Energy Quantization for Harmonic Maps

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0 Introduction

Let M, N be smooth, compact Riemannian manifolds without boundary. Suppose $u : M \to N$ is a smooth harmonic map such that the homotopy class, $[u]$, of u is not trivial. Then, it follows easily from the small energy regularity theorem of Schoen-Uhlenbeck (cf. [Sc]) that the total energy of the map u:

$$E(u) = \int_M |\nabla u|^2 \, dx \geq \varepsilon_0(M, N) > 0. \tag{0.1}$$

We should view the estimate (0.1) as the simplest energy quantization phenomena in harmonic maps. A more subtle statement is the following well-known theorem of Sacks-Uhlenbeck [SaU]: let $0 \neq \alpha \in \pi_2(N)$, and let u_i, $i = 1, 2, \ldots$, be a sequence of minimizers of $\int_{\mathbb{S}^2} (1 + |\nabla u_i|^2)^{p_i} \, dx$ with $[u_i] = \alpha$, and $p_i \to 1^+$ as $i \to \infty$. Then (by choosing subsequences if necessary) *either* $\{u_i\}$ converges strongly in $H^1(\mathbb{S}^2, N)$ to a smooth harmonic map u: $\mathbb{S}^2 \to N$ such that $[u] = \alpha$, *or* $\{u_i\}$ converges only weakly in $H^1(\mathbb{S}^2, N)$, and there is a nonconstant, smooth harmonic map $v : \mathbb{S}^2 \to N$ (obtained from a blow-up of u_i's near a point of $\mathbb{S}^2$) such that $E(v) \leq \underline{\lim}_i E(u_i)$.

In any case, one obtains, in particular, that the constant $\varepsilon_0(M, N)$ in the estimate (0.1) can be replaced by the twice of the least possible area among all (so-called incompressible) minimal spheres in N. The Sacks-Uhlenbeck's theorem has since then been generalized and improved to the case of pseudo-holomorphic curves, see [PW], [Y]; to the case of smooth harmonic maps from a Riemannian surface [J], [P]; and to the case of heat-flow and approximate harmonic maps, see [Q], [W] and [DT]. Naturally all those works require the domain dimension to be two. There is also a natural generalization of such statement for conformal invariant variational integrals in higher dimensions, see [MW] and the recent interesting work of Duzzar-Kuwert [DK] in which they obtained much refined statement than that of [SaU] for arbitrary energy minimizing sequences among continuous maps in given homotopy classes.

The situation becomes much more difficult to understand when the domain dimensions are larger than the conformal dimensions of variational integrals. For examples, for harmonic maps, the problem becomes much harder when the domain dimension $n \geq 3$. The

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only earlier works known to authors which might have something to do with this higher dimensional quantization phenomena are the work of Giaquinta-Modica-Souček [GMS] and that of Bethuel-Brezis-Coron [BBC]. In [L], one of the author established similar results as [SaU] for energy minimizing sequences among continuous maps in a given homotopy class. More precisely, let $\{u_i\} \in C^0(M,N)$ be an energy minimizing sequence in the homotopy class $\alpha = [u_i]$, $i = 1, 2, \ldots$. Then (by taking subsequences if needed) $u_i \rightharpoonup u$ weakly in $H^1(M,N)$, and $|\nabla u_i|^2\,dx \rightharpoonup \mu = |\nabla u|^2\,dx + \nu$ as Radon measures. Here u is a weakly harmonic map from M into N that is smooth away from a closed, rectifiable set of Σ of $H^{n-2}(\Sigma) < \infty$; and here $\nu \geq 0$ is a Radon measure of the form $\Theta^{n-2}(\nu,x)\,\mathcal{H}^{n-2}\lfloor\Sigma$, with $\varepsilon_0(M,N) \leq \Theta^{n-2}(\nu,x) \leq C(\alpha,M,N)$. The crucial fact proved in [L] is that, for $\mathcal{H}^{n-2}$–a.e. $x \in \Sigma$, $\Theta^{n-2}(\nu,x) = \inf\{E(u) : u \in H^1(\mathbb{S}^2,N),\ [u] = \alpha_x \in \pi_2(N)\}$ for some nontrivial class $\alpha_x \in \pi_2(N)$. Moreover,

$$\Theta^{n-2}(\nu,x) = \sum_{j=1}^{\ell_x} E(\phi_j), \tag{0.2}$$

for some smooth, nonconstant harmonic maps $\phi_j : \mathbb{S}^2 \to N$ such that ϕ_j, $j = 1, \ldots, \ell_x$, is energy minimizing in the homotopy class $[\phi_j]$, and that $[\phi_j]$'s give simply a decomposition of the class $\alpha_x \in \pi_2(N)$ (see [DK] also).

On the other hand, let $\{u_i\}$ be a sequence of stationary harmonic maps from M into N such that $u_i \rightharpoonup u$ weakly in $H^1(M,N)$. Then $|\nabla u_i|^2\,dx \rightharpoonup |\nabla u|^2\,dx + \nu$ as Radon measures. Here u is a weakly harmonic map from M into N which is smooth away from a closed, rectifiable set $\Sigma \subseteq M$ with $\mathcal{H}^{n-2}(\Sigma) < \infty$; and here

$$\nu = \Theta^{n-2}(\nu,x)\,\mathcal{H}^{n-2}\lfloor\Sigma \tag{0.3}$$

satisfies $\varepsilon_0 \leq \Theta^{n-2}(\nu,x) < \infty$ (see [L2]). Naturally, one believes that $\Theta^{n-2}(\nu,x)$, for $\mathcal{H}^{n-2}$–a.e. x, is a sum of energies of harmonic maps from $\mathbb{S}^2$ into N. This is the issue the present article will address. We shall see the analytic issues involved are rather subtle and one of it is very much like our recent work concerning quantization phenomena for 3-D Ginzburg-Landau equations [LR]. To simplify the presentation, we shall assume through out this paper that N is the standard k-dimensional sphere (and in some of the results below, $k = 2$). One of the main results of the paper is the following.

Theorem A. *Let N be the standard k-dimensional sphere, $k \geq 2$, and let $\nu = \Theta^{n-2}(\nu,x)\,\mathcal{H}^{n-2}$ be as in* (0.3). *Then for $\mathcal{H}^{n-2}$–a.e. $x \in \Sigma$,*

$$\Theta^{n-2}(\nu,x) = \sum_{j=1}^{\ell_x} E(\phi_j),$$

where, for $j = 1, \ldots, \ell_x$ ϕ_j is a smooth, nonconstant harmonic map from $\mathbb{S}^2$ into $\mathbb{S}^k = N$.

The conclusion of Theorem A can also be viewed as a higher dimensional version of the "energy identity" for a weakly convergent sequence of harmonic maps which were shown earlier by Jost [J] and Parker [P] (see also [DT], [Q], [W] and [LW]); In the case N is the standard sphere $\mathbb{S}^2$, then we have

Theorem B. *Let $u : \mathbb{R}^3 \to \mathbb{S}^2$ be a stationary harmonic map. Then*

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_r(0)} |\nabla u|^2 \, dx = 8\pi n, \tag{0.4}$$

$n = 0, 1, 2, \ldots, +\infty$. If u is, in addition, smooth, then

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_r(0)} |\nabla u|^2 \, dx = 16\pi n, \tag{0.5}$$

$n = 0, 1, 2, \ldots, +\infty$.

The above result remains true when one studies the tangent map of u at $\underline{0}$ (i.e. one studies the limit as $r \downarrow 0$).

We note that if $u : \mathbb{R}^2 \to \mathbb{S}^2$ is a smooth harmonic map, then [SaU] says that

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = 8\pi n, \quad \text{for} \quad n = 0, 1, 2, \cdots + \infty.$$

1 Proof of Theorem B

We start with the following proposition.

Proposition 1. *Let N be a smooth, compact Riemannian manifold without boundary, and let $u : \mathbb{R}^3 \to N$ be an entire stationary harmonic map with bounded normalized energy:*

$$\sup_{r>0} \frac{1}{r} \int_{B_r(0)} |\nabla u|^2 dx \le \Lambda < \infty. \tag{1.1}$$

Then any tangent map, u_0 at infinity of u can be described as follows:

(i) There is a sequence $R_i \to +\infty$ such that $u_i(x) \equiv u(R_i x) \rightharpoonup u_0(x) \equiv u_0\left(\frac{x}{|x|}\right)$ weakly in $H^1_{loc}(\mathbb{R}^3, N)$, where $u_0 : \mathbb{S}^2 \to N$ is a smooth harmonic map.

(ii) $\mu_i = |\nabla u_i|^2(x)\, dx \rightharpoonup \mu = |\nabla u_0|^2(x)\, dx + \nu$, as Radon measures on $\mathbb{R}^3$. Where the defect measure ν is a nonnegative Radon measure supported on finitely many rays emitted from the origin. Moreover on each ray ν is a constant multiple of the one-dimensional Hausdorff measure.

(iii) There is a balancing condition on energy density of u_0 and masses of ν on various rays:

$$\int_{\mathbb{S}^2} x \cdot |\nabla u_0(x)|^2 + \sum_{i=1}^{\ell} \Theta(x_i)\, x_i = 0. \tag{1.2}$$

Here $x_i \in \mathbb{S}^2$, $S = \bigcup_{i=1}^{\ell} \overline{Ox_i}$, and $\nu = \sum_{i=1}^{\ell} \Theta(x_i)\, \mathcal{H}^1 \lfloor \overline{Ox_i}$.

Proof. Let u be a stationary harmonic map from $\mathbb{R}^3$ into N such that $\sup_{r>0} \frac{1}{r} \int_{B_r(0)} |\nabla u|^2 dx = \Lambda < \infty$.

We recall first the energy monotonicity formula (cf. [Sc]) for stationary maps in $\mathbb{R}^3$:

$$\frac{1}{R}\int_{B_R(0)} |\nabla u|^2 dx - \frac{1}{r}\int_{B_r(0)} |\nabla u|^2 dx = 2\int_{B_R(0)} \frac{1}{\rho}\left|\frac{\partial u}{\partial \rho}\right|^2 dx, \tag{1.3}$$

for all $0 < r < R < \infty$. In particular,

$$\Lambda = \sup_{r>0} \frac{1}{r}\int_{B_r(0)} |\nabla u|^2 dx = \lim_{r\to+\infty} \frac{1}{r}\int_{B_r(0)} |\nabla u|^2\, dx. \tag{1.4}$$

which is finite by our assumption.

Next, for any sequence of $\lambda_i \to +\infty$, we may find a subsequence of $\{u(\lambda_i, x)\}$ that weakly converges in $H^1_{\text{loc}}(\mathbb{R}^3, N)$ to a map $u_0(x)$. By taking subsequences if needed, we may assume that $u_i(x) = u(\lambda_i\, x)$, $u_i \to u_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^3, N)$, and that $|\nabla u_i|^2 dx = \mu_i \rightharpoonup \mu = |\nabla u_0|^2 dx + \nu$ as Radon measures. It follows from (1.3)–(1.4) that

$$\int\limits_{\mathbb{R}^3\setminus B_R(0)} \frac{1}{\rho}\left|\frac{\partial u}{\partial \rho}\right|^2 dx \to 0 \quad \text{as} \quad R\to\infty. \tag{1.5}$$

We thus obtain for a.e. $0 < r < R < \infty$, that

$$\begin{aligned} \frac{1}{R}\mu\left(B_R(0)\right) - \frac{1}{r}\mu\left(B_r(0)\right) &= \lim_{i\to\infty}\left\{\frac{1}{R}\mu_i\left(B_R(0)\right) - \frac{1}{r}\mu_i\left(B_r(0)\right)\right] \\ &= \lim_{i\to\infty} \int\limits_{B_{\lambda_i R}(0)\setminus B_{\lambda_i r}(0)} \frac{1}{\rho}\left|\frac{\partial u}{\partial \rho}\right|^2 dx = 0. \end{aligned} \tag{1.6}$$

and

$$\frac{1}{R}\mu_0\left(B_R(0)\right) = \Lambda. \tag{1.7}$$

In particular, for $0 < r < R < \infty$,

$$\int\limits_{B_R(0)\setminus B_r(0)} \frac{1}{\rho}\left|\frac{\partial u_0}{\partial \rho}\right|^2 dx < \underline{\lim}_i \int\limits_{B_R(0)\setminus B_r(0)} \frac{1}{\rho}\left|\frac{\partial u_i}{\partial \rho}\right|^2 dx = 0. \tag{1.8}$$

i.e., $\frac{\partial u_0}{\partial \rho} \equiv 0$ or equivalently $u_o(x) = u_0\left(\frac{x}{|x|}\right)$.

Next, we let $\phi = \mathbb{S}^2 \to \mathbb{R}_+$ be a smooth function, and let $\psi \in C_0^\infty(0,1)$ be such that $\psi \geq 0$ and $\int_0^1 \psi(t)\, dt = 1$. We consider, for $0 < a < \infty$, $0 < \varepsilon \ll a$, the functions

$$\begin{aligned} &E(u,\phi,a,\varepsilon) \\ &= \int_0^\infty \int_{\mathbb{S}^2} \left[(r+a)^2\left|\frac{\partial u_i}{\partial r}\right|^2 + \left|\frac{\partial u_i}{\partial \theta}\right|^2\right](r+a,\theta)\ \phi(\theta)\cdot\psi_\varepsilon(r)\, d\theta dr, \end{aligned} \tag{1.9}$$

here $\psi_\varepsilon(r) = \frac{1}{\varepsilon}\,\psi\left(\frac{r}{\varepsilon}\right)$. Then a direct computation using the fact that

$$\text{div}\ \left[\delta_{kl}|\nabla u_i|^2 - 2D_k\, u_i\, D_l\, u_i\right] = 0 \text{ in the sense of distribution} \tag{1.10}$$

we obtain

$$\begin{aligned}\frac{d}{da}\,E\,(u_i,\phi,a,\varepsilon) &= 2\frac{d}{da}\int_0^\infty\int_{\mathbb{S}^2}(r+a)^2\left|\frac{\partial u_i}{\partial\, r}\right|^2(r+a,\theta)\,\phi(\theta)\,\psi_\varepsilon(r)\,d\theta dr\\ &\quad+2\int_0^\infty\int_{\mathbb{S}^2}(r+a)\left|\frac{\partial u_i}{\partial r}\right|^2(r+a\;\theta)\,\phi(\theta)\,\psi_\varepsilon(r)\,d\theta dr\\ &\quad-\int_0^\infty\int_{\mathbb{S}^2}\frac{\partial}{\partial r}\,u_i\,\frac{\partial}{\partial\theta}\,u_i\,(r+a\;\theta)\left(\frac{\partial}{\partial\theta}\,\phi(\theta)\right)\psi_\varepsilon(r)\,d\theta dr.\end{aligned} \tag{1.11}$$

Integration both side of (1.11) with respect to $a\in(r,R)$, we get

$$\begin{aligned}&E(u_i,\phi,R,\varepsilon)-E(u_i,\phi,r,\varepsilon)\\ &=\int_0^\infty\int_{\mathbb{S}^2}2(\rho+a)^2\left|\frac{\partial u_i}{\partial\rho}\right|^2(\rho+a,\theta)\,\phi(\theta)\,\psi_\varepsilon(\rho)\,d\theta d\rho\Bigg|_{a=r}^{a=R}\\ &\quad+2\int_0^\infty\int_r^R\int_{\mathbb{S}^2}(\rho+a)\left|\frac{\partial u_i}{\partial\rho}\right|^2(\rho+a,\theta)\,\phi(\theta)\,\psi_\varepsilon(\rho)\,d\theta da d\rho\\ &\quad-\int_0^\infty\int_r^R\int_{\mathbb{S}^2}2\,\frac{\partial u_i}{\partial\rho}\cdot\frac{\partial u_i}{\partial\theta}\,(\rho+a,\theta)\,\phi_\theta(\theta)\,\phi_\varepsilon(\rho)\,d\theta da d\rho\end{aligned} \tag{1.12}$$

Now let $\varepsilon\to0^+$, we obtain for a.e. $0<r<R<\infty$, that

$$\begin{aligned}&\int_{\mathbb{S}^2}\left[R^2\left|\frac{\partial u_i}{\partial\rho}\right|^2+\left|\frac{\partial u_i}{\partial\theta}\right|^2\right](R,\theta)\;\phi(\theta)\,d\theta\\ &\quad-\int_{\mathbb{S}^2}\left[r^2\left|\frac{\partial u_i}{\partial\rho}\right|^2+\left|\frac{\partial u_i}{\partial\theta}\right|^2\right](r,\theta)\,\phi(\theta)\,d\theta\\ &=2\int_{\mathbb{S}^2}R^2\left|\frac{\partial u_i}{\partial\rho}\right|^2(R,\theta)\,\phi(\theta)\,d\theta-\int_{\mathbb{S}^2}r^2\left|\frac{\partial u_i}{\partial\rho}\right|^2(r\theta)\,\phi(\theta)\,d\theta\\ &\quad+\int_r^R\int_{\mathbb{S}^2}2\rho\left|\frac{\partial}{\partial\rho}\,u_i\right|^2(\rho,\theta)\,\phi(\theta)\,d\theta d\rho\\ &\quad-\int_r^R\int_{\mathbb{S}^2}2\,\frac{\partial}{\partial\rho}\,u_i\,\frac{\partial}{\partial\theta}\,u_i(\rho,\theta)\,\frac{\partial\phi}{\partial\theta}\,d\theta d\rho.\end{aligned} \tag{1.13}$$

We write $|\nabla u_i|^2\,dx=\left[\rho^2\left|\frac{\partial u_i}{\partial\rho}\right|^2+\left|\frac{\partial u_i}{\partial\theta}\right|^2\right](\rho,\theta)\,d\theta d\rho=d\sigma_i(\rho,\theta)\,d\rho$, then (1.13) implies that

$$\int_{\mathbb{S}^{n-1}}\phi(\theta)\,d\sigma_i(R,\theta)-\int_{\mathbb{S}^{n-1}}\phi(\theta)\,d\sigma_i(r,\theta)\to0 \tag{1.14}$$

as $i\to\infty$. On the other hand, since $\mu_i=|\nabla u_i|^2\,dx\rightharpoonup\mu$ as Radon measures on $(0,\infty)\times\mathbb{S}^2$, i.e., $d\sigma_i(\rho,\theta)\,d\sigma\rightharpoonup d\mu$. We may conclude, from (1.14), the translation invariance property of the weak limit of $d\sigma_i(\rho,\theta)$ in ρ to obtain $d\mu = d\sigma(\theta)\,d\rho$. Note that the Radon measure $d\sigma(\theta)$ can be obtained also as a weak limit of measures $d\sigma_i(r_i,\theta)$ for a suitable sequence in r_i's.

What we have shown is that the Radon measure μ has the property that $\mu_\lambda \equiv \mu$, for any $\lambda > 0$. Here $\mu_\lambda(A) = \frac{\mu(\lambda A)}{\lambda}$.

Finally, since $\mu = |\nabla u_0|^2\,dx + v$, and since $v = \Theta(x)\,H^1\lfloor S$ (by [L2]), we conclude that S is consistent of finitely many rays emitted from the origin, and $\Theta(x)$ is a constant on each ray. In the next section we shall show all such constant $\Theta(x)$ must be of the form $\sum_{j=1}^{\ell} E(\phi_j)$, for some smooth harmonic maps ϕ_j, $j = 1,\ldots,\ell_x$ from $\mathbb{S}^2$ into N.

Finally, since each u_i is a stationary harmonic map, thus a simple first variation gives

$$\int_{\mathbb{S}^2} x\,|\nabla u_i|^2\,(x)\,dx = 2\int_{\mathbb{S}^2}\frac{\partial u_i}{\partial r}\cdot\nabla u_i(x). \tag{1.15}$$

Pass to the limit $i \to +\infty$, we obtain

$$\int_{\mathbb{S}^2} x\cdot|\nabla u_0(x)|^2 + \sum_{i=1}^{\ell}\Theta(x_i)\,x_i = 0.$$

Here $x_i \in \mathbb{S}^2$ and $\bigcup_{i=1}^{\ell}\overrightarrow{0x_i} = S$. This complete the proof of proposition. □

The statement of Theorem B now simply follows from the above proposition and Theorem A. Indeed, from Theorem A, one concludes that each $\Theta(x_i)$ is a sum $\sum_{j=1}^{\ell_i} E(\phi_{ij})$, for some harmonic maps $\phi_{ij} : \mathbb{S}^2 \to N = \mathbb{S}^k$. Hence if $N = \mathbb{S}^2$, then $\lim_{r\to\infty}\frac{1}{r}\int_{B_r(0)}|\nabla u|^2\,dx = E(u_0) + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell_i} E(\phi_{ij}) = 8\pi n$, for a nonnegative integer n.

Suppose the starting map u is also smooth, then it is easy to see that

$$\deg(u_0) + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell_i}\deg(\phi_{ij}) = 0.$$

Here deg denotes the degree of maps into $\mathbb{S}^2$ into itself. Thus $\sum_{i=1}^{\ell}\sum_{j=1}^{\ell_i}|\deg\phi_{ij}| + |\deg u_0|$ is an even number, and so

$$E(u_0) + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell_i} E(\phi_{ij}) = 16\pi\,n, \quad \text{for some nonnegative integer } n.$$

The remaining of the present article will be devoted to the proof of Theorem A.

2 Set up and Basic estimates

Let $\nu = \Theta^{n-1}(\nu,x)\,\mathcal{H}^{n-2}\lfloor\Sigma$ be as in (0.3). Then, by [L2, §2], one has for $\mathcal{H}^{n-2}$–a.e. $x_0 \in \Sigma$, Σ has a unique classical tangent plane P_0 at x_0, and ν has the tangent measure at x_0 given by $\Theta^{n-2}(\nu,x_0)\,\mathcal{H}^{n-2}\lfloor P_0$. For simplicity, we take one of such x_0 to be the origin, and let P_0 be the $(n-2)$-dimensional plane $\mathbb{R}^{n-2}\times\{(0,0)\} \subseteq \mathbb{R}^n$. We note that the measure $\Theta^{n-2}(\nu,x_0)\,\mathcal{H}^{n-2}\lfloor P_0$ can be obtained from a sequence of stationary harmonic maps in the sense that, there is a weakly convergent sequence of stationary maps $\{u_i\}$ such that u_i's converge to a constant map weakly in $H^1_{\text{loc}}(\mathbb{R}^n, N)$ and such that $|\nabla u_i|^2(x)\,dx \rightharpoonup \Theta^{n-2}(\nu, x_0)\,\mathcal{H}^{n-2}\lfloor P_0$ as Radon measures on $\mathbb{R}^n$. In particular, if we restrict each map to $Q_1 = \{x \in \mathbb{R}^n : |(x_1, \ldots, x_{n-2})| < 1,\ |(x_{n-1}, x_n)| < 1\}$, we have $u_i \rightharpoonup$ a constant weakly in $H^1(Q_1, N)$ and $|\nabla u_i|^2\,dx \rightharpoonup \Theta^{n-2}(\nu, x_0)\,\mathcal{H}^{n-2}\lfloor\Sigma_0$ weakly as Radon measures on Q_1. Here $\Sigma_0 = P_0 \cap Q_1$. Moreover, by discussions in [L2, §2] one has

$$\int_{Q_1} \sum_{k=1}^{n-2} \left|\frac{\partial u_i}{\partial x_k}\right|^2 (x)\,dx \to 0 \quad \text{as} \quad i \to +\infty. \tag{2.1}$$

To prove Theorem A, it suffices to verify that such $\Theta^{n-2}(\nu, x_0)$ is a finite sum of energies of smooth harmonic maps from $\mathbb{S}^2$ into N.

Let us recall how the first harmonic map from $\mathbb{S}^2$ into N is obtained from the blow-up analysis in [L2]. We let $X_1 = (x_1, \ldots, x_{n-2})$, $X_2 = (x_{n-1}, x_n)$,

$$f_i(X_1) = \int_{B_1^2(0)} \sum_{k=1}^{n-2} \left|\frac{\partial u_i}{\partial x_k}\right|^2 (X_1, X_2)\,dX_2, \quad X_1 \in B_1^{n-2}(0). \tag{2.2}$$

Then Fubini's theorem and (2.1) imply that $f_i \to 0$ in $L^1\left(B_1^{n-2}(0)\right)$. By partial regularity theorem of Bethuel [B] for stationary harmonic maps (when $N = \mathbb{S}^K$, see [E]), one has, for $\mathcal{H}^{n-2}$-a.e. $X_1 \in B_1^{n-2}(0)$, $u_i(x)$ is smooth near all points of $(X_1, X_2) \in B_1^{n-2}(0) \times B_1^2(0)$. By the weak-$L^1$ estimate for Hardy-Littlewood maximal functions, one can easily find a sequence of point $\{X_1^i\}$, $i = 1, 2, \ldots$, such that

$$u_i(x) \text{ is smooth near all points } (X_1^i, X_2) \in B_1^{n-2}(0) \times B_1^2(0); \tag{2.3}$$

and that

$$\sup_{0<r\le\frac{1}{2}} r^{2-n} \int_{B_r^{n-2}(X_1^i)} f_i(X_1)\,dX_1 \to 0 \quad \text{as } i \to +\infty. \tag{2.4}$$

As in [L2, §2], we may find, for all i sufficiently large, $\delta_i \in (0, 1/2)$ and $X_1^i \in B^2_{1/4}(0)$ that the maximum value of

$$\max_{X_2 \in B^2_{1/2}(0)} \delta_i^{n-2} \int_{B^{n-2}_{\delta_i}(X_1^i) \times B^2_{\delta_i}(X_2)} |\nabla u_i|^2(X_1, X_2)\,dX_1\,dX_2 = \varepsilon_0/c(n), \tag{2.5}$$

is achieved at X_2^i. Here ε_0 is a small constant appeared in the small energy regularity theorem of [B] and [E], and that $c(n)$ is a suitable large dimensional constant (see [L2, §3]). Note the $\delta_i \to 0^+$ as $i \to +\infty$. Consider the sequence of new maps $v_i(y) = u_i(X^i + \delta_i\,y)$, $X^i = (X_1^i, X_2^i)$, then it was shown in [L2, §3] that

$$v_i \to v \quad \text{in} \quad C^{1,\alpha}_{\text{loc}}\left(\mathbb{R}^{n-2} \times \mathbb{R}^2\right) \quad \text{as} \quad i \to +\infty. \tag{2.6}$$

Moreover, $v(y) = v(y_{n-1}, y_n)$ is a non-constant, smooth harmonic maps from $\mathbb{R}^2$ into N with a finite total energy. This gives rise to the first harmonic map ϕ_1 from $\mathbb{S}^2$ into N.

The following lemma is the step where we require N to be a standard sphere $\mathbb{S}^k$.

Lemma 1. *Let $u \in H^1(Q_1, \mathbb{S}^k)$ be a weakly harmonic map, then for any $0 < r < 1$, there is a $f \in \mathcal{H}_a\left(\mathbb{R}^n.\mathbb{R}^{k+1}\right)$ such that*

$$-\Delta u = f \quad in \quad Q_r = B_r^{n-2}(0) \cap B_r^2(0).$$

Here $\mathcal{H}_a$ denotes the Hardy space.

For the proof of Lemma 1 see [E] and [H, Lemma 3.2.10]. Note $\|f\|_{\mathcal{H}_a} \le C(r,n)\,\Lambda$ which follows from proofs in [E] and [H].

As a consequence of the above lemma is the following.

Corollary 2. *Let $\{u_i\}$ be a sequence of weakly harmonic maps from Q_1 into $\mathbb{S}^k$ such that $E(u_i) \le \Lambda$, for each i. Then*

$$\|u_i\|_{W^{2,1}(Q_{2/3})} \le C(n,k)\,\Lambda. \tag{2.7}$$

Since for each $m \ge 2$, then space $W^{1,1}(\mathbb{R}^m)$ is continuous embedded in the Lorentz space $L^{\left(\frac{m}{m-1},1\right)}(\mathbb{R}^m)$ see [H, Theorem 3.3.10] and [T]), we obtained from (2.7) and the Fubini Theorem that

$$\|\nabla u_i\|_{L^{(2,1)}(B^2_{2/3}(0))}(X_1, \cdot) \le C^*(n,k)\,\Lambda \tag{2.8}$$

for all $X_1 \in E_i \subset B^{n-2}_{2/3}(0)$, $i = 1, 2, \ldots,$. Here we may assume $\mathcal{L}^{n-2}(E_i) \ge 0.99\ \mathcal{L}^{n-2}(B^{n-2}_{2/3}(0))$.

On the other hand, we may also find a set $F_i \subset B^{n-2}_{2/3}(0)$ such that $\mathcal{L}^{n-2}(F_i) \ge 0.99\ \mathcal{L}^{n-2}(B^{n-2}_{2/3}(0))$ and that for all $X_1^i \in F_i$, X_1^i satisfies (2.3) and (2.4). From now on we should always work on "good slices" $\{X_1 = X_1^i\}$ such that (2.3), (2.4) as well as the estimate (2.8) valid on such slices, $X_1^i \in F_i \cap E_i$.

Let $\overline{u}_i(X_2) = u_i(X_1^i, X_2)$ be the restriction of u_i on a "good slice" $X_1^i \in E_i \cap F_i$. From [L2, §3], it is easy to see that for $\mathcal{L}^{n-2}$-a.e. $X_1^i \in E_i \cap F_i$,

$$\lim_i E(\overline{u}_i) = \lim_i \int_{B^2_{1/2}(0)} |\nabla u_i|^2(X_1^i, X_2)\,dX_2 = \Theta^{n-2}(\nu, x_0). \tag{2.9}$$

Thus, in order to prove Theorem A, we must verify that

$$\lim_i E(\overline{u}_i) = \sum_{j=1}^m E(\phi_j), \tag{2.10}$$

where $\phi_j : \mathbb{S}^2 \to \mathbb{S}^k$ are smooth harmonic maps, and $E(\phi_j) = \int_{\mathbb{S}^2} |\nabla \phi_j|^2$, for $j = 1, \ldots, m$.

Let δ_i be as in (2.5) , and $X_2^i \to 0$ as $i \to +\infty$ so that the maximum value (2.5) is achieved at X_2^i. Let $v_i(X_2) \equiv u_i\left(X_1^i, X_2^i + \delta_i(X_2 - X_2^i)\right)$. Then as (2.6) one has that v_i converges strongly in $H^1_{\text{loc}}(\mathbb{R}^2, N)$ to a nonconstant harmonic map $\Psi_1 : \mathbb{R}^2 \to \mathbb{S}^k$ with a finite total energy. Then $\phi_1 = \Psi_1 \circ \Pi$ is the first harmonic map from $\mathbb{S}^2$ into $\mathbb{S}^k$. Here $\Pi : \mathbb{S}^2 \setminus \{\text{north pole}\} \to \mathbb{R}^2$ is the sterographic projection.

Let $A(R,i) = \left\{X_2 \in \mathbb{R}^2 : \delta_i R \le |X_2 - X_2^i| \le \frac{1}{2}\right\}$, then (2.10) is equivalent to

$$\lim_{R\to\infty} \lim_{i\to\infty} E\left(u_i, A(R,i)\right) = \lim_{R\to\infty} \lim_{i\to\infty} \int_{A(R,i)} |\overline{\nabla}\, u_i|^2(X_1^i, X_2)\,dX_2 = \sum_{j=2}^m E(\phi_j). \tag{2.11}$$

Here $\overline{\nabla} = \nabla_{X_2}$.

For simplicity, we will assume $X_1^i = \mathbf{0} \in \mathbb{R}^{n-2}$ and $X_2^i = \mathbf{0} \in \mathbb{R}^2$. Let $(X_1, r, \theta) \in \mathbb{R}^{n-2} \times R_+ \times \mathbb{S}^1$ be the cylindrical coordinates of $\mathbb{R}^n$ and, let $W_i(X_1, t, \theta) = u_i(X_1, r, \theta)$, $r = e^{-t}$. Then $E(u_i, A(R, i)) = E(W_i, B_i)$ where $B_i = [\log 2, |\log \delta_i R|] \times \mathbb{S}^1$. we observe that $W_i = B_{1/2}^{n-2}(0) \times B_i \to \mathbb{S}^k$ is a stationary harmonic map and it satisfies the equation:

$$\frac{\partial^2}{\partial t^2} W_i + \frac{\partial^2}{\partial \theta^2} W_i + A(W_i)(\nabla W_i, \nabla W_i) = -e^{-2t} \nabla_{X_1} W_i \tag{2.12}$$

here $\nabla W_i = \left(e^{-t} \nabla_{X_1} W_i, \frac{\partial}{\partial t} W_i, \frac{\partial}{\partial \theta} W_i\right)$.

Let $Q(0, M) = [\log 2, \log 2 + M] \times \mathbb{S}^1$ and

$$Q(i, M) = [|\log \delta_i R| - M, |\log \delta_i R|] \times \mathbb{S}^2, \quad \text{for} \quad M > 0.$$

Then it is rather easy to check that

$$E(W_i, Q(0, M)) \to 0, \qquad E(W_i, Q(i, M)) \to 0, \tag{2.13}$$

as $i \to \infty$ and $R \to \infty$, for any given $M > 0$.

3 Proof of Theorem A

We are going to prove Theorem A by inducting on the numbers of harmonic spheres that arise in the blow-up analysis. That is the number m in the sum of the right-hand side of (2.11). We consider first the case $m = 1$.

In that case we must show that $E(W_i, B_i) \to 0$ as $i \to \infty$ and $R \to \infty$. We claim first that for any $\varepsilon_1 > 0$, there are sufficiently large R and I such that, for all $i \geq I$, one has

$$e^{(n-2)t} \int_{|X_1| \leq e^{-t}} \int_t^{t+1} \int_{\mathbb{S}^1} |\nabla W_i|^2 (X_1, t, \theta)\, d\theta\, dt\, dX_1 < \varepsilon_1, \tag{3.1}$$

for all $t \in [\log 2\, |\log \delta_i R| - 1]$.

Indeed, if the claim is false, then we may assume that, as $i \to \infty$, there exists t_i such that

$$e^{(n-2)t_i} \int_{|X_1| \leq e^{-t_i}} \int_{t_i}^{t_i+1} \int_{\mathbb{S}^1} |\nabla W_i|^2 (X_1, t, \theta)\, d\theta\, dt\, dX_1$$

is equal to the maximum of the left hand side of (3.1), for $t \in [\log 2, |\log \delta_i R| - 1]$, and it is larger than ε_1. In view of (2.13), we must have $t_i \to \infty$ and $|\log \delta_i R| - t_i \to \infty$.

Let $X_1 = e^{-t_i} Y_1$, and define $V_i(Y_1, t, \theta) = W_i(e^{-t_i} Y_1, t - t_i, \theta)$. Then we can view V_i as a map defined on $B_2^{n-2}(0) \times [-M_i, M_i] \times \mathbb{S}^1$ with $M_i \to \infty$. Moreover,

$$\int_{|Y_1| \leq 1} \int_0^1 \int_{\mathbb{S}^1} |\nabla V_i|^2 (Y_1, t, \theta)\, d\theta\, dt\, dY_1 \geq \varepsilon_1. \tag{3.2}$$

Here $\nabla V_i = \left(e^{-t} \nabla_{Y_1} V_i, \frac{\partial}{\partial t} V_i, \frac{\partial}{\partial \theta} V_i\right)$. From (2.12) one also sees that V_i is a stationary harmonic map on $Q_i \equiv B_2^{n-2}(0) \times [-M_i, M_i] \times \mathbb{S}^1$, and that V_i satisfies the equation

$$\frac{\partial^2}{\partial t^2} V_i + \frac{\partial^2}{\partial \theta^2} V_i + A(V_i) (\nabla V_i, \nabla V_i) = -e^{-2t} \Delta_{Y_1} V_i. \tag{3.3}$$

Furthermore, by (2.4), we have that

$$\sup_{0<r\leq 2} r^{2-n}\int_{B_r^{n-2}(0)}\int_{-M_i}^{M_i}\int_{\mathbb{S}^1}\left|e^{-t}\,\nabla_{Y_1}V_i\right|^2(Y_1,t,\theta)\,d\theta dt dY_1\to 0\quad\text{as}\quad i\to\infty. \tag{3.4}$$

Since the energy bound on $\{u_i\}$, we have the uniform energy bound on $\{V_i\}$ defined on $\{Q_i\}$. Hence, we may assume (by taking subsequences if needed) that $V_i \rightharpoonup V_\infty$ weakly in $H^1_{\text{loc}}(Q_i,\mathbb{S}^k)$, as $i\to\infty$.

(3.3)–(3.4) imply that V_∞ is a harmonic map with finite energy on $\mathbb{R}^1\times\mathbb{S}^1$. Since $\mathbb{R}^1\times\mathbb{S}^1$ is conformally equivalent to $\mathbb{S}^1\setminus\{\text{two poles}\}$, we deduce, from [SaU] and [H], that $\phi_\infty = V_\infty\circ\Pi:\mathbb{S}^2\to\mathbb{S}^k$ is a smooth harmonic map. Suppose now the convergence of V_i to V_∞ is strong on $B_i^{n-2}(0)\times[0,1]\times\mathbb{S}^1$ then (3.2)–(3.4) imply that V_∞ is a nonconstant map. This contradicts to the assumption that $m=1$.

Next, if the convergence of V_i to V_∞ is not strong on $B_1^{n-1}(0)\times[0,1]\times\mathbb{S}^1$. Then we follow the arguments in [L2, §2] and using (3.2)–(3.4) to conclude that (note V_∞ is smooth) $|\nabla V_i|^2\,d\theta dt dY_1 \rightharpoonup \mu$ as Radon measures on $B_2^{n-2}(0)\times\mathbb{R}^1\times\mathbb{S}^1$. Moreover, $\mu = |\nabla V_\infty|^2\,d\theta\,dt\,dY_1 + \sum_{j=1}^{\ell} C_j\,H^{n-2}\lfloor D_j$. Here each $C_j\geq\varepsilon_0$ is a constant, and $D_j = B_2^{n-2}(0)\times(t_j,\theta_j)$, for $j=1,\ldots,\ell$. In particular, we have non-trivial energy concentrations of V_i on point $(\mathbf{0},t_j,\theta_j)$, $j=1,2,\ldots,\ell$. Then we can again follow the blow-up arguments in [L2, §3] to prove (as the first bubble described before) that there is another constant harmonic map from $\mathbb{S}^2$ into $\mathbb{S}^k$ by a suitable scaling near each $(\mathbf{0},t_j,\theta_j)$. This again contradicts $m=1$, and hence it shows the claim (3.1) is true.

Since $W_i = B_{1/2}^{n-2}(0)\times B_i\to\mathbb{S}^k$ is a stationary map, (2.12) and (3.1) imply that

$$|\nabla W_i|^2 = e^{-2t}\,|\nabla_{X_1}W_i|^2 + \left|\frac{\partial}{\partial t}W_i\right|^2 + \left|\frac{\partial}{\partial\theta}W_i\right|^2 \leq C\,\varepsilon_1 \tag{3.5}$$

on $\mathbf{0}\times B_i$ (cf. [E]) for all i large. By taking partial derivatives with respect to $x_1,\ldots x_{n-2}$ we obtain equations for $\frac{\partial W_i}{\partial x_k}$, $k=1,\ldots,n-2$, from (2.12). Using (3.5), we then obtain

$$\left|\frac{\partial^2}{\partial t\partial\theta}W_i\right|^2 + \left|\frac{\partial^2}{\partial t^2}W_i\right|^2 + \left|\frac{\partial^2}{\partial\theta^2}W_i\right|^2 + \left|e^{-2t}\,\nabla^2_{X_1}W_i\right|^2 \leq C\,\varepsilon_1, \tag{3.6}$$

for some constant C and for all $(X_1,t,\theta)\in\mathbf{0}\times B_i$. Here near the boundary ∂B_i we may use the fact (2.13) and [E] again.

In the (X_1,r,θ)-coordinates system, estimate (3.5) should be viewed as the Lorentz space $L^{2,\infty}$-estimate on $\nabla_{X_2}\overline{u}_i(X_2)$. Indeed, (3.5) is equivalent to say

$$\left|X_2 - X_2^i\right|\left|\nabla_{X_2}\overline{u}_i(X_2)\right| \leq C\,\sqrt{\varepsilon_1}, \tag{3.7}$$

for $R\,\delta_i\leq|X_2-X_2^i|\leq\frac{1}{2}$, and for $i\geq I$, $R\geq R_0$. Similarly, (3.6) is equivalent to

$$\left|X_2 - X_2^i\right|^2\left|\nabla^2_{X_2}\overline{u}_i(X_2)\right| \leq C\,\sqrt{\varepsilon_1}, \tag{3.8}$$

for $R\,\delta_i\leq|X_2-X_2^i|\leq\frac{1}{2}$, $i\geq I$, $R\geq R_0$.

For suitably large R, it is not hard then to smoothly extend $\overline{u}_i(X_2)$ on $\{|X_2 - X_2^i| = \delta_i\, R\}$ inside $\{|X_2 - X_2^i| < \delta_i\, R\}$ in such a way that the extended map, say $u_i^*(X_2)$ on $\{|X_2 - X_2^i| \leq \delta_i\, R\}$ satisfies, for all X_2, $|X_2 - X_2^i| \leq \delta_i\, R$

$$
\begin{cases}
\text{(i)} & u_1^*(X_2) & \in & \mathbb{S}^k; \\
\text{(ii)} & \left|\overline{\nabla} u_i^*(X_2)\right| & \leq & C_*\, (\delta_i\, R)^{-1}\ \sqrt{\varepsilon_1}; \\
\text{(iii)} & \left|\overline{\nabla}^2 u_i^*(X_2)\right| & \leq & C_*\, (\delta_i\, R)^{-2}\ \sqrt{\varepsilon_1}\,.
\end{cases}
\tag{3.9}
$$

Let us denote $\tilde{u}_i(X_2)$ be such that

$$
\tilde{u}_i(X_2) = \begin{cases} \overline{u}_i(X_2), & \text{if } \frac{1}{2} \geq |X_2 - X_2^i| \geq R\,\delta_i \\ u_i^*(X_2) & \text{if } |X_2 - X_2^i| \leq \delta_i\, R. \end{cases}
$$

Then with (3.7)–(3.9) and (2.7)–(2.8) that

$$
\|\overline{\nabla} \tilde{u}_i\|_{L^{2,1}\left(B^2_{1/2}(X_2^i)\right)} \leq C_1\, \Lambda
\tag{3.10}
$$

and

$$
\|\overline{\nabla} \tilde{u}_i\|_{L^{2,\infty}\left(B^2_{1/2}(X_2^i)\right)} \leq C_*\, \sqrt{\varepsilon_1}
\tag{3.11}
$$

Indeed, (3.11) follows from a direct checking and (3.10) follows from (2.7) and (3.9).

Estimates (3.10)–(3.11) imply in particular that

$$
\|\overline{\nabla} \tilde{u}_i\|_{L^{2}\left(B^2_{1/2}(X_2^i)\right)} \leq C\,(\Lambda, C_1, C_*)\, \varepsilon_1^{1/4}.
\tag{3.12}
$$

Hence $E\,(W_i, B_i) \leq C\,(\Lambda, C_1, C_*)\, \varepsilon_1^{1/4}$. Since $\varepsilon_1 > 0$ is arbitrary, we proved (2.11) for the case $m = 1$.

Remark. In order to make the idea more transparent, we present here another proof which is modified from [SaU] and [DT] for the case $m = 1$. However, we will again need (3.10)–(3.11).

We want to estimate the energy of $W_i(\mathbf{0}, t, \theta)$ on B_i. Let $\overline{n}$ be an integer such that $d = [|\log \delta_i\, R| - \log 2]\, / \overline{n} \in \left(\frac{1}{2}, 1\right]$. Let $t_j = \log 2 + jd$ for $j = 0, 1, \ldots, \overline{n}$. We simply write $W(t, \theta)$ for $W_i(\mathbf{0}, t, \theta)$, and denote $P_j = [t_j, t_{j+1}] \times \mathbb{S}^2$, and $S_j = \{t_j\} \times \mathbb{S}^1$. Let $h(t)$ be a $\mathbb{R}^{k+1}$-valued function, linear on each interval $[t_j, t_{j+1}]$ and with $h(t_j)$ equals the mean value of W on S_j. Consider h as a map from B_i into $\mathbb{R}^{k+1}$. Then

$$
\begin{aligned}
& \frac{\partial^2}{\partial t^2}\,(W - h) + \frac{\partial^2}{\partial \theta^2}\,(W - h) \\
= \; & -e^{-2t}\,(\Delta_{X_1}\, W)\,(\mathbf{0}, t, \theta) - A(W)(\nabla W, \nabla W)(\mathbf{0}, t, \theta) \\
= \; & \alpha
\end{aligned}
\tag{3.13}
$$

on each $(t_j, t_{j+1}) \times \mathbb{S}^1$ for $j = 0, \ldots, \overline{n} - 1$. After taking the inner product of this equation with $W - h$ and integrating over P_j, we obtain

$$
\begin{aligned}
\int_{P_j} \left|\overline{\nabla}(W - h)\right|^2 d\theta dt \;\; = \;\; & \int_{P_j} (W - h)\, \alpha\; d\theta\, dt \\
& + \int_{S_{j+1}} - \int_{S_j} (W - h)\,(W_t - h_t)\, d\theta.
\end{aligned}
$$

Note that the boundary integrals of $(W-h)\,h_t$ vanish. So we get

$$\begin{aligned}\int_{P_j} \left|\overline{\nabla}(W-h)\right|^2 d\theta dt \quad &\le \quad \int_{P_j} (W-h)\,\alpha\, d\theta dt \\ &\qquad + \int_{S_{j+1}} - \int_{S_j} (W-h)\, W_t\, d\theta dt.\end{aligned}$$

Summing the inequalities over j, we obtain

$$\int_{B_i} \left|\overline{\nabla}(W-h)\right|^2 d\theta\, dt \le \int_{S_0} (W-h)\, W_t - \int_{S\overline{n}} (W-h)\, W_t + \int_{B_i} (W-h)\,\alpha\, dt d\theta. \tag{3.14}$$

Since (3.5), one has $|W-h| \le C\sqrt{\varepsilon_1}$. On the other hand

$$\begin{aligned}\int_{B_i} |\alpha|\, dt d\theta &= \int_{\delta_i R \le |X_2 - X_2^i| \le \frac{1}{2}} \left|\Delta_{X_1} u_i + |\nabla u_i|^2 u_i\right| (X_1^i, X_2)\, dX_2 \\ &\le \int_{B^2_{1/2}(X_2^i)} |\Delta_{X_1} u_i|\, (X_1^i, X_2)\, dX_2 + \int_{B^2_{1/2}(X_2^i)} |\nabla u_i|^2 \left(X_1^i, X_2\right) dX_2.\end{aligned} \tag{3.15}$$

Using (2.7)–(2.8), one has

$$\int_{B_i} |\alpha|\, dt d\theta \le C. \tag{3.16}$$

We thus obtain that

$$\int_{B_i} \left|\overline{\nabla}(W-h)\right|^2 d\theta\, dt \le C\sqrt{\varepsilon_1}. \tag{3.17}$$

In particular

$$\int_{B_i} \left|\frac{\partial}{\partial\theta} W\right|^2 d\theta\, dt \le C\sqrt{\varepsilon_1}. \tag{3.18}$$

To estimate $\frac{\partial}{\partial t} W$, we need to control the Hopf-differential. For any $u(X_2)$, $X_2 = (y_1, y_2)$, we denote $\phi(u)$ be the Hopf-differential associated with u as follows

$$\phi(u) = \left(\left|\frac{\partial u}{\partial y_1}\right|^2 - \left|\frac{\partial u}{\partial y_2}\right|^2\right) - 2\frac{\partial u}{\partial y_1} \cdot \frac{\partial u}{\partial y_2}\, i\;. \tag{3.19}$$

We note that if $u : \Omega \subseteq \mathbb{R}^2 \to N$ is a harmonic map, then $\phi(u)$ is holomorphic, and if, in addition $\Omega = \mathbb{R}^2$ and $E(u) < \infty$, then $\phi(u) \equiv 0$.

In the case $m = 1$, our previous arguments imply already that

$$\int\limits_{B^2_{1/2}(X_2^i)} |\phi(\overline{u}_i)|\, dX_2 \le C\,\varepsilon_1^{1/4}, \quad \text{for all} \quad i \ge I. \tag{3.20}$$

Indeed, we observe first that $\int_{B^2_{\delta_i\,R}(X_1^i)} |\phi(\overline{u}_i)|\, dX_2 = \int_{B^2_R(0)} |\phi(V_i)|\, dX_2 \to \int_{B^2_R(0)} |\phi(\psi_1)|\,(X_2)\, dX_2$ as $i \to \infty$. This is valid because $V_i \to \psi_1$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^k)$, as $i \to \infty$. The last term goes to zero as $R \to \infty$ since $\psi_1 : \mathbb{R}^2 \to \mathbb{S}^k$ is a finite energy harmonic map. Next, by (3.12), we have also that

$$\int\limits_{\delta_i\,R \le |X_2 - X_2^i| \le \frac{1}{2}} |\phi(\overline{u}_i)|\,(X_2)\, dX_2 \le C\,\varepsilon_1^{1/4}.$$

This proves (3.20). Therefore

$$\begin{aligned} \int_{B_i} \left|\frac{\partial}{\partial t} W\right|^2 d\theta dt &\le \int_{B_i} \left|\frac{\partial}{\partial \theta} W\right|^2 d\theta\, dt + \int_{B^2_{1/2}(X_2^i)} |\phi(\overline{u}_i)|\, dX_2 \\ &\le C\left(\varepsilon_1^{1/2} + \varepsilon_1^{1/4}\right) \le C\,\varepsilon_1^{1/4}. \end{aligned} \tag{3.21}$$

Here we note $|\phi(\overline{u}_i)|\,(X_2)\, dX_2$ is conformal invariant with respect to conformal change of variables (see [H]). In any case, we have completed the proof of the case $m = 1$.

4 Completion of the Proof of Theorem A

Let us consider the case $m \ge 2$. As in [DT], the essential part of the proof is identical to the proof for the case $m = 1$. One has to distinguish more "bubble domains" where after a blow-up gives rise to harmonic maps from $\mathbb{S}^2$ into $\mathbb{S}^k$, and "neck domains" where one has to show that energy can be made arbitrary small. The general case can be done by an induction. We should simply sketch then proof for the case $m = 2$.

For this case, $m = 2$ (3.1) can no longer be valid. Then from (3.2), we have two possibilities. First, the convergence of V_i to V_∞ is strong in H^1_{loc}. Define $a_i = t_i - \log 2$, $\quad b_i = |\log\, \delta_i\, R| - t_i$. We want to show the energy of V_i on the cylinder $[a_i, b_i] \times \mathbb{S}^1$ is close to the energy of V_∞ on $\mathbb{R} \times \mathbb{S}^1$. Indeed, we have

$$\lim_{M\to\infty} \lim_{i\to\infty} E\left(V_i, [-M, M] \times \mathbb{S}^1\right) = \lim_{M\to\infty} E\left(V_\infty, [-M, M] \times \mathbb{S}^1\right),$$

by the strong convergence of V_i to V_∞ and by [SaU]. On the other hand, we shall follow the proof for the case $m = 1$ to verify that energies of V_i on cylinders $[-a_i, -M] \times \mathbb{S}^1$ and $[M, b_i] \times \mathbb{S}^1$ can be made arbitrary small if one let $i \to \infty$, $M \to \infty$ (and also $R \to \infty$). In other words, $E\,(W_i, B) = E(\phi_2) + o_i(1)$. Here ϕ_2 is $V_\infty \circ G$, $G : \mathbb{S}^2 \to R \times \mathbb{S}^1$ is a conformal mapping with north, south poles corresponding to $\{+\infty\} \times \mathbb{S}^1$ and $\{-\infty\} \times \mathbb{S}^1$. Note via [SaU], $\phi_2 : \mathbb{S}^1 \to \mathbb{S}^k$ is a smooth harmonic map. To show $E\,(V_i, [M, b_i] \times \mathbb{S}^1) \to 0$ is the same as to show

$$E\left(\overline{u}_i, B_{r_i/\delta_i R}\right) \to 0, \quad \text{as } i \to \infty, \quad M \to \infty, \quad R \to \infty. \tag{4.1}$$

Here $r_i = e^{-t_i - M} \gg \delta_i\, R$. The proof of (4.1) is exactly the same as for the case $m = 1$ that follows from (3.12) with $\frac{1}{2}$ replacing by r_i.

To show $E\,(V_i, [-a_i, M] \times \mathbb{S}^1) \to 0$ is the same as to show

$$E\left[\overline{u}_i, B^2_{1/2}(X_2^i) \setminus B^2_{R_i}(X_2^i)\right] \to 0. \tag{4.2}$$

Here $R_i = e^{-t_i+M}$. Here we note that on $\partial B_{R_i}(X_2^i)$, $\overline{u}_i$ satisfies

$$\left|\overline{\nabla}\,\overline{u}_i\right|^2(X_2) + \left|\overline{\nabla}^2\,\overline{u}_i\right|(X_2) \le C\,R_i^{-2}\,\varepsilon_1. \tag{4.3}$$

As for the $m = 1$, we can extend $\overline{u}_i$ inside $B^2_{R_i}(X_2^i)$ properly so that similar estimates as (3.11)–(3.12) are valid. That will lead to

$$E\left[\overline{u}_i, B^2_{1/2}(X_2^i) \setminus B^2_{R_i}(X_2^i)\right] \le C\,\varepsilon_1. \tag{4.4}$$

Hence, we proved $E\,(W_i, B_i) = E(\phi_2) + o_i(1)$ for the first possibility.

The second possibility is that energy V_i may be concentrated near a point $p \in (0,1] \times \mathbb{S}^1$. In this case, we apply the arguments for the case $m = 1$ to the map $V_i : B_1^{n-2}(0) \times [-1,2] \times \mathbb{S}^1 \to \mathbb{S}^k$ to obtain $E\,(V_i, [-1,2] \times \mathbb{S}^1) = E(\phi_2) + o_i(1)$.

Then using (4.1) and (4.4), to show

$$E\left(V_i, [-a_i, -1] \times \mathbb{S}^1\right) + E\left(V_i, [2, b_i] \times \mathbb{S}^1\right) \le C\,\varepsilon_1. \tag{4.5}$$

We conclude that $E\,(V_i, [-a_i, b_i] \times \mathbb{S}^1) = E\,(W_i, B_i) = E(\phi_2) + o_i(1)$.

We would like also to note that one can also follow the arguments in [DT] but uses our estimates (3.18) and (3.20) and (3.21) to prove the same conclusion. This completes the proof of Theorem A.

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