Notes on hypersurfaces in contact manifolds

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Abstract

These notes discuss some concepts explained by Yasha Eliashberg in his lectures on 18 and 25 May 2022 at the ITS at ETH Zürich [4].

1 Nondegenerate pairs

Let Σ be a nonempty closed connected oriented $2\mathsf{n}$-manifold.

Definition 1.1. *A pair* $(\beta, \tau) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$ *is called* **nondegenerate** *iff it satisfies the condition*

$$\beta_x \wedge \tau_x^{\mathsf{n}-1} = 0 \qquad \Longrightarrow \qquad \tau_x^{\mathsf{n}} \neq 0 \tag{1.1}$$

for every $x \in \Sigma$.

Lemma 1.2. *Let* $\beta \in \Omega^1(\Sigma)$ *and* $\tau \in \Omega^2(\Sigma)$*. Then the following are equivalent.*

(i) *The pair* (β, τ) *is nondegenerate.*

(ii) *For every volume form* $\rho \in \Omega^{2\mathsf{n}}(\Sigma)$ *there exists a function* $f \in \Omega^0(\Sigma)$ *and a 1-form* $\gamma \in \Omega^1(\Sigma)$ *such that*

$$\rho = f\tau^{\mathsf{n}} + \mathsf{n}\beta \wedge \tau^{\mathsf{n}-1} \wedge \gamma. \tag{1.2}$$

(iii) *There exists a pair* $(f, \gamma) \in \Omega^0(\Sigma) \times \Omega^1(\Sigma)$ *such that the formula* (1.2) *defines a volume form* ρ *on* Σ*.*

Proof. First assume (i) and fix a volume form $\rho \in \Omega^{2\mathsf{n}}(\Sigma)$. Then Σ is the union of the open sets

$$U := \left\{x \in \Sigma \,\middle|\, (\tau_x)^{\mathsf{n}} \neq 0\right\}, \qquad V := \left\{x \in \Sigma \,\middle|\, \beta_x \wedge (\tau_x)^{\mathsf{n}-1} \neq 0\right\}.$$

Also there exists a unique function $f_U \in \Omega^0(U)$ such that $f_U\tau^{\mathsf{n}} = \rho$ on U, and there exists a 1-form $\gamma_V \in \Omega^1(V)$ such that $\mathsf{n}\beta \wedge \tau^{\mathsf{n}-1} \wedge \gamma_V = \rho$ on V. Since $U \cup V = \Sigma$, there exist smooth functions $\theta_U, \theta_V : \Sigma \to [0,1]$ such that

$$\mathrm{supp}(\theta_U) \subset U, \qquad \mathrm{supp}(\theta_V) \subset V, \qquad \theta_U + \theta_V = 1.$$

Hence $f := \theta_U f_U$ and $\gamma := \theta_V\gamma_V$ extend to all of Σ and satisfy (1.2). Thus we have proved that (i) implies (ii). That (ii) implies (iii) is obvious and that (iii) implies (i) follows directly from the definitions. This proves Lemma 1.2. □

Remark 1.3 (Stable almost complex structures). Fix a nondegenerate pair $(\beta, \tau) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$. Let $\rho \in \Omega^{2\mathsf{n}}(\Sigma)$ be a positive volume form and choose f, γ as in part (ii) of Lemma 1.2. Then, for each $x \in \Sigma$, the 2-form

$$\omega_x := \tau_x - \beta_x \wedge ds + \gamma_x \wedge dt + f(x) ds \wedge dt$$

on $T_x\Sigma \times \mathbb{R}^2$ satisfies $\omega_x^{\mathsf{n}+1} = (\mathsf{n}+1)\rho_x \wedge ds \wedge dt$ and hence is nondegenerate. Moreover, the set of all pairs (f, γ) for which (1.2) is a positive volume form is evidently a convex subset of $\Omega^0(\Sigma) \times \Omega^1(\Sigma)$. Thus each nondegenerate pair on Σ gives rise to a unique homotopy class of nondegenerate 2-forms on the vector bundle $T\Sigma \times \mathbb{R}^2$ and hence to a unique homotopy class of stable almost complex structures.

Definition 1.4 (Characteristic foliation). *Let $(\beta, \tau) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$ be a nondegenerate pair. The* **characteristic foliation** *of (β, τ) is the collection of linear subspaces $\ell_x \subset T_x\Sigma$, one for each $x \in \Sigma$, defined by*

$$\ell_x := \left\{v \in \ker(\beta_x) \,\middle|\, \tau_x(v, w) = 0 \text{ for all } w \in \ker(\beta_x)\right\}. \tag{1.3}$$

For each $x \in \Sigma$ the subspace ℓ_x has dimension zero if and only if $\beta_x = 0$ and has dimension one otherwise.

The next lemma shows that the leaves of the characteristic foliation of a nondegenerate pair (β, τ) on an oriented $2\mathsf{n}$-manifold Σ are the integral curves of a vector field X (introduced by Giroux [5]).

Lemma 1.5. *Let $(\beta, \tau) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$ be a nondegenerate pair and choose a triple ρ, f, γ as in part (ii) of Lemma 1.2. Define the function $\lambda \in \Omega^0(\Sigma)$ and the vector fields $X, Y \in \mathrm{Vect}(\Sigma)$ by*

$$\iota(X)\rho = \mathsf{n}\beta \wedge \tau^{\mathsf{n}-1}, \qquad \iota(Y)\rho = \mathsf{n}\tau^{\mathsf{n}-1} \wedge \gamma, \qquad \lambda\rho = \tau^{\mathsf{n}}. \tag{1.4}$$

Then

$$\beta(X) = \gamma(Y) = 0, \qquad \gamma(X) + \beta(Y) = 0, \tag{1.5}$$

$$\iota(X)\tau = \lambda\beta, \qquad \iota(Y)\tau = \lambda\gamma, \qquad f\lambda = \gamma(X) + 1. \tag{1.6}$$

Moreover, for all $x \in \Sigma$ we have $X(x) \in \ell_x$ and

$$X(x) = 0 \iff \beta_x = 0, \qquad Y(x) = 0 \iff \gamma_x = 0, \tag{1.7}$$

Proof. Since $\beta \wedge \rho = 0$ we have $\beta(X)\rho = \beta \wedge \iota(X)\rho = 0$, where the last equality follows from the definition of X in (1.4). Thus $\beta(X) = 0$. Likewise, it follows from the definition of Y in (1.4) that $\gamma(Y)\rho = \gamma \wedge \iota(Y)\rho = 0$, and so $\gamma(Y) = 0$. Similarly, $\beta(Y)\rho = \beta \wedge \iota(Y)\rho = \mathsf{n}\beta \wedge \tau^{\mathsf{n}-1} \wedge \gamma = (\iota(X)\rho) \wedge \gamma = -\gamma(X)\rho$ and so $\beta(Y) + \gamma(X) = 0$. This proves (1.5).

To prove (1.6), observe that every vector field $Z \in \mathrm{Vect}(\Sigma)$ satisfies

$$\begin{aligned}\tau(X, Z)\rho &= -(\iota(X)\iota(Z)\tau) \wedge \rho = -(\iota(Z)\tau) \wedge \iota(X)\rho \\ &= -\mathsf{n}(\iota(Z)\tau) \wedge \beta \wedge \tau^{\mathsf{n}-1} = \beta \wedge \iota(Z)\tau^{\mathsf{n}} = \beta(Z)\tau^{\mathsf{n}} = \beta(Z)\lambda\rho.\end{aligned}$$

This shows that $\iota(X)\tau = \lambda\beta$. Likewise,

$$\begin{aligned}\tau(Y, Z)\rho &= -(\iota(Y)\iota(Z)\tau) \wedge \rho = -(\iota(Z)\tau) \wedge \iota(Y)\rho \\ &= -\mathsf{n}(\iota(Z)\tau) \wedge \gamma \wedge \tau^{\mathsf{n}-1} = \gamma \wedge \iota(Z)\tau^{\mathsf{n}} = \gamma(Z)\tau^{\mathsf{n}} = \gamma(Z)\lambda\rho,\end{aligned}$$

for all $Z \in \mathrm{Vect}(\Sigma)$ and so $\iota(Y)\tau = \lambda\gamma$. Commbining this with (1.5) and the definition of X and λ in (1.4) we obtain

$$\rho = f\tau^{\mathsf{n}} + \mathsf{n}\beta \wedge \tau^{\mathsf{n}-1} \wedge \gamma = f\lambda\rho + (\iota(X)\rho) \wedge \gamma = \big(f\lambda - \gamma(X)\big)\rho.$$

This proves (1.6).

It follows directly from (1.5) and (1.6) that $X(x) \in \ell_x$ for all $x \in \Sigma$.

To prove (1.7), fix an element $x \in \Sigma$. If $\beta_x = 0$, then it follows directly from the definition of X in (1.4) that $X(x) = 0$. Conversely, suppose that $X(x) = 0$. Then it follows from (1.4) that the $(2\mathsf{n} - 1)$-form $\beta \wedge \tau^{\mathsf{n}-1}$ vanishes at x. Hence the nondegeneracy condition (1.1) asserts that τ_x is a nondegenerate 2-form on $T_x\Sigma$ and this implies $\beta_x = 0$. Thus we have proved that $X(x) = 0$ if and only if $\beta_x = 0$. That $Y(x) = 0$ if and only if $\gamma_x = 0$ follows by verbatim the same argument. This proves (1.7) and Lemma 1.5. $\square$

The following remark examines the homotopy classes of nondegenerate pairs in dimension two.

Remark 1.6 (Euler characteristic). Let (Σ, σ) be a nonempty closed connected symplectic 2-manifold. Then a pair $(\beta, \tau) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$ determines (and is determined by) a pair $(X, \lambda) \in \mathrm{Vect}(\Sigma) \times \Omega^0(\Sigma)$ via

$$\iota(X)\sigma = \beta, \qquad \tau = \lambda\sigma. \tag{1.8}$$

Under this correspondence the pair (β, τ) is nondegenerate if and only if the pair (X, λ) satisfies, for each $x \in \Sigma$, the condition

$$X(x) = 0 \qquad \Longrightarrow \qquad \lambda(x) \neq 0. \tag{1.9}$$

If the zeros of X are all isolated, the **Euler characteristic of the pair** (β, τ) is the integer defined by

$$\chi(\beta, \tau) := \sum_{X(x)=0} \mathrm{sign}\big(\lambda(x)\big)\iota(x, X). \tag{1.10}$$

Here the sum runs over all zeros of X and $\iota(x, X) \in \mathbb{Z}$ denotes the index of x as a zero of the vector field X (see [7, p 32]). The Euler characteristic of the pair (β, τ) is an even integer because the sum $\sum_x \iota(x, X) = \chi(\Sigma)$ is the Euler characteristic of Σ by the Poincaré–Hopf Theorem (see [7, p 35]) and the difference $\chi(\beta, \tau) - \chi(\Sigma)$ is even by definition.

Standard arguments as in [7] show that the Euler characteristic is a homotopy invariant and so is well defined for every nondegenerate pair (β, τ), regardless of whether or not the zeros of β are isolated. In fact, $\chi(\beta, \tau)$ is the pairing of the first Chern class of the nondegenerate 2-form on the vector bundle $T\Sigma \times \mathbb{R}^2$ in Remark 1.3 with the fundamental class of Σ.

Next we remark that the vector bundle $T\Sigma \times \mathbb{R}^2$ admits a trivialization and so is isomorphic to the trivial bundle $\Sigma \times \mathbb{R}^4$. Under such a trivialization, a nondegenerate 2-form on $T\Sigma \times \mathbb{R}^2$ that is compatible with the orientation is homotopic to a smooth map from Σ to the unit sphere in the 3-dimensional vector space $\Lambda^{2,+}$ of self-dual 2-forms on $\mathbb{R}^4$, and the degree of this map is half the first Chern number. Moreover, the Hopf Degree Theorem (see [7, p 51]) asserts that the homotopy class of a smooth map from Σ to the 2-sphere is uniquely determined by its degree. Hence two nondegenerate pairs on Σ are homotopic if and only if they have the same Euler characteristic.

2 Hypersurfaces in contact manifolds

Let (M, ξ) be a $(2\mathsf{n}+1)$-dimensional contact manifold with a cooriented contact structure ξ. Thus $\xi \subset TM$ is a field of hyperplanes and there exists a nowhere vanishing 1-form α on M such that $\xi = \ker(\alpha)$ and $\alpha \wedge (d\alpha)^{\mathsf{n}} \neq 0$. Any such 1-form α is called a **contact form** on (M, ξ). A contact form α is called **positive** iff it takes positive values on positive normal vectors of ξ. The orientation of M is determined by the volume form $\alpha \wedge (d\alpha)^{\mathsf{n}}$ for every positive contact form α. A vector field $\nu \in \mathrm{Vect}(M)$ is called a **contact vector field** iff its flow preserves the contact structure ξ, i.e. for each contact form α there exists a smooth function $h : M \to \mathbb{R}$ such that $\mathcal{L}_\nu \alpha = h\alpha$.

A **hypersurface** in M is a nonempty closed connected $2\mathsf{n}$-dimensional submanifold $\Sigma \subset M$. A hypersurface $\Sigma \subset M$ is called **coorientable** iff it admits a transverse vector field $\nu \in \Omega^0(\Sigma, T_\Sigma M)$, so $\nu(x) \in T_xM \setminus T_x\Sigma$ for all $x \in \Sigma$. If Σ is coorientable, then the space of such transverse vector fields has precisely two connected components, and a choice of one of these connected components is called a **coorientation of** Σ. If a coorientation of Σ has been chosen, we say that Σ is **cooriented** and call a transverse vector field ν **positive** iff it belongs to the preferred connected component. The orientation of a cooriented hypersurface $\Sigma \subset M$ is determined by the volume form $\iota(\nu)(\alpha \wedge (d\alpha)^{\mathsf{n}})|_\Sigma$, where $\alpha \in \Omega^1(M)$ is a positive contact form for ξ and ν is a positive transverse vector field along Σ.

Lemma 2.1. *Let $\Sigma \subset M$ be a cooriented hypersurface, let $\alpha \in \Omega^1(M)$ be a positive contact form for ξ, choose any positive transverse vector field ν along Σ, and define $\beta, \gamma \in \Omega^1(\Sigma)$ and $f \in \Omega^0(\Sigma)$ by*

$$\beta := \alpha|_\Sigma, \qquad f := \alpha(\nu)|_\Sigma, \qquad \gamma := -\iota(\nu)d\alpha|_\Sigma. \tag{2.1}$$

Then the volume form $\rho := \iota(\nu)(\alpha \wedge (d\alpha)^{\mathsf{n}})|_\Sigma \in \Omega^{2\mathsf{n}}(\Sigma)$ is given by

$$\rho := f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge \gamma. \tag{2.2}$$

Hence the pair $(\beta, d\beta)$ is nondegenerate (Definition 1.1).

Proof. We compute

$$\begin{aligned} \rho = \iota(\nu)\big(\alpha \wedge (d\alpha)^{\mathsf{n}}\big)\big|_\Sigma &= \alpha(\nu)\ (d\alpha)^{\mathsf{n}}|_\Sigma - \mathsf{n}\ \alpha \wedge (d\alpha)^{\mathsf{n}-1} \wedge \iota(\nu)d\alpha\big|_\Sigma \\ &= f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge \gamma. \end{aligned}$$

This proves (2.2). That the pair $(\beta, d\beta)$ is nondegenerate follows from (2.2) and Lemma 1.2. This proves Lemma 2.1. □

Definition 2.2. *Let Σ be a closed oriented $2\mathsf{n}$-manifold. A 1-form $\beta \in \Omega^1(\Sigma)$ is called a* **germ of a contact form** *iff the pair $(\beta, d\beta)$ is nondegenerate.*

Remark 2.3. That the restriction of a contact form to a cooriented hypersurface is a germ of a contact form was shown in Lemma 2.1. Conversely, let $\beta \in \Omega^1(\Sigma)$ be the germ of a contact form on an oriented $2\mathsf{n}$-manifold Σ. Then Lemma 1.2 asserts that there exists a pair $(f, \gamma) \in \Omega^0(\Sigma) \times \Omega^1(\Sigma)$ such that the $2\mathsf{n}$-form ρ in (2.2) is a positive volume form on Σ. In this situation define $\alpha \in \Omega^1(\mathbb{R} \times \Sigma)$ by

$$\alpha := \beta + t(df - \gamma) + f dt, \tag{2.3}$$

where t denotes the coordinate on $\mathbb{R}$. Then α restricts to a contact form on a neighborhood of $\{0\} \times \Sigma$ and its pullback under the inclusion $\iota : \Sigma \to \mathbb{R} \times \Sigma$, defined by $\iota(x) := (0, x)$, is the 1-form $\iota^*\alpha = \beta$. Thus Σ embeds as a cooriented hypersurface into a contact manifold (M, ξ) such that β is the restriction of a contact form for ξ to Σ. Note that, if $\gamma = df$, then $\alpha := \beta + f dt$ is a contact form on all of $\mathbb{R} \times \Sigma$.

The characteristic foliation

Let $\Sigma \subset M$ be a cooriented hypersurface, let $\alpha \in \Omega^1(M)$ be a positive contact form for ξ, and define $\beta := \alpha|_\Sigma \in \Omega^1(\Sigma)$. Then Lemma 2.1 asserts that the pair $(\beta, d\beta)$ is nondegenerate (Definition 1.1) and hence determines a characteristic foliation (Definition 1.4). In this case the characteristic foliation can be expressed in the form

$$\ell_x := \{v \in \xi_x \cap T_x\Sigma \mid d\alpha(v, w) = 0 \text{ for all } w \in \xi_x \cap T_x\Sigma\}. \tag{2.4}$$

For each $x \in \Sigma$ the subspace ℓ_x is independent of the choice of α, has dimension zero if and only if $T_x\Sigma = \xi_x$, and has dimension one otherwise.

Now let ν, γ, f, ρ be as in Lemma 2.1 and define X, Y, λ as in Lemma 1.5 by (1.4) with $\tau := d\beta$. Thus

$$\iota(X)\rho = \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1}, \qquad \iota(Y)\rho = \mathsf{n}(d\beta)^{\mathsf{n}-1} \wedge \gamma, \qquad \lambda\rho = (d\beta)^{\mathsf{n}}. \tag{2.5}$$

Then Lemma 1.5 asserts, in particular, that the leaves of the characteristic foliation are the integral curves of X. The lemma also asserts that the zeros of X are the zeros of β, the zeros of Y are the zeros of γ, and that

$$\beta(X) = \gamma(Y) = 0, \qquad \beta(Y) + \gamma(X) = 0, \tag{2.6}$$

$$\iota(X)d\beta = \lambda\beta, \qquad \iota(Y)d\beta = \lambda\gamma. \qquad f\lambda = \gamma(X) + 1. \tag{2.7}$$

The next lemma gives a formula for the Reeb vector field of α along Σ.

Lemma 2.4. *Let $\Sigma \subset M$ be a cooriented hypersurface, let $\alpha \in \Omega^1(M)$ be a positive contact form for ξ, let $R_\alpha \in \mathrm{Vect}(M)$ be the Reeb vector field of α, let $\nu, \beta, \gamma, f, \rho$ be as in Lemma 2.1, and define X, Y, λ by* (2.5)*. Then*

$$R_\alpha(x) = Y(x) + \lambda(x)\nu(x) \qquad \text{for all } x \in \Sigma. \tag{2.8}$$

Proof. Define $R := Y + \lambda\nu \in \Omega^0(\Sigma, T_\Sigma M)$. Then, by (2.6) and (2.7),

$$\alpha(R) = \beta(Y) + \lambda\alpha(\nu) = \beta(Y) + \lambda f = -\gamma(X) + \lambda f = 1.$$

Moreover, for $Z \in \mathrm{Vect}(\Sigma)$ and $s \in \Omega^0(\Sigma)$ it follows from (2.6), (2.7), and the definition of γ in Lemma 2.1 that

$$\begin{aligned} d\alpha(R, Z + s\nu) &= d\alpha(Y + \lambda\nu, Z + s\nu) \\ &= d\beta(Y, Z) + \lambda d\alpha(\nu, Z) + s d\alpha(Y, \nu) \\ &= d\beta(Y, Z) - \lambda\gamma(Z) + s\gamma(Y) = 0. \end{aligned}$$

Thus $\iota(R)d\alpha = 0$ and so R is the the restriction of the Reeb vector field of α to Σ. This proves Lemma 2.4. □

Definition 2.5 (**Orientations**). *Let $\Sigma \subset M$ be a cooriented hypersurface, let α be a positive contact form for ξ, define $\beta := \alpha|_\Sigma$, and let ρ be a positive volume form on Σ. Choose $x \in \Sigma$ such that ℓ_x has dimension one.*

(i) *The orientation of ℓ_x is determined by the vector $X(x)$ in* (2.5)*. Thus a nonzero vector $v \in \ell_x$ is positive if and only if the orientations of the space $T_x\Sigma/\ell_x$ induced by the $(2\mathsf{n}-1)$-forms $\iota(v)\rho$ and $\beta \wedge (d\beta)^{\mathsf{n}-1}$ agree.*

(ii) *The $(2\mathsf{n}-2)$-form $(d\beta)^{\mathsf{n}-1}$ descends to a volume form on the quotient space $(\xi_x \cap T_x\Sigma)/\ell_x$ and hence determines an orientation of this space.*

A local version of the following theorem, for submanifolds of any dimension, was proved by Arnol'd and Givental in [1, p 74, Theorem A]. The next theorem is a global version of their result for the special case of hypersurfaces with a slightly modified proof.

Theorem 2.6 (**Arnol'd–Givental**). *Let ξ, ξ' be cooriented contact structures on M inducing the same orientation and let $\Sigma \subset M$ be a cooriented hypersurface such that $\xi'_x \cap T_x\Sigma = \xi_x \cap T_x\Sigma$ for all $x \in \Sigma$. Assume also that ξ and ξ' induce the same orientations on the vector spaces ℓ_x and $(\xi_x \cap T_x\Sigma)/\ell_x$ whenever ℓ_x has dimension one. Then there exists an orientation preserving diffeomorphism $\phi : M \to M$ and an open neighborhood $U \subset M$ of Σ such that $\phi(x) = x$ for all $x \in \Sigma$ and $d\phi(x)\xi_x = \xi'_{\phi(x)}$ for all $x \in U$.*

The proof of Theorem 2.6 uses the following lemma.

Lemma 2.7 (Uniqueness of germs of contact forms). *Let $\beta, \beta' \in \Omega^1(\Sigma)$ be two germs of contact forms such that*

$$\ker(\beta_x) = \ker(\beta'_x) \qquad \textit{for all } x \in \Sigma. \tag{2.9}$$

Then there exists a nowhere vanishing function $h \in \Omega^0(\Sigma)$ such that $\beta' = h\beta$.

Proof. By (2.9) there exists a nowhere vanishing real valued function h on the open set $U := \{x \in \Sigma \,|\, \beta_x \neq 0\}$ such that $\beta'_x = h(x)\beta_x$ for all $x \in U$. We prove that h extends to a smooth function on all of Σ. To see this, choose coordinates $x_1, \dots, x_{2\mathsf{n}}$ on Σ in a neighborhood of a point in $\Sigma \setminus U$, write

$$\beta = \sum_i b_i dx_i, \qquad \beta' = \sum_i b'_i dx_i,$$

and suppose that $b_i(0) = b'_i(0) = 0$ for $i = 1, \dots, 2\mathsf{n}$. Then $d\beta$ is a symplectic form near the origin. Hence there exists a pair of indices $i \neq j$ such that $\partial b_i / \partial x_j(0) \neq 0$. Assume without loss of generality that $j = 1$. Then the coordinate system can be chosen such that $b_i(x_1, \dots, x_{2\mathsf{n}}) = x_1$. Since $b' = hb$ this implies $x_1 h(x_1, \dots, x_{2\mathsf{n}}) = b'_i(x_1, \dots, x_{2\mathsf{n}})$. Also $b'_i(x_1, \dots, x_{2\mathsf{n}}) = 0$ whenever $x_1 = 0$, and so

$$\begin{aligned} h(x_1, \dots, x_{2\mathsf{n}}) &= \frac{1}{x_1} b'_i(x_1, \dots, x_{2\mathsf{n}}) \\ &= \frac{1}{x_1} \int_0^{x_1} \partial_1 b'_i(s, x_2, \dots, x_{2\mathsf{n}})\, ds \\ &= \int_0^1 \partial_1 b'_i(sx_1, x_2, \dots, x_{2\mathsf{n}})\, ds. \end{aligned}$$

The right hand side of this equation extends to a smooth function in a neighborhood of the origin and agrees with h wherever h is defined. Hence h extends to a smooth function on all of Σ, still denoted by $h : \Sigma \to \mathbb{R}$. At every point $x \in \Sigma$ at which β vanishes the 2-forms $d\beta$ and $d\beta' = d(h\beta)$ are both symplectic, and this implies $h(x) \neq 0$ for all $x \in \Sigma$. This proves Lemma 2.7. □

Remark 2.8. The main argument in the proof of Lemma 2.7 is the assertion that, if $f, g : \Sigma \to \mathbb{R}$ are smooth functions such that zero is a regular value of f and $f^{-1}(0) \subset g^{-1}(0)$, then the function $h := g/f : \Sigma \setminus f^{-1}(0) \to \mathbb{R}$ extends uniquely to a smooth function on all of Σ.

Proof of Theorem 2.6. Let α be a positive contact form for ξ. We prove in four steps that there exists an orientation preserving diffeomorphism ϕ of M such that $\phi|_\Sigma = \mathrm{id}$ and $(\phi^{-1})^*\alpha$ is a contact form for ξ' near Σ.

Step 1. *There exists a positive contact form α' for ξ' such that $\alpha'|_\Sigma = \alpha|_\Sigma$.*

Choose any positive contact form α' for ξ' and define $\beta' := \alpha'|_\Sigma$, $\beta := \alpha|_\Sigma$. Then $\ker(\beta'_x) = \ker(\beta_x)$ for all $x \in \Sigma$ by assumption. Hence, by Lemma 2.7 there exists a nowhere vanishing function $h \in \Omega^0(\Sigma)$ such that $\beta' = h\beta$. To prove that h is positive, choose a positive transverse vector field ν along Σ and define $\rho := \iota(\nu)\big(\alpha \wedge (d\alpha)^{\mathsf{n}-1}\big)|_\Sigma$ and $\rho' := \iota(\nu)\big(\alpha' \wedge (d\alpha')^{\mathsf{n}-1}\big)|_\Sigma$. By assumption the volume forms $\alpha \wedge (d\alpha)^{\mathsf{n}}$ and $\alpha' \wedge (d\alpha')^{\mathsf{n}}$ induce the same orientation on M and so ρ and ρ' induce the same orientation on Σ. Define the vector fields X, X' on Σ by $\iota(X)\rho = \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1}$ and $\iota(X')\rho' = \mathsf{n}\beta' \wedge (d\beta')^{\mathsf{n}-1}$. Now let $x \in \Sigma$ such that ℓ_x has dimension one. Then, by assumption, the vectors $X(x)$ and $X'(x)$ are related by a positive factor. Since $X' = h^{\mathsf{n}}(\rho/\rho')X$, it follows that $h^{\mathsf{n}} > 0$. Also the orientations on $(\xi_x \cap T_x\Sigma)/\ell_x$ induced by the symplectic forms $d\beta_x$ and $d\beta'_x$ agree. Hence $h^{\mathsf{n}-1} > 0$ and so $h > 0$.

Now extend h to a positive function on all of M to obtain a positive contact form $\alpha'' := h^{-1}\alpha'$ for ξ' that satisfies $\alpha''|_\Sigma = \alpha|_\Sigma$. This proves Step 1.

Step 2. *Let α' be as in Step 1 and let ν be a positive transverse vector field along Σ. Then there exists a positive transverse vector field ν' along Σ such that $\alpha(\nu) = \alpha'(\nu')$.*

Define the open set $U := \{x \in \Sigma \,|\, \beta_x \neq 0\}$ and note that $f(x) := \alpha(\nu(x))$ and $f'(x) := \alpha'(\nu(x))$ are nonzero and have the same sign for each $x \in \Sigma \setminus U$. Hence there exists a positive function $s : \Sigma \to \mathbb{R}$ such that $s = f/f'$ in a neighborhood of the set $\Sigma \setminus U$. Thus there exists a vector field $Y \in \mathrm{Vect}(\Sigma)$, supported in U, such that $\beta(Y) = f - sf'$. Then $\alpha'(s\nu + Y) = f = \alpha(\nu)$ and so the transverse vector field $\nu' := s\nu + Y$ satisfies the requirements of Step 2.

Step 3. *Let α' be as in Step 1 and let ν be as in Step 2. Then there exists an orientation preserving diffeomorphism $\psi : M \to M$ such that $\psi|_\Sigma = \mathrm{id}$ and $\alpha(\nu) = (\psi^*\alpha')(\nu)$.*

Choose ν' as in Step 2 and choose a diffeomorphism $\psi : M \to M$ that satisfies

$$\psi(x) = x, \qquad d\psi(x)\nu(x) = \nu'(x) \qquad \text{for all } x \in \Sigma. \tag{2.10}$$

Then ψ is orientation preserving and it follows from Step 2 that, for all $x \in \Sigma$,

$$(\psi^*\alpha')_x(\nu(x)) = \alpha'_{\psi(x)}(d\psi(x)\nu(x)) = \alpha'_x(\nu'(x)) = \alpha_x(\nu(x)).$$

More precisely, define an isotopy ψ_s in a neighborhood of Σ by choosing a Riemannian metric and taking $\psi_s(\exp_x(t\nu(x))) := \exp_x(st\nu'(x) + (1-s)t\nu(x))$ for $x \in \Sigma$, $t \in \mathbb{R}$ sufficiently small, and $0 \le s \le 1$. Now turn it into a global isotopy with compact support by multiplying the generating vector fields with a suitable cutoff function that is equal to one near Σ, and take $\psi := \psi_1$ to obtain a diffeomorphism that satisfies (2.10). This proves Step 3.

Step 4. *We prove Theorem 2.6.*

This is a Moser isotopy argument. By Step 1, there exists a positive contact form α' for ξ' such $\alpha'|_\Sigma = \alpha|_\Sigma$. Now choose ν as in Step 2 and choose ψ as in Step 3. Then, for each $s \in [0,1]$, the 1-form $\alpha_s := (1-s)\alpha + s\psi^*\alpha' \in \Omega^1(M)$ is a contact form near Σ. These contact forms satisfy

$$\alpha_0 = \alpha, \qquad \alpha_1 = \psi^*\alpha', \qquad \alpha_s|_\Sigma = \alpha|_\Sigma, \qquad \alpha_s(\nu) = \alpha(\nu) \tag{2.11}$$

for all s. Next choose a smooth family of vector fields $X_s \in \mathrm{Vect}(M)$ and a smooth family of functions $h_s \in \Omega^0(M)$ for $0 \le s \le 1$, both with compact support, that in a neighborhood of Σ satisfy the equation

$$\mathcal{L}_{X_s}\alpha_s + \partial_s\alpha_s = h_s\alpha_s. \tag{2.12}$$

More precisely, choose X_s such that $\alpha_s(X_s) = 0$ and $(\iota(X_s)d\alpha_s + \partial_s\alpha_s)|_{\xi_s} = 0$, where $\xi_s = \ker(\alpha_s)$, and define $h_s := (\partial_s\alpha_s)(R_s)$, where R_s is the Reeb vector field of α_s (so that $\iota(R_s)d\alpha_s = 0$ and $\alpha_s(R_s) = 1$). This determines X_s and h_s uniquely near Σ. Now multiply X_s by a suitable cutoff function to obtain vector fields with compact support.

Define the isotopy $\phi_s : M \to M$ by

$$\partial_s\phi_s = X_s \circ \phi_s, \qquad \phi_0 = \mathrm{id}. \tag{2.13}$$

Then by (2.11) the 1-form $\partial_s\alpha_s$ vanishes at every point $x \in \Sigma$ for all s and hence it follows from the definition of X_s and h_s that $X_s(x) = 0$ and $h_s(x) = 0$ for all $x \in \Sigma$ and all s. This implies $\phi_s(x) = x$ for all $x \in \Sigma$ and all s and so, in particular, there exists a neighborhood $U \subset M$ of Σ such that for each $s \in [0,1]$ the image $\phi_s(U)$ is contained in the domain where (2.12) holds. Hence

$$\partial_s\phi_s^*\alpha_s = \phi_s^*\left(\mathcal{L}_{X_s}\alpha_s + \partial_s\alpha_s\right) = \phi_s^*(h_s\alpha_s) \tag{2.14}$$

in U and this implies $\phi_s^*\alpha_s = \exp\left(\int_0^s (h_r \circ \phi_r)\,dr\right)\alpha$ in U for $0 \le s \le 1$. Hence the restriction of $\phi_s^*\alpha_s$ to U is a contact form for ξ for every s. Since $\phi_1^*\alpha_1 = \phi_1^*\psi^*\alpha'$, it follows that the diffeomorphism $\phi := \psi \circ \phi_1$ satisfies $d\phi(x)\xi_x = \xi'_{\phi(x)}$ for all $x \in U$. This proves Step 4 and Theorem 2.6. $\square$

Remark 2.9. Two contact forms for ξ that agree on a hypersurface need not be locally diffeomorphic by a diffeomorphism that restricts to the identity on the hypersurface. To see this, fix any nonempty closed connected cooriented hypersurface $\Sigma \subset M$, let $\alpha \in \Omega^1(M)$ be any positive contact form for ξ, let $\nu \in \Omega^0(\Sigma, T_\Sigma M)$ be a positive transverse vector field along Σ, and choose an element $x_0 \in \Sigma$ such that $d\alpha$ restricts to a degenerate 2-form on $T_{x_0}\Sigma$. Then the Reeb vector field R_α is tangent to Σ at x_0 (Lemma 2.4).

Now let $h \in \Omega^0(M)$ be a positive smooth function such that $h|_\Sigma = 1$ and suppose that there exists a diffeomorphism $\phi : M \to M$ satisfying

$$\phi|_\Sigma = \mathrm{id}, \qquad \phi^*\alpha = h\alpha \quad \text{near } \Sigma. \tag{2.15}$$

Since $R_\alpha(x_0) \in T_{x_0}\Sigma$, we have

$$d\phi(x_0)R_\alpha(x_0) = R_\alpha(x_0), \qquad dh(x_0)R_\alpha(x_0) = 0$$

and hence

$$\begin{aligned} 0 &= (\phi^* d\alpha)_{x_0}(R_\alpha(x_0), \nu(x_0)) \\ &= d(h\alpha)_{x_0}(R_\alpha(x_0), \nu(x_0)) \\ &= -dh(x_0)\nu(x_0). \end{aligned}$$

Thus, for any positive smooth function $h : M \to \mathbb{R}$ that satisfies $h|_\Sigma = 1$ and $dh(x_0)\nu(x_0) \neq 0$ there does not exist a diffeomorphism $\phi : M \to M$ that satisfies (2.15).

The linear counterpart to this observation is the following. If $\widehat{h} : M \to \mathbb{R}$ is a smooth function that vanishes on Σ and $X \in \mathrm{Vect}(M)$ is a vectorfield that vanishes on Σ and satisfies $\mathcal{L}_X\alpha + \widehat{h}\alpha = 0$ near Σ, then the function $f := \alpha(X) : M \to \mathbb{R}$ satisfies $f(x) = 0$ and $df(x)|_{\xi_x} = 0$ for all $x \in \Sigma$ as well as $df(R_\alpha) + \widehat{h} = 0$ near Σ. At every point $x_0 \in \Sigma$ where R_α is tangent to Σ this implies $d\widehat{h}(x_0) = 0$.

Remark 2.10. Let $\Sigma \subset M$ be a cooriented hypersurface in a $(2\mathsf{n}+1)$-dimensional contact manifold (M, ξ) and let $\alpha \in \Omega^1(M)$ be a contact form for ξ. Then the Reeb vector field R_α is transverse to Σ at every point $x \in \Sigma$ with $T_x\Sigma = \xi_x$. Thus, in a neighborhood $U \subset \Sigma$ of the set where β vanishes, the transverse vector field ν in Lemma 2.1 can be chosen to agree with either plus or minus the Reeb vector field, depending on the coorientation of Σ. With this choice, we have $\gamma = 0$, $f = \pm 1$, and $\rho = \pm(d\beta)^{\mathsf{n}}$ in U, and so the vector field X in (2.5) satisfies $\mathcal{L}_X(d\beta) = \pm\beta$ in U and hence is plus or minus a Liouville vector field in U with respect to the symplectic form $d\beta$.

3 Convex hypersurfaces

Let (M, ξ) be as in Section 2. A cooriented hypersurface $\Sigma \subset M$ is called **convex** iff there exists a contact vector field ν, defined in a neighborhood of Σ,, such that $\nu|_\Sigma$ is a positive transverse vector field. The set of such transverse contact vector fields in a given neighborhood of a convex hypersurface Σ is convex. The concept of a convex hypersurface was first introduced in [2].

Lemma 3.1 (Convexity). *Let $\Sigma \subset M$ be a nonempty cooriented closed connected hypersurface. Choose a positive contact form $\alpha \in \Omega^1(M)$ for ξ and define $\beta := \alpha|_\Sigma \in \Omega^1(\Sigma)$. Then the following are equivalent.*

(i) *Σ is convex.*

(ii) *There exists a smooth function $f : \Sigma \to \mathbb{R}$ such that*

$$\rho := f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge df \tag{3.1}$$

is a positive volume form on Σ.

Proof. We prove that (i) implies (ii). Thus assume that Σ is convex, choose a contact vector field ν in a neighborhood of Σ such that $\nu|_\Sigma$ is a positive transverse vector field, and define $f := \alpha(\nu)|_\Sigma$. Since ν is a contact vector field, there exists a smooth function $h : \Sigma \to \mathbb{R}$ such that $\mathcal{L}_\nu \alpha|_\Sigma = h\beta$. Hence $\iota(\nu)d\alpha|_\Sigma = h\beta - df$. By Lemma 2.1 this implies that the positive volume form $\rho := \iota(\nu)(\alpha \wedge (d\alpha)^{\mathsf{n}})|_\Sigma$ on Σ is given by (3.1) and so Σ satisfies (ii).

Conversely, assume (ii) and choose an orientation preserving diffeomorphism $\psi : \mathbb{R} \times \Sigma \to M$ onto a neighborhood U of Σ in M such that $\psi(0, x) = x$ for all $x \in \Sigma$. Then the 1-forms $\alpha_0 := \beta + f\,dt$ and $\alpha_1 := \psi^*\alpha$ are contact forms on $\mathbb{R} \times \Sigma$ that agree on $\{0\} \times \Sigma$ and satisfy the requirements of Theorem 2.6. Hence there exists a positive smooth function $h : \mathbb{R} \times \Sigma \to \mathbb{R}$ and an orientation preserving diffeomorphism $\phi : \mathbb{R} \times \Sigma \to \mathbb{R} \times \Sigma$ such that $\phi(0, x) = (0, x)$ for all $x \in \Sigma$ and $\phi^*\alpha_1 = h\alpha_0$ near $\{0\} \times \Sigma$. Now write $h_t(x) := h(t, x)$ to obtain

$$(\psi \circ \phi)^*\alpha = h_t(\beta + f dt) \quad \in \quad \Omega^1((-\varepsilon, \varepsilon) \times \Sigma).$$

Since $\partial/\partial t$ is a contact vector field for the contact structure determined by the contact form $h_t(\beta + f dt)$ on $\mathbb{R} \times \Sigma$, it follows that the vector field ν on U defined by $(\psi \circ \phi)^*\nu = \partial/\partial t$ is a contact vector field near Σ with respect to the contact structure ξ determined by α. Since $\psi \circ \phi : \mathbb{R} \times \Sigma \to U$ is an orientation preserving diffeomorphism, ν restricts to a positive transverse vector field along Σ. Hence Σ is convex and this proves Lemma 3.1. □

The next lemma gives a sufficient condition for convexity in terms of the existence of a Lyapunov function for the vector field X that generates the characteristic foliation.

Lemma 3.2. *Let $\Sigma \subset M$ be a nonempty cooriented closed connected hypersurface, choose a positive contact form $\alpha \in \Omega^1(M)$ for ξ and a positive volume form $\rho \in \Omega^{2n}(\Sigma)$, and define $\beta \in \Omega^1(\Sigma)$, $\lambda \in \Omega^0(\Sigma)$, and $X \in \mathrm{Vect}(\Sigma)$ by*

$$\beta := \alpha|_\Sigma, \qquad \iota(X)\rho := \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1}, \qquad \lambda\rho := (d\beta)^{\mathsf{n}}.$$

Suppose that there exists a Lyapunov function $h : \Sigma \to \mathbb{R}$ that satisfies

$$\beta_x \neq 0 \qquad \Longrightarrow \qquad dh(x)X(x) < 0, \tag{3.2}$$

$$\beta_x = 0 \qquad \Longrightarrow \qquad h(x)\lambda(x) > 0 \tag{3.3}$$

for every $x \in \Sigma$. Then Σ is convex.

Proof. By (3.3) there exists an open neighborhood $U \subset \Sigma$ of the set of zeros of X such that $h\lambda > 0$ on U. By (3.2) there exists an open neighborhood $V \subset \Sigma$ of the compact set $h^{-1}(0)$ such that $h\lambda - dh(X) > 0$ on V. Again by (3.2) the function $h^2 dh(X)$ is negative on the set $\Sigma \setminus (U \cup V)$. Hence there exists a constant $c > 0$ such that $ch^2 dh(X) < h\lambda$ on $\Sigma \setminus (U \cup V)$. This implies

$$h\lambda - dh(X) - ch^2 dh(X) > 0 \tag{3.4}$$

on all of Σ. Namely, on U the first term is positive and the other two terms are nonnegative, on V the sum of the first two terms is positive and the third term is nonnegative, and on $\Sigma \setminus (U \cup V)$ the sum of the first and third terms is positive and so is the middle term. Now define

$$f := e^{ch^2/2} h. \tag{3.5}$$

Then $df = e^{ch^2/2}(1 + ch^2)dh$ and hence

$$\begin{aligned} f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge df &= f\lambda\rho + (\iota(X)\rho) \wedge df \\ &= \big(f\lambda - df(X)\big)\rho \\ &= e^{ch^2/2}\big(h\lambda - dh(X) - ch^2 dh(X)\big)\rho. \end{aligned}$$

The right hand side is a positive volume form by (3.4) and hence it follows from Lemma 3.1 that Σ is convex. This proves Lemma 3.2. $\square$

Lemma 3.3 (**Giroux [5]**). *Let $\Sigma \subset M$ be a nonempty cooriented closed connected convex hypersurface and choose a contact vector field ν in a neighborhood of Σ such that $\nu|_\Sigma$ is a positive transverse vector field. Then the following holds.*
(i) *The set*

$$S := \left\{ x \in \Sigma \,\middle|\, \nu(x) \in \xi_x \right\} \tag{3.6}$$

is a nonempty smooth $(2\mathsf{n}-1)$-dimensional submanifold of Σ.
(ii) *Let S be as in (i). Then ξ is transverse to TS and the hyperplanes*

$$\eta_x := \xi_x \cap T_xS, \qquad x \in S, \tag{3.7}$$

define a cooriented contact structure on S.
(iii) *Let (S,η) be as in (ii). Then there exists a decomposition $\Sigma = \Sigma^+ \cup \Sigma^-$ into submanifolds with the common boundary*

$$\partial\Sigma^+ = \partial\Sigma^- = S = \Sigma^+ \cap \Sigma^-$$

and there exist exact symplectic forms $\omega^\pm = d\lambda^\pm$ on $\Sigma^\pm$, such that λ^+ and λ^- agree along S and define a positive contact form on (S,η).

Proof. Choose a positive contact form $\alpha \in \Omega^1(M)$ on (M,ξ) and let

$$\rho := \iota(\nu)\big(\alpha \wedge (d\alpha)^{\mathsf{n}}\big)\big|_\Sigma \quad \in \quad \Omega^{2\mathsf{n}}(\Sigma)$$

be the volume form in Lemma 2.1 associated to α and ν. Since ν is a contact vector field, this volume form is given by equation (3.1) as in the proof of Lemma 3.1. Thus we have

$$\rho = f(d\beta)^{\mathsf{n}} + n\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge df, \tag{3.8}$$

where $\beta \in \Omega^1(\Sigma)$ and $f \in \Omega^0(\Sigma)$ are given by

$$\beta := \alpha|_\Sigma\,, \qquad f := \alpha(\nu)|_\Sigma\,. \tag{3.9}$$

By (3.6) and (3.9) an element x of Σ belongs to the subset S if and only if

$$f(x) = \alpha_x(\nu(x)) = 0$$

and so $S = f^{-1}(0)$.

We prove that S is nonempty. Suppose, by contradiction, that this is wrong. Then f does not vanish anywhere and

$$d\frac{\beta}{f} = \frac{f\,d\beta + \beta\wedge df}{f^2}.$$

Hence it follows from (3.1) that

$$\left(d\frac{\beta}{f}\right)^{\mathsf{n}} = \frac{f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta\wedge(d\beta)^{\mathsf{n}-1}\wedge df}{f^{\mathsf{n}+1}} = \frac{\rho}{f^{\mathsf{n}+1}} \neq 0.$$

Thus $d(\beta/f)$ is an exact symplectic form on Σ, in contradiction to the assumption that Σ is a nonempty closed submanifold of M. This contradiction shows that the set S is nonempty as claimed.

Next it follows from (3.1) that the $2\mathsf{n}$-form

$$\tau := \beta\wedge(d\beta)^{\mathsf{n}-1}\wedge df \quad\in\quad \Omega^{2\mathsf{n}}(\Sigma)$$

does not vanish near $f^{-1}(0)$. Hence $df(x)$ is nonzero whenever $f(x)=0$. Thus zero is a regular value of f and so $S=f^{-1}(0)$ is a smooth $(2\mathsf{n}-1)$-dimensional submanifold of Σ. This proves part (i).

Since τ does not vanish along S it follows also that $\beta\wedge(d\beta)^{\mathsf{n}-1}$ restricts to a volume form on S and so $\beta|_S$ is a contact form on S. By (3.9) this contact form is precisely the restriction of α to S and so the associated contact structure $\eta := \ker(\beta)$ on S is given by (3.7). This proves part (ii).

To prove part (iii), define

$$\Sigma^{\pm} := \{x\in\Sigma \mid \pm f(x)\geq 0\} \tag{3.10}$$

and define $\omega^{\pm}\in\Omega^2(\Sigma^{\pm})$ by

$$\omega^{\pm} := d\lambda^{\pm}, \qquad \lambda^{\pm} := \left.\frac{\delta\beta}{\delta\pm f}\right|_{\Sigma^{\pm}} \in \Omega^1(\Sigma^{\pm}). \tag{3.11}$$

Here $\delta>0$ will be chosen sufficiently small. Since zero is a regular value of $f=\iota(X)\alpha|_{\Sigma}$, it follows that $\Sigma^+=f^{-1}([0,\infty))$ and $\Sigma^-=f^{-1}((-\infty,0])$ are submanifolds of Σ with the common boundary $S=\Sigma^+\cap\Sigma^-$. Next observe that $\omega^{\pm}=\delta d(\delta\pm f)^{-1}\beta = \delta(\delta\pm f)^{-2}\big((\delta\pm f)d\beta\pm\beta\wedge df\big)$ and hence

$$(\omega^{\pm})^{\mathsf{n}} = \frac{\delta^{\mathsf{n}}}{(\delta\pm f)^{\mathsf{n}+1}}\Big(\delta(d\beta)^{\mathsf{n}} \pm \big(f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta\wedge(d\beta)^{\mathsf{n}-1}\wedge df\big)\Big).$$

By (3.1) the right hand side does not vanish on $\Sigma^{\pm}$ whenever $\delta>0$ is chosen sufficiently small. Hence $\omega^{\pm}=d\lambda^{\pm}$ is an exact symplectic form on $\Sigma^{\pm}$. Moreover λ^+ and λ^- both agree with α on S and this proves Lemma 3.3. □

A deep theorem by Honda–Huang [6] asserts that every nonempty closed connected cooriented hypersurface Σ of a contact manifold (M, ξ) admits a C^0-small deformation to a convex hypersurface. In the intrinsic language, a corollary of this result is the assertion in Theorem 3.6 below about homotopy classes of nondegenerate pairs (Definition 1.1). Thus we will now assume that Σ is a nonempty closed connected oriented $2\mathsf{n}$-manifold.

Remark 3.4 (h-Principle). An h-principle argument shows that every homotopy class of nondegenerate pairs on Σ contains a pair of the form $(\beta, d\beta)$, where β is a germ of a contact form. A parametrized version of the h-principle asserts that two germs of contact forms β_0 and β_1 on Σ can be joined by a smooth path of germs of contact forms if and only if the pairs $(\beta_0, d\beta_0)$ and $(\beta_1, d\beta_1)$ can be joined by a smooth path of nondegenerate pairs (see [3]).

The following definition is motivated by Lemma 3.1.

Definition 3.5. *A* 1*-form* $\beta \in \Omega^1(\Sigma)$ *is called a* **convex germ of a contact form** *iff there exists a smooth function* $f \in \Omega^0(\Sigma)$ *such that the* $2\mathsf{n}$*-form*

$$\rho := f(d\beta)^{\mathsf{n}} + \mathsf{n}\beta \wedge (d\beta)^{\mathsf{n}-1} \wedge df \tag{3.12}$$

is a volume form on Σ.

Theorem 3.6 (Honda–Huang [6]). *Every homotopy class of nondegenerate pairs on* Σ *contains a pair of the form* $(\beta, d\beta)$*, where* $\beta \in \Omega^1(\Sigma)$ *is a convex germ of a contact form.*

Proof. See Honda–Huang [6] and Eliashberg–Pancholi [4]. □

Example 3.7. Let β be a closed 1-form on the 2-torus $\Sigma = \mathbb{T}^2$ without zeros. Then β is a germ of a contact form. Since $f d\beta + \beta \wedge df = -d(f\beta)$ is exact for every smooth function $f : \mathbb{T}^2 \to \mathbb{R}$, β is not convex.

Example 3.8. Let (Σ, σ) be a closed symplectic 2-manifold and choose a σ-compatible complex structure J, so $\langle \cdot, \cdot \rangle := \sigma(\cdot, J\cdot)$ is a Riemannian metric. Then the Laplace–Beltrami operator is given by $\Delta f := -d^* df = \frac{-d(df \circ J)}{\sigma}$ for $f \in \Omega^0(\Sigma)$. Now let $h : \Sigma \to \mathbb{R}$ be a Morse function such that

$$h(x)\Delta h(x) < 0 \qquad \text{for all } x \in \mathrm{Crit}(h). \tag{3.13}$$

Define $\beta := dh \circ J$. Then the vector field X and the function λ in Lemma 3.2 are given by $X = -\nabla h$ and $\lambda = -\Delta h$. Thus h is a Lyapunov function for X and the condition (3.13) asserts that $h\lambda > 0$ on the zeros of β. Hence Lemma 3.2 asserts that $\beta = dh \circ J$ is a convex germ of a contact form.

Example 3.9. It was shown by Giroux [5] that in dimension two a convex germ of a contact form can be obtained by a C^∞-small perturbation of any germ of a contact form. As an example consider the 2-torus $\Sigma := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the coordinates (x, y) and for $\varepsilon \geq 0$ let $\beta_\varepsilon \in \Omega^1(\mathbb{T}^2)$ be the 1-form

$$\beta_\varepsilon := dx + \varepsilon \sin(2\pi x) dy.$$

Define the function $f : \mathbb{T}^2 \to \mathbb{R}$ by

$$f(x, y) := \cos(2\pi x).$$

Then

$$f d\beta_\varepsilon + \beta_\varepsilon \wedge df = 2\pi\varepsilon\, dx \wedge dy,$$

so β_ε is a convex germ of a contact form (without zeros) for every $\varepsilon > 0$, while β_0 is a nonconvex germ of a contact form as noted in Example 3.7. Define the volume form on $\Sigma = \mathbb{T}^2$ by $\sigma := dx \wedge dy$. and define the vector field X_ε by $\iota(X_\varepsilon)\sigma := \beta_\varepsilon$. Then

$$X_\varepsilon = \varepsilon \sin(2\pi x)\partial/\partial x - \partial/\partial y$$

and so $df(X_\varepsilon) = -2\pi\varepsilon \sin^2(2\pi x) \leq 0$. The zero set of f splits the torus into two annuli and for $\varepsilon > 0$ the characteristic foliation has precisely two periodic orbits along which f attains its extremal values.

Remark 3.10. In dimension two the corollary of the Honda–Huang Theorem stated in Theorem 3.6 is much weaker than Giroux' theorem mentioned in Example 3.9. To prove it one can use the construction of Example 3.8 to find, for each integer k, a convex germ β of a contact form that satisfies $\chi(\beta, d\beta) = 2k$ (see Remark 1.6).

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