Notes on the Axiom of Choice

Dietmar A. Salamon
ETH-Zürich

2 March 2025

1 Definitions and Axioms

The purpose of these notes is to prove that the Axiom of Choice, the Lemma of Zorn, and the Well Ordering Principle are equivalent to each other. Here are the relevant definitions.

Definition. *A relation* $\preccurlyeq$ *on a set* P *is called a* **partial order** *iff it satisfies the following conditions.*

(Reflexive) *Every element* $p \in P$ *satisfies* $p \preccurlyeq p$*.*

(Anti-Symmetric) *If* $p, q \in P$ *satisfy* $p \preccurlyeq q$ *and* $q \preccurlyeq p$*, then* $p = q$*.*

(Transitive) *If* $p, q, r \in P$ *satisfy* $p \preccurlyeq q$ *and* $q \preccurlyeq r$*, then* $p \preccurlyeq r$*.*

A partial order $\preccurlyeq$ *on a set* P *is called a* **total order** *iff any two distinct elements* $p, q \in P$ *satisfy either* $p \preccurlyeq q$ *or* $q \preccurlyeq p$*.*

Definition. *Let* $(P, \preccurlyeq)$ *be a partially ordered set.*

(i) *An element* $m \in P$ *is called* **maximal** *iff* $m \not\preccurlyeq p$ *for all* $p \in P \setminus \{m\}$*.*

(ii) *A* **chain** *in* P *is a totally ordered subset* $C \subset P$*, i.e. any two distinct elements* $p, q \in C$ *satisfy either* $p \preccurlyeq q$ *or* $q \preccurlyeq p$*.*

(iii) *Let* $C \subset P$ *be a nonempty chain. An element* $a \in P$ *is called an* **upper bound of** C *iff every element* $p \in C$ *satisfies* $p \preccurlyeq a$*. It is called a* **supremum of** C *iff it is an upper bound of* C *and every upper bound* $b \in P$ *of* C *satisfies* $a \preccurlyeq b$*. The supremum, if it exists, is unique and denoted by* $\sup C$*.*

Definition. *A total order* $\preccurlyeq$ *on a set* X *is called a* **well ordering** *iff for every nonempty subset* $A \subset X$ *there exists an element* $a_0 \in A$ *(called the* **minimum of** A *and denoted by* $\min A$*) such that* $a_0 \preccurlyeq a$ *for every* $a \in A$*.*

The Axiom of Choice. *Let I and X be two nonempty sets and, for each element $i \in I$, let $X_i \subset X$ be a nonempty subset. Then there exists a map $g : I \to X$ such that every $i \in I$ satisfies $g(i) \in X_i$.*

The Lemma of Zorn. *Let $(P, \preccurlyeq)$ be a partially ordered set such that every nonempty chain $C \subset P$ admits an upper bound. Let $p \in P$. Then there exists a maximal element $m \in P$ such that $p \preccurlyeq m$.*

The Well Ordering Principle. *Every set admits a well ordering.*

Theorem 1. *The Lemma of Zorn implies the Axiom of Choice.*

Proof. Assume the Lemma of Zorn. Let I and X be nonempty sets and, for each $i \in I$, let $X_i \subset X$ be a nonempty subset. Define

$$\mathscr{P} := \left\{ (J, g) \;\middle|\; \begin{array}{l} \emptyset \neq J \subset I,\ g : J \to X, \\ g(i) \in X_i \text{ for all } i \in J \end{array} \right\}.$$

This set is partially ordered by the relation

$$(J, g) \preccurlyeq (K, h) \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad J \subset K \quad \text{and} \quad h|_J = g$$

for $(J, g), (K, h) \in \mathscr{P}$. It is nonempty, because each pair (i_0, x_0) with $i_0 \in I$ and $x_0 \in X_{i_0}$ determines a pair $(J_0, g_0) \in \mathscr{P}$ with $J_0 := \{i_0\}$, $g_0(i_0) := x_0$. Each nonempty chain $\mathscr{C} \subset \mathscr{P}$ has a supremum $(K, h) = \sup \mathscr{C}$ given by

$$K := \bigcup_{(J,g) \in \mathscr{C}} J, \qquad h(i) := g(i) \quad \text{for } (J, g) \in \mathscr{C} \text{ and } i \in J.$$

Hence, by the Lemma of Zorn, there exists a maximal element $(J, g) \in \mathscr{P}$. This element satisfies $J = I$. Otherwise, there exists an element $i_0 \in I \setminus J$ and an element $x_0 \in X_{i_0}$, and then the pair $(J', g') \in \mathscr{P}$ with $J' := J \cup \{i_0\}$ and $g'|_J = g$, $g'(i_0) = x_0$ satisfies $(J, g) \preccurlyeq (J', g')$ and $(J, g) \neq (J', g')$, in contradiction to maximality. This shows that there exists a map $g : I \to X$ such that $g(i) \in X_i$ for all $i \in I$. Thus we have proved Theorem 1. □

Theorem 2. *The Well Ordering Principle implies the Axiom of Choice.*

Proof. Let I and X be nonempty sets and let $I \to 2^X \setminus \{\emptyset\} : i \mapsto X_i$ be a map which assigns to every $i \in I$ a nonempty subset $X_i \subset X$. By the Well Ordering Principle, choose a well ordering $\preccurlyeq$ of X. This well ordering determines a map $2^X \setminus \{\emptyset\} \to X : Y \mapsto \min Y$, which assigns to every nonempty subset $Y \subset X$ its minimum $\min Y \in Y$ with respect to the well ordering $\preccurlyeq$. The composition of these two maps gives rise to a map $g : I \to X$ which sends $i \in I$ to the element $g(i) := \min X_i \in X_i$ as required by the Axiom of Choice. This proves Theorem 2. □

2 Bourbaki–Witt

The proof that the Axiom of Choice implies the Lemma of Zorn relies on the Bourbaki–Witt Fixed Point Theorem. It is taken from [1, Appendix A] and follows the exposition by Imre Leader [3].

Theorem 3 (**Bourbaki–Witt**). *Let $(P, \preccurlyeq)$ be a nonempty partially ordered set such that every nonempty chain $C \subset P$ admits a supremum and let $f : P \to P$ be a map such that*

$$p \preccurlyeq f(p) \qquad \text{for all } p \in P.$$

Then there exists an element $p \in P$ such that $f(p) = p$.

Proof. Fix any element $p_0 \in P$ and denote by

$$\mathcal{A} \subset 2^P$$

be the set of all subsets $A \subset P$ that satisfy the following three conditions.
(I) $p_0 \in A$.
(II) If $p \in A$, then $f(p) \in A$.
(III) If $C \subset A$ is a nonempty chain, then $\sup C \in A$.
Then $\mathcal{A}$ is nonempty because $P \in \mathcal{A}$. Now let

$$E := \bigcap_{A \in \mathcal{A}} A \subset P$$

be the intersection of all subsets $A \in \mathcal{A}$. Then the set E also satisfies the conditions (I), (II), and (III) and hence is itself an element of $\mathcal{A}$. In particular, E is nonempty. We prove in five steps that E is a chain.

Step 1. *Every element $p \in E$ satisfies $p_0 \preccurlyeq p$.*

The set

$$P_0 := \{p \in P \,|\, p_0 \preccurlyeq p\}$$

satisfies the conditions (I), (II), and (III), and hence is an element of $\mathcal{A}$. Thus $E \subset P_0$ and this proves Step 1.

Step 2. *Let $F \subset E$ be the subset*

$$F := \left\{ q \in E \;\middle|\; \begin{array}{l} \text{every element } p \in E \setminus \{q\} \\ \text{with } p \preccurlyeq q \text{ also satisfies } f(p) \preccurlyeq q \end{array} \right\}.$$

Then $p_0 \in F$.

By Step 1 there is no element $p \in E \setminus \{p_0\}$ with $p \preccurlyeq p_0$. Hence $p_0 \in F$.

Step 3. *Let $p \in E$ and $q \in F$. Then $p \preccurlyeq q$ or $f(q) \preccurlyeq p$.*

Fix an element $q \in F$ and consider the set

$$E_q := \big\{p \in E \,\big|\, p \preccurlyeq q\big\} \cup \{p \in E \mid f(q) \preccurlyeq p\}.$$

We prove that $E_q \in \mathcal{A}$ satisfies (I). Since $q \in F \subset E$ we have $p_0 \preccurlyeq q$ by Step 1. Since $p_0 \in E$, this implies $p_0 \in E_q$ and so E_q satisfies condition (I).

We prove that E_q satisfies (II). Fix an element $p \in E_q$. Then $f(p) \in E$ because E satisfies (II). If $p \preccurlyeq q$ and $p \neq q$, then $f(p) \preccurlyeq q$, because q is an element of F, and this implies $f(p) \in E_q$. If $p = q$, then $f(q) \preccurlyeq f(p)$ and this implies $f(p) \in E_q$. If $p \not\preccurlyeq q$, then we must have $f(q) \preccurlyeq p$, because $p \in E_q$, and this implies again $f(q) \preccurlyeq f(p)$ and therefore $f(p) \in E_q$. This shows that E_q satisfies (II).

We prove that E_q satisfies (III). Thus let $C \subset E_q$ be a nonempty chain and $s := \sup C$. Then $s \in E$ because E satisfies (III). If $p \preccurlyeq q$ for all $p \in C$, then $s \preccurlyeq q$ und therefore $s \in E_q$. Otherwise there exists an element $p \in C$ with $p \not\preccurlyeq q$. Since $p \in E_q$, we must have $f(q) \preccurlyeq p \preccurlyeq s$ and therefore $s \in E_q$. This shows that E_q satisfies (III).

Thus $E_q \in \mathcal{A}$ and hence $E \subset E_q$. This proves Step 3.

Step 4. *$F = E$.*

By Step 2 we have $p_0 \in F$ and so F satisfies (I).

We prove that F satisfies (II). Fix an element $q \in F$. We must prove that $f(q) \in F$. To see this, note first that $f(q) \in E$ because E satisfies (II). Now let $p \in E \setminus \{f(q)\}$ with $p \preccurlyeq f(q)$. Under these assumptions we must show that $f(p) \preccurlyeq f(q)$. Since $f(q) \not\preccurlyeq p$, we have $p \preccurlyeq q$ by Step 3. If $p \neq q$, then it follows from the definition of F that $f(p) \preccurlyeq q \preccurlyeq f(q)$. If $p = q$, then we also have $f(p) \preccurlyeq f(q)$. Thus we have shown that $f(p) \preccurlyeq f(q)$ for every element $p \in E \setminus \{f(q)\}$ with $p \preccurlyeq f(q)$. Hence $f(q) \in F$ and this shows that F satisfies (II).

We prove that F satisfies (III). Let $C \subset F$ be a nonempty chain and define $s := \sup C$. We must prove that $s \in F$. To see this, note first that $s \in E$ because E satisfies (III). Now let $p \in E \setminus \{s\}$ with $p \preccurlyeq s$. Under these assumptions we must show that $f(p) \preccurlyeq s$. Since $s \neq p$, we have $s \not\preccurlyeq p$. Thus there exists an element $q \in C$ with $q \not\preccurlyeq p$, and hence also $f(q) \not\preccurlyeq p$. Since $q \in C \subset F$, this implies $p \preccurlyeq q$ by Step 3. Since $p \neq q$ and $q \in F$, this implies $f(p) \preccurlyeq q$. Since $q \in C$ and $s = \sup C$, this implies $f(p) \preccurlyeq s$. Thus we have proved that $s \in F$ and this shows that F satisfies (III).

Thus $F \in \mathcal{A}$, hence $E \subset F$, and hence $E = F$. This proves Step 4.

Step 5. *E is a chain.*

Let $p, q \in E$. Then $q \in F$ by Step 4, and so $p \preccurlyeq q$ or $f(q) \preccurlyeq p$ by Step 3. Thus $p \preccurlyeq q$ or $q \preccurlyeq p$ and this proves Step 5.

By Step 5, the set E has a supremum

$$s := \sup E \in P.$$

Since E satisfies condition (III) we have $s \in E$. Since E also satisfies (II), this implies $f(s) \in E$ and hence $f(s) \preccurlyeq s$. Since $s \preccurlyeq f(s)$ by assumption, we have $f(s) = s$ and this proves Theorem 3. □

We remark that the Lemma of Zorn implies the existence of a maximal element $m \in P$ under the assumptions of Theorem 3, and that any such maximal element must be a fixed point of f. However, the above proof of the Bourbaki–Witt Theorem does not use the Lemma of Zorn (nor does it use the Axiom of Choice) and so the result can be used to show that the Axiom of Choice implies the Lemma of Zorn.

Theorem 4. *The Axiom of Choice implies the Lemma of Zorn.*

Proof. Assume the Axiom of Choice. Under this assumption we prove the Lemma of Zorn in two steps.

Step 1. *Let $(P, \preccurlyeq)$ be a nonempty partially ordered set such that every nonempty chain $C \subset P$ has a supremum. Then P has a maximal element.*

Assume, by contradiction, that P does not have a maximal element. Then the set

$$S(p) := \{q \in P \,|\, p \preccurlyeq q,\, p \neq q\} \subset P$$

is nonempty for every element $p \in P$. Hence the Axiom of Choice asserts that there exists a map $f : P \to P$ such that

$$f(p) \in S(p) \qquad \text{for all } p \in P.$$

This map f satisfies the condition

$$p \preccurlyeq f(p) \qquad \text{for all } p \in P.$$

However, f does not have a fixed point, in contradiction to Theorem 3. This shows that our assumption, that P does not have a maximal element, must have been wrong. So P has a maximal element and this proves Step 1.

Step 2. *Let $(P, \preccurlyeq)$ be a partially ordered set such that every nonempty chain $C \subset P$ admits an upper bound. Let $p \in P$. Then there exists a maximal element $m \in P$ with $p \preccurlyeq m$.*

Let $\mathscr{P} \subset 2^P$ be the set of all chains in P that contain the point p, i.e.

$$\mathscr{P} := \left\{ C \subset P \,\middle|\, C \text{ is a chain and } p \in C \right\}.$$

Then $\mathscr{P}$ is a nonempty set, partially ordered by inclusion.

Now let $\mathscr{C} \subset \mathscr{P}$ be a nonempty chain in $\mathscr{P}$ and define the set

$$S := \bigcup_{C \in \mathscr{C}} C.$$

This set contains the point p and we claim that it is a chain in P. To see this, let $p_0, p_1 \in S$ and choose chains $C_0, C_1 \in \mathscr{C}$ such that

$$p_0 \in C_0, \qquad p_1 \in C_1.$$

Since $\mathscr{C}$ is a chain we have $C_0 \subset C_1$ or $C_1 \subset C_0$. Hence

$$C := C_0 \cup C_1 \in \mathscr{C}$$

is a chain in P that contains both p_0 and p_1, and thus $p_0 \preccurlyeq p_1$ or $p_1 \preccurlyeq p_0$. This shows that S is an element of $\mathscr{P}$ and therefore is the supremum of the chain of chains $\mathscr{C} \subset \mathscr{P}$. Thus we have proved that every nonempty chain in $\mathscr{P}$ has a supremum.

With this understood, Step 1 asserts that there exists a maximal chain

$$M \subset P$$

that contains the point p. Let $m \in P$ be an upper bound of M. Then

$$p \preccurlyeq m.$$

Moreover, m is a maximal element of P, because otherwise there would exist an element $q \in P$ with $m \preccurlyeq q$ and $m \neq q$, so $q \notin M$, and then $M' := M \cup \{q\}$ would be a larger chain containing p, in contradiction to the maximality of M. This proves Step 2 and Theorem 4. □

3 Well Ordering

Theorem 5. *The Axiom of Choice implies the Well Ordering Principle.*

Our proof follows the argument of Dag Normann [4] which in turn is based on [2]. We begin with some basic observations about well ordered sets. A subset $S \subset X$ of a well ordered set $(X, \preccurlyeq)$ is called an **initial segment** iff every element $s \in S$ satisfies $\{x \in X \mid x \preccurlyeq s\} \subset S$.

Lemma 6. *Let $(X, \preccurlyeq)$ be a nonempty well ordered set and let $g : X \to X$ be a bijective map such that all $x, y \in X$ satisfy*

$$x \preccurlyeq y \qquad \Longleftrightarrow \qquad g(x) \preccurlyeq g(y).$$

Then $g = \mathrm{id}$.

Proof. Suppose, by contradiction, that $g \neq \mathrm{id}$. Then the set

$$Y := \{y \in X \mid g(y) \neq y\}$$

is nonempty and hence admits a minimum $y_0 \in Y$. Thus $y_0 \preccurlyeq y$ for all $y \in Y$. This implies $g(y_0) \preccurlyeq g(y)$ for all $y \in Y$. Moreover, since the map g is bijective, it restricts to a bijection of Y. Thus we have $y_0 \preccurlyeq g(y_0)$ and $g(y_0) \preccurlyeq y_0$, which implies $y_0 = g(y_0)$ in contradiction to the fact that $y_0 \in Y$. Hence g is the identity map as claimed. This proves Lemma 6. □

Lemma 7. *Let $(X, \preccurlyeq)$ and $(X', \preccurlyeq')$ be nonempty well ordered sets. Then one of the following assertions holds.*

(i) *There exists an injective map $g : X \to X'$ such that $g(X)$ is an initial segment of X' and all $x, y \in X$ satisfy*

$$x \preccurlyeq y \qquad \Longleftrightarrow \qquad g(x) \preccurlyeq' g(y). \tag{1}$$

(ii) *There exists an injective map $h : X' \to X$ such that $h(X')$ is an initial segment of X and all $x', y' \in X'$ satisfy*

$$x' \preccurlyeq' y' \qquad \Longleftrightarrow \qquad h(x') \preccurlyeq h(y'). \tag{2}$$

Proof. Define

$$\mathscr{P} := \left\{ (S, g) \;\middle|\; \begin{array}{l} S \text{ is an initial segment of } X, \\ g : S \to X' \text{ is an injective map} \\ \text{that satisfies (1) for all } x, y \in S, \\ \text{and } g(S) \text{ is an initial segment of } X' \end{array} \right\}.$$

The set $\mathscr{P}$ has the following properties.

(a) $\mathscr{P}$ is nonempty.

Let x_0 be the minimum of $(X, \preccurlyeq)$ and let x_0' be the minimum of $(X', \preccurlyeq')$. Define $S := \{x_0\}$ and define $g : S \to X'$ by $g(x_0) := x_0'$. Then $(S, g) \in \mathscr{P}$.

(b) If $(S, g) \in \mathscr{P}$ and $T \subset S$ is an initial segment of S, then $(T, g|_T) \in \mathscr{P}$.

This holds because an initial segment of an initial segment of X is an initial segment of X, and likewise for X'.

(c) If $(S_0, g_0), (S_1, g_1) \in \mathscr{P}$, then $g_0(x) = g_1(x)$ for all $x \in S_0 \cap S_1$.

Assume, by contradiction, that this is wrong. Then the set

$$Y := \left\{ y \in S_0 \cap S_1 \,\middle|\, g_0(y) \neq g_1(y) \right\}$$

is nonempty and hence admits a minimum $y_0 \in Y$. Define

$$S := \left\{ x \in X \,\middle|\, x \neq y_0,\ x \preccurlyeq y_0 \right\}.$$

Then, since S_0 and S_1 are initial segments of X, so is the set $S \subset S_0 \cap S_1$, and it follows from the definition of y_0 that $g_0(x) = g_1(x)$ for all $x \in S$. Also, it follows from (b) that $(S, g_0|_S) \in \mathscr{P}$. Thus $S' := g_0(S)$ is an initial segment of X'. Moreover, $S \cup \{y_0\} \subset S_0 \cap S_1$ is an initial segment of X, and hence the sets $g_0(S \cup \{y_0\}) = S' \cup \{g_0(y_0)\}$ and $g_1(S \cup \{y_0\}) = S' \cup \{g_1(y_0)\}$ are initial segments of X'. Thus $g_0(y_0) = g_1(y_0)$, a contradiction. This proves (c).

(d) If $(S_0, g_0), (S_1, g_1) \in \mathscr{P}$, then $S_1 \subset S_0$ or $S_0 \subset S_1$.

Assume $S_1 \not\subset S_0$ and choose an element $x_1 \in S_1 \setminus S_0$. If $x \in S_0$, then we have $x_1 \not\preccurlyeq x$, hence $x \preccurlyeq x_1$, and hence $x \in S_1$. This proves (d).

Define the sets

$$S_{\max} := \bigcup_{(S,g) \in \mathscr{P}} S \subset X, \qquad S'_{\max} := \bigcup_{(S,g) \in \mathscr{P}} g(S) \subset X',$$

and define the map $g_{\max} : S_{\max} \to S'_{\max}$ by

$$g_{\max}(y) := g(y) \qquad \text{for } (S, g) \in \mathscr{P} \text{ and } y \in S.$$

By (d) the set $S_{\max}$ is an initial segment of X and the set $S'_{\max}$ is an initial segment of X'. By (c) and (d) the map $g_{\max} : S_{\max} \to S'_{\max}$ is well defined. Moreover, it is bijective and satisfies (1). Thus $(S_{\max}, g_{\max}) \in \mathscr{P}$.

If $S_{\max} = X$, then the map $g_{\max} : X \to X'$ satisfies (i). If $S_{\max} \neq X$, then $S'_{\max} = X'$, because otherwise $g_{\max}$ extends to an element of $\mathscr{P}$ with a larger domain $S_{\max} \cup \{\min(X \setminus S_{\max})\}$, a contradiction. Hence, in this case the map $g_{\max}^{-1} : X' \to S_{\max} \subset X$ satisfies (ii). This proves Lemma 7. $\square$

With these preparations in place we are ready to begin with the proof of Theorem 5. Let X be a nonempty set. By the Axiom of Choice there exists a map $f : 2^X \setminus \{X\} \to X$ that satisfies

$$f(Y) \in X \setminus Y \qquad \text{for all } Y \subsetneq X. \tag{3}$$

Let such a map f be given. We will prove that X admits a well ordering $\preccurlyeq$ (unique by Lemma 8 below) such that

$$x = f\big(\{y \in X \setminus \{x\} \,\big|\, y \preccurlyeq x\}\big) \qquad \text{for all } x \in X. \tag{4}$$

The minimum of this well ordering is the element $x_0 := f(\emptyset)$.

Definition. *An* f**-string in** X *is a nonempty subset* $A \subset X$ *equipped with a well ordering* $\preccurlyeq_A$ *such that*

$$a = f\big(\{a' \in A \setminus \{a\} \,\big|\, a' \preccurlyeq_A a\}\big) \qquad \textit{for all } a \in A. \tag{5}$$

Lemma 8. *Let* $(A, \preccurlyeq_A)$ *and* $(B, \preccurlyeq_B)$ *be* f*-strings in* X*. Then* A *is an initial segment of* B *or* B *is an initial segment of* A*. Also, if* $a, a' \in A \cap B$*, then*

$$a \preccurlyeq_A a' \qquad \Longleftrightarrow \qquad a \preccurlyeq_B a'. \tag{6}$$

Proof. By Lemma 7 there exists an isomorphism from $(A, \preccurlyeq_A)$ onto an initial segment of B, or there exists an isomorphism from $(B, \preccurlyeq_B)$ onto an initial segment of A. Assume the former and let $g : A \to B$ be an injective map such that $g(A)$ is an initial segment of B and, for all $a, a' \in A$,

$$a \preccurlyeq_A a' \qquad \Longleftrightarrow \qquad g(a) \preccurlyeq_B g(a'). \tag{7}$$

We claim that $g(a) = a$ for all $a \in A$. Assume, by contradiction, that this is not the case. Then the set $A_1 := \{a \in A \,|\, g(a) \neq a\}$ is nonempty and hence contains a minimal element a_1. This element satisfies $b_1 := g(a_1) \neq a_1$ and $g(a) = a$ for all $a \in A \setminus \{a_1\}$ such that $a \preccurlyeq_A a_1$. Hence, by (7), we have

$$\begin{aligned} \{a \in A \setminus \{a_1\} \,|\, a \preccurlyeq_A a_1\} &= \big\{g(a) \,\big|\, a \in A \setminus \{a_1\},\, g(a) \preccurlyeq_B g(a_1)\big\} \\ &= \big\{b \in B \setminus \{b_1\} \,\big|\, b \preccurlyeq_B b_1\big\}\,. \end{aligned}$$

Here the second equality uses the fact that $g(A)$ is an initial segment of B. Since $(A, \preccurlyeq_A)$ and $(B, \preccurlyeq_B)$ are f-strings, it then follows from (5) that

$$a_1 = f\left(\{a \in A \setminus \{a_1\} \,|\, a \preccurlyeq_A a_1\}\right) = f\left(\{b \in B \setminus \{b_1\} \,|\, b \preccurlyeq_B b_1\}\right) = b_1.$$

This is a contradiction and shows that $g(a) = a$ for all $a \in A$. Thus A is an initial segment of B and (6) holds. This proves Lemma 8. □

Proof of Theorem 5. Choose a map $f : 2^X \setminus \{X\} \to X$ that satisfies (3) and let $\mathscr{S}_f$ be the set of all f-strings in X. Then $\mathscr{S}_f$ is nonempty because the set $A = \{x_0\}$ with its unique order is an f-string. Define a set $S \subset X$ by

$$S := \bigcup_{(A,\preccurlyeq_A)\in\mathscr{S}_f} A \tag{8}$$

and define a relation $\preccurlyeq$ on S by

$$x \preccurlyeq y \quad \overset{\text{def}}{\iff} \quad \begin{array}{l}\text{there exists an } f\text{-string } (A, \preccurlyeq_A) \in \mathscr{S}_f \\ \text{such that } x, y \in A \text{ and } x \preccurlyeq_A y\end{array} \tag{9}$$

for $x, y \in S$. For any two elements $x, y \in S$ it follows from Lemma 8 that there exists an f-string $(A, \preccurlyeq_A)$ such that $x, y \in A$ and that, for any two f-strings $(A, \preccurlyeq_A)$ and $(B, \preccurlyeq_B)$ with $x, y \in A \cap B$, we have $x \preccurlyeq_A y$ if and only if $x \preccurlyeq_B y$. Hence the relation in (9) on the set S in (8) is well defined and $(S, \preccurlyeq)$ is a totally ordered set.

We prove that $\preccurlyeq$ is a well ordering of S. Let Y be a nonempty subset of S, choose an f-string $(A, \preccurlyeq_A)$ such that $Y \cap A \neq \emptyset$, and let $y_0 \in Y \cap A$ be the minimum of $Y \cap A$ with respect to the well ordering $\preccurlyeq_A$ on A. If $y \in Y \setminus A$, then there exists an f-string $(B, \preccurlyeq_B)$ such that $y \in B$, hence $B \not\subset A$, hence by Lemma 8 the set A is an initial segment of B, and hence $y_0 \preccurlyeq_B y$. This shows that y_0 is the minimum of Y with respect to the partial order $\preccurlyeq$ on S. Thus we have proved that $\preccurlyeq$ is a well ordering of S and hence $(S, \preccurlyeq) \in \mathscr{S}_f$.

We prove that $S = X$. Suppose not. Then the set $S' := S \cup \{f(S)\}$ with the order relation $\preccurlyeq'$ defined by

$$x \preccurlyeq' y \quad \overset{\text{def}}{\iff} \quad y = f(S) \text{ or } x, y \in S \text{ and } x \preccurlyeq y$$

for $x, y \in S'$ is an f-string, in contradiction to (8). Thus $S = X$ and so the relation $\preccurlyeq$ in (9) defines a well ordering of X. This proves Theorem 5. □

References

[1] Theo Bühler & Dietmar A. Salamon, *Functional Analysis*, AMS *Graduate Studies in Mathematics*, Volume **191**, 2018. `https://people.math.ethz.ch/~salamond/PREPRINTS/funcana-ams.pdf`

[2] Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs*. Elsevier, Studies in Logic and the Foundations of Mathematics, Volume **102**, 1980.

[3] Imre Leader, *Logic and Set Theory*. University of Cambridge, 2003.

[4] Dag Normann, The Axiom of Choice, the Well Ordering Principle, and Zorn's Lemma. Preprint, Univ. of Oslo, 2012. `https://www.mn.uio.no/math/tjenester/kunnskap/kompendier/acwozl.pdf`