Lagrangian intersections in contact geometry (First draft)

Y. Eliashberg[*], H. Hofer[†], and D. Salamon[‡]
Stanford University[*]
Ruhr-Universität Bochum[†]
Mathematics Institute[‡]
University of Warwick
Coventry CV4 7AL Great Britain

October 1994

1 Introduction

Contact geometry

Let M be a compact manifold of odd dimension $2n+1$ and $\xi \subset TM$ be a **contact structure**. This means that ξ is an oriented field of hyperplanes and there exists a 1-form α on M such that

$$\xi = \ker\,\alpha, \qquad \alpha \wedge d\alpha^{\wedge n} \neq 0.$$

The last condition means that the restriction of $d\alpha$ to the kernel of α is non-degenerate. Any such form is called a **contact form** for ξ. Any contact form determines a **contact vector field** (sometimes also called the Reeb vector field) $Y : M \to TM$ via

$$\iota(Y)\mathrm{d}\alpha = 0, \qquad \iota(Y)\alpha = 1.$$

A **contactomorphism** is a diffeomorphism $\psi : M \to M$ which preserves the contact structure ξ. This means that $\psi^*\alpha = e^h\alpha$ for some smooth function $h : M \to \mathrm{R}$. A **contact isotopy** is a smooth family of contactomorphisms $\psi_s : M \to M$ such that

$$\psi_s^*\alpha = e^{h_s}\alpha$$

for all s. Any such isotopy with $\psi_0 = \mathrm{id}$ and $h_0 = 0$ is generated by **Hamiltonian functions** $H_s : M \to \mathrm{R}$ as follows. Each Hamiltonian function H_s determines a **Hamiltonian vector field** $X_s = X_{H_s} = X_{\alpha,H_s} : M \to TM$ via

$$\iota(X_s)\alpha = -H_s, \qquad \iota(X_s)d\alpha = dH_s - \iota(Y)dH_s \cdot \alpha,$$

and these vector fields determine ψ_s and h_s via

$$\frac{d}{ds}\psi_s = X_s \circ \psi_s, \qquad \frac{d}{ds}h_s = -\iota(Y)dH_s.$$

A **Legendrian submanifold** of M is an n-dimensional integral submanifold L of the hyperplane field ξ, that is $\alpha|_{TL} = 0$. An $n+1$-dimensional submanifold $L \subset M$ is called **Lagrangeable** if there exists a positive function $f : L \to \mathrm{R}$ such that the 1-form $f^{-1}\alpha|_L$ is closed This implies that L is foliated by Legendrian leaves. This definition is somewhat too strong. There are some interesting examples in which the function f cannot be chosen positive but vanishes on a subset of codimension 2. We shall discuss this in section 4.

Remark Let $\phi : M \to M$ be a contactomorphism with $\phi^*\alpha = e^g\alpha$. Then for any Hamiltonian function $H : M \to \mathrm{R}$ we have

$$\phi^* X_{\alpha,H} = X_{\phi^*\alpha, H\circ\phi} = X_{\alpha, e^{-g}H\circ\phi}.$$

Symplectization

Associated to a contact manifold M with contact form α is the **symplectization** $M \times \mathrm{R}$ with symplectic structure

$$\omega = \omega_\alpha = d(e^\theta\alpha) = e^\theta(d\alpha - \alpha \wedge d\theta).$$

Here θ denotes the coordinate on R. If $\psi : M \to M$ is a contactomorphism with $\psi^*\alpha = e^h\alpha$ then the diffeomorphism

$$(p,\theta) \mapsto (\psi(p), \theta - h(p))$$

of $M \times \mathrm{R}$ preserves the symplectic structure ω. Moreover every contact isotopy $\psi_s : M \to M$ corresponds to a Hamiltonian isotopy on $M \times \mathrm{R}$ determined by the homogeneous Hamiltonian functions $(p,\theta) \mapsto e^\theta H_s(p)$. The corresponding Hamiltonian differential equation is of the form

$$\dot{p} = X_s(p), \qquad \dot{\theta} = \iota(Y)dH_s(p).$$

If $L \subset M$ is a Legendrian submanifold then $L \times \mathrm{R}$ is a Lagrangian submanifold of $M \times \mathrm{R}$. If $L \subset M$ is a Lagrangeable submanifold of M and $f^{-1}\alpha|_L$ is closed then

$$L_f = \{(p, -\log(f(p))) \,|\, p \in L\} \subset M \times \mathrm{R}$$

is also a Lagrangian submanifold of $M \times \mathrm{R}$. This is the motivation for the term *Lagrangeable*.

Remark The relation between contact geometry and symplectic geometry has to be handled with care. The symplectic structure $\omega = \omega_\alpha = d(e^\theta\alpha)$ is not determined by the contact structure ξ but depends explicitly on the contact form α. Given a contactomorphism $\psi : M \to M$ it is sometimes convenient to work with the diffeomorphism $(p, \theta) \mapsto (\psi(p), \theta)$ which does not preserve the symplectic structure but satisfies

$$(\psi \times \mathrm{id})^* \omega_\alpha = \omega_{\psi^* \alpha}.$$

This will especially be the case when we discuss almost complex structures and J-holomorphic curves.

Lagrangian intersections

The goal of this paper is to construct Floer homology groups

$$HF_*(L_0, L_1; \alpha)$$

for intersections of a Legendrian submanifold L_1 and a Lagrangeable submanifold L_0. These groups are invariant under contact isotopies. They should be thought of as the Floer homology groups $HF_*(M \times \mathrm{R}, (L_0)_{f_0}, L_1 \times \mathrm{R})$ for the corresponding Lagrangian intersections in the symplectic manifold $M \times \mathrm{R}$. The noncompactness at $\theta \to +\infty$ is no problem due to convexity but new difficulties arise from the *black hole* at $\theta \to -\infty$. These can be overcome if the following hypotheses are satisfied.

(H1) There no nontrivial discs in M with boundary in L_0:

$$\pi_2(M, L_0) = 0.$$

(H2) Every periodic solution $\gamma : \mathrm{R}/T\mathrm{Z} \to M$ of the contact flow $\dot{\gamma} = Y(\gamma)$ represents a nontrivial homotopy class (of loops in M).

(H3) Every solution $\gamma : [0, T] \to M$ of the boundary value problem

$$\dot{\gamma} = Y(\gamma), \qquad \gamma(0) \in L_1, \qquad \gamma(T) \in L_1$$

represents a nontrivial homotopy class (of paths in M with endpoints in L_1).

The notions of *Legendrian* and *Lagrangeable submanifold* as well as *contact isotopy* are independent of the choice of the contact form α and one would expect the Floer homology groups $HF_*(L_0, L_1; \alpha)$ to be independent of α as well. However, we were not able to prove this. In fact, the conditions (H2) and (H3) above explicitly depend on α and the set of contact forms which satisfy these hypotheses is not dense. This is somewhat misfurtunate but may be related to the fact that Floer homology is an intrinsically symplectic concept and the symplectic structure does depend on α. In applications we shall consider pairs L_0, L_1 which satisfy in addition the following condition.

(H4) L_1 is a connected submanifold of L_0.

Under this assumtion the Floer homology groups $HF_*(L_0, L_1; \alpha)$ are isomorphic to the homology of L_1 and this implies the following intersection theorem.

Theorem A *Assume (H1), (H2), (H3), (H4). Let $\psi : M \to M$ be a Hamiltonian contactomorphism such that L_0 and $\psi^{-1}(L_1)$ intersect transversally. Then*

$$\# L_0 \cap \psi^{-1}(L_1) \geq \sum_{k=0}^{n} \dim H_k(L_1; \mathbf{Z}_2).$$

In particular the intersection $L_0 \cap \psi^{-1}(L_1)$ is always nonempty.

2 J-holomorphic curves

The space of paths

Fix a contact form α_0 on M and let $L_0 \subset M$ be a Lagrangeable submanifold with $f_0 : L_0 \to (0, \infty)$ such that ${f_0}^{-1}\alpha_0|_{L_0}$ is closed. We also fix a Legendrian submanifold $L_1 \subset M$. Denote by

$$\mathcal{P} = \mathcal{P}(L_0, f_0, L_1)$$

the space of paths $(\gamma, \theta) : [0, 1] \to M \times \mathbf{R}$ such that

$$\gamma(0) \in L_0, \qquad e^{\theta(0)} = \frac{1}{f_0(\gamma(0))}, \qquad \gamma(1) \in L_1.$$

If L_0 and L_1 satisfy hypothesis (H4) then there is a natural injection $L_1 \hookrightarrow \mathcal{P}$: $x \mapsto (\gamma_x, \theta_x)$ defined by

$$\gamma_x(s) = x, \qquad \theta_x(s) = -\log(f(x)).$$

Denote by $\mathcal{P}_0$ the component of $\mathcal{P}$ which contains the constant paths (γ_x, θ_x).

Lemma 2.1 *Assume (H1) and (H4). Then $(\gamma, \theta) \in \mathcal{P}_0$ if and only if there exists an extension $\gamma : [-1, 1] \to M$ of γ such that*

(i) *$\gamma(s) \in L_0$ for $-1 \leq s \leq 0$,*

(ii) *$\gamma(-1) \in L_1$,*

(iii) *the path $\gamma : [-1, 1] \to M$ is homotopic to a constant path in the space of paths in M with endpoints in L_1.*

Any two such extensions $\gamma_0, \gamma_1 : [-1, 0] \to L_0$ are homotopic in the space of paths $\beta : [-1, 0] \to L_0$ with $\beta(0) = \gamma(0)$ and $\beta(-1) \in L_1$.

Proof: First assume $(\gamma,\theta)\in\mathcal{P}_0$. Then there exists a homotopy $[-1,0]\to\mathcal{P}_0 : \lambda\mapsto(\gamma_\lambda,\theta_\lambda)$ such that $(\gamma_0,\theta_0)=(\gamma,\theta)$ and $\gamma_{-1}(s)\equiv x$, $\theta_{-1}(s)\equiv -\log(f_0(x))$ for some $x\in L_1$. The required extension is given by $s\mapsto\gamma_s(0)$ for $-1\le s\le 0$. Conversely, if the required extension exists then the homotopy $(\gamma_\lambda,\theta_\lambda)\in\mathcal{P}_0$ to the constant path is constructed from the homotopy of condition (iii). To prove uniqueness note that if $\gamma_0,\gamma_1:[-1,0]\to L_0$ are two extensions of $\gamma:[0,1]\to M$ then condition (iii) gives rise to a disc $u:D\to M$ whose boundary $u(\partial D)$ consists of the paths γ_0 and γ_1 together with a path in L_1 which connects the base points $\gamma_0(-1)$ and $\gamma_1(-1)$. By (H1) the loop $u(\partial D)$ is contractible in L_0. □

Lemma 2.2 *Assume (H1) and (H4). Then the injection $L_1\hookrightarrow\mathcal{P}_0$ induces an isomorphism of fundamental groups.*

Proof: The map $\pi_1(L_1)\to\pi_1(\mathcal{P}_0)$ induced by $x\mapsto(\gamma_x,\theta_x)$ is obviously injective: If a loop

$$\mathrm{R}/\mathrm{Z}\to\mathcal{P}_0:\lambda\mapsto(\gamma_\lambda,\theta_\lambda)$$

is contractible then so is the loop

$$\mathrm{R}/\mathrm{Z}\to L_1:\lambda\mapsto\gamma_\lambda(1).$$

To prove that the map is surjective we must prove the converse: If the loop $\lambda\mapsto\gamma_\lambda(1)$ is contractible in L_1 then the loop $\lambda\mapsto(\gamma_\lambda,\theta_\lambda)$ is contractible in $\mathcal{P}_0$. To see this assume without loss of generality that $\gamma_\lambda(1)=x\in L_1$ for all λ. Then use $\pi_2(M,L_0)=0$ (hypothesis (H1)) to construct the required homotopy. □

The tangent space $T_{(\gamma,\theta)}\mathcal{P}$ at a path $(\gamma,\theta)\in\mathcal{P}$ consists of all vectorfields $(v(s),\tau(s))\in T_{\gamma(s)}M\times\mathrm{R}$ along (γ,θ) which satisfy the boundary condition

$$v(0)\in T_{\gamma(0)}L_0,\qquad \tau(0)=-\frac{df_0(\gamma(0))v(0)}{f_0(\gamma(0))},\qquad v(1)\in T_{\gamma(1)}L_1.$$

Symplectic action

Let

$$\alpha_s=\phi_s^*\alpha=e^{g_s}\alpha_0$$

be a smooth family of contact forms generated by a contact isotopy such that $g_0=0$ and let $H_s:M\to\mathrm{R}$ be a smooth family of Hamiltonian fumctions. Assume that L_0 and L_1 satisfy hypotheses (H1) and (H4). Then the **symplectic action functional**

$$S=S_{\alpha,H}:\mathcal{P}_0\to\mathrm{R}$$

is defined by

$$S(\gamma,\theta) = \int_{-1}^{0} \frac{\alpha_0(\dot\gamma(s))}{f_0(\gamma(s))}\,ds + \int_0^1 e^{\theta(s)}\left(\alpha_s(\dot\gamma(s)) + H_s(\gamma(s))\right)\,ds$$

for $(\gamma,\theta) \in \mathcal{P}_0$. Here the extension $\gamma : [-1,0] \to M$ is chosen as in Lemma 2.1. It follows from the uniqueness part of the lemma and hypothesis (H1) that the action $S(\gamma,\theta)$ is independent of the choice of the extension.

Denote by $Y_s = \phi_s^* Y_0$ the contact vector field determined by α_s and by $X_s = X_{\alpha_s,H_s}$ the Hamiltonian vector field generated by H_s and α_s. Then the differential of the symplectic action in the direction of a variation $(v,\tau) \in T_{(\gamma,\theta)}\mathcal{P}$ is given by

$$\begin{aligned} dS(\gamma,\theta)(v,\tau) &= \int_0^1 e^{\theta}\Big(d\alpha_s(v,\dot\gamma - X_s(\gamma)) + \tau\alpha_s(\dot\gamma - X_s(\gamma))\Big)\,ds \\ &\quad - \int_0^1 e^{\theta}\alpha_s(v)\left(\dot\theta + \dot g_s(\gamma) - dH_s(\gamma)Y_s(\gamma)\right)\,ds. \end{aligned}$$

Here we have used the notation $\dot g_s = d/ds g_s$. The formula for dS shows that the critical points of a are in one-to-one correspondence with paths $\gamma : [0,1] \to M$ which satisfy the boundary value problem

$$\dot\gamma(s) = X_s(\gamma(s)), \qquad \gamma(0) \in L_0, \qquad \gamma(1) \in L_1.$$

(The path $\theta : [0,1] \to \mathrm{R}$ is uniquely determined by γ.) Let $\psi_s = \psi_{\alpha,H,s}$ denote the contact isotopy determined by $X_s = X_{\alpha_s,H_s}$ via

$$\frac{d}{ds}\psi_s = X_s \circ \psi_s, \qquad \psi_0 = \mathrm{id}.$$

Then the critical points of a correspond to paths

$$\gamma(s) = \psi_s(x), \qquad x \in L_0 \cap {\psi_1}^{-1}(L_1).$$

Moreover, the corresponding critical point (γ,θ) of a is nondegenerate if and only if L_0 and ${\psi_1}^{-1}(L_1)$ intersect transversally at x. In other words the action functional a is a Morse function if and only if L_0 and ${\psi_1}^{-1}(L_1)$ intersect transversally.

Remark Assume that the 1-form ${f_0}^{-1}\alpha_0|_{L_0}$ is exact:

$${f_0}^{-1}\alpha_0|_{L_0} = dF_0$$

for some function $F_0 : L_0 \to \mathrm{R}$. If in addition $L_1 \subset L_0$ and L_1 is connected then we may assume that F_0 vanishes on L_1. In this case the symplectic action functional is given by

$$S(\gamma,\theta) = F_0(\gamma(0)) + \int_0^1 e^{\theta(s)}\left(\alpha_s(\dot\gamma(s)) + H_s(\gamma(s))\right)\,ds.$$

for $(\gamma, \theta) \in \mathcal{P}_0$. This formula also works when hypothesis (H4) does not hold. In most applications, however, the 1-form ${f_0}^{-1}\alpha_0|_{L_0}$ will not be exact. If neither this holds nor (H4) then the symplectic action functional may not be a well defined function but may only give rise to a closed 1-form on the space of paths. In this case the corresponding Floer homology groups will be modules over a suitable Novikov ring as in [7]. We shall, however, not consider that case.

Almost complex structures

To discuss the gradient flow of the symplectic action we must choose a suitable metric on the loop space and this will require an almost complex structure on $M \times \mathrm{R}$. Since every symplectic vector bundle over any manifold admits a compatible almost complex structure, there exists an almost complex structure J on $\ker \alpha$ which is compatible with the symplectic structure $d\alpha$. We think of this structure as an endomorphism $J : TM \to TM$ such that

$$JY = 0, \qquad J^2 v = \alpha(v)Y - v$$

and the bilinear form

$$\langle v, w \rangle = d\alpha(v, Jw) + \alpha(v)\alpha(w)$$

defines a Riemannian metric on M. We shall call such a structure **compatible** with α. For any such J the automorphism $\widetilde{J} : TM \times \mathrm{R} \to TM \times \mathrm{R}$ defined by

$$\widetilde{J}(v, \tau) = (Jv + \tau Y, -\alpha(v))$$

is an almost complex structure on $M \times \mathrm{R}$ which is compatible with the symplectic form $\omega = d(e^\theta \alpha)$. The corresponding Riemannian metric is the product metric on $M \times \mathrm{R}$ (with the above metric on M) multiplied by e^θ.

Remark Let $\psi : M \to M$ be any diffeomorphism of M and assume that J is compatible with α. Then $\psi^* Y$ is the contact vectorfield for $\psi^*\alpha$ and $\psi^* J$ is an almost complex structure compatible with $\psi^*\alpha$.

Now choose a family of almost complex structures $J_s : TM \to TM$ which are compatible with α_s. Any such structure induces a metric on the path space $\mathcal{P}$ via

$$\|(v, \tau)\|_J = \int_0^1 e^\theta \left(d\alpha_s(v, J_s v) + |\alpha_s(v)|^2 + |\tau|^2 \right) \, ds$$

for $(v, \tau) \in T_{(\gamma,\theta)}\mathcal{P}$.

Gradient flow

The gradient flow of the symplectic action with respect to the above metric on $\mathcal{P}$ is given by

$$\operatorname{grad} S(\gamma,\theta) = \begin{pmatrix} -J_s(\dot\gamma - X_s(\gamma)) - (\dot\theta + \dot g_s(\gamma) - dH_s(\gamma)Y_s(\gamma))Y_s(\gamma) \\ \alpha_s(\dot\gamma - X_s(\gamma)) \end{pmatrix}.$$

Hence a gradient flow line of the symplectic action is a smooth map (u,θ) : $[0,1]\times \mathrm{R} \to M \times \mathrm{R}$ which satisfies the first order partial differential equation

$$\begin{gathered} \partial_s u - X_s(u) - \alpha_s(\partial_s u - X_s(u))Y_s(u) + J_s(u)\partial_t u = 0, \\ \partial_s\theta + \dot g_s(u) - dH_s(u)Y_s(u) = \alpha_s(\partial_t u), \\ \partial_t\theta = -\alpha_s(\partial_s u - X_s(u)) \end{gathered} \tag{1}$$

with boundary condition

$$u(0,t)\in L_0, \qquad e^{\theta(0,t)} = \frac{1}{f_0(u(0,t))}, \qquad u(1,t)\in L_1. \tag{2}$$

The following lemma can be proved by a direct computation which we leave to the reader.

Lemma 2.3 *Let* $\chi_s : M \to M$ *be a contact isotopy such that*

$$\frac{d}{ds}\chi_s = X_{\alpha_s,K_s}\circ\chi_s, \qquad \chi_s^*\alpha_s = e^{g'_s}\alpha,$$

and define

$$u'(s,t) = {\chi_s}^{-1}(u(s,t)), \qquad \theta'(s,t) = \theta(s,t).$$

Then u' *and* θ' *satisfy the same equation (1) with* α_s, Y_s, J_s, H_s, X_s *and* $\dot g_s$ *replaced by*

$$\alpha'_s = \chi_s^*\alpha_s, \qquad Y'_s = \chi_s^* Y_s, \qquad J'_s = \chi_s^* J_s,$$

$$H'_s = (H_s - K_s)\circ\chi_s, \qquad X'_s = \chi_s^*(X_s - X_{\alpha_s,K_s}), \qquad \dot g'_s = (\dot g_s - \iota(Y_s)dK_s)\circ\chi_s.$$

Moreover, u' *and* θ' *satisfy the boundary conditions (2) with* L_0, f_0, *and* L_1 *replaced by*

$$L'_0 = {\chi_0}^{-1}(L_0), \qquad f'_0 = f_0\circ\chi_0, \qquad L'_1 = {\chi_1}^{-1}(L_1).$$

Remarks

(i) The previous lemma shows that equation (1) can be reduced to the case where either $H_s \equiv 0$ (choose $\chi_s = \psi_s$) or $\alpha_s \equiv \alpha_0$ (choose $\chi_s = \phi_s$). Both cases will be of interest and we shall therefore continue to discuss the general situation.

(ii) If $H_s \equiv 0$, $\alpha_s \equiv \alpha_0$, and $J_s \equiv J_0$ then the solutions of (1) are Gromov's pseudoholomorphic curves in $M \times \mathrm{R}$ with respect to the above almost complex structure $\widetilde{J}(v,\tau) = (J_0 v + \tau Y_0, -\alpha_0(v))$. Such J-holomorphic curves have been recently used by Hofer [6] to prove existence theorems for periodic orbits on contact manifolds.

(iii) If α_s depends on s then so does the symplectic form $\omega_s = d(e^\theta \alpha_s)$. Thus the above equation seems more general than the corresponding equation in the usual Floer homology theory for Lagrangian intersections. This, however, is due to notational convenience. The diffeomorphism $\phi_s^{-1} \times \mathrm{id}$ of $M \times \mathrm{R}$ transforms ω_s into ω_0. (Compare Lemma 2.3 with $\chi_s = \phi_s$.)

Energy

The **energy** of a solution (u,θ) of (1) and (2) is given by

$$
\begin{aligned}
E(u,\theta) &= \int_{-\infty}^{\infty}\int_0^1 e^\theta \left(\alpha_s(\partial_t u, J_s(u)\partial_t u) + |\alpha_s(\partial_t u)|^2 + |\partial_t \theta|^2\right) dsdt \\
&= \int_{-\infty}^{\infty}\int_0^1 (u,\theta)^* d\left(e^\theta \alpha_s + e^\theta H_s ds\right).
\end{aligned}
$$

Here d is the differential with respect to all variables $(p,\theta,s) \in M \times \mathrm{R} \times \mathrm{R}$ and (u,θ) is to be understood as the map $(s,t) \mapsto (u(s,t), \theta(s,t), s)$. It turns out that if the energy is finite and L_0 and ${\psi_1}^{-1}(L_1)$ intersect transversally then the limits

$$
\lim_{t\to\pm\infty} u(s,t) = \psi_s(x^\pm), \qquad x^\pm \in L_0 \cap {\psi_1}^{-1}(L_1) \tag{3}
$$

exist (see Proposition 2.5 below). In this case the energy is given by

$$
E(u,\theta) = -\int_{-\infty}^{\infty} \frac{\alpha_0(\partial_t u(0,t))}{f_0(u(0,t))}\, dt
$$

Hence the energy depends only on the homotopy class of paths in L_0 from x^- to x^+ determined by $u(0,t)$. If the hypotheses (H1) and (H4) are satisfied then the energy is uniformly bounded and is given by

$$
E(u,\theta) = S(\psi_s(x^-)) - S(\psi_s(x^+)) \tag{4}
$$

whenever $(u(\cdot,t), \theta(\cdot,t)) \in \mathcal{P}_0$ for all t.

Now a Floer complex can be defined as usual by counting the connecting orbits when the relative Morse index is 1. Under the hypotheses (H1), (H2), (H3) there are no obstructions to compactness and the usual theory carries through. The crucial point is the following compactness theorem for the solutions of (1) and (2) with bounded energy. The proof will be given in section 5.

Theorem 2.4 *Assume that M, α_0, L_0, and L_1 satisfy the hypotheses (H1), (H2), and (H3). Then for every $c > 0$ the space*

$$\mathcal{M}^c = \mathcal{M}^c(L_0, L_1; \alpha, H, J, f_0)$$

of all smooth solutions (u, θ) of the boundary value problem (1) and (2) which satisfy the energy bound

$$E(u, \theta) \leq c$$

is compact (with respect to the topology of uniform convergence with all derivatives on compact sets).

If we knew a-priori that all the solutions of (1) and (2) would take values in a compact subset of $M \times \mathrm{R}$ (that is $|\theta(s,t)| \leq C$ for some universal constant C) then the above theorem would follow directly from the usual compactness theory for Gromov's pseudoholomorphic curves (cf. [5] or [8]). To see this consider without loss of generality the case $\alpha_s \equiv \alpha_0$. Hypothesis (H1) guarantees that there are no J_0-holomorphic discs with boundary on the extended manifold $(L_0)_{f_0}$, there are obviously no such discs with boundary in $L_1 \times \mathrm{R}$, and since the symplectic form $\omega = d(e^\theta \alpha_0)$ is exact there are no J_0-holomorphic spheres. Hence there could be no bubbling (for J_0-holomorphic curves with values in a compact set) and Theorem 2.4 would follow. Now in the case $\alpha_s \equiv \alpha_0$ and $H_s \equiv 0$ The function θ is harmonic and its normal derivative vanishes on $s = 1$. Hence it follows from the maximum principle that at least in this case θ is bounded above by a universal constant $(-\log(\inf_{L_0} f_0))$. In contrast it is not at all obvious that the functions θ for $(u, \theta) \in \mathcal{M}^c$ are uniformly bounded from below. Such a bound will only exist if the hypotheses (H2) and (H3) are satisfied as well. The proof requires an analysis of J-holomorphic planes and halfplanes. In the case of planes this was recently carried out by Hofer (cf. [6]) and we shall explain his results and their generalizations to half planes in section 5.

Heuristically the argument is as follows: If there is a sequence $(u_\nu, \theta_\nu) \in \mathcal{M}^c$ and a sequence of points $z_\nu = s_\nu + it_\nu$ such that $c_\nu = \theta_\nu(z_\nu) \to -\infty$ the following happens. Assume without loss of generality that $s_\nu \to s^*$ and denote $\alpha = \alpha_{s^*}$, $J = J_{s^*}$, $Y = Y_{s^*}$. If z_ν stays clear of either boundary at $s = 0$ and $s = 1$ we may replace θ_ν by $\widetilde{\theta}_\nu = \theta_\nu - c_\nu$ to obtain another sequence of J-holomorphic curves $(u_\nu, \widetilde{\theta}_\nu)$ which stay away from $-\infty$ but whose energy diverges to ∞. However, after suitable rescaling we obtain in the limit a J-holomorphic plane

$$(u, \theta) : \mathrm{C} \to M \times \mathrm{R}$$

which has finite **contact energy**

$$E_0(u) = \int u^*(d\alpha) = T > 0 \tag{5}$$

and satisfies

$$\lim_{|z| \to \infty} \theta(z) = \infty. \tag{6}$$

Now for any such J-holomorphic plane the curves $\gamma_r(t) = u(re^{2\pi it/T})$ converge to a T-periodic solution $\gamma(t)$ of $\dot{\gamma} = Y(\gamma)$ as r tends to infinity. This periodic solution is necessarily contractible and by hypothesis (H2) no such solution exists. A similar argument in the case $s^* = 1$ (roughly speaking) shows that there would be a J-holomorphic half-plane

$$(u, \theta) : \mathrm{H} \to M \times \mathrm{R}, \qquad u(s, 0) \in L_1$$

which also satisfies (5) and (6). In this case the curves $\gamma_r(t) = u(re^{\pi it/T})$ will converge to a solution of $\dot{\gamma} = Y(\gamma)$ with $\gamma(0) \in L_1$ and $\gamma(T) \in L_1$ and this is excluded by hypothesis (H3). Finally, in the case $s^* = 0$ we would obtain a nonconstant J-holomorphic discs with boundary in $(L_0)_{f_0}$ and such discs do not exist by hypothesis (H1). It follows from these arguments that θ_ν remains bounded for any sequence $(u_\nu, \theta_\nu) \in \mathcal{M}^c$ and this will imply Theorem 2.4. The details are quite subtle and are based on the work of Hofer in [6]. They will be carried out in section 5.

We close this section with a corollary of Theorem 2.4 which guarantees the existence of the limits (3).

Proposition 2.5 *Assume that M, α_0, L_0, and L_1 satisfy the hypotheses (H1), (H2), and (H3). Assume further that L_0 and ${\psi_1}^{-1}L_1$ intersect transversally and that (u, θ) is a solution of (1) and (2) with finite energy $E(u, \theta) < \infty$. Then the limits (3) exist.*

Proof: Given any sequence $t_\nu \to \infty$ apply Theorem 2.4 to $u_\nu(s, t) = u(s, t + t_\nu)$ and $\theta_\nu(s, t) = \theta(s, t + t_\nu)$. It follows that a subsequence converges, uniformly with all derivatives on compact sets, to a J-holomorphic curve (u^*, θ^*) with zero energy. Any such function must be of the form

$$u^*(s, t) = \psi_s(x^+), \qquad x^+ \in L_0 \cap {\psi_1}^{-1}(L_1).$$

Now any sequence $u(s, t_\nu)$ with $t_\nu \to +\infty$ must converge to the same limit $\psi_s(x^+)$. Otherwise there would exist a sequence $t'_\nu \to \infty$ such that

$$d(u(0, t'_\nu), L_0 \cap {\psi_1}^{-1}(L_1)) \geq \delta > 0$$

for all ν. But this would lead to a contradiction as above. $\Box$

To prove the previous proposition it suffices to assume that the intersection points of L_0 and ${\psi_1}^{-1}(L_1)$ are isolated. In the transverse case it can in fact be proved that $u(s, t)$ converges exponentially to $\psi_s(x^\pm)$ for $t \to \pm\infty$ and $\partial_t u(s, t)$ converges to zero exponentially. This follows by standard arguments in Floer homology (cf. [2]) and we shall omit the proof.

3 Floer homology

Let M, L_0, f_0, L_1, α_s, J_s, and H_s be as above and assume that the hypotheses (H1-4) are satisfied. The Floer homology groups

$$HF^*(L_0, L_1; \alpha_s, J_s, H_s, f_0)$$

can roughly be described as the *middle dimensional homology groups* of the path space $\mathcal{P}_0 = \mathcal{P}_0(L_0, f_0, L_1)$. They are obtained from the gradient flow of the symplectic action

$$S_{\alpha,H} : \mathcal{P}_0 \to \mathrm{R}$$

as in Floer's original work on Lagrangian intersections in compact symplectic manifolds [1], [2], [3]. They can roughly be described as an infinite dimensional version of the Morse complex as described by Witten [12]. We summarize the main points of Floer's construction.

Assume that L_0 and ${\psi_1}^{-1}(L_1)$ intersect transversally. Then all the critical points of $S_{\alpha,H}$ are nondegenerate. Given two intersection points $x^\pm \in L_0 \cap {\psi_1}^{-1}(L_1)$ denote by

$$\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+, \alpha_s, H_s, J_s)$$

the space of all solutions $(u, \theta) : [0, 1] \times \mathrm{R} \to M \times \mathrm{R}$ of (1) with boundary condition (2) and limits (3). Denote by

$$\mathcal{M}_0(x^-, x^+) = \mathcal{M}_0(x^-, x^+, \alpha_s, H_s, J_s, f_0)$$

The subspace of those $(u, \theta) \in \mathcal{M}(x^-, x^+)$ such that the path $(u(\cdot, t), \theta(\cdot, t))$ is in $\mathcal{P}_0$ for every t. Linearizing the differential equation (1) gives rise to an operator

$$D_{(u,\theta)} : W^{1,2}_L((u, \theta)^*(TM \times \mathrm{R})) \to L^2((u, \theta)^*(TM \times \mathrm{R})).$$

Here $W^{1,2}_L((u, \theta)^*(TM \times \mathrm{R}))$ denotes the Sobolev space of all vector fields $(\xi(s, t), \tau(s, t)) \in T_{u(s,t)}M \times \mathrm{R}$ along (ξ, θ) which satisfy the boundary condition

$$\xi(0, t) \in T_{u(0,t)}L_0, \qquad \tau(0, t) = -\frac{df_0(u(0, t))\xi(0, t)}{f_0(u(0, t))}, \qquad \xi(1, t) \in T_{u(1,t)}L_0.$$

The space $L^2((u, \theta)^*(TM \times \mathrm{R}))$ is defined similarly and $D_{(u,\theta)}$ is a Cauchy-Riemann operator. This operator is Fredholm whenever L_0 and ${\psi_1}^{-1}(L_1)$ intersect transversally. It's index is a relative Maslov class and can be defined as follows. Given $(u, \theta) \in \mathcal{M}(x^-, x^+)$ choose a symplectic trivialization

$$\Phi(s, t) : \mathrm{R}^{2n+2} \to T_{{\psi_s}^{-1}(u(s,t))}M \times \mathrm{R}$$

of $(u,\theta)^*(TM \times \mathrm{R})$ such that

$$\Phi(s,t)^*\omega_s = \sum_{j=0}^{n} dx_j \wedge dy_j$$

where $\omega_s = d(e^\theta \psi_s^* \alpha_s)$ and

$$\lim_{t\to\pm\infty} \Phi(s,t) = \Phi^\pm : \mathrm{R}^{2n+2} \to T_{x^\pm}M \times \mathrm{R}.$$

This gives rise to two Lagrangian paths in $\mathrm{R}^{2n+2} = \mathrm{R}^{2n+1} \times \mathrm{R}$:

$$\Lambda_0(t) = \Phi(0,t)^{-1} T_{(u(0,t),\theta(0,t))}(L_0)_{f_0}$$

and

$$\Lambda_1(t) = \Phi(1,t)^{-1} \left(T_{\psi_1{}^{-1}(u(1,t))} \psi_1{}^{-1}(L_1) \times \mathrm{R} \right).$$

These paths are transverse at $t = \pm\infty$ and therefore have a relative Maslov index $\mu(\Lambda_0, \Lambda_1)$ (cf. [1] and [9]). This index is independent of the choice of the trivialization. The Fredholm index of $D_{(u,\theta)}$ agrees with this Maslov index

$$\mathrm{index} D_{(u,\theta)} = \mu(u,\theta) = \mu(\Lambda_0, \Lambda_1)$$

whenever u and θ satisfy the boundary condition (2) and the limit condition (3) (cf. [1] and [10]). Now if $x^- = x^+$ then the Maslov index is zero.

Lemma 3.1 *If $x^- = x^+$ then $\mu(u,\theta) = 0$.*

Proof: In this case $\Lambda_0(+\infty) = \Lambda_0(-\infty)$ and $\Lambda_1(+\infty) = \Lambda_1(-\infty)$. So Λ_0 and Λ_1 are loops of Lagrangian planes and the relative Maslov index agrees with the difference of the ordinary Maslov indices of these loops

$$\mu(\Lambda_0, \Lambda_1) = \mu(\Lambda_1) - \mu(\Lambda_0)$$

(cf. [9]). Now it follows from Lemma 2.2 That there exists a loop $t \mapsto \Lambda(t)$ of Lagrangian planes in R^{2n} such that

$$\Lambda_0 \sim \Lambda \times \mathrm{R} \times 0, \qquad \Lambda_1 \sim \Lambda \times 0 \times \mathrm{R}.$$

and hence $\mu(\Lambda_0) = \mu(\Lambda_1)$. This proves the lemma. □

The previous lemma shows that there exists a map $\mu : L_0 \cap \psi_1{}^{-1}(L_1) \to \mathbf{Z}$ such that

$$\mathrm{index}\, D_{(u,\theta)} = \mu(x^-) - \mu(x^+)$$

whenever u and θ satisfy (2) and (3). Now everything is as usual. A triple (α, H, J) is called **regular** if L_0 and $\psi_1{}^{-1}(L_1)$ intersect transversally and the operator $D_{(u,\theta)}$ is onto for every $(u,\theta) \in \mathcal{M}(x^-, x^+)$ and every pair of intersection points $x^\pm \in L_0 \cap {\psi_1}^{-1}(L_1)$. By the Sard-Smale theorem the set

$$\mathcal{REG} = \mathcal{REG}(L_0, L_1)$$

of regular triples is dense in the set of all triples. The argument is as in [2] or [11]. Now for every triple $(\alpha, H, J) \in \mathcal{REG}$ the space $\mathcal{M}_0(x^-, x^+)$ is a finite dimensional manifold with

$$\dim \mathcal{M}_0(x^-, x^+) = \mu(x^-) - \mu(x^+).$$

If $\mu(x^-) - \mu(x^+) = 1$ then, by Theorem 2.4, the quotient space $\mathcal{M}(x^-, x^+)/\mathrm{R}$ consists of finitely many orbits and the numbers

$$n_2(x^-, x^+) = \#\mathcal{M}(x^-, x^+)/\mathrm{R} \pmod 2$$

determine the Floer chain complex as follows. The chain groups are defined by

$$CF_k = CF_k(L_0, L_1, H) = \sum_{\substack{x \in L_0 \cap {\psi_1}^{-1}(L_1) \\ \mu(x) = k}} \mathbf{Z}_2 \langle x \rangle.$$

and the boundary operator $\partial : CF_k \to CF_{k-1}$ is given by

$$\partial\langle x \rangle = \sum_{\mu(y) = k-1} n_2(x, y) \langle y \rangle$$

for $x \in L_0 \cap {\psi_1}^{-1}(L_1)$ with $\mu(x) = k$. As in Floer's original proof one uses gluing techniques to prove that $\partial \circ \partial = 0$ (cf. [3] and [11]). The **Floer homology groups** are now defined as the homology of this chain complex:

$$HF^*(L_0, L_1; \alpha_s, J_s, H_s) := H_*(CF, \partial).$$

They are invariant under Hamiltonian isotopy.

Theorem 3.2 *Assume (H1), (H2), (H3), (H4).*

(i) *For any two triples $(\alpha, J, H), (\alpha', J', H') \in \mathcal{REG}$ there is a natural isomorphism*

$$HF_*(L_0, L_1; \alpha_s, J_s, H_s) \to HF_*(L_0, L_1; \alpha'_s, J'_s, H'_s).$$

(ii) *For any triple $(\alpha, J, H) \in \mathcal{REG}$ and any Hamiltonian isotopy χ_s generated by vector fields X_{α_s, K_s} there exists a natural isomorphism*

$$HF_*(L_0, L_1; \alpha_s, J_s, H_s) \\ \to HF_*({\chi_0}^{-1}(L_0), {\chi_1}^{-1}(L_1); \chi_s^*\alpha_s, \chi_s^* J_s, (H_s - K_s) \circ \chi_s).$$

(iii) *For any triple* $(\alpha, J, H) \in \mathcal{REG}$ *there exists a natural isomorphism*

$$HF_*(L_0, L_1; \alpha_s, J_s, H_s) \to H_*(L_1; \mathbf{Z}_2).$$

The proof of this theorem is as usual in Floer homology and we refer to [3], [4] and [11] for more details in a slightly different context. In particular statement (ii) follows directly from Lemma 2.3. An immediate consequence of Theorem 3.2 is the following extension of the Arnold conjecture to contact manifolds.

Theorem 3.3 *Assume (H1), (H2), (H3), (H4). Let* $\psi : M \to M$ *be a Hamiltonian contactomorphism such that* L_0 *and* $\psi^{-1}(L_1)$ *intersect transversally. Then*

$$\#L_0 \cap \psi^{-1}(L_1) \geq \sum_{k=0}^{n} \dim H_k(L_1; \mathbf{Z}_2).$$

In particular the intersection $L_0 \cap \psi^{-1}(L_1)$ *is always nonempty.*

4 Examples

5 Compactness

Let M be a compact contact manifold with contact form α and corresponding contact vector field Y determined by $\iota(Y)d\alpha = 0$ and $\iota(Y)\alpha = 1$. Let $J : TM \to TM$ be an almost complex structure which is compatible with α. A J-holomorphic curve in $M \times \mathbf{R}$ (strictly speaking a $\widetilde{J}$-holomorphic curve, but we shall omit the tilde) on $\Omega \subset \mathbf{C}$ is a pair of functions $u : \Omega \to M$ and $\theta : \Omega \to \mathbf{R}$ which satisfy

$$\begin{aligned} \partial_s u - \alpha(\partial_s u)Y(u) + J(u)\Big(\partial_t u - \alpha(\partial_t u)Y(u)\Big) &= 0, \\ \partial_s \theta = \alpha(\partial_t u), \qquad \partial_t \theta &= -\alpha(\partial_s u). \end{aligned} \tag{7}$$

We shall consider the case of J-holomorphic planes ($\Omega = \mathbf{C}$), J-holomorphic half-planes ($\Omega = \mathbf{H} = \{s + it \,|\, t \geq 0\}$) with Legendrian boundary condition $u(\mathbf{R}) \subset L$ where $\alpha|_{TL} = 0$, and J-holomorphic discs ($\Omega = D = \{z \in \mathbf{C} \,|\, |z| \leq 1\}$) with Lagrangeable boundary conditions $u(z) \in L$, $e^{\theta(z)} = 1/f(u(z))$ for $|z| = 1$ where $f^{-1}\alpha|_L$ is closed. The **symplectic energy** of a J-holomorphic curve is defined by

$$E(u, \theta) = \int (u, \theta)^* d(e^\theta \alpha)$$

and the **contact energy** by

$$E_0(u) = \int u^* d\alpha.$$

Remark 5.1 If $\gamma : \mathrm{R} \to M$ is any solution of $\dot{\gamma} = Y(\gamma)$ then the functions

$$u(s,t) = \gamma(t), \qquad \theta(s,t) = s$$

determine a J-holomorphic plane. This plane has infinite symplectic energy but the contact energy is zero. If γ is periodic with period T then this formula determines a J-holomorphic cylinder. Similarly, if $\gamma(0) \in L_1$ we get a J-holomorphic half-plane $(u, \theta) : \mathrm{H} \to M \times \mathrm{R}$ and $u(\mathrm{R}) \subset L_1$. If γ satisfies the boundary conditions $\gamma(0) \in L_1$ and $\gamma(T) \in L_1$ we get a J-holomorphic strip with boundary values in L_1.

Remark 5.2 Given any positive smooth function $\phi : \mathrm{R} \to \mathrm{R}$ with positive derivative $\phi'(\theta) > 0$ define the ϕ-energy of a J-holomorphic curve $(u, \theta) : \Omega \to M \times \mathrm{R}$ on an open set $\Omega \subset \mathrm{C}$ by

$$E_\phi(u, \theta) = \int_\Omega (u, \theta)^* d(\phi\alpha).$$

A simple calculation using (7) shows that

$$\begin{aligned} E_\phi(u,\theta) &= \int_\Omega \Big(\phi(\theta) \left| \partial_s u - \alpha(\partial_s u) Y(u) \right|^2 + \phi'(\theta) \Big(\left|\alpha(\partial_s u)\right|^2 + \left|\alpha(\partial_t u)\right|^2 \Big) \Big) \\ &= \int_\Omega \Big(\phi(\theta) \Delta\theta + \phi'(\theta) \left|\nabla\theta\right|^2 \Big) \\ &= \int_{\partial\Omega} \phi(\theta) u^* \alpha \\ &= \int_{\partial\Omega} \phi(\theta) \frac{\partial\theta}{\partial\nu} \end{aligned}$$

Here Δ denotes the standard Laplacian and $\partial/\partial\nu$ the outward normal derivative on $\partial\Omega$. The symplectic energy corresponds to $\phi(\theta) = e^\theta$ and the contact energy to $\phi(\theta) = 1$.

J-holomorphic planes

Denote by

$$\mathcal{F}$$

the space of all smooth functions $\phi : \mathrm{R} \to \mathrm{R}$ such that $1/2 \le \phi(\theta) \le 1$ and $0 \le \phi'(\theta) \le 1$ for all θ. Given $\phi \in \mathcal{F}$ denote the shifted function by

$$\phi_c(\theta) = \phi(\theta + c).$$

In [6] Hofer proved essentially the following result.

Theorem 5.3 *Let M be a compact contact manifold with contact form α. Let J be compatible almost complex structure and assume that $(u, \theta) : \mathrm{C} \to M \times \mathrm{R}$ is a smooth solution of (7). Then the following statements are equivalent.*

(i) *u is nonconstant and* $\sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty$ *for some nonconstant* $\phi \in \mathcal{F}$.

(ii) *u is nonconstant and* $\sup_{\phi\in\mathcal{F}} E_\phi(u,\theta) < \infty$.

(iii) $\lim_{|z|\to\infty} \theta(z) = \infty$ *and* $0 < E_0(u) < \infty$.

If these conditions are satisfied then every sequence $r_\nu \to \infty$ has a subsequence (still denoted by r_ν) such that the limit

$$\gamma(t) = \lim_{\nu\to\infty} u(r_\nu e^{2\pi it/T})$$

exists (in the C^∞-topology) and defines a periodic solution of the differential equation

$$\dot{\gamma} = Y(\gamma)$$

with period $T = E_0(u)$.

Remark 5.4 The proof of the previous theorem shows that if (iii) holds then

$$\lim_{s\to\infty} \frac{\partial}{\partial s}\theta(e^{2\pi(s+it)}) = E_0(u)$$

and hence $\theta(z)$ diverges to ∞ like $(2\pi)^{-1}E_0(u)\log|z|$.

We reproduce here Hofer's proof of this theorem and then give an extension along similar lines to J-holomorphic half planes. The proof of Theorem 5.3 is based on four lemmata. The first is an observation about complete metric spaces due to Hofer.

Lemma 5.5 *Let M be a complete metric space and $f : M \to \mathrm{R}$ be continuous. Given $x \in M$ and $\varepsilon > 0$ there exist $\delta \in (0,\varepsilon)$ and $\xi \in B_\varepsilon(x)$ such that*

$$\sup_{B_\delta(\xi)} |f| \le 2|f(\xi)|, \qquad \delta|f(\xi)| \ge \tfrac{1}{2}\varepsilon|f(x)|.$$

Lemma 5.6 *Let $\phi \in \mathcal{F}$ be nonconstant and $\theta : \mathrm{C} \to \mathrm{R}$ be a harmonic function such that*

$$\sup_{c\in\mathrm{R}} e_{\phi_c}(\theta) < \infty, \qquad e_\phi(\theta) = \int_{\mathrm{C}} \phi'(\theta)|\nabla\theta|^2.$$

Then θ is constant.

Proof: First note that for any biholomorphic function $f : \mathrm{C} \to \mathrm{C}$

$$e_\phi(\theta \circ f) = e_\phi(\theta).$$

Secondly, if θ is nonconstant and satisfies a linear growth condition of the form $|\theta(z)| \le c(1 + |z|)$ then there exists a biholomorphic function $f(z) = az + b$ such that $\theta \circ f(z) = \mathrm{Re}\, z$. Hence in this case $e_\phi(\theta) = e_\phi(\theta \circ f) = \infty$. This proves the lemma in the case of linear growth.

If θ does not satisfy a linear growth condition then there exists a sequence z_ν such that

$$R_\nu = |\nabla\theta(z_\nu)| \to \infty.$$

By Lemma 5.5 we may assume without loss of generality that for some sequence $\varepsilon_\nu \to 0$ we have

$$\sup_{B_{\varepsilon_\nu}(z_\nu)} |\nabla\theta| \le 2R_\nu, \qquad \varepsilon_\nu R_\nu \to \infty.$$

It follows that the sequence $\theta_\nu(z) = \theta(z_\nu + z/R_\nu) - \theta(z_\nu)$ satisfies

$$\theta_\nu(0) = 0, \qquad |\nabla\theta_\nu(0)| = 1, \qquad \sup_{B_{\varepsilon_\nu R_\nu}(0)} |\nabla\theta_\nu| \le 2, \qquad e_\phi(\theta_\nu) = e_{\phi_{\theta(z_\nu)}}(\theta).$$

Hence there exists a subsequence (still denoted by θ_ν) which converges, uniformly with all derivatives on compact sets, to a harmonic function $\tilde\theta$ such that

$$\left|\nabla\tilde\theta(0)\right| = 1, \qquad \sup_{z\in\mathrm{C}} \left|\nabla\tilde\theta(z)\right| \le 2, \qquad e_\phi(\tilde\theta) < \infty.$$

By the first part of the proof the function $\tilde\theta$ must be constant in contradiction to $|\nabla\tilde\theta(0)| = 1$. This proves the lemma. $\square$

Lemma 5.7 *Let $(u,\theta) : \mathrm{C} \to M \times \mathrm{R}$ be a solution of (7) such that*

$$\sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty, \qquad E_0(u) = 0$$

for some nonconstant function $\phi \in \mathcal{F}$. Then u is constant.

Proof: If $E_0(u) = 0$ then the equation

$$\begin{aligned} \Delta\theta &= \partial_s\alpha(\partial_t u) - \partial_t\alpha(\partial_s u) \\ &= d\alpha(\partial_s u, \partial_t u) \\ &= |\partial_s u - \alpha(\partial_s u)Y(u)|^2 \end{aligned}$$

shows that $\theta : \mathrm{C} \to \mathrm{R}$ is harmonic. By Remark 5.2

$$\sup_{c\in\mathrm{R}} e_{\phi_c}(\theta) = \sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty$$

and, by Lemma 5.6, θ is constant. By (7) it follows that $\alpha(\partial_s u) \equiv 0$ and $\alpha(\partial_t u) \equiv 0$. Hence u is constant. $\square$

Lemma 5.8 *Let* $(u, \theta) : \mathbb{C}/iT\mathbb{Z} \to M \times \mathbb{R}$ *be a solution of (7) such that*

$$\sup_{c\in\mathbb{R}} E_{\phi_c}(u, \theta) < \infty$$

for some nonconstant function $\phi \in \mathcal{F}$. *Then*

$$\sup_{z\in\mathbb{C}} |du(z)| < \infty.$$

Proof: Suppose otherwise that there exists a sequence $z_\nu \in \mathbb{C}$ such that

$$R_\nu = |du(z_\nu)| \to \infty.$$

By Lemma 5.5 we may assume without loss of generality that for some sequence $\varepsilon_\nu \to 0$ we have

$$\sup_{B_{\varepsilon_\nu}(z_\nu)} |du| \le 2R_\nu, \qquad \varepsilon_\nu R_\nu \to \infty.$$

It follows that the sequences $u_\nu(z) = u(z_\nu + z/R_\nu)$ and $\theta_\nu(z) = \theta(z_\nu + z/R_\nu) - \theta(z_\nu)$ satisfy (7) and

$$|du_\nu(0)| = 1, \qquad \sup_{B_{\varepsilon_\nu R_\nu}(0)} |du_\nu| \le 2, \qquad \sup_{B_{\varepsilon_\nu R_\nu}(0)} |\nabla\theta_\nu| \le 2\, \|\alpha\|_{L^\infty},$$

and

$$\sup_{c\in\mathbb{R}} E_{\phi_c}(u_\nu, \theta_\nu) = \sup_{c\in\mathbb{R}} E_{\phi_c}(u, \theta).$$

Hence, by the usual elliptic bootstrapping argument for J-holomorphic curves in compact symplectic manifolds, there exists a subsequence (still denoted by u_ν and θ_ν) which converges, uniformly with all derivatives on compact sets, to a J-holomorphic curve $(\tilde{u}, \tilde{\theta}) : \mathbb{C} \to M \times \mathbb{R}$ such that

$$|d\tilde{u}(0)| = 1, \qquad \sup_{c\in\mathbb{R}} E_{\phi_c}(\tilde{u}, \tilde{\theta}) < \infty, \qquad E_0(\tilde{u}) = 0.$$

By Lemma 5.7 such a function $\tilde{u}$ cannot exist. This contradiction proves the lemma. □

Proof of Theorem 5.3: We prove that (iii) implies (ii). The function u is obviously nonconstant since θ is nonconstant. Moreover we shall prove that if $\theta(z)$ converges to infinity as $|z| \to \infty$ then

$$\sup_{\phi\in\mathcal{F}} E_\phi(u, \theta) = E_0(u). \tag{8}$$

To see this fix $\phi \in \mathcal{F}$ and $\varepsilon > 0$ and choose a sufficiently large regular value $c > 0$ of θ such that

$$\int_{\mathbb{C}\setminus\Omega_c} (u, \phi)^* d(\phi\alpha) \le \varepsilon, \qquad \Omega_c = \{z \in \mathbb{C} \,|\, \theta(z) \le c\}.$$

Then $\theta(z) = c = \sup_{\Omega_c} \theta$ for every $z \in \partial\Omega_c$ and hence,

$$\frac{\partial\theta}{\partial\nu}(z) \geq 0, \qquad z \in \partial\Omega_c.$$

This implies

$$\begin{aligned} E_\phi(u,\theta) - \varepsilon &\leq \int_{\Omega_c} (u,\theta)^* d(\phi\alpha) \\ &= \int_{\partial\Omega_c} \phi(\theta) \frac{\partial\theta}{\partial\nu} \\ &\leq \int_{\partial\Omega_c} \frac{\partial\theta}{\partial\nu} \\ &= \int_{\Omega_c} u^* d\alpha \\ &\leq E_0(u). \end{aligned}$$

This proves (8). Thus we have proved that (iii) implies (ii). Obviously (ii) implies (i).

Now assume (i). Then obviously $E_0(u) \leq 2E_\phi(u,\theta) < \infty$ and, by Lemma 5.7 $E_0(u) > 0$. We shall prove that every sequence $r_\nu \to \infty$ has a subsequence (still denoted by r_ν) such that

$$\lim_{\nu\to\infty} u(r_\nu e^{2\pi it/T}) = \gamma(t)$$

where $T = E_0(u)$, the limit is in the C^∞-topology, and $\gamma(t) = \gamma(t+T)$ is a periodic solution of $\dot\gamma = Y(\gamma)$.

To construct the subsequence define

$$\tilde u(s,t) = u(e^{2\pi(s+it)/T}), \qquad \tilde\theta(s,t) = \theta(e^{2\pi(s+it)/T}).$$

Then

$$\sup_{c\in\mathbf{R}} E_{\phi_c}(\tilde u, \tilde\theta) = \sup_{c\in\mathbf{R}} E_{\phi_c}(u,\theta) < \infty$$

and hence, by Lemma 5.8,

$$\sup_{z\in\mathbf{C}} |d\tilde u(z)| < \infty.$$

Now choose $s_\nu \to \infty$ such that $r_\nu = e^{2\pi s_\nu/T}$ and define

$$u_\nu(s,t) = \tilde u(s+s_\nu, t), \qquad \theta_\nu(s,t) = \tilde\theta(s+s_\nu,t) - \tilde\theta(s_\nu, 0).$$

Then $\sup_\nu \|du_\nu\|_{L^\infty} < \infty$ and it follows from the usual elliptic bootstrapping arguments for J-holomorphic curves (cf. [8]) that there exists a subsequence (still denoted by u_ν and θ_ν) which converges, uniformly with all derivatives on compact sets, to a J-holomorphic curve $(u^*, \theta^*) : \mathrm{C}/iT\mathrm{Z} \to M \times \mathrm{R}$ satisfying

$$E_0(u^*) = 0, \qquad \int_0^T \alpha(\partial_t u^*(0,t))\, dt = E_0(u) > 0.$$

It follows that $\Delta\theta^* = 0$ and hence there exists a unique function $\tau : \mathrm{C} \to \mathrm{R}$ such that $\theta^* + i\tau : \mathrm{C} \to \mathrm{C}$ is holomorphic and $\tau(0) = 0$. By (7) this implies

$$\alpha(\partial_s u^*) = -\partial_t\theta^* = \partial_s\tau, \qquad \alpha(\partial_t u^*) = \partial_s\theta^* = \partial_t\tau.$$

Since $E_0(u^*) = 0$ it follows that

$$u^*(s,t) = \gamma(\tau(s,t))$$

where $\gamma : \mathrm{R} \to M$ is the unique solution of

$$\dot\gamma(\tau) = Y(\gamma(\tau)), \qquad \gamma(0) = u^*(0).$$

Now recall that du^* is uniformly bounded and hence so is $\nabla\theta^*$. This implies that the holomorphic function $\theta^* + i\tau : \mathrm{C} \to \mathrm{C}$ satisfies a linear growth condition and hence there exist complex numbers a and b such that

$$\theta^*(z) + i\tau(z) = az + b.$$

Since $\theta^*(z + iT) = \theta^*(z)$ we have $a \in \mathrm{R}$ and since $\theta^*(0) = \tau(0) = 0$ we have $b = 0$. It follows that

$$\theta^*(s,t) = as, \qquad u^*(s,t) = \gamma(at)$$

for some real number $a \in \mathrm{R}$. Now the formula

$$T = E_0(u) = \int_0^T \alpha(\partial_t u^*(0,t))\, dt = \int_0^T \alpha(aY(\gamma(at)))\, dt = aT$$

shows that $a = 1$.

As a byproduct of this proof we obtain that every sequence $r_\nu \to \infty$ has a subsequence (still denoted by r_ν) such that

$$\lim_{\nu\to\infty} \left.\frac{\partial}{\partial s}\right|_{s=0} \theta(r_\nu e^{2\pi(s+it)/T}) = 1.$$

This implies the formula of Remark 5.4 and hence $\lim_{|z|\to\infty} \theta(z) = \infty$. In particular we have proved that (i) implies (iii). This completes the proof of Theorem 5.3. □

J-holomorphic half planes

Theorem 5.9 *Let M be a compact contact manifold with contact form α and $L \subset M$ be a compact Legendrian submanifold. Let J be compatible almost complex structure and assume that $(u, \theta) : \mathrm{H} \to M \times \mathrm{R}$ is a smooth solution of (7) such that $u(\mathrm{R}) \subset L$. Then the following statements are equivalent.*

(i) *u is nonconstant and $\sup_{c \in \mathrm{R}} E_{\phi_c}(u, \theta) < \infty$ for some nonconstant $\phi \in \mathcal{F}$.*

(ii) *u is nonconstant and $\sup_{\phi \in \mathcal{F}} E_\phi(u, \theta) < \infty$.*

(iii) *$\lim_{|z| \to \infty} \theta(z) = \infty$ and $0 < E_0(u) < \infty$.*

If these conditions are satisfied then every sequence $r_\nu \to \infty$ has a subsequence (still denoted by r_ν) such that the limit

$$\gamma(t) = \lim_{\nu \to \infty} u(r_\nu e^{\pi i t/T})$$

exists (in the C^∞-topology) and defines a solution of the boundary value problem

$$\dot{\gamma} = Y(\gamma), \qquad \gamma(0) \in L, \qquad \gamma(T) \in L$$

with $T = E_0(u)$.

Remark 5.10 The proof of the previous theorem shows that if (iii) holds then

$$\lim_{s \to \infty} \frac{\partial}{\partial s} \theta(e^{\pi(s+it)}) = E_0(u)$$

and hence $\theta(z)$ diverges to ∞ like $\pi^{-1} E_0(u) \log |z|$.

The proof of Theorem 5.9 is completely analogous to that of Theorem 5.3. It relies on the following three lemmata.

Lemma 5.11 *Let $\theta : \mathrm{H} \to \mathrm{R}$ be a solution of the Neumann boundary value problem*

$$\Delta\theta = 0, \qquad \partial_t \theta(s, 0) = 0.$$

If there exists a nonconstant function $\phi \in \mathcal{F}$ such that

$$\sup_{c \in \mathrm{R}} e_{\phi_c}(\theta) < \infty, \qquad e_\phi(\theta) = \int_{\mathrm{H}} \phi'(\theta) |\nabla\theta|^2,$$

then θ is constant.

Proof: The proof is analogous to that of Lemma 5.6. First, for any biholomorphic function $f : \mathrm{H} \to \mathrm{H}$

$$e_\phi(\theta \circ f) = e_\phi(\theta).$$

Secondly, if θ is nonconstant and satisfies a linear growth condition of the form $|\theta(z)| \leq c(1+|z|)$ then there exists a biholomorphic function $f(z) = (az+b)/(cz+d)$ such that $\theta \circ f(z) = \mathrm{Re}\, z$. Hence in this case $e_\phi(\theta) = e_\phi(\theta \circ f) = \infty$. This proves the lemma in the case of linear growth.

If θ does not satisfy a linear growth condition then, as before, there exist sequences $z_\nu \in \mathrm{H}$ and $\varepsilon_\nu \to 0$ such that

$$\sup_{B_{\varepsilon_\nu}(z_\nu)\cap \mathrm{H}} |\nabla\theta| \leq 2R_\nu, \qquad \varepsilon_\nu R_\nu \to \infty, \qquad R_\nu = |\nabla\theta(z_\nu)|.$$

If $R_\nu \mathrm{Im}\, z_\nu \to \infty$ we may assume without loss of generality that $\varepsilon_\nu < \mathrm{Im}\, z_\nu$ and argue as in the proof of Lemma 5.6. Otherwise we may assume that the sequence $R_\nu \mathrm{Im}\, z_\nu$ converges (to t^* for some $T^* > 0$) and consider the sequence $\theta_\nu(z) = \theta(\mathrm{Re}\, z_\nu + z/R_\nu) - \theta(\mathrm{Re}\, z_\nu)$. It satisfies

$$|\nabla\theta_\nu(iR_\nu \mathrm{Im}\, z_\nu)| = 1, \qquad \sup_{B_{\varepsilon_\nu R_\nu}(iR_\nu \mathrm{Im}\, z_\nu)\cap \mathrm{H}} |\nabla\theta_\nu| \leq 2,$$

and

$$\theta_\nu(iR_\nu \mathrm{Im}\, z_\nu) = 0, \qquad e_\phi(\theta_\nu) = e_{\phi_{\theta(\mathrm{Re}\, z_\nu)}}(\theta).$$

Hence there exists a subsequence (still denoted by θ_ν) which converges, uniformly with all derivatives on compact sets, to a harmonic function $\tilde\theta : \mathrm{H} \to \mathrm{R}$ such that $\partial_t\tilde\theta(s,0) = 0$ and

$$\left|\nabla\tilde\theta(it^*)\right| = 1, \qquad \sup_{z\in\mathrm{H}}\left|\nabla\tilde\theta(z)\right| \leq 2, \qquad e_\phi(\tilde\theta) < \infty.$$

By the first part of the proof the function $\tilde\theta$ must be constant in contradiction to $|\nabla\tilde\theta(it^*)| = 1$. This proves the lemma. □

Lemma 5.12 *Let* $(u,\theta) : \mathrm{H} \to M \times \mathrm{R}$ *be a solution of (7) such that* $u(s,0) \in L$ *and*

$$\sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty, \qquad E_0(u) = 0$$

for some nonconstant function $\phi \in \mathcal{F}$. *Then* u *is constant.*

Proof: Firstly, since $E_0(u) = 0$, the argument in the proof of Lemma 5.7 shows that $\theta : \mathrm{H} \to \mathrm{R}$ is harmonic. Secondly, since $u(s,0) \in L$ and $\alpha|_{TL} = 0$ we have

$$\partial_t\theta(s,0) = -\alpha(\partial_s u(s,0)) = 0.$$

Thirdly, by Remark 5.2,

$$\sup_{c\in\mathrm{R}} e_{\phi_c}(\theta) = \sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty.$$

Hence it follows from Lemma 5.11 that θ is constant. Since $E_0(u) = 0$ this implies that u is constant. □

Lemma 5.13 *Let $(u,\theta) : \mathrm{R} + i[0,T] \to M \times \mathrm{R}$ be a solution of (7) such that*

$$u(\mathrm{R} + i\{0,T\}) \subset L$$

and

$$\sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty$$

for some nonconstant function $\phi \in \mathcal{F}$. Then

$$\sup_{z\in\mathrm{R}+i[0,T]} |du(z)| < \infty.$$

Proof: If the conclusion does not hold choose a sequence $z_\nu = s_\nu + it_\nu$ such that $R_\nu = |du_\nu(s_\nu,t_\nu)| \to \infty$ and distinguish the cases (i) $R_\nu t_\nu$ bounded, (ii) $R_\nu(T - t_\nu)$ bounded, (iii) both sequences converge to infinity. In the latter case argue as in the proof of Lemma 5.8. In the first two cases use a similar argument and obtain a contradiction from Lemma 5.12 (Compare the proof of Lemma 5.11). The details are left to the reader. □

Proof of Theorem 5.9: The proof is analogous to that of Theorem 5.3 and we shall only explain the main points. To prove that (iii) implies (ii) choose a regular value $c \in \mathrm{R}$ of θ as before and consider the domain

$$\Omega_c = \{z \in \mathrm{H} \,|\, \theta(z) \le c\}.$$

Then the boundary of Ω has two parts

$$\partial\Omega_c = \Gamma_0 \cup \Gamma_1, \qquad \Gamma_0 = \Omega_c \cap \mathrm{R}, \qquad \Gamma_1 = \{z \in \partial\Omega \,|\, \mathrm{Im}\, z > 0\},$$

and we have $\partial\theta/\partial\nu = 0$ on Γ_0 and $\theta = c$ on Γ_1. As before we obtain

$$\begin{aligned}\int_{\Omega_c} (u,\theta)^* d(\phi\alpha) &= \int_{\Gamma_1} \phi(\theta)\frac{\partial\theta}{\partial\nu} \\ &\le \int_{\Gamma_1} \frac{\partial\theta}{\partial\nu} \\ &= \int_{\Omega_c} u^* d\alpha \\ &\le E_0(u).\end{aligned}$$

Since c can be chosen arbitrarily large this implies (8) as before.

Now assume (i). Then as before it follows from Lemma 5.12 that $T = E_0(u) > 0$. Define

$$\tilde{u}(s,t) = u(e^{\pi(s+it)/T}), \qquad \tilde{\theta}(s,t) = \theta(e^{\pi(s+it)/T}).$$

for $s \in \mathrm{R}$ and $0 \le t \le T$. Then

$$\sup_{c\in\mathrm{R}} E_{\phi_c}(\tilde{u},\tilde{\theta}) = \sup_{c\in\mathrm{R}} E_{\phi_c}(u,\theta) < \infty$$

and hence, by Lemma 5.13,

$$\sup_{z\in\mathbf{R}+i[0,T]} |d\tilde{u}(z)| < \infty.$$

Now choose $s_\nu \to \infty$ such that $r_\nu = e^{2\pi s_\nu/T}$ and define

$$u_\nu(s,t) = \tilde{u}(s+s_\nu,t), \qquad \theta_\nu(s,t) = \tilde{\theta}(s+s_\nu,t) - \tilde{\theta}(s_\nu,0).$$

Then $\sup_\nu \|du_\nu\|_{L^\infty} < \infty$ and it follows from the usual elliptic bootstrapping arguments for J-holomorphic curves with Lagrangian boundary conditions (cf. [8]) that there exists a subsequence (still denoted by u_ν and θ_ν) which converges, uniformly with all derivatives on compact sets, to a J-holomorphic curve $(u^*,\theta^*) : \mathbf{R}+i[0,T] \to M\times\mathbf{R}$ satisfying

$$E_0(u^*) = 0, \qquad u^*(\mathbf{R}+i\{0,T\}) \subset L, \qquad \int_0^T \alpha(\partial_t u^*(0,t))\,dt = E_0(u) > 0.$$

It follows that $\Delta\theta^* = 0$ and hence there exists a unique function $\tau : \mathbf{R}+i[0,T] \to \mathbf{R}$ such that $\theta^* + i\tau$ is holomorphic and $\tau(0) = 0$. As before it follows (7) and $E_0(u^*) = 0$ that

$$u^*(s,t) = \gamma(\tau(s,t))$$

where $\gamma : \mathbf{R} \to M$ is the unique solution of

$$\dot{\gamma}(\tau) = Y(\gamma(\tau)), \qquad \gamma(0) = u^*(0).$$

Again as before, $\nabla\theta^*$ is uniformly bounded and hence the holomorphic function $\theta^* + i\tau : \mathbf{R}+i[0,T] \to \mathbf{C}$ satisfies a linear growth condition. Since θ^* satisfies Neumann boundary conditions at $t=0$ and $t=T$ and $\theta^*(0) = \tau(0) = 0$ there exists a real number $a\in\mathbf{R}$ such that $\theta^*(z) + i\tau(z) = az$. Hence

$$\theta^*(s,t) = as, \qquad u^*(s,t) = \gamma(at)$$

and the formula

$$T = E_0(u) = \int_0^T \alpha(\partial_t u^*(0,t))\,dt = \int_0^T \alpha(aY(\gamma(at)))\,dt = aT$$

shows that $a = 1$. Moreover, the boundary conditions for u imply that $\gamma(0)\in L$ and $\gamma(T)\in L$.

As before we obtain as a byproduct of this proof the formula of Remark 5.10 and hence $\lim_{|z|\to\infty}\theta(z) = \infty$. This shows that (i) implies (iii) and completes the proof of Theorem 5.9. □

J-holomorphic discs

In this section we shall prove a removable singularity theorem for J-holomorphic discs in $M \times \mathbb{R}$ with boundary on a $\widetilde{L}$ where $L \subset M$ is a compact Lagrangeable submanifold. Here we assume as before that $f : L \to \mathbb{R}$ is a positive smooth function such that $f^{-1}\alpha|_L$ is closed. We also fix an almost complex structure J on M which is compatible with α.

Theorem 5.14 *Let $(u, \theta) : \mathbb{H} \to M \times \mathbb{R}$ be a smooth solution of (7) such that*

$$u(s,0) \in L, \qquad e^{\theta(s,0)} = \frac{1}{f(u(s,0))}$$

for $s \neq 0$ and

$$E(u,\theta) < \infty.$$

Then (u, θ) extends to a smooth J-holomorphic disc $\mathbb{H} \cup \{\infty\} \to M \times \mathbb{R}$. (This means that the map $z \mapsto (u(-1/z), \theta(-1/z))$ extends to a J-holomorphic half-plane at $z = 0$.)

Lemma 5.15 *Let $\theta : \mathbb{H} \to \mathbb{R}$ be a harmonic function such that*

$$\sup_{s \in \mathbb{R}} |\theta(s,0)| < \infty, \qquad e(\theta) = \int_{\mathbb{H}} e^{\theta} |\nabla \theta|^2 < \infty.$$

Then θ is bounded and, in fact,

$$\inf_{\mathbb{R}} \theta \leq \theta(z) \leq \sup_{\mathbb{R}} \theta$$

for all $z \in \mathbb{H}$.

Proof: The function $\tilde{\theta} : \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by

$$\tilde{\theta}(s,t) = \theta(e^{\pi(s+it)})$$

is harmonic, bounded on $\mathbb{R} \times \{0,1\}$, and satisfies

$$\int_{-\infty}^{\infty} \int_0^1 e^{\tilde{\theta}} \left|\nabla \tilde{\theta}\right|^2 \, dsdt < \infty.$$

Hence there exists a sequence $s_\nu \to \infty$ such that

$$\int_0^1 e^{\tilde{\theta}(s_\nu,t)} \left|\partial_t \tilde{\theta}(s_\nu,t)\right|^2 = {\varepsilon_\nu}^2 \to 0.$$

Since $\partial_t e^{\tilde{\theta}/2} = \frac{1}{2} e^{\tilde{\theta}/2} \partial_t \tilde{\theta}$ this implies

$$\begin{aligned} \left|e^{\tilde{\theta}(s_\nu,t)/2} - e^{\tilde{\theta}(s_\nu,0)/2}\right|^2 &\leq \int_0^1 \left|\partial_t e^{\tilde{\theta}(s_\nu,t)/2}\right|^2 \, ds \\ &= \tfrac{1}{4} \int_0^1 e^{\tilde{\theta}(s_\nu,t)} \left|\partial_t \tilde{\theta}(s_\nu,t)\right|^2 \, ds \\ &\leq \tfrac{1}{4} {\varepsilon_\nu}^2 \end{aligned}$$

and hence

$$\limsup_{\nu\to\infty} \sup_t \tilde{\theta}(s_\nu, t) \le \sup_{\mathrm{R}} \theta, \qquad \liminf_{\nu\to\infty} \inf_t \tilde{\theta}(s_\nu, t) \ge \inf_{\mathrm{R}} \theta.$$

A similar estimate holds for a suitable sequence $s'_\nu \to -\infty$. By the maximum principle for harmonic functions $\tilde{\theta}$ is bounded. □

Lemma 5.16 *Let $(u, \theta) : \mathrm{H} \to M \times \mathrm{R}$ be a J-holomorphic half plane such that $u(\mathrm{R}) \subset L$, $e^{\theta(s,0)} = 1/f(u(s, 0))$ and*

$$E(u, \theta) < \infty, \qquad E_0(u) = 0.$$

Then u is constant.

Proof: By Lemma 5.15 θ is bounded. Hence it follows from the usual removable singularity theorem for J-holomorphic discs in compact symplectic manifolds that (u, θ) extends to a J-holomorphic disc $\mathrm{H} \cup \{\infty\} \to M \times \mathrm{R}$ (see for example [8]). Hence

$$\gamma(t) = \gamma(t + 1) = u\left(\frac{\sin 2\pi t}{1 + \cos 2\pi t}\right) \in L$$

defines a loop in L such that

$$\dot{\gamma} = \alpha(\dot{\gamma})Y(\gamma).$$

Here we have used the fact that $E_0(u) = 0$ and hence each partial derivative of u is parallel to $Y(u)$. The identity

$$\int_0^1 \alpha(\dot{\gamma}(t))\, dt = E_0(u) = 0$$

shows that γ is contractible in L via

$$\gamma_\lambda(t) = \phi^{\lambda \int_0^t \alpha(\dot{\gamma}(s))\, ds}(\gamma(0)), \qquad 0 \le \lambda \le 1.$$

Here $\phi^t : M \to M$ denotes the flow of the contact vector field Y and $\gamma_\lambda([0, 1]) \subset \gamma([0, 1]) \subset L$. Now the symplectic energy of (u, θ) is given by

$$E(u, \theta) = \int_0^1 \frac{\alpha(\dot{\gamma}(t))}{f(\gamma(t))}\, dt.$$

Since γ is contractible in L and $f^{-1}\alpha|_L$ is closed it follows that $E(u, \theta) = 0$ and hence u is constant. □

Lemma 5.17 *If u and θ satisfy the assumptions of Theorem 5.14 then*

$$E_\phi(u,\theta) \le e^{c_1} E(u,\theta), \qquad c_1 = -\log\left(\sup_L f\right),$$

for every $\phi \in \mathcal{F}$.

Proof: By assumption $\theta(s,0) \ge -c_1$ for every $s \in \mathbb{R}$. Now choose $\varepsilon > 0$ such that $-c_1 - \varepsilon$ is a regular value of θ and denote

$$\Omega_\varepsilon = \{z \in \mathbb{H} \,|\, \theta(z) \le -c_1 - \varepsilon\}.$$

An argument as in the proof of Lemma 5.15 shows that Ω_ε consistes of (possibly infinitely many) bounded components and, moreover, $\partial\Omega_\varepsilon \cap \mathbb{R} = \emptyset$. By Stokes' theorem

$$\begin{aligned}\int_{\Omega_\varepsilon} (u,\theta)^* d(\phi\alpha) &= \int_{\partial\Omega_\varepsilon} \phi(\theta)\frac{\partial\theta}{\partial\nu} \\ &\le e^{c_1+\varepsilon}\int_{\partial\Omega_\varepsilon} e^\theta \frac{\partial\theta}{\partial\nu} \\ &= e^{c_1+\varepsilon}\int_{\Omega_\varepsilon} (u,\theta)^* d(e^\theta\alpha)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{C}\setminus\Omega_\varepsilon} (u,\theta)^* d(\phi\alpha) &\le \int_{\mathbb{C}\setminus\Omega_\varepsilon} (\Delta\theta + |\nabla\theta|^2) \\ &\le e^{c_1+\varepsilon}\int_{\mathbb{C}\setminus\Omega_\varepsilon} e^\theta(\Delta\theta + |\nabla\theta|^2) \\ &= e^{c_1+\varepsilon}\int_{\mathbb{C}\setminus\Omega_\varepsilon} (u,\theta)^* d(e^\theta\alpha)\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small the statement follows. □

Proof of Theorem 5.14: Consider the J-holomorphic strip $(\tilde{u},\tilde{\theta}) : \mathbb{R} \times [0,1] \to M \times \mathbb{R}$ defined by

$$\tilde{u}(s,t) = u(e^{\pi(s+it)}), \qquad \tilde{\theta} = \theta(e^{\pi(s+it)}).$$

We must prove that $d\tilde{u}$ is bounded. Then $\tilde{\theta}$ is bounded as well and hence the result follows from the usual removable singularity theorem for J-holomorphic discs in compact symplectic manifolds (cf. [8]). Assume, by contradiction, that there exists a sequence $z_\nu = s_\nu + it_\nu$ with $s_\nu \to \infty$ such that

$$R_\nu = |d\tilde{u}(z_\nu)| \to \infty.$$

By Lemma 5.5 we may assume without loss of generality that

$$\sup_{B_{\varepsilon_\nu}(z_\nu)\cap \mathbb{R}\times[0,1]} |d\tilde{u}| \le 2R_\nu, \qquad \varepsilon_\nu R_\nu \to \infty.$$

If $t_\nu R_\nu \to \infty$ and $(1-t_\nu)R_\nu \to \infty$ we may assume that $\varepsilon_\nu \le \min\{t_\nu, 1-t_\nu\}$ and argue as in the proof of Theorem 5.3 to obtain a nonconstant J-holomorphic plane $(u^*,\theta^*): \mathbb{C} \to M \times \mathbb{R}$ such that

$$\sup_{\phi\in\mathcal{F}} E_\phi(u^*,\theta^*) < \infty, \qquad E_0(u^*) = 0.$$

By Lemma 5.7 such planes do not exist. This contradiction shows that either of the sequences $t_\nu R_\nu$ or $(1-t_\nu)R_\nu$ is bounded. In the former case we may assume without loss of generality that $t_\nu R_\nu$ converges to $t^* \ge 0$. Consider the sequence

$$u_\nu(s,t) = \tilde{u}(s_\nu + s/R_\nu, t/R_\nu), \qquad \theta_\nu(s,t) = \tilde{\theta}(s_\nu + s/R_\nu, t/R_\nu).$$

This sequence satisfies

$$\begin{array}{rcl} |du_\nu(0,t_\nu R_\nu)| & = & 1, \\ \sup\limits_{B_{\varepsilon_\nu R_\nu}(0,t_\nu R_\nu)} |du_\nu| & \le & 2, \\ E(u_\nu,\theta_\nu; B_{\varepsilon_\nu R_\nu}(0,t_\nu R_\nu)) & \le & E(u,\theta). \end{array}$$

Hence a subsequence (still denoted by (u_ν,θ_ν)) converges, uniformly with all derivatives on compact sets, to a J-holomorphic half plane $(u^*,\theta^*): \mathbb{H} \to M\times\mathbb{R}$ such that $u^*(\mathbb{R}) \subset L$, $e^{\theta^*(s,0)} = 1/f(u^*(s,0))$ for $s\in\mathbb{R}$ and

$$|du^*(0,t^*)| = 1, \qquad E(u^*,\theta^*) < \infty, \qquad E_0(u^*) = 0.$$

By Lemma 5.16 such a half plane does not exist. A similar argument leads to a contradiction when the sequence $(1-t_\nu)R_\nu$ is bounded and this proves the theorem. □

Proof of Theorem 2.4

We shall now return to the gradient flow of the symplectic action $a: \mathcal{P} \to \mathbb{R}$ discussed in section 2. For every function $\phi \in \mathcal{F}$ and every solution (u,θ) of (1) and (2) define the ϕ-energy of (u,θ) by

$$\begin{aligned} E_\phi(u,\theta) &= \int_{-\infty}^{\infty}\int_0^1 \phi(\theta)\alpha_s(\partial_t u, J_s(u)\partial_t u) + \phi'(\theta)\left(|\alpha_s(\partial_t u)|^2 + |\partial_t\theta|^2\right) dsdt \\ &= \int_{-\infty}^{\infty}\int_0^1 (u,\theta)^* d\left(\phi\alpha_s + \phi H_s ds\right). \end{aligned}$$

The next lemma is the analogue of Lemma 5.17.

Lemma 5.18 *If (u,θ) is a solution of (1) and (2) with finite energy then*

$$E_\phi(u,\theta) \le e^{c_1} E(u,\theta), \qquad c_1 = -\log\left(\sup_{L_0} f_0\right),$$

for every $\phi \in \mathcal{F}$.

Proof: The proof is similar to that of Lemma 5.17. By (2) we have $\theta(0,t) \ge -c_1$ for every $t \in \mathrm{R}$. As before, denote

$$\Omega_\varepsilon = \{(s,t) \in [0,1] \times \mathrm{R} \,|\, \theta(s,t) \le -c_1 - \varepsilon\}$$

where $\varepsilon > 0$ and $-c_1 - \varepsilon$ is a regular value of θ. Since $E(u,\theta) < \infty$ it follows from an argument as in the proof of Lemma 5.15 that Ω_ε consistes of (possibly infinitely many) bounded components. Moreover, $\partial\Omega_\varepsilon \cap 0 \times \mathrm{R} = \emptyset$ and hence $\partial\Omega_\varepsilon$ consists of two parts $\partial\Omega_\varepsilon = \Gamma_0 \cup \Gamma_1$ where

$$\Gamma_0 = \{(s,t) \in \partial\Omega \,:\, 0 < s < 1\}, \qquad \Gamma_1 = \{(s,t) \in \partial\Omega \,:\, s = 1\}.$$

Now on Γ_1 we have $\alpha_s(\partial_t u) = 0$ and hence

$$\int_{\Gamma_1} (u,\theta)^*(\phi\alpha_s + \phi H_s ds) = 0.$$

This continues to hold with $\phi(\theta)$ replaced by e^θ. Hence

$$\begin{aligned}
E_\phi(u,\theta;\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} (u,\theta)^* d(\phi\alpha_s + \phi H_s ds) \\
&= \int_{\partial\Omega_\varepsilon} (u,\theta)^*(\phi\alpha_s + \phi H_s ds) \\
&= \int_{\Gamma_0} (u,\theta)^*(\phi\alpha_s + \phi H_s ds) \\
&= \phi(-c_1-\varepsilon) \int_{\Gamma_0} u^*(\alpha_s + H_s ds) \\
&= \phi(-c_1-\varepsilon) e^{c_1+\varepsilon} \int_{\Gamma_0} (u,\theta)^*(e^\theta\alpha_s + e^\theta H_s ds) \\
&= \phi(-c_1-\varepsilon) e^{c_1+\varepsilon} \int_{\Omega_\varepsilon} (u,\theta)^* d(e^\theta\alpha_s + e^\theta H_s ds) \\
&\le e^{c_1+\varepsilon} E(u,\theta;\Omega_\varepsilon).
\end{aligned}$$

Moreover, as in the proof of Lemma 5.17

$$E_\phi(u,\theta;\mathrm{C}\setminus\Omega_\varepsilon) \le e^{c_1+\varepsilon} E(u,\theta;\mathrm{C}\setminus\Omega_\varepsilon).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small this proves the lemma. □

Proof of Theorem 2.4: We shall prove that

$$\sup_{(u,\theta)\in\mathcal{M}^c} \|du\|_{L^\infty} < \infty \tag{9}$$

Here the norm of the linear map $du(s,t) : \mathrm{R}^2 \to T_{u(s,t)}M$ is to be understood with respect to the metric on M induces by α_s and J_s. (Of course all these norms are equivalent.) It then follows that

$$\sup_{(u,\theta)\in\mathcal{M}^c} \|\nabla\theta\|_{L^\infty} < \infty$$

and hence the gradient flow lines $(u,\theta) \in \mathcal{M}^c$ of a with $E(u,\theta) \le c$ all stay in a compact subset of $M \times \mathrm{R}$ and have uniformly bounded derivatives. Thus Theorem 2.4 follows from the usual elliptic bootstrapping arguments (see for example [8]).

The proof of the L^∞-estimate (9) relies on a standard bubbling argument already employed in the proof of Lemma 5.8, Lemma 5.13, and Theorem 5.14. Assume by contradiction that (9) fails. Then there exists a sequence $(u_\nu, \theta_\nu) \in \mathcal{M}^c$ and a sequence $s_\nu \in [0,1]$ such that

$$R_\nu = |du_\nu(s_\nu, 0)| \to \infty.$$

By Lemma 5.5 we may assume without loss of generality that there exists a sequence $\varepsilon_\nu \to 0$ such that

$$\sup_{B_{\varepsilon_\nu}(s_\nu,0)\cap[0,1]\times\mathrm{R}} |du_\nu| \le 2R_\nu, \qquad \varepsilon_\nu R_\nu \to \infty.$$

We may also assume without loss of generality that

$$s_\nu \to s^*.$$

Denote $\alpha = \alpha_{s^*}$, $Y = Y_{s^*}$ and $J = J_{s^*}$.

There are three cases. First assume that $s_\nu R_\nu \to \infty$ and $(1-s_\nu)R_\nu \to \infty$. Then we may assume $\varepsilon_\nu < \min\{s_\nu, 1-s_\nu\}$ and a rescaling argument as in the proof of Lemma 5.8 shows that there exists a nonconstant J-holomorphic plane $(u^*, \theta^*) : \mathrm{C} \to M \times \mathrm{R}$ such that

$$\sup_{\phi\in\mathcal{F}} E_\phi(u^*, \theta^*) < \infty.$$

This last inequality follows from Lemma 5.18. Hence it follows from Theorem 5.3 that there exists a periodic solution $\gamma(t) = \gamma(t+T)$ of $\dot\gamma = Y(\gamma)$ which represents a contractible loop in M. By hypothesis (H2) such a solution does not exist. This contradiction shows that either $s_\nu R_\nu$ or $(1-s_\nu)R_\nu$ is bounded.

If $(1-s_\nu)R_\nu$ is bounded then a rescaling argument as in the proof of Lemma 5.13 shows that there exists a nonconstant J-holomorphic half plane $(u^*,\theta^*):\mathrm{H}\to M\times\mathrm{R}$ such that $u^*(\mathrm{R})\subset L_1$ and

$$\sup_{\phi\in\mathcal{F}} E_\phi(u^*,\theta^*)<\infty.$$

Hence it follows from Theorem 5.9 that there exists a solution $\gamma:[0,T]\to M$ of the boundary value problem

$$\dot\gamma = Y(\gamma),\qquad \gamma(0)\in L_1,\qquad \gamma(1)\in L_1,$$

which is contractible in the space of paths in M with endpoints in L_1. By hypothesis (H3) such a solution does not exist. This contradiction shows that $s_\nu R_\nu\to 0$ is bounded.

If the sequence $s_\nu R_\nu$ is bounded a rescaling argument as in the proof of Theorem 5.14 shows that there exists a nonconstant J-holomorphic half plane $(u^*,\theta^*):\mathrm{H}\to M\times\mathrm{R}$ such that

$$u^*(\mathrm{R})\subset L_0,\qquad e^{\theta^*(s,0)}=\frac{1}{f_0(u^*(s,0))},\qquad E(u^*,\theta^*)<\infty.$$

By Theorem 5.14 this map extends to a J-holomorphic disc $(u^*,\theta^*):\mathrm{H}\cup\{\infty\}\to M\times\mathrm{R}$. The energy of this disc is given by

$$E(u^*,\theta^*)=\int_0^1\frac{\alpha_0(\dot\gamma(t))}{f_0(\gamma(t))}\,dt$$

where

$$\gamma(t)=\gamma(t+1)=u^*\left(\frac{\sin 2\pi t}{1+\cos 2\pi t}\right).$$

(Compare the proof of Lemma 5.16.) By hypothesis (H1) we have $\pi_2(M,L_0)=0$ and hence the loop γ is contractible in L_0. Since the form ${f_0}^{-1}\alpha_0|_{L_0}$ is closed it follows that $E(u^*,\theta^*)=0$ in contradiction to the fact that u^* is nonconstant. This final contradiction proves the L^∞-estimate (9) and this completes the proof of Theorem 2.4. □

References

[1] A. Floer, A relative Morse index for the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 393–407.

[2] A. Floer, The unregularized gradient flow of the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 775–813.

[3] A. Floer, Morse theory for the symplectic action, *J. Diff. Geom.* **28** (1988), 513–547.

[4] A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989), 575–611.

[5] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Inv. Math.* **82** (1985), 307–347.

[6] H. Hofer, Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three, Preprint, Ruhr-Universität Bochum, 1993.

[7] H. Hofer and D.A. Salamon, Floer homology and Novikov rings, to appear in Floer memorial volume.

[8] D. McDuff and D.A. Salamon, Symplectic Topology II, in preparation.

[9] J.W. Robbin and D.A. Salamon, The Maslov index for paths, to appear in *Topology.*

[10] J.W. Robbin and D.A. Salamon, The Spectral flow and the Maslov index, to appear in *Bulletin L.M.S.*

[11] D.A. Salamon and E. Zehnder, Morse theory for periodic orbits of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.* **45** (1992), 1303–1360.

[12] E. Witten, Morse theory and supersymmetry, *J. Diff. Equations* **17** (1982), 661–692.