Uniqueness of tangent cones for calibrated 2-cycles

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Abstract

In this paper we prove that the tangent cones to calibrated 2-cycles are unique. Furthermore, using this result we prove a rate of convergence for the mass of the blow-up of a calibrated integral 2-cycle C towards the limiting density $\Theta(\|C\|, x_0)$, i.e. there exist constants $C_1 > 0$, $\gamma > 0$ such that

$$\frac{M(C \llcorner B_r(x_0))}{\pi r^2} - \Theta(\|C\|, x_0) \leq C_1 r^\gamma \ .$$

We also obtain such a rate for J-holomorphic maps between almost complex manifolds and deduce that their tangent maps are unique.

1 Introduction

Let M be a smooth compact m-dimensional manifold and let $\Omega_0^k(M)$ denote the smooth compactly supported k-forms on M. Then a k-dimensional current C in M is a distribution on $\Omega_0^k(M)$. The boundary of such a k-current is the $(k-1)$-current defined by $\partial C(\omega) := C(d\omega)$, where $\omega \in \Omega_0^{k-1}(M)$ and we say that a k-current C is a k-cycle if $\partial C = 0$.
We can put an arbitrary smooth Riemannian metric g on M and define the comass of a k-form ω to be

$$\|\omega\|_* := \sup_{x\in M} \sup_{e_1,\ldots,e_k \in S_xM} |\langle \omega, e_1 \wedge \ldots \wedge e_k\rangle| \ ,$$

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where S_xM means the unit sphere in T_xM with respect to g_x. Then the mass $M(C)$ of a k-current C is defined by

$$M(C) := \sup_{\omega\in\Omega_0^k(M), \|\omega\|_*\leq 1} |C(\omega)| \ .$$

A k-cycle C is called a normal cycle if C satisfies $M(C) < +\infty$. Furthermore, we call a current an integer multiplicity rectifiable k-current, if C is a normal k-current and there are k-Hausdorff measurable subsets $\mathcal{N}_j$ of oriented k-dimensional C^1-submanifolds N_j with $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $i \neq j$, and a multiplicity $\Theta : \bigcup_{j=1}^{\infty} \mathcal{N}_j \to \mathbf{Z}$ such that

$$\langle C; \psi\rangle = \sum_{j=1}^{\infty} \int \psi\, \Theta \ d\mathcal{H}^k \llcorner \mathcal{N}_j \ .$$

In this case the mass of C is given by $M(C) = \sum_{j=1}^{\infty} \int |\Theta| \ d\mathcal{H}^k \llcorner \mathcal{N}_j$. In this paper we will only work with k-cycles and abbreviate integer multiplicity rectifiable k-cycles by calling them integral k-cycles (usually integral currents are currents for which both C and ∂C are integer multiplicity rectifiable currents).

We also need to define the notion of a smooth calibration on M:

Definition 1 *Let ω be a smooth k-form on (M^m, g). Then we say that a k-current is calibrated by ω if for all open subsets $U \subset M$ we have*

$$\langle C \llcorner U; \omega\rangle = M(C \llcorner U) \ ,$$

where $C \llcorner U$ means the restriction of C to U.

If ω has comass equal to 1 and satisfies $d\omega = 0$, it is easy to see, that a current calibrated by ω is homologically mass-minimizing (see Harvey and Lawson [10] for details). In this paper we will always assume our calibrations to have comass equal to 1, but for reasons which will become apparent later, we do not assume the calibration to be closed. Thus calibrated currents will not always be (homologically) mass-minimizing.
We now want to give some well-known and important examples of calibrated currents. In complex geometry (complex) p-dimensional algebraic sub-varieties of $\mathbf{CP}^n$ can be seen as $2p$-dimensional integral currents calibrated by $\omega^p_{\mathbf{CP}^n}$, where $\omega_{\mathbf{CP}^n}$ is the standard Kähler form on $\mathbf{CP}^n$. More generally, when (M^{2m}, J, ω) is an almost Kähler manifold where ω is the closed Kähler form compatible with J (this means that $\omega(\cdot, J\cdot)$ is a Riemannian metric on M), integral $p{-}p$-currents are calibrated by ω^p — here a $p{-}p$-cycle is an integral cycle where the C^1-manifolds N_j satisfy $J_x(T_xN_j) = T_xN_j$

for any $x \in N_j$. Note that in this case $p-p$-currents are a subclass of the area-minimizing $2p$-currents.
On a general compact almost complex manifold (M^{2m}, J) one can still define $p-p$-currents as above. Notice that the case of $1-1$-cycles is particularly interesting, since they arise as perturbations of J_0-holomorphic graphs and are generic from the existence point of view — see the introduction of [17]. If one can find a symplectic form which is compatible with J, then we are back in the almost Kähler case and $1-1$-cycles are area-minimizing. It turns out that locally one can indeed find a 2-form ω which is compatible with J but in general this ω will not be closed (hence not symplectic). In real dimensions ≤ 4 locally the form ω can be constructed to be closed — see the appendix of [18] — but for higher dimensions R. Bryant [4] constructed an almost complex structure on S^6 which does not admit any compatible ω even locally. Hence in such a case $1-1$-cycles are still calibrated by ω but no longer area-minimizing. However, $1-1$-cycles are of geometric interest even in the absence of a symplectic form when results for area-minimizing currents no longer apply. This is one of the reasons, why we do not want to assume the calibration to be closed in this paper.
Another important example of calibrated currents is connected to Special Lagrangian currents in Calabi-Yau p-folds. A Calabi-Yau p-fold is a $2p$-dimensional Kähler manifold (M^{2p}, J, ω) for which the holonomy is $SU(p)$. As one can see in the lecture notes by D. Joyce [12] , for such a manifold there exists a $(p,0)$-holomorphic form Ω (called the holomorphic volume form) satisfying

$$\frac{\omega^p}{p!} = (-1)^{\frac{p(p-1)}{2}} \left(\frac{i}{2}\right)^p \Omega \wedge \bar{\Omega} \ .$$

Then a p-current C is called Special Lagrangian if it is calibrated by $\mathrm{Re}\,\Omega$ — the real part of Ω. Tangent cones C_∞ to such currents (this will be explained below) are calibrated by $\mathrm{Re}(dz_1 \wedge \ldots \wedge dz_p)$ (where we use usual normal coordinates) and are of the form $C_\infty = 0 \divideontimes T$ (see [7] for the notation), where T is a Special Legendrian $p-2$-current in S^{2p-2}. These $p-2$-currents are calibrated by $\mathrm{Re}\left(\sum z_i dz_{i+1} \wedge dz_{i-1}\right)$ and it is important to note that again this calibration is not closed.

Our goal is to study the regularity of such calibrated cycles so it is natural to analyze their blow-up around points first, i.e. to prove whether such a blow-up exists or not and when it exists whether it is unique or not. The blow-up analysis of C around a point $x_0 \in M$ is done as follows: consider a dilation of C around x_0 which in normal coordinates near x_0 is given by the push-forward of the current C under the map $\frac{x-x_0}{r}$ — here we mean that

$\frac{x-x_0}{r}{}_* C(\psi) := C(\frac{x-x_0}{r}{}^* \psi)$. To analyze the behavior of these dilations as $r \to 0$ we need the monotonicity formula which says that $r^{-k} M(C \llcorner B_r(x_0))$ is increasing and the compactness theorem by Federer-Fleming (see [7] 4.2.17). The monotonicity formula gives that $M(\frac{x-x_0}{r}{}_* C \llcorner B_1(0))$ is bounded independent of r since $M(\frac{x-x_0}{r}{}_* C \llcorner B_1(0)) \cong r^{-k} M(C \llcorner B_r(x_0))$. Using this and the fact that C is a cycle we apply the compactness theorem to deduce that there exists $r_n \to 0$ and a normal k-current C_∞ such that weakly

$$\frac{x-x_0}{r_n}{}_* C \llcorner B_1 \rightharpoonup C_\infty \ .$$

It turns out that C_∞ is a cone — called a tangent cone to C at x_0 — which is calibrated by ω_{x_0} (see section 3.1 of this paper). The main questions related to such a construction are whether the blow-up limit is unique (i.e. independent of the chosen subsequence of radii) or not and whether the dilated currents converge to the limiting object at a certain rate.

These questions are directly related to rate of convergence results for elliptic systems obtained by C. Morrey (see [15] or [8]). For example, if one considers stationary harmonic maps $u : B^m \to N^n \subset \mathbf{R}^N$ from the unit ball in $\mathbf{R}^m$ into an n-dimensional submanifold N of $\mathbf{R}^N$ — for a detailed discussion we refer to the books by F. Hélein [11] and L. Simon [20] —, one obtains that $\Theta_u(x_0) := \lim_{r\to 0} \frac{1}{r^{m-2}} \int_{B_r(x_0)} |\nabla u|^2$ exists for all $x_0 \in B^m$. This implies that the dilated maps $u_{x_0,r}(x) := u(x_0 + rx)$ are uniformly bounded in $W^{1,2}$ and hence that there exists a sequence $r_n \to 0$ and a $W^{1,2}$-map $u_{x_0,\infty}$ — a tangent map to u at x_0 — such that $u_{x_0,r_n} \rightharpoonup u_{x_0,\infty}$. In case $\Theta_u(x_0) = 0$, C. Evans [6] and F. Bethuel [3] showed that there exist constants $C_1 > 0$, $\gamma > 0$ such that

$$\frac{1}{r^{m-2}} \int_{B_r(x_0)} |\nabla u|^2 \leq C_1 r^\gamma \ .$$

Using results of Morrey, this rate implies that in a small neighborhood U of x_0 we have that $u \in C^{0,\frac{\gamma}{2}}(U, N)$. Then the $u_{x_0,r}$ converge to a unique tangent map (independent of the sequence r_n) $u_{x_0,\infty}$ which is constant.
However, it would also be interesting to know, if for $\Theta_u(x_0) > 0$ there are constants $C_1 > 0$, $\gamma > 0$ such that the following more general estimate holds:

$$\frac{1}{r^{m-2}} \int_{B_r(x_0)} |\nabla u|^2 - \Theta_u(x_0) \leq C_1 r^\gamma \ .$$

In other words, even at singular points one would like to have a rate of convergence for $r^{2-m} \int_{B_r(x_0)} |\nabla u|^2$ towards the limiting density $\Theta_u(x_0)$. Note that again such a rate of convergence is very strong information as it implies

the weak convergence of the $u_{x_0,r}$ to a unique blow-up limit $u_{x_0,\infty}$ as shown in proposition 2.
Unfortunately, Gulliver and White showed in [9] that such a rate does not always hold for stationary harmonic maps. However, in the first section of this paper we show, that such a rate in fact exists for locally approximable J-holomorphic maps. For a compact almost complex manifold (M, J_M) and an almost Kähler manifold (N, J_N, ω_N) a $W^{1,2}$-map $u : M \to N$ is called a locally approximable J-holomorphic map, if for a.e. $x \in M$ and all $X \in T_xM$ we have $du_x(J_M(x)(X)) = J_N(u(x))(du_x(X))$ and u locally is in the strong $W^{1,2}$-closure of the C^∞-maps in $W^{1,2}(M,N)$ (note that in special cases J-holomorphic maps are stationary harmonic — see [18] or [23]). For such a map we can prove the following theorem:

Theorem 1 *Let $u : M^{2m} \to N^{2n}$ be a locally approximable J-holomorphic map from an almost complex manifold (M, J) into an almost Kähler manifold (N, J_N, ω_N), such that $u \in W^{1,2}(M,N)$. Given any $x_0 \in M$ there exist $r_0 > 0$, $C_1 > 0$ and $\gamma \in (0,1]$ all independent of u such that for $r \in (0, r_0)$*

$$\frac{1}{r^{2m-2}} \int_{B_r(x_0)} |\nabla u|^2 - \Theta_u(x_0) \leq C_1\big(1 + \|\nabla u\|^2_{L^2(B_1)}\big) r^\gamma \ ,$$

where $\Theta_u(x_0) := \lim_{\rho\to 0} \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2$.

Now we go back to the case of area-minimizing integral k-cycles. From the monotonicity formula we know that the limiting density $\Theta(\|C\|, x_0) := \lim_{r\to 0} \alpha(k)^{-1} r^{-k} M(C \llcorner B_r(x_0))$ exists at all point $x_0 \in M$ (here $\alpha(k)$ means the volume of the k-dimensional unit ball). B. White showed that a rate of convergence for $\alpha(k)^{-1} r^{-k} M(C \llcorner B_r(x_0))$ to the limiting density $\Theta(\|C\|, x_0)$ implies the uniqueness of the tangent cone to C at x_0 — see theorem 3 in [24] —, i.e. if there exist constants $C_1 > 0$ and $\gamma > 0$ such that

$$\frac{M(C \llcorner B_r(x_0))}{\alpha(k) r^k} - \Theta(\|C\|, x_0) \leq C_1 r^\gamma \ ,$$

then the blow-up limit is independent of the subsequence chosen. Using an idea of E. Reifenberg he deduced such a rate from a comparison argument in the case of area-minimizing integral 2-cycles. In section 2 of this paper we will show that the tangent cones to normal 2-cycles calibrated by a not necessarily closed form are unique:

Theorem 2 *Let C be a normal 2-cycle in a compact manifold M which is calibrated by a C^2-form ω (not necessarily closed). Then for any $x_0 \in M$ there exists a unique tangent cone C_∞ to C at x_0.*

The proof we give only depends on geometric observations and the monotonicity formula, so we do not use the approach through a comparison argument. The monotonicity we use will be proved in proposition 3. It is similar to the one for area-minimizing integral k-cycles obtained by H. Federer in chapter 5 of [7] (5.4.3 (2)) and was proven for calibrated currents (in case the calibration is a closed form) by R. Harvey and B. Lawson [10]. In fact the monotonicity formula we obtain is only an almost monotonicity formula but turns out to be good enough to study blow-ups.
We then use theorem 2 to deduce a rate of convergence for integral 2-cycles:

Theorem 3 *If, in addition to the hypotheses of theorem 2, C is also integer rectifiable, then for any $x_0 \in M$ there exist $r_0 > 0$, $C_1 > 0$ and $\gamma \in (0,1]$ such that for $r \in (0, r_0)$ we have*

$$\frac{M(C \llcorner B_r(x_0))}{\pi r^2} - \Theta(\|C\|, x_0) \leq C_1 r^\gamma \ .$$

As we already mentioned above, uniqueness of tangent cones has been proven in several related situations — for area-minimizing integral 2-cycles we refer the reader to the paper by B. White [24]. The problem of uniqueness was also studied by W. Allard and F. Almgren in [1] and [2], L. Simon in [19] and J. Taylor in [21] and [22].

Our paper is organized as follows: in section 2 we prove a rate of convergence for J-holomorphic maps and deduce the uniqueness of their tangent maps. In section 3 we prove theorems 2 and 3 first for constant calibrations and then for general calibrations through a perturbation argument.

2 Proof of theorem 1

Since the argument is local, we can assume that the domain manifold is in fact the open unit ball $B^{2m} \subset \mathbf{R}^{2m}$ and that $x_0 = 0 \in B^{2m}$. Also assume that the target is isometrically embedded in $\mathbf{R}^{2N}$ for some N large enough (see [16] for details). To make the presentation clearer we split the proof of theorem 1 into two cases. First we prove the theorem in the case where the almost complex structure is the standard complex structure J_0 of $\mathbf{R}^{2m}$ to illustrate the ideas. The remaining case will be handled by a perturbation argument.

2.1 The case of the standard complex structure

In this case we know that J-holomorphic maps are stationary harmonic (in fact energy minimising in their homotopy class — see for instance [18]). Therefore we have the usual monotonicity formula:

$$\frac{d}{dr}\left(\frac{1}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2\right) = 2\int_{\partial B_r(x_0)} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 ,$$

where $\frac{\partial u}{\partial R}$ means the derivative of u in the direction of $R = |x - x_0|$. Integrating this formula from 0 to r we obtain that

$$2\int_0^r d\rho \int_{\partial B_\rho(x_0)} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 = \frac{1}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2 - \Theta_u(x_0) < +\infty$$

i.e. that $\int_{B_r}|x-x_0|^{2-2m}|\frac{\partial u}{\partial R}|^2 < +\infty$. Next we want to show that given almost any J-holomorphic plane in $\mathbf{R}^{2m}$ the restriction of u to such a plane is a J-holomorphic $W^{1,2}$-map. To see this first note that given any unit vector X it lies in the plane spanned by $\{X, JX\}$ and that

$$\int_{B_r(x_0)}|x-x_0|^{2-2m}|u\cdot X|^2 \leq \int_{B_r(x_0)}|x-x_0|^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 < +\infty$$

by the monotonicity formula. For the slicing let π be the map parametrising the J-holomorphic curves through x_0, i.e. $\pi : B^{2m}(x_0) \to \mathbf{CP}^{m-1}$. Then this map satisfies $|\nabla\pi(x)| = O\big(\frac{1}{R}\big)$, so that there are constants $C_1, C_2 > 0$ with $C_1 - C_2|x-x_0| \leq |x-x_0|^{2m-2}|J_{2m-2}\pi| \leq C_1 + C_2|x-x_0|$ (here $J_{2m-2}\pi$ denotes the $(2m-2)$-Jacobian of π). We use this fact when we now apply the co-area formula:

$$\int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)} \frac{|x-x_0|^{2-2m}|u\cdot X|^2}{|J_{2m-2}\pi|} = \int_{B^{2m}(x_0)}|x-x_0|^{2-2m}|u\cdot X|^2 < +\infty .$$

By Fubini's theorem this implies that for almost every $p \in \mathbf{CP}^{m-1}$, $\int_{\pi^{-1}(p)}|u\cdot X|^2 < +\infty$. Since the map u and $\pi^{-1}(p)$ are J-holomorphic we conclude that for such p we also have $\int_{\pi^{-1}(p)}|u\cdot JX|^2 < +\infty$ and hence that for almost every $p \in \mathbf{CP}^{m-1}$, $\int_{\pi^{-1}(p)}|\nabla u|^2 < +\infty$. Therefore $u|_{\pi^{-1}(p)}$ is a J-holomorphic $W^{1,2}$-map from $(\pi^{-1}(p), J_0)$ to (N, J_N). From the regularity theorems for J-holomorphic maps from Riemann surfaces we conclude that the restricted map is smooth and that there exists $\delta > 0$ such that

$$\int_{\pi^{-1}(p)\cap B_r(x_0)}|\nabla u|^2 < \delta\int_{\pi^{-1}(p)\cap B_{2r}(x_0)}|\nabla u|^2 .$$

Integrating this estimate over $p \in \mathbf{CP}^{m-1}$ yields

$$\int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_r(x_0)} |u \cdot X|^2 < \delta \int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_{2r}(x_0)} |u \cdot X|^2$$

since $|\nabla u|^2 = |u \cdot X|^2 + |u \cdot JX|^2 = 2|u \cdot X|^2$. Using the co-area formula again this gives

$$\begin{aligned} \int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_r(x_0)} |u \cdot X|^2 &= \int_{B_r(x_0)} |u \cdot X|^2 |J_{2m-2}\pi| \\ &< \delta \int_{B_{2r}(x_0)} |u \cdot X|^2 |J_{2m-2}\pi| \ . \end{aligned}$$

Next we take $r > 0$ sufficiently small so that $\frac{C_1+C_2r}{C_1-C_2r}$ is close enough to 1 for $\frac{C_1+C_2r}{C_1-C_2r}\delta < 1$. Then for some $\tilde{\delta} \in (0,1)$ the above estimate becomes

$$\int_{B_r(x_0)} |x - x_0|^{2-2m} |u \cdot X|^2 < \tilde{\delta} \int_{B_{2r}(x_0)} |x - x_0|^{2-2m} |u \cdot X|^2 \ .$$

Therefore we get that there exist $\gamma \in (0,1]$ and $C > 0$ such that

$$\int_{B_r(x_0)} |x - x_0|^{2-2m} |u \cdot X|^2 \leq C r^\gamma \ .$$

Furthermore, as above, we get that

$$\int_{B_r(x_0)} |x - x_0|^{2-2m} \Big|\frac{\partial u}{\partial R}\Big|^2 \leq \int_{B_r(x_0)} |x - x_0|^{2-2m} |u \cdot X|^2 \leq C r^\gamma \ ,$$

which together with the monotonicity formula yields

$$\begin{aligned} \frac{1}{2}\int_0^r d\rho \frac{d}{d\rho}\Big(\rho^{2-2m} \int_{B_\rho(x_0)} |\nabla u|^2\Big) &= \int_0^r d\rho \int_{\partial B_\rho(x_0)} R^{2-2m} \Big|\frac{\partial u}{\partial R}\Big|^2 \\ &= \int_{B_r(x_0)} |x - x_0|^{2-2m} \Big|\frac{\partial u}{\partial R}\Big|^2 \\ &\leq C r^\gamma \ , \end{aligned}$$

i.e.

$$\frac{1}{r^{2m-2}} \int_{B_r(x_0)} |\nabla u|^2 - \Theta_u(x_0) \leq C r^\gamma \ ,$$

establishing is the desired estimate in the easy case.

2.2 Perturbing the proof

In this subsection we show that the above proof can still be carried out when the domain of u is the unit ball $B^{2m} \subset \mathbf{R}^{2m}$ together with an arbitrary complex structure J (the proof for an arbitrary metric is similar). Without loss of generality (through a change of coordinates) we can always assume that this complex structure is the standard complex structure at the origin, i.e. that $J(0) = J_0$, and that J satisfies $\|J(r) - J_0\|_{C^2} = O(r)$ on the whole ball. We will show that these assumptions suffice to carry out the above proof. To do this we first prove that a locally approximable J-holomorphic map satisfies an equation similar to the equation for stationary harmonic maps (see for example L. Simon [20]). We then proceed to deduce an almost monotonicity formula which will give us that the density exists at any point. Such a monotonicity formula was first proved by P. DeBartholomeis and G. Tian [5]. However, they did not show the precise error term which is crucial in the proof below. Finally, we show that all the arguments in the J_0-case above can still be carried out introducing only a small extra error.

2.2.1 A second order elliptic equation for u

We begin by showing that a locally approximable J-holomorphic map is almost a critical point of the Dirichlet energy $E(u) := \frac{1}{2}\int_B |\nabla u|^2$ for perturbations in the domain. Given an arbitrary smooth 1-parameter family F_t of diffeomorphisms of B^{2m} we want to compute $\frac{d}{dt}E(u \circ F_t)\Big|_{t=0}$. First note that one can easily verify the following alternative expression for the energy:

$$E(u) = \frac{1}{4}\int_B |\nabla u + J_N \circ \nabla u \circ J_0|^2 + \frac{1}{2}\int_B \langle J_N \nabla u, \nabla u \circ J_0 \rangle \ .$$

To simplify the notation we write $u_t := u \circ F_t$. Note that for any map u_t in the second term above we can write $\langle J_N \circ \nabla u_t, \nabla u_t \circ J_0 \rangle = \langle \omega_0, u_t^* \omega_N \rangle$ where ω_0, ω_N are the symplectic structures compatible with the metrics on domain and target respectively. For smooth perturbations of the domain we claim that the second term in the above expression satisfies

$$\frac{d}{dt}\int_{B^{2m}} \langle \omega_0, u_t^* \omega_N \rangle \Big|_{t=0} = 0 \ .$$

To see this note that on B^{2m} there exists a smooth 1-form ϕ such that $d\phi = \omega_0$ and that $u_t^* \omega_N = F_t^* u^* \omega_N$, whence $d(u_t^* \omega_N) = F_t^* d(u^* \omega_N) = 0$ as u is locally approximable (i.e. $d(u^* \omega_N) = u^* d\omega_N = 0$). Using Stoke's theorem and the fact that u_t has compact support in B^{2m} we get

$$\begin{aligned}
\int_{B^{2m}} \langle \omega_0, u_t^*\omega_N \rangle &= \int_{B^{2m}} \langle d\phi, u_t^*\omega_N \rangle \\
&= C \int_{B^{2m}} d(\phi \wedge \omega_0^{2m-2}) \wedge u_t^*\omega_N \\
&= C \int_{B^{2m}} d(\phi \wedge \omega_0^{2m-2} \wedge u_t^*\omega_N) \\
&= 0 \ .
\end{aligned}$$

Thus it remains to compute the derivative for the first term. We will show that since J is close to J_0 in B^{2m}, u is close to being J_0-holomorphic. Note that

$$\nabla u + J_N \circ \nabla u \circ J_0 = \nabla u + J_N \circ \nabla u \circ J + J_N \circ \nabla u \circ (J - J_0)$$

so that for u_t (using the fact that J_N is compatible with g_N) this means that

$$\begin{aligned}
\int_{B^{2m}} |\nabla u_t + J_N \circ \nabla u_t \circ J_0|^2 &= \int_{B^{2m}} |\nabla u_t + J_N \circ \nabla u_t \circ J|^2 && (1) \\
&+ 2\int_{B^{2m}} \langle \nabla u_t, J_N \circ \nabla u_t \circ (J - J_0) \rangle && (2) \\
&+ \int_{B^{2m}} \langle \nabla u_t \circ (J_0 + J), \nabla u_t \circ (J - J_0) \rangle && (3)
\end{aligned}$$

For the first term on the right-hand side (1) note that since u is J-holomorphic we have $|\nabla u + J_N \circ \nabla u \circ J_0|^2 = 0$ pointwise a.e. in B^{2m}, i.e. $\int_{B^{2m}} |\nabla u + J_N \circ \nabla u \circ J_0|^2 = 0$. Also

$$\int_{B^{2m}} |\nabla u_t + J_N \circ \nabla u_t \circ J|^2 \geq \int_{B^{2m}} |\nabla u + J_N \circ \nabla u \circ J|^2 = 0 \ ,$$

which implies that

$$\frac{d}{dt} \int_{B^{2m}} |\nabla u_t + J_N \circ \nabla u_t \circ J|^2 \bigg|_{t=0} = 0 \ .$$

For the remaining terms (2) and (3) we work in local coordinates. Since any 1-parameter group F_t is generated by a vector field ξ having compact support in B^{2m}, we can write $u_t(x) = u(x + t\xi)$. Writing $(J - J_0)(x) = [a_{kl}](x)$ and $J_N(u(x)) = [b_{\alpha\beta}](u(x))$ — throughout this part of the paper we use latin indices to denote coordinates on the domain and greek ones for the target — we get the following expression for the derivative of the term in (2):

$$\frac{d}{dt}\int_{B^{2m}} \langle \nabla u_t, J_N \circ \nabla u_t \circ (J - J_0)\rangle \Big|_{t=0} =$$

$$= \int_{B^{2m}} (-\operatorname{div}\xi) \frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_l}[a_{kl}][b_{\beta\alpha}] \quad (4)$$

$$+ \; 2\int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_{l'}}\frac{\partial \xi^{l'}}{\partial x_l}[a_{kl}][b_{\beta\alpha}] \; , \quad (5)$$

where we sum over repeated indices. For the last term (3) we write $(J + J_0)(x) = ([a_{kl}](x) + 2[\tilde{\delta}_{kl}])$, where $[\tilde{\delta}_{kl}]$ denotes the matrix for J_0 in the standard coordinates on B^{2m}. Then a computation shows that

$$\frac{d}{dt}\int_{B^{2m}} \langle \nabla u_t \circ (J_0 + J), \nabla u_t \circ (J - J_0)\rangle \Big|_{t=0} =$$

$$= \int_{B^{2m}} (-\operatorname{div}\xi) \frac{\partial u^\alpha}{\partial x_{k'}}\frac{\partial u^\alpha}{\partial x_{l'}}[a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}]) \quad (6)$$

$$+ \; 2\int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_{k'}}\frac{\partial u^\alpha}{\partial x_{l''}}\frac{\partial \xi^{l''}}{\partial x_{l'}}[a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}]) \; . \quad (7)$$

Therefore we obtain the following equation for u:

$$\int_{B^{2m}} \sum_{i,j} (|\nabla u|^2 \delta_{ij} - 2\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\rangle)\frac{\partial \xi^j}{\partial x_i} = \quad (8)$$

$$\begin{aligned} &= -\frac{1}{2}\int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_l}[a_{kl}][b_{\beta\alpha}]\delta_{ij}\frac{\partial \xi^j}{\partial x_i} \\ &+ \int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_j}[a_{ki}][b_{\beta\alpha}]\frac{\partial \xi^j}{\partial x_i} \\ &- \frac{1}{2}\int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_{k'}}\frac{\partial u^\alpha}{\partial x_{l'}}[a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}])\delta_{ij}\frac{\partial \xi^j}{\partial x_i} \\ &+ \int_{B^{2m}} \frac{\partial u^\alpha}{\partial x_{k'}}\frac{\partial u^\alpha}{\partial x_j}[a_{k'l}]([a_{ik}] + 2[\tilde{\delta}_{ik}])\frac{\partial \xi^j}{\partial x_i} \; , \end{aligned}$$

which is valid for all $\xi \in C_c^\infty(B^{2m}, \mathbf{R}^{2m})$. Note that this equation is very similar to the equation for stationary harmonic maps where the right-hand side vanishes identically, whereas in our case the right-hand side is $O(r)\|\nabla u\|^2_{L^2}$. It is clear that for small enough radii (depending only on J) the second order operator involved is only a small perturbation of the Laplacian, and hence elliptic.

2.2.2 Monotonicity formula for the energy of u

In this subsection we show that from equation (8) we can deduce an almost monotonicity formula for the energy of u. We will show that for locally approximable J-holomorphic maps $\frac{e^{C\rho}-\rho}{\rho^{2m-2}}\int_{B^{2m}_\rho}|\nabla u|^2$ is an increasing function of ρ provided that ρ is smaller than some fixed r_0 independent of u. Precisely, we will prove the following proposition:

Proposition 1 *Let $u : B^{2m} \to N \subset \mathbf{R}^{2N}$ be a locally approximable J-holomorphic map. Then there exist a constant $C > 0$ and $r_0 \in (0, 1]$ both independent of u such that*

$$\frac{e^{C\tau}-\tau}{\tau^{2m-2}}\int_{B^{2m}_\tau}|\nabla u|^2 - \frac{e^{C\sigma}-\sigma}{\sigma^{2m-2}}\int_{B^{2m}_\sigma}|\nabla u|^2 \geq 2\int_{B_\tau\setminus B_\sigma} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2$$

for all σ, $\tau \in (0, r_0)$ with $\sigma < \tau$ (here $R = |x|$ and $\frac{\partial u}{\partial R}$ denotes the derivative in the direction of R).

Proof:
The proof will follow the proof for stationary harmonic maps given in [20]. We use the fact that if $\int a_j \frac{\partial \xi}{\partial x_j} = 0$ for all $\xi \in C^\infty_c(B)$, then for a.e. $\rho \in (0, 1)$ we have $\int_{B_\rho} a_j \frac{\partial \xi}{\partial x_j} = \int_{\partial B_\rho} \eta \cdot a\xi$. Using this we test the above equation for u (8) with $\xi^j(x) = x^j$ (so that $\frac{\partial \xi^j}{\partial x_i} = \delta_{ij}$). This yields the following identity for u:

$$\begin{aligned}(2-2m)\rho^{1-2m}\int_{B_\rho}|\nabla u|^2 + \rho^{2-2m}\int_{\partial B_\rho}|\nabla u|^2 &= \\ = \rho^{1-2m}\int_{B_\rho}A(x)|\nabla u|^2 - \rho^{2-2m}\int_{\partial B_\rho}B(x)|\nabla u|^2 + 2\int_{\partial B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2,\end{aligned}$$

where $\int_{B_\rho} A(x)|\nabla u|^2$ and $\int_{\partial B_\rho} B(x)|\nabla u|^2$ denote terms which can be bounded by $C\rho\int_{B_\rho}|\nabla u|^2$ and $C\rho\int_{\partial B_\rho}|\nabla u|^2$ respectively. This leads to the chain of inequalities below

$$\begin{aligned}\left|\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2 - 2\int_{B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2\right)\right| &\leq \\ &\leq C\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2 + C\frac{1}{\rho^{2m-3}}\int_{\partial B_\rho}|\nabla u|^2 \\ &\leq C\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2 + C\frac{d}{d\rho}\left(\rho^{2m-3}\int_{B_\rho}|\nabla u|^2\right).\end{aligned} \tag{9}$$

Using this estimate we conclude that

$$\begin{aligned}
\frac{d}{d\rho}\left(\frac{e^{C\rho}}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right) &= e^{C\rho}\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right)+\frac{Ce^{C\rho}}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2 \\
&\geq 2\frac{d}{d\rho}\left(\int_{B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2\right)+\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-3}}\int_{B_\rho}|\nabla u|^2\right),
\end{aligned}$$

and integration from σ to τ yields the desired result.

Remark:
From equation (9) in the proof of the monotonicity formula we also get the estimate

$$\begin{aligned}
\left|\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right)\right| &\leq C\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2+C\frac{1}{\rho^{2m-3}}\int_{\partial B_\rho}|\nabla u|^2 \\
&+ 2\int_{\partial B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 \\
&\leq C\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2+C\rho\left|\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right)\right| \\
&+ 2\int_{\partial B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 .
\end{aligned}$$

Therefore for ρ so small that $\frac{1}{2}\leq 1-C\rho$ we obtain that

$$\frac{d}{d\rho}\left(\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right)\leq C\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2+C\int_{\partial B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2,$$

i.e. that

$$\frac{d}{d\rho}\left(\frac{e^{-C\rho}}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2\right)\leq C\int_{\partial B_\rho}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 .$$

Integrating this inequality from σ to τ we obtain the estimate

$$\frac{e^{-C\tau}}{\tau^{2m-2}}\int_{B_\tau}|\nabla u|^2-\frac{e^{-C\sigma}}{\sigma^{2m-2}}\int_{B_\sigma}|\nabla u|^2\leq C\int_{B_\tau\setminus B_\sigma}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2, \qquad (10)$$

which we will use in the proof below.

2.2.3 Obtaining a rate of convergence for u

We now give the proof of theorem 1 in the perturbed situation. As above we use the monotonicity formula to deduce that

$$C\int_{B_r}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2\leq\frac{e^{Cr}}{r^{2m-2}}\int_{B_r}|\nabla u|^2-\Theta_u(x_0)<\infty .$$

Contrary to the easy case, this time we cannot slice by J_0-holomorphic planes but we have to slice by J-holomorphic curves passing through $x_0 = 0$. The fact that such curves exist for any almost complex structure J was proven in [18] by the second author and G.Tian (see their Appendix A). There they also proved that such curves form a singular foliation of $B^{2m} \subset \mathbf{R}^{2m}$. It is important to note that the tangent plane to such a J-holomorphic curve is spanned by two vectors X and JX, where near the origin X is only a small perturbation of $\frac{\partial}{\partial R}$. More precisely, we have that any tangent plane is spanned by a vector X for which we have

$$X = \frac{\partial}{\partial R} + O(r)\frac{\partial}{\partial x_i} \ ,$$

i.e. for a map u we get that

$$u \cdot X = \frac{\partial u}{\partial R} + O(r)\frac{\partial u}{\partial x_i} \ .$$

For a J-holomorphic map we will use the monotonicity formula to show that

$$\int_{B_r} R^{2-2m}|u \cdot X|^2 < \infty \ .$$

First note that the remark on X implies that we get the estimate

$$\begin{aligned} \int_0^r d\rho \int_{\partial B_\rho} R^{2-2m}|u \cdot X|^2 \ &\leq \ \int_0^r d\rho \int_{\partial B_\rho} \left[R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 + CR^{3-2m}|\nabla u|^2 \right] \\ &= \ \int_{B_\rho} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 + C\int_{B_\rho} R^{3-2m}|\nabla u|^2 \ . \end{aligned}$$

The bound on the first term follows directly from the monotonicity formula, whence it remains only to bound the second one. This can be done through integration by parts and applying the monotonicity formula to each term:

$$\begin{aligned} \int_0^r d\rho\ \rho^{3-2m} \int_{\partial B_\rho} |\nabla u|^2 \ &= \\ &= \ \left[\rho^{3-2m} \int_{B_\rho} |\nabla u|^2 \right]_0^r + (2m-3)\int_0^r d\rho\ \rho^{2-2m} \int_{B_\rho} |\nabla u|^2 \\ &\leq \ Cr\left[\sup_{0<\rho<1} \frac{1}{\rho^{2m-2}} \int_{B_\rho} |\nabla u|^2 \right] \\ &- \ \lim_{\rho \to 0} \left(\rho \left[\sup_{0<\rho<1} \frac{1}{\rho^{2m-2}} \int_{B_\rho} |\nabla u|^2 \right] \right) < \infty \ . \end{aligned}$$

In the paper by the second author and G. Tian mentioned above it was shown that the J-holomorphic curves foliating B^{2m} are smoothly parametrised by $\mathbf{CP}^{m-1}$. We again take $\pi : B^{2m} \to \mathbf{CP}^{m-1}$ to be this parametrisation. This map π has the property that its gradient is $O(\frac{1}{r})$; precisely, there are constants C_1, $C_2 > 0$ such that first

$$C_1 - C_2 r \leq R^{2m-2}|J_{2m-2}\pi| \leq C_1 + C_2 r$$

(again $J_{2m-2}\pi$ denotes the $(2m-2)$-Jacobian of π) and second $1 \leq \frac{C_1+C_2 r}{C_1-C_2 r} \leq 1+O(r)$. Therefore we can apply the co-area formula to the slicing by π and obtain that

$$\int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)} \frac{R^{2-2m}|u\cdot X|^2}{|J_{2m-2}\pi|} = \int_{B^{2m}} R^{2-2m}|u\cdot X|^2 < \infty \ ,$$

which is what we wanted to show since by Fubini's theorem we now know that for a.e. $p \in \mathbf{CP}^{m-1}$ we have

$$\int_{\pi^{-1}(p)} |u\cdot X|^2 \leq C_2 \int_{\pi^{-1}(p)} \frac{R^{2-2m}|u\cdot X|^2}{|J_{2m-2}\pi|} < \infty \ .$$

Since the metric on the target g_N is compatible with the target complex structure J_N we have that $|u\cdot JX|^2 = |J_N \circ \nabla u(X)|^2 = |\nabla u(X)|^2$ and hence from the above argument we also deduce that $\int_{\pi^{-1}(p)} |u\cdot JX|^2 < \infty$.
Next we want to show how the above implies that $\int_{\pi^{-1}(p)} |\nabla u|^2 < \infty$ for a.e. $p \in \mathbf{CP}^{m-1}$. To see this first note that as X and JX span the tangent space to $\pi^{-1}(p)$

$$\nabla u|_{\pi^{-1}(p)} = (\nabla u\cdot X)X + (\nabla u\cdot JX)JX$$

from which we conclude that

$$|u\cdot X|^2 \leq \frac{1}{2-Cr}|\nabla u|_{\pi^{-1}(p)}|^2 \leq \frac{2+Cr}{2-Cr}|u\cdot X|^2 \tag{11}$$

for some $C > 0$. Therefore we obtain that for r small enough $\int_{\pi^{-1}(p)} |\nabla u|^2$ is bounded by some constant times the integrals in the X and JX directions. Hence we have shown that as in the easy case $u|_{\pi^{-1}(p)}$ is a J-holomorphic $W^{1,2}$-map from $(\pi^{-1}(p), J)$ into (N, J_N).
Since $(\pi^{-1}(p), J)$ is a Riemann surface for each $p \in \mathbf{CP}^{m-1}$ we get that $u|_{\pi^{-1}(p)}$ is smooth and, since $u|_{\pi^{-1}(p)}$ satisfies an equation similar to the $\bar\partial$-equation, that there exists $\delta > 0$ such that for small enough r

$$\int_{\pi^{-1}(p)\cap B_r} |\nabla u|^2 < \delta \int_{\pi^{-1}(p)\cap B_{2r}} |\nabla u|^2 \ . \tag{12}$$

Together with equation (11) this gives the following estimate

$$\begin{aligned}\int_{\pi^{-1}(p)\cap B_r} |u\cdot X|^2 &\le \frac{1}{2-Cr}\int_{\pi^{-1}(p)\cap B_r} |\nabla u|^2 < \frac{\delta}{2-Cr}\int_{\pi^{-1}(p)\cap B_{2r}} |\nabla u|^2 \\ &\le \delta\frac{2+Cr}{2-Cr}\int_{\pi^{-1}(p)\cap B_{2r}} |u\cdot X|^2 \ ,\end{aligned}$$

where we can take r so small that $\delta\frac{2+Cr}{2-Cr} < 1$. Integrating this over $\mathbf{CP}^{m-1}$ we get

$$\int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)\cap B_r} |u\cdot X|^2 < \tilde{\delta}\int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)\cap B_{2r}} |u\cdot X|^2 \tag{13}$$

for some $\tilde{\delta} \in (0,1)$. Applying the co-area formula again we obtain the estimate

$$\begin{aligned}\int_{B_r} |u\cdot X|^2 |J_{2m-2}\pi| &= \int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)\cap B_r} |u\cdot X|^2 \qquad (14)\\ &< \tilde{\delta}\int_{\mathbf{CP}^{m-1}} dp \int_{\pi^{-1}(p)\cap B_{2r}} |u\cdot X|^2 = \tilde{\delta}\int_{B_{2r}} |u\cdot X|^2 |J_{2m-2}\pi| \ .\end{aligned}$$

As in the easy case this gives

$$\int_{B_r} R^{2-2m}|u\cdot X|^2 < \tilde{\delta}\int_{B_{2r}} R^{2-2m}|u\cdot X|^2$$

which implies that there exist $C > 0$ and $\gamma \in (0,1]$ such that when r is sufficiently small

$$\int_{B_r} R^{2-2m}|u\cdot X|^2 \le Cr^\gamma \ .$$

From this we will deduce a rate of convergence for $\int_{B_r} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2$. Note that from $X = \frac{\partial}{\partial R} + O(r)\frac{\partial}{\partial x_i}$ we deduce that

$$\begin{aligned}\int_{B_r} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 &\le \int_{B_r} R^{2-2m}|u\cdot X|^2 \le Cr^\gamma + C\int_{B_r} R^{3-2m}|\nabla u|^2 \\ &\le Cr^\gamma + Cr\sup_{0<\rho<1}\frac{1}{\rho^{2m-2}}\int_{B_\rho}|\nabla u|^2 \\ &\le C(1+\|\nabla u\|^2_{L^2(B_1)})r^\gamma \ ,\end{aligned}$$

where for the last estimate we used the monotonicity formlua.
To finish the proof we will use the remark at the end of the proof of the monotonicity formula. From estimate (10) we get that

$$\frac{e^{-Cr}}{r^{2m-2}}\int_{B_r}|\nabla u|^2 - \Theta_u(x_0) \le C\int_{B_r} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 \le C\Big(1+\|\nabla u\|^2_{L^2(B_1)}\Big)r^\gamma$$

which for r small enough so that $e^{2Cr} \leq 2$ becomes

$$\frac{e^{Cr}}{r^{2m-2}} \int_{B_r} |\nabla u|^2 - e^{2Cr}\Theta_u(x_0) \leq 2C\Big(1 + \|\nabla u\|^2_{L^2(B_1)}\Big) r^\gamma \ .$$

Therefore we get

$$\begin{aligned}
\frac{e^{Cr} - r}{r^{2m-2}} \int_{B_r} |\nabla u|^2 - \Theta_u(x_0) &\leq 2C\Big(1 + \|\nabla u\|^2_{L^2(B_1)}\Big) r^\gamma + \Big(e^{2Cr} - 1\Big)\Theta_u(x_0) \\
&\leq C\Big(1 + \|\nabla u\|^2_{L^2(B_1)}\Big) r^\gamma + C\|\nabla u\|^2_{L^2(B_1)} r \\
&\leq C\Big(1 + \|\nabla u\|^2_{L^2(B_1)}\Big) r^\gamma \ ,
\end{aligned}$$

which completes the proof of theorem 1.

2.3 Uniqueness of tangent maps

In this subsection we show how theorem 1 implies the uniqueness of tangent maps at all $x_0 \in M$.

Proposition 2 *Let u, M, N be as in theorem 1. Then u has a unique tangent map u_{∞,x_0} at all $x_0 \in M$. In particular,*

$$\lim_{r\to 0} u(x_0 + rx)$$

exists weakly in $W^{1,2}$.

Proof:
The existence of tangent maps for u follows with only minor modifications from a paper by J. Li and G. Tian [13]. Now fix $x_0 \in M$ and without loss of generality we can assume that we have chosen coordinates so that $x_0 = 0$. From the proof of theorem 1 we know that there are constants $C_1 > 0$ and $\gamma > 0$ such that

$$\int_{B_r(x_0)} R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 \leq C_1 r^\gamma \ ,$$

where we absorbed the term $(1 + \|\nabla u\|^2_{L^2(B_1)})$ in the constant C_1. Setting $u_{0,r}(x) := u(rx)$ we note that $\int_{B_1} R^{2-2m}\left|\frac{\partial u_{0,r}}{\partial R}\right|^2 = \int_{B_r} R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2$. Thus

$$\int_{B_1} R^{2-2m} \left|\frac{\partial u_{0,r}}{\partial R}\right|^2 \leq C_1 r^\gamma \ .$$

From this we deduce that for any $0 < \sigma < \tau$ sufficiently small we get

$$\begin{aligned}
\|u_{0,\tau} - u_{0,\sigma}\|_{L^2(S^{2m-1})} &\le \int_\sigma^\tau \left\| \frac{\partial u_{0,r}}{\partial r} \right\|_{L^2(S^{2m-1})} dr \\
&\le \left(\int_\sigma^\tau r^{1-\frac{\gamma}{2}} \left\| \frac{\partial u_{0,r}}{\partial r} \right\|^2_{L^2(S^{2m-1})} dr \right)^{\frac{1}{2}} \left(\int_\sigma^\tau r^{-(1-\frac{\gamma}{2})} dr \right)^{\frac{1}{2}} \\
&\le C_2 \tau^{\frac{\gamma}{4}} ,
\end{aligned}$$

where the fact that $\int_\sigma^\tau r^{1-\frac{\gamma}{2}} \left\| \frac{\partial u_{0,r}}{\partial r} \right\|^2_{L^2(S^{2m-1})} dr \le C_2$ follows from an integration by parts. If we now take a subsequence $\sigma_j \to 0$ as $j \to \infty$ such that u_{0,σ_j} converges to some tangent map u_∞ weakly in $W^{1,2}$ but strongly in $L^2(S^{2m-1})$, then by the triangle inequality and the above estimates for any other sequence $\tau_j \to 0$ we have $u_{0,\tau_j} \to u_\infty$ strongly in $L^2(S^{2m-1})$. Hence u_{0,τ_j} converges weakly to u_∞ in $W^{1,2}$ and therefore u_∞ is the unique tangent map.

3 Proofs of theorems 2 and 3

As in the proof of theorem 1, the argument will be local and we can thus assume to be working in a domain in $\mathbf{R}^m$ with the standard Euclidean metric. The proof for a general Riemannian metric only introduces a small perturbation term and the extension of the proof to this case is standard.

3.1 Preliminaries

We will adopt the standard notation of geometric measure theory (see [7]). In this section we will recall some facts about calibrated cycles (of arbitrary dimension and co-dimension in $\mathbf{R}^m$), compute a monotonicity formula for them and investigate the structure of their tangent cones.

Since throughout the paper we are assuming that the calibration ω is at least C^2 (but not necessarily closed), we know that for any point $x_0 \in \mathbf{R}^m$ there exists a neighborhood depending only on the C^2-norm of ω such that in this neighborhood we can write $\omega(x) = \omega_0(x) + \omega_1(x)$ with $\omega_0(x) = \omega(x_0)$ and $\|\omega_1(x)\|_{C^2} = O(|x - x_0|)$.

In the special case where ω is closed and the cycle is integer rectifiable from [7] theorem 5.4.3 (2) we know that C satisfies a monotonicity formula. In the case where C is a normal cycle calibrated by a constant calibration, Harvey and Lawson proved a monotonicity formula depending only on this first order information (see [10] theorem 5.7). Based on their proof we will now show an almost monotonicity formula in the general case.

Proposition 3 *Let C be a k-dimensional normal cycle in $\mathbf{R}^m$. Assume that C is calibrated by a comass 1 k-form ω. Then there exist $C_1 > 0$, $r_0 > 0$ depending only on the C^2-norm of ω such that given $x_0 \in \mathbf{R}^m$ for any $0 < s < r \leq r_0$ we have*

$$\begin{aligned}\frac{e^{C_1 r} + C_1 r}{r^k} M(C \llcorner B_r(x_0)) \quad &- \quad \frac{e^{C_1 s} + C_1 s}{s^k} M(C \llcorner B_s(x_0)) \\ &\geq \int_{B_r \setminus B_s(x_0)} \frac{1}{|x - x_0|^k} \sum_{i=1}^{N(x)} \lambda_i(x) \left| \xi_i(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\|(x) \ ,\end{aligned}$$

where C is represented by $\int \langle \tau(x), \cdot \rangle d\|C\|$ and $\sum_{i=1}^{N(x)} \lambda_i(x) \xi_i(x)$ is the decomposition of $\tau(x)$ into a convex sum of calibrated simple k-vectors.

Proof:
Using the setting established above we write $\omega(x) = \omega_0(x) + \omega_1(x)$ where $\omega_0(x) = \omega(x_0)$ and $\|\omega_1(x)\|_{C^2} = O(|x - x_0|)$. Note that since ω_0 is a constant k-form we know that $\omega_0 = \frac{1}{k} d\left(\frac{\partial}{\partial r} \lrcorner \omega_0\right) = \frac{1}{k} L_{\frac{\partial}{\partial r}} \omega_0$ — here $L_{\frac{\partial}{\partial r}}$ denotes the Lie derivative in the direction of $\frac{\partial}{\partial r}$. Next we take a smooth cut-off function $\phi : \mathbf{R} \to \mathbf{R}$ such that $\phi(t) = 1$ for $t \leq \frac{1}{2}$, $\phi(t) = 0$ for $t \geq 1$ and $\phi'(t) \leq 0$. Setting $\gamma(x) := \phi\left(\frac{r}{\rho}\right)$ (here and subsequently $r = |x|$), $I(\rho) := \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega \rangle d\|C\|$ and

$$\omega^t := \frac{\partial}{\partial r} \lrcorner \left(\frac{\partial}{\partial r} \wedge \omega \right)$$

we make the following computation:

$$\begin{aligned}
kI(\rho) &= k\int_{\mathbf{R}^m} \gamma(x)\langle \tau, \omega_0\rangle d\|C\| + k\int_{\mathbf{R}^m} \gamma(x)\langle \tau, \omega_1\rangle d\|C\| \\
&= \int_{\mathbf{R}^m} \Big\langle \tau, \gamma(x) d\Big(\frac{\partial}{\partial r} \lrcorner \omega_0\Big)\Big\rangle d\|C\| + k\int_{\mathbf{R}^m} \gamma(x)\langle \tau, \omega_0\rangle d\|C\| \\
&= \int_{\mathbf{R}^m} \Big\langle \tau, d\Big(\gamma(x)\Big(\frac{\partial}{\partial r} \lrcorner \omega_0\Big)\Big)\Big\rangle d\|C\| \\
&- \int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{dr}(\gamma(x)) dr \wedge d\Big(\frac{\partial}{\partial r} \lrcorner \omega_0\Big)\Big\rangle d\|C\| + k\int_{\mathbf{R}^m} \gamma(x)\langle \tau, \omega_0\rangle d\|C\| \\
&= \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\Big[\frac{dr}{r} \wedge \frac{\partial}{\partial r} \lrcorner \omega_0\Big]\Big\rangle d\|C\| + k\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\rho}\Big)\langle \tau, \omega_1\rangle d\|C\| \\
&= \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)(\omega_0 - \omega_0^t)\Big\rangle d\|C\| + k\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\rho}\Big)\langle \tau, \omega_1\rangle d\|C\| \\
&= \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\omega\Big\rangle d\|C\| - \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\omega^t\Big\rangle d\|C\| \\
&- \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\omega_1\Big\rangle d\|C\| + \rho\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\omega_1^t\Big\rangle d\|C\| \\
&+ k\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\rho}\Big)\langle \tau, \omega_1\rangle d\|C\| \ .
\end{aligned}$$

Setting $J(\rho) := \int_{\mathbf{R}^m} \langle \tau, \phi(\frac{r}{\rho})\omega^t\rangle d\|C\|$ the above computation can be summarised as

$$\begin{aligned}
-\frac{kI(\rho)}{\rho^{k+1}} + \frac{I'(\rho)}{\rho^k} - \frac{J'(\rho)}{\rho^k} &= \frac{1}{\rho^k}\int_{\mathbf{R}^m} \Big\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)(\omega_1 - \omega_1^t)\Big\rangle d\|C\| \\
&- \frac{k}{\rho^{k+1}}\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\tau}\Big)\langle \tau, \omega_1\rangle d\|C\| \ .
\end{aligned}$$

If $\rho > 0$ is chosen small enough we obtain the following estimate

$$\begin{aligned}
\Big| -\frac{kI(\rho)}{\rho^{k+1}} + \frac{I'(\rho)}{\rho^k} - \frac{J'(\rho)}{\rho^k}\Big| &\leq \\
&\leq \frac{C_1}{\rho^k}\int_{\mathbf{R}^m} \Big|\frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)\Big| \cdot |x| d\|C\| + \frac{C_2}{\rho^{k+1}}\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\tau}\Big)|x| d\|C\| \\
&\leq \frac{C_1}{\rho^{k-1}}\int_{\mathbf{R}^m} \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big) d\|C\| + \frac{C_2}{\rho^k}\int_{\mathbf{R}^m} \phi\Big(\frac{r}{\tau}\Big) d\|C\| \\
&= \frac{C_1}{\rho^{k-1}} I'(\rho) + \frac{C_2}{\rho^k} I(\rho) \\
&= \frac{C_3}{\rho^{k-1}} I(\rho) + C_3\frac{d}{d\rho}\Big(\frac{I(\rho)}{\rho^{k-1}}\Big) \ .
\end{aligned}$$

Using this estimate we have that

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^k}\right) + C_3\frac{I(\rho)}{\rho^k} \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) - C_3\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^{k-1}}\right) ,$$

which for $\rho > 0$ possibly chosen even smaller becomes

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}I(\rho)}{\rho^k}\right) \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) - C_3\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^{k-1}}\right) .$$

This implies that for small ρ we have

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}+C_3\rho}{\rho^k}I(\rho)\right) \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) .$$

Letting ϕ increase to the characteristic function of $(-\infty, 1)$, we obtain that the above inequality continues to hold in the sense of distributions, i.e.

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}+C_3\rho}{\rho^k}M(C\llcorner B_\rho(x_0))\right) \geq \frac{d}{d\rho}\int_{B_\rho(x_0)}\frac{1}{|x|^k}\langle\tau,\omega^t\rangle d\|C\| .$$

Since $\tau(x)$ is calibrated by $\omega(x)$ for $\|C\|$-a.e. $x \in B_\rho(x_0)$, with the help of lemma 5.11 in [10] we can express the integrand on the right-hand side above as a positive quantity:

$$\langle\tau(x), \omega^t(x)\rangle = \sum_{i=1}^{N(x)}\lambda_i(x)\left|\xi_i(x)\wedge\frac{\partial}{\partial r}\right|^2 ,$$

where $\tau(x) = \sum_{i=1}^{N(x)}\lambda_i(x)\xi_i(x)$ and the $\xi_j(x)$ are simple k-vectors calibrated by $\omega(x)$. Integration from $0 < s < r \leq r_0$ finishes the proof of the proposition.

We now look at some implications for tangent cones of calibrated currents. First of all note that the monotonicity formula implies that $r^{-k}M(C\llcorner B_r(x_0))$ is almost increasing and bounded for small enough radii. Hence we know that the density $\Theta(\|C\|, x_0) := \lim_{r\to 0}\alpha(k)^{-1}r^{-k}M(C\llcorner B_r(x_0))$ exists at all $x_0 \in \mathbf{R}^m$. Also, if we dilate the current C around x_0 by setting

$$C_{r,x_0} := (\lambda_*^{r,x_0}T)\llcorner B_1(x_0) ,$$

where λ_*^{r,x_0} means the push-forward by $\lambda^{r,x_0}(x) = \frac{x-x_0}{r}$, then from $M(C_{r,x_0}) = r^{-k}M(C\llcorner B_r(x_0))$ we conclude that $M(C_{r,x_0})$ is uniformly bounded as r tends to 0. From the cycle condition we get that $\partial C_{r,x_0}\llcorner B_1(x_0) = 0$ and hence that $\mathbf{N}(C_{r,x_0})$ is uniformly bounded (see [7] for the notation). Therefore, from the compactness theorem of Federer-Fleming we get that for any sequence of radii $\{r_n\}$ tending to 0, there exists a subsequence $\{r_{n'}\}$ such that as $n' \to \infty$

$$C_{r_{n'},x_0} \xrightarrow{\mathbf{F}} C_{\infty,x_0} \ ,$$

for some normal k-current C_{∞,x_0}. For reasons that will become apparent later, we call such a limiting current C_{∞,x_0} a *tangent cone* to C at x_0. Note that a-priori the limiting object might very well depend on the subsequence chosen and the rest of the paper is devoted to showing that this is not the case, i.e. that there is a unique tangent cone.

We now show that the tangent cones to a calibrated current are still calibrated k-currents. To see this, we first note that the lower semi-continuity of mass under weak convergence implies that

$$\lim_{r\to 0} \frac{M(C \llcorner B_r(x_0))}{r^k} = \lim_{r\to 0} M(C_{r,x_0}) \geq M(C_{\infty,x_0}) \ . \tag{15}$$

However, for currents calibrated by ω the above inequality can be improved to an equality: since $M(C \llcorner B_r(x_0)) = C \llcorner B_r(x_0)(\omega)$ we have

$$M(C_{r,x_0}) = \frac{1}{r^k} C \llcorner B_r(x_0)(\omega) = C_{r,x_0}(r^k (\lambda^{r,x_0})^* \omega) \ ,$$

i.e. that C_{r,x_0} is calibrated by $r^k(\lambda^{r,x_0})^*\omega$. Since $M(C_{r,x_0}) \leq C_1 < \infty$ we conclude that as $r \to 0$

$$|C_{r,x_0}(r^k(\lambda^{r,x_0})^*\omega - \omega_0)| \leq C_1 \| r^k(\lambda^{r,x_0})^*\omega - \omega_0\|_\infty \to 0$$

(recall that ω is in C^2). Therefore we obtain that

$$\lim_{r\to 0} M(C_{r,x_0}) = \lim_{n'\to\infty} C_{r_{n'},x_0}(\omega_0) = C_{\infty,x_0}(\omega_0) \ . \tag{16}$$

Since ω_0 has comass equal to 1, we conclude that $C_{\infty,x_0}(\omega_0) \leq M(C_{\infty,x_0})$. Combining (15) and (16) we deduce that

$$\lim_{n'\to\infty} M(C_{r_{n'},x_0}) = M(C_{\infty,x_0}) = C_{\infty,x_0}(\omega_0) \ ,$$

i.e. that C_{∞,x_0} is calibrated by ω_0. Note that ω_0 is a closed form, so even though calibrated k-cycles may not be (homologically) mass-minimizing their tangent cones always are.

We continue our discussion of tangent cones by looking at the density of a tangent cone at the origin. From the discussion above we get that

$$\begin{aligned}\frac{1}{r^k}\|C_{\infty,x_0}\|(B_r(0)) &= \lim_{n'\to\infty}\frac{1}{r^k}\|C_{r_{n'},x_0}\|(B_r(x_0))\\ &= \lim_{n'\to\infty}\frac{1}{(rr_{n'})^k}\|C\|(B_{rr_{n'}}(x_0))\\ &= \alpha(k)\Theta(\|C\|,x_0)\ ,\end{aligned}$$

where $\alpha(k)$ again denotes the volume of the unit ball in $\mathbf{R}^k$. Thus we conclude that

$$\frac{1}{\alpha(k)r^k}\|C_{\infty,x_0}\|(B_r(0)) = \Theta(\|C_{\infty,x_0}\|,0) = \Theta(\|C\|,x_0)\ .$$

To justify the notion *"tangent cone"* note that currents calibrated by a constant form satisfy a simpler monotonicity formula for $0 < s < r$ (see [10] theorem 5.7 for a proof of this, or go through the proof of proposition 3 without the perturbation term):

$$\begin{aligned}&\frac{1}{r^k}\|C_{\infty,x_0}\|(B_r(0)) - \frac{1}{s^k}\|C_{\infty,x_0}\|(B_s(0))\\ &\qquad = \int_{B_r(0)\setminus B_s(0)}\frac{1}{|x|^k}\sum_{j=1}^{N(x)}\lambda_j(x)\left|\xi_j(x)\wedge\frac{\partial}{\partial r}\right|^2 d\|C_{\infty,x_0}\|\ ,\end{aligned}$$

where as above $\tau^\infty(x) = \sum_{j=1}^{N(x)}\lambda_j(x)\xi_j(x)$. From the identity for the density above we conclude that the right-hand side in the monotonicity formula is equal to 0, i.e. that $\sum_{j=1}^{N(x)}\lambda_j(x)\left|\xi_j(x)\wedge\frac{\partial}{\partial r}\right|^2 = 0$ at $\|C_{\infty,x_0}\|$-a.e. x. Therefore we get that $\tau^\infty(x)\wedge\frac{\partial}{\partial r} = 0$, $\|C_{\infty,x_0}\|$-a.e., which by the homotopy formula (applied to the affine homotopy from λ^{1,x_0} to λ^{r,x_0}) implies that C_{∞,x_0} is a cone.

3.2 Tangent cones of calibrated 2-cycles

In this section we will investigate the support and structure of a tangent cone calibrated by ω_0, where the calibration is 2-dimensional. To do this we recall the structure theorem for constant 2-forms of unit comass on $\mathbf{R}^m$ (see [10] Theorem 7.16, page 79). For such a 2-form ω_0 we know that there are coordinates and an almost complex structure J on a subspace $\mathbf{R}^{2n}$ for some n, such that J is compatible with the Euclidean metric and such that ω_0 is the standard symplectic form for J.
This implies that, without loss of generality, for any $x_0 \in \operatorname{spt}\|C\|$ we can assume that the coordinates are chosen so that $x_0 = 0$, $J = J_0$ and ω_0 is the

standard symplectic form on $\mathbf{R}^{2n} \subset \mathbf{R}^m$ for some n. Therefore, calibrated 2-vectors are 0 in the $\mathbf{R}^{m-2n}$-direction. Using this together with the fact that $\tau^\infty(x) \wedge \frac{\partial}{\partial r}(x) = 0$ for $\|C_{\infty,x_0}\|$-a.e. $x \in \mathbf{R}^m$, we deduce that the set of $x \in \operatorname{spt} \|C_{\infty,x_0}\| \cap \mathbf{R}^{n-2m}$ has $\|C_{\infty,x_0}\|$-measure 0 (because of the above for these x, $\tau^\infty(x) \wedge \frac{\partial}{\partial r}(x) \neq 0$). Thus the support of $\|C_{\infty,x_0}\|$ can be assumed to be contained in $\mathbf{R}^{2n} \subset \mathbf{R}^m$.
Furthermore, since $\tau^\infty(x)$ is calibrated by ω_0, we know that the approximate tangent planes are J_0-holomorphic and thus from $\tau^\infty(x) \wedge \frac{\partial}{\partial r}(x) = 0$ we deduce that $\tau^\infty(x) \wedge J_0 \frac{\partial}{\partial r}(x) = 0$. Hence $\tau^\infty(x) = \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r}(x)$ for $\|C_\infty\|$-a.e. $x \in \mathbf{R}^m$. Thus we immediately conclude the following proposition:

Proposition 4 *Let C_∞ be a tangent cone to C at x_0 which is calibrated by the 2-form ω_{x_0}. Then there are coordinates centered at x_0 such that for any 2-form ψ, $C_\infty(\psi)$ is of the form:*

$$C_\infty(\psi) = \left\langle \phi\,;\psi\left(\frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r}\right)\right\rangle ,$$

where ϕ is a distribution in $\mathcal{D}'(\mathbf{R}^m)$ with support in $\mathbf{R}^{2n} \subset \mathbf{R}^m$.

From the fact that C_∞ is a J_0-holomorphic cone we can deduce more information on the structure of the distribution ϕ above:

Proposition 5 *Let ϕ be the distribution given by the previous proposition. Then ϕ is of the form*

$$\langle \phi\,; f\rangle = \int_0^1 \frac{1}{t}\left\langle \Gamma\,; \int_{H^{-1}(H(\theta))} f(t,\theta)\ d\mathcal{H}^1(\theta)\right\rangle d\mathcal{H}^1(t) ,$$

where $H : S^{2n-1} \to \mathbf{CP}^{n-1}$ is the Hopf fibration and Γ is a distribution on $\mathbf{CP}^{n-1}$ determining ϕ.

Remark:
Note that the two above propositions combined imply that a tangent cone C_∞ to C at x_0 is completely determined by $H_*\partial[C_\infty]$, i.e. if two tangent cones C^1_∞ and C^2_∞ satisfy $H_*\partial[C^1_\infty] = H_*\partial[C^2_\infty]$, then $C^1_\infty = C^2_\infty$. We will make use of this fact when proving that the tangent cone to C at x_0 is unique.
In the case of tangent cones to integral area-minimizing cycles, F. Morgan proved in [14] that then C_∞ is a union of 2-dimensional disks. For J-holomorphic integral $1-1$-cycles a more direct proof of this fact was given by the second author and G. Tian in section 2 of [17].

Proof of proposition 5:
From the fact that C_∞ is a cone we know that $\lambda_{0,r_*}C_\infty = C_\infty$ for any positive r. Thus ϕ also satisfies $\langle \phi\,;f(t,\theta)\rangle = \langle\phi\,;f(rt,\theta)\rangle$ and hence as a distribution ϕ is independent of r, i.e. $\frac{\partial\phi}{\partial r}=0$. From this one immediately deduces that

$$\langle\phi\,;f(t,\theta)\rangle = \int_0^1 \frac{1}{t}\langle\Sigma\,;f(t,\theta)\rangle d\mathcal{H}^1(t)\ ,$$

where Σ is a distribution on $S^{2n-1}\subset\mathbf{R}^{2n}$ and for fixed t we view $f(t,\theta)$ as a function on S^{2n-1}.
It remains to use the fact that C_∞ is also J_0-holomorphic. From this we get that $\langle\Sigma\,;f(t,\theta)\rangle = \langle\Sigma\,;f(t,J_0\theta)\rangle$ and hence that for any $s\in[0,2\pi]$, $\langle\Sigma\,;f(t,\theta)\rangle = \langle\Sigma\,;f(t,e^{is}\cdot\theta)\rangle$, where by $e^{is}\cdot\theta$ we mean the multiplication of each component by e^{is}. Thus Σ is invariant along the fibers of the Hopf fibration given as $H^{-1}(p)$ for $p\in\mathbf{CP}^{n-1}$. It is then easy to check that Σ defines a distribution Γ on $\mathbf{CP}^{n-1}$ given by

$$\left\langle\Gamma\,;\int_{H^{-1}(p)} f(t,\tilde\theta)d\mathcal{H}^1(\tilde\theta)\right\rangle := \left\langle\Sigma\,;\int_{H^{-1}(H(\theta))} f(t,\tilde\theta)d\mathcal{H}^1(\tilde\theta)\right\rangle$$

and the proposition holds.

3.3 Proof of theorem 2 for calibrated J_0-holomorphic 2-cycles

In this part of the paper we prove theorems 2 and 3 in an important special case which we will refer back to when proving the theorems in full generality. The setting we consider now is that C is a 2-dimensional normal cycle in $\mathbf{R}^{2m}$ which is calibrated by the standard symplectic form ω_0 of $\mathbf{R}^{2m}$ — here we order the coordinates x_i in $\mathbf{R}^{2m}$ so that for the standard complex structure J_0 on $\mathbf{R}^{2m}$ we get

$$J_0\cdot\frac{\partial}{\partial x_{2i-1}} = \frac{\partial}{\partial x_{2i}}\ ,\qquad \omega_0 = \sum_{i=1}^m dx_{2i-1}\wedge dx_{2i}\ .$$

For a calibrated cycle C with tangent vector τ we thus have that $\tau(x) = \sum_{j=1}^{N(x)}\lambda_j(x)\xi_j(x)$, where the ξ_j are simple 2-vectors calibrated by ω_0, i.e. where the $\xi_j(x)$ can be written as $\xi_j(x) = v_j(x)\wedge J_0 v_j(x)$ (this is an immediate consequence of Wirtinger's inequality, see [7] or [10]).
The proof of the theorems heavily depends on the map $\pi:(\mathbf{R}^{2m},J_0)\to(\mathbf{CP}^{m-1},j_0)$ which we already used for the proof of theorem 1 (note that π is the radial extension of the Hopf fibration). We begin by proving the following lemma:

Lemma 1 *Let τ be in the convex hull of simple vectors calibrated by ω_0 on $\mathbf{R}^{2m}$, i.e. $\tau = \sum_{j=1}^{N} \lambda_j \xi_j$ with ξ_j calibrated by ω and $\sum_{j=1}^{N} \lambda_j = 1$, $0 \leq \lambda_j \leq 1$. Then there exists a constant $C_{2m} > 0$ depending only on the dimension $2m$ such that for any vector $\zeta \in \mathbf{R}^{2m}$ we have*

$$\sum_{j=1}^{N} \lambda |\xi_j \wedge \zeta|^2 \leq \sum_{j=1}^{N} \lambda |\xi_j \wedge \zeta \wedge J_0 \zeta| \leq C_{2m} \sum_{j=1}^{N} \lambda |\xi_j \wedge \zeta|^2 \ .$$

Proof:
Clearly it suffices to prove the lemma when τ is a single simple vector calibrated by ω_0. In this case we know that $\tau = \xi_1 \wedge J_0 \xi_1$ for ξ_1 of unit length by Wirtinger's inequality (see [7]). Since ξ_1 and $J_0\xi_1$ are orthonormal we can extend them to an ordered orthonormal basis $\{\xi_1, J_0\xi_1, \xi_2, J_0\xi_2, \ldots, \xi_m, J_0\xi_m\}$ of $\mathbf{R}^{2m}$. Writing an arbitrary vector $\zeta = \sum_{i=1}^{2m} a_i \xi_i$ in this basis we get that

$$|\tau \wedge \xi|^2 = \left| \sum_{l=1}^{2m} a_l \tau \wedge \xi_l \right|^2 = \left| \sum_{l=3}^{2m} a_l \xi_1 \wedge J_0 \xi_1 \wedge \xi_l \right|^2 = \sum_{l=3}^{2m} |a_l|^2 \ .$$

Next we compute $|\tau \wedge \zeta \wedge J_0\zeta|$. Writing $\zeta = \sum_{k=1}^{m} (a_{2k-1}\xi_{2k-1} + a_{2k}\xi_{2k})$ and $J_0\zeta = \sum_{l=1}^{m} (a_{2l-1}\xi_{2l} - a_{2l}\xi_{2l-1})$ we obtain

$$\begin{aligned}
|\tau \wedge \zeta \wedge J_0\zeta| &= \Bigg| \xi_1 \wedge J_0\xi_1 \wedge \Bigg[\sum_{k,l=1}^{m} (a_{2k-1}a_{2l-1}\xi_{2k-1} \wedge \xi_{2l} - \\
&\quad -(a_{2k-1}a_{2l})\xi_{2k-1} \wedge \xi_{2l-1} + (a_{2k}a_{2l-1})\xi_{2k} \wedge \xi_{2l} - \\
&\quad -(a_{2k}a_{2l})\xi_{2k} \wedge \xi_{2l-1} \Bigg]\Bigg) \\
&\leq C_{2m} \left| \sum_{3 \leq l < k}^{2m} a_l^2 \xi_1 \wedge J_0\xi_1 \wedge \xi_l \wedge \xi_k \right| \\
&\leq C_{2m} \sum_{3=l}^{2m} |a_l|^2 \ .
\end{aligned}$$

The other estimate is done in a similar way.

In the next lemma we apply the above result to estimate the mass of $\pi_*[C \llcorner B_r \setminus B_s(x_0)]$ for small enough radii.

Lemma 2 *Let C be a normal 2-cycle in $\mathbf{R}^{2m}$ which is calibrated by the standard symplectic form ω_0. Then there exist constants $C_1 > 0$ and $r_0 > 0$ such* *that for* $0 < s < r \leq r_0$ *and* $x_0 \in \mathrm{spt}\, \|C\|$ *we have*

$$M(\pi_*[C \llcorner B_r \setminus B_s(x_0)]) \leq C_1 \left[\frac{M(C \llcorner B_r(x_0))}{r^2} - \frac{M(C \llcorner B_s(x_0))}{s^2} \right] .$$

Proof:
Without loss of generality we can assume that the coordinates are centered at x_0. Let $r_0 > 0$ be the one obtained in the monotonicity formula. Applying the previous lemma for $\zeta = \frac{\partial}{\partial r}$ we deduce that

$$\begin{aligned}
\int_{B_r(0) \setminus B_s(0)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \\
&\leq \int_{B_r(x) \setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j(x) \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right| d\|C\| \\
&\leq C_{2m} \int_{B_r(x) \setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \ .
\end{aligned}$$

Combining this with the monotonicity formula we obtain that

$$\begin{aligned}
\int_{B_r(x) \setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j(x) \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right| d\|C\| \\
&\leq C_{2m} \left[\frac{M(C \llcorner B_r(0))}{r^2} - \frac{M(C \llcorner B_s(0))}{s^2} \right] .
\end{aligned}$$

Since π equals $H \circ \frac{x}{|x|}$ we deduce that

$$\langle \tau(x), \bigwedge^2 \pi(x) \rangle \cong \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right|$$

and the proof of the lemma is completed.

We now show how we can prove uniqueness of tangent cones (i.e. theorem 2) in this special case with the help of the above two lemmas. Note that since $r^{-2} M(C \llcorner B_r(x_0))$ is increasing in r, the above lemma implies that

$$\lim_{r \to 0} \lim_{s \to 0} M(\pi_*[C \llcorner B_r \setminus B_s(x_0)]) = 0 \ . \tag{17}$$

Now suppose that $\{r_i\}$ and $\{s_i\}$ are two sequences converging to 0, where we can assume that $s_i < r_i \leq r_0$ for all i. Also assume that the sequences are chosen so that for two tangent cones C^1_{∞,x_0}, C^2_{∞,x_0}

$$C_{r_i,x_0} \xrightarrow[i \to \infty]{\mathbf{F}} C^1_{\infty,x_0} \quad \text{and} \quad C_{s_i,x_0} \xrightarrow[i \to \infty]{\mathbf{F}} C^2_{\infty,x_0} \ .$$

We now want to show that $C^1_{\infty,x_0} = C^2_{\infty,x_0}$. To see this we use that the map π can be seen as a composition of the maps $\frac{x}{|x|} : \mathbf{R}^{2m} \to S^{2m-1}$ and $H : S^{2m-1} \to \mathbf{CP}^{m-1}$ (the Hopf map), i.e. $\pi = H \circ \frac{x}{|x|}$. Then (17) implies that

$$\lim_{i\to\infty} M\left(H_*\left(\frac{x}{|x|}_* [C \llcorner B_{r_i} \setminus B_{s_i}(x_0)]\right)\right) = 0 \ ,$$

which immediately gives

$$H_*\left(\frac{x}{|x|}_* [C \llcorner B_{r_i} \setminus B_{s_i}(x_0)]\right) \xrightarrow[i\to\infty]{w} 0$$

and hence that

$$H_*\left(\partial\frac{x}{|x|}_* [C \llcorner B_{r_i} \setminus B_{s_i}(x_0)]\right) \xrightarrow[i\to\infty]{w} 0 \ . \tag{18}$$

Since for almost all $0 < s < r$ we have

$$\partial\frac{x}{|x|}_* [C \llcorner B_r \setminus B_s(x_0)] = \partial\lambda_*^{r,x_0} C \llcorner B_r(x_0) - \partial\lambda_*^{s,x_0} C \llcorner B_s(x_0) \ ,$$

from line (18) we deduce that

$$H_*\left(\partial\lambda_*^{r_i,x_0} C \llcorner B_{r_i}(x_0) - \partial\lambda_*^{s_i,x_0} C \llcorner B_{s_i}(x_0)\right) \xrightarrow[i\to\infty]{w} 0 \ .$$

From the choice of sequences $\{r_i\}$ and $\{s_i\}$ it now follows that for the tangent cones C^1_{∞,x_0} and C^2_{∞,x_0}we have

$$H_*\big(\partial C^1_{\infty,x_0}\big) = H_*\big(\partial C^2_{\infty,x_0}\big) \ .$$

For integer rectifiable currents the result now follows immediately, since the boundaries of their tangent cones are unions of great circles contained in J-holomorphic planes (see the remark after proposition 5). For normal currents this follows from proposition 5. Thus the proof of theorem 2 is completed for this special case.

3.4 Obtaining a rate for integer rectifiable 2-cycles calibrated by ω_0

This section is devoted to the proof of theorem 3 for integer rectifiable 2-cycles calibrated by ω_0. From lemma 2 above we know that it suffices to prove a rate for $M(\pi_* C \llcorner B_r(x_0))$ since

$$\begin{aligned}\frac{M(C\llcorner B_r(x_0))}{r^2} - \pi\Theta(\|C\|, x_0) &= \int_{B_r(x_0)} \frac{1}{|x-x_0|^2}\left|\tau\wedge\frac{\partial}{\partial r}\right|^2 d\|C\| \\ &\leq M(\pi_* C\llcorner B_r(x_0))\ .\end{aligned}$$

From now on, by ω we will denote the standard symplectic 2-form which is compatible with the metric on $(\mathbf{CP}^{m-1}, j_0)$. We will deduce the rate for $M(\pi_* C\llcorner B_r(x_0))$ from a rate for

$$\int_{B_r(x_0)} \pi^*\omega|_C \, d\|C\|\ .$$

First note that, from the monotonicity formula, for all $0 < r \leq r_0$

$$\begin{aligned}\int_{B_r(x_0)} \pi^*\omega|_C \, d\|C\| &= \pi_*[C\llcorner B_r(x_0)](\omega) \\ &\leq \|\omega\|\, M(\pi_* C\llcorner B_r(x_0)) \\ &\leq C_1\left[\frac{M(C\llcorner B_{r_0}(x_0))}{{r_0}^2} - \pi\Theta(\|C\|, x_0)\right] < +\infty\ .\end{aligned}$$

Furthermore, since the map π is J_0-j_0-holomorphic and C is J_0 holomorphic, we know that pointwise ($\|C\|$-a.e.) we have $|\nabla\pi|_C|^2(x) = \pi^*\omega|_C(x)$, i.e. that for all $0 < r \leq r_0$ we have

$$\int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq C_1 < +\infty\ .$$

As mentioned in the remark after proposition 5, since C is an integral cycle the tangent cones to C at x_0 are unions of Q J_0-holomorphic disks (here $Q = \Theta(\|C\|, x_0)$). Arguing like in the proof of lemma III.1 (the part to obtain equation III.3) in the paper [17] by the second author and G. Tian from our theorem 2 one can deduce the following lemma proved in [17]:

Lemma 3 *Let C be an integer rectifiable 2-cycle calibrated by ω_0. Let $x_0 \in \operatorname{spt}\|C\|$ and let $D_1, \ldots, D_Q$ be 2-disks calibrated by ω_0 such that $C_{\infty,x_0} = \bigoplus_{i=1}^Q D_i$. Then given $\epsilon > 0$ there exists $r_\epsilon > 0$ such that for any $0 < r \leq r_\epsilon$ and any $\psi \in C_0^\infty\big(\bigwedge^2\left(B_1^{2m}\setminus\{x\in B_1^{2m}\ :\ \operatorname{dist}(x, \cup_i D_i) \leq \epsilon|x|\}\right)\big)$ we have*

$$C_{r,x_0}(\psi) = 0\ .$$

Remark:
It is important to note that the proof of this lemma in [17] only depends on

uniqueness of tangent cones, the structure of tangent cones and the monotonicity formula and not on the epiperimetric inequality by B. White [24]. We will give a proof of this lemma in a more general setting in the last section of this paper.

From the lemma it immediately follows that there exists a cone K centered at x_0 such that for $r > 0$ small enough we have

$$M(\pi_*[C \llcorner B_r(x_0) \cap K]) = 0$$

and hence there exists a small ball B in $\mathbf{CP}^{m-1}$ such that $B \subset\subset \pi(K)$ and $(\pi_*[C \llcorner B_r(x_0)]) \llcorner B = 0$. Therefore we can find a smooth 1-form α on $\mathbf{CP}^{m-1}$ such that $\omega = d\alpha$ on $\mathbf{CP}^{m-1} \setminus B$ — recall that ω generates $H^2(\mathbf{CP}^{m-1}; \mathbf{R})$, i.e. that the above steps are necessary to obtain the existence of α.

Before proving the theorem we need one more lemma:

Lemma 4 *For C as in the previous lemma, let $\omega = d\alpha$ on $\mathbf{CP}^{m-1} \setminus B$. Given $0 < r \leq r_0$, there exists $\rho_0 \in \left[\frac{r}{2}, r\right]$ such that the following hold:*

1. $M(\langle C, |\cdot|, \rho_0\rangle) \leq C_1 \rho_0$;
2. $\int_{\mathbf{R}^{2m}} |\nabla \pi|_C|^2 d\langle C, |\cdot|, \rho_0\rangle \leq \frac{1}{\rho_0} \int_{B_r(x_0)\setminus B_{\frac{r}{2}}(x_0)} |\nabla \pi|_C|^2 d\|C\|$,

where $\langle C, |\cdot|, \rho_0\rangle$ denotes the slice current of C (see [7] chapter 4.3).

Proof:
From [7] 4.2.1 and the monotonicity formula we know that there exists $C_1 > 0$ such that

$$\begin{aligned} \int_{\frac{r}{2}}^{r} \frac{1}{\rho} M(\langle C, |\cdot|, \rho\rangle) d\rho &\leq \frac{2}{r} M(C \llcorner B_r \setminus B_{\frac{r}{2}}) \leq \frac{2}{r} M(C \llcorner B_r) \\ &\leq C_1 \frac{r}{2} , \end{aligned}$$

i.e.

$$⨍_{\frac{r}{2}}^{r} \frac{1}{\rho} M(\langle C, |\cdot|, \rho\rangle) d\rho \leq C_1 .$$

Also, setting $\phi(r) := \int_{B_r(x_0)\setminus B_{\frac{r}{2}}(x_0)} |\nabla \pi|_C|^2 d\|C\|$ we get that

$$\int_{\frac{r}{2}}^{r} \frac{r}{\phi(r)} \int_{\mathbf{R}^{2m}} |\nabla \pi|_C|^2 d\langle C, |\cdot|, \rho\rangle d\rho \leq \frac{r}{\phi(r)} \phi(r) = r .$$

Therefore we obtain an estimate for the following average integral:

$$\fint_{\frac{r}{2}}^{r}\left[\frac{1}{\rho}M(\langle C,|\cdot|,\rho\rangle)+\frac{r}{\phi(r)}\int_{\mathbf{R}^{2m}}|\nabla\pi|_C|^2 d\langle C,|\cdot|,\rho\rangle\right]d\rho\leq C_2\ ,$$

which implies that there exists $\rho_0\in\left[\frac{r}{2},r\right]$ with

$$\frac{1}{\rho_0}M(\langle C,|\cdot|,\rho_0\rangle)+\frac{r}{\phi(r)}\int_{\mathbf{R}^{2m}}|\nabla\pi|_C|^2 d\langle C,|\cdot|,\rho_0\rangle\leq C_2$$

and the lemma follows immediately.

Now we are in position to prove the theorem. From the fact that $\omega=d\alpha$ on the support of $\pi_*[C\llcorner B_r(x_0)]$, $\partial C=0$, the construction of ρ_0 above and the Cauchy-Schwarz inequality we derive the following chain of inequalities:

$$\begin{aligned}
\int_{B_{\frac{r}{2}}(x_0)}|\nabla\pi|_C|^2 d\|C\| \ &\leq\ \int_{B_{\rho_0}(x_0)}|\nabla\pi|_C|^2 d\|C\|=\int_{B_{\rho_0}(x_0)}\pi^*\omega|_C d\|C\|\\
&=\ \int_{\mathbf{R}^{2m}}\pi^*\alpha|_C\, d\langle C,|\cdot|,\rho_0\rangle\\
&\leq\ C_1\int_{\mathbf{R}^{2m}}|\pi|_C-\bar{\pi}|_C|\ |\nabla\pi|_C|\, d\langle C,|\cdot|,\rho_0\rangle\\
&\leq\ C_1\left(\int_{\mathbf{R}^{2m}}|\pi|_C-\bar{\pi}|_C|^2 d\langle C,|\cdot|,\rho_0\rangle\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^{2m}}|\nabla\pi|_C|^2 d\langle C,|\cdot|,\rho_0\rangle\right)^{\frac{1}{2}}.
\end{aligned}$$

Having obtained this inequality we now use the fact that $\langle C,|\cdot|,\rho_0\rangle$ is a 1-dimensional integral cycle. From the decomposition theorem for integral 1-cycles (see [7] 4.2.25) we deduce that one can write $\langle C,|\cdot|,\rho_0\rangle$ as $\langle C,|\cdot|,\rho_0\rangle=\sum_{i=1}^{\infty}T_i$ where each T_i is an indecomposable 1-cycle and $M(\langle C,|\cdot|,\rho_0\rangle)=\sum_{i=1}^{\infty}M(T_i)$. Furthermore, for each $i\in\mathbf{N}$ there exists $f_i:\mathbf{R}\to\mathbf{R}^{2m}$ with $\mathrm{Lip}(f_i)\leq 1$ such that $T_i=f_{i*}[[0,M(T_i)]]$. Using these facts one can apply Poincaré's inequality on each of the pieces, where because of $\mathrm{Lip}(f_i)\leq 1$ we can take the constants in the estimate equal to 1 independent of i. Thus we get

$$\begin{aligned}
\left(\int_{\mathbf{R}^{2m}}|\pi|_C-\bar{\pi}|_C|^2 d\langle C,|\cdot|,\rho_0\rangle\right)^{\frac{1}{2}}\ &=\ \left(\sum_{i=0}^{\infty}\int_{\mathbf{R}^{2m}}|\pi|_C-\bar{\pi}|_C|^2 d\|T_i\|\right)^{\frac{1}{2}}\\
&\leq\ \left(\sum_{i=0}^{\infty}M(T_i)^2\int_{\mathbf{R}^{2m}}|\nabla\pi|_C|^2 d\|T_i\|\right)^{\frac{1}{2}}\\
&\leq\ M(\langle C,|\cdot|,\rho_0\rangle)\left(\int_{\mathbf{R}^{2m}}|\nabla\pi|_C|^2 d\langle C,|\cdot|,\rho_0\rangle\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining the above inequalities with lemma 4 we get that

$$\begin{aligned}\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| &\leq M(\langle C, |\cdot|, \rho_0\rangle) \int_{\mathbf{R}^{2m}} |\nabla\pi|_C|^2 d\langle C, |\cdot|, \rho_0\rangle \\ &\leq C_1\rho_0 \frac{1}{\rho_0} \int_{B_r(x_0)\setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| ,\end{aligned}$$

which immediately implies that

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq \frac{C}{C+1} \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| .$$

Now a standard iteration argument (see the book by M. Giaquinta [8] for details) shows that there exist $C_1 > 0$ and $\gamma \in (0, 1]$ with

$$\int_{B_r(x_0)} \pi^*\omega|_C \, d\|C\| = \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq C_1 r^\gamma .$$

Recall that π is J_0-j_0-holomorphic and hence that $\pi_*[C \llcorner B_r(x_0)]$ is a j_0-holomorphic 2-current (but not necessarily a cycle). Therefore $\pi_*[C \llcorner B_r(x_0)]$ is calibrated by ω so that the above estimate implies

$$M(\pi_*[C \llcorner B_r(x_0)]) = \pi_*[C \llcorner B_r(x_0)](\omega) = \int_{B_r(x_0)} \pi^*\omega|_C \, d\|C\| \leq C_1 r^\gamma ,$$

which together with the monotonicity formula completes the proof of theorem 3 in this special case.

Remark:
There are two steps in the above proof which (for the time being) cannot be carried out for calibrated normal cycles in general. The one is lemma 3, the other the decomposition theorem for integer rectifiable 1-cycles which we used to show that Poincaré's inequality was valid. It would be interesting to know whether one can nevertheless still use the above strategy to prove a rate of convergence.

3.5 Uniqueness of tangent cones for calibrated 2-cycles

The aim of this section is to prove theorem 2 in the general case. The setting we work in is described as follows: first fix $x_0 \in \operatorname{spt}\|C\|$ and assume that we have chosen coordinates so that $\omega_0 = \omega(x_0)$ is the standard symplectic form on $\mathbf{R}^{2n} \subset \mathbf{R}^m$ (see section 3.2). Furthermore, we can assume that we work in a small ball $B_{r_0}(0)$ so that on this ball we have $\|\omega(x) - \omega_0\|_{C^2} = O(|x|)$ and the almost monotonicity formula holds true.

Using this setting we prove the following lemma:

Lemma 5 *If $x \in \operatorname{spt}\|C\|$ and if $\tau(x)$ is calibrated by $\omega(x) = \omega_0 + \omega_1(x)$, with ω_0 as above, then we can write $\tau(x)$ as $\tau_0(x)+\tau_1(x)$ so that $\|\tau_0(x)\| = 1$, $\omega_0(\tau_0(x)) = 1$ and $\|\tau_1(x)\| = O(|x|^{\frac{1}{2}})$.*

Proof:
Since $\tau(x)$ is calibrated by $\omega(x)$ we can write $\tau(x) = \sum_{j=1} \lambda_j(x)\xi_j(x)$, where each $\xi_j(x)$ is again calibrated by $\omega(x)$. To show the lemma it therefore clearly suffices to show the lemma for calibrated simple vectors $\xi(x)$. From the fact that $\omega(x)$ calibrates $\xi(x)$ and $\omega(x) = \omega_0 + \omega_1(x)$ we deduce that $\omega_0(\xi(x)) = 1 + O(|x|)$. Hence ξ is not perpendicular to $\mathbf{R}^{2n} \subset \mathbf{R}^m$. To construct a simple 2-vector ξ_0 close to ξ we first orthogonally project $\xi(x)$ onto $\mathbf{R}^{2n} \subset \mathbf{R}^m$ to obtain a simple 2-vector $\tilde{\xi}(x)$. Then we get that $\omega_0(\tilde{\xi}(x)) = 1 + O(|x|)$ and that therefore $\|\xi(x) - \tilde{\xi}(x)\| = O\Big(|x|^{\frac{1}{2}}\Big)$. Setting $\bar{\xi}(x) := \frac{\tilde{\xi}}{\|\tilde{\xi}\|}$ we know that there are orthonormal vectors $v(x)$, $w(x) \in \mathbf{R}^{2n} \subset \mathbf{R}^m$ such that $\bar{\xi}(x) = v(x) \wedge w(x)$. Then the 2-vector $\xi_0(x) := v(x) \wedge J_0 v(x)$ is of mass 1 and calibrated by ω_0. Thus it remains to show that $\|\xi(x) - \xi_0\| = O\Big(|x|^{\frac{1}{2}}\Big)$. To see this note that it suffices to estimate $\|\tilde{\xi}(x) - \xi_0\|^2 = \|[\|\tilde{\xi}\| w(x) - J_0 v(x)]\|^2$. This follows from $\omega_0(\tilde{\xi}(x)) = 1 + O(|x|)$ where we deduce that $\|[\|\tilde{\xi}\| w(x) - J_0 v(x)]\|^2 = O(|x|)$ and the proof of the lemma is completed.

Combining this lemma with lemma 1 we immediately obtain the following estimate:

Corollary 1 *For x and $\tau(x) = \tau_0(x) + \tau_1(x)$ as in the lemma above there exists $C_1 > 0$ such that for any vector $\zeta \in \mathbf{R}^{2n} \subset \mathbf{R}^m$ we have*

$$\begin{aligned} |\tau \wedge \zeta|^2 &\leq 2|\tau_0 \wedge \zeta|^2 + 2|\tau_1 \wedge \zeta|^2 \\ &\leq 2|\tau_0 \wedge \zeta \wedge J_0\zeta| + 2|\tau_1 \wedge \zeta|^2 \\ &\leq C_1|\tau_0 \wedge \zeta|^2 + 2|\tau_1 \wedge \zeta|^2 \\ &\leq C_1|\tau \wedge \zeta|^2 + (C_1 + 2)|\tau_1 \wedge \zeta|^2 \ . \qquad (19) \end{aligned}$$

As in the easy case, from line 19 in the corollary we get the following estimate:

$$\int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j^0(x) \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right| d\|C\|$$
$$\leq C_{2m} \int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left[\left| \xi_j(x) \wedge \frac{\partial}{\partial r} \right|^2 + \left| \xi_j^1(x) \wedge \frac{\partial}{\partial r} \right|^2 \right] d\|C\|$$

(here $\tau_i(x) = \sum_{j=0}^{N(x)} \lambda(x) \xi_j^i(x)$ for $i = 1, 2$ are as in lemma 5). The first term on the right-hand side can again be estimated by the almost monotonicity formula. For the term on the right-hand side we will use lemma 5 to show:

Lemma 6 *For C and $\tau^1(x)$ as above there exist $C_1 > 0$ and $r_0 > 0$ such that $0 < s < r \leq r_0$ implies*

$$\int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j^1(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \leq C_1 r \ .$$

Proof:
First note that by lemma 5 there exists $C_1 > 0$ with

$$\int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|^2} \sum_{j=1}^{N(x)} \lambda(x) \left| \xi_j^1(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \leq C_1 \int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|} d\|C\| \ ,$$

for $0 < s < r \leq r_0$. From 2.5.18 (3) in [7] and the almost monotonicity formula we get the following inequalities:

$$\begin{aligned}
\int_{B_r(x)\setminus B_s(x)} \frac{1}{|x|} d\|C\| &\leq \int_s^r \frac{1}{\rho} \frac{d}{d\rho} \left[\|C\|(B_\rho(0)) \right] d\rho \\
&\leq \frac{1}{\rho} \|C\|(B_\rho(0)) \Big|_s^r + \int_s^r \frac{1}{\rho^2} \left[\|C\|(B_\rho(0)) \right] d\rho \\
&\leq \rho \sup_{0<\rho\leq r_0} \frac{\|C\|(B_\rho(0))}{\rho^2} \Big|_s^r + \sup_{0<\rho\leq r_0} \frac{\|C\|(B_\rho(0))}{\rho^2} (r-s) \\
&\leq C_1 (r-s) \ ,
\end{aligned}$$

establishing the lemma.

Therefore, we again deduce that

$$\lim_{r\to 0} \lim_{s\to 0} M(\pi_*[C \llcorner B_r \setminus B_s(x_0)]) = 0 \ ,$$

where this time we extended the map π to $\pi : \mathbf{R}^{2n} \times \mathbf{R}^{m-2n} \to \mathbf{CP}n-1 \times \mathbf{R}^{m-2n}$ by sending $(x_1, \ldots, x_{2n}, x_{2n+1}, \ldots, x_m)$ to $(\pi(x_1, \ldots, x_{2n}), 0)$. From the last part of section 3.2 we know that tangent cones to C are entirely determined by their $\mathbf{R}^{2n}$-components and hence the uniqueness of tangent cones follows by exactly the same arguments as in section 3.3.

3.6 Proof of theorem 3

Using arguments similar to the ones from the previous section we now prove theorem 3 in the full generality stated. In section 3.4 we used the exact monotonicity formula to deduce a rate of convergence from a rate of convergence for $M(\pi_* C \llcorner B_r(x_0))$. In the general case we have to use an estimate like estimate (10) in the perturbation argument for J-holomorphic maps, i.e. there exist $C_1 > 0$ and $r_0 > 0$ with

$$\begin{aligned} \frac{e^{-C_1 r}}{r^2} M(C \llcorner B_r(x_0)) &- \frac{e^{-C_1 s}}{s^2} M(C \llcorner B_s(x_0)) \\ &\leq C_1 \int_{B_r(x_0)} \frac{1}{|x-x_0|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \ . \qquad (20) \end{aligned}$$

This estimate follows from the proof of the monotonicity formula in the same fashion as estimate (10) for J-holomorphic maps. Using lemmas 5 and 6 we can see that

$$\int_{B_r(x_0)} \frac{1}{|x-x_0|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \leq M(\pi_* C \llcorner B_r(x_0)) + C_1 r \ ,$$

and hence that a rate of convergence would still follow from a rate of convergence for $M(\pi_* C \llcorner B_r(x_0))$.

Since lemma 4 is still valid in this case (its proof depended only on the monotonicity formula and general estimates from [7]), it remains only to show two steps. The one is to prove lemma 3 in this context, the other to show that the fact that C is not exactly J_0-holomorphic near x_0 does not matter. We begin by deducing the latter from lemma 7 stated below; in fact, denoting C

with orientation $\tau_0(x)$ by C_0, we have

$$\begin{aligned}\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| &\leq \int_{B_{\rho_0}(x_0)} |\nabla\pi|_C|^2 d\|C\| \\ &\leq \int_{B_{\rho_0}(x_0)} \pi^*\omega|_{C_0}\, d\|C\| + C_1'\rho_0 \int_{B_{\rho_0}(x_0)} |\nabla\pi|_C|^2 d\|C\| \\ &\leq \int_{B_{\rho_0}(x_0)} d\pi^*\alpha|_{C_0}\, d\|C\| + C_1' r \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| \\ &\leq \int_{B_{\rho_0}(x_0)} d\pi^*\alpha|_C\, d\|C\| + C_1 r \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| \ .\end{aligned}$$

Now the same strategy as in section 3.4 applies to show

$$\int_{B_{\frac{r}{2}(x_0)}} |\nabla\pi|_C|^2\, d\|C\| \leq \frac{C_1 + C_1 r}{C_1 + 1} \int_{B_r(x_0)} |\nabla\pi|_C|^2\, d\|C\| \ ,$$

which (having chosen r small enough) implies the theorem the same way as before.
To complete the proof of theorem 3 it remains to show the following lemma which is a part of lemma III.1 in [17]. Their proof, although given for calibrated J-holomorphic currents, also works in the general case and we include it for the sake of completeness.

Lemma 7 *Let C be an integer rectifiable 2-cycle calibrated by ω. Let $x_0 \in \operatorname{spt}\|C\|$ and let $D_1, \ldots, D_Q$ be 2-disks calibrated by ω_0 such that $C_{\infty,x_0} = \bigoplus_{i=1}^Q D_i$. Then given $\epsilon > 0$ there exists $r_\epsilon > 0$ such that for any $0 < r \leq r_\epsilon$ and any $\psi \in C_0^\infty\big(\bigwedge^2 \left(B_1^m \setminus \{x \in B_1^m \ : \ \operatorname{dist}(x, \cup_i D_i) \leq \epsilon|x|\}\right)\big)$ we have*

$$C_{r,x_0}(\psi) = 0 \ .$$

Proof:
For an argument by contradiction suppose the above statement is false. Then there exists $\varepsilon_0 > 0$, a sequence $r_n \to 0$ and $\psi_n \in \bigwedge^2(B_1^m)$ with

$$\operatorname{spt}\psi_n \subset E_0^c \ ,$$

where $E_0 := \{x \in B_1^m \ : \ \operatorname{dist}\cup_i D_i \leq \varepsilon_0|x|\}$, and yet

$$C_{r_n,0}(\psi_n) \neq 0 \ .$$

Then there exist $x_n \in E_0^c$ with $\lim_{r\to 0} M(C_{r,x_n}) \neq 0$ and from the monotonicity formula we get that $M\big(C_{r_n|x_n|,0} \llcorner B_{\frac{\varepsilon_0}{2}}\big(\frac{x_n}{|x_n|}\big)\big) \geq \frac{\varepsilon_0^2}{4}\pi$. Taking a subsequence so that $\frac{x_n}{|x_n|} \to x_\infty$ (with $x_\infty \in E_0^c$) we have that

$$M\big(C_{r_n|x_n|,0} \llcorner B_{\frac{3\varepsilon_0}{4}}(x_\infty)\big) \geq \frac{\varepsilon_0^2}{4}\pi \ .$$

Since we also have

$$\Big|C_{r_n|x_n|,0} \llcorner B_{\frac{3\varepsilon_0}{4}}(x_\infty)\big((r_n|x_n|)^2\lambda^{r_n|x_n|,0^*}\omega - \omega_0)\big)\Big|$$
$$\leq\ M(C_{r_n|x_n|,0} \llcorner B_{\frac{3\varepsilon_0}{4}}(x_\infty))\Big\|(r_n|x_n|)^2\lambda^{r_n|x_n|,0^*}\omega - \omega_0\Big\|_\infty \to 0\ ,$$

we conclude that we would get

$$C_{\infty,0} \llcorner B_{\frac{3\varepsilon_0}{4}}(x_\infty)(\omega_0) \geq \frac{\varepsilon_0^2}{4}\pi\ ,$$

which contradicts the fact that $C_{\infty,0}$ is the union of Q ω_0-calibrated disks, i.e. has support in E_0. Hence the lemma holds true.

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