote with a subscript k every quantity pertaining to the immersion Φ_k (e.g. the Gauss map of Φ_k will be simply denoted by N_k).

Lemma 6.5.3. *Let $(\Phi_k)_k$ be a sequence in $\mathscr{F}(B_1,\mathbb{R}^n,\Gamma,N_0)$ with $E=\sup_k\mathcal{W}_2(\Phi_k)<+\infty$. Then:*

(i) the L^1-norm of the Gauss curvature $\|K_k\mathrm{e}^{2\lambda_k}\|_{L^1(B_1)}$ is uniformly bounded,

(ii) the number of branch points of Φ_k and their multiplicity is uniformly bounded on k,

(iii) The $L^{(2,\infty)}$-norm of the gradient of the conformal factor $\|\nabla\lambda_k\|_{L^{(2,\infty)}(B_1)}$ is uniformly bounded on k,

(iv) for any fixed $E_0>0$, the set of points $x\in\overline{B_1}$ such that the Willmore energy concentrates above the level E_0, i.e.:

$$\liminf_{k\to+\infty}\Big(\inf\{r>0:\|\nabla N_k\|^2_{L^2(B_1\cap B_r(x))}\geq E_0\}\Big)=0$$

is finite and its cardinality is uniformly bounded on k.

All such bounds depend only on E and on $\|k_g\|_{L^1(\Gamma)}$.

Proof of Lemma 6.5.3. *Proof of (i).* The bound follows from the pointwise a.e. relation (see §6.4 for more details)

$$K_k e^{2\lambda_k} \le \frac{|\nabla N_k|^2}{2}. \tag{6.5.1}$$

Proof of (ii). Since λ_k is a weak solution to the Liouville's equation

$$\begin{cases} -\Delta\lambda_k = K_k e^{2\lambda_k} - 2\pi \sum_{i_k=1}^{\ell_k} n_{i_k}\delta_{a_{i_k}} & \text{in } B_1, \\ \partial_\nu \lambda_k = k_g(\sigma_k)e^{\lambda_k} - 1 & \text{on } \partial B_1, \end{cases} \tag{6.5.2}$$

there must hold

$$-2\pi \sum_{i_k=1}^{\ell_k} n_{i_k} + \int_{B_1} K_k e^{2\lambda_k}\, dx + \int_{\partial B_1} (k_g(\sigma_k)e^{\lambda_k} - 1)\, d\mathcal{H}^1 = 0,$$

hence

$$\sum_{i_k=1}^{\ell_k} |n_{i_k}| \le C\Big(\|K_k e^{2\lambda_k}\|_{L^1(B_1)} + \|k_g\|_{L^1(\Gamma)} + 1\Big),$$

and the result then follows from (i).

Proof of (iii). Using Green's representation formula (the green function $\mathcal{G}$ is given in (6.3.3)), we have

$$\begin{aligned} \nabla\lambda_k(x) = &\int_{B_1} \nabla_x \mathcal{G}(x,y) K_k(y) e^{2\lambda_k(y)}\, dy - 2\pi \sum_{i_k=1}^{\ell_k} n_{i_k} \nabla_x \mathcal{G}(x, a_{i_k}) \\ &+ \int_{\partial B_1} \nabla_x \mathcal{G}(x,y) k_g(\sigma_k(y)) e^{\lambda_k(y)}\, d\mathcal{H}^1(y), \end{aligned}$$

and since there holds

$$\sup_{y\in B_1} \|\nabla_x \mathcal{G}(\cdot, y)\|_{L^{(2,\infty)}(B_1)} < +\infty,$$

we can estimate

$$\|\nabla\lambda_k\|_{L^{(2,\infty)}(B_1)} \le C\Big(\|K_k e^{2\lambda_k}\|_{L^1(B_1)} + \sum_{i_k=1}^{\ell_k} |n_{i_k}| + \|k_g\|_{L^1(\Gamma)} + 1\Big), \tag{6.5.3}$$

hence deduce that the right-hand-side of (6.5.3) does not depend on k thanks to (i) and (ii).

Proof of (iv). The proof of (iv) follows from standard concentration-compactness arguments and we omit it. □

Combining Lemmas 6.4.1 and 6.5.3, the following estimate is obtained for points in B_1.

Lemma 6.5.4. *There exists an $\varepsilon_0 > 0$ with the following property. Let $(\Phi_k)_k$ be a sequence in $\overline{\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0)}$ with $E = \sup_k \mathcal{W}_2(\Phi_k) < +\infty$ and let $x_0 \in B_1$ and $0 < r < 1$ be so that $\overline{B_r(x_0)} \subset B_1$. If, for every $k \in \mathbb{N}$, $\overline{B_r(x_0)}$ contains no branch points of Φ_k and there holds*

$$\|\nabla N_k\|^2_{L^2(B_r(x_0))} \le \varepsilon$$

for some $0 < \varepsilon < \varepsilon_0$*, then* $\lambda_k \in C^0(\overline{B_r(x_0)}) \cap W^{1,2}(B_r(x_0))$ *and there exist a constant* $C = C(\Gamma, N_0, E, \varepsilon_0, r) > 0$ *and a sequence* $(c_k)_k \subset \mathbb{R}$ *so that*

$$\sup_k \left(\|\nabla \lambda_k\|_{L^2(B_r(x_0))} + \|\lambda_k - c_k\|_{L^\infty(B_{r/2}(x_0))} \right) \leq C. \tag{6.5.4}$$

The proof is essentially as in [Riv12, Theorem 4.5]). It is given below for the reader's convenience.

Proof of Lemma 6.5.4. Thanks to Lemma 6.4.1, we may find a ortho-normal frame $e_k = (e_{k,1}, e_{k,2})$ spanning N_k with controlled energy: $\|\nabla e_k\|^2_{L^2(B_r(x_0))} \leq C\|\nabla N_k\|_{L^2(B_r(x_0))}$, and from Liouville's equation ((6.1.4) and (6.4.3)) we may write:

$$-\Delta \lambda_k = \langle \nabla^\perp e_{k,1}, \nabla e_{k,2} \rangle \quad \text{in } B_r(x_0).$$

We then decompose $\lambda_k = \mu_k + \nu_k$, where μ_k solves

$$\begin{cases} -\Delta \mu_k = \langle \nabla^\perp e_{k,1}, \nabla e_{k,2} \rangle & \text{in } B_r(x_0), \\ \quad \mu_k = 0 & \text{on } \partial B_r(x_0), \end{cases}$$

and $\nu_k = \lambda_k - \mu_k$ is the harmonic rest. Thanks to Wente's inequality (Theorem 2.1.1 of Chapter 2), there holds

$$\|\mu_k\|_{L^\infty(B_r(x_0))} + \|\nabla \mu_k\|_{L^\infty(B_r(x_0))} \leq C\|\nabla N_k\|^2_{L^2(B_r(x_0))},$$

while using the properties of traces, Poincaré's inequality and Dirichlet's principle we deduce that

$$\|\nu_k - c_k\|_{L^\infty(B_{r/2}(x_0))} + \|\nabla \nu_k\|_{L^2(B_r(x_0))} \leq C\|\nabla \lambda_k\|_{L^2(B_r(x_0))},$$

where $c_k = \fint_{B_r(x_0)} \nu_k \, dx$ is the average of ν_k over $B_r(x_0)$. Together with the continuous embedding $L^2(B_r(x_0)) \to L^{2,\infty}(B_r(x_0))$ and Lemma 6.5.3-(iii), these estimates for μ_k and ν_k lead to (6.5.4). □

We have the following analogue result for boundary points.

Lemma 6.5.5. *There exists an* $\varepsilon_0 > 0$ *with the following property. Let* $(\Phi_k)_k$ *be a sequence of conformal maps in* $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0)$ *with* $E = \sup_k \mathcal{W}_2(\Phi_k) < +\infty$*, and let* $x_0 \in \partial B_1$ *and* $0 < r < 1$*. If, for every* $k \in \mathbb{N}$*,* $B_1 \cap B_r(x_0)$ *contains no branch points of* Φ_k *and, having denoted* $e_k = \mathrm{e}^{-\lambda_k}(\partial_1 \Phi_k, \partial_2 \Phi_k)$ *the ortho-normal frame associated with* Φ_k*, there holds*

$$\|\nabla N_k\|^2_{L^2(B_1 \cap B_r(x_0))} + \|k_g(\sigma_k)\mathrm{e}^{\lambda_k}\|_{L^1(\partial B_1 \cap B_r(x_0))} \leq \varepsilon \quad \text{and} \quad [e_k]^2_{W^{1/2,2}(\partial B_1 \cap B_r(x_0))} \leq \varepsilon \tag{6.5.5}$$

for some $0 < \varepsilon < \varepsilon_0$*, then* $\lambda_k \in C^0(\overline{B_1 \cap B_r(x_0)}) \cap W^{1,2}(B_1 \cap B_r(x_0))$ *and there exist constant a constant* $C = C(\Gamma, N_0, E, \varepsilon_0, r) > 0$ *independent of* ε *and a sequence* $(c_k)_k \subset \mathbb{R}$ *so that*

$$\sup_k \left(\|\nabla \lambda_k\|_{L^2(B_1 \cap B_{r/4}(x_0))} + \|\lambda_k - c_k\|_{L^\infty(B_1 \cap B_{r/4}(x_0))} \right) \leq C.$$

Proof of Lemma 6.5.5. On the one hand, from the pointwise a.e. relation (6.5.1) and (6.5.5) we have

$$\|k_g(\sigma_k)\mathrm{e}^{\lambda_k}\|_{L^1(\partial B_1\cap B_r(x_0))}+2\|K_k\mathrm{e}^{2\lambda_k}\|_{L^1(B_1\cap B_r(x_0))}\leq\varepsilon$$

consequently since λ_k is a weak solution to Liouville's equation (6.5.2), by choosing an ε_0 small enough, by Lemma 6.3.3 and Lemma 6.5.3 we may find a $p=p(\varepsilon_0)>1$ so that, uniformly on k, there holds

$$\|\mathrm{e}^{\lambda_k-\overline{\lambda}_k}\|_{L^p(\partial B_1\cap B_{r/2}(x_0))}\leq C. \tag{6.5.6}$$

On the other hand, possibly after reducing ε_0 we can invoke Lemma 6.4.3 and deduce the existence of Coulomb ortho-normal frames $g_k=(g_{k,1},g_{k,2})\in W^{1,2}(B_1\cap B_r(x_0))$ lifting N_k in $B_1\cap B_r(x_0)$, coinciding with e_k on $\partial B_1\cap B_r(x_0)$ and so that uniformly in k there holds

$$\|\nabla g_k\|^2_{L^2(B_1\cap B_r(x_0))}\leq C. \tag{6.5.7}$$

In particular, we may write:

$$\begin{aligned}K_k\mathrm{e}^{2\lambda_k}&=\langle\nabla^\perp g_{k,1},\nabla g_{k,2}\rangle\text{ in }B_1\cap B_r(x_0),\\ k_g(\sigma_k)\mathrm{e}^{\lambda_k}-1&=\langle\partial_\tau g_{k,1},g_{k,2}\rangle\text{ on }\partial B_1\cap B_r(x_0).\end{aligned}$$

From Lemma 2.1.8 of Chapter 2, we deduce that $\lambda_k\in C^0(\overline{B_1\cap B_{r/4}(x_0)})\cap W^{1,2}(B_1\cap B_{r/4}(x_0))$ and that for some constant $c_k\in\mathbb{R}$ there holds

$$\begin{aligned}\|\lambda_k-c_k\|_{L^\infty(B_1\cap B_{r/4}(x_0))}&+\|\nabla\lambda_k\|_{L^2(B_1\cap B_{r/4}(x_0))}\\ &\leq C\Bigg(\|K_k\mathrm{e}^{2\lambda}\|_{L^1(B_1)}+\|k_g(\sigma_k)\mathrm{e}^{\lambda_k}\|_{L^1(\partial B_1)}+\sum_{i_k=1}^{\ell}|\alpha_{i_k}|\\ &+\|\nabla g_{k,1}\|_{L^2(B_1\cap B_{r/2}(x_0))}\|g_{k,2}\|_{W^{1,2}(B_1\cap B_{r/2}(x_0))}\\ &+\|\partial_\tau g_{k,1}|_{\partial B_1}\|_{L^p(\partial B_1\cap B_{r/2}(x_0))}\|g_{k,2}|_{\partial B_1}\|_{W^{1,p}(\partial B_1\cap B_{r/2}(x_0))}\Bigg).\end{aligned} \tag{6.5.8}$$

The first line on the right hand side of (6.5.8) can be estimated uniformly on k by means of lemma 6.5.3-(i)-(ii). The second line can be estimated uniformly on k with (6.5.7). Finally the third line is estimated uniformly on k with (6.5.6) since, for $i=1,2$, we have

$$\begin{aligned}\|g_{k,i}|_{\partial B_1}\|_{W^{1,p}(\partial B_1\cap B_{r/2}(x_0))}&\leq C\Big(\|(|k_g(\sigma_k)|+|\dot N_0(\sigma_k)|)\mathrm{e}^{\lambda_k}\|_{L^p(\partial B_1\cap B_{r/2}(x_0))}+1\Big)\\ &\leq C\Big(\|\mathrm{e}^{\lambda_k-\overline{\lambda}_k}\|_{L^p(\partial B_1\cap B_{r/2}(x_0))}+1\Big),\end{aligned}$$

where we used Jensen's inequality to deduce that, uniformly on k, there holds:

$$\mathrm{e}^{\overline{\lambda}_k}=\exp\fint_{\partial B_1}\lambda_k\leq\fint_{\partial B_1}\exp(\lambda_k)=\frac{\mathcal{H}^1(\Gamma)}{2\pi}.$$

This concludes the proof of the lemma. $\square$

Definition 6.5.6. *Let P_1,P_2,P_3 be three distinct, fixed, consecutive points in P_1,P_2,P_3 on Γ that is, $\gamma(s_j)=P_j$ for some $0\leq s_1<s_2<s_3<\mathcal{H}^1(\Gamma)$. We denote by $\mathscr{F}^*(B_1,\mathbb{R}^n,\Gamma,N_0,a)$ is the set of maps $\mathscr{F}(B_1,\mathbb{R}^n,\Gamma,N_0,a)$ so that*

$$\Phi\left(\mathrm{e}^{\frac{2\pi i}{3}j}\right)=P_j\quad\textit{for } j=1,2,3. \tag{6.5.9}$$

Remark 6.5.7 We note that:

(i) if σ_Φ defines the boundary parametrization of Φ, that is $\Phi|_{\partial B_1} = \gamma \circ \sigma_\Phi$, condition (6.5.9) is equivalent to $\sigma_\Phi\left(\frac{2\pi}{3} j\right) = s_j$ for $j = 1, 2, 3$.

(ii) For every $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$, there is a unique Möbius transformation of B_1 so that $\Phi \circ \phi \in \mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)$. Moreover, the invariance by diffeomorphisms of the Willmore energy implies that $\mathcal{W}_2(\Phi) = \mathcal{W}_2(\Phi \circ \phi)$, hence the infimum of $\mathcal{W}_2$ over $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ will equal that over $\mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)$.

The following lemma is a consequence of the Courant-Lebesgue lemma, a key tool in the analysis of Plateau's problem (see e.g. [CI11, Lemma 4.14]).

Lemma 6.5.8. *For any sequence $\Phi_k)_k$ in $\mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)$, the sequence of boundary curves $(\Phi_k|_{\partial B_1})_k$ is equicontinuous.*

Equicontinuity of the boundary curves is equivalent to the equicontinuity of the σ_k's. As a consequence of lemma 6.5.8, we have:

Lemma 6.5.9. *Let $(\Phi_k)_k$ be a sequence in $\mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ and let $x_0 \in \partial B_1$ be fixed. Then, possibly passing to a subsequence, for any any $\varepsilon > 0$, there always exists an $r = r(\Gamma, N_0) > 0$ so that:*

$$\sup_{k\in\mathbb{N}} [e_k]_{W^{1/2,2}(\partial B_1 \cap B_r(x_0))} \leq \varepsilon \quad \textit{and} \quad \sup_{k\in\mathbb{N}} \|k_g(\sigma_k) e^{\lambda_k}\|_{L^1(\partial B_1 \cap B_r(x_0))} \leq \varepsilon. \tag{6.5.10}$$

Proof of Lemma 6.5.9. Possibly after extracting a subsequence, thanks to lemma 6.5.8 and the Arzelà-Ascoli theorem, we may suppose that σ_k converges uniformly on ∂B_1 to some continuous map σ. As a consequence, we have the pointwise convergence away from the diagonal:

$$\lim_k \left(\frac{\sigma_k(\theta_1) - \sigma_k(\theta_2)}{e^{i\theta_1} - e^{i\theta_2}} \right) = \frac{\sigma(\theta_1) - \sigma(\theta_2)}{e^{i\theta_1} - e^{i\theta_2}} \quad \text{for } \theta_1 \neq \theta_2,$$

and, possibly after extracting another subsequence, the bound:

$$\frac{|\sigma_k(\theta_1) - \sigma_k(\theta_2)|^2}{|e^{i\theta_1} - e^{i\theta_2}|^2} \leq 2 \frac{|\sigma(\theta_1) - \sigma(\theta_2)|^2}{|e^{i\theta_1} - e^{i\theta_2}|^2}.$$

Hence, integrating both sides, we deduce that for every $\rho > 0$ and $x \in \partial B_1$ there holds:

$$[\sigma_k]^2_{W^{1/2,2}(\partial B_1 \cap B_\rho(x))} \leq 2[\sigma]^2_{W^{1/2,2}(\partial B_1 \cap B_\rho(x))}. \tag{6.5.11}$$

Let us assume without loss of generality that $x_0 = 1$ and that $r < 1$, so that we may identify $[-\theta_0, \theta_0] \simeq \partial B_1 \cap B_r(x_0)$ for some $0 < \theta_0 < \pi$. Writing in complex notation (see §6.4)

$$e^{i\theta}(e_{k,1} + i e_{k,2}) = \star(\mathbf{t} \wedge N_0)(\sigma_k) + i\mathbf{t}(\sigma_k) \quad \text{on } \partial B_1,$$

thanks to (6.5.11) we deduce that[3] that

$$\begin{aligned}[e_k]^2_{W^{1/2,2}((-\theta_0,\theta_0))} &\leq 2\left([\mathbf{t}(\sigma_k)]^2_{W^{1/2,2}((-\theta_0,\theta_0))} + [\star(\mathbf{t}\wedge N_0)(\sigma_k)]^2_{W^{1/2,2}((-\theta_0,\theta_0))}\right) + 4\theta_0^2 \\ &\leq 2\left([\mathbf{t}]^2_{C^{0,1}} + [\star(\mathbf{t}\wedge N_0)]^2_{C^{0,1}}\right)[\sigma_k]^2_{W^{1/2,2}((-\theta_0,\theta_0))} + 4\theta_0^2 \\ &\leq C\left([\sigma]^2_{W^{1/2,2}((-\theta_0,\theta_0))} + \theta_0^2\right),\end{aligned}$$

and so the first inequality in (6.5.10) follow by choosing a sufficiently small θ_0. As for the second inequality in (6.5.10), if a and b denote the extrema of $\partial B_1 \cap B_r(x_0)$, from the point-wise convergence of σ_k to σ we have:

$$\lim_k \int_{\partial B_1\cap B_r(x_0)} |k_g(\sigma_k)|\mathrm{e}^{\lambda_k(\mathrm{e}^{i\theta})}\,d\theta = \lim_k \int_{\sigma_k(a)}^{\sigma_k(b)} |k_g(s)||\dot\gamma(s)|\,ds = \int_{\sigma(a)}^{\sigma(b)} |k_g(s)||\dot\gamma(s)|\,ds,$$

hence, possibly after extracting a subsequence, there holds:

$$\|k_g(\sigma_k)\sigma_k'\|_{L^1(\partial B_1\cap B_r(x_0))} \leq 2\|k_g(\sigma)\sigma'\|_{L^1(\partial B_1\cap B_r(x_0))},$$

and the results then follows by choosing r sufficiently small. This concludes the proof of the lemma. □

Definition 6.5.10 (Weak Sequential convergence). *Given a sequence $(\Phi_k)_k$ in $\mathscr{F}(B_1,\mathbb{R}^n,\Gamma,N_0)$, and a conformal map $\Phi : B_1 \to \mathbb{R}^n$, we say that Φ_k* weakly converges *to Φ if:*

(i) $\Phi_k \rightharpoonup \Phi \quad$ *in $W^{1,2}(B_1,\mathbb{R}^n)$ and a.e. on B_1,*

(ii) $\Phi_k|_{\partial B_1} \to \Phi|_{\partial B_1} \quad$ *uniformly in $C^0(\partial B_1)$,*

(iii) $\nabla\lambda_k \overset{*}{\rightharpoonup} \nabla\lambda$ *in $L^{(2,\infty)}(B_1)$,*

and there exists a finite, possibly empty set $\underline{\eta} = \{\eta_1,\dots,\eta_\ell\} \subset \overline{B_1}$ so that, for every open set $\Omega \subset \mathbb{R}^2$ with compact closure in $\overline{B_1}\setminus\underline{\eta}$, there holds:

(iv) $\lambda_k \overset{*}{\rightharpoonup} \lambda$ *in $L^\infty(\Omega,\mathbb{R}^n)$ and ,*

(v) $\Phi_k \rightharpoonup \Phi \quad$ *in $W^{2,2}(\Omega,\mathbb{R}^n)$,*

where $\lambda = \log(|\nabla\Phi|/\sqrt{2})$.

We are now in the position to prove the following compactness result.

Lemma 6.5.11. *Let $(\Phi_k)_k$ be a sequence in $\mathscr{F}^*(B_1,\mathbb{R}^n,\Gamma,N_0,a)$ with $\sup_k \mathcal{W}_2(\Phi_k) < +\infty$. Then $(\Phi_k)_k$ contains a subsequence weakly converging in the sense of Definition 6.5.10 to an element $\mathscr{F}^*(B_1,\mathbb{R}^n,\Gamma,N_0,a)$.*

[3] Recall the elementary inequality:

$$[ab]^2_{W^{1/2,2}} \leq 2\left(\|a\|^2_{L^\infty}[b]^2_{W^{1/2,2}} + \|b\|^2_{L^\infty}[a]^2_{W^{1/2,2}}\right).$$

Also recall that if $a : (l_1,l_2) \to \mathbb{C}$ is a Lipschitz function and $b : (-\theta_0,\theta_0) \to (l_1,l_2)$ is a function in $H^{1/2}((-\theta_0,\theta_0))$ we have:

$$[a\circ b]^2_{W^{1/2,2}((-\theta_0,\theta_0))} \leq [a]^2_{C^{0,1}((l_1,l_2))}[b]^2_{W^{1/2,2}((-\theta_0,\theta_0))}.$$

Proof of Lemma 6.5.11. *Step 1.* Since for every k we have that $\|\nabla\Phi_k\|^2_{L^2(B_1)} = 2a$ and the three-point condition (6.5.9) holds, we deduce that $\sup_k(\|\Phi_k\|_{W^{1,2}(B_1)})$ is finite and so by the Rellich–Kondrachov theorem, possibly passing to a subsequence, condition (i) of Definition 6.5.10 is satisfied.

Step 2. From lemma 6.5.8, by Arzelà–Ascoli theorem we deduce that, possibly passing to a subsequence, condition (ii) of Definition 6.5.10 is satisfied.

Step 3. From lemma 6.5.3-(iii), we deduce that, possibly passing to a subsequence, condition (iii) of Definition 6.5.10 is satisfied.

Step 4. If $\{a_{k,1},\dots a_{k,\ell_k}\}$ is the set of branch points of Φ_k, from Lemma 6.5.3, possibly after extracting a subsequence, we may suppose that ℓ_k is independent of k and that, for each $j = 1,\dots\ell$, $\lim_{k\to\infty} a_{k,j} = a_j$ for some $a_j \in \overline{B_1}$.

We say that a point p belongs to $\underline{\eta}$ if either:

- $p = a_k$ for some $k = 1,\dots,\ell$, or
- there holds
$$\liminf_{k\to\infty}\left(\inf\left\{r>0 : \|\nabla N_k\|^2_{L^2(D\cap B_r(p))} \geq \varepsilon_0\right\}\right) = 0, \tag{6.5.12}$$
with ε_0 is as in Lemma 6.5.4 if $p \in B_1$, or as in Lemma 6.5.5 if $p \in \partial B_1$, or
- $p = \mathrm{e}^{\frac{2\pi i}{3}j}$ for $j = 1, 2$ or 3.

Note that the set of points satisfying (6.5.12) is, due to Lemma 6.5.3, finite and uniformly bounded in k. Let $\Omega \subset \mathbb{R}^2$ an open set with compact closure in $\overline{B_1}\setminus\underline{\eta}$ and let Ω' be a closed set contained in $\overline{B_1}\setminus\underline{\eta}$ and with smooth boundary so that $\overline{\Omega}\subset\Omega'$ and for some small $\delta > 0$ there holds

$$\overline{B_1}\setminus\cup_{p\in\underline{\eta}}B_{2\delta}(p) \subset\subset \Omega' \subset\subset \overline{B_1}\setminus\cup_{p\in\underline{\eta}}B_{\delta}(p).$$

Possibly passing to a further subsequence, we may suppose that, for every k, the set of branch points $\{a_{k,1},\dots a_{k,\ell_k}\}$ of Φ_k, lies in $\cup_{p\in\underline{\eta}}B_\delta(p)$. Now, for every $x\in\overline{\Omega'}$, we can choose an $r_x > 0$ so that, if $x \in B_1$, then $\overline{B_{r_x}(x)}\subset\Omega'$ and $\|\nabla N_k\|^2_{L^2(B_{r_x}(x))} < \varepsilon_0$, and, if $x\in\partial B_1$, then

$$\begin{aligned}\|\nabla N_k\|^2_{L^2(B_1\cap B_{r_x}(x_0))} + \|k_g(\sigma_k)\mathrm{e}^{\lambda_k}\|_{L^1(\partial B_1\cap B_{r_x}(x))} &< \varepsilon_0,\\ [e_k]^2_{W^{1/2,2}(\partial B_1\cap B_{r_x}(x))} &< \varepsilon_0,\end{aligned}$$

(this can be done uniformly on k thanks to Lemma 6.5.9). The family $\{B_{r_x/4}(x)\}_{x\in\overline{\Omega'}}$ forms an open cover of $\overline{\Omega'}$, from which we may extract a finite sub-cover $\{B_{r_j}(x_j)\}_{j=1}^M$. From Lemmas 6.5.4 and 6.5.5, we deduce that $\lambda\in C^0(\overline{B_1\cap B_{r_j}(x_j)})\cap W^{1,2}(B_1\cap B_{r_j}(x_j))$ and there exists constants $l_k(x_j)$, so that, for $j = 1,\dots,M$,

$$\sup_k\left(\|\nabla\lambda_k\|_{L^2(B_1\cap B_{r_j}(x_j))} + \|\lambda_k - l_k(x_j)\|_{L^\infty(B_1\cap B_{r_j}(x_j))}\right) \leq C. \tag{6.5.13}$$

Notice that, for every i, j the bound $|l_k(x_i)-l_k(x_j)|\leq MC$ holds. Indeed, if $B_\rho(x_i)\cap B_\rho(x_j)\neq\emptyset$, then from (6.5.13) and the triangle inequality we have $|l_k(x_i)-l_k(x_j)|\leq 2C$. and general i and j pick a collection from the covering which connects x_j and x_j and reach a similar conclusion.

We can then assume that such constants do not depend on x_j and consequently that, for some l_k, there holds:

$$\sup_k \left(\|\nabla \lambda_k\|_{L^2(\Omega')} + \|\lambda_k - l_k\|_{L^\infty(\Omega')} \right) \leq C. \tag{6.5.14}$$

We claim that the sequence $(l_k)_k$ is uniformly bounded on k. To see that $(l_k)_k$ is bounded from below, note that, if we had $\limsup_{k\to\infty} l_k = -\infty$, possibly after extracting a subsequence condition (6.5.14) would imply that $\lim_{k\to\infty} \lambda_k = +\infty$ uniformly on Ω. Consequently we would have

$$\lim_{k\to\infty} \int_{\Omega'} \mathrm{e}^{2\lambda_k}\, dx = +\infty,$$

which contradicts condition $\|\nabla \Phi_k\|^2_{L^2(B_1)} = 2a$. Suppose now $(l_k)_k$ is not bounded from above, that is $\liminf_{k\to\infty} l_k = +\infty$. Let α be an arbitrary closed, connected sub-arc of ∂B_1 which does not contain any point in $\underline{\eta}$. We add if necessary a finite number of balls $B_{r_j}(x_j)$, to the above finite cover of Ω' so that $\{B_{r_j}(x_j)\}_j$ also covers α. Since λ_k is continuous, possibly passing to a subsequence, from (6.5.14), that $\lim_{k\to\infty} \lambda_k = -\infty$ uniformly in α. Then,

$$0 = \lim_{k\to+\infty} \int_\alpha \mathrm{e}^{\lambda_k}\, d\sigma = \lim_{k\to+\infty} \mathcal{H}^1(\Phi_k|_{\partial B_1}(\alpha)),$$

and thus, by the weak lower-semicontinuity of the Hausdorff measure with respect to the uniform convergence, that $\mathcal{H}^1(\Phi(\alpha)) = 0$. Since the arc α was arbitrarily chosen, from the Borel-regularity of the Hausdorff measure (see [EG15]) we have $\mathcal{H}^1(\Phi(\partial B_1 \setminus \underline{\eta})) = 0$, but then, Φ being continuous and $\underline{\eta}$ consisting of a finite set, this gives $\mathcal{H}^1(\Phi(\partial B_1)) = 0$. This contradicts the three-point normalization condition. This proves that condition (6.5.14) can actually be strengthened to

$$\sup_k \left(\|\nabla \lambda_k\|_{L^2(\Omega')} + \|\lambda_k\|_{L^\infty(\Omega')} \right) \leq C. \tag{6.5.15}$$

and thus, possibly passing to a subsequence, condition (iv) of definition 6.5.10 is satisfied.

Step 5. From (6.5.15), we can estimate

$$\|\nabla \Phi_k\|_{L^\infty(\Omega')} \leq C, \tag{6.5.16}$$

$$\|\Delta \Phi_k\|^2_{L^2(\Omega')} \leq \frac{1}{4} \|\mathrm{e}^{2\lambda_k}\|_{L^\infty(\Omega')} W(\Phi_k) \leq C, \tag{6.5.17}$$

moreover,

$$\begin{aligned}
\|\mathrm{e}^{\lambda_k}\|_{W^{1/2,2}(\partial B_1 \cap \Omega')} &\leq \|\mathrm{e}^{\lambda_k}\|_{W^{1/2,2}(\partial \Omega')} \\
&\leq C \|\mathrm{e}^{\lambda_k}\|_{W^{1,2}(\Omega')} \\
&\leq C\, \mathrm{e}^{\|\lambda_k\|_{L^\infty(\Omega')}} (1 + \|\nabla \lambda_k\|_{L^2(\Omega')}) \leq C,
\end{aligned}$$

hence

$$\begin{aligned}
\|\partial_\tau \Phi_k\|_{W^{1/2,2}(\partial B_1 \cap \Omega')} &= \|\mathrm{e}^{\lambda_k} e_{k,1}(\sigma_k)\|_{W^{1/2,2}(\partial B_1 \cap \Omega')} \\
&\leq C \big(\|\mathrm{e}^{\lambda_k}\|_{W^{1/2,2}(\partial B_1 \cap \Omega')} \\
&\qquad + \|\mathrm{e}^{\lambda_k}\|_{L^\infty(\partial B_1 \cap \Omega')} [e_{k,1}]_{W^{1/2,2}(\partial B_1 \cap \Omega')} \big) \\
&\leq C.
\end{aligned} \tag{6.5.18}$$

From (6.5.16)-(6.5.17)-(6.5.18), elliptic regularity theory yields $\sup_k \|\Phi_k\|_{W^{2,2}(\Omega)} < +\infty$. Thus, possibly passing to a subsequence, also condition (iv) of definition 6.5.10 holds.

Step 6. As in [Riv12, Lemma 5.1], we have that the Gauss map of Φ extends to a map in $W^{1,2}(B_1, \mathrm{Gr}_{n-2}(\mathbb{R}^n))$, and consequently, from lemma 6.5.1 in appendix 6.5.1, the structure near points of $\underline{\eta}$ lying in B_1 is that of a (possibly removable) branch point. Finally as shown in lemma 6.5.2 singular points $a \in \underline{\eta}$ lying on the boundary are always removable, and the limiting map Φ extends to a conformal Lipschitz immersion near ∂B_1. This concludes the proof of the lemma. □

The proof of the following lemma can be easily deduced from its analogue in the closed case (see [Riv12, Theorem 5.9]).

Lemma 6.5.12. *The Willmore energy $\mathcal{W}_2$ is sequentially lower semi-continuous in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ with respect to weak convergence in the sense of Definition 6.5.10, that is, if $(\Phi_k)_k$ is a sequence in $\mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ weakly converging to Φ, then*

$$\liminf_{k\to\infty} \mathcal{W}_2(\Phi_k) \geq \mathcal{W}_2(\Phi).$$

Proof of Theorem 6.1.2. Since the triple (Γ, N_0, a) is admissible, the set $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ is not empty and we can consider a sequence $(\Phi_k)_k$ in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ minimising the Willmore energy:

$$\lim_{k\to\infty} \mathcal{W}_2(\Phi_k) = \inf\{\mathcal{W}_2(\Psi) : \Psi \in \mathscr{F}^*(B_1, \mathbb{R}^n, \Gamma, N_0, a)\}.$$

By Remark 6.5.7, we may assume that each Φ_k satisfies the three-point normalisation condition given by Definition 6.5.6. From Lemma 6.5.11 we can then extract a weakly converging subsequence in the sense of Definition 6.5.10 to a conformal map Φ in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$. Finally, because of Lemma 6.5.12 we have

$$\mathcal{W}_2(\Phi) = \lim_{k\to\infty} \mathcal{W}_2(\Phi_k) = \inf\{E(\Psi) : \Psi \in \mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)\}.$$

This concludes the proof of the theorem. □

6.6 Regularity of Minimizers

This section is devoted to prove that any element in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ which minimizes the Willmore energy satisfies all the regularity statements of Theorem 6.1.3.

We need first some preparatory results regarding Lipschitz $W^{2,2}$ immersions and estimates on suitable competitors for the Germain–Poisson problem. In this section, we denote with $D\,\mathrm{Area}(\Phi)w$ and $D\mathcal{W}_2(\Phi)w$ the directional derivative at Φ along w of the area and Willmore energy functional, namely

$$D\,\mathrm{Area}(\Phi)w = \left.\frac{d\,\mathrm{Area}(\Phi + tw)}{dt}\right|_{t=0}, \quad D\mathcal{W}_2(\Phi)w = \left.\frac{d\mathcal{W}_2(\Phi + tw)}{dt}\right|_{t=0},$$

and, if Ω is the domain of Φ (a ball or a half–ball) we set

$$\|D\,\mathrm{Area}(\Phi)\| := \sup\{|D\,\mathrm{Area}(\Phi)w| : w \in W^{1,\infty}(\Omega, \mathbb{R}^n),\ \|w\|_{W^{1,\infty}(\Omega)} \leq 1,\ \mathrm{supp}\, w \subset\subset \Omega\}.$$

6.6.1 Lemmas on Conformal Immersions

Lemma 6.6.1 (Interior Estimates for λ)**.** *There exists an $\varepsilon_0 > 0$ such that, if $\Phi : B_1 \to \mathbb{R}^n$ is a conformal Lipschitz $W^{2,2}$ immersion satisfying*

$$\mathcal{W}_2(\Phi) = \int_{B_1} |A|^2 \, d\sigma_g < \varepsilon_0,$$

then,

(i) for any $0 < r < 1$ there holds

$$\int_{B_r} |\nabla\lambda|^2 \, dx \leq \left(\frac{r^2}{2} + C\varepsilon_0\right) \int_{B_1} |\nabla^2\Phi|_g^2 \, d\sigma_g \tag{6.6.1}$$

(ii) for any compact set $K \subset\subset B_1$, there holds

$$\|\lambda - (\lambda)_{B_1}\|_{L^\infty(K)} \leq \frac{C}{\mathrm{dist}(K, \partial B_1)^2} \|\nabla\lambda\|_{L^{(2,\infty)}(B_1)} + C\varepsilon_0, \tag{6.6.2}$$

where $(\lambda)_{B_1} = \fint_{B_1} \lambda \, dx$ denotes the average of λ on B_1 and $C > 0$ is a constant independent of Φ.

Proof of Lemma 6.6.1. Thanks to Lemma 6.4.1, if ε_0 is small enough we may find a Coulomb orthonormal frame f defined on B_1 satisfying the estimate

$$\|\nabla f\|_{L^2(B_1)}^2 \leq C\|\nabla N\|_{L^2(B_1)}^2 \leq C\varepsilon_0.$$

We write $\lambda = \mu + h$, where μ is the solution be the solution to

$$\begin{cases} -\Delta\mu = \langle \nabla^\perp f_1, \nabla f_2 \rangle & \text{in } B_1, \\ \quad \mu = 0 & \text{on } \partial B_1, \end{cases}$$

and h is the harmonic rest. By Wente's lemma, μ belongs to $C^0(\overline{B_r}) \cap W^{1,2}(B_r)$ with

$$\|\nabla\mu\|_{L^2(B_1)} + \|\mu\|_{L^\infty(B_1)} \leq C\|\nabla N\|_{L^2(B_1)}^2 \leq C\varepsilon_0. \tag{6.6.3}$$

As for h, since it is harmonic, for any $0 < r < 1$ it satisfies

$$\int_{B_r} |\nabla h|^2 \, dx \leq r^2 \int_{B_1} |\nabla h|^2 \, dx. \tag{6.6.4}$$

Using successively (6.6.4), the Dirichlet principle, (6.6.3) and identity $|\nabla^2\Phi|_g^2 = 4e^{-2\lambda}|\nabla\lambda|^2 + |A|^2$ we then deduce

$$\begin{aligned}
\int_{B_r(0)} |\nabla\lambda|^2 \, dx &= \int_{B_r(0)} |\nabla(\mu + h)|^2 \, dx \\
&\leq 2r^2 \int_{B_1} |\nabla h|^2 + 2\int_{B_1} |\nabla\mu|^2 \, dx \\
&\leq 2r^2 \int_{B_1} |\nabla\lambda|^2 \, dx + C\|\nabla N\|_{L^2(B_1)}^4 \\
&\leq \frac{1}{2} r^2 \int_{B_1} |\nabla^2\Phi|_g^2 \, d\sigma_g + C\|\nabla N\|_{L^2(B_1)}^4 \\
&= \left(\frac{r^2}{2} + C\varepsilon_0\right)\left(\int_{B_1} |\nabla^2\Phi|_g^2 \, d\sigma_g\right),
\end{aligned}$$

which proves (6.6.1). As for (6.6.2), we note that h may be written by means of the Poisson kernel[4] as

$$h(x) = \int_{\partial B_1} K(x,y)\,\lambda(y)\,d\mathcal{H}^1(y), \quad x \in B_1,$$

consequently we deduce that, for any $x \in K$, using the trace theorem and Poincaré's inequality, there holds

$$\begin{aligned}
|h(x) - (\lambda)_{B_1}| &\leq \frac{1-|x|^2}{2\pi}\int_{\partial B_1}\frac{1}{|x-y|^2}|\lambda(y) - (\lambda)_{B_1}|\,d\mathcal{H}^1(y)\\
&\leq \frac{C}{\operatorname{dist}(K,\partial B_1)^2}\|\lambda - (\lambda)_{B_1}\|_{L^1(\partial B_1)}\\
&\leq \frac{C}{\operatorname{dist}(K,\partial B_1)^2}\|\nabla\lambda\|_{L^1(B_1)}\\
&\leq \frac{C}{\operatorname{dist}(K,\partial B_1)^2}\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}.
\end{aligned}$$

We may then conclude with (6.6.3) that

$$\begin{aligned}
\|\lambda_\Phi - (\lambda)_{B_1}\|_{L^\infty(K)} &\leq \|h - (\lambda)_{B_1}\|_{L^\infty(K)} + \|\mu\|_{L^\infty(B_1)}\\
&\leq \frac{C}{\operatorname{dist}(K,\partial B_1)^2}\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)} + C\varepsilon_0,
\end{aligned}$$

as desired. This concludes the proof of the lemma. □

Lemma 6.6.2 (Boundary Estimates for λ)**.** *There exists an $\varepsilon_0 > 0$ so that the following holds. Let $\Phi : B_1^+(0) \to \mathbb{R}^n$ be a conformal Lipschitz $W^{2,2}$ immersion and let $e = (e_1, e_2)$ be its coordinate frame. If for some $p > 1$ we have $\partial_\tau e \in L^p(I, \mathbb{R}^n \times \mathbb{R}^n)$ and*

$$\int_{B_1^+(0)} |A|^2\,d\sigma_g + [e]^2_{W^{1/2,2}(I)} < \varepsilon_0,$$

then:

(i) for any $0 < r < 1$ there holds

$$\begin{aligned}
\int_{B_r^+}|\nabla\lambda|^2\,dx &\leq \left(\frac{r^2}{2}\right)\int_{B_1^+}|\nabla^2\Phi|^2\,d\sigma_g\\
&+\Big(C\varepsilon_0 + C(p)\|\partial_\tau e_1\|_{L^p(I)}(1+\|\partial_\tau e_2\|_{L^p(I)})\Big)\left(\int_{B_1^+}|A|^2\,d\sigma_g + \|\langle\partial_\tau e_1, e_2\rangle\|_{L^1(I)}\right),
\end{aligned} \tag{6.6.5}$$

(ii) for any compact set $K \subset \overline{B_1^+(0)}$ so that $\operatorname{dist}(K,S) > 0$ there holds

$$\inf_{c\in\mathbb{R}}\|\lambda - c\|_{L^\infty(K)} \leq \frac{C}{\operatorname{dist}(K,S)^2}\|\nabla\lambda\|_{L^{(2,\infty)}(B_1^+)} + C(p)(\varepsilon_0 + \|\partial_\tau e_1\|_{L^p(I)}(1+\|\partial_\tau e_2\|_{L^p(I)})), \tag{6.6.6}$$

where $C > 0$ is an universal constant and $C(p) > 0$ is a constant depending only on p.

[4] the explicit formula is (see [Eva10, §2.2.4]): $K(x,y) = \frac{1-|x|^2}{2\pi}\frac{1}{|x-y|^2}$.

Proof of Lemma 6.6.2. *Step 1.* Thanks to Lemma 6.4.4 (see also Remark 6.4.5), we can consider an extension f of $e|_I$ to all of ∂B_1^+ such that

$$[f\,]_{W^{1/2,2}(\partial B_1^+)} \leq 2[f\,]_{W^{1/2,2}(I)},$$

Step 2. We choose ε_0 sufficiently small so that thanks to lemma 6.4.2 we may find a frame $g = (g_1, g_2)$ lifting N on B_1^+ that coincides with f on ∂B_1^+ and satisfies

$$\|\nabla g\|_{L^2(B_1^+)} \leq C\left(\|\nabla N\|_{L^2(B_1^+)} + [f]_{W^{1/2,2}(\partial B_1^+)}\right) \leq C\varepsilon_0.$$

Step 3. We write $\lambda = \mu + h$, where:

$$\begin{cases} -\Delta\mu = \langle\nabla^\perp g_1, \nabla g_1\rangle & \text{in } B_1^+, \\ \partial_\nu\mu = \langle\partial_\tau g_1, g_2\rangle & \text{on } I, \\ \mu = 0 & \text{on } S, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta h = 0 & \text{in } B_1^+, \\ \partial_\nu h = 0 & \text{on } I, \\ h = \lambda & \text{on } S. \end{cases}$$

Step 4: estimate of μ. Since μ satisfies a homogeneous Dirichlet condition on S, its extension to $\mathbb{R}^2_+$ by means of odd inversion along S (given by the conformal map $x \mapsto x/|x|^2$):

$$\widehat{\mu}(x) = \begin{cases} \mu(x) & \text{for } x \in B_1^+, \\ -\mu(x/|x|^2) & \text{for } x \in \mathbb{R}^2_+ \setminus B_1^+, \end{cases}$$

satisties (also thanks to the transformation law under conformal maps of the Laplace operator and of the determinant):

$$\begin{cases} -\Delta\widehat{\mu} = \langle\nabla^\perp\widetilde{g}_1, \nabla\widetilde{g}_2\rangle & \text{in } \mathbb{R}^2_+, \\ \partial_\nu\widehat{\mu} = \langle\partial_\tau\widetilde{g}_1, \widetilde{g}_2\rangle & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{6.6.7}$$

where for $i = 1, 2$,

$$\widetilde{g}_i(x) = \begin{cases} g_i(x) & \text{for } x \in B_1^+, \\ g_i(x/|x|^2) & \text{for } x \in \mathbb{R}^2_+ \setminus B_1^+, \end{cases}$$

denote extensions of g_i by means of even inversion along S.

We then consider the Cayley map $\phi(z) = i\frac{-z+1}{z+1}$ mapping biholomorphically B_1 onto $\mathbb{R}^2_+$ and by simplicity of notation we continue to denote by $\widehat{\mu}$ and $\widetilde{g}_i$ the composition $\widehat{\mu} \circ \phi$ and $\widetilde{g}_i \circ \phi$. By the conformal invariance of the problem (6.6.7), $\widehat{\mu} \circ \phi$ satisfies

$$\begin{cases} -\Delta\widehat{\mu} = \langle\nabla^\perp\widetilde{g}_1, \nabla\widetilde{g}_2\rangle & \text{in } B_1, \\ \partial_\nu\widehat{\mu} = \langle\partial_\tau\widetilde{g}_1, \widetilde{g}_2\rangle & \text{on } \partial B_1, \end{cases}$$

Thanks to Theorem 2.1.7 of Chapter 2, we have

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|\widehat{\mu} - c\|_{L^\infty(B_1)} &\leq C\|\nabla\widetilde{g}_1\|_{L^2(B_1)}(1 + \|\nabla\widetilde{g}_2\|_{L^2(B_1)}) \\ &\quad + C(p)\|\partial_\tau\widetilde{g}_1\|_{L^p(\partial B_1)}(1 + \|\partial_\tau\widetilde{g}_2\|_{L^p(\partial B_1)}). \end{aligned}$$

Because of the conformal invariance of the Dirichlet energy there holds

$$\|\nabla\widetilde{g}_i\|_{L^2(B_1)} = \|\nabla g_i\|_{L^2(B_1^+)} \quad i = 1, 2,$$

on the other hand a direct computation shows that

$$\|\partial_\tau \widetilde{g}_i\|_{L^p(\partial B_1)} \leq 2\|\partial_\tau g_i\|_{L^p(I)} \quad i = 1,2,$$

hence we deduce (assuming without loss of generality that $\varepsilon_0 < 1$ and recalling that $g = e$ on I):

$$\inf_{c\in\mathbb{R}} \|\mu - c\|_{L^\infty(B_1^+)} \leq C\varepsilon_0 + C(p)\|\partial_\tau e_1\|_{L^p(I)}(1 + \|\partial_\tau e_2\|_{L^p(I)}). \tag{6.6.8}$$

Consequently, through integration by parts and by using (6.6.8) we can estimate

$$\begin{aligned}\int_{B_1^+} |\nabla\mu|^2\, dx &= -\int_{B_1^+} (\mu - c)\Delta u\, dx + \int_I (\mu - c)\partial_\nu \lambda\, d\mathcal{H}^1 \\ &\leq \inf \|\mu - c\|_{L^\infty(B_1^+)}\Big(\|\langle\nabla^\perp g_1, \nabla g_2\rangle\|_{L^1(B_1)} + \|\langle\partial_\tau e_1, e_2\rangle\|_{L^1(I)}\Big) \\ &\leq (C\varepsilon_0 + C(p)\|\partial_\tau e_1\|_{L^p(I)}(1 + \|\partial_\tau e_2\|_{L^p(I)}))\Big(\int_{B_1^+} |A|^2\, d\sigma_g + \|\langle\partial_\tau e_1, e_2\rangle\|_{L^1(I)}\Big).\end{aligned} \tag{6.6.9}$$

Step 5: estimate of the harmonic rest h. We observe that the existence of h can be deduced by variational methods. Since h satisfies a homogeneous Neumann condition along I, its extension to B_1 by even reflection along I:

$$\widetilde{h}(x) = \begin{cases} h(x^1, x^2) & \text{in } B_1^+, \\ h(x^1, -x^2) & \text{in } B_1^- = B_1 \cap \mathbb{R}^2_-, \end{cases}$$

will then satisfy

$$\begin{cases} -\Delta\widetilde{h} = 0 & \text{in } B_1, \\ \quad\ \tilde{h} = \widetilde{\lambda} & \text{on } \partial B_1, \end{cases}$$

where $\widetilde{\lambda}$ similarly denotes the extension of λ to B_1 by even reflection along I. From the classical estimate for harmonic function we will then deduce

$$\int_{B_r} |\nabla\widetilde{h}|^2\, dx \leq r^2 \int_{B_1} |\nabla\widetilde{\lambda}|^2\, dx,$$

and consequently,

$$\int_{B_r^+} |\nabla h|^2\, dx \leq r^2 \int_{B_1^+} |\nabla\lambda|^2\, dx. \tag{6.6.10}$$

By joining estimates (6.6.9)-(6.6.10) we then deduce

$$\begin{aligned}\int_{B_r^+} |\nabla\lambda|^2\, dx &\leq 2\int_{B_1^+} |\nabla\mu|^2\, dx + 2\int_{B_r^+} |\nabla h|^2\, dx \\ &\leq (C\varepsilon_0 + C(p)\|\partial_\tau e_1\|_{L^p(I)}(1 + \|\partial_\tau e_2\|_{L^p(I)}))\Big(\int_{B_1^+} |A|^2\, d\sigma_g + \|\langle\partial_\tau e_1, e_2\rangle\|_{L^1(I)}\Big) \\ &\quad + r^2 \int_{B_1^+} |\nabla\lambda|^2\, dx,\end{aligned}$$

which then yields estimate (6.6.5).

As far as the estimate (6.6.6) is concerned, similarly as in Lemma 6.6.1, we deduce that

$$\|\widetilde{h} - (\widetilde{\lambda})_{B_1}\|_{L^\infty(\widetilde{K})} \le \frac{C}{\operatorname{dist}(\widetilde{K}, \partial_1 B)} \|\nabla \widetilde{\lambda}\|_{L^{(2,\infty)}(B_1)},$$

where we denoted by $\widetilde{K} = K \cup \{(x^1, -x^2) : (x^1, x^2) \in K\}$, and consequently that

$$\|h - (\lambda)_{B_1^+}\|_{L^\infty(K)} \le \frac{C}{\operatorname{dist}(K, \partial_1 B)^2} \|\nabla \lambda\|_{L^{(2,\infty)}(B_1^+)}. \tag{6.6.11}$$

We may then write

$$\inf_{c' \in \mathbb{R}} \|\lambda - c'\|_{L^\infty(K)} \le \|h - (\lambda)_{B_1}\|_{L^\infty(K)} + \inf_{c \in \mathbb{R}} \|\mu - c\|_{L^\infty(B_1^+)},$$

and with estimates (6.6.8)-(6.6.11) we deduce the validity of (6.6.6). This concludes the proof of the lemma. □

Lemma 6.6.3 (Affine approximation)**.** *For every $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for every conformal Lipschitz $W^{2,2}$ immersion $\Phi : B_1 \to \mathbb{R}^n$ satisfying*

$$\int_{B_1} |\nabla^2 \Phi|_g^2 \, d\sigma_g < \varepsilon_0,$$

there exists a conformal affine immersion $L = L_0 + x^1 X_1 + x^2 X_2 = L_0 + \langle x, X \rangle$ so that

$$\|\Phi - L\|_{W^{2,2}(B_{1/2})} < \delta \|\nabla \Phi\|_{L^2(B_1)}$$

and, if $\mathrm{e}^\nu = |X_1| = |X_2|$ denotes the conformal factor of L,

$$\|\lambda - \nu\|_{L^\infty(B_{1/2}(0))} < \delta.$$

Proof of Lemma 6.6.3. We argue by contradiction and suppose that there exists a $\delta > 0$ such that, for every $k \in \mathbb{N}$, there is a conformal Lipschitz $W^{2,2}$ immersion $\Phi_k : B_1 \to \mathbb{R}^n$ such that (writing as usual $\mathrm{e}^{\lambda_{\Phi_k}} = \mathrm{e}^{\lambda_k}$),

$$\int_{B_1} |\nabla^2 \Phi_k|_{g_{\Phi_k}}^2 \, d\sigma_{g_{\Phi_k}} = \int_{B_1} \mathrm{e}^{-2\lambda_k} |\nabla^2 \Phi_k|^2 dx \le \frac{1}{k}, \tag{6.6.12}$$

and for every conformal affine immersion L there holds

$$\|\Phi_k - L\|_{W^{2,2}(B_{1/2}(0))} > \delta \|\nabla \Phi_k\|_{L^2(B_1)}. \tag{6.6.13}$$

or, if e^ν denotes the conformal factor of L,

$$\|\lambda_k - \nu\|_{L^\infty(B_{1/2}(0))} \ge \delta. \tag{6.6.14}$$

Since (6.6.12) is invariant under translations and dilations in $\mathbb{R}^n$, writing for short $c_k = -⨍_{B_1} \lambda_k \, dx$ if we set

$$\widetilde{\Phi}_k(x) = \mathrm{e}^{c_k} (\Phi_k(x) - \Phi_k(0)), \quad x \in B_1, \tag{6.6.15}$$

then $\widetilde{\Phi}_k : B_1 \to \mathbb{R}^n$ defines for every k a conformal Lipschitz $W^{2,2}$ immersion such that, if $\mathrm{e}^{\widetilde{\lambda}_k}$ denotes its conformal factor, there holds

$$\widetilde{\lambda}_k = \lambda_k + c_k, \qquad \widetilde{\Phi}_k(0) = 0, \qquad \fint_{B_1} \widetilde{\lambda}_k \, dx = 0,$$

and

$$\int_{B_1} \mathrm{e}^{-2\widetilde{\lambda}_k} |\nabla^2 \widetilde{\Phi}_k|^2 \, dx \le \frac{1}{k}. \tag{6.6.16}$$

From the identity $|\nabla^2\Phi|^2 = 4\mathrm{e}^{-2\lambda}|\nabla\lambda|^2 + |A|^2$, Lemma 6.6.1 and (6.6.16) it follows

$$\begin{aligned} \|\widetilde{\lambda}_k - (\widetilde{\lambda}_k)_{B_1}\|_{L^\infty(B_{1/2}(0))} &= \|\widetilde{\lambda}_k\|_{L^\infty(B_{1/2}(0))} \\ &\le C\|\nabla\widetilde{\lambda}_k\|_{L^2(B_1)} + C\|\nabla\widetilde{N}_k\|^2_{L^2(B_1)} \le \frac{C}{k} \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

so

$$\widetilde{\lambda}_k \to 0 \quad \text{uniformly in } B_{1/2}, \tag{6.6.17}$$

and consequently we infer that

$$\begin{aligned} \|\nabla\widetilde{\Phi}_k\|_{L^2(B_{1/2})} &= \int_{B_{1/2}} 2\mathrm{e}^{2\widetilde{\lambda}_k} \, dx \to \pi/2 \quad \text{as } k \to \infty, \\ \|\nabla^2\widetilde{\Phi}_k\|_{L^2(B_{1/2})} &\le \mathrm{e}^{C/k} \int_{B_{1/2}} \mathrm{e}^{-2\widetilde{\lambda}_k} |\nabla^2\widetilde{\Phi}_k|^2 \, dx \le \frac{C}{k} \to 0 \quad \text{as } k \to \infty, \end{aligned} \tag{6.6.18}$$

from which we deduce, with Poincaré's inequality,

$$\|\Phi_k\|_{W^{2,2}(B_{1/2})} = \|\Phi_k - \Phi_k(0)\|_{W^{2,2}(B_{1/2})} \le C.$$

We deduce that, up to extraction of subsequences, for a map $\widetilde{\Phi}_\infty \in W^{2,2}(B_{1/2}, \mathbb{R}^n)$ we have

$$\begin{aligned} \widetilde{\Phi}_k &\rightharpoonup \widetilde{\Phi}_\infty \quad \text{in } W^{2,2}(B_{1/2}, \mathbb{R}^n), \\ \nabla^2\widetilde{\Phi}_k &\to 0 \quad \text{in } L^2(B_{1/2}, \mathbb{R}^n), \\ \widetilde{\Phi}_k &\to \widetilde{\Phi}_\infty \quad \text{in } W^{1,p}(B_{1/2}, \mathbb{R}^n) \text{ for every } 1 \le p < \infty \text{ and a.e. in } B_{1/2}. \end{aligned}$$

consequently, $\Phi_\infty : B_{1/2}(0) \to \mathbb{R}^n$ is a conformal map and, from the uniform convergence of $\widetilde{\lambda}_k$ above, up to a further subsequence, its conformal factor is 1 (that is, $\widetilde{\Phi}_\infty$ is a isometric linear immersion). Being $\nabla^2\Phi_\infty = 0$, there actually holds

$$\widetilde{\Phi}_k \to \widetilde{\Phi}_\infty \quad \text{in } W^{2,2}(B_{1/2}, \mathbb{R}^n). \tag{6.6.19}$$

Note now that from the definition (6.6.15), (6.6.13) is equivalent to

$$\|\mathrm{e}^{-c_k}\widetilde{\Phi}_k + \Phi_k(0) - L\|_{W^{2,2}(B_{1/2})} \ge \delta\|\nabla\Phi_k\|_{L^2(B_{1/2})},$$

hence

$$\|\widetilde{\Phi}_k + \mathrm{e}^{c_k}(\Phi_k(0) - L)\|_{W^{2,2}(B_{1/2})} \ge \delta\|\nabla(\mathrm{e}^{c_k}\Phi_k)\|_{L^2(B_{1/2})} = \delta\|\nabla(\widetilde{\Phi}_k)\|_{L^2(B_{1/2})},$$

Since L is arbitrary, we may consider the sequence $L_k = \Phi_k(0) + \mathrm{e}^{-c_k}\widetilde{\Phi}_\infty$ and deduce from (6.6.18) that

$$\begin{aligned}\|\widetilde{\Phi}_k - \widetilde{\Phi}_\infty\|_{W^{2,2}(B_{1/2})} &\geq \delta\|\nabla(\widetilde{\Phi}_k)\|_{L^2(B_{1/2})}\\ &= \delta(\pi/2 + o(1)) \quad \text{as } k \to \infty,\end{aligned}$$

which is a contradiction with (6.6.19). Similarly, if (6.6.14) holds, since the conformal factor of L_k is e^{-c_k}, we have

$$\|\lambda_k - \nu_k\|_{L^\infty(B_{1/2})} = \|\lambda_k + c_k\|_{L^\infty(B_{1/2})} = \|\widetilde{\lambda}_k\|_{L^\infty(B_{1/2})} \geq \delta \quad \text{for every } k \in \mathbb{N},$$

which contradicts (6.6.17). □

6.6.2 Construction of Suitable Competitors

Lemma 6.6.4 (Interior Competitors). *Let $\Phi : B_1 \to \mathbb{R}^n$ be a conformal Lipschitz $W^{2,2}$ immersion. For every $\delta > 0$ there there exists an $\varepsilon_0 > 0$ such that if*

$$\int_{B_{4r}} |\nabla^2\Phi|_g^2\, d\sigma_g < \varepsilon_0,$$

for some $0 < r \leq 1/4$, then there exists a $\rho \in [r/2, r]$ such that the solution to

$$\begin{cases} \Delta^2\psi = 0 & \text{in } B_\rho,\\ \psi = \Phi & \text{on } \partial B_\rho,\\ \nabla\psi = \nabla\Phi & \text{on } \partial B_\rho,\end{cases} \tag{6.6.20}$$

defines an immersion which satisfies:

$$\int_{B_\rho} |\nabla^2\psi|_{g_\psi}^2\, d\sigma_{g_\psi} \leq C\Big(1 + C_0(\delta + o(\delta))\Big)\int_{B_r\setminus B_{r/2}} |\nabla^2\Phi|_g^2\, d\sigma_g, \tag{6.6.21}$$

and

$$|\operatorname{Area}(\Phi|_{B_\rho}) - \operatorname{Area}(\psi)| \leq C_0(\delta + o(\delta))\|\nabla\Phi\|_{L^2(B_\rho)}^2, \tag{6.6.22}$$

and

$$\|D\operatorname{Area}(\Phi|_{B_\rho}) - D\operatorname{Area}(\psi)\| \leq C_0(\delta + o(\delta))\|\nabla\Phi\|_{L^2(B_r)}, \tag{6.6.23}$$

where $C > 0$ independent of r and Φ, $C_0 > 0$ depends only on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$ and $o(\delta)/\delta \to 0$ as $\delta \to 0$.

Remark 6.6.5 We note that:

(i) For every ρ, the existence and uniqueness of a solution to (6.6.20) in $W^{2,2}$ is given, for example, by the fact that such problem is the Euler-Lagrange equation for the biharmonic energy functional $\mathcal{B}(\sigma) = \int_{B_\rho} |\Delta\sigma|^2\, d\mathcal{L}^2$ (or, equivalently, of the Hessian energy $\int_{B_\rho} |\nabla^2\sigma|^2\, d\mathcal{L}^2$), subject to the prescribed boundary data. Since Φ is of class $W^{2,2}$, existence and uniqueness by an argument similar to the one for the Dirichlet problem.

(ii) An elementary fact that will be used in the proof of lemma 6.6.4 is the following. For a function $f \in L^1(B_R)$ ($R > 0$ is arbitrary) and a constant $C > 0$ we say that $\rho \in [R/2, R]$ defines a *C-good slice for f* (in $B_R(0) \setminus B_{R/2}$) if $f|_{\partial B_\rho}$ is in $L^1(\partial B_\rho)$ and there holds

$$\rho \int_{\partial B_\rho} |f(r,\theta)|\, d\sigma \leq C \int_{B_R \setminus B_{R/2}} |f|\, dx.$$

The existence of C-good slices for some C and any f is a consequence of Fubini's theorem. Moreover, one can check that, for every $0 < \delta < R/2$, there exists a $C_\delta > 0$ so that, for every $f \in L^1(B_R)$, the radii $\rho \in [R/2, R]$ defining C_δ- good slices for f have Lebesgue measure at least $R/2 - \delta$.

Proof of Lemma 6.6.4. It is sufficient to prove the thesis only for any $\delta > 0$ sufficiently small. We first treat the case $r = 1/4$ and argue through rescalings at the end. In what follows, we denote by C a positive constant (possibly varying line to line) which is independent of Φ, and with C_0 a positive constant depending only on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$.

Step 1. For ε_0 sufficiently small as in Lemma 6.6.1, we have that

$$\|\lambda - (\lambda)_{B_1}\|_{L^\infty(B_{3/4})} \leq C\left(\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)} + \varepsilon_0\right) = C_0, \tag{6.6.24}$$

where $(\lambda)_{B_1}$ denotes the average of λ over B_1. Also, for every $\delta > 0$, if ε_0 is sufficiently small as in Lemma 6.6.3, then there exists a conformal affine immersion L whose conformal factor we denote by e^ν, satisfying the following estimate

$$\|\Phi - L\|_{W^{2,2}(B_{1/4})} < \delta\|\nabla\Phi\|_{L^2(B_{1/2})}, \tag{6.6.25}$$

$$\|\lambda - \nu\|_{L^\infty(B_{1/4})} < \delta. \tag{6.6.26}$$

By combining (6.6.24)-(6.6.26), we deduce

$$\begin{aligned}\|\lambda - \nu\|_{L^\infty(B_{3/4})} &\leq \|\lambda - (\lambda)_{B_1}\|_{L^\infty(B_{3/4})} + |(\lambda)_{B_1} - \nu| \\ &\leq \|\lambda - (\lambda)_{B_1}\|_{L^\infty(B_{3/4})} + \|\lambda - (\lambda)_{B_1}\|_{L^\infty(B_{1/2})} + \|\lambda - \nu\|_{L^\infty(B_{1/2})} \\ &\leq C_0 + \delta.\end{aligned}$$

It follows that

$$C_0^{-1}(1 - \delta - o(\delta))\mathrm{e}^\nu \leq \mathrm{e}^{\lambda(x)} \leq C_0(1 + \delta + o(\delta))\mathrm{e}^\nu \quad \text{for } x \in B_{3/4}, \tag{6.6.27}$$

and in particular

$$C_0^{-1}(1 - \delta - o(\delta))\mathrm{e}^{2\nu} \leq \|\nabla\Phi\|^2_{L^2(B_{3/4})} \leq C_0(1 + \delta + o(\delta))\mathrm{e}^{2\nu}. \tag{6.6.28}$$

We then consider a good-slice choice $\rho \in [1/8, 1/4]$ so that Φ and $\Phi - L$ belong to $W^{2,2}(\partial B_\rho, \mathbb{R}^n)$ with

$$\begin{aligned}\|\Phi\|_{W^{2,2}(\partial B_\rho)} &\leq C\|\Phi\|_{W^{2,2}(B_{1/4} \setminus B_{1/8})}, \\ \|\Phi - L\|_{W^{2,2}(\partial B_\rho)} &\leq C\|\Phi - L\|_{W^{2,2}(B_{1/4} \setminus B_{1/8})},\end{aligned}$$

hence we consider the solution to (6.6.20) for such choice of ρ. Elliptic regularity theory (see for instance [LM72, Remark 7.2, Chapter 2]) implies that

$$\|\psi - L\|_{W^{5/2,2}(B_\rho)} \leq C\|\Phi - L\|_{W^{2,2}(\partial B_\rho)},$$

while Sobolev embedding $W^{5/2,2} \hookrightarrow C^{1,\alpha}$ implies that, for every $0 < \alpha < 1/2$,

$$\|\psi - L\|_{C^{1,\alpha}(\overline{B_\rho})} \le C\|\psi - L\|_{W^{5/2,2}(B_\rho)}.$$

Hence we have

$$\begin{aligned}
\|\nabla\psi - \nabla L\|_{L^\infty(B_\rho)} &\le C\|\psi - L\|_{W^{5/2,2}(B_\rho)} && \text{(Sobolev embedding)}\\
&\le C\|\Phi - L\|_{W^{2,2}(\partial B_\rho)} && \text{(elliptic estimates)}\\
&\le C\|\Phi - L\|_{W^{2,2}(B_{1/4}\setminus B_{1/8})} && \text{(good-slice choice)}\\
&\le C\delta\|\nabla\Phi\|_{L^2(B_{1/4})} && \text{(by (6.6.25))}\\
&\le C_0 \mathrm{e}^{\nu}(\delta + o(\delta)) && \text{(by (6.6.27) and (6.6.28)).}
\end{aligned}$$

Hence for $i = 1, 2$, we deduce the pointwise estimates in B_ρ

$$\begin{aligned}
||\partial_i\psi|^2 - \mathrm{e}^{2\nu}| &= ||\partial_i\psi| - \mathrm{e}^{\nu}|\,||\partial_i\psi| + \mathrm{e}^{\nu}|\\
&\le |\partial_i\psi - \partial_i L|\,||\partial_i\psi| + \mathrm{e}^{\nu}|\\
&\le C_0\mathrm{e}^{2\nu}(\delta + o(\delta)),
\end{aligned}$$

and similarly

$$\begin{aligned}
|\langle\partial_1\psi, \partial_2\psi\rangle| &= |\langle\partial_1\psi, \partial_2\psi\rangle - \langle\partial_1 L, \partial_2 L\rangle|\\
&= |\langle(\partial_1\psi - \partial_1 L), \partial_2\psi\rangle + \langle\partial_1 L, (\partial_2\psi - \partial_2 L)\rangle|\\
&\le ||\partial_2\psi| + |\partial_1 L||\big(|\partial_1\psi - \partial_1 L| + |\partial_2\psi - \partial_2 L|\big)\\
&\le C_0\mathrm{e}^{2\nu}(\delta + o(\delta)).
\end{aligned}$$

This implies that, if $g_\psi = (\langle\partial_i\psi, \partial_j\psi\rangle)_{ij}$ denotes the metric associated with ψ, for every vector $X = (X^1, X^2) \in \mathbb{R}^2$, we have

$$\mathrm{e}^{2\nu}\left(1 - C_0(\delta + o(\delta))\right)|X|^2 \le g_\psi(X, X) \le \mathrm{e}^{2\nu}\left(1 + C_0(\delta + o(\delta))\right)|X|^2, \tag{6.6.29}$$

where $|X| = \sqrt{(X^1)^2 + (X^2)^2}$ is the Euclidean norm of X. As a consequence, we deduce that for $\delta > 0$ small enough, ψ defines an immersion, and in such case we have

$$\mathrm{e}^{2\nu}\left(1 - C_0(\delta + o(\delta))\right) \le \sqrt{\det g_\psi} \le \mathrm{e}^{2\nu}\left(1 + C_0(\delta + o(\delta))\right), \tag{6.6.30}$$

and

$$\mathrm{e}^{-4\nu}\left(1 + C_0(\delta + o(\delta))\right)^{-2}|\nabla^2\psi|^2 \le |\nabla^2\psi|^2_{g_\psi} \le \mathrm{e}^{-4\nu}\left(1 - C_0(\delta + o(\delta))\right)^{-2}|\nabla^2\psi|^2$$

consequently wee that, point-wise in $B_\rho(0)$,

$$|\nabla^2\psi|^2_{g_\psi}\sqrt{\det g_\psi} \le \frac{1 + C_0(\delta + o(\delta))}{\left(1 - C_0(\delta + o(\delta))\right)^2}\mathrm{e}^{-2\nu}|\nabla^2\psi|^2. \tag{6.6.31}$$

Step 2: estimate for the curvature energy. Since ψ solves (6.6.20), from elliptic regularity theory, we have that for any affine function $M(x) = M_0 + \langle Y, x\rangle$ there holds

$$\begin{aligned}
\|\nabla^2\psi\|_{L^2(B_\rho)} &= \|\nabla^2(\psi - M)\|_{L^2(B_\rho)}\\
&\le C\left(\|\Phi - M\|_{W^{2,2}(\partial B_\rho)} + \|\nabla(\Phi - M)\|_{W^{1,2}(\partial B_\rho)}\right)\\
&\le C\left(\|\Phi - M\|_{L^2(\partial B_\rho)} + \|\nabla(\Phi - M)\|_{L^2(\partial B_\rho)} + \|\nabla^2\Phi\|_{L^2(\partial B_\rho)}\right),
\end{aligned}$$

and hence if we suitably choose M so that

$$\|\nabla(\Phi - M)\|_{L^2(\partial B_\rho)} \leq C\|\nabla^2\Phi\|_{L^2(\partial B_\rho)},$$
$$\|\Phi - M\|_{L^2(\partial B_\rho)} \leq C\|\nabla(\Phi - M)\|_{L^2(\partial B_\rho)}$$

(if $M(x) = M_0 + \langle Y, x\rangle$, it is sufficient to choose $Y = (\nabla\Phi)_{\partial B_\rho}$ and $M_0 = (\Phi - M)_{\partial B_\rho}$), we actually deduce that

$$\|\nabla^2\psi\|_{L^2(B_\rho)} \leq C\|\nabla^2\Phi\|_{L^2(\partial B_\rho)},$$

and so, from the choice of ρ we made, we have

$$\|\nabla^2\psi\|_{L^2(B_\rho)} \leq C\|\nabla^2\Phi\|_{L^2(\partial B_\rho)} \leq C\|\nabla^2\Phi\|_{L^2(B_{1/4}\setminus B_{1/8})}.$$

Note also that

$$\begin{aligned}\int_{B_{1/4}\setminus B_{1/8}} |\nabla^2\Phi|^2\,dx &\leq \int_{B_{1/4}\setminus B_{1/8}} \mathrm{e}^{2\lambda}\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\,dx \\ &\leq \mathrm{e}^{2\delta}\mathrm{e}^{2\nu}\int_{B_{1/4}\setminus B_{1/8}} \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\,dx \\ &\leq \mathrm{e}^{2\nu}(1 + 2(\delta + o(\delta)))\int_{B_{1/4}\setminus B_{1/8}} \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\,dx,\end{aligned} \tag{6.6.32}$$

By joining estimates (6.6.31) – (6.6.32), we deduce that

$$\int_{B_\rho} |\nabla^2\psi|_{g_\psi}\,d\sigma_{g_\psi} \leq C(1 + C_0(\delta + o(\delta)))\int_{B_{1/4}\setminus B_{1/8}} |\nabla^2\Phi|_g^2\,d\sigma_g.$$

Step 3: estimate on the area. From (6.6.26) and (6.6.30)[5] we deduce that (recall that $\rho \in [1/8, 1/4]$), we have

$$|\sqrt{\det g_\psi} - \mathrm{e}^{2\lambda}| \leq \mathrm{e}^{2\nu}C_0(\delta + o(\delta)) \leq \mathrm{e}^{2\lambda}C_0(\delta + o(\delta)) \quad \text{in } B_\rho,$$

hence by integrating over B_ρ we deduce

$$\begin{aligned}|\operatorname{Area}(\psi) - \operatorname{Area}(\Phi|_{B_\rho})| &= \left|\int_{B_\rho}\left(\sqrt{\det g_\psi} - \mathrm{e}^{2\lambda}\right)dx\right| \\ &\leq \int_{B_\rho}\left|\sqrt{\det g_\psi} - \mathrm{e}^{2\lambda}\right|dx \\ &\leq C_0\|\nabla\Phi\|^2_{L^2(B_\rho)}(\delta + o(\delta)).\end{aligned}$$

Step 4: estimate on the derivative of the area. For $w \in W^{1,\infty}(B_\rho, \mathbb{R}^n)$ with compact support in B_ρ, we have

$$D\operatorname{Area}(\Phi)w = \int_{B_\rho}\langle\nabla\Phi, \nabla w\rangle\,dx,$$
$$D\operatorname{Area}(\psi)w = \int_{B_\rho} g_\psi(\nabla\psi, \nabla w)dg_{g_\psi} = \int_{B_\rho} g_\psi^{ij}\langle\partial_i\psi, \partial_j w\rangle\sqrt{\det g_\psi}\,dx,$$

[5] (6.6.26) implies $(1 - 2(\delta + o(\delta)))\mathrm{e}^{2\nu} \leq \mathrm{e}^{2\lambda} \leq (1 + 2(\delta + o(\delta)))\mathrm{e}^{2\nu} \quad \text{in } B_{1/4}$.

and from (6.6.29) and (6.6.30) we deduce that

$$\delta^{ij}(1 - C_0(\delta + o(\delta))) \le g_\psi^{ij}\sqrt{\det g_\psi} \le \delta^{ij}(1 + C_0(\delta + o(\delta))). \tag{6.6.33}$$

We also observe that

$$\|\nabla\psi - \nabla\Phi\|_{L^2(B_\rho)} \le C\delta\|\nabla\Phi\|_{L^2(B_{1/4})}.$$

Moreover

$$\begin{aligned} g_\psi^{ij}\langle\partial_i\psi, \partial_j w\rangle\sqrt{\det g_\psi} - \delta^{ij}\langle\partial_i\Phi, \partial_j w\rangle &= g_\psi^{ij}\langle\partial_i\psi, \partial_j w\rangle\sqrt{\det g_\psi} - g_\psi^{ij}\langle\partial_i\Phi, \partial_j w\rangle\sqrt{\det g_\psi} \\ &\quad + g_\psi^{ij}\langle\partial_i\Phi, \partial_j w\rangle\sqrt{\det g_\psi} - \delta^{ij}\langle\partial_i\Phi, \partial_j w\rangle. \end{aligned}$$

By using (6.6.33) we get that

$$\int_{B_\rho}\left|g_\psi^{ij}\langle\partial_i\Phi, \partial_j w\rangle\sqrt{\det g_\psi} - \delta^{ij}\langle\partial_i\Phi, \partial_j w\rangle\right| dx \le C_0(\delta + o(\delta))\int_{B_\rho}|\langle\nabla\Phi, \nabla w\rangle| dx \tag{6.6.34}$$

and

$$\begin{aligned} &\int_{B_\rho}\left|g_\psi^{ij}\langle\partial_i\psi, \partial_j w\rangle\sqrt{\det g_\psi} - g_\psi^{ij}\langle\partial_i\Phi, \partial_j w\rangle\sqrt{\det g_\psi}\right| dx \\ &\le C_0\|\nabla\psi - \nabla\Phi\|_{L^2(B_\rho)} \le C_0\delta\|\nabla\Phi\|_{L^2(B_{1/4})}. \end{aligned} \tag{6.6.35}$$

By combining estimates (6.6.34) and (6.6.35) we get

$$\begin{aligned} |D\operatorname{Area}(\psi)w - D\operatorname{Area}(\Phi)w| &\le C_0(\delta + o(\delta))\int_{B_\rho}|\langle\partial_i\Phi, \partial_i w\rangle| dx + C_0\|\nabla\psi - \nabla\Phi\|_{L^2(B_\rho)} \\ &\le C_0\delta\|\nabla\Phi\|_{L^2(B_{1/4})} \end{aligned}$$

Step 5: the estimates for a general r. If $0 < r < 1/4$, we may reduce to the case $r = 1/4$: indeed, if we consider the rescaling $\widetilde{\Phi}(x) = \Phi(4rx)$ for $x \in B_1$, by conformal invariance we have

$$\int_{B_{4r}}|\nabla^2\Phi|_g^2\, d\sigma_g = \int_{B_1}|\nabla^2\Phi|_g^2\, d\sigma_g,$$

and the area functional, the Dirichlet energy, solution to the problem (6.6.20) and the $L^{(2,\infty)}$–seminorm of $\nabla\lambda$ are invariant by rescalings as well [6]. We may apply the previous steps to $\widetilde{\Phi}$, estimate $\|\nabla\lambda\|_{L^{(2,\infty)}(B_{4r})}$ with $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$ and then rescale back. □

Definition 6.6.6. *An immersion $\Phi : B_1^+ \to \mathbb{R}^n$ is said to have* flat geometric boundary data *on the base diameter I if there holds*

$$\Phi(x^1, 0) \in \operatorname{span}_{\mathbb{R}}\{\boldsymbol{\varepsilon}_1\} \quad \text{and} \quad N(x^1, 0) = \boldsymbol{\varepsilon}_3 \wedge \ldots \wedge \boldsymbol{\varepsilon}_{n-2} \quad \text{for } (x^1, 0) \in I.$$

For a conformal immersion $\Phi : B_1^+(0) \to \mathbb{R}^n$ with flat geometric boundary data on I, its geometric reflection *along I, $\widehat{\Phi} : B_1 \to \mathbb{R}^3$ is defined as*

$$\widehat{\Phi}(x^1, x^2) = \begin{cases} \Phi(x^1, x^2) & \text{if } x^2 \ge 0, \\ \Phi^1(x^1, -x^2)\boldsymbol{\varepsilon}_1 - \sum_{l=2}^m \Phi^l(x^1, -x^2)\boldsymbol{\varepsilon}_l & \text{if } x^2 < 0. \end{cases}$$

[6] to see this last fact, note that, if $d_{\nabla\lambda}(t) = \mathcal{L}^2(\{x : |\nabla\lambda(x)| > t\})$ denotes the distribution function of $\nabla\lambda$, and $\sigma > 0$, $\lambda_\sigma(x) = \lambda(\sigma x)$ has distribution function $d_{\nabla\lambda_\sigma}(t) = \sigma^{-2}d_{\nabla\lambda}(t/\sigma)$. Consequently

$$\|\nabla\lambda_\sigma\|_{L^{(2,\infty)}} = \sup_{t>0} t\sqrt{d_{\nabla\lambda_\sigma}(t)} = \sup_{u>0} u\sqrt{d_{\nabla\lambda}(u)} = \|\nabla\lambda\|_{L^{(2,\infty)}}.$$

Note that if Φ as in Definition 6.6.6 is conformal and $W^{2,2}$, there holds

$$\partial_1\Phi(x^1,0)=\mathrm{e}^{\lambda(x^1,0)}\boldsymbol{\varepsilon}_1, \quad \text{and} \quad \partial_2\Phi(x^1,0)=\mathrm{e}^{\lambda(x^1,0)}\boldsymbol{\varepsilon}_2.$$

Hence, provided that $\|\lambda\|_{L^\infty(B_1^+)}<+\infty$, the geometric nature of the reflection and conformality imply that $\widehat{\Phi}$ defines a conformal immersion of class $(W^{1,\infty}\cap W^{2,2})(B_1,\mathbb{R}^3)$, hence a Lipschitz $W^{2,2}$ immersion with

$$\begin{aligned}
|\nabla\widehat{\Phi}(x^1,x^2)|^2 &= |\nabla\Phi(x^1,x^2)|^2\chi_{\{x^2\geq 0\}}+|\nabla\Phi(x^1,-x^2)|^2\chi_{\{x^2<0\}},\\
|\nabla^2\widehat{\Phi}(x^1,x^2)|^2 &= |\nabla^2\Phi(x^1,x^2)|^2\chi_{\{x^2\geq 0\}}+|\nabla^2\Phi(x^1,-x^2)|^2\chi_{\{x^2<0\}},\\
|\Delta\widehat{\Phi}(x^1,x^2)|^2 &= |\Delta\Phi(x^1,x^2)|^2\chi_{\{x^2\geq 0\}}+|\Delta\Phi(x^1,-x^2)|^2\chi_{\{x^2<0\}},\\
\mathrm{e}^{\lambda_{\widehat{\Phi}}(x^1,x^2)} &= \mathrm{e}^{\lambda(x^1,x^2)}\chi_{\{x^2\geq 0\}}+\mathrm{e}^{\lambda(x^1,-x^2)}\chi_{\{x^2<0\}}.
\end{aligned}$$

The following is a boundary analogue of Lemma 6.6.4, where additionally we have flat geometric boundary data on I in the sense of Definition 6.6.6.

Lemma 6.6.7. *There exists an ε_0 with the following property. Let $\Phi: B_1^+\to\mathbb{R}^n$ be a conformal Lipschitz $W^{2,2}$ immersion and flat geometric boundary data on I such that $\|\lambda\|_{L^\infty(B_1^+)}<+\infty$. For every $\delta>0$ there there exists an $\varepsilon_0>0$ such that, if*

$$\int_{B_{4r}^+}|\nabla^2\Phi|_g^2\,d\sigma_g<\varepsilon_0,$$

for $0<r\leq 1/4$, then there exists a $\rho\in[r/2,r]$ and an immersion $\psi\in C^{1,\alpha}(\overline{B_\rho^+},\mathbb{R}^n)$ which satisfies

$$\begin{aligned}
\psi &= \Phi \quad \text{on } \partial B_\rho\cap B_1^+,\\
\nabla\psi &= \nabla\Phi \quad \text{on } \partial B_\rho\cap B_1^+,
\end{aligned}$$

has flat geometric boundary data on ρI and satisfies

$$\begin{aligned}
\int_{B_\rho^+}|\nabla^2\psi|_{g_\psi}^2\,d\sigma_{g_\psi} \leq{}& C\Big(1+C_0(\delta+o(\delta))\Big)\int_{B_r^+\setminus B_{r/2}^+}|\nabla^2\Phi|_g^2\,d\sigma_g\\
&+C_0(\delta+o(\delta))\,\mathrm{Area}(\Phi|_{B_\rho^+}),
\end{aligned}\tag{6.6.36}$$

and

$$|\,\mathrm{Area}(\Phi|_{B_\rho^+})-\mathrm{Area}(\psi)|\leq C_0(\delta+o(\delta))\,\mathrm{Area}(\Phi|_{B_\rho^+}),\tag{6.6.37}$$

and

$$\|D\,\mathrm{Area}(\Phi|_{B_\rho^+})-D\,\mathrm{Area}(\psi)\|\leq C_0(\delta+o(\delta))\,\mathrm{Area}(\Phi|_{B_r^+}),\tag{6.6.38}$$

where $C>0$ is independent of r and Φ, $C_0>0$ may depend on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1^+)}$ and $o(\delta)\to 0$ as $\delta\to 0$.

Parts of the proof of this lemma are similar to the proof of Lemma 6.6.4, so we will focus on the differences. The overall idea is first to reflect geometrically Φ as in Definition 6.6.6, then consider the biharmonic competitor as in Lemma 6.6.4 and finally to smoothly "correct" it so that it has flat geometric boundary data on ρI. Such "correction" will be essentially constructed by means of the 1st order Taylor polynomials of such biharmonic comparison at the points $(\rho,0)$ and $(-\rho,0)$ respectively.

Proof of Lemma 6.6.7. As in the case of Lemma 6.6.4, it sufficient to prove the thesis only for any $\delta > 0$ sufficiently small. We first treat the case $r = 1/4$ and argue through rescalings at the end. In what follows, we denote by C a positive constant which is independent of Φ, and with C_0 a positive constant depending only on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1^+)}$.

Step 1. We consider the geometric reflection of Φ, $\widehat{\Phi} : B_1 \to \mathbb{R}^n$, according to Definition 6.6.6. For ε_0 sufficiently small as in Lemma 6.6.1, we have that

$$
\begin{aligned}
\|\lambda_{\widehat{\Phi}} - (\lambda_{\widehat{\Phi}})_{B_1}\|_{L^\infty(B_{3/4})} &\leq C\left(\|\nabla\lambda_{\widehat{\Phi}}\|_{L^{(2,\infty)}(B_1)} + \varepsilon_0\right) \\
&\leq C\left(\|\nabla\lambda\|_{L^{(2,\infty)}(B_1^+)} + \varepsilon_0\right) \leq C_0,
\end{aligned} \tag{6.6.39}
$$

where $(\lambda_{\widehat{\Phi}})_{B_1}$ denotes the average of $\lambda_{\widehat{\Phi}}$ over B_1. Also, for every $\delta > 0$, if ε_0 is sufficiently small as in Lemma 6.6.3, then there exists a conformal affine immersion L whose conformal factor e^ν is so that the estimates

$$
\|\Phi - L\|_{W^{2,2}(B_{1/4}^+)} < \delta\|\nabla\Phi\|_{L^2(B_{1/2}^+)} \quad \Longleftrightarrow \quad \|\widehat{\Phi} - L\|_{W^{2,2}(B_{1/4})} < \delta\|\nabla\widehat{\Phi}\|_{L^2(B_{1/2})} \tag{6.6.40}
$$

$$
\|\lambda - \nu\|_{L^\infty(B_{1/4}^+(0))} < \delta \quad \Longleftrightarrow \quad \|\lambda_{\widehat{\Phi}} - \nu\|_{L^\infty(B_{1/4}(0))} < \delta \tag{6.6.41}
$$

are satisfied. By combining (6.6.39) -(6.6.41), we deduce

$$
\begin{aligned}
\|\lambda_{\widehat{\Phi}} - \nu\|_{L^\infty(B_{3/4})} &\leq \|\lambda_{\widehat{\Phi}} - (\lambda_{\widehat{\Phi}})_{B_1}\|_{L^\infty(B_{3/4})} + |(\lambda_{\widehat{\Phi}})_{B_1} - \nu| \\
&\leq \|\lambda_{\widehat{\Phi}} - (\lambda_{\widehat{\Phi}})_{B_1}\|_{L^\infty(B_{3/4})} + \|\lambda_{\widehat{\Phi}} - (\lambda_{\widehat{\Phi}})_{B_1}\|_{L^\infty(B_{1/2})} + \|\lambda_{\widehat{\Phi}} - \nu\|_{L^\infty(B_{1/2})} \\
&\leq C_0 + \delta,
\end{aligned}
$$

consequently we point-wise estimate from above and below

$$
C_0^{-1}(1 - \delta - o(\delta))\mathrm{e}^\nu \leq \mathrm{e}^{\lambda_{\widehat{\Phi}}(x)} \leq C_0(1 + \delta + o(\delta))\mathrm{e}^\nu \quad \text{for } x \in B_{3/4}.
$$

We then consider a good-slice choice $\rho \in [1/8, 1/4]$ so that $\widehat{\Phi}$ and $\widehat{\Phi} - L$ belong to $W^{2,2}(\partial B_\rho, \mathbb{R}^n)$ (equivalently, so that Φ and $\Phi - L$ belong to $W^{2,2}(\partial B_\rho \cap B_1^+, \mathbb{R}^n)$) with

$$
\begin{aligned}
\|\widehat{\Phi}\|_{W^{2,2}(\partial B_\rho)} &\leq C\|\widehat{\Phi}\|_{W^{2,2}(B_{1/4}\setminus B_{1/8})}, \\
\|\widehat{\Phi} - L\|_{W^{2,2}(\partial B_\rho)} &\leq C\|\widehat{\Phi} - L\|_{W^{2,2}(B_{1/4}\setminus B_{1/8})},
\end{aligned} \tag{6.6.42}
$$

hence we consider the solution for such choice of ρ to

$$
\begin{cases}
\Delta^2\psi_0 = 0 & \text{in } B_\rho, \\
\psi_0 = \widehat{\Phi} & \text{on } \partial B_\rho, \\
\nabla\psi_0 = \nabla\widehat{\Phi} & \text{on } \partial B_\rho,
\end{cases}
$$

which satisfies, as in lemma 6.6.4, the estimates

$$
\|\psi_0 - L\|_{W^{5/2,2}(B_\rho)} \leq C_0\mathrm{e}^\nu(\delta + o(\delta)), \tag{6.6.43}
$$

$$
\|\nabla^2\psi\|_{L^2(B_\rho)} \leq C\|\nabla^2\widehat{\Phi}\|_{L^2(B_{1/4}\setminus B_{1/8})} \leq C\|\nabla^2\Phi\|_{L^2(B_{1/4}\setminus B_{1/8})}, \tag{6.6.44}
$$

and consequently, by Sobolev embedding $W^{5/2,2} \hookrightarrow C^{1,\alpha}$ for every $0 < \alpha < 1/2$ we have,

$$
\|\nabla^2\psi_0\|_{C^{1,\alpha}(\overline{B_\rho})} \leq C_0\mathrm{e}^\nu(\delta + o(\delta)). \tag{6.6.45}
$$

The estimates (6.6.43) and (6.6.45) will be crucial for what follows. If $T^1_{\psi_0,(-\rho,0)}(x)$ and $T^1_{\psi_0,(\rho,0)}(x)$ denote the Taylor polynomial of ψ_0 at the points $(-\rho,0)$ $(\rho,0)$ respectively, we may write, for $x \in B_\rho(0)$,

$$T^1_{\psi_0,(-\rho,0)}(x) = T^1_{\Phi,(-\rho,0)}(x) := \Phi(-\rho,0) + \left\langle \nabla\Phi(-\rho,0), \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right\rangle,$$
$$T^1_{\psi_0,(\rho,0)}(x) = T^1_{\Phi,(\rho,0)}(x) := \Phi(\rho,0) + \left\langle \nabla\Phi(\rho,0), \begin{pmatrix} x^1-\rho \\ x^2 \end{pmatrix} \right\rangle,$$

where the expressions on the right-hand sides have a well-defined meaning because of our good-slice choice of ρ. Note moreover that $T^1_{\psi_0,(-\rho,0)}(x)$ and $T^1_{\psi_0,(\rho,0)}(x)$ are conformal, they define a parametrization of the plane $\operatorname{span}\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\}$ and are so that $T^1_{\psi_0,(-\rho,0)}(x^1,0), T^1_{\psi_0,(\rho,0)}(x^1,0) \in \operatorname{span}\{\boldsymbol{\varepsilon}_1\}$ (in particular they have flat geometric boundary data on ρI according to definition 6.6.6). For every $x \in B_\rho(0)$, by virtue of Taylor's theorem there exists $\xi \in ((-\rho,0),x) \subset B_\rho(0)$ so that

$$\psi_0(x) - T^1_{\psi_0,(-\rho,0)}(x) = \left\langle (\nabla\psi_0(\xi) - \nabla\psi_0(-\rho,0)), \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right\rangle,$$

consequently we deduce that for every $x \in B_\rho(0)$ there holds

$$|\psi_0(x) - T^1_{\psi_0,(-\rho,0)}(x)| \le [\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho(0)})} \left| \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right|^{1+\alpha}, \tag{6.6.46}$$
$$|\nabla\psi_0(x) - \nabla T^1_{\psi_0,(-\rho,0)}| \le [\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho(0)})} \left| \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right|^{\alpha}, \tag{6.6.47}$$

and similarly that

$$|\psi_0(x) - T^1_{\psi_0,(\rho,0)}(x)| \le [\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho(0)})} \left| \begin{pmatrix} x^1-\rho \\ x^2 \end{pmatrix} \right|^{1+\alpha}, \tag{6.6.48}$$
$$|\nabla\psi_0(x) - \nabla T^1_{\psi_0,(\rho,0)}| \le [\nabla\psi_0]_{C^{1,\alpha}(\overline{B_\rho(0)})} \left| \begin{pmatrix} x^1-\rho \\ x^2 \end{pmatrix} \right|^{\alpha}. \tag{6.6.49}$$

Note also that for every $x \in B_\rho(0)$ we may estimate

$$\begin{aligned} |T^1_{\Phi,(-\rho,0)}(x) - L(x)| &= \left| T^1_{\Phi,(-\rho,0)}(x) - L(-\rho,0) - \left\langle \nabla L(-\rho,0), \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right\rangle \right| \\ &\le |\Phi(-\rho,0) - L(-\rho,0)| + |\nabla\Phi(-\rho,0) - \nabla L(-\rho,0)| \left| \begin{pmatrix} x^1+\rho \\ x^2 \end{pmatrix} \right|, \\ |\nabla T^1_{\Phi,(-\rho,0)} - \nabla L| &= |\nabla\Phi(-\rho,0) - \nabla L(-\rho,0)|, \end{aligned}$$

and similarly

$$|T^1_{\Phi,(\rho,0)}(x) - L(x)| \le |\Phi(\rho,0) - L(\rho,0)| + |\nabla\Phi(\rho,0) - \nabla L(\rho,0)| \left| \begin{pmatrix} x^1-\rho \\ x^2 \end{pmatrix} \right|,$$
$$|\nabla T^1_{\Phi,(\rho,0)} - \nabla L(\rho,0)| = |\nabla\Phi(\rho,0) - \nabla L(\rho,0)|,$$

hence thanks to (6.6.40)–(6.6.42),

$$\|T^1_{\Phi,(-\rho,0)} - L\|_{C^1(\overline{B_\rho})} \le C_0 \mathrm{e}^\nu(\delta + o(\delta)), \tag{6.6.50}$$
$$\|T^1_{\Phi,(\rho,0)} - L\|_{C^1(\overline{B_\rho})} \le C_0 \mathrm{e}^\nu(\delta + o(\delta)). \tag{6.6.51}$$

Consequently, with (6.6.45) we deduce

$$\begin{aligned}\|\psi_0 - T^1_{\Phi,(-\rho,0)}\|_{C^{1,\alpha}(\overline{B_\rho})} &\le \|\psi_0 - L\|_{C^{1,\alpha}(\overline{B_\rho})} + \|T^1_{\Phi,(-\rho,0)} - L\|_{C^{1,\alpha}(\overline{B_\rho})}\\ &\le C_0 \mathrm{e}^\nu(\delta + o(\delta)),\end{aligned}$$

and similarly

$$\|\psi_0 - T^1_{\Phi,(\rho,0)}\|_{C^{1,\alpha}(\overline{B_\rho})} \le C_0 \mathrm{e}^\nu(\delta + o(\delta)).$$

Step 2. We now let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function so that

$$f(t) = \begin{cases} 0 & \text{for } t \le -\rho/2,\\ 1 & \text{for } t \ge \rho/2,\end{cases}$$

and we set, for $x \in B_\rho(0)$,

$$\phi(x) = T^1_{\Phi,(-\rho,0)}(x) + f(x^1)(T_{\Phi,(\rho,0)}(x) - T^1_{\Phi,(-\rho,0)}(x)).$$

Such function has range in the plane $\mathrm{span}\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ and is so that $\phi(x^1, 0) \in \mathrm{span}\{\boldsymbol{\varepsilon}_1\}$. Moreover, since

$$\begin{aligned}\partial_1\phi(x) &= \partial_1 T^1_{\Phi,(-\rho,0)}(x) + f'(x^1)(T^1_{\Phi,(\rho,0)}(x) - T^1_{\Phi,(-\rho,0)}(x))\\ &\quad + f(x^1)(\partial_1 T^1_{\Phi,(\rho,0)}(x) - \partial_1 T^1_{\Phi,(-\rho,0)}(x))\\ &= \mathrm{e}^{\lambda_\Phi(-\rho,0)}\boldsymbol{\varepsilon}_1 + f'(x^1)(T^1_{\Phi,(\rho,0)}(x) - T^1_{\Phi,(-\rho,0)}(x))\\ &\quad + f(x^1)(\mathrm{e}^{\lambda_\Phi(\rho,0)} - \mathrm{e}^{\lambda_\Phi(-\rho,0)})\boldsymbol{\varepsilon}_1,\\ &= \Big(\mathrm{e}^{\lambda_\Phi(-\rho,0)} + f(x^1)(\mathrm{e}^{\lambda_\Phi(\rho,0)} - \mathrm{e}^{\lambda_\Phi(-\rho,0)})\Big)\boldsymbol{\varepsilon}_1\\ &\quad + f'(x^1)(T^1_{\Phi,(\rho,0)}(x) - T^1_{\Phi,(-\rho,0)}(x))\\ \partial_2\phi(x) &= \Big(\mathrm{e}^{\lambda_\Phi(-\rho,0)} + f(x^1)(\mathrm{e}^{\lambda_\Phi(\rho,0)} - \mathrm{e}^{\lambda_\Phi(-\rho,0)})\Big)\boldsymbol{\varepsilon}_2,\end{aligned}$$

we have that, if $\delta > 0$ is chosen small enough, it defines an immersion. Indeed, on the one hand from (6.6.50)-(6.6.51) we can estimate

$$\begin{aligned}\|T^1_{\Phi,(-\rho,0)} - T^1_{\Phi,(\rho,0)}\|_{C^0(\overline{B_\rho})} &\le \|T^1_{\Phi,(-\rho,0)} - L\|_{C^0(\overline{B_\rho})} + \|T^1_{\Phi,(\rho,0)} - L\|_{C^0(\overline{B_\rho})}\\ &\le C_0 \mathrm{e}^\nu(\delta + o(\delta)),\end{aligned}$$

on the other hand, from (6.6.41) and the mean value theorem we may estimate

$$\begin{aligned}|\mathrm{e}^{\lambda_\Phi(-\rho,0)} - \mathrm{e}^{\lambda_\Phi(\rho,0)}| &\le |\lambda_\Phi(-\rho,0) - \lambda_\Phi(\rho,0)| \sup\{\mathrm{e}^\xi : \xi \in [\lambda_\Phi(\pm\rho,0), \lambda_\Phi(\mp\rho]\}\\ &\le 2\delta \sup\{\mathrm{e}^\xi : \xi \in [\nu - \delta, \nu + \delta]\}\\ &\le C\delta\mathrm{e}^\nu,\end{aligned}$$

hence we have the estimates, uniformly in $x \in B_\rho(0)$,

$$
\begin{aligned}
|\partial_1\phi(x)| &\geq e^{\lambda_\Phi(-\rho,0)} - C\delta e^\nu - C_0\|f'\|_{L^\infty((-\rho,\rho))}e^\nu(\delta + o(\delta))\\
&\geq e^\nu\Big(e^{-\delta} - C\delta - C_0\|f'\|_{L^\infty((-\rho,\rho))}(\delta + o(\delta))\Big),
\end{aligned}
$$

and similarly

$$
|\partial_2\phi(x)| \geq e^\nu\Big(e^{-\delta} - C\delta\Big),
$$

and

$$
|\langle \partial_1(x), \partial_2\phi(x)\rangle| \leq e^{2\nu}\Big(\|f'\|_{L^\infty(-\rho,\rho)}C_0(\delta + o(\delta))\Big)(e^\delta + C\delta).
$$

These inequalities imply the immersive nature of ϕ if $\delta > 0$ is chosen small enough. Note also that thanks to (6.6.50)–(6.6.51) there holds

$$
\begin{aligned}
|\nabla^2\phi(x)| &\leq |f''(x^1)||T^1_{\Phi,(\rho,0)}(x) - T^1_{\Phi,(-\rho,0)}(x)| + 2|f'(x^1)||\nabla T^1_{\Phi,(\rho,0)} - \nabla T^1_{\Phi,(-\rho,0)}|\\
&\leq C_0 e^\nu(\delta + o(\delta)).
\end{aligned}
$$

Since we may write

$$
\psi_0(x) - \phi(x) = (1 - f(x^1))(T^1_{\Phi,(-\rho,0)}(x) - \psi_0(x)) + f(x^1)(T^1_{\Phi,(\rho,0)}(x) - \psi_0(x)),
$$

we deduce thanks to (6.6.46) – (6.6.47) – (6.6.48) – (6.6.49) that

$$
|\psi_0(x) - \phi(x)| \leq [\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}\left((1 - f(x^1))\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f(x^1)\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{1+\alpha}\right), \tag{6.6.52}
$$

$$
\begin{aligned}
|\nabla\psi_0(x) - \nabla\phi(x)| \leq [\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}\Bigg(&|f'(x^1)|\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + (1 - f(x^1))\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{\alpha}\\
&+ |f'(x^1)|\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f(x^1)\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{\alpha}\Bigg),
\end{aligned} \tag{6.6.53}
$$

$$
|\nabla^2\psi_0(x) - \nabla^2\phi(x)| \leq |\nabla^2\psi_0(x)| + C_0 e^\nu(\delta + o(\delta)). \tag{6.6.54}
$$

Step 3. In this step we construct a function $\chi : B_\rho(0) \to \mathbb{R}$, which we will use in a moment, with the following properties: it is supported in $B_\rho(0) \setminus \{(-\rho,0),(\rho,0)\}$, it is smooth away from $(-\rho,0),(\rho,0)$ and is so that

$$
\chi \equiv 1 \text{ in a neighbourhood of } (-\rho,\rho)\times\{0\} \text{ in } B_\rho(0) \text{ which shrinks at } (\pm\rho,0),
$$

$$
|\nabla\chi(x)| \sim \left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{-1} \text{ as } x \to (-\rho,0), \quad |\nabla\chi(x)| \sim \left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{-1} \text{ as } x \to (\rho,0),
$$

$$
|\nabla^2\chi(x)| \sim \left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{-2} \text{ as } x \to (-\rho,0), \quad |\nabla^2\chi(x)| \sim \left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{-2} \text{ as } x \to (\rho,0).
$$

Such function may be constructed as follows. Let $k_0 : S^1 \to \mathbb{R}$ be a smooth function so that, for an angle β to be specified below, it satisfies

$$
k_0(\theta) = \begin{cases} 1 & \text{if } -\beta \leq \theta \leq \beta,\\ 0 & \text{if } \theta \in (-\pi,\pi]\setminus[-\beta,\beta].\end{cases}
$$

We extend it by homogeneity to $\mathbb{R}^2 \setminus \{0\}$, we choose $\beta = \arccos(2/\sqrt{5})$ and we rescale it of a factor $r = \rho\sqrt{5}/4$ so to match the construction indicated in figure 6.6.2:

$$\chi_0(x) = r\, k_0\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

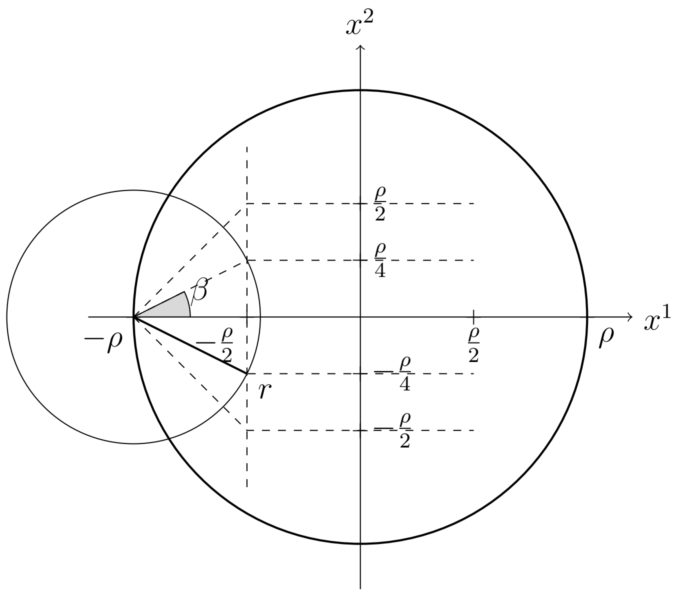

Figure 6.1: definition of α and r in the construction of χ

Such function will satisfy

$$|\nabla\chi_0(x)| \le \left|\chi_0'\left(\frac{x}{|x|}\right)\right|\left|\nabla\left(\frac{x}{|x|}\right)\right| \le \frac{C}{|x|},$$
$$|\nabla^2\chi_0(x)| \le \left|\chi_0''\left(\frac{x}{|x|}\right)\right|\left|\nabla\left(\frac{x}{|x|}\right)\right|^2 + \left|\chi_0'\left(\frac{x}{|x|}\right)\right|\left|\nabla^2\left(\frac{x}{|x|}\right)\right| \le \frac{C}{|x|^2}.$$

We then let $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be a triple of smooth functions so that $f_1 + f_2 + f_3 \equiv 1$ and

$$f_1(t) = \begin{cases} 1 & \text{if } t \le -3\rho/4, \\ 0 & \text{if } t \ge -\rho/2, \end{cases} \quad f_2(t) = \begin{cases} 1 & \text{if } -\rho/2 \le t \le \rho/2 \\ 0 & \text{if } t \ge 3\rho/4, \end{cases} \quad f_3(t) = \begin{cases} 0 & \text{if } t \le 3\rho/4, \\ 1 & \text{if } t \ge \rho, \end{cases}$$

and let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function so that

$$\eta(x^1, x^2) = \begin{cases} 1 & \text{if } x^2 \in [-\rho/4, \rho/4], \\ 0 & \text{if } x^2 \in \mathbb{R}^2 \setminus [-\rho/2, \rho/2]. \end{cases}$$

The required function χ is then given by

$$\chi(x) = f_1(x^1)\chi_0(x + (\rho, 0)) + f_2(x^1)\eta(x) + f_3(x^1)\chi_0(x - (\rho, 0)), \quad x \in B_\rho(0).$$

Step 4. We claim that the function ψ in the statement of the lemma is the given by (the restriction to $B_1^+(0)$ of) the interpolation between ψ and ψ_0 through χ, namely

$$\psi(x) = \chi(x)\phi(x) + (1 - \chi(x))\psi_0(x), \quad x \in B_\rho(0).$$

Indeed, by construction we have that ψ has flat boundary data on ρI according to Definition 6.6.6 and that

$$\psi = \hat{\Phi}, \quad \nabla\psi = \nabla\hat{\Phi} \quad \text{on } \partial B_\rho(0).$$

As the proof of Lemma 6.6.4 shows, to prove all the estimates in the statement, we have to verify that

$$\|\nabla\psi - \nabla L\|_{L^\infty(B_\rho)} \le C_0 \mathrm{e}^\nu(\delta + o(\delta)), \tag{6.6.55}$$

$$\|\nabla^2\psi\|_{L^2(B_\rho)} \le C\|\nabla^2\Phi\|_{L^2(B_{1/4}\setminus B_{1/8})} + C_0(\delta + o(\delta))\|\nabla\Phi\|_{L^2(B_\rho)}. \tag{6.6.56}$$

We may write

$$\psi(x) - L(x) = \psi(x) - \psi_0(x) + \psi_0(x) - L(x) = \chi(x)(\phi(x) - \psi_0(x)) + \psi_0(x) - L(x).$$

To see that (6.6.55) holds, note first of all that from the definition of χ we have

$$\begin{aligned}\nabla\chi(x) = \quad & f_1'(x^1)\chi_0(x + (\rho,0))\boldsymbol{\varepsilon}_1 + f_1(x^1)\nabla\chi_0(x + (\rho,0)) \\ & + f_2'(x^1)\eta(x))\boldsymbol{\varepsilon}_1 + f_2(x^1)\nabla\eta(x) \\ & + f_3'(x^1)\chi_0(x - (\rho,0))\boldsymbol{\varepsilon}_1 + f_3(x^1)\nabla\chi_0(x - (\rho,0)),\end{aligned}$$

while from the definition of the functions f, f_1 and f_3 we have for every $x^1 \in [-\rho,\rho]$ that

$$\begin{aligned} f_1(x^1)f(x_1) &\equiv 0, & f_1'(x^1)f(x_1) &\equiv 0, \\ f_1(x^1)(1 - f(x_1)) &\equiv f_1(x^1), & f_1'(x^1)(1 - f(x_1)) &\equiv f_1'(x^1), \\ f_3(x^1)f(x_1) &\equiv f_3(x^1), & f_3'(x^1)f(x_1) &\equiv f_3'(x^1), \\ f_3(x^1)(1 - f(x_1)) &\equiv 0, & f_3'(x^1)(1 - f(x_1)) &\equiv 0,\end{aligned}$$

consequently from (6.6.52) we deduce the estimate

$$\begin{aligned} &|\nabla\chi(x)(\phi(x) - \psi_0(x))| \\ &\quad \le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}\Bigg(|f_1'(x^1)|\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f_1(x^1)\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{\alpha} \\ &\qquad + |f_2'(x^1)|\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f_2(x^1)\left|\begin{pmatrix} x^1+\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} \\ &\qquad + |f_2'(x^1)|\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f_2(x^1)\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} \\ &\qquad + |f_3'(x^1)|\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{1+\alpha} + f_3(x^1)\left|\begin{pmatrix} x^1-\rho \\ x^2\end{pmatrix}\right|^{\alpha}\Bigg),\end{aligned}$$

which then implies

$$\|\nabla\chi(\phi - \psi_0)\|_{L^\infty(B_\rho)} \le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}.$$

From estimate (6.6.53) we immediately deduce that

$$\|\chi(\nabla\phi - \nabla\psi_0)\|_{L^\infty(\overline{B_\rho})} \le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})},$$

consequently using (6.6.45) we can estimate

$$\begin{aligned}\|\nabla\psi - \nabla L\|_{L^\infty(B_\rho)} &= \|\nabla\chi(\phi - \psi_0)\|_{L^\infty(B_\rho)} + \|\chi(\nabla\phi - \nabla\psi_0)\|_{L^\infty(B_\rho)} + \|\psi_0 - L\|_{L^\infty(B_\rho)} \\ &\le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})} + C_0 \mathrm{e}^\nu(\delta + o(\delta)) \\ &\le C_0\mathrm{e}^\nu(\delta + o(\delta)),\end{aligned}$$

which proves (6.6.55). To establish (6.6.56), the argument is similar: from the properties of χ one deduces the estimates

$$|\nabla^2\chi(x)(\phi(x) - \psi_0(x))| \le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}\left(f_1(x^1)\left|\begin{pmatrix}x^1+\rho\\ x^2\end{pmatrix}\right|^{-1+\alpha} + f_3(x^1)\left|\begin{pmatrix}x^1-\rho\\ x^2\end{pmatrix}\right|^{-1+\alpha}\right),$$

$$|\nabla\chi(x)(\nabla\phi(x) - \nabla\psi_0(x))| \le C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})}\left(f_1(x^1)\left|\begin{pmatrix}x^1+\rho\\ x^2\end{pmatrix}\right|^{-1+\alpha} + f_3(x^1)\left|\begin{pmatrix}x^1-\rho\\ x^2\end{pmatrix}\right|^{-1+\alpha}\right),$$

and, since $\alpha > 0$, the right-hand sides of these last two inequalities are in $L^2(B_\rho)$. Consequently also thanks to (6.6.45) and (6.6.54) we estimate

$$\begin{aligned}\|\nabla^2\psi\|_{L^2(B_\rho)} &\le \|\nabla^2\chi(\phi - \psi_0)\|_{L^2(B_\rho)} + 2\|\nabla\chi(\nabla\phi - \nabla\psi_0)\|_{L^2(B_\rho)} + \|\chi(\nabla^2\phi - \nabla^2\psi_0)\|_{L^2(B_\rho)} \\ &\le \|\nabla^2\psi_0\|_{L^2(B_\rho)} + C[\nabla\psi_0]_{C^{0,\alpha}(\overline{B_\rho})} + C_0\mathrm{e}^\nu(\delta + o(\delta)) \\ &\le \|\nabla^2\psi_0\|_{L^2(B_\rho)} + C_0\mathrm{e}^\nu(\delta + o(\delta)),\end{aligned}$$

which then implies, thanks to (6.6.44), the estimate (6.6.56). The rest of the proof now follows the same lines as in the proof of Lemma 6.6.4. □

6.6.3 Morrey–Type Estimates and Conclusion For a conformal map $\Phi : B_1 \to \mathbb{R}^n$ which is a minimizer for the Willmore energy in the class $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ as that in theorem 6.1.2, there are two possibilities.

The first one is that Φ is a minimal surface, that is

$$D\,\mathrm{Area}(\Phi)w = 0 \quad \text{for all } w \in C_c^\infty(B_1, \mathbb{R}^n),$$

and this implies that Φ satisfies

$$\Delta\Phi^i = 0 \quad \text{in } \mathcal{D}'(B_1) \quad \text{for } i = 1, \dots m.$$

The regularity up to the boundary for the first case is classic, and is essentially the one for the Plateau problem, for which we refer to [DHT10, DHS10].

We now study the second possibility.

Lemma 6.6.8 (Interior Morrey-type Estimates). *Let $\Phi : B_1 \to \mathbb{R}^n$ be a conformal Lipschitz $W^{2,2}$ immersion which is an interior minimizer for the Willmore energy in $\mathscr{F}(B_1, \mathbb{R}^n)$ at a fixed area value, that is*

$$\mathcal{W}_2(\Phi) \le \mathcal{W}_2(\Psi)$$

for every map $\Psi \in \mathscr{F}(B_1, \mathbb{R}^n)$ coinciding with Φ outside some compact subset of B_1 and so that $\mathrm{Area}(\Psi) = \mathrm{Area}(\Phi)$. *Assume that Φ is not a minimal surface and set*

$$\zeta = \|D\,\mathrm{Area}(\Phi)\| > 0. \tag{6.6.57}$$

Then, there exists some $r_0 > 0$ such that

$$\sup\left\{ r^{-\gamma} \int_{B_r(p)} (|\nabla^2\Phi|_g^2 + 1)d\sigma_g : p \in \overline{B_{1/2}},\, 0 < r < r_0 \right\} \leq C_0,$$

for constants $\gamma > 0$ and $C_0 > 0$ depending only on ζ and $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$.

Proof of Lemma 6.6.8. In what follows, we denote by C a positive constant (possibly varying line to line) which is independent of Φ, and with C_0 a positive constant which depend on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$ and ζ.

Step 1: constructing a suitable competitor. Since Φ is not a minimal surface, we may choose some non-zero $w \in C_c^\infty(B_1, \mathbb{R}^n)$ such that $D\mathrm{Area}(\Phi)w > \zeta/2$. We let $\delta, \varepsilon_0 > 0$ to be as in Lemma 6.6.4 and whose size is specified in what follows. Let $r_0 > 0$ be sufficiently small so that

$$\sup\left\{ \int_{B_{4r_0}(p)} |\nabla^2\Phi|_g^2\, d\sigma_g : p \in \overline{B_{1/2}(0)} \right\} < \varepsilon_0.$$

We now fix arbitrary $p \in \overline{B_{1/2}(0)}$ and $0 < r < r_0$ and (similarly as done as in the proof of Lemma 6.6.4) for ε_0 sufficently small as in Lemma 6.6.1, we have that

$$\|\lambda - (\lambda)_{B_{4r}(p)}\|_{L^\infty(B_{3r}(p))} \leq C_0, \tag{6.6.58}$$

where $(\lambda)_{B_{4r}(p)}$ denotes the average of λ over $B_{4r}(p)$. If ε_0 is sufficiently small as in Lemma 6.6.3, then there exists a conformal affine immersion L whose conformal factor we denote by e^ν, such that the estimates

$$\begin{aligned} &\|\Phi - L\|_{W^{2,2}(B_r(p))} < \delta\|\nabla\Phi\|_{L^2(B_{2r}(p))}, \\ &\quad\|\lambda - \nu\|_{L^\infty(B_r(p))} < \delta, \end{aligned} \tag{6.6.59}$$

are satisfied.

By combining (6.6.58)–(6.6.59), we deduce

$$\begin{aligned} \|\lambda - \nu\|_{L^\infty(B_{3r}(p))} &\leq \|\lambda - (\lambda)_{B_{4r}(p)}\|_{L^\infty(B_{3r}(p))} + |(\lambda)_{B_{4r}(p)} - \nu| \\ &\leq \|\lambda - (\lambda)_{B_{4r}(p)}\|_{L^\infty(B_{3r}(p))} + \|\lambda - (\lambda)_{B_{4r}(p)}\|_{L^\infty(B_r(p))} + \|\lambda - \nu\|_{L^\infty(B_r(p))} \\ &\leq C_0 + \delta, \end{aligned}$$

consequently we pointwise estimate from above and below

$$C_0^{-1}(1 - \delta - o(\delta))\mathrm{e}^\nu \leq \mathrm{e}^{\lambda(x)} \leq C_0(1 + \delta + o(\delta))\mathrm{e}^\nu \quad \text{for } x \in B_{3r}(p). \tag{6.6.60}$$

Hence we consider

$$\Psi = \begin{cases} \psi & \text{in } B_\rho(p), \\ \Phi & \text{in } B_1 \setminus B_\rho(p). \end{cases}$$

where $\rho \in [r/2, r]$ and ψ are given as in Lemma 6.6.4.

Thanks to (6.6.23), we have that, if δ is chosen sufficiently small, there holds

$$|D\operatorname{Area}(\Psi)w - D\operatorname{Area}(\Phi)w| = |D\operatorname{Area}(\psi)w - D\operatorname{Area}((\Phi|_{B_\rho(p)})w| < \frac{\zeta}{4},$$

and consequently,

$$D\operatorname{Area}(\Psi)w > \frac{\zeta}{4} > 0.$$

We consider the function given by

$$a(t) = \operatorname{Area}(\Psi + tw), \quad t \in \mathbb{R}.$$

Let $\varepsilon > 0$ be sufficiently small so that, for every $t \in [-\varepsilon, \varepsilon]$, $\Psi + tw$ defines a Lipschitz $W^{2,2}$ immersion. Then a is continuously differentiable in $[-\varepsilon, \varepsilon]$ with

$$a'(t) = D\operatorname{Area}(\Psi + tw)w = -2\int_{B_1} \langle H_{\Psi+tw}, w\rangle, \, d\sigma g_{\Psi+tw}, \quad \text{for } t \in [-\varepsilon, \varepsilon],$$

and in particular

$$a'(0) > \frac{\zeta}{4} > 0.$$

By the inverse function theorem, we deduce that, after possibly shrinking ε, a defines a C^1-diffeomorphism of $[-\varepsilon, \varepsilon]$ onto $[\operatorname{Area}(\Psi) - \varepsilon, \operatorname{Area}(\Psi) + \varepsilon]$ and

$$\frac{\zeta}{8} \le a'(t) \le \frac{\zeta}{2} \quad \text{for } t \in [-\varepsilon, \varepsilon].$$

Thanks to (6.6.23), we have that, if δ is chosen sufficiently small, there holds

$$|\operatorname{Area}(\Psi) - \operatorname{Area}(\Phi)| = |\operatorname{Area}(\psi) - \operatorname{Area}((\Phi|_{B_\rho(p)})| \le \frac{\varepsilon}{2},$$

so we may find a unique $\bar{t} \in [-\varepsilon, \varepsilon]$ so that

$$\operatorname{Area}(\Psi + \bar{t}w) = \operatorname{Area}(\Phi).$$

We then set

$$\overline{\Psi} = \Psi(x) + \bar{t}w(x) \quad \text{for } x \text{ in } B_1.$$

Then $\overline{\Psi}$ is a Lipschitz $W^{2,2}$ immersion, and by construction there holds $\operatorname{Area}(\Psi) = \operatorname{Area}(\Phi)$.

Step 2: comparison of Φ *with* $\overline{\Psi}$. By the minimality of Φ we then have

$$\frac{1}{4}\int_{B_1} |A|_g^2 \, d\sigma_g \le \frac{1}{4}\int_{B_1} |A_{\overline{\Psi}}|^2_{g_{\overline{\Psi}}} \, d\sigma_{g_{\overline{\Psi}}}, \tag{6.6.61}$$

Following a computation analogous to ([MR13, Lemma A.5]), the term on the right-hand-side can be expanded to

$$\frac{1}{4}\int_{B_1} |A_{\overline{\Psi}}|^2_{g_{\overline{\Psi}}} \, d\sigma_{g_{\overline{\Psi}}} = \mathcal{W}_2(\Psi + \bar{t}w) = \mathcal{W}_2(\Psi) + \bar{t}D\mathcal{W}_2(\Psi)w + R_w^\Psi(\bar{t}),$$

where $R_w^\Psi(\bar{t})$ is a remainder term satisfying

$$|R_w^\Psi(\bar{t})| \leq C_{\Psi,w}\bar{t}^2,$$

and, since Φ is a minimiser for the Willmore energy with prescribed area, we may write (we use the divergence form of the Willmore equation, valid for weak immersions), there holds for some $c \in \mathbb{R}$,

$$D\mathcal{W}_2(\Psi)w = \int_{B_\rho(p)} \left\langle \nabla H_\Psi + \langle A_\Psi^\circ, H_\Psi\rangle^{\sharp_{g_\Psi}} + \langle A_\Psi, H_\Psi\rangle^{\sharp_{g_\Psi}} + cH_\Psi, \nabla w\right\rangle_{g_\Psi} d\sigma_{g_\Psi},$$

so that we can simply estimate: $|D\mathcal{W}_2(\Psi)w| \leq C_{\Phi,w}$.

By the mean value theorem and the estimates (6.6.22) and (6.6.60) it holds

$$\begin{aligned}
|\bar{t}| &= |a^{-1}(a(\bar{t})) - a^{-1}(a(0))| \\
&\leq \sup_{\xi\in J} |(a^{-1})'(\xi)||a(\bar{t}) - a(0)| \\
&\leq \frac{8}{\zeta}|\operatorname{Area}(\Psi + tw) - \operatorname{Area}(\Psi)| \\
&\leq C|\operatorname{Area}(\Phi) - \operatorname{Area}(\Psi)| \\
&\leq C(\delta + o(\delta))\operatorname{Area}(\Phi|_{B_\rho(0)}) \\
&\leq C_0(\delta + o(\delta))\mathrm{e}^{2\nu},
\end{aligned}$$

It follows that

$$\int_{B_1} |A_{\overline{\Psi}}|^2_{g_{\overline{\Psi}}}\, d\sigma_{\overline{\Psi}} \leq \int_{B_1} |A_\Psi|^2_{g_\Psi}\, d\sigma_{g_\Psi} + C_0\mathrm{e}^{2\nu}(\delta + o(\delta)).$$

Thanks to (6.6.21), the above estimate and (6.6.61) then imply

$$\int_{B_\rho(p)} |A|^2\, d\sigma_g \leq C_0 \int_{B_r(p)\setminus B_{r/2}(p)} |\nabla^2\Phi|^2_g\, d\sigma_g + C_0\mathrm{e}^{2\nu}(\delta + o(\delta)). \tag{6.6.62}$$

Step 3: monotonicity of Area. Thanks to (6.6.60), for every $0 < s < r$ we can estimate

$$\begin{aligned}
\operatorname{Area}(\Phi|_{B_s(p)}) &= \int_{B_s(p)} |\nabla\Phi|^2\, dx \\
&= 2\int_{B_s(p)} \mathrm{e}^{2\lambda}\, dx \leq C_0\mathrm{e}^{2\nu}s^2 \leq C_0\mathrm{e}^{4\nu}\frac{s^2}{r^2}\int_{B_r(p)} |\nabla\Phi|^2 dx.
\end{aligned} \tag{6.6.63}$$

Step 4: obtaining the Morrey decrease. For any $0 < \eta < 1/2$, thanks to the identity $|\nabla^2\Phi|^2 = 4\mathrm{e}^{-2\lambda}|\nabla\lambda|^2 + |A|^2$ and Lemma 6.6.1, there exists $C > 0$ independent of p and r so that

$$\begin{aligned}
\int_{B_{\eta r}(p)} (|\nabla^2\Phi|^2_g\, d\sigma_g + 1)\, d\sigma_g &\leq \left(\frac{\eta^2}{2} + C\varepsilon_0\right)\int_{B_r(p)} |\nabla^2\Phi|^2_g\, d\sigma_g \\
&\quad + \int_{B_{\eta r}(p)} |A|^2\, d\sigma_g + \operatorname{Area}(\Phi|_{B_{\eta r}(p)}),
\end{aligned}$$

and so by estimate (6.6.62) we deduce that there holds

$$\begin{aligned}\int_{B_{\eta r}(p)}(|\nabla^2\Phi|_g^2+1)d\sigma_g &\leq \left(\frac{\eta^2}{2}+C\varepsilon_0\right)\int_{B_r(p)}|\nabla^2\Phi|_g^2\,d\sigma_g\\ &\quad + C_0\int_{B_{2\eta r}(p)\setminus B_{\eta r}(p)}|\nabla^2\Phi|_g^2\,d\sigma_g\\ &\quad + C_0\left((\delta+o(\delta))+\eta^2 r^2\right)\mathrm{e}^{2\nu}.\end{aligned}$$

By using (6.6.60) and (6.6.63) and adding $C_0\int_{B_{\eta r}(p)}|\nabla^2\Phi|_g^2\,d\sigma_g$ to both hand-sides and dividing by $1+C_0$ yields

$$\begin{aligned}\int_{B_{\eta r}(p)}(|\nabla^2\Phi|_g^2+1)d\sigma_g &\leq \left(\frac{\eta^2/2+C\varepsilon_0+C_0}{C_0+1}\right)\int_{B_r(p)}|\nabla^2\Phi|_g^2\,d\sigma_g\\ &\quad + \left(\frac{C_0(\delta+o(\delta)+\eta^2)}{C_0+1}\right)\mathrm{Area}(\Phi|_{B_r(p)}).\end{aligned}$$

If η and δ are chosen sufficiently small so that

$$\beta := \max\left\{\frac{\eta^2/2+C\varepsilon_0+C_0}{C_0+1},\frac{C_0(\delta+o(\delta)+\eta^2)}{C_0+1}\right\}<1,$$

we deduce that

$$\int_{B_{\eta r}(p)}(|\nabla^2\Phi|_g^2+1)\,d\sigma_g \leq \beta\int_{B_r(p)}(|\nabla^2\Phi|_g^2+1)\,d\sigma_g$$

for any $p\in\overline{B_{1/2}(0)}$ and any $0<r<r_0$ where $0<\beta<1$ does not depend on r or p. This inequality can be now iterated and interpolated to yield

$$\int_{B_r(p)}(|\nabla^2\Phi|_g^2+1)\,d\sigma_g \leq r^{\log_{1/\eta}(1/\beta)}\frac{1}{r_0^{\log_{1/\eta}(1/\beta)}}\int_{B_{r_0}(p)}(|\nabla^2\Phi|_g^2+1)\,d\sigma_g$$

for any $r<\eta r_0$, where β and η depend only on ζ and $\|\nabla\lambda_\Phi\|_{L^{(2,\infty)}(B_1)}$. After relabelling r_0, and setting $\gamma:=\log_{1/\eta}(1/\beta)$ we can concludes of the proof of the lemma. □

For boundary points, we have the following. Recall that we write $\partial B_1^+ = I+S$, where I is the base diameter and S is the upper semi-circle, and when we say that a Lipschitz immersion "has geometric boundary data of class $C^{1,1}$" if its boundary curve and boundary Gauss map are $C^{1,1}$ up to re-parametrization and not in a point-wise sense, see definition 6.1.5–(ii).

Lemma 6.6.9 (Boundary Morrey–type Estimates). *Let $\Phi : {B_1}^+\to\mathbb{R}^n$ be a conformal Lipschitz $W2,2$ immersion so that*

$$\mathcal{W}_2(\Phi)\leq\mathcal{W}_2(\Psi)$$

for every map $\Psi\in\mathscr{F}(B_1^+,\mathbb{R}^n)$ that coincides with Φ outside some subset K of $\overline{B_1^+}$ with $\mathrm{dist}(K,S)>0$ *and having the same geometric boundary data of Ψ in I, that is*

$$\Psi(I)=\Phi(I)\quad \textit{and}\quad N_\Psi(I)=N(I),$$

and the same area, that is $\mathrm{Area}(\Phi) = \mathrm{Area}(\Phi)$*. Assume that the boundary geometric data of* Φ *on* I *are of class* $C^{1,1}$ *and that* Φ *is not a minimal surface. Let* $\zeta > 0$ *be as in* (6.6.57)*. Then, there exist some* $0 < \overline{r} < r_0 < 1$ *so that*

$$\sup\left\{ r^{-\gamma} \int_{B_r^+(p)} |\nabla^2 \Phi|_g^2 \, d\sigma_g : p \in r_0 I,\, 0 < r < \overline{r} \right\} < +\infty,$$

for some constant $\gamma > 0$*.*

Remark 6.6.10 Two elementary facts that will be used in the proof of lemma 6.6.9 are the following:

(i) If $e = (e_1, e_2)$ denotes the coordinate frame of the map Φ, then we have

$$e(x) = (\mathbf{t}, \star(\mathbf{t} \wedge N_0)(\sigma(x^1)), \quad x^1 \simeq (x^1, 0) \in I,$$

where $\mathbf{t}$ denotes the tangent vector of the boundary curve and σ_Φ is some homeomorphism with $\sigma'(x^1) = \mathrm{e}^{\lambda(x^1,0)}$. In particular, since the boundary data are assumed of class $C^{1,1}$, we see that for $i = 1, 2$ we can estimate, for every $1 < p < \infty$,

$$\|\partial_\tau e_i\|_{L^p(I)} \le C_0 \|\mathrm{e}^{\lambda(\cdot,0)}\|_{L^p(I)},$$

where C_0 depends only on the geometric boundary data. Furthermore, for every $1 < p < \infty$, if we set $\Phi_r(x) = \Phi(rx)$, $x \in B_1^+$ one can compute that $\|\mathrm{e}^{\lambda_{\Phi_r}(\cdot,0)}\|_{L^p(I)} = r^{\frac{p-1}{p}} \|\mathrm{e}^{\lambda(\cdot,0)}\|_{L^p(rI)}$ and thus deduce that the L^p-norm of $\mathrm{e}^{\lambda(\cdot,0)}$ is *decreasing* with respect to rescaling in the domain, for $0 < r < 1$.

(ii) For a generic immersion of an open domain $\Omega \subset \mathbb{R}^n$, $X : \Omega \to \mathbb{R}^n$ and a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, denoting for brevity denoting $\vartheta = O(\|\nabla f - \mathbb{1}_{m\times m}\|_{L^\infty(\mathbb{R}^n)})$ and $\eta = O(\|\nabla^2 f\|_{L^\infty(\mathbb{R}^n)})$, we can deduce the point-wise estimates

$$\begin{aligned}
(1-\vartheta)\, g_X \le g_{f\circ X} &\le (1+\vartheta) g_X,\\
(1-\vartheta) d\sigma_{g_X} \le d\sigma_{g_{f\circ X}} &\le (1+\vartheta) d\sigma_{g_X},\\
|\nabla^2 (f\circ X)|^2_{g_{f\circ X}} &\le (1+\vartheta)(\eta + \vartheta |\nabla^2 X|^2_{g_X}) \le C_0 (1 + |\nabla^2 X|^2_{g_X}),
\end{aligned}$$

where C_0 is a constant depending only on f.

Proof of Lemma 6.6.9. In what follows, we denote by C a positive constant (possibly varying line to line) which is independent of Φ, and with C_0 a positive constant dependeding only on $\|\nabla\lambda\|_{L^{(2,\infty)}(B_1)}$, and on the geometric boundary data at I of Φ.

Step 1: preliminaries and reductions. We fix $p > 1$ and a suitably small ε_0 that will be specified below. Since the boundary data of Φ along I are of class $C^{1,1}$, we may find some $0 < r_0 < 1$ and a $C^{1,1}$-homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ (that is, f and its inverse belong to $C^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$), so that $(f \circ \Phi)|_{B^+_{r_0}}$ has flat boundary data along $r_0 I$ as in the sense of Definition 6.6.6. Up to further reducing r_0, we also assume that

$$\int_{B^+_{r_0}} (|\nabla^2\Phi|^2_{g_\Phi} + 1)\, d\sigma_g + \|\mathrm{e}^\lambda\|_{L^p(r_0 I)} < \varepsilon_0. \tag{6.6.64}$$

Note that, since the Lagrangian $\Phi \mapsto \int (1 + |\nabla^2\Phi^2|^2_g)\, d\sigma_g$ is invariant with respect to reparametrizations and the conformal factor is decreasing with respect to rescalings (see Remark

6.6.10–(i)), (6.6.64) implies in particular that, having set for brevity $\Phi_{r_0}(x) = \Phi(r_0 x)$, there holds

$$\sup\left\{ \int_{B_{4r}^+(x)} (|\nabla^2 \Phi_{r_0}|^2_{g_{\Phi_{r_0}}} + 1)\, d\sigma_{g_{\Phi_{r_0}}} + \|e^{\lambda_{\Phi_{r_0}}}\|_{L^p(x+rI)} : x \in \frac{1}{2}I,\ 0 < r \leq \frac{1}{16} \right\} < \varepsilon_0.$$

We will work from now on, omitting the subscript, with $\Phi_{r_0}(x) = \Phi(r_0 x)$ in place of Φ. Also, since Φ is not a minimal surface, there exists some $w \in C_c^\infty(B_1^+, \mathbb{R}^n)$ so that

$$D\,\mathrm{Area}(\Phi) w \geq \zeta/2 > 0. \tag{6.6.65}$$

Finally, thanks to Lemma 6.1.6 there exist some bi-Lipschitz homeomorphism $\phi : B_1^+ \to B_1^+$ so that $f \circ \Phi \circ \phi$ is conformal moreover up to further composing with a conformal self-map of B_1^+ we may suppose that $\phi(\pm 1) = \pm 1$ and $\phi(0) = 0$, hence that the geometric boundary data on I are sent (globally) onto themselves: $f(\Phi(I)) = f(\Phi(\phi(I)))$.

Step 2. For simplicity of notation we will prove the Morrey-type decay at $x = 0$; the one for other points in $(1/2)I$ is analogous. Since ϕ is bi-Lipschitz, we may find a sufficiently big $P \in \mathbb{N}$ and a sufficiently big $M = M(P)$ so that, for every $0 < r < 1$, we have

$$B^+_{r/2^M} \subset \phi(B_{r/2^P}) \subset B_r^+. \tag{6.6.66}$$

Let $0 < r \leq 1/16$ be fixed. For a sufficiently small ε_0, thanks to Lemma 6.6.7 we may find a $\rho \in [r/2^{P+1}, r/2^P]$ and an immersion $\psi \in C^{1,\alpha}(\overline{B_\rho^+}, \mathbb{R}^n)$ which satisfies

$$\begin{aligned} \psi &= f \circ \Phi \circ \phi && \text{on } \partial B_\rho \cap B_1^+, \\ \nabla \psi &= \nabla(f \circ \Phi \circ \phi) && \text{on } \partial B_\rho \cap B_1^+, \end{aligned}$$

has flat boundary data on ρI and satisfies the estimates (6.6.36), (6.6.37) and (6.6.38) with $r/2^P$ in place of r and $f \circ \Phi \circ \phi$ in place of Φ, and in particular

$$\begin{aligned} \int_{B^+_{r/2^{P+1}}} |\nabla^2 \psi|^2_{g_\psi}\, d\sigma_{g_\psi} \leq C_0 &\int_{B^+_{r/2^P} \setminus B_{r/2^{P+1}}} |\nabla^2 (f \circ \Phi \circ \phi)|^2 d\sigma_{g_{f\circ\Phi\circ\phi}} \\ &+ C_0 \int_{B^+_{r/2^P}} d\sigma_{g_{f\circ\Phi\circ\phi}}. \end{aligned} \tag{6.6.67}$$

Hence we set

$$\Psi = \begin{cases} f^{-1} \circ \psi \circ \phi^{-1} & \text{in } \phi(B_\rho^+), \\ \Phi & \text{in } B_1^+ \setminus \phi(B_\rho^+) \end{cases}$$

From (6.6.37) and (6.6.38) we deduce

$$\begin{aligned} |\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+)}) - \mathrm{Area}(f^{-1} \circ \psi \circ \phi^{-1})| &\leq C_0(\delta + o(\delta))\, \mathrm{Area}(\Phi|_{\phi(B_\rho^+)}), \\ \|D\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+)}) - D\,\mathrm{Area}(f^{-1} \circ \psi \circ \phi^{-1})\| &\leq C_0(\delta + o(\delta))\, \mathrm{Area}(\Phi|_{\phi(B^+_{r/2^N})}), \end{aligned}$$

where $C_0 > 0$ depends on $\|\nabla \lambda\|_{L^{2\infty}(B_1^+)}$ and on f. Thanks to (6.6.65), we have that, if δ (and accordingly ε_0) is chosen sufficiently small, there holds

$$|D\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+)}) w - D\,\mathrm{Area}(f^{-1} \circ \psi \circ \phi^{-1}) w| < \frac{\zeta}{4},$$

and consequently, $D\,\mathrm{Area}(\Psi)w > \zeta/4$. We then consider the function given by

$$a(t) = \mathrm{Area}(\Psi + tw), \quad t \in \mathbb{R}.$$

Let $\varepsilon > 0$ be sufficiently small so that, for every $t \in [-\varepsilon, \varepsilon]$, $\Psi + tw$ defines a Lipschitz $W^{2,2}$ immersion. Then a is continuously differentiable in $[-\varepsilon, \varepsilon]$ with

$$a'(t) = D\,\mathrm{Area}(\Psi + tw)w = -2\int_{B_1^+} \langle H_{\Psi+tw}, w\rangle,\, d\sigma_{g_{\Psi+tw}}, \quad \text{for } t \in [-\varepsilon, \varepsilon],$$

and in particular $a'(0) > \zeta/4 > 0$. By the inverse function theorem, we deduce that, after possibly shrinking ε, a defines a C^1-diffeomorphism of $[-\varepsilon, \varepsilon]$ onto $[\mathrm{Area}(\Psi) - \varepsilon, \mathrm{Area}(\Psi) + \varepsilon]$ and

$$\frac{\zeta}{8} \le a'(t) \le \frac{\zeta}{2} \quad \text{for } t \in [-\varepsilon, \varepsilon].$$

By choosing δ sufficiently small we may suppose that

$$|\,\mathrm{Area}(\Phi) - \mathrm{Area}(\Psi)| = |\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+(p))}) - \mathrm{Area}(f^{-1} \circ \psi \circ \phi^{-1})| \le \frac{\varepsilon}{2},$$

so we may find a unique $\bar{t} \in [-\varepsilon, \varepsilon]$ so that $\mathrm{Area}(\Psi + \bar{t}w) = \mathrm{Area}(\Phi)$. We then set

$$\overline{\Psi} = \Psi(x) + \bar{t}w(x) \quad \text{for } x \text{ in } B_1^+.$$

Then $\overline{\Psi}$ is a Lipschitz $W^{2,2}$ immersion , and by construction there holds $\mathrm{Area}(\overline{\Psi}) = \mathrm{Area}(\Phi)$. Similarly as done in the proof of Lemma 6.6.8, following a computation analogous to ([MR13, Lemma A.5]), the Willmore energy of $\overline{\Psi}$ can be expanded with respect to $\bar{t}$ as

$$\frac{1}{4}\int_{B_1^+} |A_{\overline{\Psi}}|^2_{g_{\overline{\Psi}}}\, d\sigma_{g_{\overline{\Psi}}} = \mathcal{W}_2(\Psi + \bar{t}w) = \mathcal{W}_2(\Psi) + \bar{t}D\mathcal{W}_2(\Psi)w + R_w^\Psi(\bar{t}),$$

with $|DE(\Psi)w| \le C_{\Phi,w}$ and $R_w^\Psi(\bar{t})$ satisfies $|R_w^\Psi(\bar{t})| \le C_{\Psi,w}\bar{t}^2$. By the mean value theorem, we have the estimate

$$\begin{aligned}
|\bar{t}| &= |a^{-1}(a(\bar{t})) - a^{-1}(a(0))| \\
&\le \sup_{\xi \in J} |(a^{-1})'(\xi)||a(\bar{t}) - a(0)| \\
&\le \frac{8}{\zeta}|\,\mathrm{Area}(\overline{\Psi}) - \mathrm{Area}(\Psi)| \\
&= \frac{8}{\zeta}|\,\mathrm{Area}(\Phi) - \mathrm{Area}(\Psi)| \\
&\le C_0(\delta + o(\delta))\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+)}),
\end{aligned}$$

where C_0 depends on $\|\nabla\lambda\|_{L^{2\infty}(B_1^+)}$, on f and on ζ and this yields the estimate

$$\int_{B_1} |A_{\overline{\Psi}}|^2_{g_{\overline{\Psi}}}\, d\sigma_{g_{\overline{\Psi}}} \le \int_{B_1} |A_\Psi|^2_{g_\Psi}\, d\sigma_{g_\Psi} + C_0(\delta + o(\delta))\,\mathrm{Area}(\Phi|_{\phi(B_\rho^+)}).$$

We write

$$\int_{\phi(B_{r/2^{P+1}})} |\nabla^2\Phi|_g^2\, d\sigma_g = \int_{\phi(B_{r/2^{P+1}})} |A|^2\, d\sigma_g + 4\int_{\phi(B^+_{r/2^{P+1}})} |\nabla\lambda|^2\, dx.$$

We have, on the one hand, thanks to Lemma 6.6.2 and the choice of P, the estimate

$$\int_{\phi(B^+_{r/2^{P+1}})} |\nabla\lambda|^2\, dx \le \int_{B^+_{r/2}} |\nabla\lambda|^2\, dx \tag{6.6.68}$$
$$\le \frac{1}{8}\int_{B^+_r(0)} |\nabla^2\Phi|^2_g\, d\sigma_g + C_0\varepsilon_0\left(\int_{B^+_r(0)} |\nabla^2\Phi|^2_g\, d\sigma_g + C_0\|\mathrm{e}^\lambda\|_{L^p(rI)}\right),$$

where C_0 depends only on ζ and N_0; on the other hand, the comparison of Φ with $\overline{\Psi}$ yields (we use the pointwise a.e. estimates in remark 6.6.10–(ii))

$$\begin{aligned}
&\int_{\phi(B^+_{r/2^{P+1}})} |A_\Phi|^2_{g_\Phi}\, d\sigma_g\\
&\le \int_{\phi(B^+_{r/2^{P+1}})} |A_{f^{-1}\circ\psi\circ\phi}|^2_{g_{f^{-1}\circ\psi\circ\phi}}\, d\sigma_{g_{f^{-1}\circ\psi\circ\phi}}\\
&= \int_{B^+_{r/2^{P+1}}} |A_{f^{-1}\circ\psi}|^2_{g_{f^{-1}\circ\psi}}\, d\sigma_{g_{f^{-1}\circ\psi}}\\
&\le \int_{B^+_{r/2^{P+1}}} |\nabla^2(f^{-1}\circ\psi)|^2_{g_{f^{-1}\circ\psi}}\, d\sigma_{g_{f^{-1}\circ\psi}}\\
&\le C_0\int_{B^+_{r/2^{P+1}}} (1+|\nabla^2\psi|^2_{g_\psi})\, d\sigma_{g_\psi} && \text{(by (6.6.67))}\\
&\le C_0\int_{B^+_{r/2^{P+1}}} d\sigma_{g_\psi}\\
&\quad + C_0\int_{B^+_{r/2^P}\setminus B^+_{r/2^{P+1}}} |\nabla^2(f\circ\Psi\circ\phi)|^2_{g_{f\circ\Psi\circ\phi}}\, d\sigma_{g_\psi} + C_0\int_{B^+_{r/2^P}} d\sigma_{g_{f\circ\Psi\circ\phi}}\\
&\le C_0\int_{\phi(B^+_{r/2^P})} d\sigma_g + C_0\int_{\phi(B^+_{r/2^P}\setminus B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_{g_\Phi}\, d\sigma_g,
\end{aligned}$$

and so all in all,

$$\int_{\phi(B^+_{r/2^{P+1}})} |A|^2\, d\sigma_g \le C_0\int_{\phi(B^+_{r/2^P})} d\sigma_g + C_0\int_{\phi(B^+_{r/2^P}\setminus B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_g\, d\sigma_g \tag{6.6.69}$$

By combining (6.6.68) and (6.6.69) we deduce that

$$\begin{aligned}
\int_{\phi(B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_g\, d\sigma_g &\le C_0\int_{\phi(B^+_{r/2^P})} d\sigma_g + C_0\int_{\phi(B^+_{r/2^P}\setminus B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_g\, d\sigma_g\\
&\quad + \frac{1}{2}\int_{B^+_r} |\nabla^2\Phi|^2_g\, d\sigma_g + C_0\varepsilon_0\left(\int_{B^+_r} |\nabla^2\Phi|^2_g\, d\sigma_g + C_0\|\mathrm{e}^\lambda\|_{L^p(rI)}\right)
\end{aligned}$$

and so adding $C_0\int_{\phi(B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_g\, d\sigma_g$ to both hand–sides

$$\begin{aligned}
\int_{\phi(B^+_{r/2^{P+1}})} |\nabla^2\Phi|^2_g\, d\sigma_g &\le \left(\frac{1/2 + C_0\varepsilon_0 + C_0}{C_0+1}\right)\int_{B^+_r} |\nabla^2\Phi|^2_g\, d\sigma_g\\
&\quad + C_0\int_{B^+_r} d\sigma_g + C_0\|\mathrm{e}^\lambda\|_{L^p(rI)}.
\end{aligned}$$

Choosing ε_0 sufficiently small so that $C_0\varepsilon_0 \le 1/4$, with (6.6.66) and the above inequality we deduce that, for every $0 < r < 1/4$ there holds

$$\int_{B^+_{r/2^{M+1}}} |\nabla^2 \Phi|^2_g \, d\sigma_g \le \beta \int_{B^+_r} |\nabla^2 \Phi|^2_g \, d\sigma_g + C_0 \|\mathrm{e}^{\lambda}\|_{L^\infty(B_1^+)} r,$$

where $0 < \beta < 1$ depends only on C_0. As in the proof of lemma 6.6.8, this equality can now be iterated and interpolated to yield the existence of some $\gamma > 0$ so that

$$\sup_{r < r < \bar{r}} \left\{ r^{-\gamma} \int_{B^+_r} |\nabla^2 \Phi|^2_g \, d\sigma_g \right\} < +\infty,$$

for some suitably small $\bar{r} < 1/16$. After going back to the original scale, this yields to the conclusion of the proof of the lemma. $\square$

The last ingredient we are going to use concerns the vanishing of first residues for minimizers. If Φ is any conformal map in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$, which is Willmore outside its branch points $a_1, \ldots, a_\ell \in B_1$ then it satisfies the Willmore equation in divergence form, plus a Lagrange multiplier term for the area constraint, in the sense of distributions, away from such points. Being however $H = \mathrm{e}^{-2\lambda}\Delta\Phi/2$ in $L^2(B_1)$, at each a_i the distributional equation can gain, at most, a contribution consisting in a Dirac mass. In other words:

$$\operatorname{div}\left(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} + c\nabla\Phi \right) = \sum_{i=1}^{\ell} \alpha_i \delta_{a_i}, \quad \text{in } \mathcal{D}'(B_1, \mathbb{R}^n),$$

for some $\alpha_1, \ldots \alpha_\ell \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Each α_i is called first residue of Φ at a_i, and needs not to vanish, in general. For minimizers however a simple implicit function theorem argument reveals that this is true.

Note that the vanishing of the first residues means Φ satisfies $\mathcal{W}(\Phi + t\varphi) = \mathcal{W}(\Phi) + o(t)$ for every $\varphi \in C^\infty_c(B_1, \mathbb{R}^n)$, that is, Φ is a“true”, possibly branched, Willmore immersion, (using a terminology from [Riv21, RM]).

Lemma 6.6.11 (Vanishing of the first residue for area-constrained minimisers)**.** *Let $\Phi \in \mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$ be a conformal immersion, possibly branched at 0, which minimises the Willmore energy in this class. Then Φ satisfies the Willmore equation*

$$\operatorname{div}\left(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} + c\nabla\Phi \right) = 0 \quad \textit{in } \mathcal{D}'(B_1, \mathbb{R}^n),$$

which is to say, the first residue at 0 *vanishes.*

Proof. We may assume that Φ is not a minimal surface (otherwise there is nothing to prove), and in particular that there exists some $\gamma > 0$ and some $v \in C^\infty_c(B_1, \mathbb{R}^n)$ so that

$$D\,\mathrm{Area}(\Phi)v > \gamma/2 > 0, \quad \text{and} \quad 0 \notin \operatorname{supp} v.$$

We denote by α the first residue of Φ at 0. Let also $w \in C^\infty_c(B_1, \mathbb{R}^n)$ and $\varepsilon > 0$ be so that

$$w(x) = \begin{cases} \alpha & \text{in } B_\varepsilon, \\ 0 & \text{in } B_1 \setminus B_{2\varepsilon}, \end{cases} \qquad \text{and} \qquad \operatorname{supp} w \cap \operatorname{supp} v = \emptyset.$$

We then define:

$$\Phi_{t,u}(x) := \Phi(x) + tw(x) + uv(x) \quad x \in B_1,$$

where $t \in [-t_0, t_0]$ and $u \in [-u_0, u_0]$ are parameters whose range will be specified below. We note first of all that

$$\begin{aligned} N_{\Phi_{t,u}} &\equiv N \quad \text{in } B_\varepsilon, \\ \nabla\Phi_{t,u} &\equiv \nabla\Phi \quad \text{in } B_\varepsilon, \\ A_{\Phi_{t,u}} &\equiv A \quad \text{in } B_\varepsilon, \\ H_{\Phi_{t,u}} &\equiv H \quad \text{in } B_\varepsilon, \end{aligned}$$

consequently, $\Phi_{t,u}$ and in particular, if t_0, u_0 are sufficiently small $\Phi_{t,u}$ defines an element of $\mathscr{F}(B_1, \mathbb{R}^n)$, possibly branched only at 0. Let us now consider the function

$$a(t,u) = \mathrm{Area}(\Phi_{t,u}) \quad (t,u) \in [-t_0, t_0] \times [-u_0, u_0].$$

Then a is of class C^1 (it has continuous first derivatives) and

$$\frac{\partial a}{\partial u}(0,0) = D\,\mathrm{Area}(\Phi)v \neq 0,$$

so by the Implicit Function Theorem, possibly reducing t_0 there will be some C^1-diffemorphism $\varphi : [-t_0, t_0] \to \mathbb{R}$ with $\phi(0) = 0$ so that

$$a(t, \varphi(t)) \equiv a(0,0) = \mathrm{Area}(\Phi) \quad t \in [-t_0, t_0].$$

As usual, differentiating in 0 this equation yields

$$0 = \partial a(0,0) + \partial_u a(0,0)\varphi'(0) = D\,\mathrm{Area}(\Phi)w + (D\,\mathrm{Area}(\Phi)v)\varphi'(0).$$

For every such value of t, the immersion $\Phi_{t,\varphi(t)}$ has the same area as Φ and is then a suitable competitor for Φ:

$$\mathcal{W}_2(\Phi) \leq \mathcal{W}_2(\Phi_{t,\varphi(t)}).$$

Now, we see that we may write

$$\begin{aligned} \int_{B_1} |A_{\Phi_{t,\varphi(t)}}|^2 \, d\sigma_{g_{\Phi_{t,\varphi(t)}}} &= \int_{B_\varepsilon} |A|^2 \, d\sigma_g && (\text{since } A_{\Phi_{t,u}} \equiv A) \\ &+ \int_{B_{2\varepsilon} \setminus B_\varepsilon} |A_{\Phi_{t,0}}|^2 \, d\sigma_{g_{\Phi_{t,0}}} && (\text{since } \operatorname{supp} v \cap B_{2\varepsilon} = \emptyset) \\ &+ \int_{\operatorname{supp} v} |A_{\Phi_{0,\varphi(t)}}|^2 \, d\sigma_{g_{\Phi_{0,\varphi(t)}}} && (\text{since } \operatorname{supp} v \cap \operatorname{supp} w = \emptyset) \\ &+ \int_{B_1 \setminus (B_{2\varepsilon} \cup \operatorname{supp} v)} |A|^2 \, d\sigma_g. \end{aligned}$$

Let us analyse the terms on the right-hand-side in detail. If we expand the second term in t, we get as $t \to 0$,

$$\int_{B_{2\varepsilon} \setminus B_\varepsilon} |A_{\Phi_{t,0}}|^2 \, d\sigma_{g_{\Phi_{t,0}}} = \int_{B_{2\varepsilon} \setminus B_\varepsilon} |A|^2 \, d\sigma_g + \left(\frac{\partial}{\partial t} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |A_{\Phi_{t,0}}|^2 \, d\sigma_{g_{\Phi_{t,0}}} \bigg|_{t=0} \right) t + o(t),$$

but, by construction, it is

$$\frac{\partial}{\partial t}\int_{B_{2\varepsilon}(0)\setminus B_\varepsilon}|A_{\Phi_{t,0}}|^2\,d\sigma_{g_{\Phi_{t,0}}}\bigg|_{t=0} = -|\alpha|^2 - cD\,\mathrm{Area}(\Phi)w,$$

so that we deduce that as $t\to 0$,

$$\int_{B_{2\varepsilon}\setminus B_\varepsilon}|A_{\Phi_{t,0}}|^2\,d\sigma_{g_{\Phi_{t,0}}} = \int_{B_{2\varepsilon}\setminus B_\varepsilon}|A|^2\,d\sigma_g - \bigg(|\alpha|^2 + cD\,\mathrm{Area}(\Phi)w\bigg)t + o(t).$$

If we expand the third term (using the mean value theorem: $\phi(t)=\varphi'(\xi)t$ for some $\xi\in(0,t)$) we get as $t\to 0$

$$\begin{aligned}
&\int_{\mathrm{supp}\,v}|A_{\Phi_{0,\varphi(t)}}|^2\,d\sigma_{g_{\Phi_{0,\varphi(t)}}}\\
&=\int_{\mathrm{supp}\,v}|A|^2\,d\sigma_g + \left(\frac{\partial}{\partial u}\int_{\mathrm{supp}\,v}|A_{\Phi_{0,u}}|^2\,d\sigma_{g_{\Phi_{0,u}}}\bigg|_{u=\phi(0)=0}\right)\varphi(t) + o(\varphi(t))\\
&=\int_{\mathrm{supp}\,v}|A|^2\,d\sigma_g + \left(\frac{\partial}{\partial u}\int_{\mathrm{supp}\,v}|A_{\Phi_{0,u}}|^2\,d\sigma_{g_{\Phi_{0,u}}}\bigg|_{u=\phi(0)=0}\right)\varphi'(\xi)t + o(t),
\end{aligned}$$

but

$$\begin{aligned}
&\frac{\partial}{\partial u}\int_{\mathrm{supp}\,v}|A_{\Phi_{0,u}}|^2\,d\sigma_{g_{\Phi_{0,u}}}\bigg|_{u=0}\\
&=\int_{\mathrm{supp}\,v}\langle\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}s, \nabla v\rangle\,dx\\
&=-c\int_{\mathrm{supp}\,v}\langle\nabla\Phi,\nabla v\rangle\,dx\\
&=-cD\,\mathrm{Area}(\Phi)v,
\end{aligned}$$

so we deduce that at $t\to 0$

$$\int_{\mathrm{supp}\,v}|A_{\Phi_{0,\varphi(t)}}|^2\,d\sigma_{g_{\Phi_{0,\varphi(t)}}} = \int_{\mathrm{supp}\,v}|A|^2\,d\sigma_g - c\,D\,\mathrm{Area}(\Phi)v\,\phi'(\xi)t + o(t).$$

All in all, we have that, as $t\to 0$,

$$\mathcal{W}_2(\Phi_{t,\varphi(t)} = \mathcal{W}_2(\Phi) + \frac{t}{4}\bigg(-|\alpha|^2 - c(D\,\mathrm{Area}(\Phi)w + \varphi'(\xi)\,D\,\mathrm{Area}(\Phi)v)\bigg) + o(t)$$

From the minimality of Φ: $\mathcal{W}_2(\Phi)\le\mathcal{W}_2(\Phi_{t,\varphi(t)})$, we then deduce:

$$0\le t\bigg(-|\alpha|^2 - c(D\,\mathrm{Area}(\Phi)w + \varphi'(\xi)\,D\,\mathrm{Area}(\Phi)v)\bigg) + o(t)$$

so, dividing by t and letting $t\to 0^+$ (hence also $\xi\to 0$),

$$0\le -|\alpha|^2 - c\,\underbrace{(D\,\mathrm{Area}(\Phi)w + \varphi'(0)\,D\,\mathrm{Area}(\Phi)v)}_{=0} = -|\alpha|^2,$$

and this necessarily implies $\alpha = 0$. $\square$

Proof of Theorem 6.1.3. Let Φ be a conformal map which is a minimizer Willmore energy in $\mathscr{F}(B_1, \mathbb{R}^n, \Gamma, N_0, a)$, and let $a_1, \dots, a_\ell$ be its branch points (recall that none of them lays on the boundary). For every sufficiently small $\delta > 0$, the conformal factor of Φ, $\mathrm{e}^\lambda = |\nabla \Phi|^2/\sqrt{2}$ is then uniformly bounded from above and below in $\overline{B_1} \setminus \cup_{i=1}^\ell B_\delta(a_i)$, and consequently, covering $\overline{B_1} \setminus \cup_{i=1}^\ell B_\delta(a_i)$ with finitely many balls, thanks to Lemma 6.6.8 and Lemma 6.6.9, we deduce that its Hessian $\nabla^2\Phi$ belongs to the Morrey space $L^{2,a}(B_1 \setminus \cup_{i=1}^\ell B_\delta(a_i))$ for some $a > 0$ (see e.g. [Gia83, GM12]), and consequently, by Morrey's Dirichlet growth theorem ([Mor66], see also [GM12, Theorem 5.7]) that $\Phi \in C^{1,a/2}(\overline{B_1} \setminus \cup_{i=1}^\ell B_\delta(a_i))$.

Thanks to Lemma 6.6.11, Φ satisfies the Euler-Lagrange equation for the Poisson problem (i.e. the Willmore equation in divergence form plus a Lagrange multiplier term for the area constraint), through the branch points, i.e. each of the first residues vanishes. From the analysis of singularities for Willmore surfaces [Riv08, BR13, KS04, KS07], this implies that Φ is of class $C^{1,\alpha}$ for every $0 < \alpha < 1$ through the branch points.

We have thus proved that Φ is of class $C^{1,\alpha}$-up to the boundary, for some $0 < \alpha < 1$ (and accordingly the Gauss map N extends to a map of class $C^{0,\alpha}$-up to the boundary) and this concludes the proof of the theorem. $\square$

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Personal

- Italian citizenship.
- Languages: Italian (native), English (fluent), German (intermediate)

Research

Partial Differential Equations and Calculus of Variations, with particular interest to problems pertaining Geometric Analysis and Differential Geometry.

Research Articles:

1. with F. Da Lio: *Remarks on Neumann boundary problems involving Jacobians.* Comm. Partial Differential Equations 42 (2017), no. 10, 1497–1509.
2. with F. Da Lio & T. Rivière: *A Resolution of the Poisson Problem for Elastic Plates.* Arch. Rational. Mech. Anal. 236 (2020), 1593–1676.
3. with F. Boarotto & R. Monti: *Third Order Open Mapping Theorems and Applications to the End-Point Map.* Nonlinearity, 33 (2020), 4539–4567.
4. with T. Rivière: *The Parametric Approach to the Willmore Flow*, preprint arxiv.org/abs/2012.11419.

Education

- 2015 – 2021 *PhD in Mathematics,* ETH Zürich
  Thesis: *The Germain-Poisson Problem and Some Aspects of the Variations of the Willmore Lagrangian.* Advisors: Francesca Da Lio & Tristan Rivière
- 2012 – 2014 *MSc in Mathematics*, Università degli Studi di Padova
  Thesis: *Length-Minimizing Curves on Sub-Riemannian Manifolds: Necessary Conditions Involving the End-Point Mapping.* Supervisor: Roberto Monti
- 2009 – 2012 *BSc in Mathematics*, Università degli Studi di Padova
  Thesis: *Coomologia di De Rham e Dualità di Poincaré.* Supervisor: Giuseppe De Marco