INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Preface

These are notes for the lecture course *"Differential Geometry I"* held by the second author at ETH Zürich in the fall semester 2010. They are based on a lecture course held by the first author at the University of Wisconsin–Madison in the fall semester 1983.

In the present manuscript the sections are roughly in a one-to-one correspondence with the 26 lectures at ETH, each lasting two times 45 minutes. (Exceptions: Of Section 2.6 only the existence of a Riemannian metric was covered in one of the earlier lectures; Sections 1.9/1.10 together were two lectures, as well as 4.4/4.5; some of the material in the longer Sections like 1.6 and 2.4 was left as exercises.)

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Chapter 1

Foundations

In very rough terms, the subject of differential topology is to study spaces up to diffeomorphisms and the subject of differential geometry is to study spaces up to isometries. Thus in differential geometry our spaces are equipped with an additional structure, a (Riemannian) metric, and some important concepts we encounter are distance, geodesics, the Levi-Civita connection, and curvature. In differential topology important concepts are the degree of a map, intersection theory, differential forms, and deRham cohomology. In both subjects the spaces we study are smooth manifolds and the goal of this first chapter is to introduce the basic definitions and properties of smooth manifolds. We begin with (extrinsic) manifolds that are embedded in Euclidean space and their tangent bundles and later examine the more general (intrinsic) definition of a manifold.

1.1 Manifolds

1.1.1 Some examples

Let k, m, n be positive integers. Throughout we denote by

$$|x| := \sqrt{x_1^2 + \cdots + x_n^2}$$

the Euclidean norm of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The basic example of a manifold of dimension m is the Euclidean space $M = \mathbb{R}^m$ itself. Another example is the unit sphere in $\mathbb{R}^{m+1}$:

$$S^m := \left\{ x \in \mathbb{R}^{m+1} \mid |x| = 1 \right\}.$$

In particular, for $m = 1$ we obtain the unit circle

$$S^1 \subset \mathbb{R}^2 \cong \mathbb{C}.$$

The m**-torus** is the product

$$\mathbb{T}^m := S^1 \times \cdots \times S^1 = \{z = (z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_1| = \cdots |z_m| = 1\}.$$

A noncompact example is the space

$$M := \left\{x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \,|\, x_1^2 + \cdots + x_m^2 = 1 + x_0^2\right\}.$$

Other examples are the groups

$$\mathrm{SL}(n, \mathbb{R}) := \left\{A \in \mathbb{R}^{n \times n} \mid \det(A) = 1\right\}, \quad \mathrm{O}(n) := \left\{A \in \mathbb{R}^{n \times n} \,|\, A^T A = \mathbb{1}\right\}.$$

Here $\mathbb{1}$ denotes the identity matrix and A^T denotes the transposed matrix of A. An example of a manifold that is not (in an obvious way) embedded in some Euclidean space is the **complex Grassmannian**

$$\mathrm{G}_k(\mathbb{C}^n) := \{E \subset \mathbb{C}^n \,|\, E \text{ is a complex linear subspace of dimension } k\}$$

of k-planes in $\mathbb{C}^n$.

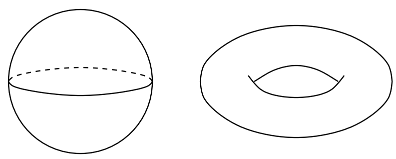


Figure 1.1: The 2-sphere and the 2-torus.

1.1.2 Recollections about smooth maps and derivatives

To define what we mean by a manifold $M \subset \mathbb{R}^k$ we recall some basic concepts from analysis [12]. Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of nonnegative integers. Let $k, \ell \in \mathbb{N}_0$ and let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$ be open sets. A map $f : U \to V$ is called **smooth** (or C^∞) if all its partial derivatives

$$\partial^\alpha f = \frac{\partial^{\alpha_1 + \cdots + \alpha_k} f}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}, \qquad \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k,$$

exist and are continuous. For a smooth map $f = (f_1, \ldots, f_\ell) : U \to V$ and a point $x \in U$ we denote by

$$df(x) : \mathbb{R}^k \to \mathbb{R}^\ell$$

the **derivative of f at x** defined by

$$df(x)\xi := \frac{d}{dt}\bigg|_{t=0} f(x+t\xi) = \lim_{t\to 0}\frac{f(x+t\xi)-f(x)}{t}, \qquad \xi\in\mathbb{R}^k.$$

This is a linear map represented by the **Jacobi matrix** of f at x which will also be denoted by

$$df(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_k}(x) \\ \vdots & & \vdots \\ \frac{\partial f_\ell}{\partial x_1}(x) & \cdots & \frac{\partial f_\ell}{\partial x_k}(x) \end{pmatrix} \in \mathbb{R}^{\ell\times k}.$$

The derivative satisfies the **chain rule**. Namely, if $U\subset\mathbb{R}^k$, $V\subset\mathbb{R}^\ell$, $W\subset\mathbb{R}^m$ are open sets and $f:U\to V$ and $g:V\to W$ are smooth maps then $g\circ f:U\to W$ is smooth and

$$d(g\circ f)(x) = dg(f(x))\circ df(x):\mathbb{R}^k\to\mathbb{R}^m$$

for every $x\in U$. Moreover the identity map is always smooth and its differential at every point is the identity matrix. This implies that, if $f:U\to V$ is a **diffeomorphism** (i.e. f is bijective and f and f^{-1} are both smooth) then $k=\ell$ and the Jacobi matrix $df(x)\in\mathbb{R}^{k\times k}$ is nonsingular for every $x\in U$. A partial converse is the inverse function theorem which we restate below. For a proof see [12] or any textbook on first year analysis.

Inverse Function Theorem. *Let $\Omega\subset\mathbb{R}^k$ be an open set, $f:\Omega\to\mathbb{R}^k$ be a smooth map, and $x_0\in\Omega$. If $\det(df(x_0))\neq 0$ then there is an open neighborhood $U\subset\Omega$ of x_0 such that $V:=f(U)$ is an open subset of $\mathbb{R}^k$ and $f|_U:U\to V$ is a diffeomorphism.*

Implicit Function Theorem. *Let $\Omega\subset\mathbb{R}^k\times\mathbb{R}^\ell$ be open and $f:\Omega\to\mathbb{R}^\ell$ be a smooth map. Let $(x_0,y_0)\in\Omega$ such that*

$$f(x_0,y_0)=0, \qquad \det\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)\neq 0.$$

(Here $\frac{\partial f}{\partial y}(x_0,y_0)\in\mathbb{R}^{\ell\times\ell}$ denotes the Jacobi matrix of the map $y\mapsto f(x_0,y)$ at the point $y=y_0$.) Then there are open sets $V\subset\mathbb{R}^k$ and $W\subset\mathbb{R}^\ell$ and a smooth map $g:V\to W$ such that $(x_0,y_0)\in V\times W\subset\Omega$, $g(x_0)=y_0$, and

$$f(x,y)=0 \qquad\Longleftrightarrow\qquad y=g(x).$$

for all $(x,y)\in V\times W$.

Now let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$ be arbitrary subsets, not necessarily open. A map $f : X \to Y$ is called **smooth** if for every $x_0 \in X$ there is an open neighborhood $U \subset \mathbb{R}^k$ of x_0 and a smooth map $F : U \to \mathbb{R}^\ell$ that agrees with f on $U \cap X$. A map $f : X \to Y$ is called a **diffeomorphism** if f is bijective and f and f^{-1} are smooth. If there exists a diffeomorphism $f : X \to Y$ then X and Y are called **diffeomorphic**.

Exercise 1.1. (i) Let $k, \ell, m \in \mathbb{N}_0$ and $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^\ell$, $Z \subset \mathbb{R}^m$ be arbitrary subsets. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps then so is the composition $g \circ f : X \to Z$. The identity map $\mathrm{id} : X \to X$ is smooth.
(ii) Let $E \subset \mathbb{R}^k$ be an m-dimensional linear subspace and let $v_1, \dots, v_m$ be a basis of E. Then the map $f : \mathbb{R}^m \to E$ defined by $f(x) := \sum_{i=1}^m x_i v_i$ is a diffeomorphism.

We also recall that any subset $M \subset \mathbb{R}^k$ inherits a topology from $\mathbb{R}^k$, called the **relative topology** of M. A subset $U_0 \subset M$ is called **relatively open** (or in short M**-open**) if there is an open set $U \subset \mathbb{R}^k$ such that $U_0 = U \cap M$. A subset $A_0 \subset M$ is called **relatively closed** (or in short M**-closed**) if there is a closed set $A \subset \mathbb{R}^k$ such that $A_0 = A \cap M$.

Exercise 1.2. Show that the relative topology satisfies the axioms of a topology (i.e. arbitrary unions and finite intersections of M-open sets are M-open, and the empty set and M itself are M-open). Show that the complement of an M-open set in M is M-closed and vice versa.

1.1.3 Submanifolds of Euclidean space

Definition 1.3. *Let $k, m \in \mathbb{N}_0$. A subset $M \subset \mathbb{R}^k$ is called a* **smooth m-dimensional submanifold of $\mathbb{R}^k$** *(or a* **smooth m-manifold***) if every point $p \in M$ has an open neighborhood $U \subset \mathbb{R}^k$ such that $U \cap M$ is diffeomorphic to an open subset $\Omega \subset \mathbb{R}^m$. A diffeomorphism $\phi : U \cap M \to \Omega$ is called a* **coordinate chart** *of M and its inverse $\psi := \phi^{-1} : \Omega \to U \cap M$ is called a* **(smooth) parametrization** *of $U \cap M$.*

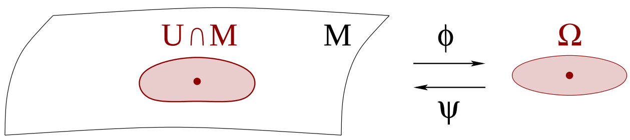


Figure 1.2: A coordinate chart $\phi : U \cap M \to \Omega$.

Lemma 1.4. *If $M \subset \mathbb{R}^k$ is a nonempty smooth m-manifold then $m \le k$.*

Proof. Let $\phi : U \cap M \to \Omega$ be a coordinate chart of M onto an open subset $\Omega \subset \mathbb{R}^m$, denote its inverse by $\psi := \phi^{-1} : \Omega \to U \cap M$, and let $p \in U \cap M$. Shrinking U, if necessary, we may assume that ϕ extends to a smooth map $\Phi : U \to \mathbb{R}^m$. This extension satisfies $\Phi(\psi(x)) = \phi(\psi(x)) = x$ and, by the chain rule, we have

$$d\Phi(\psi(x))d\psi(x) = \mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m$$

for every $x \in \Omega$. Hence $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective for $x \in \Omega$ and, since $\Omega \ne \emptyset$, this implies $m \le k$. $\square$

Example 1.5. Consider the 2-sphere

$$M := S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\right\}$$

depicted in Figure 1.1 and let $U \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^2$ be the open sets

$$U := \left\{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\right\}, \qquad \Omega := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\right\}.$$

The map $\phi : U \cap M \to \Omega$ given by

$$\phi(x, y, z) := (x, y)$$

is bijective and its inverse $\psi := \phi^{-1} : \Omega \to U \cap M$ is given by

$$\psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

Since both ϕ and ψ are smooth, the map ϕ is a coordinate chart on S^2. Similarly, we can use the open sets $z < 0$, $y > 0$, $y < 0$, $x > 0$, $x < 0$ to cover S^2 by six coordinate charts. Hence S^2 is a manifold. A similar argument shows that the unit sphere $S^m \subset \mathbb{R}^{m+1}$ is a manifold for every integer $m \ge 0$.

Example 1.6. Let $\Omega \subset \mathbb{R}^m$ be an open set and $h : \Omega \to \mathbb{R}^{k-m}$ be a smooth map. Then the graph of h is a smooth submanifold of $\mathbb{R}^m \times \mathbb{R}^{k-m} = \mathbb{R}^k$:

$$M := \mathrm{graph}(h) = \{(x, y) \,|\, x \in \Omega,\ y = h(x)\}.$$

It can be covered by a single coordinate chart $\phi : U \cap M \to V$ where $U := \Omega \times \mathbb{R}^{k-m}$, ϕ is the projection onto Ω, and $\psi := \phi^{-1} : \Omega \to U$ is given by $\psi(x) = (x, h(x))$.

Exercise 1.7 (The case $m = 0$). Show that a subset $M \subset \mathbb{R}^k$ is a 0-dimensional submanifold if and only if M is discrete, i.e. for every $p \in M$ there is an open set $U \subset \mathbb{R}^k$ such that $U \cap M = \{p\}$.

Exercise 1.8 (The case $m = k$). Show that a subset $M \subset \mathbb{R}^m$ is an m-dimensional submanifold if and only if M is open.

Exercise 1.9 (Products). If $M_i \subset \mathbb{R}^{k_i}$ is an m_i-manifold for $i = 1, 2$ show that $M_1 \times M_2$ is an $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{k_1+k_2}$. Prove by induction that the n-torus $\mathbb{T}^n$ is a smooth submanifold of $\mathbb{C}^n$.

The next theorem characterizes smooth submanifolds of Euclidean space. In particular condition (iii) will be useful in many cases for verifying the manifold condition.

Theorem 1.10 (Manifolds). *Let m and k be integers with $0 \le m \le k$. Let $M \subset \mathbb{R}^k$ be a set and $p \in M$. Then the following are equivalent.*

(i) *There is an M-open neighborhood $U_0 \subset M$ of p, an open set $\Omega_0 \subset \mathbb{R}^m$, and a diffeomorphism $\phi_0 : U_0 \to \Omega_0$.*

(ii) *There are open sets $U, \Omega \subset \mathbb{R}^k$ and a diffeomorphism $\phi : U \to \Omega$ such that $p \in U$ and*

$$\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \{0\}).$$

(iii) *There is an open set $U \subset \mathbb{R}^k$ and a smooth map $f : U \to \mathbb{R}^{k-m}$ such that $p \in U$, the differential $df(q) : \mathbb{R}^k \to \mathbb{R}^{k-m}$ is surjective for every $q \in U \cap M$, and*

$$U \cap M = f^{-1}(0) = \{q \in U \mid f(q) = 0\}.$$

Moreover, if (i) holds then the diffeomorphism $\phi : U \to \Omega$ in (ii) can be chosen such that $U \cap M \subset U_0$ and $\phi(p) = (\phi_0(p), 0)$ for every $p \in U \cap M$.

Proof. We prove that (i) implies (ii). Let $\phi_0 : U_0 \to \Omega_0$ be the coordinate chart in (i), let $\psi_0 := \phi_0^{-1} : \Omega_0 \to U_0$ be its inverse, and denote $x_0 := \phi_0(p) \in \Omega_0$. Then, by Lemma 1.4, the differential $d\psi_0(x_0) : \mathbb{R}^m \to \mathbb{R}^k$ is injective. Hence there is a matrix $B \in \mathbb{R}^{k\times(k-m)}$ such that

$$\det(d\psi_0(x_0)\, B) \neq 0.$$

Define the map $\psi : \Omega_0 \times \mathbb{R}^{k-m} \to \mathbb{R}^k$ by

$$\psi(x, y) := \psi_0(x) + By.$$

Then the $k \times k$-matrix

$$d\psi(x_0, 0) = [d\psi_0(x_0)\, B] \in \mathbb{R}^{k\times k}$$

is nonsingular, by choice of B. Hence, by the inverse function theorem, there is an open neighborhood $\widetilde{\Omega} \subset \Omega_0 \times \mathbb{R}^{k-m}$ of $(x_0, 0)$ such that $\widetilde{U} := \psi(\widetilde{\Omega})$ is open and $\psi|_{\widetilde{\Omega}} : \widetilde{\Omega} \to \widetilde{U}$ is a diffeomorphism. In particular, the restriction of ψ to $\widetilde{\Omega}$ is injective. Now the set

$$\widetilde{U}_0 := \left\{\psi_0(x) \,|\, (x, 0) \in \widetilde{\Omega}\right\} = \left\{q \in U_0 \,|\, (\phi_0(q), 0) \in \widetilde{\Omega}\right\} \subset M$$

is M-open and contains p. Hence, by the definition of the relative topology, there is an open set $W \subset \mathbb{R}^k$ such that $\widetilde{U}_0 = W \cap M$. Define

$$U := \widetilde{U} \cap W, \qquad \Omega := \widetilde{\Omega} \cap \psi^{-1}(W).$$

Then ψ restricts to a diffeomorphism from Ω to U and, for $(x, y) \in \Omega$, we claim that

$$\psi(x, y) \in M \qquad \Longleftrightarrow \qquad y = 0. \tag{1.1}$$

If $y = 0$ then obviously $\psi(x, y) = \psi_0(x) \in M$. Conversely, let $(x, y) \in \Omega$ and suppose that $q := \psi(x, y) \in M$. Then

$$q \in U \cap M = \widetilde{U} \cap W \cap M = \widetilde{U}_0 \subset U_0$$

and hence $(\phi_0(q), 0) \in \widetilde{\Omega}$, by definition of $\widetilde{U}_0$. This implies that

$$\psi(\phi_0(q), 0) = \psi_0(\phi_0(q)) = q = \psi(x, y).$$

Since the pairs (x, y) and $(\phi_0(q), 0)$ both belong to the set $\widetilde{\Omega}$ and the restriction of ψ to $\widetilde{\Omega}$ is injective we obtain $x = \phi_0(q)$ and $y = 0$. This proves (1.1). It follows from (1.1) that the map $\phi := (\psi|_{\Omega})^{-1} : U \to \Omega$ satisfies (ii) and agrees with the map $q \mapsto (\phi_0(q), 0)$ on $U \cap M$. Thus we have proved that (i) implies (ii).

That (ii) implies (iii) is obvious. Just define $f : U \to \mathbb{R}^{k-m}$ as the composition of ϕ with the projection of $\mathbb{R}^k$ onto the last $k - m$ coordinates.

We prove that (iii) implies (i). Let $f : U \to \mathbb{R}^{k-m}$ be as in (iii) and denote

$$X := \ker df(p) \subset \mathbb{R}^k.$$

By (iii) this is an m-dimensional linear subspace of $\mathbb{R}^k$ and we choose a $(k - m)$-dimensional linear subspace $Y \subset \mathbb{R}^k$ such that

$$\mathbb{R}^k = X \oplus Y.$$

Then the restriction of $df(p)$ to Y is a vector space isomorphism from Y to $\mathbb{R}^{k-m}$. Hence, by the implicit function theorem, there are open neighborhoods $V \subset X$ and $W \subset Y$ of the origin and a smooth map $g : V \to W$ such that

$$Q := \{p + x + y \,|\, x \in V,\, y \in W\} \subset U, \qquad g(0) = 0,$$

and, for all $x \in V$ and $y \in W$, we have

$$f(p + x + y) = 0 \qquad \iff \qquad y = g(x).$$

Hence assertion (i) holds with

$$U_0 := Q \cap M = \{p + x + g(x) \,|\, x \in V\}, \qquad \Omega_0 := V \subset X \cong \mathbb{R}^m,$$

and the diffeomorphism $\phi_0 : U_0 \to \Omega_0$ given by $\phi_0(p + x + y) := x$ with inverse $\phi_0^{-1}(x) := p + x + g(x)$. This proves the theorem. □

Definition 1.11. *Let $U \subset \mathbb{R}^k$ be an open set and $f : U \to \mathbb{R}^\ell$ be a smooth function. An element $c \in \mathbb{R}^\ell$ is called a* **regular value** *of f if, for all $p \in U$, we have*

$$f(p) = c \qquad \Longrightarrow \qquad df(p) : \mathbb{R}^k \to \mathbb{R}^\ell \text{ is surjective.}$$

Otherwise c is called a **singular value** *of f. The set of singular values of f will be denoted by*

$$S_f := \{f(p) \,|\, p \in U,\, \operatorname{rank} df(p) < \ell\}.$$

and the set of regular values by

$$R_f := \mathbb{R}^\ell \setminus S_f.$$

Theorem 1.10 asserts that, if c is a regular value of f the preimage

$$M := f^{-1}(c) = \{p \in U \,|\, f(p) = c\}$$

is a smooth $k - \ell$-dimensional submanifold of $\mathbb{R}^k$. Sard's theorem asserts that every smooth map $f : U \to \mathbb{R}^\ell$ has a regular value.

Sard's theorem. *For every smooth map $f : U \to \mathbb{R}^\ell$, defined on an open set $U \subset \mathbb{R}^k$, the set S_f of singular values of f has Lebesgue measure zero.*

Proof. §3 in Milnor's book [8]. □

Sard's theorem implies that the set of singular values does not contain any open set and hence the set $\mathbb{R}^\ell \setminus S_f$ of regular values is everywhere dense in $\mathbb{R}^\ell$. In other words, for every $c \in \mathbb{R}^\ell$ there is a sequence of regular values converging to c. In fact, it follows from Sard's theorem that the set of regular values is **residual** in the sense of Baire, i.e. it contains a countable intersection of open and dense sets. By **Baire's category theorem** residual sets (in a complete metric space) are always dense and, by definition, a countable intersection of residual sets is still residual. Thus any countable collection of smooth maps into the same target space $\mathbb{R}^\ell$ still has a dense set of common regular values.

We also emphasize that, if $\ell > k$ the differential $df(p) : \mathbb{R}^k \to \mathbb{R}^\ell$ can never be surjective and so the set of singular values is just the image of f. In this case Sard's theorem asserts that the image of f has Lebesgue measure zero and this remains valid for every continuously differentiable function. When $k \geq \ell$ Sard's theorem continues to hold for maps of class $C^{k-\ell+1}$ (see Abraham–Robbin [1]).

1.1.4 Examples and exercises

Example 1.12. Let $A = A^T \in \mathbb{R}^{k\times k}$ be a nonzero symmetric matrix and define $f : \mathbb{R}^k \to \mathbb{R}$ by

$$f(x) := x^T Ax.$$

Then $df(x)\xi = 2x^T A\xi$ for $x, \xi \in \mathbb{R}^k$ and hence the linear map $df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective if and only if $Ax \neq 0$. Thus $c = 0$ is the only singular value of f and, for $c \in \mathbb{R} \setminus \{0\}$, the set

$$M := f^{-1}(c) = \left\{ x \in \mathbb{R}^k \,|\, x^T Ax = c \right\}$$

is a smooth manifold of dimension $m = k - 1$.

Example 1.13 (The sphere). As a special case of Example 1.12 take $k = m + 1$, $A = \mathbb{1}$, and $c = 1$. Then $f(x) = |x|^2$ and so we have another proof that the unit sphere

$$S^m = \left\{ x \in \mathbb{R}^{m+1} \mid |x|^2 = 1 \right\}$$

in $\mathbb{R}^{m+1}$ is a smooth m-manifold. (See Example 1.5.)

Example 1.14. Define the map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y) := |x - y|^2$. This is another special case of Example 1.12 and so, for every $r > 0$, the set

$$M := \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x - y| = r \right\}$$

is a smooth 5-manifold.

Example 1.15 (The 2-torus). Let $0 < r < 1$ and define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) := (x^2 + y^2 + r^2 - z^2 - 1)^2 - 4(x^2 + y^2)(r^2 - z^2).$$

This map has zero as a regular value and $M := f^{-1}(0)$ is diffeomorphic to the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. An explicit diffeomorphism is given by

$$(e^{\mathbf{i}s}, e^{\mathbf{i}t}) \mapsto \big((1 + r\cos(s))\cos(t), (1 + r\cos(s))\sin(t), r\sin(s)\big).$$

This example corresponds to the diagram in Figure 1.1.
Exercise: Show that $f(x, y, z) = 0$ if and only if $(\sqrt{x^2 + y^2} - 1)^2 + z^2 = r^2$. Verify that zero is a regular velue of f.

Example 1.16 (The real projective plane). The set

$$M := \big\{(x^2, y^2, z^2, yz, zx, xy)\,|\,x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 = 1\big\}$$

is a smooth 2-manifold in $\mathbb{R}^6$. To see this, define an equivalence relation on the unit sphere $S^2 \subset \mathbb{R}^3$ by $p \sim q$ iff $q = \pm p$. The quotient space (the set of equivalence classes) is called the **real projective plane** and is denoted by

$$\mathbb{R}P^2 := S^2/\{\pm 1\}.$$

It is equipped with the quotient topology, i.e. a subset $U \subset \mathbb{R}P^2$ is open, by definition, iff its preimage under the obvious projection $S^2 \to \mathbb{R}P^2$ is an open subset of S^2. Now the map $f : S^2 \to \mathbb{R}^6$ defined by

$$f(x, y, z) := (x^2, y^2, z^2, yz, zx, xy)$$

descends to a homeomorphism from $\mathbb{R}P^2$ onto M. An atlas on M is given by the local smooth parametrizations

$$\begin{aligned}
\Omega \to M &: (x, y) \mapsto f(x, y, \sqrt{1 - x^2 - y^2}),\\
\Omega \to M &: (x, z) \mapsto f(x, \sqrt{1 - x^2 - z^2}, z),\\
\Omega \to M &: (y, z) \mapsto f(\sqrt{1 - y^2 - z^2}, y, z),
\end{aligned}$$

defined on the open unit disc $\Omega \subset \mathbb{R}^2$. We remark the following.
(a) M is *not* the preimage of a regular value under a smooth map $\mathbb{R}^6 \to \mathbb{R}^4$.
(b) M is *not* diffeomorphic to a submanifold of $\mathbb{R}^3$.
(c) The projection $\Sigma := \big\{(yz, zx, xy)\,|\,x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 = 1\big\}$ of M onto the last three coordinates is called the **Roman surface** and was discovered by Jakob Steiner. The Roman surface can also be represented as the set of solutions $(\xi, \eta, \zeta) \in \mathbb{R}^3$ of the equation $\eta^2\zeta^2 + \zeta^2\xi^2 + \xi^2\eta^2 = \xi\eta\zeta$. It is not a submanifold of $\mathbb{R}^3$.
Exercise: Prove this. Show that M is diffeomorphic to a submanifold of $\mathbb{R}^4$.

Exercise 1.17. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (kinetic plus potential energy) by

$$H(x,y) := \frac{1}{2}|y|^2 + V(x).$$

Prove that c is a regular value of H if and only if it is a regular value of V.

Exercise 1.18. Consider the **general linear group**

$$\mathrm{GL}(n,\mathbb{R}) = \left\{g \in \mathbb{R}^{n\times n} \,|\, \det(g) \neq 0\right\}$$

Prove that the derivative of the function $f = \det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is given by

$$df(g)v = \det(g)\,\mathrm{trace}(g^{-1}v)$$

for every $g \in \mathrm{GL}(n,\mathbb{R})$ and every $v \in \mathbb{R}^{n\times n}$. Deduce that the **special linear group**

$$\mathrm{SL}(n,\mathbb{R}) := \{g \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(g) = 1\}$$

is a smooth submanifold of $\mathbb{R}^{n\times n}$.

Example 1.19. The **orthogonal group**

$$\mathrm{O}(n) := \left\{g \in \mathbb{R}^{n\times n} \,|\, g^T g = \mathbb{1}\right\}$$

is a smooth submanifold of $\mathbb{R}^{n\times n}$. To see this, denote by

$$\mathscr{S}_n := \left\{S \in \mathbb{R}^{n\times n} \,|\, S^T = S\right\}$$

the vector space of symmetric matrices and define $f : \mathbb{R}^{n\times n} \to \mathscr{S}_n$ by

$$f(g) := g^T g.$$

Its derivative $df(g) : \mathbb{R}^{n\times n} \to \mathscr{S}_n$ is given by $df(g)v = g^T v + v^T g$. This map is surjective for every $g \in \mathrm{O}(n)$: if $g^T g = \mathbb{1}$ and $S = S^T \in \mathscr{S}_n$ then the matrix $v := \frac{1}{2}gS$ satisfies $df(g)v = S$. Hence $\mathbb{1}$ is a regular value of f and so $\mathrm{O}(n)$ is a smooth manifold. It has the dimension

$$\dim \mathrm{O}(n) = n^2 - \dim \mathscr{S}_n = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Exercise 1.20. Prove that the set

$$M := \left\{(x,y) \in \mathbb{R}^2 \,|\, xy = 0\right\}$$

is not a submanifold of $\mathbb{R}^2$. **Hint:** If $U \subset \mathbb{R}^2$ is a neighborhood of the origin and $f : U \to \mathbb{R}$ is a smooth map such that $U \cap M = f^{-1}(0)$ then $df(0,0) = 0$.

1.2 Tangent spaces and derivatives

The main reason for first discussing the extrinsic notion of embedded manifolds in Euclidean space (and postponing the more general intrinsic definition of a manifold via a system of coordinate charts on a topological space) is that the concept of a tangent vector is much easier to digest in the embedded case: it is simply the derivative of a curve in M, understood as a vector in the ambient Euclidean space in which M is embedded.

1.2.1 Tangent spaces

Definition 1.21. *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. A vector $v \in \mathbb{R}^k$ is called a* **tangent vector** *of M at p if there is a smooth curve $\gamma : \mathbb{R} \to M$ such that*

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = v.$$

The set

$$T_pM := \{\dot{\gamma}(0) \,|\, \gamma : \mathbb{R} \to M \text{ is smooth}, \gamma(0) = p\}$$

of tangent vectors of M at p is called the **tangent space** *of M at p (see Figure 1.3).*

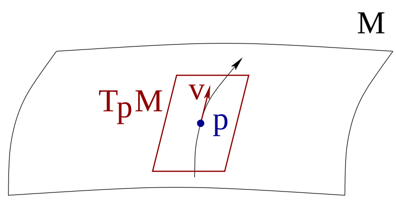


Figure 1.3: The tangent space T_pM.

Remark 1.22. Let $p \in M \subset \mathbb{R}^k$ be as in Definition 1.21 and let $v \in \mathbb{R}^k$. Then

$$v \in T_pM \qquad \Longleftrightarrow \qquad \begin{array}{l} \exists \varepsilon > 0 \ \exists \gamma : (-\varepsilon, \varepsilon) \to M \ \ni \\ \gamma \text{ is smooth}, \gamma(0) = p, \dot{\gamma}(0) = v. \end{array}$$

To see this suppose that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Define $\widetilde{\gamma} : \mathbb{R} \to M$ by

$$\widetilde{\gamma}(t) := \gamma\left(\frac{\varepsilon t}{\sqrt{\varepsilon^2 + t^2}}\right), \qquad t \in \mathbb{R}.$$

Then $\widetilde{\gamma}$ is smooth and satisfies $\widetilde{\gamma}(0) = p$ and $\dot{\widetilde{\gamma}}(0) = v$. Hence $v \in T_pM$.

Theorem 1.23 (Tangent spaces). *Let $M \subset \mathbb{R}^k$ de a smooth m-dimensional manifold and fix a point $p \in M$. Then the following holds.*

(i) *Let $U_0 \subset M$ be an M-open set with $p \in U_0$ and $\phi_0 : U_0 \to \Omega_0$ be a diffeomorphism onto an open subset $\Omega_0 \subset \mathbb{R}^m$. Let $x_0 := \phi_0(p)$ and let $\psi_0 := \phi_0^{-1} : \Omega_0 \to U_0$ be the inverse map. Then*

$$T_pM = \mathrm{im}\left(d\psi_0(x_0) : \mathbb{R}^m \to \mathbb{R}^k\right).$$

(ii) *Let $U, \Omega \subset \mathbb{R}^k$ be open sets and $\phi : U \to \Omega$ be adiffeomorphism such that $p \in U$ and $\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \{0\})$. Then*

$$T_pM = d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right).$$

(iii) *Let $U \subset \mathbb{R}^k$ be an open neighborhood of p and $f : U \to \mathbb{R}^{k-m}$ be a smooth map such that 0 is a regular value of f and $U \cap M = f^{-1}(0)$. Then*

$$T_pM = \ker df(p).$$

(iv) *T_pM is an m-dimensional linear subspace of $\mathbb{R}^k$.*

Proof. We prove that

$$\mathrm{im}\, d\psi_0(x_0) \subset T_pM \subset d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right). \tag{1.2}$$

To prove the first inclusion in (1.2) we choose a nonzero vector $\xi \in \mathbb{R}^m$ and choose $\varepsilon > 0$ such that

$$B_\varepsilon(x_0) := \{x \in \mathbb{R}^m \mid |x - x_0| < \varepsilon\} \subset V_0.$$

Define the curve $\gamma : (-\varepsilon/|\xi|, \varepsilon/|\xi|) \to M$ by

$$\gamma(t) := \psi_0(x_0 + t\xi), \qquad |t| < \frac{\varepsilon}{|\xi|}.$$

Then γ is a smooth curve in M with

$$\gamma(0) = \psi_0(x_0) = p, \qquad \dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \psi_0(x_0 + t\xi) = d\psi_0(x_0)\xi.$$

Hence it follows from Remark 1.22 that $d\psi_0(x_0)\xi \in T_pM$, as claimed. To prove the second inclusion in (1.2) we fix a vector $v \in T_pM$. Then, by definition of the tangent space, there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $U \subset \mathbb{R}^k$ be as in (ii) and choose $\varepsilon > 0$ so small that $\gamma(t) \in U$ for $|t| < \varepsilon$. Then

$$\phi(\gamma(t)) \in \phi(U \cap M) \subset \mathbb{R}^m \times \{0\}$$

for $|t| < \varepsilon$ and hence

$$d\phi(p)v = d\phi(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \phi(\gamma(t)) \in \mathbb{R}^m \times \{0\}.$$

This shows that $v \in d\phi(p)^{-1}(\mathbb{R}^m \times \{0\})$ and thus we have proved (1.2).

Now the sets $\mathrm{im}\, d\psi_0(x_0)$ and $d\phi(p)^{-1}(\mathbb{R}^m \times \{0\})$ are both m-dimensional linear subspaces of $\mathbb{R}^k$. Hence it follows from (1.2) that these subspaces agree and that they both agree with T_pM. Thus we have proved assertions (i), (ii), and (iv).

We prove (iii). If $v \in T_pM$ then there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. For t sufficiently small we have $\gamma(t) \in U$, where $U \subset \mathbb{R}^k$ is the open set in (iii), and hence $f(\gamma(t)) = 0$. This implies

$$df(p)v = df(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0.$$

Thus we have proved that $T_pM \subset \ker\, df(p)$. Since the kernel of $df(p)$ and T_pM are both m-dimensional linear subspaces of $\mathbb{R}^k$ we deduce that $T_pM = \ker\, df(p)$. This proves (iii) and the theorem. □

Example 1.24. Let $A = A^T \in \mathbb{R}^{k\times k}$ be a nonzero matrix as in Example 1.12 and let $c \neq 0$. Then, by Theorem 1.23 (iii), the tangent space of the manifold

$$M = \left\{x \in \mathbb{R}^k \,|\, x^TAx = c\right\}$$

at a point $x \in M$ is the $k-1$-dimensional linear subspace

$$T_xM = \left\{\xi \in \mathbb{R}^k \,|\, x^TA\xi = 0\right\}.$$

Example 1.25. As a special case of Example 1.24 with $A = \mathbb{1}$ and $c = 1$ we find that the tangent space of the unit sphere $S^m \subset \mathbb{R}^{m+1}$ at a point $x \in S^m$ is the orthogonal complement of x:

$$T_xS^m = x^\perp = \left\{\xi \in \mathbb{R}^{m+1} \,|\, \langle x, \xi\rangle = 0\right\}.$$

Here $\langle x, \xi\rangle = \sum_{i=0}^m x_i\xi_i$ denotes the standard inner product on $\mathbb{R}^{m+1}$.

Exercise 1.26. What is the tangent space of the 5-manifold

$$M := \left\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, |x-y| = r\right\}$$

at a point $(x,y) \in M$? (See Exercise 1.14.)

Example 1.27. Let $H(x,y) := \frac{1}{2}|y|^2 + V(x)$ be as in Exercise 1.17 and let c be a regular value of H. If $(x,y) \in M := H^{-1}(c)$ Then

$$T_{(x,y)}M = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \langle y,\eta\rangle + \langle \nabla V(x), \xi\rangle = 0\}.$$

Here $\nabla V := (\partial V/\partial x_1, \dots, \partial V/\partial x_n) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the gradient of V.

Exercise 1.28. The tangent space of $\mathrm{SL}(n,\mathbb{R})$ at the identity matrix is the space

$$\mathfrak{sl}(n,\mathbb{R}) := T_{\mathbb{1}}\mathrm{SL}(n,\mathbb{R}) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \mathrm{trace}(\xi) = 0\right\}$$

of traceless matrices. (Prove this, using Exercise 1.18.)

Example 1.29. The tangent space of $\mathrm{O}(n)$ at g is

$$T_g\mathrm{O}(n) = \left\{v \in \mathbb{R}^{n\times n} \,|\, g^Tv + v^Tg = 0\right\}.$$

In particular, the tangent space of $\mathrm{O}(n)$ at the identity matrix is the space of skew-symmetric matrices

$$\mathfrak{o}(n) := T_{\mathbb{1}}\mathrm{O}(n) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \xi^T + \xi = 0\right\}$$

To see this, choose a smooth curve $\mathbb{R} \to \mathrm{O}(n) : t \mapsto g(t)$. Then we have $g(t)^Tg(t) = \mathbb{1}$ for every $t \in \mathbb{R}$ and, differentiating this identity with respect to t, we obtain $g(t)^T\dot{g}(t) + \dot{g}(t)^Tg(t) = 0$ for every t. Hence every matrix $v \in T_g\mathrm{O}(n)$ satisfies the equation $g^Tv + v^Tg = 0$. The claim follows from the fact that $g^Tv + v^Tg = 0$ if and only if the matrix $\xi := g^{-1}v$ is skew-symmetric and that the space of skew-symmetric matrices in $\mathbb{R}^{n\times n}$ has dimension $n(n-1)/2$.

Exercise 1.30. Let $\Omega \subset \mathbb{R}^m$ be an open set and $h : \Omega \to \mathbb{R}^{k-m}$ be a smooth map. Prove that the tangent space of the graph of h at a point $(x, h(x))$ is the graph of the differential $dh(x) : \mathbb{R}^m \to \mathbb{R}^{k-m}$:

$$M = \{(x,h(x)) \,|\, x \in \Omega\}, \qquad T_{(x,h(x))}M = \{(\xi, dh(x)\xi) \,|\, \xi \in \mathbb{R}^m\}.$$

1.2.2 The derivative

A key purpose behind the concept of a smooth manifold is to carry over the notion of a smooth map and its derivatives from the realm of first year analysis to the present geometric setting. Here is the basic definition.

Definition 1.31. *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and $f : M \to \mathbb{R}^\ell$ be a smooth map. The* **derivative** *of f at a point $p \in M$ is the map*

$$df(p) : T_pM \to \mathbb{R}^\ell$$

defined as follows. Given a tangent vector $v \in T_pM$ choose a smooth curve $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$; now define the vector

$$df(p)v \in \mathbb{R}^\ell$$

by

$$df(p)v := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \lim_{h\to 0} \frac{f(\gamma(h)) - f(p)}{h}. \tag{1.3}$$

That the limit on the right in equation (1.3) exists follows from our assumptions. We must prove, however, that the derivative is well defined, i.e. that the right hand side of (1.3) depends only on the tangent vector v but not on the choice of the curve γ used in the definition. This is the content of the first assertion in the next theorem.

Theorem 1.32 (Derivatives). *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and $f : M \to \mathbb{R}^\ell$ be a smooth map. Fix a point $p \in M$. Then the following holds.*

(i) *The right hand side of (1.3) is independent of γ.*

(ii) *The map $df(p) : T_pM \to \mathbb{R}^\ell$ is linear.*

(iii) *If $N \subset \mathbb{R}^\ell$ is a smooth n-manifold and $f(M) \subset N$ then*

$$df(p)T_pM \subset T_{f(p)}N.$$

(iv) (Chain Rule) *Let N be as in (iii), suppose that $f(M) \subset N$, and let $g : N \to \mathbb{R}^d$ be a smooth map. Then*

$$d(g \circ f)(p) = dg(f(p)) \circ df(p) : T_pM \to \mathbb{R}^d.$$

(v) *If $f = \mathrm{id} : M \to M$ then $df(p) = \mathrm{id} : T_pM \to T_pM$.*

Proof. We prove (i). Let $v \in T_pM$ and $\gamma : \mathbb{R} \to M$ be as in Definition 1.31. By definition there is an open neighborhood $U \subset \mathbb{R}^k$ of p and a smooth map $F : U \to \mathbb{R}^\ell$ such that $F(q) = f(q)$ for every $q \in U \cap M$. Let $dF(p) \in \mathbb{R}^{\ell \times k}$ denote the Jacobi matrix (i.e. the matrix of all first partial derivatives) of F at p. Then, since $\gamma(t) \in U \cap M$ for t sufficiently small, we have

$$dF(p)v = dF(\gamma(0))\dot{\gamma}(0) = \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

The right hand side of this identity is independent of the choice of F while the left hand side is independent of the choice of γ. Hence the right hand side is also independent of the choice of γ and this proves (i). Assertion (ii) follows immediately from the identity $df(p)v = dF(p)v$ just established.

Assertion (iii) follows directly from the definitions. Namely, if γ is as in Definition 1.31 then

$$\beta := f \circ \gamma : \mathbb{R} \to N$$

is a smooth curve in N satisfying

$$\beta(0) = f(\gamma(0)) = f(p) =: q, \qquad \dot{\beta}(0) = df(p)v =: w.$$

Hence $w \in T_qN$. Assertion (iv) also follows directly from the definitions. If $g : N \to \mathbb{R}^d$ is a smooth map and β, q, w are as above then

$$\begin{aligned} d(g \circ f)(p)v &= \left.\frac{d}{dt}\right|_{t=0} g(f(\gamma(t))) \\ &= \left.\frac{d}{dt}\right|_{t=0} g(\beta(t)) \\ &= dg(q)w \\ &= dg(f(p))df(p)v. \end{aligned}$$

Assertion (v) is obvious. This proves the theorem. □

Corollary 1.33 (Diffeomorphisms). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold. If $f : M \to N$ is a diffeomorphism then $m = n$ and the differential $df(p) : T_pM \to T_{f(p)}N$ is a vector space isomorphism for every $p \in M$.*

Proof. Denote $g := f^{-1} : N \to M$. Then $g \circ f$ is the identity on M and $f \circ g$ is the identity on N. Hence it follows from Theorem 1.32 that, for $p \in M$ and $q := f(p) \in N$, we have

$$dg(q) \circ df(p) = \mathrm{id} : T_pM \to T_pM, \qquad df(p) \circ dg(q) = \mathrm{id} : T_qN \to T_qN.$$

Hence $df(p)$ is a vector space isomorphism with inverse $dg(q)$. □

1.2.3 The inverse and implicit function theorems

Corollary 1.33 is analogous to the corresponding assertion for smooth maps between open subsets of Euclidean space. Likewise, the inverse function theorem for manifolds is a partial converse of Corollary 1.33.

Theorem 1.34 (Inverse Function Theorem). *Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be smooth n-manifolds and $f : M \to N$ be a smooth map. Let $p_0 \in M$ and suppose that the differential $df(p_0) : T_{p_0}M \to T_{f(p_0)}N$ is a vector space isomorphism. Then there is a relatively open neighborhood $U \subset M$ of p_0 such that $V := f(U) \subset N$ is a relatively open subset of N and the restriction of f to U is a diffeomorphism from U to V.*

Proof. Choose a coordinate charts $\phi_0 : U_0 \to \widetilde{U}_0$, defined on an M-open neighborhood $U_0 \subset M$ of p_0 onto an open set $\widetilde{U}_0 \subset \mathbb{R}^n$, and $\psi_0 : V_0 \to \widetilde{V}_0$, defined on an N-open neighborhood $V_0 \subset N$ of $q_0 := f(p_0)$ onto an open set $\widetilde{V}_0 \subset \mathbb{R}^n$. Shrinking U_0, if necessary we may assume that $f(U_0) \subset V_0$. Then the map

$$\widetilde{f} := \psi_0 \circ f \circ \phi_0^{-1} : \widetilde{U}_0 \to \widetilde{V}_0$$

is smooth and its differential $d\widetilde{f}(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective at the point $x_0 := \phi_0(p_0)$. Hence it follows from the inverse function theorem for smooth maps on open subsets of Euclidean space that there is an open set $\widetilde{U} \subset \widetilde{U}_0$ such that $\widetilde{V} := \widetilde{f}(\widetilde{U})$ is an open subset of $\widetilde{V}_0$ and the restriction of $\widetilde{f}$ to $\widetilde{U}$ is a diffeomorphism from $\widetilde{U}$ to $\widetilde{V}$. Hence the assertion of the theorem holds with $U := \phi_0^{-1}(\widetilde{U})$ and $V := \psi_0^{-1}(\widetilde{V})$. □

Definition 1.35 (Regular value). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold. Let $f : M \to N$ be a smooth map. A point $q \in N$ is called a* **regular value** *of f if the differential*

$$df(p) : T_pM \to T_{f(p)}N$$

is surjective for every $p \in M$ with $f(p) = q$.

Theorem 1.36 (Implicit function theorem). *Let $f : M \to N$ be as in Definition 1.35 and let $q \in N$ be a regular value of f. Then the set*

$$P := f^{-1}(q) = \{p \in M \mid f(p) = q\}$$

is a manifold of dimension $m - n$ and its tangent space is

$$T_pP = \ker df(p) = \{v \in T_pM \mid df(p)v = 0\}$$

for every $p \in P$.

Proof. Fix a point $p_0 \in P \subset M$ and choose a coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ on an M-open neighborhood $U_0 \subset M$ of p_0. Let $U \subset \mathbb{R}^k$ be an open set such that $U \cap M = U_0$. Likewise, choose a coordinate chart $\psi_0 : V_0 \to \mathbb{R}^n$ on an N-open neighborhood $V_0 \subset N$ of q. Shrinking U (and hence U_0), if necessary, we may assume that $f(U_0) \subset V_0$. Then the point $c_0 := \psi_0(q)$ is a regular value of the map

$$f_0 := \psi_0 \circ f \circ \phi_0^{-1} : \phi_0(U_0) \to \mathbb{R}^n.$$

Hence, by Theorem 1.10, the set

$$f_0^{-1}(c_0) = \left\{ x \in \phi_0(U_0) \,|\, f(\phi_0^{-1}(x)) = q \right\} = \phi_0(U_0 \cap P)$$

is a manifold of of dimension $m - n$ contained in the open set $\phi_0(U_0) \subset \mathbb{R}^m$. It now follows directly from Definition 1.3 that $U_0 \cap P = U \cap P$ is also a manifold of dimension $m - n$. Thus every point $p_0 \in P$ has an open neighborhood $U \subset \mathbb{R}^k$ such that $U \cap P$ is a manifold of dimension $m - n$. Hence P is a manifold of dimension $m - n$. The assertion about the tangent space is now an easy exercise. □

1.3 Submanifolds and embeddings

The implicit function theorem deals with subsets of a manifold M that are themselves manifolds in the sense of Definition 1.3. Such subsets are called submanifolds of M.

Definition 1.37. *Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold. A subset $L \subset M$ is called a* **submanifold** *of M of dimension ℓ, if L itself is an ℓ-manifold.*

Definition 1.38. *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold. A smooth map $f : N \to M$ is called an* **immersion** *if its differential $df(q) : T_qN \to T_{f(q)}M$ is injective for every $q \in N$. It is called* **proper** *if, for every compact subset $K \subset f(N)$, the preimage $f^{-1}(K) = \{q \in N \,|\, f(q) \in K\}$ is compact. The map f is called an* **embedding** *if it is a proper injective immersion.*

Remark 1.39. In our definition of proper maps it is important that the compact set K is required to be contained in the image of f. The litura-ture contains sometimes a stronger definition of *proper* which requires that $f^{-1}(K)$ is a compact subset of M for every compact subset $K \subset N$, whether or not K is contained in the image of f. If this holds the image of f is nec-essarily M-closed. (Exercise!)

Theorem 1.40 (Submanifolds). *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold.*

(i) *If $f : N \to M$ is an embedding then $f(N)$ is a submanifold of M.*

(ii) *If $P \subset M$ is a submanifold then the inclusion $P \to M$ is an embedding.*

(iii) *A subset $P \subset M$ is a submanifold of dimension n if and only if for every $p_0 \in P$ there is a coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ defined on an M-open neighborhood $U_0 \subset M$ of p_0 such that*

$$\phi_0(U_0 \cap P) = \phi_0(U_0) \cap (\mathbb{R}^n \times \{0\}).$$

(See Figure 1.4.)

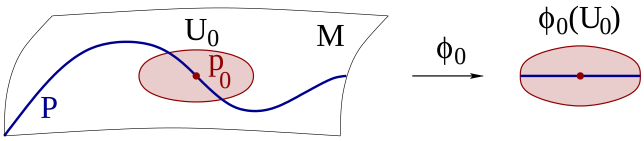


Figure 1.4: A coordinate chart adapted to a submanifold.

Lemma 1.41 (Embeddings). *Let M and N be as in Theorem 1.40 and let $f : N \to M$ be an embedding. Denote $P := f(N)$ and let $q_0 \in N$ and $p_0 := f(q_0) \in P$. Then there is an M-open neighborhood $U \subset M$ of p_0, an N-open neighborhood $V \subset N$ of q_0, an open neighborhood $W \subset \mathbb{R}^{m-n}$ of the origin, and a diffeomorphism $F : V \times W \to U$ such that, for all $q \in V$ and $z \in W$, we have*

$$F(q, 0) = f(q) \tag{1.4}$$

and

$$F(q, z) \in P \qquad \iff \qquad z = 0. \tag{1.5}$$

Proof. Choose a coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ on an M-open neighborhood $U_0 \subset M$ of p_0. Then the differential

$$d(\phi_0 \circ f)(q_0) = d\phi_0(f(q_0)) \circ df(q_0) : T_{q_0}N \to \mathbb{R}^m$$

is injective. Hence there is a linear map $B : \mathbb{R}^{m-n} \to \mathbb{R}^m$ such that the map

$$T_{q_0}N \times \mathbb{R}^{m-n} \to \mathbb{R}^m : (w, \zeta) \mapsto d(\phi_0 \circ f)(q_0)w + B\zeta \tag{1.6}$$

is a vector space isomorphism. Define the set

$$\Omega := \left\{(q, z) \in N \times \mathbb{R}^{m-n} \,|\, f(q) \in U_0,\ \phi_0(f(q)) + Bz \in \phi_0(U_0)\right\}.$$

This is an open subset of $N \times \mathbb{R}^{m-n}$ and we define $F : \Omega \to M$ by

$$F(q,z) := \phi_0^{-1}\left(\phi_0(f(q)) + Bz\right).$$

This map is smooth, it satisfies $F(q,0) = f(q)$ for all $q \in f^{-1}(U_0)$, and the derivative $dF(q_0,0) : T_{q_0}N \times \mathbb{R}^{m-n} \to T_{p_0}M$ is the composition of the map (1.6) with $d\phi_0(p_0)^{-1} : \mathbb{R}^m \to T_{p_0}M$ and hence is a vector space isomorphism. Thus the Inverse Function Theorem 1.34 asserts that there is an N-open neighborhood $V_0 \subset N$ of q_0 and an open neighborhood $W_0 \subset \mathbb{R}^{m-n}$ of the origin such that $V_0 \times W_0 \subset \Omega$, the set $U_0 := F(V_0 \times W_0) \subset M$ is M-open, and the restriction of F to $V_0 \times W_0$ is a diffeomorphism onto U_0. Thus we have constructed a diffeomorphism $F : V_0 \times W_0 \to U_0$ that satisfies (1.4). We claim that its restriction to the product $V \times W$ of sufficiently small open neighborhoods $V \subset N$ of q_0 and $W \subset \mathbb{R}^{m-n}$ of the origin also satisfies (1.5). Otherwise there are sequences $q_i \in V_0$ converging to q_0 and $z_i \in W_0 \setminus \{0\}$ converging to zero such that $F(q_i, z_i) \in P$. Hence there is a sequence $q_i' \in N$ such that

$$F(q_i, z_i) = f(q_i').$$

This sequence converges to $f(q_0)$. Since f is proper we may assume, passing to a suitable subsequence, that q_i' converges to a point $q_0' \in N$. Since f is continuous we have

$$f(q_0') = \lim_{i\to\infty} f(q_i') = \lim_{i\to\infty} F(q_i, z_i) = f(q_0).$$

Since f is injective we have $q_0' = q_0$. Hence $(q_i', 0) \in V_0 \times W_0$ for i sufficiently large and $F(q_i', 0) = F(q_i, z_i)$. This contradicts the fact that the map $F : V_0 \times W_0 \to M$ is injective. Thus we have proved the lemma. □

Proof of Theorem 1.40. We prove (i). Let $q_0 \in N$, denote $p_0 := f(q_0) \in P$, and choose a diffeomorphism $F : V \times W \to U$ as in Lemma 1.41. Then the set $U \cap P$ is P-open, the set $V \subset N$ is diffeomorphic to an open subset of $\mathbb{R}^n$ (after schrinking V if necessry), and the map $V \to U \cap P : q \mapsto F(q,0)$ is a diffeomorphism. Hence a P-open neighborhood of p_0 is diffeomorphic to an open subset of $\mathbb{R}^n$. Since $p_0 \in P$ was chosen arbitrarily, this shows that P is an n-dimensional submanifold of M.

We prove (ii). The inclusion $\iota : P \to M$ is obviously smooth and injective (it extends to the identity map on $\mathbb{R}^k$). Moreover, $T_pP \subset T_pM$ for every $p \in P$ and the differential $d\iota(p) : T_pP \to T_pM$ is the obvious inclusion for every $p \in P$. That ι is proper follows immediately from the definition. Hence ι is an embedding.

We prove (iii). If a coordinate chart ϕ_0 as in (iii) exists then the set $U_0 \cap P$ is P-open and is diffeomorphic to an open subset of $\mathbb{R}^n$. Since the point $p_0 \in P$ was chosen arbitrarily this proves that P is an n-dimensional submanifold of M. Conversely, suppose that P is an n-dimensional submanifold of M and let $p_0 \in P$. Choose any coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ of M defined on an M-open neighborhood $U_0 \subset M$ of p_0. Then $\phi_0(U_0 \cap P)$ is an n-dimensional submanifold of $\mathbb{R}^m$. Hence it follows from Theorem 1.10 that there are open sets $V, W \subset \mathbb{R}^m$ and a diffeomorphism $\psi : V \to W$ such that

$$\phi_0(p_0) \in V, \qquad \psi(V \cap \phi_0(U_0 \cap P)) = W \cap (\mathbb{R}^n \times \{0\}).$$

Shrinking U_0 if necessary, we may assume that $\phi_0(U_0) \subset V$. Then the coordinate chart $\psi \circ \phi_0 : U_0 \to \mathbb{R}^m$ has the required properties. □

Example 1.42. Let $S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$ be the unit circle and consider the map $f : S^1 \to \mathbb{R}^2$ given by

$$f(x, y) := (x, xy).$$

This map is a proper immersion but is not injective (the points $(0, 1)$ and $(0, -1)$ have the same image under f). The image $f(S^1)$ is a figure 8 in $\mathbb{R}^2$ and is not a submanifold: see Figure 1.5.

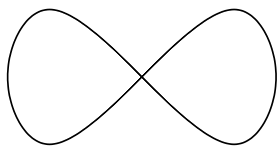


Figure 1.5: A proper immersion.

Example 1.43. Consider the restriction of the map f in Example 1.42 to the submanifold $N := S^1 \setminus \{(0, -1)\}$. The resulting map $f : N \to \mathbb{R}^2$ is an injective immersion but it is not proper. It has the same image as before and hence $f(N)$ is not a manifold.

Example 1.44. The map $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) := (t^2, t^3)$$

is proper and injective, but is not an embedding (its differential at $x = t$ is not injective). The image of f is the set

$$f(\mathbb{R}) = C := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^3 = y^2\right\}$$

and is not a submanifold: see Figure 1.6. (Prove this!)

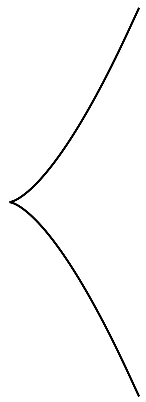


Figure 1.6: A proper injection.

Example 1.45. Define the map $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) := (\cos(t), \sin(t))$. This map is an immersion but is neither injective nor proper. However, its image is the unit circle and hence is a submanifold of $\mathbb{R}^2$. The map $\mathbb{R} \to \mathbb{R}^2 : t \mapsto f(t^3)$ is not an immersion and is neither injective nor proper, but its image is still the unit circle.

1.4 Vector fields and flows

1.4.1 Vector fields

Definition 1.46 (Vector fields). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold. A* **(smooth) vector field** *on M is a smooth map $X : M \to \mathbb{R}^k$ such that $X(p) \in T_pM$ for every $p \in M$. The set of smooth vector fields on M will be denoted by*

$$\mathrm{Vect}(M) := \left\{ X : M \to \mathbb{R}^k \,|\, X \text{ is smooth, } X(p) \in T_pM \;\forall\, p \in M \right\}.$$

Exercise 1.47. Prove that the set of smooth vector fields on M is a real vector space.

Example 1.48. Denote the standard cross product on $\mathbb{R}^3$ by

$$x \times y := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

For $x, y \in \mathbb{R}^3$. Fix a vector $\xi \in S^2$ and define the maps $X, Y : S^2 \to \mathbb{R}^3$ by

$$X(p) := \xi \times p, \qquad Y(p) := (\xi \times p) \times p.$$

These are vector fields with zeros $\pm\xi$. Their integral curves (see Definition 1.51 below) are illustrated in Figure 1.7.

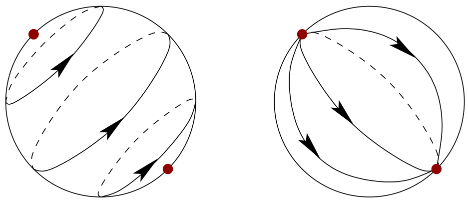


Figure 1.7: Two vector fields on the 2-sphere.

Example 1.49. Let $M := \mathbb{R}^2$. A vector field on M is then any smooth map $X : \mathbb{R}^2 \to \mathbb{R}^2$. As an example consider the vector field

$$X(x, y) := (x, -y).$$

This vector field has a single zero at the origin and its integral curves are illustrated in Figure 1.8.

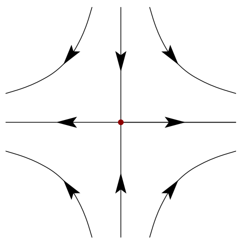


Figure 1.8: A hyperbolic fixed point.

Example 1.50. Every smooth function $f : \mathbb{R}^m \to \mathbb{R}$ determines a gradient vector field

$$X = \nabla f := \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^m.$$

Definition 1.51 (Integral curves). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, $X \in \mathrm{Vect}(M)$ be a smooth vector field on M, and $I \subset \mathbb{R}$ be an open interval. A smooth map $\gamma : I \to M$ is called an* **integral curve** *of X if it satisfies the equation $\dot\gamma(t) = X(\gamma(t))$ for every $t \in I$.*

Theorem 1.52. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Fix a point $p_0 \in M$. Then the following holds.*
(i) *There is an open interval $I \subset \mathbb{R}$ containing 0 and a smooth curve $\gamma : I \to M$ satisfying the equation*

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \gamma(0) = p_0 \tag{1.7}$$

for every $t \in I$.
(ii) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are two solutions of (1.7) on open intervals I_1 and I_2 containing 0, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.*

Proof. We prove (i). Let $\phi_0 : U_0 \to \mathbb{R}^m$ be a coordinate chart on M, defined on an M-open neighborhood $U_0 \subset M$ of p_0. The image of ϕ_0 is an open set

$$\Omega := \phi_0(U_0) \subset M$$

and we denote the inverse map by $\psi_0 := \phi_0^{-1} : \Omega \to M$. Then, by Theorem 1.23, the differential $d\psi_0(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is the tangent space $T_{\psi_0(x)}M$ for every $x \in \Omega$. Define $f : \Omega \to \mathbb{R}^m$ by

$$f(x) := d\psi_0(x)^{-1} X(\psi_0(x)), \qquad x \in \Omega.$$

This map is smooth and hence, by the basic existence and uniqueness theorem for ordinary differential equations in $\mathbb{R}^m$ (see [11]), the equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0 := \phi_0(p_0), \tag{1.8}$$

has a solution $x : I \to \Omega$ on some open interval $I \subset \mathbb{R}$ containing 0. Hence the function $\gamma := \psi_0 \circ x : I \to U_0 \subset M$ is a smooth solution of (1.7). This proves (i).

The local unqueness theorem asserts that two solutions $\gamma_i : I_i \to M$ of (1.7) for $i = 1, 2$ agree on the interval $(-\varepsilon, \varepsilon) \subset I_1 \cap I_2$ for $\varepsilon > 0$ sufficiently small. This follows immediately from the standard uniqueness theorem for the solutions of (1.8) in [11] and the fact that $x : I \to \Omega$ is a solution of (1.8) if and only if $\gamma := \psi_0 \circ x : I \to U_0$ is a solution of (1.7).

To prove (ii) we observe that the set $I := I_1 \cap I_2$ is an open interval containing zero and hence is connected. Now consider the set

$$A := \{ t \in I \,|\, \gamma_1(t) = \gamma_2(t) \}.$$

This set is nonempty, because $0 \in A$. It is closed, relative to I, because the maps $\gamma_1 : I \to M$ and $\gamma_2 : I \to M$ are contionuous. Namely, if $t_i \in I$ is a sequence converging to $t \in I$ then $\gamma_1(t_i) = \gamma_2(t_i)$ for every i and, taking the limit $i \to \infty$, we obtain $\gamma_1(t) = \gamma_2(t)$ and hence $t \in A$. The set A is also open by the local uniqueness theorem. Since I is connected it follows that $A = I$. This proves (ii) and the theorem. □

1.4.2 The flow of a vector field

Definition 1.53 (The flow of a vector field). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. For $p_0 \in M$ the* **maximal existence interval of** p_0 *is the open interval*

$$I(p_0) := \bigcup \left\{ I \;\middle|\; \begin{array}{l} I \subset \mathbb{R} \text{ is an open interval containing } 0 \\ \text{and there is a solution } x : I \to M \text{ of (1.7)} \end{array} \right\}.$$

By Theorem 1.52 equation (1.7) has a solution $\gamma : I(p_0) \to M$. The **flow of** X *is the map*

$$\phi : \mathcal{D} \to M$$

defined by

$$\mathcal{D} := \{(t, p_0) \,|\, p_0 \in M,\, t \in I(p_0)\}$$

and

$$\phi(t, p_0) := \gamma(t), \qquad \text{where } \gamma : I(p_0) \to M \text{ is the unique solution of (1.7).}$$

Theorem 1.54. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Let $\phi : \mathcal{D} \to M$ be the flow of X. Then the following holds.*

(i) *$\mathcal{D}$ is an open subset of $\mathbb{R} \times M$.*

(ii) *The map $\phi : \mathcal{D} \to M$ is smooth.*

(iii) *Let $p_0 \in M$ and $s \in I(p_0)$. Then*

$$I(\phi(s, p_0)) = I(p_0) - s \tag{1.9}$$

and, for every $t \in \mathbb{R}$ with $s + t \in I(p_0)$, we have

$$\phi(s + t, p_0) = \phi(t, \phi(s, p_0)). \tag{1.10}$$

Lemma 1.55. *Let M, X, and $\phi : \mathcal{D} \to M$ be as in Theorem 1.54. Let $K \subset M$ be a compact set. Then there is an M-open set $U \subset M$ and an $\varepsilon > 0$ such that $K \subset U$, $(-\varepsilon, \varepsilon) \times U \subset \mathcal{D}$, and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$.*

Proof. In the case where $M = \Omega$ is an open subset of $\mathbb{R}^m$ this was proved in [12]. Using local coordinates we deduce (as in the proof of Theorem 1.52) that, for every $p \in M$, there is an M-open neighborhood $U_p \subset M$ of p and an $\varepsilon_p > 0$ such that $(-\varepsilon_p, \varepsilon_p) \times U_p \subset \mathcal{D}$ and the restriction of ϕ to $(-\varepsilon_p, \varepsilon_p) \times U_p$ is smooth. Using this observation for every point $p \in K$ (and the axiom of choice) we obtain an M-open cover $K \subset \bigcup_{p \in K} U_p$. Since K is compact there is a finite subcover $K \subset U_{p_1} \cup \cdots \cup U_{p_N} =: U$. With $\varepsilon := \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_N}\}$ we obtain that ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$. This proves the lemma. □

Proof of Theorem 1.54. We prove (iii). The map $\gamma : I(p_0) - s \to M$ defined by $\gamma(t) := \phi(s+t, p_0)$ is a solution of the initial value problem $\dot\gamma(t) = X(\gamma(t))$ with $\gamma(0) = \phi(s, p_0)$. Hence $I(p_0) - s \subset I(\phi(s, p_0))$ and (1.10) holds for every $t \in \mathbb{R}$ with $s + t \in I(p_0)$. In particular, with $t = -s$, we have $p_0 = \phi(-s, \phi(s, p_0))$. Thus we obtain equality in (1.9) by the same argument with the pair $(s.p_0)$ replaced by $(-s, \phi(s, p_0))$.

We prove (i) and (ii). Fix a pair $(t_0, p_0) \in \mathcal{D}$ so that $p_0 \in M$ and $t_0 \in I(p_0)$. Suppose $t_0 \geq 0$. Then the set

$$K := \{\phi(t, p_0) \,|\, 0 \leq t \leq t_0\}$$

is a compact subset of M. (It is the image of the compact interval $[0, t_0]$ under the unique solution $\gamma : I(p_0) \to M$ of (1.7).) Hence, by Lemma 1.55, there is an M-open set $U \subset M$ and an $\varepsilon > 0$ such that

$$K \subset U, \qquad (-\varepsilon, \varepsilon) \times U \subset \mathcal{D},$$

and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$. Choose N so large that $t_0/N < \varepsilon$. Define $U_0 := U$ and, for $k = 1, \ldots, N$, define the sets $U_k \subset M$ inductively by

$$U_k := \{p \in U \,|\, \phi(t_0/N, p) \in U_{k-1}\}.$$

These sets are open in the relative topology of M.

We prove by induction on k that $(-\varepsilon, kt_0/N + \varepsilon) \times U_k \subset \mathcal{D}$ and ϕ is smooth on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. For $k = 0$ this holds by definition of ε and U. If $k \in \{1, \ldots, N\}$ and the assertion holds for $k - 1$ then we have

$$\begin{aligned} p \in U_k &\implies p \in U,\ \phi(t_0/N, p) \in U_{k-1} \\ &\implies (-\varepsilon, \varepsilon) \subset I(p),\ (-\varepsilon, (k-1)t_0/N + \varepsilon) \subset I(\phi(t_0/N, p)) \\ &\implies (-\varepsilon, kt_0/N + \varepsilon) \subset I(p). \end{aligned}$$

Here the last implication follows from (1.9). Moreover, for $p \in U_k$ and $t_0/N - \varepsilon < t < kt_0/N + \varepsilon$, we have, by (1.10), that

$$\phi(t, p) = \phi(t - t_0/N, \phi(t_0/N, p))$$

Since $\phi(t_0/N, p) \in U_{k-1}$ for $p \in U_k$ the right hand side is a smooth map on the open set $(t_0/N - \varepsilon, kt_0/N + \varepsilon) \times U_k$. Since $U_k \subset U$, ϕ is also a smooth map on $(-\varepsilon, \varepsilon) \times U_k$ and hence on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. This completes the induction. With $k = N$ we have found an open neighborhood of (t_0, p_0) contained in $\mathcal{D}$, namely the set $(-\varepsilon, t_0 + \varepsilon) \times U_N$, on which ϕ is smooth. The case $t_0 \leq 0$ is treated similarly. This proves (i) and (ii) and the theorem. □

Definition 1.56. *A vector field* $X \in \mathrm{Vect}(M)$ *is called* **complete** *if, for each* $p_0 \in M$*, there is an integral curve* $\gamma : \mathbb{R} \to M$ *of* X *with* $\gamma(0) = p_0$.

Lemma 1.57. *If* $M \subset \mathbb{R}^k$ *is a compact manifold then every vector field on* M *is complete.*

Proof. Let X be a smooth vector field on M. It follows from Lemma 1.55 with $K = M$ that there is a constant $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset I(p)$ for every $p \in M$. By Theorem 1.54 (iii) this implies $I(p) = \mathbb{R}$ for every $p \in M$. Hence X is complete. □

Let $M \subset \mathbb{R}^k$ be a smooth manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Then

$$X \text{ is complete} \quad \Longleftrightarrow \quad I(p) = \mathbb{R} \; \forall \, p \in M \quad \Longleftrightarrow \quad \mathcal{D} = \mathbb{R} \times M.$$

If X is complete and $\phi : \mathbb{R} \times M \to M$ is the flow of X we define $\phi^t : M \to M$ by

$$\phi^t(p) := \phi(t, p)$$

for $t \in \mathbb{R}$ and $p \in M$. Then Theorem 1.54 says, in particular, that the map $\phi^t : M \to M$ is smooth for every $t \in \mathbb{R}$ and that

$$\phi^{s+t} = \phi^s \circ \phi^t, \qquad \phi^0 = \mathrm{id} \tag{1.11}$$

for all $s, t \in \mathbb{R}$. In particular this implies that $\phi^t \circ \phi^{-t} = \phi^{-t} \circ \phi^t = \mathrm{id}$. Hence ϕ^t is bijective and $(\phi^t)^{-1} = \phi^{-t}$, so each ϕ^t is a diffeomorphism.

Exercise 1.58. Let $M \subset \mathbb{R}^k$ be a smooth manifold. A vector field X on M is said to have **compact support** if there is a compact subset $K \subset M$ such that $X(p) = 0$ for every $p \in M \setminus K$. Prove that every vector field with compact support is complete.

1.4.3 The group of diffeomorphisms

Let us denote the space of diffeomorphisms of M by

$$\mathrm{Diff}(M) := \{\phi : M \to M \,|\, \phi \text{ is a diffeomorphism}\}.$$

This is a group. The group operation is composition and the neutral element is the identity. Now equation (1.11) asserts that the flow of a complete vector field $X \in \mathrm{Vect}(M)$ is a group homomorphism

$$\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t.$$

This homomorphism is smooth and is characterized by the equation

$$\frac{d}{dt}\phi^t(p) = X(\phi^t(p)), \qquad \phi^0(p) = p$$

for all $p \in M$ and $t \in \mathbb{R}$. We will often abbreviate this equation in the form

$$\frac{d}{dt}\phi^t = X \circ \phi^t, \qquad \phi^0 = \mathrm{id}. \tag{1.12}$$

Exercise 1.59 (Isotopy). Let $M \subset \mathbb{R}^k$ be a compact manifold and $I \subset \mathbb{R}$ be an open interval containing 0. Let

$$I \times M \to \mathbb{R}^k : (t, p) \mapsto X_t(p)$$

be a smooth map such that $X_t \in \mathrm{Vect}(M)$ for every t. Prove that there is a smooth family of diffeomorphisms $I \times M \to M : (t, p) \mapsto \phi_t(p)$ satisfying

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t, \qquad \phi_0 = \mathrm{id} \tag{1.13}$$

for every $t \in I$. Such a family of diffeomorphisms $I \to \mathrm{Diff}(M) : t \mapsto \phi_t$ is called an **isotopy** of M. Conversely prove that every smooth isotopy $I \to \mathrm{Diff}(M) : t \mapsto \phi_t$ is generated (uniquely) by a smooth family of vector fields $I \to \mathrm{Vect}(M) : t \mapsto X_t$.

1.5 The Lie bracket

Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be smooth m-manifolds and $X \in \mathrm{Vect}(M)$ be smooth vector field on M. If $\psi : N \to M$ is a diffeomorphism, the **pullback** of X under ψ is the vector field on N defined by

$$(\psi^* X)(q) := d\psi(q)^{-1} X(\psi(q)) \tag{1.14}$$

for $q \in N$. If $\phi : M \to N$ is a diffeomorphism then the **pushforward** of X under ψ is the vector field on N defined by

$$(\phi_* X)(q) := d\phi(\phi^{-1}(q)) X(\phi^{-1}(q)) \tag{1.15}$$

for $q \in N$.

Exercise 1.60. Prove that $\phi_* X = (\phi^{-1})^* X$. Prove that for two diffeomorphism $\phi : M \to N$ and $\psi : N \to P$ and two vector fields $X \in \mathrm{Vect}(M)$ and $Z \in \mathrm{Vect}(P)$ we have $(\psi \circ \phi)_* X = \psi_* \phi_* X$ and $(\psi \circ \phi)^* Z = \phi^* \psi^* Z$.

We think of a vector field on M as a smooth map $X : M \to \mathbb{R}^k$ that satisfies the condition $X(p) \in T_pM$ for every $p \in M$. Ignoring this condition temporarily, we can differentiate X as a map from M to $\mathbb{R}^k$ and its differential at p is then a linear map $dX(p) : T_pM \to \mathbb{R}^k$. In general, this differential will no longer take values in the tangent space T_pM. However, if we have two vector fields X and Y on M the next lemma shows that the difference of the derivative of X in the direction Y and of Y in the direction X does take values in the tangent spaces of M.

Lemma 1.61. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X, Y \in \mathrm{Vect}(M)$ be complete vector fields. Denote by*

$$\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t, \qquad \mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \psi^t$$

the flows of X and Y, respectively. Fix a point $p \in M$ and consider the smooth map $\gamma : \mathbb{R} \to M$ defined by

$$\gamma(t) := \phi^t \circ \psi^t \circ \phi^{-t} \circ \psi^{-t}(p).$$

Then $\dot\gamma(0) = 0$ and

$$\begin{aligned} \frac{1}{2}\ddot\gamma(0) &= \left.\frac{d}{ds}\right|_{s=0} \left((\phi^s)_* Y\right)(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left((\psi^t)^* X\right)(p) \\ &= dX(p)Y(p) - dY(p)X(p) \qquad \in \quad T_pM. \end{aligned}$$

Exercise 1.62. Let $\gamma : \mathbb{R} \to \mathbb{R}^k$ be a C^2-curve and assume

$$\dot\gamma(0) = 0.$$

Prove that the curve $[0, \infty) \to \mathbb{R}^k : t \mapsto \gamma(\sqrt{t})$ is differentiable at $t = 0$ and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt{t}) = \frac{1}{2}\ddot\gamma(0).$$

Proof of Lemma 1.61. Define the map $\beta : \mathbb{R}^2 \to M$ by

$$\beta(s,t) := \phi^s \circ \psi^t \circ \phi^{-s} \circ \psi^{-t}(p)$$

for $s, t \in \mathbb{R}$. Then

$$\gamma(t) = \beta(t,t)$$

and

$$\frac{\partial\beta}{\partial s}(0,t) = X(p) - d\psi^t(\psi^{-t}(p))X(\psi^{-t}(p)), \tag{1.16}$$

$$\frac{\partial\beta}{\partial t}(s,0) = d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) - Y(p). \tag{1.17}$$

Hence

$$\dot\gamma(0) = \frac{\partial\beta}{\partial s}(0,0) + \frac{\partial\beta}{\partial t}(0,0) = 0.$$

Moreover,

$$\beta(s,0) = \beta(0,t) = p$$

for all s and t. This implies

$$\frac{\partial^2\beta}{\partial s^2}(0,0) = \frac{\partial^2\beta}{\partial t^2}(0,0) = 0$$

and hence

$$\ddot\gamma(0) = 2\frac{\partial^2\beta}{\partial s\partial t}(0,0). \tag{1.18}$$

Combining (1.17) and (1.18) we find

$$\begin{aligned}
\frac{1}{2}\ddot\gamma(0) &= \left.\frac{d}{ds}\right|_{s=0} \frac{\partial\beta}{\partial t}(s,0) \\
&= \left.\frac{d}{ds}\right|_{s=0} d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left((\phi^s)_* Y\right)(p)).
\end{aligned}$$

Note that the right hand side is the derivative of a smooth curve in the tangent space T_pM and hence is itself an element of T_pM. Likewise, combining equations (1.16) and (1.18) we find

$$\begin{aligned}
\frac{1}{2}\ddot\gamma(0) &= \left.\frac{d}{dt}\right|_{t=0} \frac{\partial\beta}{\partial s}(0,t) \\
&= -\left.\frac{d}{dt}\right|_{t=0} d\psi^t(\psi^{-t}(p))X(\psi^{-t}(p)) \\
&= \left.\frac{d}{dt}\right|_{t=0} d\psi^{-t}(\psi^t(p))X(\psi^t(p)) \\
&= \left.\frac{d}{dt}\right|_{t=0} d\psi^t(p)^{-1}X(\psi^t(p)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left((\psi^t)^* X\right)(p)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{1}{2}\ddot{\gamma}(0) &= \frac{\partial}{\partial s}\Big|_{s=0} d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) \\ &= \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0} \phi^s\circ\psi^t\circ\phi^{-s}(p) \\ &= \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0} \phi^s\circ\psi^t\circ\phi^{-s}(p) \\ &= \frac{\partial}{\partial t}\Big|_{t=0}\big(X(\psi^t(p)) - d\psi^t(p)X(p)\big) \\ &= dX(p)Y(p) - \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}\psi^t\circ\phi^s(p) \\ &= dX(p)Y(p) - \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\psi^t\circ\phi^s(p) \\ &= dX(p)Y(p) - \frac{\partial}{\partial s}\Big|_{s=0} Y(\phi^s(p)) \\ &= dX(p)Y(p) - dY(p)X(p).\end{aligned}$$

This proves the lemma. □

Definition 1.63. *Let $M \subset \mathbb{R}^k$ be a smooth manifold and $X, Y \in \mathrm{Vect}(M)$ be smooth vector fields on M. The* **Lie bracket** *of X and Y is the vector field $[X,Y] \in \mathrm{Vect}(M)$ defined by*

$$[X,Y](p) := dX(p)Y(p) - dY(p)X(p). \tag{1.19}$$

Warning: In the literature on differential geometry the Lie bracket of two vector fields is often (but not always) defined with the opposite sign. The rationale behind the present choice of the sign will be explained below.

Lemma 1.64. *Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be a smooth manifolds. Let X, Y, Z be smooth vector fields on M and let $\psi : N \to M$ be a diffeomorphism. Then*

$$\psi^*[X,Y] = [\psi^*X, \psi^*Y], \tag{1.20}$$

$$[X,Y] + [Y,X] = 0, \tag{1.21}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{1.22}$$

The last equation is called the **Jacobi identity**

Proof. Denote by ϕ^s the flow of X. Then the map $s \mapsto \psi^{-1} \circ \phi^s \circ \psi$ is the flow of the vector field $\psi^* X$. Hence it follows from Lemma 1.61 that

$$\begin{aligned}[][\psi^* X, \psi^* Y] &= \left.\frac{d}{ds}\right|_{s=0} \left(\psi^{-1} \circ \phi^s \circ \psi\right)_* \psi^* Y \\ &= \left.\frac{d}{ds}\right|_{s=0} \psi^* \left(\phi^s\right)_* Y \\ &= \psi^* \left.\frac{d}{ds}\right|_{s=0} \left(\phi^s\right)_* Y \\ &= \psi^*[X, Y].\end{aligned}$$

This proves equation (1.20). Equation (1.21) is obvious and the proof of (1.22) is left to the reader. (**Hint:** Prove this in local coordinates.) □

Definition 1.65. *A* **Lie algebra** *is a real vector space* $\mathfrak{g}$ *equipped with a skew symmetric bilinear map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ *that satisfies the Jacobi identity.*

Example 1.66. The Vector fields on a smooth manifold $M \subset \mathbb{R}^k$ form a Lie algebra with the Lie bracket defined by (1.19). The space $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$ of real $n \times n$-matrices is a Lie algebra with the Lie bracket

$$[\xi, \eta] := \xi\eta - \eta\xi.$$

It is also interesting to consider subspaces of $\mathfrak{gl}(n, \mathbb{R})$ that are invariant under this Lie bracket. An example is the space

$$\mathfrak{o}(n) := \left\{\xi \in \mathfrak{gl}(n, \mathbb{R}) \,|\, \xi^T + \xi = 0\right\}$$

of skew-symmetric $n \times n$-matrices. It is a nontrivial fact that every finite dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n. For example, the cross product defines a Lie algebra structure on $\mathbb{R}^3$ and the resulting Lie algebra is isomorphic to $\mathfrak{o}(3)$.

Remark 1.67. There is a linear map

$$\mathbb{R}^{m \times m} \to \mathrm{Vect}(\mathbb{R}^m) : \xi \mapsto X_\xi$$

which assigns to a matrix $\xi \in \mathfrak{gl}(m, \mathbb{R})$ the linear vector field $X_\xi : \mathbb{R}^m \to \mathbb{R}^m$ given by $X_\xi(x) := \xi x$ for $x \in \mathbb{R}^m$. This map preserves the Lie bracket, i.e. $[X_\xi, X_\eta] = X_{[\xi,\eta]}$, and hence is a **Lie algebra homomorphism**.

To understand the geometric interpretation of the Lie bracket consider again the curve

$$\gamma(t) := \phi^t \circ \psi^t \circ \phi^{-t} \circ \psi^{-t}(p)$$

in Lemma 1.61, where ϕ^t and ψ^t are the flows of the vector fields X and Y, respectively. Since $\dot{\gamma}(0) = 0$ we have

$$[X,Y](p) = \frac{1}{2}\ddot{\gamma}(0) = \frac{d}{dt}\bigg|_{t=0} \phi^{\sqrt{t}} \circ \psi^{\sqrt{t}} \circ \phi^{-\sqrt{t}} \circ \psi^{-\sqrt{t}}(p). \tag{1.23}$$

(See Exercise 1.62.) Geometrically this means following first the backward flow of Y for time ε, then the backward flow of X for time ε, then the forward flow of Y for time ε, and finally the forward flow of X for time ε, we will not, in general, get back to the original point p where we started but approximately obtain an *"error"* $\varepsilon^2[X,Y](p)$. An example of this (which I learned from Donaldson) is the mathematical formulation of parking a car.

Example 1.68 (Parking a car). The configuration space for driving a car in the plane is the manifold $M := \mathbb{C} \times S^1$, where $S^1 \subset \mathbb{C}$ denotes the unit circle. Thus a point in M is a pair $p = (z, \lambda) \in \mathbb{C} \times \mathbb{C}$ with $|\lambda| = 1$. The point $z \in \mathbb{C}$ represents the position of of the car and the unit vector $\lambda \in S^1$ represents the direction in which it is pointing. The *left turn* is represented by a vector field X and the *right turn* by a vector field Y on M. These vector field are given by

$$X(z,\lambda) := (\lambda, \mathbf{i}\lambda), \qquad Y(z,\lambda) := (\lambda, -\mathbf{i}\lambda).$$

Their Lie bracket is the vector field

$$[X,Y](z,\lambda) = (-2\mathbf{i}\lambda, 0).$$

This vector field represents a sideways move of the car to the right. And a sideways move by $2\varepsilon^2$ can be achieved by following a backward right turn for time ε, then a backward left turn for time ε, then a forward right turn for time ε, and finally a forward left turn for time ε.

This example can be reformulated by identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + \mathbf{i}y$ and representing a point in the unit circle by the angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ via $\lambda = e^{\mathbf{i}\theta}$. In this formulation the manifold is $M = \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$, a point in M is represented by a triple $(x, y, \theta) \in \mathbb{R}^3$, the vector fields X and Y are

$$X(x,y,\theta) := (\cos(\theta), \sin(\theta), 1), \qquad Y(x,y,\theta) := (\cos(\theta), \sin(\theta), -1),$$

and their Lie bracket is $[X,Y](x,y,\theta) = 2(\sin(\theta), -\cos(\theta), 0)$.

Lemma 1.69. *Let* $X, Y \in \mathrm{Vect}(M)$ *be complete vector fields on a manifold* M *and* $\phi^t, \psi^t \in \mathrm{Diff}(M)$ *be the flows of* X *and* Y*, respectively. Then the Lie bracket* $[X, Y]$ *vanishes if and only if the flows of* X *and* Y *commute, i.e.*

$$\phi^s \circ \psi^t = \psi^t \circ \phi^s \qquad \forall\, s, t \in \mathbb{R}.$$

Proof. If the flows of X and Y commute then the Lie bracket $[X, Y]$ vanishes by Lemma 1.61. Conversely, suppose that $[X, Y] = 0$. Then we have

$$\frac{d}{ds}\left(\phi^s\right)_* Y = \left(\phi^s\right)_* \frac{d}{dr}\left(\phi^r\right)_* Y = \left(\phi^s\right)_* [X, Y] = 0$$

for every $s \in \mathbb{R}$ and hence

$$(\phi^s)_* Y = Y. \tag{1.24}$$

Now fix a real number s and consider the curve $\gamma : \mathbb{R} \to M$ given by

$$\gamma(t) := \phi^s\left(\psi^t(p)\right).$$

This curve satisfies $\gamma(0) = \phi^s(p)$ and the differential equation

$$\dot\gamma(t) = d\phi^s(\psi^t(p))Y(\psi^t(p)) = \left(\left(\phi^s\right)_* Y\right)(\gamma(t)) = Y(\gamma(t)).$$

Here the last equation follows from (1.24). Since ψ^t is the flow of Y we obtain $\gamma(t) = \psi^t(\phi^s(p))$ for every $t \in \mathbb{R}$ and this proves the lemma. □

1.6 Lie groups

Combining the concept of a group and a manifold it is interesting to consider groups which are also manifolds and have the property that the group operation and the inverse define smooth maps. We shall only consider groups of matrices.

1.6.1 Definition and examples

Definition 1.70 (Lie groups). *A nonempty subset* $\mathrm{G} \subset \mathbb{R}^{n\times n}$ *is called a* **Lie group** *if it is a submanifold of* $\mathbb{R}^{n\times n}$ *and a subgroup of* $\mathrm{GL}(n, \mathbb{R})$*, i.e.*

$$g, h \in \mathrm{G} \qquad \Longrightarrow \qquad gh \in \mathrm{G}$$

(where gh *denotes the product of the matrices* g *and* h*) and*

$$g \in \mathrm{G} \qquad \Longrightarrow \qquad \det(g) \neq 0 \text{ and } g^{-1} \in \mathrm{G}.$$

(Since $\mathrm{G} \neq \emptyset$ *it follows from these conditions that the identity matrix* $\mathbb{1}$ *is an element of* G*.)*

Example 1.71. The general linear group $\mathrm{G} = \mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ and hence is a Lie group. By Exercise 1.18 the special linear group

$$\mathrm{SL}(n, \mathbb{R}) = \{g \in \mathrm{GL}(n, \mathbb{R}) \,|\, \det(g) = 1\}$$

is a Lie group and, by Example 1.19, the special orthogonal group

$$\mathrm{SO}(n) := \left\{g \in \mathrm{GL}(n, \mathbb{R}) \,|\, g^T g = 1\!\!1,\ \det(g) = 1\right\}$$

are Lie groups. In fact every orthogonal matrix has determinant ± 1 and so $\mathrm{SO}(n)$ is an open subset of $\mathrm{O}(n)$ (in the relative topology).

In a similar vein the group

$$\mathrm{GL}(n, \mathbb{C}) := \left\{g \in \mathbb{C}^{n\times n} \,|\, \det(g) \neq 0\right\}$$

of complex matrices with nonzero (complex) determinant is an open subset of $\mathbb{C}^{n\times n}$ and hence is a Lie group. As in the real case one can prove that the subgroups

$$\mathrm{SL}(n, \mathbb{C}) := \{g \in \mathrm{GL}(n, \mathbb{C}) \,|\, \det(g) = 1\},$$
$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n, \mathbb{C}) \,|\, g^* g = 1\!\!1\},$$
$$\mathrm{SU}(n) := \{g \in \mathrm{GL}(n, \mathbb{C}) \,|\, g^* g = 1\!\!1,\ \det(g) = 1\}$$

are submanifolds of $\mathrm{GL}(n, \mathbb{C})$ and hence are Lie groups. Here $g^* := \bar{g}^T$ denotes the conjugate transpose of a complex matrix.

Exercise 1.72. Prove that $\mathrm{SO}(n)$ is connected and deduce that $\mathrm{O}(n)$ has two connected components.

Exercise 1.73. Prove that the group $\mathrm{GL}(n, \mathbb{C})$ can be identified with the group

$$\mathrm{G} := \{\Phi \in \mathrm{GL}(2n, \mathbb{R}) \,|\, \Phi J_0 = J_0 \Phi\}, \qquad J_0 := \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}.$$

Hint: Use the isomophism $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n : (x, y) \mapsto x + \mathbf{i}y$. Show that a matrix $\Phi \in \mathbb{R}^{2n\times 2n}$ commutes with J_0 if and only if it has the form

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \qquad X, Y \in \mathbb{R}^{n\times n}.$$

What is the relation between the real determinant of Φ and the complex determinant of $X + \mathbf{i}Y$?

Exercise 1.74. Prove that $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are Lie groups.

Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group. Then the maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g, h) \mapsto gh$$

and

$$\mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$$

are smooth. This was proved in Analysis II [12]. Fixing an element $h \in \mathrm{G}$ we find that the derivative of the map $\mathrm{G} \to \mathrm{G} : g \mapsto gh$ at $g \in \mathrm{G}$ is given by the linear map

$$T_g\mathrm{G} \to T_{gh}\mathrm{G} : v \mapsto vh. \tag{1.25}$$

Here v and h are both matrices in $\mathbb{R}^{n\times n}$ and vh denotes the matrix product. In fact, if $v \in T_g\mathrm{G}$ then, since G is a manifold, there is a smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ with $\gamma(0) = g$ and $\dot\gamma(0) = v$. Since G is a group we obtain a smooth curve $\beta : \mathbb{R} \to \mathrm{G}$ given by $\beta(t) := \gamma(t)h$. It satisfies $\beta(0) = gh$ and so $vh = \dot\beta(0) \in T_{gh}\mathrm{G}$.

The linear map (1.25) is obviously a vector space isomorphism with inverse $T_{gh}\mathrm{G} \to T_g\mathrm{G} : w \mapsto wh^{-1}$. It is sometimes convenient to define the map $R_h : \mathrm{G} \to \mathrm{G}$ by $R_h(g) := gh$ (*right multiplication* by h). This is a diffeomorphism and the linear map (1.25) is just the derivative of R_h at g so that

$$dR_h(g)v = vh$$

for every $v \in T_g\mathrm{G}$. Similarly, for every $g \in \mathrm{G}$ we have a diffeomorphism $L_g : \mathrm{G} \to \mathrm{G}$ given by $L_g(h) := gh$ for $h \in \mathrm{G}$ and its derivative at a point $h \in \mathrm{G}$ is again given by matrix multiplication:

$$T_h\mathrm{G} \to T_{gh}\mathrm{G} : w \mapsto gw = dL_g(h)w. \tag{1.26}$$

Since L_g is a diffeomorphism its differential $dL_g(h) : T_h\mathrm{G} \to T_{gh}\mathrm{G}$ is again a vector space isomorphism for every $h \in \mathrm{G}$.

Exercise 1.75. Prove that the map $\mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$ is a diffeomorphism and that its derivative at $g \in \mathrm{G}$ is the vector space isomorphism

$$T_g\mathrm{G} \to T_{g^{-1}}\mathrm{G} : v \mapsto -g^{-1}vg^{-1}.$$

Hint: Use [12] or any textbook on first year analysis.

1.6.2 The Lie algebra of a Lie group

Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group. Its tangent space at the identity is called the **Lie algebra** of G and will be denoted by

$$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) := T_{\mathbb{1}}\mathrm{G}.$$

This terminology is justified by the fact that $\mathfrak{g}$ is in fact a Lie algebra, i.e. it is invariant under the standard Lie bracket operation $[\xi, \eta] := \xi\eta - \eta\xi$ on the space $\mathbb{R}^{n\times n}$ of square matrices (see Lemma 1.76 below). The proof requires the notion of the **exponential matrix**. For $\xi \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$ we define

$$\exp(t\xi) := \sum_{k=0}^{\infty} \frac{t^k \xi^k}{k!}. \tag{1.27}$$

It was proved in Analysis II (see [12]) that this series converges absolutely (and uniformly on compact t-intervals), that the map $t \mapsto \exp(t\xi)$ is smooth and satisfies the differential equation

$$\frac{d}{dt} \exp(t\xi) = \xi \exp(t\xi) = \exp(t\xi)\xi, \tag{1.28}$$

and that

$$\exp((s+t)\xi) = \exp(s\xi)\exp(t\xi), \qquad \exp(0\xi) = \mathbb{1} \tag{1.29}$$

for all $s, t \in \mathbb{R}$.

Lemma 1.76. *Let* $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ *be a Lie group and denote by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *its Lie algebra. Then the following holds.*

(i) *If* $\xi \in \mathfrak{g}$ *then* $\exp(t\xi) \in \mathrm{G}$ *for every* $t \in \mathbb{R}$.

(ii) *If* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *then* $g\eta g^{-1} \in \mathfrak{g}$.

(iii) *If* $\xi, \eta \in \mathfrak{g}$ *then* $[\xi, \eta] = \xi\eta - \eta\xi \in \mathfrak{g}$.

Proof. We prove (i). For every $g \in \mathrm{G}$ we have a vector space isomorphism $\mathfrak{g} = T_{\mathbb{1}}\mathrm{G} \to T_g\mathrm{G} : \xi \mapsto \xi g$ as in (1.25). Hence every element $\xi \in \mathfrak{g}$ determines a vector field $X_\xi \in \mathrm{Vect}(\mathrm{G})$ defined by

$$X_\xi(g) := \xi g \in T_g\mathrm{G}, \qquad g \in \mathrm{G}. \tag{1.30}$$

By Theorem 1.52 there is an integral curve $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{G}$ satisfying

$$\dot\gamma(t) = X_\xi(\gamma(t)) = \xi\gamma(t), \qquad \gamma(0) = \mathbb{1}.$$

By (1.28), the curve $(-\varepsilon,\varepsilon)\to\mathbb{R}^{n\times n}:t\mapsto\exp(t\xi)$ satisfies the same initial value problem and hence, by uniqueness, we have $\exp(t\xi)=\gamma(t)\in\mathrm{G}$ for $|t|<\varepsilon$. Now let $t\in\mathbb{R}$ and choose $N\in\mathbb{N}$ such that $\left|\frac{t}{N}\right|<\varepsilon$. Then $\exp(\frac{t}{N}\xi)\in\mathrm{G}$ and hence it follows from (1.29) that

$$\exp(t\xi)=\exp\left(\frac{t}{N}\xi\right)^N\in\mathrm{G}.$$

This proves (i).

We prove (ii). Consider the smooth curve $\gamma:\mathbb{R}\to\mathbb{R}^{n\times n}$ defined by

$$\gamma(t):=g\exp(t\eta)g^{-1}.$$

By (i) we have $\gamma(t)\in\mathrm{G}$ for every $t\in\mathbb{R}$. Since $\gamma(0)=\mathbb{1}$ we have

$$g\eta g^{-1}=\dot\gamma(0)\in\mathfrak{g}.$$

This proves (ii).

We prove (iii). Define the smooth map $\eta:\mathbb{R}\to\mathbb{R}^{n\times n}$ by

$$\eta(t):=\exp(t\xi)\eta\exp(-t\xi).$$

By (i) we have $\exp(t\xi)\in\mathrm{G}$ and, by (ii), we have $\eta(t)\in\mathfrak{g}$ for every $t\in\mathbb{R}$. Hence $[\xi,\eta]=\dot\eta(0)\in\mathfrak{g}$. This proves (iii) and the lemma. □

By Lemma 1.76 the curve $\gamma:\mathbb{R}\to\mathrm{G}$ defined by $\gamma(t):=\exp(t\xi)g$ is the integral curve of the vector field X_ξ in (1.30) with initial condition $\gamma(0)=g$. Thus X_ξ is complete for every $\xi\in\mathfrak{g}$.

Lemma 1.77. *If $\xi\in\mathfrak{g}$ and $\gamma:\mathbb{R}\to\mathrm{G}$ is a smooth curve satisfying*

$$\gamma(s+t)=\gamma(s)\gamma(t),\qquad\gamma(0)=\mathbb{1},\qquad\dot\gamma(0)=\xi,\tag{1.31}$$

then $\gamma(t)=\exp(t\xi)$ for every $t\in\mathbb{R}$.

Proof. For every $t\in\mathbb{R}$ we have

$$\dot\gamma(t)=\left.\frac{d}{ds}\right|_{s=0}\gamma(s+t)=\left.\frac{d}{ds}\right|_{s=0}\gamma(s)\gamma(t)=\dot\gamma(0)\gamma(t)=\xi\gamma(t).$$

Hence γ is the integral curve of the vector field X_ξ in (1.30) with $\gamma(0)=\mathbb{1}$. This implies $\gamma(t)=\exp(t\xi)$ for every $t\in\mathbb{R}$, as claimed. □

Example 1.78. Since the general linear group $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ its Lie algebra is the space of all real $n\times n$-matrices

$$\mathfrak{gl}(n,\mathbb{R}) := \mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) = \mathbb{R}^{n\times n}.$$

The Lie algebra of the special linear group is

$$\mathfrak{sl}(n,\mathbb{R}) := \mathrm{Lie}(\mathrm{SL}(n,\mathbb{R})) = \{\xi\in\mathfrak{gl}(n,\mathbb{R})\,|\,\mathrm{trace}(\xi)=0\}$$

(see Exercise 1.28) and the Lie algebra of the special orthogonal group is

$$\mathfrak{so}(n) := \mathrm{Lie}(\mathrm{SO}(n)) = \left\{\xi\in\mathfrak{gl}(n,\mathbb{R})\,|\,\xi^T+\xi=0\right\} = \mathfrak{o}(n)$$

(see Example 1.29).

Exercise 1.79. Prove that the Lie algebras of the general linear group over $\mathbb{C}$, the special linear group over $\mathbb{C}$, the unitary group, and the special unitary group are given by

$$\mathfrak{gl}(n,\mathbb{C}) := \mathrm{Lie}(\mathrm{GL}(n,\mathbb{C})) = \mathbb{C}^{n\times n},$$

$$\mathfrak{sl}(n,\mathbb{C}) := \mathrm{Lie}(\mathrm{SL}(n,\mathbb{C})) = \{\xi\in\mathfrak{gl}(n,\mathbb{C})\,|\,\mathrm{trace}(\xi)=0\},$$

$$\mathfrak{u}(n) := \mathrm{Lie}(\mathrm{U}(n)) = \{\xi\in\mathfrak{gl}(n,\mathbb{R})\,|\,\xi^*+\xi=0\},$$

$$\mathfrak{su}(n) := \mathrm{Lie}(\mathrm{SU}(n)) = \{\xi\in\mathfrak{gl}(n,\mathbb{C})\,|\,\xi^*+\xi=0,\ \mathrm{trace}(\xi)=0\}.$$

These are vector spaces over the reals. Determine their real dimensions. Which of these are also complex vector spaces?

Exercise 1.80. Let $\mathrm{G}\subset\mathrm{GL}(n,\mathbb{R})$ be a subgroup. Prove that G is a Lie group if and only if it is a closed subset of $\mathrm{GL}(n,\mathbb{R})$ in the relative topology.

1.6.3 Lie group homomorphisms

Let G and H be Lie groups and $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A **Lie group homomorphism** from G to H is a smooth map $\rho:\mathrm{G}\to\mathrm{H}$ that is a group homomorphism. A **Lie algebra homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map that preserves the Lie bracket.

Lemma 1.81. *Let* G *and* H *be Lie groups and denote their Lie algebras by* $\mathfrak{g}:=\mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{h}:=\mathrm{Lie}(\mathrm{H})$. *Let* $\rho:\mathrm{G}\to\mathrm{H}$ *be a Lie group homomorphism and denote its derivative at* $\mathbb{1}\in\mathrm{G}$ *by*

$$\dot\rho := d\rho(\mathbb{1}):\mathfrak{g}\to\mathfrak{h}.$$

Then $\dot\rho$ *is a Lie algebra homomorphism.*

Proof. The proof has three steps.

Step 1. *For all* $\xi \in \mathfrak{g}$ *and* $t \in \mathbb{R}$ *we have* $\rho(\exp(t\xi)) = \exp(t\dot\rho(\xi))$.

Fix an element $\xi \in \mathfrak{g}$. Then, by Lemma 1.76, we have $\exp(t\xi) \in \mathrm{G}$ for every $t \in \mathbb{R}$. Thus we can define a map $\gamma : \mathbb{R} \to \mathrm{H}$ by $\gamma(t) := \rho(\exp(t\xi))$. Since ρ is smooth, this is a smooth curve in H and, since ρ is a group homomorphism and the exponential map satisfies (1.29), our curve γ satisfies the conditions

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot\gamma(0) = d\rho(\mathbb{1})\xi = \dot\rho(\xi).$$

Hence it follows from Lemma 1.77 that $\gamma(t) = \exp(t\dot\rho(\xi))$. This proves Step 1.

Step 2. *For all* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *we have* $\dot\rho(g\eta g^{-1}) = \rho(g)\dot\rho(\eta)\rho(g^{-1})$.

Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by $\gamma(t) := g\exp(t\eta)g^{-1}$. By Lemma 1.76 this curve takes values in G. By Step 1 we have

$$\rho(\gamma(t)) = \rho(g)\rho(\exp(t\eta))\rho(g)^{-1} = \rho(g)\exp(t\dot\rho(\eta))\rho(g)^{-1}$$

for every t. Since $\gamma(0) = \mathbb{1}$ and $\dot\gamma(0) = g\eta g^{-1}$ we obtain

$$\begin{aligned}
\dot\rho(g\eta g^{-1}) &= d\rho(\gamma(0))\dot\gamma(0) \\
&= \frac{d}{dt}\Big|_{t=0} \rho(\gamma(t)) \\
&= \frac{d}{dt}\Big|_{t=0} \rho(g)\exp(t\dot\rho(\eta))\rho(g^{-1}) \\
&= \rho(g)\dot\rho(\eta)\rho(g^{-1}).
\end{aligned}$$

This proves Step 2.

Step 3. *For all* $\xi, \eta \in \mathfrak{g}$ *we have* $\dot\rho([\xi,\eta]) = [\dot\rho(\xi), \dot\rho(\eta)]$.

Define the curve $\eta : \mathbb{R} \to \mathfrak{g}$ by $\eta(t) := \exp(t\xi)\eta\exp(-t\xi)$. By Lemma 1.76 this curve takes values in the Lie algebra of G and $\dot\eta(0) = [\xi, \eta]$. Hence

$$\begin{aligned}
\dot\rho([\xi,\eta]) &= \frac{d}{dt}\Big|_{t=0} \dot\rho\left(\exp(t\xi)\eta\exp(-t\xi)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \rho\left(\exp(t\xi)\right)\dot\rho(\eta)\rho\left(\exp(-t\xi)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \exp\left(t\dot\rho(\xi)\right)\dot\rho(\eta)\exp\left(-t\dot\rho(\xi)\right) \\
&= \left[\dot\rho(\xi), \dot\rho(\eta)\right].
\end{aligned}$$

Here the first equation follows from the fact that $\dot\rho$ is linear, the second equation follows from Step 2 with $g = \exp(t\xi)$, and the third equation follows from Step 1. This proves Step 3 and the lemma. □

Example 1.82. The complex determinant defines a Lie group homomorphism $\det : \mathrm{U}(n) \to S^1$. The associated Lie algebra homomorphism is

$$\mathrm{trace} = \dot{\det} : \mathfrak{u}(n) \to \mathbf{i}\mathbb{R} = \mathrm{Lie}(S^1).$$

Example 1.83 (The unit quaternions). The Lie group SU(2) is diffeomorphic to the 3-sphere. Namely, every matrix in SU(2) can be written in the form

$$g = \begin{pmatrix} x_0 + \mathbf{i}x_1 & x_2 + \mathbf{i}x_3 \\ -x_2 + \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}, \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{1.32}$$

Here the x_i are real numbers. They can be interpreted as the coordinates of a quaternion

$$x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \in \mathbb{H}.$$

There is a product structure on the quaternions defined by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

This product structure is associative but not commutative. Conjugation on the quaternions reverses the sign of the coordinates x_1, x_2, x_3. Thus

$$\bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$$

and we have

$$\overline{xy} = \bar{y}\bar{x}, \qquad x\bar{x} = |x|^2, \qquad |xy| = |x|\,|y|$$

for $x, y \in \mathbb{H}$, where $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{H} \cong \mathbb{R}^4$. Thus the unit quaternions form a group

$$\mathrm{Sp}(1) := \{x \in \mathbb{H} \,|\, |x| = 1\}$$

with inverse map $x \mapsto \bar{x}$ and the map $\mathrm{Sp}(1) \to \mathrm{SU}(2) : x \mapsto g$ in (1.32) is a Lie group isomorphism.

Exercise 1.84 (The double cover of SO(3)**).** Identify the imaginary part of $\mathbb{H}$ with $\mathbb{R}^3$. Thus we write a vector $\xi \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ as a purely imaginary quaternion $\xi = \mathbf{i}\xi_1 + \mathbf{j}\xi_1 + \mathbf{k}\xi_3$. Define the map $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ by

$$\rho(x)\xi := x\xi\bar{x}, \qquad x \in \mathrm{Sp}(1), \qquad \xi \in \mathrm{Im}(\mathbb{H}).$$

Prove that $\rho(x)$ is represented by the 3×3-matrix

$$\rho(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 + x_2^2 - x_3^2 - x_1^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 + x_3^2 - x_1^2 - x_2^2 \end{pmatrix}.$$

Show that ρ is a Lie group homomorphism. For $x, y \in \mathrm{Sp}(1)$ prove that $\rho(x) = \rho(y)$ if and only if $y = \pm x$.

Example 1.85. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the set

$$\mathrm{Aut}(\mathfrak{g}) := \left\{ \Phi : \mathfrak{g} \to \mathfrak{g} \,\middle|\, \begin{array}{l} \Phi \text{ is a bijective linear map,} \\ \Phi[\xi,\eta] = [\Phi\xi, \Phi\eta] \;\forall\, \xi,\eta \in \mathfrak{g} \end{array} \right\}$$

of **Lie algebra automorphisms** of $\mathfrak{g}$ is a Lie group. Its Lie algebra is the space of **derivations** on $\mathfrak{g}$ denoted by

$$\mathrm{Der}(\mathfrak{g}) := \left\{ A : \mathfrak{g} \to \mathfrak{g} \,\middle|\, \begin{array}{l} A \text{ is a linear map,} \\ A[\xi,\eta] = [A\xi, \eta] + [\xi, A\eta] \;\forall\, \xi,\eta \in \mathfrak{g} \end{array} \right\}.$$

Now suppose that $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ is the Lie algebra of a Lie group G. Then there is a map

$$\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{ad}(g)\eta := g\eta g^{-1}, \tag{1.33}$$

for $g \in \mathrm{G}$ and $\eta \in \mathfrak{g}$. Lemma 1.76 (ii) asserts that $\mathrm{ad}(g)$ is indeed a linear map from $\mathfrak{g}$ to itself for every $g \in \mathrm{G}$. The reader may verify that the map $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism for every $g \in \mathrm{G}$ and that the map $\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ is a Lie group homomorphism. The associated Lie algebra homomorphism is the map

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \qquad \mathrm{Ad}(\xi)\eta := [\xi, \eta], \tag{1.34}$$

for $\xi, \eta \in \mathfrak{g}$. To verify the claim $\mathrm{Ad} = \dot{\mathrm{ad}}$ we compute

$$\dot{\mathrm{ad}}(\xi)\eta = \left.\frac{d}{dt}\right|_{t=0} \mathrm{ad}(\exp(t\xi))\eta = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi) = [\xi,\eta].$$

Exercise 1.86. Let $\mathfrak{g}$ be any Lie algebra and define $\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by (1.34). Prove that $\mathrm{Ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ is a derivation for every $\xi \in \mathfrak{g}$ and that $\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism. If $\mathfrak{g}$ is finite dimensional, prove that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$.

Example 1.87. Consider the map

$$\mathrm{GL}(n,\mathbb{R}) \to \mathrm{Diff}(\mathbb{R}^n) : g \mapsto \phi_g$$

which assigns to every nonsingular matrix $g \in \mathrm{GL}(n,\mathbb{R})$ the linear diffeomorphism $\phi_g : \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi_g(x) := gx$. This map $g \to \phi_g$ is a group homomorphism. The group $\mathrm{Diff}(\mathbb{R}^n)$ is infinite dimensional and thus cannot be a Lie group. However, it has many properties in common with Lie groups. For example one can define what is meant by a smooth path in $\mathrm{Diff}(\mathbb{R}^n)$ and extend formally the notion of a tangent vector (as the derivative of a path through a given element of $\mathrm{Diff}(\mathbb{R}^n)$) to this setting. In particular, the tangent space of $\mathrm{Diff}(\mathbb{R}^n)$ at the identity can then be identified with the space of vector fields $T_{\mathrm{id}}\mathrm{Diff}(\mathbb{R}^n) = \mathrm{Vect}(\mathbb{R}^n)$. Differentiating the map $g \to \phi_g$ one then obtains a linear map

$$\mathfrak{gl}(n, \mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n) : \xi \mapsto X_\xi$$

which assigns to every matrix $\xi \in \mathfrak{gl}(n, \mathbb{R})$ the vector field $X_\xi : \mathbb{R}^n \to \mathbb{R}^n$ given by $X_\xi(x) := \xi x$. We have already seen in Remark 1.67 that this map is a Lie algebra homomorphism.

1.6.4 Lie groups and diffeomorphisms

There is a natural correspondence between Lie groups and Lie algebras on the one hand and diffeomorphisms and vector fields on the other hand. We summarize this correspondence in the following table

Lie groups	**Diffeomorphisms**
$\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$	$\mathrm{Diff}(M)$
$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$	$\mathrm{Vect}(M) = T_{\mathrm{id}}\mathrm{Diff}(M)$
exponential map	flow of a vector field
$t \mapsto \exp(t\xi)$	$t \mapsto \phi^t =$ "$\exp(tX)$"
adjoint representation	pushforward
$\xi \mapsto g\xi g^{-1}$	$X \mapsto \phi_* X$
Lie bracket on $\mathfrak{g}$	Lie bracket of vector fields
$[\xi, \eta] = \xi\eta - \eta\xi$	$[X, Y] = dX \cdot Y - dY \cdot X.$

To understand the correspondence between the exponential map and the flow of a vector field compare equation (1.12) with equation (1.28). To understand the correspondence between adjoint representation and pushforward observe that

$$\phi_* Y = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \psi^t \circ \phi^{-1}, \qquad g\eta g^{-1} = \left.\frac{d}{dt}\right|_{t=0} g \exp(t\eta) g^{-1},$$

where ψ^t denotes the flow of Y. To understand the correspondence between the Lie brackets recall that

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} (\phi^t)_* Y, \qquad [\xi, \eta] = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta \exp(-t\xi),$$

where ϕ^t denotes the flow of X. We emphasize that the analogy between Lie groups and Diffeomorphisms only works well when the manifold M is compact so that every vector field on M is complete. The next exercise gives another parallel between the Lie bracket on the Lie algebra of a Lie group and the Lie bracket of two vector fields.

Exercise 1.88. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by

$$\gamma(t) := \exp(t\xi) \exp(t\eta) \exp(-t\xi) \exp(-t\eta).$$

Prove that $\dot{\gamma}(0) = 0$ and $\frac{1}{2}\ddot{\gamma}(0) = [\xi, \eta]$. Compare this with Lemma 1.61.

Exercise 1.89. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Show that $[\xi, \eta] = 0$ if and only if $\exp(s\xi) \exp(t\eta) = \exp(t\eta) \exp(s\xi)$ for all $s, t \in \mathbb{R}$. How can this result be deduced from Lemma 1.69?

1.6.5 Vector fields and derivations

Let M be a compact smooth manifold and denote by

$$\mathscr{F}(M) := C^\infty(M, \mathbb{R})$$

the space of smooth real valued functions $f : M \to \mathbb{R}$. This is an algebra with addition and multiplication of functions. An **automorphism** of $\mathscr{F}(M)$ is a bijective linear map $\Phi : \mathscr{F}(M) \to \mathscr{F}(M)$ that satisfies

$$\Phi(fg) = \Phi(f)\Phi(g), \qquad \Phi(1) = 1.$$

A **derivation** of $\mathscr{F}(M)$ is a linear map $\delta : \mathscr{F}(M) \to \mathscr{F}(M)$ that satisfies

$$\delta(fg) = \delta(f)g + f\delta(g).$$

The automorphisms of $\mathscr{F}(M)$ form a group denoted by $\mathrm{Aut}(\mathscr{F}(M))$ and the derivations form a Lie algebra denoted by $\mathrm{Der}(\mathscr{F}(M))$. We may think of $\mathrm{Der}(\mathscr{F}(M))$ as the Lie algebra of $\mathrm{Aut}(\mathscr{F}(M))$ with the Lie bracket given by the commutator. Now there is a natural map

$$\mathrm{Diff}(M) \to \mathrm{Aut}(\mathscr{F}(M)) : \phi \mapsto \phi^*. \tag{1.35}$$

Here the pullback automorphism $\phi^* : \mathscr{F}(M) \to \mathscr{F}(M)$ is defined by

$$\phi^* f := f \circ \phi.$$

The map $\phi \mapsto \phi^*$ can be thought of as a Lie group anti-homomorphism. Differentiating it at the identity $\phi = \mathrm{id}$ we obtain a linear map

$$\mathrm{Vect}(M) \to \mathrm{Der}(\mathscr{F}(M)) : X \mapsto \mathcal{L}_X. \tag{1.36}$$

Here the operator $\mathcal{L}_X : \mathscr{F}(M) \to \mathscr{F}(M)$ is given by the derivative of a function f in the direction of the vector field X, i.e.

$$\mathcal{L}_X f := df \cdot X = \left.\frac{d}{dt}\right|_{t=0} f \circ \phi^t,$$

where ϕ^t denotes the flow of X. Since the map (1.36) is the derivative of the "Lie group" anti-homomorphism (1.35) we expect it to be a Lie algebra anti-homomorphism. Indeed, one can show that

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_X \mathcal{L}_Y = -[\mathcal{L}_X, \mathcal{L}_Y] \tag{1.37}$$

for $X, Y \in \mathrm{Vect}(M)$. This confirms that our sign in the definition of the Lie bracket is consistent with the standard conventions in the theory of Lie groups. In the literature the difference between a vector field and the associated derivation $\mathcal{L}_X$ is sometimes neglected in the notation and many authors write $Xf := df \cdot X = \mathcal{L}_X f$, thus thinking of a vector field on a manifold M as an operator on the space of functions. With this notation one obtains the equation $[X, Y]f = Y(Xf) - X(Yf)$ and here lies the origin for the use of the opposite sign for the Lie bracket in many books on differential geometry.

Exercise 1.90. Prove that the map (1.35) is bijective. **Hint:** An **ideal** in $\mathscr{F}(M)$ is a linear subspace $\mathscr{J} \subset \mathscr{F}(M)$ satisfying the condition

$$f \in \mathscr{F}(M),\ g \in \mathscr{J} \qquad \Longrightarrow \qquad fg \in \mathscr{J}.$$

A **maximal ideal** in $\mathscr{F}(M)$ is an ideal $\mathscr{J} \subset \mathscr{F}(M)$ such that every ideal $\mathscr{J}' \subsetneq \mathscr{F}(M)$ containing $\mathscr{J}$ must be equal to $\mathscr{J}$. Prove that if $\mathscr{J} \subset \mathscr{F}(M)$ is an ideal with the property that, for every $p \in M$, there is an $f \in \mathscr{J}$ with $f(p) \neq 0$ then $\mathscr{J} = \mathscr{F}(M)$. Deduce that every maximal ideal in $\mathscr{F}(M)$ has the form $\mathscr{J}_p := \{f \in \mathscr{F}(M) \,|\, f(p) = 0\}$ for some $p \in M$. If $\Phi \in \mathrm{Aut}(M)$ and $p \in M$ show that $\Phi^{-1}\mathscr{J}_p$ is a maximal ideal and hence there is a unique element $\phi(p) \in M$ such that $\Phi^{-1}\mathscr{J}_p = \mathscr{J}_{\phi(p)}$. Show that $\phi : M \to M$ is a diffeomorphism and that $\Phi = \phi^*$.

Exercise 1.91. Prove that the map (1.36) is bijective. **Hint:** Fix a derivation $\delta \in \mathrm{Der}(\mathscr{F}(M))$ and prove the following. **Fact 1:** If $U \subset M$ is an open set and $f \in \mathscr{F}(M)$ vanishes on U then $\delta(f)$ vanishes on U. **Fact 2:** If $p \in M$ and the derivative $df(p) : T_pM \to \mathbb{R}$ is zero then $(\delta(f))(p) = 0$. (By Fact 1 the proof of Fact 2 can be reduced to an argument in local coordinates.)

Exercise 1.92. Verify the formula (1.37).

1.7 Vector bundles and submersions

1.7.1 Submersions

Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold. A smooth map $f : N \to M$ is called a **submersion** if $df(q) : T_qN \to T_{f(q)}M$ is surjective for every $q \in M$.

Lemma 1.93. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, $N \subset \mathbb{R}^\ell$ be a smooth n-manifold, and $f : N \to M$ be a smooth map. The following are equivalent.*

(i) *f is a submersion.*

(ii) *For every $q_0 \in N$ there is an M-open neighborhood U of $p_0 := f(q_0)$ and a smooth map $g : U \to N$ such that $g(f(q_0)) = q_0$ and $f \circ g = \mathrm{id} : U \to U$. Thus f has a local right inverse near every point in N (see Figure 1.9).*

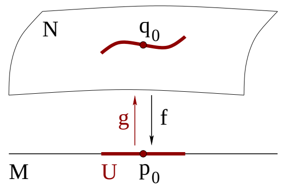


Figure 1.9: A local right inverse of a submersion.

Proof. We prove that (i) implies (ii). Since $df(q_0) : T_{q_0}N \to T_{p_0}M$ is surjective we have $n \geq m$ and $\dim\ker df(q_0) = n - m$. Hence there is a linear map $A : \mathbb{R}^\ell \to \mathbb{R}^{n-m}$ whose restriction to the kernel of $df(q_0)$ is bijective. Now define the map $\psi : N \to M \times \mathbb{R}^{n-m}$ by

$$\psi(q) := (f(q), A(q - q_0))$$

for $q \in N$. Then $\psi(q_0) = (p_0, 0)$ and $d\psi(q_0) : T_{q_0}N \to T_{p_0}M \times \mathbb{R}^{n-m}$ sends $w \in T_{q_0}N$ to $(df(q_0)w, Aw)$ and is therefore bijective. Hence it follows from the inverse function theorem for manifolds (Theorem 1.34) that there is an N-open neighborhood $V \subset N$ of q_0 such that $W := \psi(N) \subset M \times \mathbb{R}^{n-m}$ is an open neighborhood of $(p_0, 0)$ and $\psi|_V : V \to W$ is a diffeomorphism. Let

$$U := \{p \in M \mid (p, 0) \in W\}$$

and define the map $g : U \to N$ by $g(p) := \psi^{-1}(p, 0)$. Then $p_0 \in U$, g is smooth, and $(p, 0) = \psi(g(p)) = (f(g(p)), A(g(p) - q_0))$. This implies $f(g(p)) = p$ for all $p \in U$ and $g(p_0) = \psi^{-1}(p_0, 0) = q_0$. Thus we have proved that (i) implies (ii). The converse is an easy consequence of the chain rule and is left to the reader. □

Corollary 1.94. *The image of a submersion $f : N \to M$ is open.*

Proof. If $p_0 = f(q_0) \in f(N)$ then the neighborhood $U \subset M$ of p_0 in Lemma 1.93 (ii) is contained in the image of f. □

Corollary 1.95. *If N is a nonempty compact manifold, M is a connected manifold, and $f : N \to M$ is a submersion then f is surjective and M is compact.*

Proof. The image $f(M)$ is an open subset of M by Corollary 1.94, it is a relatively closed subset of M because N is compact, and it is nonempty because N is nonempty. Since M is connected this implies that $f(N) = M$. In particular, M is compact. □

Exercise 1.96. Let $f : N \to M$ be a smooth map. Prove that the sets $\{q \in N \,|\, df(q) \text{ is injective}\}$ and $\{q \in N \,|\, df(q) \text{ is surjective}\}$ are open (in the relative topology of N).

1.7.2 Vector bundles

Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold. A **(smooth) vector bundle (over M of rank n)** is a smooth submanifold $E \subset M \times \mathbb{R}^\ell$ such that, for every $p \in M$, the set

$$E_p := \left\{ v \in \mathbb{R}^\ell \,|\, (p, v) \in E \right\}$$

is an n-dimensional linear subspace of $\mathbb{R}^\ell$ (called the **fiber of E over** p). If $E \subset M \times \mathbb{R}^\ell$ is a vector bundle then a **(smooth) section of** E is smooth map $s : M \to \mathbb{R}^\ell$ such that $s(p) \in E_p$ for every $p \in M$. A vector bundle $E \subset M \times \mathbb{R}^\ell$ is equipped with a smooth map

$$\pi : E \to M$$

defined by $\pi(p, v) := p$, called the **projection**. A section $s : M \to \mathbb{R}^\ell$ of E determines a smooth map $\sigma : M \to E$ which sends the point $p \in M$ to the pair $(p, s(p)) \in E$. This map satisfies

$$\pi \circ \sigma = \mathrm{id}.$$

It is sometimes convenient to abuse notation and eliminate the distinction between s and σ. Thus we will sometimes use the same letter s for the map from M to E.

Example 1.97. Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional submanifold. The set

$$TM := \{(p, v) \,|\, p \in M,\, v \in T_pM\}$$

is called the **tangent bundle** of M. This is a subset of $M \times \mathbb{R}^k$ and its fiber T_pM is an m-dimensional linear subspace of $\mathbb{R}^k$ by Theorem 1.23. However, it is not immediately clear from the definition that TM is a submanifold of $M \times \mathbb{R}^k$. This will be proved below. The sections of TM are the vector fields on M.

Exercise 1.98. Let $f : M \to N$ be a smooth map between manifolds. Prove that the tangent map $TM \to TN : (p, v) \mapsto (f(p), df(p)v)$ is smooth.

Exercise 1.99. Let $V \subset \mathbb{R}^\ell$ be an n-dimensional linear subspace. The **orthogonal projection** of $\mathbb{R}^\ell$ onto V is the matrix $\Pi \in \mathbb{R}^{\ell\times\ell}$ that satisfies

$$\Pi = \Pi^2 = \Pi^T, \qquad \operatorname{im}\Pi = V. \tag{1.38}$$

Prove that there is a unique matrix $\Pi \in \mathbb{R}^{\ell\times\ell}$ satisfying (1.38). Prove that, for every symmetric matrix $S = S^T \in \mathbb{R}^{\ell\times\ell}$, the kernel of S is the orthogonal complement of the image of S. If $D \in \mathbb{R}^{\ell\times n}$ is any injective matrix whose image is V, prove that $\det(D^TD) \neq 0$ and

$$\Pi = D(D^TD)^{-1}D^T. \tag{1.39}$$

Theorem 1.100. *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and let $E \subset M \times \mathbb{R}^\ell$ be a subset. Assume that, for every $p \in M$, the set*

$$E_p := \left\{v \in \mathbb{R}^\ell \,|\, (p, v) \in E\right\} \tag{1.40}$$

is an n-dimensional linear subspace of $\mathbb{R}^\ell$. Let $\Pi : M \to \mathbb{R}^{\ell\times\ell}$ be the map that assigns to every $p \in M$ the orthogonal projection of $\mathbb{R}^\ell$ onto E_p, i.e.

$$\Pi(p) = \Pi(p)^2 = \Pi(p)^T, \qquad \operatorname{im}\Pi(p) = E_p. \tag{1.41}$$

Then the following are equivalent.

(i) *E is a vector bundle.*

(ii) *The map $\Pi : M \to \mathbb{R}^{\ell\times\ell}$ is smooth.*

(iii) *For every $p_0 \in M$ and every $v_0 \in E_{p_0}$ there is a smooth map $s : M \to \mathbb{R}^\ell$ such that $s(p) \in E_p$ for every $p \in M$ and $s(p_0) = v_0$.*

Condition (i) implies that the projection $\pi : E \to M$ is a submersion. In (iii) the section s can be chosen to have compact support, i.e. there is a compact subset $K \subset M$ such that $s(p) = 0$ for $p \notin K$.

Corollary 1.101. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold. Then TM is a vector bundle over M and hence is a smooth $2m$-manifold in $\mathbb{R}^k \times \mathbb{R}^k$.*

Proof. Let $\phi : U \to \Omega$ be a coordinate chart on an M-open set $U \subset M$ with values in an open subset $\Omega \subset \mathbb{R}^m$. Denote its inverse by $\psi := \phi^{-1} : \Omega \to M$. By Theorem 1.23 the linear map $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is $T_{\psi(x)}M$ for every $x \in \Omega$. Hence the map $D : U \to \mathbb{R}^{k\times m}$ defined by

$$D(p) := d\psi(\phi(p)) \in \mathbb{R}^{k\times m}$$

is smooth and, for every $p \in U$, the linear map $D(p) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is T_pM. Thus the function $\Pi^{TM} : M \to \mathbb{R}^{k\times k}$ defined by (1.41) with $E_p = T_pM$ is given by

$$\Pi^{TM}(p) = D(p)\left(D(p)^T D(p)\right)^{-1} D(p)^T$$

for $p \in U$ (see Exercise 1.99). Hence the restriction of Π to U is smooth. Since M can be covered by coordinate charts it follows that Π^{TM} is smooth and hence, by Theorem 1.100, TM is a smooth vector bundle. This proves the corollary. □

Let $M \subset \mathbb{R}^k$ be an m-manifold, $N \subset \mathbb{R}^\ell$ be an n-manifold, $f : N \to M$ be a smooth map, and $E \subset M \times \mathbb{R}^d$ be a vector bundle. The **pullback bundle** is the vector bundle $f^*E \to N$ defined by

$$f^*E := \left\{(q,v) \in N \times \mathbb{R}^d \,|\, v \in E_{f(q)}\right\}$$

and the **normal bundle of** E is the vector bundle $E^\perp \to M$ defined by

$$E^\perp := \left\{(p,w) \in M \times \mathbb{R}^d \,|\, \langle v,w\rangle = 0 \;\forall\, v \in E_p\right\}.$$

Corollary 1.102. *The pullback and normal bundles are vector bundles.*

Proof. Let $\Pi = \Pi^E : M \to \mathbb{R}^{d\times d}$ be the map defined by (1.41). This map is smooth by Theorem 1.100. Moreover, the corresponding maps for f^*E and $E^\perp$ are given by

$$\Pi^{f^*E} = \Pi^E \circ f : N \to \mathbb{R}^{d\times d}, \qquad \Pi^{E^\perp} = \mathbb{1} - \Pi^E : M \to \mathbb{R}^{d\times d}.$$

These maps are smooth and hence it follows again from Theorem 1.100 that f^*E and $E^\perp$ are vector bundles. □

Proof of Theorem 1.100. We first assume that E is a vector bundle and prove that $\pi : E \to M$ is a submersion. Let $\sigma : M \to E$ denote the zero section given by $\sigma(p) := (p, 0)$. Then $\pi \circ \sigma = \mathrm{id}$ and hence it follows from the chain rule that the derivative $d\pi(p,0) : T_{(p,0)}E \to T_pM$ is surjective. Now it follows from Exercise 1.96 that for every $p \in M$ there is an $\varepsilon > 0$ such that the derivative $d\pi(p,v) : T_{(p,v)}E \to T_pM$ is surjective for every $v \in E_p$ with $|v| < \varepsilon$. Consider the map $f_\lambda : E \to E$ defined by $f_\lambda(p,v) := (p, \lambda v)$. This map is a diffeomorphism for every $\lambda > 0$. It satisfies $\pi = \pi \circ f_\lambda$ and hence

$$d\pi(p,v) = d\pi(p,\lambda v) \circ df_\lambda(p,v) : T_{(p,v)}E \to T_pM.$$

Since $df_\lambda(p,v)$ is bijective and $d\pi(p,\lambda v)$ is surjective for $\lambda < \varepsilon/|v|$ it follows that $d\pi(p,v)$ is surjective for every $p \in M$ and every $v \in E_p$. Thus the projection $\pi : E \to M$ is a submersion for every vector bundle E over M.

We prove that (i) implies (iii). Let $p_0 \in M$ and $v_0 \in E_{p_0}$. We have already proved that π is a submersion. Hence it follows from Lemma 1.93 that there is an M-open neighborhood $U \subset M$ of p_0 and a smooth map $\sigma_0 : U \to E$ over U such that $\pi \circ \sigma_0 = \mathrm{id} : U \to U$ and $\sigma_0(p_0) = (p_0, v_0)$. Define the map $s_0 : U \to \mathbb{R}^\ell$ by $(p, s_0(p)) := \sigma_0(p))$. Then $s_0(p_0) = v_0$ and $s_0(p) \in E_p$ for all $p \in U$. Now choose $\varepsilon > 0$ such that $\{p \in M \mid |p - p_0| < \varepsilon\} \subset U$ and choose a smooth cutoff function $\beta : \mathbb{R}^k \to [0,1]$ such that $\beta(p_0) = 1$ and $\beta(p) = 0$ for $|p - p_0| \geq \varepsilon$. Define $s : M \to \mathbb{R}^\ell$ by

$$s(p) := \begin{cases} \beta(p)s_0(p), & \text{if } p \in U, \\ 0, & \text{if } p \notin U. \end{cases}$$

This map satisfies the requirements of (iii).

That (ii) implies (iii) follows by choosing $s(p) := \Pi(p)v_0$ for every $p \in M$.

We prove that (iii) and (ii). Thus we assume that E satisfies (ii). Choose a point $p_0 \in M$ and a basis $v_1, \dots, v_n$ of E_{p_0}. By (ii) there are smooth sections $s_1, \dots, s_n : M \to \mathbb{R}^\ell$ of E such that $s_i(p_0) = v_i$ for $i = 1, \dots, n$. Now there is an M-open neighborhood $U \subset M$ of p_0 such that the vectors $s_1(p), \dots, s_n(p)$ are linearly independent, and hence form a basis of E_p, for every $p \in U$. Hence, for every $p \in U$, we have

$$E_p = \mathrm{im} D(p), \qquad D(p) := [s_1(p) \cdots s_n(p)] \in \mathbb{R}^{\ell \times n}.$$

By Exercise 1.99 this implies

$$\Pi(p) = D(p)\left(D(p)^T D(p)\right)^{-1} D(p)^T$$

for every $p \in U$. Thus we have proved that every point $p_0 \in M$ has a neighborhood U such that the restriction of Π to U is smooth. This shows that (iii) implies (ii).

To prove that (iii) also implies (i) we fix a point $p_0 \in M$ and choose $D : U \to \mathbb{R}^{\ell \times n}$ as above. Shrinking U if necessary, we may assume that there is a coordinate chart $\phi : U \to \Omega$ with values in an open set $\Omega \subset \mathbb{R}^m$. Consider the open subset

$$\pi^{-1}(U) = \{(p, v) \,|\, p \in U,\, v \in E_p\}$$

of E and define the map $\Phi : \pi^{-1}(U) \to \Omega \times \mathbb{R}^n$ by

$$\Phi(p, v) := \Big(\phi(p), \big(D(p)^T D(p)\big)^{-1} D(p)^T v\Big).$$

This map is a diffeomorphism with $\Phi^{-1}(x, \xi) = \big(\phi^{-1}(x), D(\phi^{-1}(x))\xi\big)$ for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Hence Φ is a coordinate chart on $\pi^{-1}(U)$. Thus we have proved that (iii) implies (i) and this completes the proof of the theorem. □

Remark 1.103. Let $D : U \to \mathbb{R}^{\ell \times n}$ be as in the above proof. Then the map $\Phi : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ in the above proof is called a **local trivialization** of the vector bundle E. It fits into a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & \Omega \times \mathbb{R}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U & \xrightarrow{\ \phi\ } & \Omega \end{array}.$$

It is sometimes convenient to consider local trivializations that leave the first coordinate unchanged. For example in our setting we could take the map $\pi^{-1}(U) \to U \times \mathbb{R}^n : (p, v) \mapsto \big(p, (D(p)^T D(p))^{-1} D(p)^T v\big)$. The restriction of this map to each fiber is an isomorphism $E_p \to \mathbb{R}^n$.

Exercise 1.104. Construct a vector bundle $E \subset S^1 \times \mathbb{R}^2$ of rank 1 that does not admit a *global trivialization*, i.e. that is not isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. Such a vector bundle is called a **Möbius strip**. Define the notion of an isomorphism between two vector bundles E and F over a manifold M.

1.8 The theorem of Frobenius

Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and n be a nonnegative integer. A **subbundle of rank** n of the tangent bundle TM is a subset $E \subset TM$ that is itself a vector bundle of rank n over M, i.e. it is a submanifold of TM and the fiber $E_p = \{v \in T_pM \,|\, (p, v) \in E\}$ is an n-dimensional linear subspace of T_pM for every $p \in M$. Note that the rank n of a subbundle is necssarily less than or equal to m. In the literature a subbundle of the tangent bundle is sometimes called a *distribution* on M. We shall, however, not use this terminology in order to avoid confusion with the concept of a distribution in the functional analytic setting.

Definition 1.105. *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $E \subset TM$ be a subbundle of rank n. E is called* **involutive** *if, for any two vector fields $X, Y \in \mathrm{Vect}(M)$, we have*

$$X(p), Y(p) \in E_p \; \forall \, p \in M \qquad \Longrightarrow \qquad [X,Y](p) \in E_p \; \forall \, p \in M. \tag{1.42}$$

E is called **integrable** *if, for every $p_0 \in M$, there exists a submanifold $N \subset M$ such that $p_0 \in N$ and $T_pN = E_p$ for every $p \in N$. A* **foliation box for** *E (see Figure 1.10) is a coordinate chart $\phi : U \to \Omega$ on an M-open subset $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ such that the set $\Omega \cap (\mathbb{R}^n \times \{y\})$ is connected for every $y \in \mathbb{R}^{m-n}$ and, for every $p \in U$ and every $v \in T_pM$, we have*

$$v \in E_p \qquad \Longleftrightarrow \qquad d\phi(p)v \in \mathbb{R}^n \times \{0\}.$$

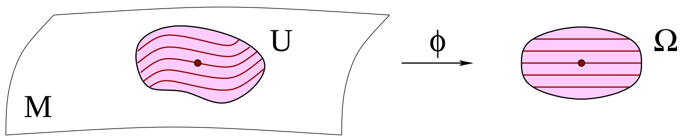


Figure 1.10: A foliation box.

Theorem 1.106 (Frobenius). *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $E \subset TM$ be a subbundle of rank n. Then the following are equivalent.*

(i) *E is involutive.*

(ii) *E is integrable.*

(iii) *For every $p_0 \in M$ there is a foliation box $\phi : U \to \Omega$ with $p_0 \in U$.*

It is easy to show that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) (see below). The hard part of the theorem is to prove that (i) $\Longrightarrow$ (iii). This requires the following lemma.

Lemma 1.107. *Let $E \subset TM$ be an involutive subbundle and $X \in \mathrm{Vect}(M)$ be a complete vector field such that $X(p) \in E_p$ for every $p \in M$. Denote by $\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t$ the flow of X. Then, for all $t \in \mathbb{R}$ and $p \in M$, we have*

$$d\phi^t(p)E_p = E_{\phi^t(p)}. \tag{1.43}$$

Lemma 1.107 implies Theorem 1.106. We prove that (iii) implies (ii). Let $p_0 \in M$, choose a foliation box $\phi : U \to \Omega$ for E with $p_0 \in U$, and define

$$N := (p \in U \mid \phi(p) \in \mathbb{R}^n \times \{y_0\}\}$$

where $(x_0, y_0) := \phi(p_0) \in \Omega$. Then N satisfies the requirements of (ii).

We prove that (ii) implies (i). Choose two vector fields $X, Y \in \mathrm{Vect}(M)$ that satisfy $X(p), Y(p) \in E_p$ for all $p \in M$ and fix a point $p_0 \in M$. By (ii) there is a submanifold $N \subset M$ containing p_0 such that $T_pN = E_p$ for every $p \in N$. Hence the restrictions $X|_N$ and $Y|_N$ are vector fields on N and so is the restriction of the Lie bracket $[X, Y]$ to N. Hence $[X, Y](p_0) \in T_{p_0}N = E_{p_0}$ as claimed.

We prove that (i) implies (iii). Thus we assume that E is an involutive subbundle of TM and fix a point $p_0 \in M$. By Theorem 1.100 there exist vector fields $X_1, \dots, X_n \in \mathrm{Vect}(M)$ such that $X_i(p) \in E_p$ for all i and p and the vectors $X_1(p_0), \dots, X_n(p_0)$ form a basis of $T_{p_0}E$. Using Theorem 1.100 again we find vector fields $Y_1, \dots, Y_{m-n} \in \mathrm{Vect}(M)$ such that the vectors

$$X_1(p_0), \dots, X_n(p_0), Y_1(p_0), \dots, Y_{m-n}(p_0)$$

form a basis of $T_{p_0}M$. Using cutoff functions as in the proof of Theorem 1.100 we may assume without loss of generality that the vector fields X_i and Y_j have compact support and hence are complete (see Exercise 1.58). Denote by $\phi_1^t, \dots, \phi_n^t$ the flows of the vector fields $X_1, \dots, X_n$, respectively, and by $\psi_1^t, \dots, \psi_{m-n}^t$ the flows of the vector fields $Y_1, \dots, Y_{m-n}$. Define the map $\psi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to M$ by

$$\psi(x, y) := \phi_1^{x_1} \circ \cdots \circ \phi_n^{x_n} \circ \psi_1^{y_1} \circ \cdots \circ \psi_{m-n}^{y_{m-n}}(p_0).$$

By Lemma 1.107, this map satisfies

$$\frac{\partial \psi}{\partial x_i}(x, y) \in E_{\psi(x,y)} \tag{1.44}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{m-n}$. Moreover,

$$\frac{\partial \psi}{\partial x_i}(0, 0) = X_i(p_0), \qquad \frac{\partial \psi}{\partial y_j}(0, 0) = Y_j(p_0).$$

Hence the derivative $d\psi(0, 0) : \mathbb{R}^n \times \mathbb{R}^{m-n} \to T_{p_0}M$ is bijective. By the inverse function theorem 1.34 it follows that there is an open neighborhood $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ of the origin such that the set $U := \psi(\Omega) \subset M$ is an M-open neighborhood of p_0 and $\psi|_\Omega : \Omega \to U$ is a diffeomorphism. Thus the vectors $\partial\psi/\partial x_i(x, y)$ are linearly independent for every $(x, y) \in \Omega$ and, by (1.44), form a basis of $E_{\psi(x,y)}$. Hence the inverse map $\phi := (\psi|_\Omega)^{-1} : U \to \Omega$ is a foliation box. This proves the theorem. □

To complete the proof of the Frobenius theorem it remains to prove Lemma 1.107. This requires the following result.

Lemma 1.108. *Let $E \subset TM$ be an involutive subbundle. If $\beta : \mathbb{R}^2 \to M$ is a smooth map such that*

$$\frac{\partial \beta}{\partial s}(s,0) \in E_{\beta(s,0)}, \qquad \frac{\partial \beta}{\partial t}(s,t) \in E_{\beta(s,t)}, \tag{1.45}$$

for all $s,t \in \mathbb{R}$ then

$$\frac{\partial \beta}{\partial s}(s,t) \in E_{\beta(s,t)}, \tag{1.46}$$

for all $s,t \in \mathbb{R}$.

Lemma 1.108 implies Lemma 1.107. Let $X \in \mathrm{Vect}(M)$ be a complete vector field satisfying $X(p) \in E_p$ for every $p \in M$ and let ϕ^t be the flow of X. Choose a point $p_0 \in M$ and a vector $v_0 \in E_{p_0}$. By Theorem 1.100 there is a vector field $Y \in \mathrm{Vect}(M)$ with values in E such that $Y(p_0) = v_0$. Moreover this vector field may be chosen to have compact support and hence it is complete (see Exercise 1.58). Thus there is a solution $\gamma : \mathbb{R} \to M$ of the initial value problem

$$\dot\gamma(s) = Y(\gamma(s)), \qquad \gamma(0) = p_0.$$

Define $\beta : \mathbb{R}^2 \to M$ by

$$\beta(s,t) := \phi^t(\gamma(s))$$

for $s,t \in \mathbb{R}$. Then

$$\frac{\partial \beta}{\partial s}(s,0) = \dot\gamma(s) = Y(\gamma(s)) \in E_{\beta(s,0)}$$

and

$$\frac{\partial \beta}{\partial t}(s,t) = X(\beta(s,t)) \in E_{\beta(s,t)}$$

for all $s,t \in \mathbb{R}$. Hence it follows from Lemma 1.108 that

$$d\phi^t(p_0)v_0 = \frac{\partial \beta}{\partial s}\phi^t(\gamma(0))\dot\gamma(0) = \frac{\partial \beta}{\partial s}(0,t) \in E_{\phi^t(p_0)}$$

for every $t \in \mathbb{R}$. This proves the lemma. □

Proof of Lemma 1.108. Given any point $p_0 \in M$ we choose a coordinate chart $\phi : U \to \Omega$, defined on an M-open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, such that

$$p_0 \in U, \qquad d\phi(p_0)E_{p_0} = \mathbb{R}^n \times \{0\}.$$

Shrinking U, if necessary, we obtain that $d\phi(p)E_p$ is the graph of a matrix $A \in \mathbb{R}^{(m-n)\times n}$ for every $p \in U$. Thus there is a map $A : \Omega \to \mathbb{R}^{(m-n)\times n}$ such that, for every $p \in U$, we have

$$d\phi(p)E_p = \{(\xi, A(x,y)\xi) \,|\, \xi \in \mathbb{R}^n\}, \qquad (x,y) := \phi(p) \in \Omega. \tag{1.47}$$

For $(x,y) \in \Omega$ we define the linear maps

$$\frac{\partial A}{\partial x}(x,y) : \mathbb{R}^n \to \mathbb{R}^{(m-n)\times n}, \qquad \frac{\partial A}{\partial y}(x,y) : \mathbb{R}^{m-n} \to \mathbb{R}^{(m-n)\times n}$$

by

$$\frac{\partial A}{\partial x}(x,y)\cdot\xi := \sum_{i=1}^n \xi_i \frac{\partial A}{\partial x_i}(x,y), \qquad \frac{\partial A}{\partial y}(x,y)\cdot\eta := \sum_{j=1}^{m-n} \eta_j \frac{\partial A}{\partial y_j}(x,y),$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $\eta = (\eta_1, \dots, \eta_{m-n}) \in \mathbb{R}^{m-n}$. We prove the following.

Claim 1. *Let $(x,y) \in \Omega$, $\xi, \xi' \in \mathbb{R}^n$ and define $\eta, \eta' \in \mathbb{R}^{m-n}$ by $\eta := A(x,y)\xi$ and $\eta' := A(x,y)\xi'$. Then*

$$\left(\frac{\partial A}{\partial x}(x,y)\cdot\xi + \frac{\partial A}{\partial y}(x,y)\cdot\eta\right)\xi' = \left(\frac{\partial A}{\partial x}(x,y)\cdot\xi' + \frac{\partial A}{\partial y}(x,y)\cdot\eta'\right)\xi.$$

The graphs of the matrices $A(z)$ determine a subbundle $\widetilde{E} \subset \Omega \times \mathbb{R}^m$ with fibers

$$\widetilde{E}_z := \left\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \,|\, \eta = A(x,y)\xi\right\}$$

for $z = (x,y) \in \Omega$. This subbundle is the image of

$$E|_U := \{(p,v) \,|\, p \in U,\, v \in E_p\}$$

under the diffeomorphism $TM|_U \to \Omega \times \mathbb{R}^m : (p,v) \mapsto (\phi(p), d\phi(p)v)$ and hence it is involutive. Now define the vector fields $\zeta, \zeta' : \Omega \to \mathbb{R}^m$ by

$$\zeta(z) := (\xi, A(z)\xi), \qquad \zeta'(z) := (\xi', A(z)\xi'), \qquad z \in \Omega.$$

Then ζ and ζ' are sections of $\widetilde{E}$ and their Lie bracket $[\zeta,\zeta']$ is given by

$$[\zeta,\zeta'](z) = \big(0, \big(dA(z)\zeta'(z)\big)\,\xi(z) - \left(dA(z)\zeta(z)\right)\xi'(z)\big).$$

Since $\widetilde{E}$ is involutive the Lie bracket $[\zeta,\zeta']$ must take values in the graph of A. Hence the right hand side vanishes and this proves Claim 1.

Claim 2. *Let $I, J \subset \mathbb{R}$ be open intervals and $z = (x, y) : I^2 \to \Omega$ be a smooth map. Fix two points $s_0 \in I$ and $t_0 \in J$ and assume that*

$$\frac{\partial y}{\partial s}(s_0, t_0) = A\big(x(s_0, t_0), y(s_0, t_0)\big)\frac{\partial x}{\partial s}(s_0, t_0), \tag{1.48}$$

$$\frac{\partial y}{\partial t}(s, t) = A\big(x(s, t), y(s, t)\big)\frac{\partial x}{\partial t}(s, t) \tag{1.49}$$

for all $s \in I$ and $t \in J$. Then

$$\frac{\partial y}{\partial s}(s_0, t) = A\big(x(s_0, t), y(s_0, t)\big)\frac{\partial x}{\partial s}(s_0, t) \tag{1.50}$$

for all $t \in J$.

Equation (1.50) holds by assumption for $t = t_0$. Moreover, dropping the argument $z(s_0, t) = z = (x, y)$ for notational convenience we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial s} - A\cdot\frac{\partial x}{\partial s}\right) &= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial s} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\cdot\left(A\frac{\partial x}{\partial t}\right)\right)\frac{\partial x}{\partial s} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial A}{\partial y}\cdot\left(A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} \\
&\quad + \left(\frac{\partial A}{\partial y}\cdot\left(\frac{\partial y}{\partial s} - A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t} \\
&= \left(\frac{\partial A}{\partial y}\cdot\left(\frac{\partial y}{\partial s} - A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t}
\end{aligned}$$

Here the second equation follows from (1.49), the third equation follows from Claim 1, and the last equation follows by differentiating equation (1.49) with respect to s. Define $\eta : J \to \mathbb{R}^{m-n}$ by

$$\eta(t) := \frac{\partial y}{\partial s}(s_0, t) - A\big(x(s_0, t), y(s_0, t)\big)\frac{\partial x}{\partial s}(s_0, t).$$

By (1.48) and what we have just proved, the function η satisfies the linear differential equation

$$\dot\eta(t) = \left(\frac{\partial A}{\partial y}\big(x(s_0, t), y(s_0, t)\big)\cdot\eta(t)\right)\frac{\partial x}{\partial t}(s_0, t), \qquad \eta(t_0) = 0.$$

Hence $\eta(t) = 0$ for all $t \in J$. This proves (1.50) and Claim 2.

Now let $\beta : \mathbb{R}^2 \to M$ be a smooth map satisfying (1.45) and fix a real number s_0. Consider the set $W := \left\{t \in \mathbb{R} \,|\, \frac{\partial\beta}{\partial s}(s_0,t) \in E_{\beta(s_0,t)}\right\}$. By going to local coordinates, we obtain from Claim 2 that W is open. Moreover, W is obviously closed, and $W \neq \emptyset$ because $0 \in W$ by (1.45). Hence $W = \mathbb{R}$. Since $s_0 \in \mathbb{R}$ was chosen arbitrarily, this proves (1.46) and the lemma. □

Any subbundle $E \subset TM$ determines an equivalence relation on M via

$$p_0 \sim p_1 \quad \iff \quad \begin{array}{l} \text{there is a smooth curve } \gamma : [0,1] \to M \text{ such that} \\ \gamma(0) = p_0,\ \gamma(1) = p_1,\ \dot\gamma(t) \in E_{\gamma(t)}\ \forall\, t \in [0,1]. \end{array} \tag{1.51}$$

If E is integrable this equivalence relation is called a **foliation** and the equivalence class of $p_0 \in M$ is called the **leaf** of the foliation through p_0. The next example shows that the leaves do not need to be submanifolds

Example 1.109. Consider the torus $M := S^1 \times S^1 \subset \mathbb{C}^2$ with the tangent bundle

$$TM = \left\{(z_1, z_2, \mathbf{i}\lambda_1 z_1, \mathbf{i}\lambda_2 z_2) \in \mathbb{C}^4 \,|\, |z_1| = |z_2| = 1,\ \lambda_1, \lambda_2 \in \mathbb{R}\right\}.$$

Let ω_1, ω_2 be real numbers and consider the subbundle

$$E := \left\{(z_1, z_2, \mathbf{i}t\omega_1 z_1, \mathbf{i}t\omega_2 z_2) \in \mathbb{C}^4 \,|\, |z_1| = |z_2| = 1,\ t \in \mathbb{R}\right\}.$$

The leaf of this subbundle through $z = (z_1, z_2) \in \mathbb{T}^2$ is given by

$$L = \left\{\left(e^{\mathbf{i}t\omega_1} z_1, e^{\mathbf{i}t\omega_2} z_2\right) \,\Big|\, t \in \mathbb{R}\right\}.$$

It is a submanifold if and only if the quotient ω_1/ω_2 is a rational number (or $\omega_2 = 0$). Otherwise each leaf is a dense subset of $\mathbb{T}^2$.

Exercise 1.110. Prove that (1.51) defines an equivalence relation for every subbundle $E \subset TM$.

Exercise 1.111. Each subbundle $E \subset TM$ of rank 1 is integrable.

Exercise 1.112. Consider the manifold $M = \mathbb{R}^3$. Prove that the subbundle $E \subset TM = \mathbb{R}^3 \times \mathbb{R}^3$ with fiber $E_p = \left\{(\xi, \eta, \zeta) \in \mathbb{R}^3 \,|\, \zeta - y\xi = 0\right\}$ over $p = (x, y, z) \in \mathbb{R}^3$ is not integrable and that any two points in $\mathbb{R}^3$ can be connected by a path tangent to E.

Exercise 1.113. Consider the manifold $M = S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ and define

$$E := \left\{(z, \zeta) \in \mathbb{C}^2 \times \mathbb{C}^2 \,|\, |z| = 1,\ \zeta \perp z,\ \mathbf{i}\zeta \perp z\right\}.$$

Thus the fiber $E_z \subset T_zS^3 = z^\perp$ is the maximal complex linear subspace of T_zS^3. Prove that E has real rank 2 and is not integrable.

Exercise 1.114. Let $E \subset TM$ be an integrable subbundle of rank n and let $L \subset M$ be a leaf of the foliation determined by E. Call a subset $V \subset L$ **L-open** if it can be written as a union of submanifolds N of M with tangent spaces $T_pN = E_p$ for $p \in N$. Prove that the L-open sets form a topology on L (called the **intrinsic topology**) Prove that the obvious inclusion $\iota : L \to M$ is continuous with respect to the intrinsic topology on L. Prove that the inclusion $\iota : L \to M$ is proper if and only if the intrinsic topology on L agrees with the relative topology inherited from M (called the **extrinsic topology**).

Remark 1.115. It is surprisingly difficult to prove that each closed leaf L of a foliation is a submanifold of M. A proof due to David Epstein [4] is sketched in Subsection 1.9.7 below.

1.9 The intrinsic definition of a manifold

It is somewhat restrictive to only consider manifolds that are embedded in some Euclidean space. Although we shall see that (at least) every compact manifold admits an embedding into a Euclidean space, such an embedding is in many cases not a natural part of the structure of a manifold. In particular, we encounter manifolds that are described as quotient spaces and there are manifolds that are embedded in certain infinite dimensional Hilbert spaces. For this reason it is convenient, at this point, to introduce a more general *intrinisc* definition of a manifold. This requires some background from point set topology that is not covered in the first year analysis courses. We shall then see that all the definitions and results of this first chapter carry over in a natural manner to the intrinsic setting.

1.9.1 Definition and examples

Definition 1.116. *Let m be a nonnegative integer. A* **smooth m-manifold** *is a topological space M equipped with an open cover $\{U_\alpha\}_\alpha$ and a collection of homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ (one for each open set in the cover) onto open subsets $\phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ such that the transition maps*

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.52}$$

are diffeomorphisms for all α, β (see Figure 1.11). The maps $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ are called **coordinate charts** *of M. The collection $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_\alpha$ is called an* **atlas** *on M. It is also sometimes called a* **smooth structure***.*

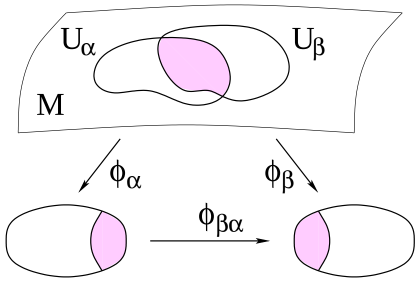


Figure 1.11: Coordinate charts and transition maps.

Remark 1.117. The coordinate charts ϕ_α determine the topology on M. Namely, for every subset $U \subset M$, we have

$$U \text{ is open} \quad \iff \quad \phi_\alpha(U \cap U_\alpha) \text{ is open for all } \alpha. \tag{1.53}$$

This observation gives rise to a slightly different, but equivalent, formulation of Definition 1.116. A manifold can be defined as a pair $(M, \mathscr{A})$ where M is a set and $\mathscr{A}$ is a collection of bijective maps $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ onto open subsets $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^m$, the sets U_α cover M, and the transition maps (1.52) are diffeomorphisms between open subsets of $\mathbb{R}^m$. One can then define a topology $\mathscr{U} \subset 2^M$ (the collection of open sets) by (1.53). It is easy to verify that (1.53) indeed defines a topology on M and that the coordinate charts ϕ_α are homeomorphisms with respect to this topology. (Prove this!)

The atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_\alpha$ is an intrinsic part of the structure of a manifold. This allows for a great deal of freedom as we may add more coordinate charts without changing anything else about the manifold M. Thus we shall identify two manifolds $(M, \mathscr{A})$ and $(M, \mathscr{A}')$ if each coordinate chart of $\mathscr{A}$ is **compatible** with each coordinate chart of $\mathscr{A}'$ in the sense that the transition maps (1.52) are all diffeomorphisms. In this case the union $\mathscr{A} \cup \mathscr{A}'$ is also an atlas on M. Thus we can turn an atlas $\mathscr{A}$ into a **maximal atlas** $\mathscr{A}_{\max}$ by adding to $\mathscr{A}$ each homeomorphism $\phi : U \to \Omega$ from an open subset $U \subset M$ onto an open set $\Omega \subset \mathbb{R}^m$ such that the map $\phi \circ \phi_\alpha^{-1} : \phi_\alpha(U \cap U_\alpha) \to \phi(U \cap \phi_\alpha)$ is a diffeomorphism for every coordinate chart ϕ_α of $\mathscr{A}$. The resulting atlas is maximal in the sense that each atlas $\mathscr{A}'$ on M whose coordinate charts are compatible with the coordinate charts in $\mathscr{A}_{\max}$ is already contained in $\mathscr{A}_{\max}$. Thus we can also define a manifold as a set M equipped with a maximal atlas. Although the coordinate charts are part of the structure of a manifold we will often not mention them explicitly and just say *"let M be a manifold"*, assuming the atlas as given.

Exercise 1.118. Prove that for every atlas $\mathscr{A}$ there is a unique maximal atlas $\mathscr{A}_{\max}$ containing $\mathscr{A}$ and that $\mathscr{A}_{\max}$ induces the same topology as $\mathscr{A}$.

Remark 1.119. One can consider manifolds where the transition maps $\phi_{\beta\alpha}$ in (1.52) satisfy additional conditions. For example we can require the transition maps to be real analytic or even polynomials, leading to the subject of **real algebraic geometry**. Or we can consider the case where $m = 2n$ is even, identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and require the transition maps to be holomorphic, leading to the subject of **complex geometry**. Still in the case $m = 2n$ we could require the transition maps to be *canonical transformations* in the sense of classical mechanics and this is the subject of **symplectic geometry**. One can also weaken the requirements on the transition maps and only ask that they are of class C^k. In the case $k = 0$ there can be dramatic differences. For example, there are 4-dimensional C^0-manifolds that do not admit any smooth structure.

Example 1.120. The **complex projective space** $\mathbb{C}\mathrm{P}^n$ is the set

$$\mathbb{C}\mathrm{P}^n = \left\{\ell \subset \mathbb{C}^{n+1} \,|\, \ell \text{ is a 1-dimensional complex subspace}\right\}$$

of complex lines in $\mathbb{C}^{n+1}$. It can be identified with the quotient space

$$\mathbb{C}\mathrm{P}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*$$

of nonzero vectors in $\mathbb{C}^{n+1}$ modulo the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers. The equivalence class of a nonzero vector $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ will be denoted by

$$[z] = [z_0 : z_1 : \cdots : z_n] := \{\lambda z \,|\, \lambda \in \mathbb{C}^*\}$$

and the associated line is $\ell = \mathbb{C}z$. This space is equipped with the quotient topology. Namely, if $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ denotes the obvious projection, a subset $U \subset \mathbb{C}\mathrm{P}^n$ is open by definition if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$. The atlas on $\mathbb{C}\mathrm{P}^n$ is given by the open cover

$$U_i := \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\}$$

and the coordinate charts $\phi_i : U_i \to \mathbb{C}^n$ are

$$\phi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \tag{1.54}$$

for $i = 0, 1, \ldots, n$. **Exercise:** Prove that each ϕ_i is a homeomorphism and the transition maps are holomorphic.

Example 1.121. The **real projective space** $\mathbb{R}\mathrm{P}^n$ is the set

$$\mathbb{R}\mathrm{P}^n = \left\{\ell \subset \mathbb{R}^{n+1} \,|\, \ell \text{ is a 1-dimensional linear subspace}\right\}$$

of real lines in $\mathbb{R}^{n+1}$. It can again be identified with the quotient space

$$\mathbb{R}\mathrm{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \mathbb{R}^*$$

of nonzero vectors in $\mathbb{R}^{n+1}$ modulo the action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers, and the equivalence class of a nonzero vector $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ will be denoted by

$$[x] = [x_0 : x_1 : \dots : x_n] := \{\lambda x \,|\, \lambda \in \mathbb{R}^*\}.$$

As before this space is equipped with the quotient topology. Namely, if $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n$ denotes the obvious projection, a subset $U \subset \mathbb{R}\mathrm{P}^n$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{R}^{n+1} \setminus \{0\}$. An atlas on $\mathbb{R}\mathrm{P}^n$ is given by the open cover

$$U_i := \{[x_0 : \dots : x_n] \,|\, x_i \neq 0\}$$

and the coordinate charts $\phi_i : U_i \to \mathbb{R}^n$ are again given by (1.54), with z_j replaced by x_j. **Exercise:** Prove that each ϕ_i is a homeomorphism and that the transition maps are real analytic.

Example 1.122. The **real n-torus** is the topological space

$$\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$$

equipped with the quotient topology. Thus two vectors $x, y \in \mathbb{R}^n$ are equivalent if their difference $x - y \in \mathbb{Z}^n$ is an integer vector and we denote by $\pi : \mathbb{R}^n \to \mathbb{T}^n$ the obvious projection which assigns to each vector $x \in \mathbb{R}^n$ its equivalence class

$$\pi(x) := [x] := x + \mathbb{Z}^n.$$

Then a set $U \subset \mathbb{T}^n$ is open if and only if the set $\pi^{-1}(U)$ is an open subset of $\mathbb{R}^n$. An atlas on $\mathbb{T}^n$ is given by the open cover

$$U_\alpha := \{[x] \,|\, x \in \mathbb{R}^n, \, |x - \alpha| < 1/2\},$$

parametrized by vectors $\alpha \in \mathbb{R}^n$, and the coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ defined by $\phi_\alpha([x]) := x$ for $x \in \mathbb{R}^n$ with $|x - \alpha| < 1/2$. **Exercise:** Show that each transition map for this atlas is a translation by an integer vector.

Example 1.123. Consider the **complex Grassmannian**

$$\mathrm{G}_k(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \,|\, v \text{ is a } k-\text{dimensional complex linear subspace}\}.$$

This set can again be described as a quotient space $\mathrm{G}_k(\mathbb{C}^n) \cong \mathcal{F}_k(\mathbb{C}^n)/\mathrm{U}(k)$. Here

$$\mathcal{F}_k(\mathbb{C}^n) := \left\{D \in \mathbb{C}^{n\times k} \,|\, D^*D = \mathbb{1}\right\}$$

denotes the set of unitary k-frames in $\mathbb{C}^n$ and the group $\mathrm{U}(k)$ acts on $\mathcal{F}_k(\mathbb{C}^n)$ contravariantly by $D \mapsto Dg$ for $g \in \mathrm{U}(k)$. The projection

$$\pi : \mathcal{F}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$$

sends a matrix $D \in \mathcal{F}_k(\mathbb{C}^n)$ to its image $V := \pi(D) := \mathrm{im}\, D$. A subset $U \subset \mathrm{G}_k(\mathbb{C}^n)$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathcal{F}_k(\mathbb{C}^n)$. Given a k-dimensional subspace $V \subset \mathbb{C}^n$ we can define an open set $U_V \subset \mathrm{G}_k(\mathbb{C}^n)$ as the set of all k-dimensional subspaces of $\mathbb{C}^n$ that can be represented as graphs of linear maps from V to $V^\perp$. This set of graphs can be identified with the complex vector space $\mathrm{Hom}^{\mathbb{C}}(V, V^\perp)$ of complex linear maps from V to $V^\perp$ and hence with $\mathbb{C}^{(n-k)\times k}$. This leads to an atlas on $\mathrm{G}_k(\mathbb{C}^n)$ with holomorphic transition maps and shows that $\mathrm{G}_k(\mathbb{C}^n)$ is a manifold of complex dimension $kn - k^2$. **Exercise:** Verify the details of this construction. Find explicit formulas for the coordinate charts and their transition maps. Carry this over to the real setting. Show that $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$ are special cases.

Example 1.124 (The real line with two zeros). A topological space M is called **Hausdorff** if any two points in M can be separated by disjoint open neighborhoods. This example shows that a manifold need not be a Hausdorff space. Consider the quotient space

$$M := \mathbb{R} \times \{0, 1\}/\equiv$$

where $[x, 0] \equiv [x, 1]$ for $x \neq 0$. An atlas on M consists of two coordinate charts $\phi_0 : U_0 \to \mathbb{R}$ and $\phi_1 : U_1 \to \mathbb{R}$ where $U_i := \{[x, i] \,|\, x \in \mathbb{R}\}$ and $\phi_i([x, i]) := x$ for $i = 0, 1$. Thus M is a 1-manifold. But the topology on M is not Hausdorff, because the points $[0, 0]$ and $[0, 1]$ cannot be separated by disjoint open neighborhoods.

Example 1.125 (A 2-manifold without a countable atlas). Consider the vector space $X = \mathbb{R} \times \mathbb{R}^2$ with the equivalence relation

$$[t_1, x_1, y_2] \equiv [t_2, x_2, y_2] \iff \begin{array}{l} \text{either } y_1 = y_2 \neq 0,\ t_1 + x_1y_1 = t_2 + x_2y_2 \\ \text{or } y_1 = y_2 = 0,\ t_1 = t_2,\ x_1 = x_2. \end{array}$$

For $y \neq 0$ we have $[0, x, y] \equiv [t, x - t/y, y]$, however, each point $(x, 0)$ on the x-axis gets replaced by the uncountable set $\mathbb{R} \times \{(x, 0)\}$. Our manifold is the quotient space $M := X/\equiv$. This time we do not use the quotient topology but the topology induced by our atlas via (1.53). The coordinate charts are parametrized by the reals: for $t \in \mathbb{R}$ the set $U_t \subset M$ and the coordinate chart $\phi_t : U_t \to \mathbb{R}^2$ are given by

$$U_t := \{[t, x, y] \,|\, x, y \in \mathbb{R}\}, \qquad \phi_t([t, x, y]) := (x, y).$$

A subset $U \subset M$ is open, by definition, if $\phi_t(U \cap U_t)$ is an open subset of $\mathbb{R}^2$ for every $t \in \mathbb{R}$. With this topology each ϕ_t is a homeomorphism from U_t onto $\mathbb{R}^2$ and M admits a countable dense subset $S := \{[0, x, y] \,|\, x, y \in \mathbb{Q}\}$. However, there is no atlas on M consisting of countably many charts. (Each coordinate chart can contain at most countably many of the points $[t, 0, 0]$.) The function $f : M \to \mathbb{R}$ given by $f([t, x, y]) := t + xy$ is smooth and each point $[t, 0, 0]$ is a critical point of f with value t. Thus f has no regular value. **Exercise:** Show that M is a path-connected Hausdorff space.

1.9.2 Paracompactness

The existence of a countable atlas is of fundamental importance for almost everything we will prove about manifolds. The next two remarks describe several equivalent conditions.

Remark 1.126. Let M be a smooth manifold and denote by $\mathscr{U} \subset 2^M$ the topology induced by the atlas via (1.53). Then the following are equivalent.

(a) M admits a countable atlas.

(b) M is **σ-compact**, i.e. there is a sequence of compact subsets $K_i \subset M$ such that $K_i \subset \mathrm{int}(K_{i+1})$ for every $i \in \mathbb{N}$ and $M = \bigcup_{i=1}^{\infty} K_i$.

(c) Every open cover of M has a countable subcover.

(d) M is **second countable**, i.e. there is a countable collection of open sets $\mathscr{B} \subset \mathscr{U}$ such that every open set $U \in \mathscr{U}$ is a union of open sets from the collection $\mathscr{B}$. ($\mathscr{B}$ is then called a **countable base** for the topology of M.)

That (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a) and (a) $\Longrightarrow$ (d) follows directly from the definitions. The proof that (d) implies (a) requires the construction of a countable refinement and the axiom of choice. (A **refinement** of an open cover $\{U_i\}_{i \in I}$ is an open cover $\{V_j\}_{j \in J}$ such that each set V_j is contained in one of the sets U_i.)

Remark 1.127. Let M and $\mathscr{U}$ be as in Remark 1.126 and suppose that M is a connected Hausdorff space. Then the existence of a countable atlas is also equivalent to each of the following conditions.

(e) M is **metrizable**, i.e. there is a distance function $d : M \times M \to [0, \infty)$ such that $\mathscr{U}$ is the topology induced by d.

(f) M is **paracompact**, i.e. every open cover of M has a locally finite refinement. (An open cover $\{V_j\}_{j \in J}$ is called **locally finite** if every $p \in M$ has a neighborhood that intersects only finitely many V_j.)

That (a) implies (e) follows from the **Urysohn metrization theorem** which asserts (in its original form) that every normal second countable topological space is metrizable [9, Theorem 34.1]. A topological space M is called **normal** if points are closed and, for any two disjoint closed sets $A, B \subset M$, there are disjoint open sets $U, V \subset M$ such that $A \subset U$ and $B \subset V$. It is called **regular** if points are closed and, for every closed set $A \subset M$ and every $b \in M \setminus A$, there are disjoint open sets $U, V \subset M$ such that $A \subset U$ and $b \in V$. It is called **locally compact** if, for every open set $U \subset M$ and every $p \in U$, there is a compact neighborhood of p contained in U. It is easy to show that every manifold is locally compact and every locally compact Hausdorff space is regular. **Tychonoff's Lemma** asserts that a regular topological space with a countable base is normal [9, Theorem 32.1]. Hence it follows from the Urysohn metrization theorem that every Hausdorff manifold with a countable base is metrizable. That (e) implies (f) follows from a more general theorem which asserts that every metric space is paracompact (see [9, Theorem 41.4] and [10]). Conversely, the **Smirnov metrization theorem** asserts that a paracompact Hausdorff space is metrizable if and only it is locally metrizable, i.e. every point has a metrizable neighborhood (see [9, Theorem 42.1]). Since every manifold is locally metrizable this shows that (f) implies (e). Thus we have (a) $\implies$ (e) $\iff$ (f) for every Hausdorff manifold.

The proof that (f) implies (a) does not require the Hausdorff property but we do need the assumption that M is connected. (A manifold with uncountably many connected components, each of which is paracompact, is itself paracompact but does not admit a countable atlas.) Here is a sketch. If M is a paracompact manifold then there is a locally finite open cover $\{U_\alpha\}_{\alpha \in A}$ by coordinate charts. Since each set U_α has a countable dense subset, the set $\{\alpha \in A \,|\, U_\alpha \cap U_{\alpha_0} \neq \emptyset\}$ is at most countable for each $\alpha_0 \in A$. Since M is connected we can reach each point from U_{α_0} through a finite sequence of sets $U_{\alpha_1}, \dots, U_{\alpha_\ell}$ with $U_{\alpha_{i-1}} \cap U_{\alpha_i} \neq \emptyset$. This implies that the index set A is countable and hence M admits a countable atlas.

Remark 1.128. A **Riemann surface** is a 1-dimensional complex manifold (i.e. the coordinate charts take values in the complex plane $\mathbb{C}$ and the transition maps are holomorphic) with a Hausdorff topology. It is a deep theorem in the theory of Riemann surfaces that every connected Riemann surface is necessarily second countable (see [2]). Thus pathological examples of the type discussed in Example 1.125 cannot be constructed with holomorphic transition maps.

Exercise 1.129. Prove that every manifold is locally compact. Find an example of a manifold M and a point $p_0 \in M$ such that every closed neighborhood of p_0 is non-compact. **Hint:** The example is necessarily non-Hausdorff.

Exercise 1.130. Prove that a manifold M admits a countable atlas if and only if it is σ-compact if and only if every open cover of M has a countable subcover if and only if it is second countable. **Hint:** Every open set in $\mathbb{R}^m$ has a countable base and is σ-compact.

Exercise 1.131. Prove that every manifold $M \subset \mathbb{R}^k$ (as in Definition 1.3) is second countable.

Exercise 1.132. Prove that every connected component of a manifold M is an open subset of M and is path-connected.

Our next goal is to carry over all the definitions from embedded manifolds in Euclidean space to the intrinsic setting.

1.9.3 Smooth maps and diffeomorphisms

Definition 1.133. *Let $(M, \{U_\alpha, \phi_\alpha\}_{\alpha \in A})$ and $(N, \{V_\beta, \psi_\beta\}_{\beta \in B})$ be smooth manifolds. A map $f : M \to N$ is called* **smooth** *if it is continuous and the map*

$$f_{\beta\alpha} := \psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta) \tag{1.55}$$

is smooth for every $\alpha \in A$ and every $\beta \in B$. It is called a **diffeomorphism** *if it is bijective and f and f^{-1} are smooth. The manifolds M and N are called* **diffeomorphic** *if there exists a diffeomorphism $f : M \to N$.*

The reader may check that the notion of a smooth map is independent of the atlas used in the definition, that compositions of smooth maps are smooth, and that sums and products of smooth maps from M to $\mathbb{R}$ are smooth.

Exercise 1.134. Let M be a smooth m-dimensional manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. Consider the quotient space

$$\widetilde{M} := \bigcup_{\alpha \in A} \{\alpha\} \times \phi_\alpha(U_\alpha) / \sim$$

where $(\alpha, x) \sim (\beta, y)$ iff $\phi_\alpha^{-1}(x) = \phi_\beta^{-1}(y)$. Define an atlas on $\widetilde{M}$ by

$$\widetilde{U}_\alpha := \{[\alpha, x] \,|\, x \in \phi_\alpha(U_\alpha)\}, \qquad \widetilde{\phi}_\alpha([\alpha, x]) := x.$$

Prove that $\widetilde{M}$ is a smooth m-manifold and that it is diffeomorphic to M.

Exercise 1.135. Prove that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to S^2. **Hint:** Stereographic projection.

Remark 1.136. Sometimes one encounters a situation where a topological space M admits in a natural way two atlases $\mathscr{A}$ and $\mathscr{A}'$ such that the coordinate charts of $\mathscr{A}$ are not compatible with the coordinate charts of $\mathscr{A}'$. However, this does not necessarily mean that $(M, \mathscr{A})$ and $(M, \mathscr{A}')$ are not diffeomorphic; it only means that the identity map on M is not a diffeomorphism between these different smooth structures. For example on $M = \mathbb{R}$ the coordinate charts $\phi, \psi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) := x$ and $\psi(x) := x^3$ define different smooth structures but the resulting manifolds are diffeomorphic.

A fundamental question in differential topology is to decide if two given smooth manifolds are diffeomorphic. For example, one can ask if a given manifold that is homeomorphic to $\mathbb{R}^m$ is in fact diffeomorphic to $\mathbb{R}^m$. A surprising and deep fact in dimension $m = 4$ is that there are uncountably many smooth manifolds that are all homeomorphic to $\mathbb{R}^4$ but no two of them are diffeomorphic to each other.

1.9.4 Submanifolds

Definition 1.137. *Let M be an m-manifold and $n \in \{0, 1, \ldots, m\}$. A subset $N \subset M$ is called an n-dimensional* **submanifold** *of M if, for every $p \in N$, there is a local coordinate chart $\phi : U \to \Omega$ for M, defined on an an open neighborhood $U \subset M$ of p and with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, such that $\phi(U \cap N) = \Omega \cap (\mathbb{R}^n \times \{0\})$.*

By Theorem 1.10 an m-manifold $M \subset \mathbb{R}^k$ in the sense of Definition 1.3 is a submanifold of $\mathbb{R}^k$ in the sense of Definition 1.137. By Theorem 1.40 the notion of a submanifold $N \subset M$ of a manifold $M \subset \mathbb{R}^k$ in Definition 1.37 agrees with the notion of a submanifold in Definition 1.137.

Exercise 1.138. Let N be a submanifold of M. Show that if M is Hausdorff so is N, and if M is paracompact so is N.

Exercise 1.139. Let N be a submanifold of M. Prove that there is an open set $U \subset M$ such that N is closed in the relative topology of U.

Exercise 1.140. Let N be a submanifold of M and P be a submanifold of N. Prove that P is a submanifold of M. **Hint:** Use Theorem 1.10.

1.9.5 Tangent spaces and derivatives

If M is a submanifold of Euclidean space and $p \in M$ we have defined the tangent space of M at p as the set of all derivatives $\dot\gamma(0)$ of smooth curves $\gamma : \mathbb{R} \to M$ that pass through $p = \gamma(0)$. We cannot do this for manifolds in the intrinsic sense, as the derivative of a curve has yet to be defined. In fact, the purpose of introducing a tangent space of M is precisely to allow us to define what we mean by the derivative of a smooth map. There are two approaches. One is to introduce an appropriate equivalence relation on the set of curves through p and the other is to use local coordinates.

Definition 1.141. *Let M be a manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ and let $p \in M$. Two smooth curves $\gamma_0, \gamma_1 : \mathbb{R} \to M$ with $\gamma_0(0) = \gamma_1(0) = p$ are called* p**-equivalent** *if for some (and hence every) $\alpha \in A$ with $p \in U_\alpha$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma_0(t)) = \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma_1(t)).$$

We write $\gamma_0 \stackrel{p}{\sim} \gamma_1$ if γ_0 is p-equivalent to γ_1 and denote the equivalence class of a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ by $[\gamma]_p$. The **tangent space** *of M at p is defined as the set of equivalence classes*

$$T_pM := \{[\gamma]_p \,|\, \gamma : \mathbb{R} \to M \text{ is smooth and } \gamma(0) = p\}. \tag{1.56}$$

Definition 1.142. *Let M be a manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ and let $p \in M$. The* **tangent space** *of M at p is defined as the quotient space*

$$T_pM := \bigcup_{p\in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \stackrel{p}{\sim} \tag{1.57}$$

where the union runs over all $\alpha \in A$ with $p \in U_\alpha$ and

$$(\alpha, \xi) \stackrel{p}{\sim} (\beta, \eta) \qquad \Longleftrightarrow \qquad d\left(\phi_\beta \circ \phi_\alpha^{-1}\right)(x)\xi = \eta, \quad x := \phi_\alpha(p).$$

The equivalence class will be denoted by $[\alpha, \xi]_p$.

In Definition 1.141 it is not immediately obvious that T_pM is a vector space. However, the quotient space (1.57) is obviously a vector space of dimension m and there is a natural bijection given by

$$[\gamma]_p \mapsto \left[\alpha, \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma(t))\right]_p \tag{1.58}$$

for a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$. Hence the set (1.56) is a vector space as well. The reader may check that the map (1.58) is well defined and is indeed a bijection between the quotient spaces (1.56) and (1.57). Moreover, for each smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ we can now define the derivative $\dot\gamma(0) \in T_pM$ simply as the equivalence class

$$\dot\gamma(0) := [\gamma]_p \cong \left[\alpha, \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma(t))\right]_p \in T_pM.$$

If $f : M \to N$ is a smooth map between two manifolds $(M, \{U_\alpha, \phi_\alpha\}_{\alpha\in A})$ and $(N, \{V_\beta, \psi_\beta\}_{\beta\in B})$ we define the derivative

$$df(p) : T_pM \to T_{f(p)}N$$

by the formula

$$df(p)[\gamma]_p := [f \circ \gamma]_{f(p)} \tag{1.59}$$

for each smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$. Here we use (1.56). Under the isomorphism (1.58) this corresponds to the linear map

$$df(p)[\alpha, \xi]_p := [\beta, df_{\beta\alpha}(x)\xi]_{f(p)}, \qquad x := \phi_\alpha(p), \tag{1.60}$$

for $\alpha \in A$ with $p \in U_\alpha$ and $\beta \in B$ with $f(p) \in V_\beta$, where $f_{\beta\alpha}$ is given by (1.55).

Remark 1.143. Think of $N = \mathbb{R}^n$ as a manifold with a single coordinate chart, namely the identity map $\psi_\beta = \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. For every $q \in N = \mathbb{R}^n$ the tangent space T_qN is then canonically isomorphic to $\mathbb{R}^n$ via (1.57). Thus for every smooth map $f : M \to \mathbb{R}^n$ the derivative of f at $p \in M$ is a linear map $df(p) : T_pM \to \mathbb{R}^n$, and the formula (1.60) reads

$$df(p)[\alpha, \xi]_p = d(f \circ \phi_\alpha^{-1})(x)\xi, \qquad x := \phi_\alpha(p).$$

This formula also applies to maps defined on some open subset of M. In-particular, with $f = \phi_\alpha : U_\alpha \to \mathbb{R}^m$ we have

$$d\phi_\alpha(p)[\alpha, \xi]_p = \xi.$$

Thus the map $d\phi_\alpha(p) : T_pM \to \mathbb{R}^m$ is the canonical vector space isomorphism determined by α.

With these definitions the derivative of f at p is a linear map and we have the chain rule for the composition of two smooth maps as in Theorem 1.32. In fact, all the theorems we have proved for embedded manifolds and their proofs carry over almost word for word to the present setting. For example we have the inverse function theorem, the notion of a regular value, the implicit function theorem, the notion of an immersion, the notion of an embedding, and the fact from Theorem 1.40 that a subset $P \subset M$ is a submanifold if and only if it is the image of an embedding.

Example 1.144 (Veronese embedding). The map

$$\mathbb{C}P^2 \to \mathbb{C}P^5 : [z_0 : z_1 : z_2] \mapsto [z_0^2 : z_1^2 : z_2^2 : z_1 z_2 : z_2 z_0 : z_0 z_1]$$

is an embedding. (**Exercise:** Prove this.) It restricts to an embedding of the real projective plane into $\mathbb{R}P^5$ and also gives rise to embeddings of $\mathbb{R}P^2$ into $\mathbb{R}^4$ as well as to the Roman surface: an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$. (See Example 1.16.) There are similar embeddings

$$\mathbb{C}P^n \to \mathbb{C}\mathrm{P}^{N-1}, \qquad N := \binom{n+d}{d},$$

for all n and d, defined in terms of monomials of degree d in $n+1$ variables. These are the **Veronese embeddings**.

Example 1.145 (Plücker embedding). The Grassmannian $\mathrm{G}_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$ is a smooth 4-manifold and can be expressed as the quotient of the space $\mathcal{F}_2(\mathbb{R}^4)$ of orthonormal 2-frames in $\mathbb{R}^4$ by the orthogonal group $\mathrm{O}(2)$. (See Example 1.123.) Write an orthonormal 2-frame in $\mathbb{R}^4$ as a matrix

$$D = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}, \qquad D^T D = \mathbb{1}.$$

Then the map $f : \mathrm{G}_2(\mathbb{R}^4) \to \mathbb{R}P^5$, defined by

$$f([D]) := [p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}], \qquad p_{ij} := x_i y_j - x_j y_i.,$$

is an embedding and its image is the quadric

$$X := f(\mathrm{G}_2(\mathbb{R}^4)) = \left\{ p \in \mathbb{R}P^5 \,|\, p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \right\}.$$

(**Exercise:** Prove this.) There are analogous embeddings

$$f : \mathrm{G}_k(\mathbb{R}^n) \to \mathbb{R}P^{N-1}, \qquad N := \binom{n}{k},$$

for all k and n, defined in terms of the $k \times k$-minors of the (orthonormal) frames. These are the **Plücker embeddings**.

1.9.6 The tangent bundle and vector fields

Let M be a m-manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. The **tangent bundle** of M is defined as the disjoint union of the tangent spaces, i.e.

$$TM := \bigcup_{p \in M} \{p\} \times T_pM = \{(p, v) \,|\, p \in M,\ v \in T_pM\}.$$

Denote by $\pi : TM \to M$ the projection given by $\pi(p, v) := p$.

Lemma 1.146. *The tangent bundle of M is a smooth $2m$-manifold with coordinate charts*

$$\widetilde{\phi}_\alpha : \widetilde{U}_\alpha := \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m, \qquad \widetilde{\phi}_\alpha(p, v) := (\phi_\alpha(p), d\phi_\alpha(p)v).$$

The projection $\pi : TM \to M$ is a submersion (a smooth map with surjective derivative at each point). If M is second countable and Hausdorff so is TM.

Proof. For each pair $\alpha, \beta \in A$ the set $\widetilde{\phi}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) = \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m$ is open in $\mathbb{R}^m \times \mathbb{R}^m$ and the transition map

$$\widetilde{\phi}_{\beta\alpha} := \widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1} : \widetilde{\phi}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) \to \widetilde{\phi}_\beta(\widetilde{U}_\alpha \cap \widetilde{U}_\beta)$$

is given by

$$\widetilde{\phi}_{\beta\alpha}(x, \xi) = (\phi_{\beta\alpha}(x), d\phi_{\beta\alpha}(x)\xi)$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$ and $\xi \in \mathbb{R}^m$ where $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}$. Thus the transition maps are all diffeomorphisms and so the coordinate charts $\widetilde{\phi}_\alpha$ define an atlas on TM. The topology on TM is determined by this atlas via (1.53). If M has a countable atlas so does TM. The remaining assertions are easy exercises. □

Definition 1.147. *Let M be a smooth m-manifold. A* **(smooth) vector field** *on M is a collection of tangent vectors $X(p) \in T_pM$, one for each point $p \in M$, such that the map $M \to TM : p \mapsto (p, X(p))$ is smooth. The set of smooth vector fields on M will be denoted by* $\mathrm{Vect}(M)$.

Associated to a vector field is a smooth map $M \to TM$ whose composition with the projection $\pi : TM \to M$ is the identity map on M. Strictly speaking this map should be denoted by a symbol other than X, for example by $\widetilde{X}$. However, it is convenient at this point, and common practice, to slightly abuse notation and denote the map from M to TM also by X. Thus a vector field can be defined as a smooth map $X : M \to TM$ such that

$$\pi \circ X = \mathrm{id} : M \to M.$$

Such a map is also called a *section of the tangent bundle.*

Now suppose $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ is an atlas on M and $X : M \to TM$ is a vector field on M. Then X determines a collection of smooth maps $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ given by

$$X_\alpha(x) := d\phi_\alpha(p)X(p), \qquad p := \phi_\alpha^{-1}(x), \tag{1.61}$$

for $x \in \phi_\alpha(U_\alpha)$. We can think of each X_α as a vector field on the open set $\phi_\alpha(U_\alpha) \subset \mathbb{R}^m$, representing the vector field X on the coordinate patch U_α. These local vector fields X_α satisfy the condition

$$X_\beta(\phi_{\beta\alpha}(x)) = d\phi_{\beta\alpha}(x)X_\alpha(x) \tag{1.62}$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. This equation can also be expressed in the form

$$X_\alpha|_{\phi_\alpha(U_\alpha\cap U_\beta)} = \phi_{\beta\alpha}^* X_\beta|_{\phi_\beta(U_\alpha\cap U_\beta)}. \tag{1.63}$$

Conversely, any collection of smooth maps $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ satisfying (1.62) determines a unique vectorfield X on M via (1.61). Thus we can define the Lie bracket of two vector fields $X, Y \in \mathrm{Vect}(M)$ by

$$[X,Y]_\alpha(x) := [X_\alpha, Y_\alpha](x) = dX_\alpha(x)Y_\alpha(x) - dY_\alpha(x)X_\alpha(x) \tag{1.64}$$

for $\alpha \in A$ and $x \in \phi_\alpha(U_\alpha)$. It follows from equation (1.20) in Lemma 1.64 that the local vector fields $[X,Y]_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ satisfy (1.63) and hence determine a unique vector field $[X,Y]$ on M via

$$[X,Y](p) := d\phi_\alpha(p)^{-1}[X_\alpha, Y_\alpha](\phi_\alpha(p)), \qquad p \in U_\alpha. \tag{1.65}$$

Thus the **Lie bracket** of X and Y is defined on U_α as the pullback of the Lie bracket of the vector fields X_α and Y_α under the coordinate chart ϕ_α. With this understood all the results in Section 1.4 about vector fields and flows along with their proofs carry over word for word to the intrinsic setting whenever M is a Hausdorff space. This includes the existence and uniquess result for integral curves in Theorem 1.52, the concept of the flow of a vector field in Definition 1.53 and its properties in Theorem 1.54, the notion of completeness of a vector field (that the integral curves exist for all time), and the various properties of the Lie bracket such as the Jacobi identity (1.22), the formulas in Lemma 1.61, and the fact that the Lie bracket of two vector fields vanishes if and only if the corresponding flows commute (see Lemma 1.69). We can also introduce the notion of a **subbundle** $E \subset TM$ **of rank** n by the condition that E is a smooth submanifold of TM and intersects each fiber T_pM in an n-dimensional linear subspace $E_p := \{v \in T_pM \,|\, (p,v) \in E\}$. Then the characterization of subbundles in Theorem 1.100 and the theorem of Frobenius 1.106 including their proofs also carry over to the intrinsic setting.

1.9.7 Leaves of a foliation

Let M be an m-dimensional paracompact Hausdorff manifold and $E \subset TM$ be an integrable subbundle of rank n. Let $L \subset M$ be a closed leaf of the foliation determined by E. Then L is a smooth n-dimensional submanifold of M. Here is a sketch of David Epstein's proof of this fact in [4].

(a) *The space L with the intrinsic topology admits the structure of a manifold such that the obvious inclusion $\iota : L \to M$ is an injective immersion.* This is an easy exercise. For the definition of the intrinsic topology see Exercise 1.114. The dimension of L is n.

(b) *If $f : X \to Y$ is a continuous map between topological spaces such that Y is paracompact and there is an open cover $\{V_j\}_{j\in J}$ of Y such that $f^{-1}(V_j)$ is paracompact for each j, then X is paracompact.* To see this, we may assume that the cover $\{V_j\}_{j\in J}$ is locally finite. Now let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then the sets $U_\alpha \cap f^{-1}(V_j)$ define an open cover of $f^{-1}(V_j)$. Choose a locally finite refinement $\{W_{ij}\}_{i\in I_j}$ of this cover. Then the open cover $\{W_{ij}\}_{j\in J, i\in I_j}$ of M is a locally finite refinement of $\{U_\alpha\}_{\alpha\in A}$.

(c) *The intrinsic topology of L is paracompact.* This follows from (b) and the fact that the intersection of L with every foliation box is paracompact in the intrinsic topology.

(d) *The intrinsic topology of L is second countable.* This follows from (a) and (c) and the fact that every connected paracompact manifold is second countable (see Remark 1.127).

(e) *The intersection of L with a foliation box consists of at most countably many connected components.* This follows immediately from (d).

(f) *If L is a closed subset of M then the intersection of L with a foliation box has only finitely many connected components.* To see this, we choose a transverse slice of the foliation at $p_0 \in L$, i.e. a connected submanifold $T \subset M$ through p_0, diffeomorphic to an open ball in $\mathbb{R}^{m-n}$, whose tangent space at each point $p \in T$ is a complement of E_p. By (d) we have that $T \cap L$ is at most countable. If this set is not finite, even after shrinking T, there must be a sequence $p_i \in (T\cap L)\setminus\{p_0\}$ converging to p_0. Using the holonomy of the leaf (obtained by transporting transverse slices along a curve via a lifting argument) we find that every point $p \in T \cap L$ is the limit point of a sequence in $(T\cap L)\setminus\{p\}$. Hence the one-point set $\{p\}$ has empty interior in the relative topology of $T\cap L$ for each $p \in T\cap L$. Thus $T\cap L$ is a countable union of closed subsets with empty interior. Since $T\cap L$ admits the structure of a complete metric space, this contradicts the Baire category theorem.

(g) It follows immediately from (f) that L is a submanifold of M.

1.9.8 Coordinate notation

Fix a coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ on an m-manifold M. The components of ϕ_α are smooth real valued functions on the open subset U_α of M and it is customary to denote them by

$$x^1, \ldots, x^m : U_\alpha \to \mathbb{R}.$$

The derivatives of these functions at $p \in U_\alpha$ are linear functionals

$$dx^i(p) : T_pM \to \mathbb{R}, \qquad i = 1, \ldots, m. \tag{1.66}$$

They form a basis of the dual space

$$T_p^*M := \mathrm{Hom}(T_pM, \mathbb{R}).$$

(A coordinate chart on M can in fact be characterized as an m-tuple of real valued functions on an open subset of M whose derivatives are everywhere linearly independent and which, taken together, form an injective map.) The dual basis of T_pM will be denoted by

$$\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^m}(p) \in T_pM \tag{1.67}$$

so that

$$dx^i(p)\frac{\partial}{\partial x^j}(p) = \delta^i_j := \left\{ \begin{array}{ll} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{array} \right.$$

Thus $\partial/\partial x^i$ is a vector field on the coordinate patch U_α. For each $p \in U_\alpha$ it is the canonical basis of T_pM determined by ϕ_α. In the notation of (1.57) and Remark 1.143 we have

$$\frac{\partial}{\partial x^i}(p) = [\alpha, e_i]_p = d\phi_\alpha(p)^{-1}e_i$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the ith place) denotes the standard basis vector of $\mathbb{R}^m$. In other words, for every vector

$$\xi = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$$

and every $p \in U_\alpha$, the tangent vector $v := d\phi_\alpha(p)^{-1}\xi \in T_pM$ is given by

$$v = [\alpha, \xi]_p = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i}(p). \tag{1.68}$$

Thus the restriction of a vector field $X \in \mathrm{Vect}(M)$ to U_α has the form

$$X|_{U_\alpha} = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i}$$

where $\xi^1, \dots, \xi^m : U_\alpha \to \mathbb{R}$ are smooth real valued functions. If the map $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ is defined by (1.61) then $X_\alpha \circ \phi_\alpha^{-1} = (\xi^1, \dots, \xi^m)$. The above notation is motivated by the observation that the derivative of a smooth function $f : M \to \mathbb{R}$ in the direction of a vector field X on a coordinate patch U_α is given by

$$\mathcal{L}_X f|_{U_\alpha} = \sum_{i=1}^m \xi^i \frac{\partial f}{\partial x^i}.$$

Here the term $\partial f/\partial x^i$ is understood as first writing f as a function of $x^1, \dots, x^m$, then taking the partial derivative, and afterwards expressing this partial derivative again as a function of p. Thus $\partial f/\partial x^i$ is the shorthand notation for the function $\left(\frac{\partial}{\partial x^i}(f \circ \phi_\alpha^{-1})\right) \circ \phi_\alpha : U_\alpha \to \mathbb{R}$.

1.10 Partitions of unity

In geometry it is often necessary to turn a construction in local coordinates into a global geometric object. A key technical tool for such *"local to global"* constructions is an existence theorem for partitions of unity.

1.10.1 Definition and existence

Definition 1.148. *Let M be a smooth manifold. A* **partition of unity** *on M is a collection of smooth functions $\theta_\alpha : M \to [0,1]$ for $\alpha \in A$ such that each point $p \in M$ has an open neighborhood $V \subset M$ on which only finitely many θ_α do not vanish, i.e.*

$$\#\{\alpha \in A \,|\, \theta_\alpha|_V \not\equiv 0\} < \infty, \tag{1.69}$$

and, for every $p \in M$, we have

$$\sum_{\alpha \in A} \theta_\alpha(p) = 1. \tag{1.70}$$

If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M then a partition of unity $\{\theta_\alpha\}_{\alpha \in A}$ (indexed by the same set A) is called **subordinate to the cover** *if each θ_α is supported in U_α, i.e.*

$$\mathrm{supp}(\theta_\alpha) := \overline{\{p \in M \,|\, \theta_\alpha(p) \neq 0\}} \subset U_\alpha.$$

Theorem 1.149 (Partitions of unity). *Let M be a manifold whose topology is paracompact and Hausdorff. For every open cover of M there exists a partition of unity subordinate to that cover.*

Lemma 1.150. *Let M be a Hausdorff manifold. For every open set $V \subset M$ and every compact set $K \subset V$ there is a smooth function $\kappa : M \to [0, \infty)$ wich compact support such that $\mathrm{supp}(\kappa) \subset V$ and $\kappa(p) > 0$ for every $p \in K$.*

Proof. Assume first that $K = \{p_0\}$ is a single point. Since M is a manifold it is locally compact. Hence there is a compact neighborhood $C \subset V$ of p_0. Since M is Hausdorff C is closed and hence the set $U := \mathrm{int}(C)$ is a neighborhood of p_0 whose closure $\overline{U} \subset C$ is compact and contained in V. Shrinking U, if necessary, we may assume that there is a coordinate chart $\phi : U \to \Omega$ with values in some open neighborhood $\Omega \subset \mathbb{R}^m$ of the origin such that $\phi(p_0) = 0$. (Here m is the dimension of M.) Now choose a smooth function $\kappa_0 : \Omega \to [0, \infty)$ with compact support such that $\kappa_0(0) > 0$. Then the function $\kappa : M \to [0, 1]$ defined by $\kappa|_U := \kappa_0 \circ \phi$ and $\kappa(p) := 0$ for $p \in M \setminus U$ is supported in V and satisfies $\kappa(p_0) > 0$. This proves the lemma in the case where K is a point.

Now let K be any compact subset of V. Then, by the first part of the proof, there is a collection of smooth functions $\kappa_p : M \to [0, \infty)$, one for every $p \in K$, such that $\kappa_p(p) > 0$ and $\mathrm{supp}(\kappa_p) \subset V$. Since K is compact there are finitely many points $p_1, \ldots, p_k \in K$ such that the sets $\{p \in M \,|\, \kappa_{p_j}(p) > 0\}$ cover K. Hence the function $\kappa := \sum_j \kappa_{p_j}$ is supported in V and is everywhere positive on K. This proves the lemma. □

Lemma 1.151. *Let M be a topological space. If $\{V_i\}_{i \in I}$ is a locally finite collection of open sets in M then*

$$\overline{\bigcup_{i \in I_0} V_i} = \bigcup_{i \in I_0} \overline{V}_i$$

for every subset $I_0 \subset I$.

Proof. The set $\bigcup_{i \in I_0} \overline{V}_i$ is obviously contained in the closure of $\bigcup_{i \in I_0} V_i$. To prove the converse choose a point $p_0 \in M \setminus \bigcup_{i \in I_0} \overline{V}_i$. Since the collection $\{V_i\}_{i \in I}$ is locally finite there is an open neighborhood U of p_0 such that

$$I_1 := \{i \in I \,|\, V_i \cap U \neq \emptyset\}$$

is a finite set. Hence the set $U_0 := U \setminus \bigcup_{i \in I_0 \cap I_1} \overline{V}_i$ is an open neighborhood of p_0 and we have $U_0 \cap V_i = \emptyset$ for every $i \in I_0$. Hence $p_0 \notin \overline{\bigcup_{i \in I_0} V_i}$. □

Proof of Theorem 1.149. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. We prove in four steps that there is a partition of unity subordinate to this cover. The proofs of steps one and two are taken from [9, Lemma 41.6].

Step 1. *There is a locally finite open cover $\{V_i\}_{i\in I}$ of M such that, for every $i \in I$, the closure $\overline{V}_i$ is compact and contained in one of the sets U_α.*

Denote by $\mathscr{V} \subset 2^M$ the set of all open sets $V \subset M$ such that $\overline{V}$ is compact and $\overline{V} \subset U_\alpha$ for some $\alpha \in A$. Since M is a locally compact Hausdorff space the collection $\mathscr{V}$ is an open cover of M. (If $p \in M$ then there is an $\alpha \in A$ such that $p \in U_\alpha$; since M is locally compact there is a compact neighborhood $K \subset U_\alpha$ of p; since M is Hausdorff K is closed and thus $V := \mathrm{int}(K)$ is an open neighborhood of p with $\overline{V} \subset K \subset U_\alpha$.) Since M is paracompact the open cover $\mathscr{V}$ has a locally finite refinement $\{V_i\}_{i\in I}$. This cover satisfies the requirements of Step 1.

Step 2. *There is a collection of compact sets $K_i \subset V_i$, one for each $i \in I$, such that $M = \bigcup_{i\in I} K_i$.*

Denote by $\mathscr{W} \subset 2^M$ the set of all open sets $W \subset M$ such that $\overline{W} \subset V_i$ for some i. Since M is a locally compact Hausdorff space, the collection $\mathscr{W}$ is an open cover of M. Since M is paracompact $\mathscr{W}$ has a locally finite refinement $\{W_j\}_{j\in J}$. By the axiom of choice there is a map

$$J \to I : j \mapsto i_j$$

such that

$$\overline{W}_j \subset V_{i_j} \qquad \forall\, j \in J.$$

Since the collection $\{W_j\}_{j\in J}$ is locally finite, we have

$$K_i := \overline{\bigcup_{i_j=i} W_j} = \bigcup_{i_j=i} \overline{W}_j \subset V_i$$

by Lemma 1.151. Since $\overline{V}_i$ is compact so is K_i.

Step 3. *There is a partition of unity subordinate to the cover $\{V_i\}_{i\in I}$.*

Choose a collection of compact sets $K_i \subset V_i$ for $i \in I$ as in Step 2. Then, by Lemma 1.150 and the axiom of choice, there is a collection of smooth functions $\kappa_i : M \to [0,\infty)$ with compact support such that

$$\mathrm{supp}(\kappa_i) \subset V_i, \qquad \kappa_i|_{K_i} > 0 \qquad \forall\, i \in I.$$

Since the cover $\{V_i\}_{i\in I}$ is locally finite the sum

$$\kappa := \sum_{i\in I} \kappa_i : M \to \mathbb{R}$$

is **locally finite** (i.e. each point in M has a neighborhood in which only finitely many terms do not vanish) and thus defines a smooth function on M. This function is everywhere positive, because each summand is nonnegative and, for each $p \in M$, there is an $i \in I$ with $p \in K_i$ so that $\kappa_i(p) > 0$. Thus the funtions $\chi_i := \kappa_i/\kappa$ define a partition of unity satisfying $\mathrm{supp}(\chi_i) \subset V_i$ for every $i \in I$ as required.

Step 4. *There is a partition of unity subordinate to the cover* $\{U_\alpha\}_{\alpha\in A}$.

Let $\{\chi_i\}_{i\in I}$ be the partition of unity constructed in Step 3. By the axiom of choice there is a map $I \to A : i \mapsto \alpha_i$ such that $V_i \subset U_{\alpha_i}$ for every $i \in I$. For $\alpha \in A$ define $\theta_\alpha : M \to [0,1]$ by

$$\theta_\alpha := \sum_{\alpha_i=\alpha} \chi_i.$$

Here the sum runs over all indices $i \in I$ with $\alpha_i = \alpha$. This sum is locally finite and hence is a smooth function on M. Moreover, each point in M has an open neighborhood in which only finitely many of the θ_α do not vanish. Hence the sum of the θ_α is a well defined function on M and

$$\sum_{\alpha\in A} \theta_\alpha = \sum_{\alpha\in A} \sum_{\alpha_i=\alpha} \chi_i = \sum_{i\in I} \chi_i \equiv 1.$$

This shows that the functions θ_α form a partition of unity. To prove the inclusion $\mathrm{supp}(\theta_\alpha) \subset U_\alpha$ we consider the open sets

$$W_i := \{p \in M \mid \chi_i(p) > 0\}$$

for $i \in I$. Since $W_i \subset V_i$ this collection is locally finite. Hence, by Lemma 1.151, we have

$$\mathrm{supp}(\theta_\alpha) = \overline{\bigcup_{\alpha_i=\alpha} W_i} = \bigcup_{\alpha_i=\alpha} \overline{W}_i = \bigcup_{\alpha_i=\alpha} \mathrm{supp}(\chi_i) \subset \bigcup_{\alpha_i=\alpha} V_i \subset U_\alpha.$$

This proves the theorem. □

1.10.2 Embedding in Euclidean space

Theorem 1.152. *For every compact m-manifold M with a Hausdorff topology there is an integer $k \in \mathbb{N}$ and an embedding $f : M \to \mathbb{R}^k$.*

Proof. Since M is compact it can be covered by finitely many coordinate charts $\phi_i : U_i \to \Omega_i$, $i = 1, \dots, \ell$, onto open subset $\Omega_i \subset \mathbb{R}^m$. By Theorem 1.149, there is a partition of unity subordinate to the cover $\{U_i\}_{i=1,\dots,\ell}$. Thus there are smooth maps $\theta_1, \dots, \theta_\ell : M \to [0,1]$ such that $\mathrm{supp}(\theta_i) \subset U_i$ for all i and $\bigcup_i U_i = M$. Let $k := \ell(m+1)$ and define $f : M \to \mathbb{R}^k$ by

$$f(p) := \begin{pmatrix} \theta_1(p) \\ \theta_1(p)\phi_1(p) \\ \vdots \\ \theta_\ell(p) \\ \theta_\ell(p)\phi_\ell(p) \end{pmatrix}.$$

This map is injective. Namely, if $p_0, p_1 \in M$ satisfy $f(p_0) = f(p_1)$ then

$$I := \{i \,|\, \theta_i(p_0) > 0\} = \{i \,|\, \theta_i(p_1) > 0\}$$

and for $i \in I$ we have $\theta_i(p_0) = \theta_i(p_1)$, hence $\phi_i(p_0) = \phi_i(p_1)$, and hence $p_0 = p_1$. Moreover, for every $p \in M$ the derivative $df(p) : T_pM \to \mathbb{R}^k$ is injective, and f is proper because M is compact. Hence f is an embedding as claimed. □

The number ℓ in the proof of Theorem 1.152 can actually be chosen less than or equal to $m+1$. However, this is a deep fact in algebraic topology and we shall not address this question here. Assuming this, the proof of Theorem 1.152 shows that every compact m-manifold M can be embedded in $\mathbb{R}^k$ with $k = (m+1)^2$. Using Sard's theorem one can in fact reduce the dimension of the ambient space to $k = 2m+1$ and a further trick, due to Whitney, shows that a compact m-manifold can always be embedded into $\mathbb{R}^{2m}$. Moreover, Theorem 1.152 can in fact be extended to noncompact manifolds and one can show that a manifold admits an embedding into a finite dimensional vector space if and only if it is Hausdorff and second countable. However, we will not address this issue here.

Remark 1.153. The manifold $\mathbb{R}\mathrm{P}^2$ cannot be embedded into $\mathbb{R}^3$. The same is true for the the **Klein bottle** $K := \mathbb{R}^2/\equiv$ where the equivalence relation is given by $[x, y] \equiv [x+k, \ell - y]$ for $x, y \in \mathbb{R}$ and $k, \ell \in \mathbb{Z}$.

Standing assumption

We have seen that all the results in the first chapter carry over to the intrinsic setting, assuming that the topology of M is Hausdorff and paracompact. In fact, in most cases it is enough to assume the Hausdorff property. However, these results mainly deal with introducing the basic concepts like smooth maps, embeddings, submersions, vector fields, flows, and verifying their elementary properties, i.e. with setting up the language for differential geometry and topology. When it comes to the substance of the subject we shall deal with Riemannian metrics and they only exist on paracompact Hausdorff manifolds. Another central ingredient in differential topology is the theorem of Sard and that requires second countability. To quote Moe Hirsch [6]: "Manifolds that are not paracompact are amusing, but they never occur naturally and it is difficult to prove anything about them." Thus we will set the following convention for the remaining chapters.

We assume from now on that each intrinsic manifold M is Hausdorff and second countable and hence is also paracompact.

For most of this text we will in fact continue to develop the theory for submanifolds of Euclidean space and indicate, wherever necessary, how to extend the definitions, theorems, and proofs to the intrinsic setting.

Chapter 2

Geodesics

2.1 The length of a curve

The **length of a smooth curve** $\gamma : [0,1] \to \mathbb{R}^n$ is the number

$$L(\gamma) := \int_0^1 |\dot{\gamma}(t)| \, dt, \tag{2.1}$$

where $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$. More generally, the length of a continuous function $\gamma : [0,1] \to \mathbb{R}^n$ can be defined as the supremum of the expressions $\sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1}|$ over all partitions $0 = t_0 < t_1 < \cdots < t_N = 1$ of the unit interval. By a theorem in first year analysis [12] this supremum is finite whenever γ is continuously differentiable and is given by (2.1).

Remark 2.1. Every smooth curve $\gamma : [0,1] \to \mathbb{R}^n$ with endpoints $\gamma(0) = p$ and $\gamma(1) = q$ satisfies the inequality

$$L(\gamma) \geq \left| \int_0^1 \dot{\gamma}(t) \, dt \right| = |p - q| \, .$$

For $\gamma(t) := p + t(q-p)$ we have equality and hence the straight lines minimize the length among all curves from p to q.

Remark 2.2 (Reparametrization). If $\gamma : [0,1] \to \mathbb{R}^n$ is a smooth curve and $\alpha : [0,1] \to [0,1]$ is a smooth function such that $\alpha(0) = 0$, $\alpha(1) = 1$, and $\dot{\alpha}(t) \geq 0$ for every $t \in [0,1]$ then

$$L(\gamma \circ \alpha) = L(\gamma).$$

To see this, we compute

$$L(\gamma \circ \alpha) = \int_0^1 \left| \frac{d}{dt}(\gamma(\alpha(t)) \right| dt = \int_0^1 |\dot{\gamma}(\alpha(t))| \, \dot{\alpha}(t) \, dt = L(\gamma).$$

Here second equation follows from the chain rule and the fact that $\dot{\alpha}(t) \geq 0$ for all t and the third equation follows from change of variables formula for the Riemann integral. This proves the lemma.

Remark 2.3. Choosing α in Remark 2.2 such that $\alpha(t) = 0$ for t sufficiently close to 0 and $\alpha(t) = 1$ for t sufficiently close to 1, we can reparametrize γ to obtain a curve that is constant near $t = 0$ and near $t = 1$. And the reparametrized curve has the same endpoints, the same image, and the same length as the original curve.

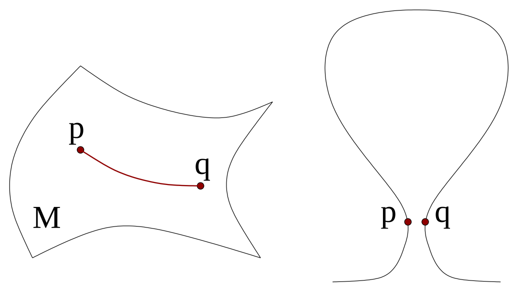


Figure 2.1: Curves in M.

Let us now assume that $M \subset \mathbb{R}^n$ is a connected smooth m-dimensional submanifold. We examine the lengths of curves $\gamma : [0, 1] \to M$ with fixed endpoints. Thus it may happen that two points on M have a very short distance in $\mathbb{R}^n$ but can only be connected by very long curves in M (see Figure 2.1). This leads to the notion of *intrinsic distance in* M. For $p, q \in M$ we denote the space of smooth paths in M connecting p and q by

$$\Omega_{p,q} := \{\gamma : [0, 1] \to M \,|\, \gamma \text{ is smooth and } \gamma(0) = p,\ \gamma(1) = q\}. \qquad (2.2)$$

Since M is connected the set $\Omega_{p,q}$ is nonempty for all $p, q \in M$. (Prove this!) We define the **intrinsic distance function** $d : M \times M \to [0, \infty)$ by

$$d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma) \qquad (2.3)$$

for $p, q \in M$. The inequality $d(p, q) \geq 0$ holds because each curve has nonnegative length and the inequality $d(p, q) < \infty$ holds because $\Omega_{p,q} \neq \emptyset$.

Lemma 2.4. *The function* $d : M \times M \to [0, \infty)$ *defines a metric on* M*:*
(i) *If* $p, q \in M$ *with* $d(p, q) = 0$ *then* $p = q$.
(ii) *For all* $p, q \in M$ *we have* $d(p, q) = d(q, p)$.
(iii) *For all* $p, q, r \in M$ *we have* $d(p, r) \leq d(p, q) + d(q, r)$.

Proof. By Remark 2.1 we have

$$d(p, q) \geq |p - q|$$

for all $p, q \in \mathbb{R}$ and this proves (i). Assertion (ii) follows from the fact that the curve $\widetilde{\gamma}(t) := \gamma(1 - t)$ has the same length as γ and belongs to $\Omega_{q,p}$ whenever $\gamma \in \Omega_{p,q}$. To prove (iii) fix a constant $\varepsilon > 0$ and choose $\gamma_0 \in \Omega_{p,q}$ and $\gamma_1 \in \Omega_{q,r}$ such that

$$L(\gamma_0) < d(p, q) + \varepsilon, \qquad L(\gamma_1) < d(q, r) + \varepsilon.$$

By Remark 2.3 we may assume without loss of generality that

$$\gamma_0(1 - t) = \gamma_1(t) = q$$

for t sufficiently small. Under this assumption the curve

$$\gamma(t) := \begin{cases} \gamma_0(2t), & \text{for } 0 \leq t \leq 1/2, \\ \gamma_1(2t - 1), & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

is smooth and has endpoints $\gamma(0) = p$ and $\gamma(1) = r$. Thus $\gamma \in \Omega_{p,r}$ and

$$L(\gamma) = L(\gamma_0) + L(\gamma_1) < d(p, q) + d(q, r) + 2\varepsilon.$$

Hence $d(p, r) < d(p, q) + d(q, r) + 2\varepsilon$ for every $\varepsilon > 0$. This proves (iii) and the lemma. □

Example 2.5. Let $M = S^2$ be the unit sphere in $\mathbb{R}^3$ and fix two points $p, q \in S^2$. Then $d(p, q)$ is the length of the shortest curve on the 2-sphere connecting p and q. Such a curve is a segment on a great circle through p and q (see Figure 2.2) and its length is

$$d(p, q) = \cos^{-1}(\langle p, q \rangle), \tag{2.4}$$

where $\langle p, q \rangle$ denotes the standard inner product, and we have

$$0 \leq d(p, q) \leq \pi.$$

(See Example 2.23 below for details.) By Lemma 2.4 this defines a metric on S^2. **Exercise:** Prove directly that (2.4) is a distance function on S^2.

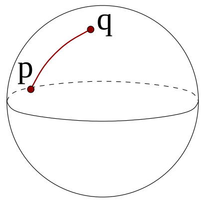


Figure 2.2: A geodesic on the 2-sphere.

We now have two topologies on our manifold $M \subset \mathbb{R}^n$, namely the topology determined by the metric d in Lemma 2.4 and the relative topology inherited from $\mathbb{R}^n$. The latter is also determined by a distance function, namely the *extrinsic distance function* defined as the restriction of the Euclidean distance function on $\mathbb{R}^n$ to the subset M. We denote it by

$$d_0 : M \times M \to [0, \infty), \qquad d_0(p, q) := |p - q| \,.$$

A natural question is now if these two metrics d and d_0 induce the same topology on M. In other words is a subset $U \subset M$ open with respect to d_0 if and only if it is open with respect to d? Or, equivalently, does a sequence $p_\nu \in M$ converge to $p_0 \in M$ with respect to d if and only if it converges to p_0 with respect to d_0? Lemma 2.7 answers this question in the affirmative.

Exercise 2.6. Prove that every translation of $\mathbb{R}^n$ and every orthogonal transformation preserves the lengths of curves.

Lemma 2.7. *For every $p_0 \in M$ we have*

$$\lim_{p,q \to p_0} \frac{d(p, q)}{|p - q|} = 1.$$

In other words, for every $p_0 \in M$ and every $\varepsilon > 0$ there is an open neighborhood $U \subset M$ of p_0 such that, for all $p, q \in U$, we have

$$(1 - \varepsilon)\, |p - q| \le d(p, q) \le (1 + \varepsilon)\, |p - q| \,.$$

Proof. We have already observed that

$$|p - q| \le d(p, q)$$

for all $p, q \in M$ (see Remark 2.1). Now fix a point $p_0 \in M$. By Exercise 2.6 we may assume without loss of generality that

$$p_0 = 0, \qquad T_0M = \mathbb{R}^m \times \{0\}.$$

Then there is a smooth function $f : \Omega \to \mathbb{R}^{n-m}$, defined on an open neighborhood $\Omega \subset \mathbb{R}^m$ of the origin such that

$$M \supset \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \,|\, x \in \Omega,\, y = f(x)\right\}, \quad f(0) = 0, \quad df(0) = 0.$$

Moreover, the graph of f is an open subset of M in the relative topology. Choose $\delta > 0$ such that, for all $x \in \mathbb{R}^m$, we have

$$|x| < \delta \qquad \Longrightarrow \qquad x \in \Omega \text{ and } |df(x)| < \varepsilon$$

and define

$$U := \{(x, f(x)) \,|\, x \in \mathbb{R}^m,\, |x| < \delta\}.$$

This is an open subset M. We prove that δ and U satisfy the assertion of the lemma. Thus let $p = (x_0, f(x_0))$ and $q = (x_1, f(x_1))$ be two points in U and consider the curve $\gamma : [0,1] \to U$ defined by

$$\gamma(t) := (x(t), f(x(t)), \qquad x(t) := (1-t)x_0 + tx_1.$$

This curve satisfies $\gamma(0) = p$, $\gamma(1) = q$, and

$$\begin{aligned} |\dot\gamma(t)|^2 &= |\dot x(t)|^2 + |df(x(t))\dot x(t)|^2 \\ &\le |\dot x(t)|^2 + |df(x(t))|^2 |\dot x(t)|^2 \\ &\le (1+\varepsilon^2) |\dot x(t)|^2 \\ &= (1+\varepsilon^2) |x_0 - x_1|^2 \\ &\le (1+\varepsilon)^2 |p - q|^2. \end{aligned}$$

This implies

$$d(p,q) \le L(\gamma) \le (1+\varepsilon) |p-q|$$

and the lemma is proved. □

A next question one might ask is: Can we choose a coordinate chart

$$\phi : U \to \Omega$$

on M with values in an open set $\Omega \subset \mathbb{R}^m$ so that the length of each smooth curve $\gamma : [0,1] \to U$ is equal to the length of the curve $c := \phi \circ \gamma : [0,1] \to \Omega$? We examine this question by considering the inverse map

$$\psi := \phi^{-1} : \Omega \to U.$$

We denote the components of x and $\psi(x)$ by

$$x = (x^1, \ldots, x^m) \in \Omega, \qquad \psi(x) = (\psi^1(x), \ldots, \psi^n(x)) \in U.$$

Given a smooth curve $[0,1] \to \Omega : t \mapsto c(t) = (c^1(t), \ldots, c^m(t))$ we can write the length of the composition $\gamma = \psi \circ c : [0,1] \to M$ in the form

$$\begin{aligned} L(\psi \circ c) &= \int_0^1 \left| \frac{d}{dt} \psi(c(t)) \right| dt \\ &= \int_0^1 \sqrt{\sum_{\nu=1}^n \left(\frac{d}{dt} \psi^\nu(c(t)) \right)^2} \, dt \\ &= \int_0^1 \sqrt{\sum_{\nu=1}^n \left(\sum_{i=1}^m \frac{\partial \psi^\nu}{\partial x^i}(c(t)) \dot{c}^i(t) \right)^2} \, dt \\ &= \int_0^1 \sqrt{\sum_{\nu=1}^n \sum_{i,j=1}^m \frac{\partial \psi^\nu}{\partial x^i}(c(t)) \frac{\partial \psi^\nu}{\partial x^j}(c(t)) \dot{c}^i(t) \dot{c}^j(t)} \, dt \\ &= \int_0^1 \sqrt{\sum_{i,j=1}^m \dot{c}^i(t) g_{ij}(c(t)) \dot{c}^j(t)} \, dt, \end{aligned}$$

where the functions $g_{ij} : \Omega \to \mathbb{R}$ are defined by

$$g_{ij}(x) := \sum_{\nu=1}^n \frac{\partial \psi^\nu}{\partial x^i}(x) \frac{\partial \psi^\nu}{\partial x^j}(x) = \left\langle \frac{\partial \psi}{\partial x^i}(x), \frac{\partial \psi}{\partial x^j}(x) \right\rangle. \tag{2.5}$$

Thus we have a smooth function $g = (g_{ij}) : \Omega \to \mathbb{R}^{m \times m}$ with values in the positive definite matrices given by

$$g(x) = d\psi(x)^T d\psi(x)$$

such that

$$L(\psi \circ c) = \int_0^1 \dot{c}(t)^T g(c(t)) \dot{c}(t) \, dt \tag{2.6}$$

for every smooth curve $c : [0,1] \to \Omega$. Thus the condition $L(\psi \circ c) = L(c)$ for every such curve is equivalent to

$$g_{ij}(x) = \delta_{ij}$$

for all $x \in \Omega$ or, equivalently,

$$d\psi(x)^T d\psi(x) = \mathbb{1}. \tag{2.7}$$

This means that ψ preserves angles and areas. The next example shows that for $M = S^2$ it is impossible to find such coordinates.

Example 2.8. Consider the manifold $M = S^2$. If there is a diffeomorphism $\psi : \Omega \to U$ from an open set $\Omega \subset S^2$ onto an open set $U \subset S^2$ that satisfies (2.7) it has to map straight lines onto arcs of great circles and it preserves the area. However, the area A of a spherical triangle bounded by three arcs on great circles satisfies the angle sum formula

$$\alpha + \beta + \gamma = \pi + A.$$

(See Figure 2.3.) Hence there can be no such map ψ.

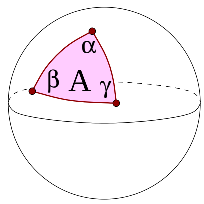


Figure 2.3: A spherical triangle.

2.2 Geodesics

Let $M \subset \mathbb{R}^n$ be an m-dimensional connected submanifold. We will address the following question. Given two points $p, q \in M$ is there is a smooth curve $\gamma : [0, 1] \to M$ with endpoints $\gamma(0) = p$ and $\gamma(1) = q$ satisfying

$$L(\gamma) = d(p, q)$$

so that γ minimizes the **length functional** $L : \Omega_{p,q} \to \mathbb{R}$ among all curves in $\Omega_{p,q}$? To address this question it is convenient to modify the problem and consider instead the **energy functional** $E : \Omega_{p,q} \to M$ defined by

$$E(\gamma) := \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt.$$

The advantage is that the function E is *smooth* in the appropriate sense (to be made more precise below) while the length fnctional L is only smooth at those points $\gamma \in \Omega_{p,q}$ that satisfy $\dot{\gamma}(t) \neq 0$ for all t. Our goal is to examine the critical points of the functionals L and E. This requires some preparation.

2.2.1 The orthogonal projection onto T_pM

For $p \in M$ let $\Pi(p) \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto T_pM. Thus

$$\Pi(p) = \Pi(p)^2 = \Pi(p)^T \tag{2.8}$$

and

$$\Pi(p)v = v \qquad \iff \qquad v \in T_pM \tag{2.9}$$

for $p \in M$ and $v \in \mathbb{R}^n$ (see Exercise 1.99). We have seen in Theorem 1.100 and Corollary 1.101 that the map $\Pi : M \to \mathbb{R}^{n\times n}$ is smooth.

Example 2.9. Consider the case $n = m + 1$. Thus $M \subset \mathbb{R}^{m+1}$ is a submanifold of codimension 1. By Corollary 1.102 the normal bundle $TM^\perp$ is a vector bundle of rank 1 over M and hence each fiber $E_p = T_pM^\perp$ is spanned by a single unit vector $\nu(p) \in \mathbb{R}^m$. Note that the vector $\nu(p) \in S^m$ is determined by the tangent space T_pM up to a sign. By Theorem 1.100 each point $p_0 \in M$ has an open neighborhood $U \subset M$ on which there is a smooth function $\nu : U \to \mathbb{R}^{m+1}$ satisfying

$$\nu(p) \perp T_pM, \qquad |\nu(p)| = 1 \tag{2.10}$$

for $p \in U$ (see Figure 2.4). Such a map ν is called a **Gauss map**. The function $\Pi : M \to \mathbb{R}^{n\times n}$ is in this case given by

$$\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^T, \qquad p \in U. \tag{2.11}$$

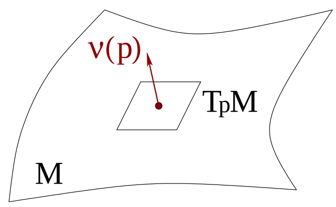


Figure 2.4: A unit normal vector field.

Example 2.10. In the case $M = S^2 \subset \mathbb{R}^3$ we have $\nu(p) = p$ and hence

$$\Pi(p) = \mathbb{1} - pp^T = \begin{pmatrix} 1-x^2 & -xy & -xz \\ -yx & 1-y^2 & -yz \\ -zx & -zy & 1-z^2 \end{pmatrix}$$

for every $p = (x, y, z) \in S^2$.

Example 2.11. If $f : U \to \mathbb{R}^{n-m}$ is a smooth function, $0 \in \mathbb{R}^{n-m}$ is a regular value of f, and $U \cap M = f^{-1}(0)$ then $T_pM = \ker df(p)$ and

$$\Pi(p) = \mathbb{1} - df(p)^T \left(df(p)df(p)^T\right)^{-1} df(p)$$

for every $p \in U \cap M$. (Prove this!)

Example 2.12. If $\psi : \Omega \to M$ is a smooth embedding of an open set $\Omega \subset \mathbb{R}^m$ then

$$\Pi(\psi(x)) = d\psi(x) \left(d\psi(x)^T d\psi(x)\right)^{-1} d\psi(x)^T$$

for every $x \in \Omega$. (See the proof of Corollary 1.101.)

Example 2.13 (The Möbius strip). Consider the submanifold $M \subset \mathbb{R}^3$ given by

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = (1 + r\cos(\theta/2))\cos(\theta), \\ y = (1 + r\cos(\theta/2))\sin(\theta), \\ z = r\sin(\theta/2),\ r, \theta \in \mathbb{R},\ |r| < \varepsilon \end{array} \right\}$$

for $\varepsilon > 0$ sufficiently small. Show that there does not exist a global smooth function $\nu : M \to \mathbb{R}^3$ satisfying (2.10).

2.2.2 The covariant derivative

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. A **vector field along** γ is a smooth map $X : I \to \mathbb{R}^n$ such that $X(t) \in T_{\gamma(t)}M$ for every $t \in I$ (see Figure 2.5). The set of smooth vector fields along γ is a real vector space and will be denoted by

$$\mathrm{Vect}(\gamma) := \left\{ X : I \to \mathbb{R}^n \,|\, X \text{ is smooth and } X(t) \in T_{\gamma(t)}M \ \forall\, t \in I \right\}.$$

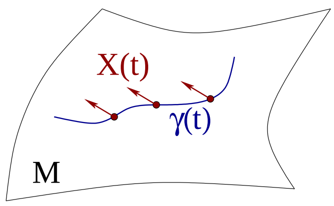


Figure 2.5: A vector field along a curve.

The first derivative $\dot{X}(t)$ of a vector field along γ at $t \in I$ will, in general, not be tangent to M. We may decompose it as a sum of a tangent vector and a normal vector in the form

$$\dot{X}(t) = \Pi(\gamma(t))\dot{X}(t) + \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot{X}(t),$$

where $\Pi : M \to \mathbb{R}^{n \times n}$ is defined by (2.8) and (2.9). The tangential component of in this decomposition plays an important geometric role. It is called the **covariant derivative of** X **at** $t \in I$ and will be denoted by

$$\nabla X(t) := \Pi(\gamma(t))\dot{X}(t) \in T_{\gamma(t)}M. \tag{2.12}$$

This is a smooth vector field along γ. Thus the covariant derivative defines a linear operator $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$. A vector field $X \in \mathrm{Vect}(\gamma)$ is called **parallel** if it belongs to the kernel of this operator so that $\nabla X \equiv 0$.

Remark 2.14. For every vector field $X \in \mathrm{Vect}(\gamma)$ and every $t \in I$ we have

$$\nabla X(t) = 0 \qquad \Longleftrightarrow \qquad \dot{X}(t) \perp T_{\gamma(t)}M.$$

In particular, $\dot{\gamma}$ is a vector field along γ and $\nabla\dot{\gamma}(t) = \Pi(\gamma(t))\ddot{\gamma}(t)$. Hence, for every $t \in I$, we have $\nabla\dot{\gamma}(t) = 0$ if and only if $\ddot{\gamma}(t) \perp T_{\gamma(t)}M$.

Remark 2.15. For any two vector fields $X, Y \in \mathrm{Vect}(\gamma)$ along γ we have

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle. \tag{2.13}$$

2.2.3 The space of paths

Fix two points $p, q \in M$. We may think of the space $\Omega_{p,q}$ of smooth paths in M connecting p to q as a kind of *"infinite dimensional manifold"*. This is to be understood in a heuristic sense and we use these terms here to emphasize an analogy. Of course, the space $\Omega_{p,q}$ is not a manifold in the strict sense of the word. To begin with it is not embedded in any finite dimensional Euclidean space. However, it has many features in common with manifolds. The first is that we can speak of *smooth curves in* $\Omega_{p,q}$. Of course $\Omega_{p,q}$ is itself a space of curves in M. Thus a smooth curve in $\Omega_{p,q}$ would then be a curve of curves, namly a map

$$\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$$

that assigns to each real number a smooth curve $\gamma_s : [0, 1] \to M$ satisfying $\gamma_s(0) = p$ and $\gamma_s(1) = q$. We shall call such a curve of curves **smooth** if the associated map $\mathbb{R} \times [0, 1] \to M : (s, t) \mapsto \gamma_s(t)$ is smooth.

Having defined what we mean by a smooth curve in $\Omega_{p,q}$ we can also differentiate such a curve with respect to s. Here we can simply recall that, since $M \subset \mathbb{R}^n$, we have a smooth map $\mathbb{R} \times [0,1] \to \mathbb{R}^n$ and the derivative of the curve $s \mapsto \gamma_s$ in $\Omega_{p,q}$ can simply be understood as the partial derivative of the map $(s,t) \mapsto \gamma_s(t)$ with respect to s. Thus, in analogy with embedded manifolds, we define the **tangent space** of the space of curves $\Omega_{p,q}$ at γ as the set of all derivatives of smooth curves $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ passing through γ:

$$T_\gamma \Omega_{p,q} := \left\{ \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s \,\middle|\, \mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s \text{ is smooth and } \gamma_0 = \gamma \right\}.$$

Let us denote such a partial derivative by

$$X(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s(t) \in T_{\gamma(t)}M.$$

Thus we obtain a smooth vector field along γ. Since $\gamma_s(0) = p$ and $\gamma_s(1) = q$ for all s, this vector field must vanish at $t = 0, 1$. This suggests the formula

$$T_\gamma \Omega_{p,q} = \{ X \in \mathrm{Vect}(\gamma) \,|\, X(0) = 0,\ X(1) = 0 \}. \tag{2.14}$$

That every tangent vector of the path space $\Omega_{p,q}$ at γ is a vector field along γ vanishing at the endpoints follows from the above discussion. The converse inclusion is the content of the next exercise.

Exercise 2.16. Verify equation (2.14) for $\gamma \in \Omega_{p,q}$. Namely, for every smooth vector field $X \in \mathrm{Vect}(\gamma)$ along γ satisfying

$$X(0) = 0, \qquad X(1) = 0,$$

there is a smooth curve $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ satisfying

$$\gamma_0 = \gamma, \qquad \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s = X. \tag{2.15}$$

Hint: Construct a smooth map $[0,1] \times \Omega \to M : (t,x) \mapsto \psi_t(x)$, for some open neighborhood $\Omega \subset \mathbb{R}^m$ of zero, such that each map $\psi_t : \Omega \to M$ is a diffeomorphism onto an open set $U_t \subset M$ satisfying $\psi_t(0) = \gamma(t)$.

2.2.4 Critical points of E and L

We can now define the **derivative of the energy functional** $E : \Omega_{p,q} \to \mathbb{R}$ at γ in the direction of a tangent vector $X \in T_\gamma\Omega_{p,q}$ by

$$dE(\gamma)X := \left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) \tag{2.16}$$

for $X \in T_\gamma\Omega_{p,q}$, where $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ is a smooth curve of curves satisfying (2.15). This defines rise to a linear map

$$dE(\gamma) : T_\gamma\Omega_{p,q} \to \mathbb{R}.$$

Similarly, the **derivative of the length functional** $L : \Omega_{p,q} \to \mathbb{R}$ at γ is the linear map

$$dL(\gamma) : T_\gamma\Omega_{p,q} \to \mathbb{R},$$

defined by

$$dL(\gamma)X := \left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) \tag{2.17}$$

for $X \in T_\gamma\Omega_{p,q}$ where $s \mapsto \gamma_s$ is chosen as before. However, care must be taken. To even define the expressions (2.16) and (2.17) the functions $s \mapsto E(\gamma_s)$ and $s \mapsto L(\gamma_s)$ must be differentiable at the origin. This is no problem in the case of E but it only holds for L when $\dot\gamma(t) \neq 0$ for all $t \in [0,1]$. Second we must shos that the right hand sides of (2.16) and (2.17) depend only on X and not on the choice of the curve of curves $s \mapsto \gamma_s$ satisfying (2.15). Third one needs to verify that the maps $dE(\gamma)$ and $dL(\gamma)$ are indeed linear. All this is an exercise in first year analysis which we leave to the reader (see also the proof of Theorem 2.17 below). A curve $\gamma \in \Omega_{p,q}$ is called a **critical point of** E if $dE(\gamma) = 0$ and, in the case $\dot\gamma(t) \neq 0$ for all t, it is called a **critical point of** L if $dL(\gamma) = 0$.

Theorem 2.17. *Let $\gamma \in \Omega_{p,q}$. Then the following are equivalent.*

(i) *γ is a critical point of E.*

(ii) *Either $\gamma(t) \equiv p = q$ or*

$$|\dot\gamma(t)| \equiv c \neq 0$$

and γ is a critical point of L.

(iii) $\nabla\dot\gamma(t) \equiv 0$.

Definition 2.18. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and $I \subset \mathbb{R}$ be an interval. A smooth curve $\gamma : I \to M$ is called a* **geodesic** *if $\nabla\dot\gamma \equiv 0$.*

Proof of Theorem 2.17. We prove that (i) is equivalent to (iii). Let

$$X \in T_\gamma \Omega_{p,q}$$

be given and choose a smooth curve of curves $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ that satisfies (2.15). Then the function $(s,t) \mapsto |\dot\gamma_s(t)|^2$ is smooth and hence we can interchange differentiation with respect to the s-variable and integration with respect to the t-variable. Thus

$$\begin{aligned}
dE(\gamma)X &= \left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) \\
&= \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_0^1 |\dot\gamma_s(t)|^2 \, dt \\
&= \frac{1}{2}\int_0^1 \left.\frac{\partial}{\partial s}\right|_{s=0} |\dot\gamma_s(t)|^2 \, dt \\
&= \int_0^1 \left\langle \dot\gamma(t), \left.\frac{\partial}{\partial s}\right|_{s=0} \dot\gamma_s(t) \right\rangle dt \\
&= \int_0^1 \left\langle \dot\gamma(t), \dot X(t) \right\rangle dt \\
&= -\int_0^1 \langle \ddot\gamma(t), X(t) \rangle \, dt.
\end{aligned}$$

That (iii) implies (i) follows immediately from this identity. Conversely, suppose that γ is a critical point of E and that there is a point $t_0 \in [0,1]$ such that $\nabla\dot\gamma(t_0) \neq 0$. Then, by Remark 2.14, $\ddot\gamma(t_0)$ is not orthogonal to $T_{\gamma(t_0)}M$. We assume without loss of generality that $0 < t_0 < 1$. and choose a vector $v_0 \in T_{\gamma(t_0)}M$ such that $\langle \ddot\gamma(t_0), v_0 \rangle > 0$. Hence there is a constant $\varepsilon > 0$ such that $0 < t_0 - \varepsilon < t_0 + \varepsilon < 1$ and

$$t_0 - \varepsilon < t < t_0 + \varepsilon \qquad \Longrightarrow \qquad \langle \ddot\gamma(t), \Pi(\gamma(t))v_0 \rangle > 0.$$

Choose a smooth cutoff function $\beta : [0,1] \to [0,1]$ such that $\beta(t) = 0$ for $|t - t_0| \geq \varepsilon$ and $\beta(t_0) = 1$. Define $X \in T_\gamma\Omega_{p,q}$ by

$$X(t) := \beta(t)\Pi(\gamma(t))v_0.$$

Then $\langle \ddot\gamma(t), X(t) \rangle \geq 0$ for all t and $\langle \ddot\gamma(t_0), X(t_0) \rangle > 0$. Hence

$$dE(\gamma)X = -\int_0^1 \langle \ddot\gamma(t), X(t) \rangle \, dt < 0$$

and this contradicts (i). Thus we have proved that (i) is equivalent to (iii).

We prove that (i) is equivalent to (ii). Assume first that γ satisfies (i). Then γ also satisfies (iii) and hence $\ddot{\gamma}(t) \perp T_{\gamma(t)}M$ for all $t \in [0,1]$. This implies

$$0 = \langle \ddot{\gamma}(t), \dot{\gamma}(t) \rangle = \frac{1}{2}\frac{d}{dt}\left|\dot{\gamma}(t)\right|^2.$$

Hence the function $t \mapsto |\dot{\gamma}(t)|^2$ is constant. Choose $c \geq 0$ such that $|\dot{\gamma}(t)| \equiv c$. If $c = 0$ then $\gamma(t)$ is constant and thus $\gamma(t) \equiv p = q$. If $c > 0$ we have

$$\begin{aligned}
dL(\gamma)X &= \left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) \\
&= \left.\frac{d}{ds}\right|_{s=0} \int_0^1 |\dot{\gamma}_s(t)| \, dt \\
&= \int_0^1 \left.\frac{\partial}{\partial s}\right|_{s=0} |\dot{\gamma}_s(t)| \, dt \\
&= \int_0^1 \frac{\left\langle \dot{\gamma}(t), \left.\frac{\partial}{\partial s}\right|_{s=0} \dot{\gamma}_s(t) \right\rangle}{|\dot{\gamma}(t)|} \, dt \\
&= \frac{1}{c}\int_0^1 \left\langle \dot{\gamma}(t), \dot{X}(t) \right\rangle \, dt \\
&= \frac{1}{c} dE(\gamma)X.
\end{aligned}$$

Thus, in the case $c > 0$, γ is a critical point of E if and only if it is a critical point of L. Hence (i) is equivalent to (ii) and this proves the theorem. □

As we have seen in Remark 2.2 the length of a curve remains unchanged under reparametrization. This implies that if a path $\gamma \in \Omega_{p,q}$ (satisfying $\dot{\gamma}(t) \neq 0$ for all t) is a critical point of L then each reparametrization of γ is still a critical point of L; only the unique reparametrization for which $|\dot{\gamma}(t)|$ is constant is also a critical point of E.

Exercise 2.19. Let $\gamma : [0,1] \to M$ be a smooth curve satisfying $\dot{\gamma}(t) \neq 0$ for every t. Prove that there is a unique smooth map $\alpha : [0,1] \to [0,1]$ satisfying $\alpha(0) = 0$, $\alpha(1) = 1$, $\dot{\alpha}(t) > 0$, such that the derivative of $\gamma \circ \alpha$ has constant norm.

Theorem 2.17 still does not answer the question of the existence of a curve that minimizes the length among all curves with the same endpoints. However it characterizes the critical points of the energy and length functionals and thus gives us a necessary condition that energy-minimizing curves must satisfy.

2.3 Christoffel symbols

We examine the covariant derivative and the geodesic equation in local coordinates on an embedded manifold $M \subset \mathbb{R}^n$ of dimension m. Let $\phi : U \to \Omega$ be a coordinate chart on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$ and denote its inverse by $\psi : \Omega \to M$. Let

$$c = (c^1, \dots, c^m) : I \to \Omega$$

be a smooth curve in Ω, defined on an interval $I \subset \mathbb{R}$, and consider the curve

$$\gamma = \psi \circ c : I \to M$$

(see Figure 2.6). Our goal is to understand the equation $\nabla\dot{\gamma} = 0$ or, more generally, to describe the operator $X \mapsto \nabla X$ on the space of vector fields along γ in local coordinates.

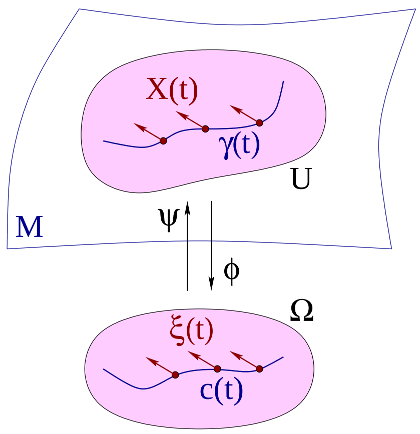


Figure 2.6: A vector field along a curve in local coordinates.

Let $X : I \to \mathbb{R}^n$ be a vector field along γ. Then

$$X(t) \in T_{\gamma(t)}M = T_{\psi(c(t))}M = \mathrm{im}\left(d\psi(c(t)) : \mathbb{R}^m \to \mathbb{R}^n\right)$$

for every $t \in I$ and hence there is a unique smooth function

$$\xi = (\xi^1, \dots, \xi^m) : I \to \mathbb{R}^m$$

such that

$$X(t) = d\psi(c(t))\xi(t) = \sum_{i=1}^{m} \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)). \tag{2.18}$$

Differentiating this identity we obtain

$$\dot{X}(t) = \sum_{i=1}^{m} \dot{\xi}^i(t) \frac{\partial \psi}{\partial x^i}(c(t)) + \sum_{i,j=1}^{m} \xi^i(t) \dot{c}^j(t) \frac{\partial^2 \psi}{\partial x^i \partial x^j}(c(t)). \tag{2.19}$$

We examine the projection $\nabla X(t) = \Pi(\gamma(t))\dot{X}(t)$ of this vector onto the tangent space of M at $\gamma(t)$. The first summand on the right in (2.19) is already tangent to M. For the second summand we simply observe that the vector $\Pi(\psi(x))\partial^2\psi/\partial x^i \partial x^j$ lies in tangent space $T_{\psi(x)}M$ and can therefore be expressed as a linear combination of the basis vectors $\partial\psi/\partial x^1, \ldots, \partial\psi/\partial x^m$. The coefficients will be denoted by $\Gamma_{ij}^k(x)$. Thus there are smooth functions $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ for $i, j, k = 1, \ldots, m$ defined by

$$\Pi(\psi(x)) \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x) = \sum_{k=1}^{m} \Gamma_{ij}^k(x) \frac{\partial \psi}{\partial x^k}(x) \tag{2.20}$$

for $x \in \Omega$ and $i, j \in \{1, \ldots, m\}$. The coefficients Γ_{ij}^k are called the **Christoffel symbols**. To sum up we have proved the following.

Lemma 2.20. *Let $c : I \to \Omega$ be a smooth curve and denote $\gamma := \psi \circ c : I \to M$. If $\xi : I \to \mathbb{R}^m$ is a smooth map and $X \in \mathrm{Vect}(\gamma)$ is given by* (2.18) *then its covariant derivative at time $t \in I$ is given by*

$$\nabla X(t) = \sum_{k=1}^{m} \left(\dot{\xi}^k(t) + \Gamma_{ij}^k(c(t)) \xi^i(t) \dot{c}^j(t) \right) \frac{\partial \psi}{\partial x^k}(c(t)), \tag{2.21}$$

where the Γ_{ij}^k are the Christoffel symbols defined by (2.20).

Corollary 2.21. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and $\phi : U \to \Omega$ be a coordinate chart on M with inverse $\psi := \phi^{-1} : \Omega \to U$. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be the Christoffel symbols defined by* (2.20). *Let $c : I \to \Omega$ be a smooth curve. Then the curve $\gamma := \psi \circ c : I \to M$ is a geodesic if and only if c satisfies the 2nd order differential equation*

$$\ddot{c}^k + \sum_{i,j=1}^{m} \Gamma_{ij}^k(c) \dot{c}^i \dot{c}^j = 0 \tag{2.22}$$

for $k = 1, \ldots, m$.

Proof. This follows immediately from the definition of geodesics and equation (2.21) in Lemma 2.20 with $X = \dot{\gamma}$ and $\xi = \dot{c}$. □

Corollary 2.22. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold.*
(i) *For every $p \in M$ and every $v \in T_pM$ there is an $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that*

$$\nabla\dot{\gamma} \equiv 0, \qquad \gamma(0) = p, \qquad \dot{\gamma}(0) = v. \tag{2.23}$$

(iI) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are geodesics and $t_0 \in I_1 \cap I_2$ with*

$$\gamma_1(t_0) = \gamma_2(t_0), \qquad \dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$$

then $\gamma_1(t) = \gamma_2(t)$ for all $t \in I_1 \cap I_2$.

Proof. Let $\phi : U \to \Omega$ be a coordinate chart on an open neighborhood $U \subset M$ of p and let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be the Christoffel symbols defined by (2.20) with $\psi := \phi^{-1}$. Let $x_0 := \phi(p)$ and $\xi_0 := d\phi(p)v$. Then $\gamma : (-\varepsilon, \varepsilon) \to U$ is a solution of (2.23) if and only if $c := \phi \circ \gamma$ is a solution of the second order differential equation (2.22) with the initial condition $c(0) = x_0$ and $\dot{c}(0) = \xi_0$. Hence it follows from the existence and uniqueness theorem for solutions of ordinary differential equation that the initial velue problem (2.23) has a solution $\gamma : (-\varepsilon, \varepsilon) \to U$ for some $\varepsilon > 0$ and that any two solutions of (2.23) with images in U agree on the intersection of their domains. To prove the global uniquess statement in Corollary 2.22 we define the set

$$A := \{t \in I_1 \cap I_2 \,|\, \gamma_1(t) = \gamma_2(t),\, \dot{\gamma}_1(t) = \dot{\gamma}_2(t)\}.$$

This set is obviously closed and nonempty because $t_0 \in A$. Moreover, it is open by the local uniqueness result just proved. Since $I_1 \cap I_2$ is connected we obtain $A = I_1 \cap I_2$ and this proves the corollary. □

Example 2.23. Let $S^m \subset \mathbb{R}^{m+1}$ be the unit sphere. Given a point $p \in S^m$ and a nonzero tangent vector $v \in T_pS^m = p^\perp$, the geodesic $\gamma : \mathbb{R} \to S^m$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ is given by

$$\gamma(t) = \cos(t\,|v|)p + \frac{\sin(t\,|v|)}{|v|}v$$

for $t \in \mathbb{R}$. Indeed, $\dot{\gamma}(t)$ is perpendicular to $\gamma(t)$ for every t. With $q := \gamma(1)$ we have $L(\gamma|_{[0,1]}) = |v|$ and, in the case $0 \le |v| \le \pi$ there is no shorter curve in S^m connecting p and q. (**Exercise:** Prove this!) Hence the intrinsic distance on S^m is given by

$$d(p, q) = \cos^{-1}(\langle p, q\rangle) \tag{2.24}$$

for $p, q \in S^m$.

Our next goal is to understand how the Christoffel symbols are determined by the metric in local coordinates. Recall from equation (2.5) that the inner products on the tangent spaces inherited from the standard Euclidean inner product on the ambient space $\mathbb{R}^n$ are in local coordinates represented by the matrix function

$$g = (g_{ij})_{i,j=1}^m : \Omega \to \mathbb{R}^{m \times m}$$

given by

$$g_{ij} := \left\langle \frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle_{\mathbb{R}^n}. \tag{2.25}$$

We shall see that the Christoffel symbols are completely determined by the functions $g_{ij} : \Omega \to \mathbb{R}$. Here are first some elementary observations.

Remark 2.24. The matrix $g(x) \in \mathbb{R}^{m \times m}$ is symmetric and positive definite for every $x \in \Omega$. This follows from the fact that the matrix $d\psi(x) \in \mathbb{R}^{n \times m}$ has rank m and

$$\langle \xi, g(x)\eta \rangle_{\mathbb{R}^m} = \langle d\psi(x)\xi, d\psi(x)\eta \rangle_{\mathbb{R}^n}$$

for all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^m$.

Remark 2.25. For $x \in \Omega$ we denote the entries of the inverse matrix $g(x)^{-1} \in \mathbb{R}^{m \times m}$ by $g^{k\ell}(x)$. They are determined by the condition

$$\sum_{j=1}^m g_{ij}(x) g^{jk}(x) = \delta_i^k = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Since $g(x)$ is symmetric and positive definite, so is its inverse matrix $g(x)^{-1}$. In particular we have $g^{k\ell}(x) = g^{\ell k}(x)$ for all $x \in \Omega$ and $k, \ell \in \{1, \dots, m\}$.

Remark 2.26. Suppose $X, Y \in \mathrm{Vect}(\gamma)$ are vector fields along our curve $\gamma = \psi \circ c : I \to M$ and $\xi, \eta : I \to \mathbb{R}^m$ are defined by

$$X(t) = \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)), \qquad Y(t) = \sum_{j=1}^m \eta^j(t) \frac{\partial \psi}{\partial x^j}(c(t)).$$

Then the inner product of X and Y is given by

$$\langle X(t), Y(t) \rangle = \sum_{i,j=1}^m g_{ij}(c(t)) \xi^i(t) \eta^j(t).$$

Theorem 2.27 (Christoffel symbols). *Let $\Omega \subset \mathbb{R}^m$ be an open set and*

$$g_{ij} : \Omega \to \mathbb{R}, \qquad i,j = 1,\dots,m,$$

be smooth functions such that each matrix $(g_{ij}(x))_{i,j=1}^m$ is symmetric and positive definite. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be smooth functions for $i,j,k = 1,\dots,m$. Then the Γ_{ij}^k satisfy the conditions

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \qquad \frac{\partial g_{ij}}{\partial x^k} = \sum_{\ell=1}^m \left(g_{i\ell}\Gamma_{jk}^\ell + g_{j\ell}\Gamma_{ik}^\ell \right) \tag{2.26}$$

for $i,j,k = 1,\dots,m$ if and only if they are given by

$$\Gamma_{ij}^k = \sum_{\ell=1}^m g^{k\ell} \frac{1}{2} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right). \tag{2.27}$$

If the Γ_{ij}^k are defined by (2.20) *and the g_{ij} by* (2.25)*, then the Γ_{ij}^k satisfy* (2.26) *and hence are given by* (2.27)*.*

Proof. Suppose that the Γ_{ij}^k are given by (2.20) and the g_{ij} by (2.25). Let $c : I \to \Omega$ and $\xi, \eta : I \to \mathbb{R}^m$ be smooth functions and suppose that the vector fields X, Y along the curve

$$\gamma := \psi \circ c : I \to M$$

are given by

$$X(t) := \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)), \qquad Y(t) := \sum_{j=1}^m \eta^j(t) \frac{\partial \psi}{\partial x^j}(c(t)).$$

Then, by Remark 2.26 and Lemma 2.20, we have

$$\begin{aligned}
\langle X, Y \rangle &= \sum_{i,j} g_{ij}(c) \xi^i \eta^j, \\
\langle X, \nabla Y \rangle &= \sum_{i,\ell} g_{i\ell}(c) \xi^i \left(\dot\eta^\ell + \Gamma_{jk}^\ell(c) \eta^j \dot c^k \right), \\
\langle \nabla X, Y \rangle &= \sum_{j,\ell} g_{\ell j}(c) \left(\dot\xi^\ell + \Gamma_{ik}^\ell(c) \xi^i \dot c^k \right) \eta^j.
\end{aligned}$$

Here we have dropped the argument t in each term. Hence it follows from equation (2.13) in Remark 2.15 that

$$\begin{aligned}
0 &= \frac{d}{dt}\langle X, Y\rangle - \langle X, \nabla Y\rangle - \langle \nabla X, Y\rangle \\
&= \sum_{i,j}\left(g_{ij}\dot{\xi}^i\eta^j + g_{ij}\xi^i\dot{\eta}^j + \sum_k \frac{\partial g_{ij}}{\partial x^k}\xi^i\eta^j\dot{c}^k\right) \\
&\quad -\sum_{i,\ell} g_{i\ell}\xi^i\dot{\eta}^\ell - \sum_{i,j,k,\ell} g_{i\ell}\Gamma_{jk}^\ell\xi^i\eta^j\dot{c}^k \\
&\quad -\sum_{j,\ell} g_{\ell j}\dot{\xi}^\ell\eta^j - \sum_{i,j,k,\ell} g_{\ell j}\Gamma_{ik}^\ell\xi^i\eta^j\dot{c}^k \\
&= \sum_{i,j,k}\left(\frac{\partial g_{ij}}{\partial x^k} - \sum_\ell g_{i\ell}\Gamma_{jk}^\ell - \sum_\ell g_{j\ell}\Gamma_{ik}^\ell\right)\xi^i\eta^j\dot{c}^k.
\end{aligned}$$

Since this equation holds for all smooth maps $c : I \to \Omega$ and $\xi, \eta : I \to \mathbb{R}^m$ we obtain that the Γ_{ij}^k satisfy the second equation in (2.26). That they are symmetric in i and j is obvious.

To prove that (2.26) is equivalent to (2.27) we define

$$\Gamma_{kij} := \sum_{\ell=1}^m g_{k\ell}\Gamma_{ij}^\ell. \tag{2.28}$$

Then equation (2.26) is equivalent to

$$\Gamma_{kij} = \Gamma_{kji}, \qquad \frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}. \tag{2.29}$$

and equation (2.27) is equivalent to

$$\Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{2.30}$$

If the Γ_{kij} are given by (2.30) then they obviously satisfy $\Gamma_{kij} = \Gamma_{kji}$ and

$$2\Gamma_{ijk} + 2\Gamma_{jik} = \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} = 2\frac{\partial g_{ij}}{\partial x^k}.$$

Conversely, if the Γ_{kij} satisfy (2.29) then we have

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial x^k} &= \Gamma_{ijk} + \Gamma_{jik}, \\
\frac{\partial g_{ki}}{\partial x^j} &= \Gamma_{kij} + \Gamma_{ikj} = \Gamma_{kij} + \Gamma_{ijk}, \\
\frac{\partial g_{kj}}{\partial x^i} &= \Gamma_{kji} + \Gamma_{jki} = \Gamma_{kij} + \Gamma_{jik}.
\end{aligned}$$

Taking the sum of the last two minus the first of these equations we obtain

$$\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} = 2\Gamma_{kij}.$$

Thus we have proved that (2.29) is equivalent to (2.30) and hence (2.26) is equivalent to (2.27). This proves the theorem. □

Exercise 2.28. Let $\Omega \subset \mathbb{R}^m$ be an open set and $g = (g_{ij}) : \Omega \to \mathbb{R}^{m \times m}$ be a smooth map with values in the space of positive definite symmetric matrices. Consider the energy functional

$$E(c) := \int_0^1 L(c(t), \dot{c}(t))\, dt$$

on the space of paths $c : [0,1] \to \Omega$, where $L : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$L(x, \xi) := \frac{1}{2} \sum_{i,j=1}^m \xi^i g_{ij}(x) \xi^j. \tag{2.31}$$

The **Euler–Lagrange equations** of this variational problem have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \xi^k}(c(t), \dot{c}(t)) = \frac{\partial L}{\partial x^k}(c(t), \dot{c}(t)), \qquad k = 1, \ldots, m. \tag{2.32}$$

Prove that the Euler–Lagrange equations (2.32) are equivalent to the geodesic equations (2.22), where the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are given by (2.27).

Exercise 2.29. Consider the case $m = 2$. Let $\Omega \subset \mathbb{R}^2$ be an open set and $\lambda : \Omega \to (0, \infty)$ be a smooth function. Suppose that the metric $g : \Omega \to \mathbb{R}^{2\times 2}$ is given by

$$g(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}.$$

Compute the Christoffel symbols Γ_{ij}^k via (2.27).

Exercise 2.30. Let $\phi : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection, given by

$$\phi(p) := \left(\frac{p_1}{1 - p_3}, \frac{p_2}{1 - p_3} \right)$$

Prove that the metric $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ has the form $g(x) = \lambda(x)\mathbb{1}$ where the function $\lambda : \mathbb{R}^2 \to (0, \infty)$ is given by

$$\lambda(x) := \frac{4}{(1 + |x|^2)^2}$$

for $x = (x^1, x^2) \in \mathbb{R}^2$.

2.4 The geodesic flow

2.4.1 The second fundamental form

Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. We explain how geodesics can be viewed as integral curves of a vector field on the tangent bundle TM. This requires some preparation. Recall that, for each $p \in M$, the orthogonal projection $\Pi(p) \in \mathbb{R}^{n\times n}$ of $\mathbb{R}^n$ onto the tangent space T_pM is characterized by the equations

$$\Pi(p)^2 = \Pi(p) = \Pi(p)^T, \qquad \operatorname{im}\Pi(p) = T_pM \tag{2.33}$$

and that the resulting map $\Pi : M \to \mathbb{R}^{n\times n}$ is smooth (see Theorem 1.100 and Corollary 1.101). Differentiating this map at a point $p \in M$ we obtain a linear map

$$d\Pi(p) : T_pM \to \mathbb{R}^{n\times n}$$

which, as usual, is defined by

$$d\Pi(p)v := \left.\frac{d}{dt}\right|_{t=0} \Pi(\gamma(t)) \in \mathbb{R}^{n\times n}$$

for $v \in T_pM$, where $\gamma : \mathbb{R} \to M$ is chosen such that $\gamma(0) = p$ and $\dot\gamma(0) = v$ (see Definition 1.31). We emphasize that the expression $d\Pi(p)v$ is a matrix and can therefore be multiplied by a vector in $\mathbb{R}^n$.

Lemma 2.31. *For all $p \in M$ and $v, w \in T_pM$ we have*

$$\big(d\Pi(p)v\big)w = \big(d\Pi(p)w\big)v \in T_pM^\perp.$$

Proof. Choose a smooth path $\gamma : \mathbb{R} \to M$ and a vector field $X : \mathbb{R} \to \mathbb{R}^n$ along γ such that $\gamma(0) = p$, $\dot\gamma(0) = v$, and $X(0) = w$. For example we can choose $X(t) := \Pi(\gamma(t))w$. Then we have

$$X(t) = \Pi(\gamma(t))X(t)$$

for every $t \in \mathbb{R}$. Differentiating this equation we obtain

$$\dot X(t) = \Pi(\gamma(t))\dot X(t) + \big(d\Pi(\gamma(t))\dot\gamma(t)\big)X(t). \tag{2.34}$$

Hence

$$\big(d\Pi(\gamma(t))\dot\gamma(t)\big)X(t) = \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot X(t) \in T_{\gamma(t)}M^\perp \tag{2.35}$$

for every $t \in \mathbb{R}$ and, with $t = 0$, we obtain $(d\Pi(p)v)\,w \in T_pM^\perp$.

Now choose a smooth map $\mathbb{R}^2 \to M : (s,t) \mapsto \gamma(s,t)$ satisfying

$$\gamma(0,0) = p, \qquad \frac{\partial\gamma}{\partial s}(0,0) = v, \qquad \frac{\partial\gamma}{\partial t}(0,0) = w,$$

(for example by doing this in local coordinates) and denote

$$X(s,t) := \frac{\partial\gamma}{\partial s}(s,t) \in T_{\gamma(s,t)}M, \qquad Y(s,t) := \frac{\partial\gamma}{\partial t}(s,t) \in T_{\gamma(s,t)}M.$$

Then $\frac{\partial Y}{\partial s} = \frac{\partial^2\gamma}{\partial s\partial t} = \frac{\partial X}{\partial t}$ and hence, using (2.35), we obtain

$$\begin{aligned}
\left(d\Pi(\gamma)\frac{\partial\gamma}{\partial t}\right)\frac{\partial\gamma}{\partial s} &= \left(d\Pi(\gamma)\frac{\partial\gamma}{\partial t}\right)X \\
&= \big(\mathbb{1} - \Pi(\gamma)\big)\frac{\partial X}{\partial t} \\
&= \big(\mathbb{1} - \Pi(\gamma)\big)\frac{\partial Y}{\partial s} \\
&= \left(d\Pi(\gamma)\frac{\partial\gamma}{\partial s}\right)Y \\
&= \left(d\Pi(\gamma)\frac{\partial\gamma}{\partial s}\right)\frac{\partial\gamma}{\partial t}.
\end{aligned}$$

With $s = t = 0$ we obtain $(d\Pi(p)w)\,v = (d\Pi(p)v)\,w$. This proves the lemma. □

Definition 2.32. *The collection of symmetric bilinear maps*

$$h_p : T_pM \times T_pM \to T_pM^\perp,$$

defined by

$$h_p(v,w) := (d\Pi(p)v)w = (d\Pi(p)w)v \tag{2.36}$$

for $p \in M$ and $v,w \in T_pM$ is called the **second fundamental form** *on M. The* **first fundamental form** *on M is the family of inner products on the tangent spaces T_pM induced by the Euclidean inner product on $\mathbb{R}^n$.*

Example 2.33. Let $M \subset \mathbb{R}^{m+1}$ be an m-manifold and $\nu : M \to S^m$ be a Gauss map so that $T_pM = \nu(p)^\perp$ for every $p \in M$. (See Example 2.9.) Then $\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^T$ and hence

$$h_p(v,w) = -\nu(p)\langle d\nu(p)v, w\rangle$$

for $p \in M$ and $v,w \in T_pM$.

Exercise 2.34. Choose a splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and write the elements of $\mathbb{R}^n$ as tuples $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$ Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold such that $p = 0 \in M$ and

$$T_0M = \mathbb{R}^m \times \{0\}, \qquad T_0M^\perp = \{0\} \times \mathbb{R}^{n-m}.$$

By the implicit function theorem, there are open neighborhoods $\Omega \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-m}$ of zero and a smooth map $f : \Omega \to V$ such that

$$M \cap (\Omega \times V) = \mathrm{graph}(f) = \{(x, f(y)) \,|\, x \in \Omega\}.$$

Thus $f(0) = 0$ and $df(0) = 0$. Prove that the second fundamental form $h_p : T_pM \times T_pM \to T_pM^\perp$ is given by the second derivatives of f, i.e.

$$h_p(v, w) = \left(0, \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right)$$

for $v, w \in T_pM = \mathbb{R}^m \times \{0\}$).

Exercise 2.35. Let $M \subset \mathbb{R}^n$ be an m-manifold. Fix a point $p \in M$ and a unit tangent vector $v \in T_pM$ so that $|v| = 1$ and define

$$L := \{p + tv + w \,|\, t \in \mathbb{R},\ w \perp T_pM\}.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to M \cap L$ be a smooth curve such that $\gamma(0) = p$, $\dot\gamma(0) = v$, and $|\dot\gamma(t)| = 1$ for all t. Prove that

$$\ddot\gamma(0) = h_p(v, v).$$

Draw a picture of M and L in the case $n = 3$ and $m = 2$.

Equation (2.34) in the proof of Lemma 2.31 shows that the derivative of a vector field X along a curve γ is given by **Gauss–Weingarten formula**

$$\dot X(t) = \nabla X(t) + h_{\gamma(t)}(\dot\gamma(t), X(t)). \tag{2.37}$$

Here the first summand is tangent to M and the second summand is orthogonal to the tangent space of M at $\gamma(t)$. Applying the Gauss–Weingarten formula to the vector field $X = \dot\gamma$ we obtain

$$\ddot\gamma = \nabla\dot\gamma + h_\gamma(\dot\gamma, \dot\gamma). \tag{2.38}$$

By definition γ is a geodesic if and only if $\nabla\dot\gamma = 0$ and, by equation (2.38), this means that γ satisfies the equation

$$\ddot\gamma = h_\gamma(\dot\gamma, \dot\gamma). \tag{2.39}$$

2.4.2 The tangent bundle of the tangent bundle

The tangent bundle TM is a smooth $2m$-dimensional manifold in $\mathbb{R}^n \times \mathbb{R}^n$. The tangent space of TM at a point $(p, v) \in TM$ can be expressed in terms of the second fundamental form as

$$T_{(p,v)}TM = \left\{(\hat{p}, \hat{v}) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \hat{p} \in T_pM,\ \big(\mathbb{1} - \Pi(p)\big)\hat{v} = h_p(\hat{p}, v)\right\}. \quad (2.40)$$

By the Gauss–Weingarten formula the derivative of a curve $t \mapsto (\gamma(t), X(t))$ in TM satisfies $\big(\mathbb{1}-\Pi(\gamma(t))\big)\dot{X}(t) = h_{\gamma(t)}(\dot{\gamma}(t), X(t))$ for every t. This proves the inclusion "$\subset$" in (2.40). Equality follows from the fact that both sides of the equation are $2m$-dimensional linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$. Now it follows from (2.40) that the formula

$$Y(p, v) := (v, h_p(v, v)) \in T_{(p,v)}TM$$

for $p \in M$ and $v \in T_pM$ defines a vector field on TM. By (2.38) the integral curves of Y have the form $(\gamma, \dot{\gamma})$, where $\gamma : I \to M$ is a geodesic. Hence, using Theorem 1.52, we obtain another proof of Corollary 2.22 which asserts that, for every $p \in M$ and every $v \in T_pM$, there is a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$, for some $\varepsilon > 0$, satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. In fact, we also obtain smooth dependence of the geodesic on the initial condition.

2.4.3 The exponential map

In general the geodesic through p in the direction $v \in T_pM$ can be defined on a **maximal existence interval**

$$I_{p,v} := \bigcup \left\{ I \subset \mathbb{R} \,\middle|\, \begin{array}{l} I \text{ is an open interval containing } 0 \text{ and there is a} \\ \text{geodesic } \gamma : I \to M \text{ satisfying } \gamma(0) = p,\ \dot{\gamma}(0) = v \end{array} \right\}.$$

Define the set $V_p \subset T_pM$ by

$$V_p := \{v \in T_pM \,|\, 1 \in I_{p,v}\}. \quad (2.41)$$

The **exponential map**

$$\exp_p : V_p \to M$$

of M at p assigns to every tangent vector $v \in V_p$ the point $\exp_p(v) := \gamma(1)$, where $\gamma : I_{p,v} \to M$ is the unique geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. It follows from Theorem 1.54 that the disjoint union of the V_p is an open subset of TM and that the exponential map

$$\bigcup_{p \in M} \{p\} \times V_p \to M : (p, v) \mapsto \exp_p(v)$$

is smooth.

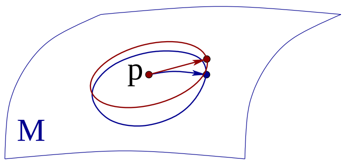


Figure 2.7: The exponential map.

Lemma 2.36. *Let $p \in M$ and $v \in V_p$. Then*

$$I_{p,v} = \{t \in \mathbb{R} \,|\, tv \in V_p\}$$

and the geodesic $\gamma : I_{p,v} \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ is given by

$$\gamma(t) = \exp_p(tv), \qquad t \in I_{p,v}.$$

Proof. Let $\gamma : I_{p,v} \to M$ be the unique geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Fix a nonzero real number λ and define the map $\gamma_\lambda : \lambda^{-1}I_{p,v} \to M$ by

$$\gamma_\lambda(t) := \gamma(\lambda t).$$

Then $\dot\gamma_\lambda(t) = \lambda\dot\gamma(\lambda t)$ and $\ddot\gamma_\lambda(t) = \lambda^2\ddot\gamma(\lambda t)$. Hence

$$\nabla\dot\gamma_\lambda(t) = \Pi(\gamma_\lambda(t))\ddot\gamma_\lambda(t) = \lambda^2\Pi(\gamma(\lambda t))\ddot\gamma(\lambda t) = \lambda^2\nabla\dot\gamma(\lambda t) = 0$$

for every $t \in \lambda^{-1}I_{p,v}$. Hence γ_λ is a geodesic and $\gamma_\lambda(0) = p$ and $\dot\gamma_\lambda(0) = \lambda v$. In particular, we have $\lambda^{-1}I_{p,v} \subset I_{p,\lambda v}$. Interchanging the roles of v and λv we obtain $\lambda^{-1}I_{p,v} = I_{p,\lambda v}$. Thus

$$\lambda \in I_{p,v} \qquad \iff \qquad 1 \in I_{p,\lambda v} \qquad \iff \qquad \lambda v \in V_p$$

and $\gamma(\lambda) = \gamma_\lambda(1) = \exp_p(\lambda v)$ for $\lambda \in I_{p,v}$. This proves the lemma. □

Corollary 2.37. *The map $\exp_p : V_p \to M$ is smooth and its derivative at the origin is*

$$d\exp_p(0) = \mathrm{id} : T_pM \to T_pM.$$

Proof. The set V_p is an open subset of the linear subspace $T_pM \subset \mathbb{R}^n$, with respect to the relative topology, and hence is a manifold. The tangent space of V_p at each point is T_pM. That the exponential map is smooth follows from Theorem 1.54. By Lemma 2.36 its derivative at the origin is

$$d\exp_p(0)v = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \dot\gamma(0) = v,$$

where $\gamma : I_{p,v} \to M$ is once again the unique geodesic through p in the direction v. This proves the corollary. □

Corollary 2.38. *Let $p \in M$ and, for $r > 0$, denote*

$$B_r(p) := \{v \in T_pM \mid |v| < r\}.$$

If $r > 0$ is sufficiently small then $B_r(p) \subset V_p$, the set

$$U_r(p) := \exp_p(B_r(p))$$

is an open subset of M, and the restriction of the exponential map to $B_r(p)$ is a diffeomorphism from $B_r(p)$ to $U_r(p)$.

Proof. This follows from Corollary 2.37 and Theorem 1.34. □

Definition 2.39. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. The* **injectivity radius of** M **at** p *is the supremum of all $r > 0$ such that the restriction of the exponential map $\exp_p$ to $B_r(p)$ is a diffeomorphism onto its image $U_r(p) := \exp_p(B_r(p))$. It will be denoted by*

$$\mathrm{inj}(p) := \mathrm{inj}(p; M) := \sup \left\{ r > 0 \,\middle|\, \begin{array}{c} \exp_p : B_r(p) \to U_r(p) \\ \textit{is a diffeomorphism} \end{array} \right\}.$$

The **injectivity radius of** M *is the infimum of the injectivity radii of M at p over all $p \in M$. It will be denoted by*

$$\mathrm{inj}(M) := \inf_{p \in M} \mathrm{inj}(p; M).$$

2.4.4 Examples and exercises

Example 2.40. The exponential map on $\mathbb{R}^m$ is given by

$$\exp_p(v) = p + v$$

for $p, v \in \mathbb{R}^m$. For every $p \in \mathbb{R}^m$ this map is a diffeomorphism from $T_p\mathbb{R}^m = \mathbb{R}^m$ to $\mathbb{R}^m$ and hence the injectivity radius of $\mathbb{R}^m$ is infinity.

Example 2.41. The exponential map on S^m is given by

$$\exp_p(v) = \cos(|v|)p + \frac{\sin(|v|)}{|v|} v$$

for every $p \in S^m$ and every nonzero tangent vector $v \in T_pS^m = p^\perp$. (See Exercise 2.23.) The restriction of this map to the open ball of radius r in T_pM is a diffeomorphism onto its image if and only if $r \leq \pi$. Hence the injectivity radius of S^m at every point is π.

Example 2.42. Consider the orthogonal group $\mathrm{O}(n) \subset \mathbb{R}^{n\times n}$ with the standard inner product

$$\langle v, w\rangle := \mathrm{trace}(v^T w)$$

on $\mathbb{R}^{n\times n}$. The orthogonal projection $\Pi(g) : \mathbb{R}^{n\times n} \to T_g\mathrm{O}(n)$ is given by

$$\Pi(g)v := \frac{1}{2}(v - gv^T g)$$

and the second fundamental form by $h_g(v, v) = -gv^T v$. Hence a curve $\gamma : \mathbb{R} \to \mathrm{O}(n)$ is a geodesic if and only if $\gamma^T\ddot{\gamma} + \dot{\gamma}^T\dot{\gamma} = 0$ or, equivalently, $\gamma^T\dot{\gamma}$ is constant. So geodesics in $\mathrm{O}(n)$ have the form $\gamma(t) = g\exp(t\xi)$ for $g \in \mathrm{O}(n)$ and $\xi \in \mathfrak{o}(n)$. It follows that the exponential map is given by

$$\exp_g(v) = g\exp(g^{-1}v) = \exp(vg^{-1})g$$

for $g \in \mathrm{O}(n)$ and $v \in T_g\mathrm{O}(n)$. In particular, for $g = \mathbb{1}$ the exponential map $\exp_{\mathbb{1}} : \mathfrak{o}(n) \to \mathrm{O}(n)$ agrees with the exponential matrix.

Exercise 2.43. What is the injectivity radius of the 2-torus $\mathbb{T}^2 = S^1 \times S^1$, the punctured 2-plane $\mathbb{R}^2 \setminus \{(0, 0)\}$, and the orthogonal group $\mathrm{O}(n)$?

2.4.5 Geodesics minimize the length

Theorem 2.44. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold, fix a point $p \in M$, and let $r > 0$ be smaller than the injectivity radius of M at p. Let $v \in T_pM$ such that $|v| < r$. Then*

$$d(p, q) = |v|, \qquad q := \exp_p(v),$$

and a curve $\gamma \in \Omega_{p,q}$ has minimal length $L(\gamma) = |v|$ if and only if there is a smooth map $\beta : [0, 1] \to [0, 1]$ satisfying $\beta(0) = 0$, $\beta(1) = 1$, $\dot{\beta} \geq 0$ such that

$$\gamma(t) = \exp_p(\beta(t)v).$$

Lemma 2.45 (Gauss Lemma). *Let M, p, and r be as in Theorem 2.44. Let $I \subset \mathbb{R}$ be an interval and $w : I \to V_p$ be a smooth map whose norm $|w(t)| =: r$ is constant. Define*

$$\alpha(s, t) := \exp_p(sw(t))$$

for $(s, t) \in \mathbb{R} \times I$ with $sw(t) \in V_p$. Then

$$\left\langle \frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t} \right\rangle \equiv 0.$$

Thus the geodesics through p are orthogonal to the boundaries of the embedded balls $U_r(p)$ in Corollary 2.38 (see Figure 2.8).

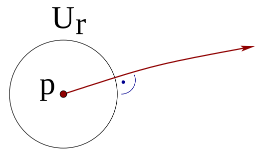


Figure 2.8: The Gauss Lemma.

Proof. For every $t \in I$ we have $\alpha(0,t) = \exp_p(0) = p$ and so the assertion holds for $s = 0$:

$$\left\langle \frac{\partial \alpha}{\partial s}(0,t), \frac{\partial \alpha}{\partial t}(0,t) \right\rangle = 0.$$

Moreover, each curve $s \mapsto \alpha(s,t)$ is a geodesic, i.e.

$$\nabla_s \frac{\partial \alpha}{\partial s} = \Pi(\alpha)\frac{\partial^2 \alpha}{\partial s^2} \equiv 0.$$

By Theorem 2.17, the function $s \mapsto \left|\frac{\partial \alpha}{\partial s}(s,t)\right|$ is constant for every t, so that

$$\left|\frac{\partial \alpha}{\partial s}(s,t)\right| = \left|\frac{\partial \alpha}{\partial s}(0,t)\right| = |w(t)| = r$$

for all s and t. This implies

$$\begin{aligned}
\frac{\partial}{\partial s}\left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle &= \left\langle \nabla_s \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle + \left\langle \frac{\partial \alpha}{\partial s}, \nabla_s \frac{\partial \alpha}{\partial t} \right\rangle \\
&= \left\langle \frac{\partial \alpha}{\partial s}, \Pi(\alpha)\frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \left\langle \Pi(\alpha)\frac{\partial \alpha}{\partial s}, \frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial \alpha}{\partial s}\right|^2 \\
&= 0.
\end{aligned}$$

Since the function $\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \rangle$ vanishes for $s = 0$ we obtain

$$\left\langle \frac{\partial \alpha}{\partial s}(s,t), \frac{\partial \alpha}{\partial t}(s,t) \right\rangle = 0$$

for all s and t. This proves the lemma. □

Proof of Theorem 2.44. Let $r > 0$ be as in Corollary 2.38 and let $v \in T_pM$ such that $0 < |v| =: \varepsilon < r$. Denote $q := \exp_p(v)$ and let $\gamma \in \Omega_{p,q}$. Assume first that

$$\gamma(t) \in \exp_p\left(\overline{B}_\varepsilon(p)\right) = \overline{U}_\varepsilon \qquad \forall\, t \in [0,1].$$

Then there is a unique smooth function $[0,1] \to T_pM : t \mapsto v(t)$ such that $|v(t)| \le \varepsilon$ and $\gamma(t) = \exp_p(v(t))$ for every t. The set

$$I := \{t \in [0,1] \,|\, \gamma(t) \ne p\} = \{t \in [0,1] \,|\, v(t) \ne 0\} \subset (0,1]$$

is open in the relative topology of $(0,1]$. Thus I is a union of open intervals in $(0,1)$ and one half open interval containing 1. Define $\beta : [0,1] \to [0,1]$ and $w : I \to T_pM$ by

$$\beta(t) := \frac{|v(t)|}{\varepsilon}, \qquad w(t) := \frac{v(t)}{\beta(t)}.$$

Then β is continuous, both β and w are smooth on I, $\beta(0) = 0$, $\beta(1) = 1$, $w(1) = v$, and

$$|w(t)| = \varepsilon, \qquad \gamma(t) = \exp_p(\beta(t)w(t)) \qquad \forall\, t \in I.$$

We prove that $L(\gamma) \ge \varepsilon$. To see this let $\alpha : [0,1] \times I \to M$ be the map of Lemma 2.45:

$$\alpha(s,t) := \exp_p(sw(t)).$$

Then $\gamma(t) = \alpha(\beta(t), t)$ and hence

$$\dot\gamma(t) = \dot\beta(t)\frac{\partial\alpha}{\partial s}(\beta(t),t) + \frac{\partial\alpha}{\partial t}(\beta(t),t)$$

for every $t > 0$. Hence it follows from Lemma 2.45 that

$$|\dot\gamma(t)|^2 = \dot\beta(t)^2\left|\frac{\partial\alpha}{\partial s}(\beta(t),t)\right|^2 + \left|\frac{\partial\alpha}{\partial t}(\beta(t),t)\right|^2 \ge \dot\beta(t)^2\varepsilon^2$$

for every $t \in I$. Hence

$$L(\gamma) = \int_0^1 |\dot\gamma(t)| = \int_I |\dot\gamma(t)|\, dt \ge \varepsilon \int_I \left|\dot\beta(t)\right|\, dt \ge \varepsilon \int_I \dot\beta(t)\, dt = \varepsilon.$$

The last equation follows by applying the fundamental theorem of calculus to each interval in I and using the fact that $\beta(0) = 0$ and $\beta(1) = 1$. If $L(\gamma) = \varepsilon$ we must have

$$\frac{\partial\alpha}{\partial t}(\beta(t),t) = 0, \qquad \dot\beta(t) \ge 0 \qquad \forall\, t \in I.$$

Thus I is a single half open interval containing 1 and on this interval the condition $\frac{\partial\alpha}{\partial t}(\beta(t),t) = 0$ implies $\dot{w}(t) = 0$. Since $w(1) = v$ we have $w(t) = v$ for every $t \in I$. Hence $\gamma(t) = \exp_p(\beta(t)v)$ for every $t \in [0,1]$. It follows that β is smooth on the closed interval $[0,1]$ (and not just on I). Thus we have proved that every $\gamma \in \Omega_{p,q}$ with values in $\overline{U}_\varepsilon$ has length $L(\gamma) \geq \varepsilon$ with equality if and only if γ is a reparametrized geodesic. But if $\gamma \in \Omega_{p,q}$ does not take values only in $\overline{U}_\varepsilon$, there must be a $T \in (0,1)$ such that $\gamma([0,T]) \subset \overline{U}_\varepsilon$ and $\gamma(T) \in \partial U_\varepsilon$. Then $L(\gamma|_{[0,T]}) \geq \varepsilon$, by what we have just proved, and $L(\gamma|_{[T,1]}) > 0$ because the restriction of γ to $[T,1]$ cannot be constant; so in this case we have $L(\gamma) > \varepsilon$. This proves the theorem. □

The next corollary gives a first answer to our problem of finding length minimizing curves. It asserts that geodesics minimize the length *locally.*

Corollary 2.46. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold, $I \subset \mathbb{R}$ be an open interval, and $\gamma : I \to M$ be a geodesic. Fix a point $t_0 \in I$. Then there is a constant $\varepsilon > 0$ such that*

$$t_0 - \varepsilon < s < t < t_0 + \varepsilon \qquad \Longrightarrow \qquad L(\gamma|_{[s,t]}) = d(\gamma(s),\gamma(t)).$$

Proof. Since γ is a geodesic its derivative has constant norm

$$|\dot{\gamma}(t)| \equiv c$$

(see Theorem 2.17). Choose $\delta > 0$ so small that the interval $[t_0 - \delta, t_0 + \delta]$ is contained in I. Then there is a constant $r > 0$ such that $r \leq \mathrm{inj}(\gamma(t))$ whenever $|t - t_0| \leq \delta$. Choose $\varepsilon > 0$ such that

$$\varepsilon < \delta, \qquad 2\varepsilon c < r.$$

If $t_0 - \varepsilon < s < t < t_0 + \varepsilon$ then

$$\gamma(t) = \exp_{\gamma(s)}\left((t-s)\dot{\gamma}(s)\right)$$

and

$$|(t-s)\dot{\gamma}(s)| = |t-s|\,c < 2\varepsilon c < r \leq \mathrm{inj}(\gamma(s)).$$

Hence it follows from Theorem 2.44 that

$$L(\gamma|_{[s,t]}) = |t-s|\,c = d(\gamma(s),\gamma(t)).$$

This proves the corollary. □

Exercise 2.47. How large can you choose the constant ε in Corollary 2.46 in the case $M = S^2$? Compare this with the injectivity radius.

2.4.6 Convexity

Definition 2.48. *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional manifold. A subset $U \subset M$ is called* **geodesically convex** *if, for any two points $p_0, p_1 \in U$, there is a unique geodesic $\gamma : [0,1] \to U$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.*

Exercise 2.49. **(a)** Find a geodesically convex set U in a manifold M and points $p_0, p_1 \in U$ such that the unique geodesic $\gamma : [0,1] \to U$ with $\gamma(0) = p_0$ and $\gamma(1) = p_1$ has length $L(\gamma) > d(p_0, p_1)$.
(b) Find a set U in a manifold M such that any two points in U can be connected by a minimal geodesic in U, but U is not geodesically convex.

Theorem 2.50 (Convex neighborhoods). *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold and fix a point $p_0 \in M$. Let $\phi : U \to \Omega$ be any coordinate chart on an open neighborhood $U \subset M$ of p_0 with values in an open set $\Omega \subset \mathbb{R}^m$. Then the set*

$$U_r := \{p \in U \mid |\phi(p) - \phi(p_0)| < r\}$$

is geodesically convex for $r > 0$ sufficiently small.

Proof. Assume without loss of generality that $\phi(p_0) = 0$. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be the Christoffel symbols of the coordinate chart and, for $x \in \Omega$, define the quadratic function $Q_x : \mathbb{R}^m \to \mathbb{R}$ by

$$Q_x(\xi) := \sum_{k=1}^m \left(\xi^k\right)^2 + \sum_{i,j,k=1}^m x^k \Gamma_{ij}^k(x) \xi^i \xi^j.$$

Then $Q_0(\xi) = |\xi|^2$ and so Q_x continues to be positive definite for $|x|$ sufficiently small. In other words there is a constant $\rho > 0$ such that, for all $x, \xi \in \mathbb{R}^m$, we have

$$|x| \le \rho,\ \xi \neq 0 \qquad \Longrightarrow \qquad x \in \Omega,\ \ Q_x(\xi) > 0.$$

Let $\gamma : [0,1] \to \overline{U}_\rho$ be a geodesic and define $c(t) := \phi(\gamma(t))$ for $0 \le t \le 1$. Then $|c(t)| \le \rho$ for every t and, by Corollary 2.21, c satisfies the differential equation $\ddot{c}^k + \sum_{i,j} \Gamma_{ij}^k(c) \dot{c}^i \dot{c}^j = 0$. Hence

$$\frac{d^2}{dt^2} \frac{|c|^2}{2} = \frac{d}{dt} \langle \dot{c}, c \rangle = |\dot{c}|^2 + \langle \ddot{c}, c \rangle = Q_c(\dot{c}) \ge 0$$

and so the function $t \mapsto |\phi(\gamma(t))|^2$ is convex.

Now choose $\delta > 0$ and $r_0 > 0$ so small that, for all $p, q \in M$, we have

$$p \in U_{r_0},\ \ d(p,q) < \delta \qquad \Longrightarrow \qquad q \in U_\rho, \tag{2.42}$$

$$p \in \overline{U}_\rho \qquad \Longrightarrow \qquad \delta < \mathrm{inj}(p), \tag{2.43}$$

$$p, q \in U_{r_0} \qquad \Longrightarrow \qquad d(p,q) < \delta. \tag{2.44}$$

We prove that U_r is geodesically convex for $0 < r \le r_0$. Fix two points $p, q \in U_r$. Then $p, q \in U_\rho$ by (2.42) and, by (2.43) and (2.44), we have

$$d(p,q) < \delta < \mathrm{inj}(p).$$

Hence, by Theorem 2.44, there is a vector $v \in T_pM$ such that $|v| = d(p,q)$ and $\exp_p(v) = q$. Define

$$\gamma(t) := \exp_p(tv), \qquad 0 \le t \le 1.$$

Then γ is a geodesic with endpoints $\gamma(0) = p$ and $\gamma(1) = q$. Moreover, by Theorem 2.44, we have

$$d(p, \gamma(t)) = t\,|v| = td(p,q) < \delta$$

for $0 \le t \le 1$ and hence $\gamma(t) \in U_\rho$ by (2.42). This implies that the function $t \mapsto |\phi(\gamma(t))|^2$ is convex and therefore

$$|\phi(\gamma(t))|^2 \le (1-t)\,|\phi(\gamma(p))|^2 + t\,|\phi(\gamma(q))|^2 \le r^2$$

for $0 \le t \le 1$. Hence $\gamma([0,1]) \subset U_r$.

We prove that there is no other geodesic from p to q whose image is contained in U_r. To see this, let $\gamma' : [0,1] \to U_r$ be any geodesic connecting p to q. Then there is a vector $v' \in T_pM$ such that

$$\exp_p(v') = q, \qquad \gamma'(t) = \exp_p(tv')$$

for $0 \le t \le 1$. Since $\gamma'(t) \in U_r \subset U_{r_0}$ it follows from (2.43) and (2.44) that

$$d(p, \gamma'(t)) < \delta < \mathrm{inj}(p)$$

for all t. If $|v'| \ge \mathrm{inj}(p)$ then there is a $t \in [0,1]$ such that $\delta < t\,|v'| < \mathrm{inj}(p)$ and then, by Theorem 2.44, we have

$$d(p, \gamma'(t)) = d(p, \exp_p(tv')) = t\,|v'| > \delta,$$

a contradiction. Hence $|v'| < \mathrm{inj}(p)$. Since

$$\exp_p(v') = q = \exp_p(v)$$

it then follows from the definition of the injectivity radius that $v' = v$ and so $\gamma'(t) = \gamma(t)$ for all t. This proves the theorem. $\square$

Remark 2.51. Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Fix a point $p \in M$ and a real number $0 < r < \mathrm{inj}(p)$ and denote

$$\Omega_r := \{x \in \mathbb{R}^m \mid |x| < r\}, \qquad U_r := \{q \in M \mid d(p,q) < r\}.$$

By Corollary 2.38 and Theorem 2.44 the map

$$\exp_p : \{v \in T_pM \mid |v| < r\} \to U_r$$

is a diffeomorphism. Hence any orthonormal basis $e_1, \dots, e_m$ of T_pM gives rise to a coordinate chart

$$\phi : U_r \to \Omega_r, \qquad \phi^{-1}(x) := \exp_p\left(\sum_{i=1}^m x^i e_i\right)$$

This coordinate chart sends geodesics through p to straight lines through the origin. One calls the components $x^1, \dots, x^m$ of ϕ **geodesically normal coordinates** at p.

Exercise 2.52. By Theorem 2.50 the set U_r in Remark 2.51 is geodesically convex for r sufficiently small. How large can you choose r in the cases

$$M = S^2, \qquad M = \mathbb{T}^2 = S^1 \times S^1, \qquad M = \mathbb{R}^2, \qquad M = \mathbb{R}^2 \setminus \{0\}.$$

Compare this with the injectivity radius. If the set U_r in these examples is geodesically convex, does it follows that every geodesic in U_r is minimizing?

2.5 The Hopf–Rinow theorem

For a Riemannian manifold there are different notions of completeness. First there is a distance function $d : M \times M \to [0, \infty)$ defined by (2.3) so that we can speak of completeness of the metric space (M, d) in the sense that every Cauchy sequence converges. Second there is the question if geodesics through any point in any direction exist for all time; if so we call a Riemannian manifold geodesically complete. The remarkable fact is that these two rather different notions of completeness are actually equivalent and that, in the complete case, any two points in M can be connected by a shortest geodesic. This is the content of the Hopf–Rinow theorem. We will spell out the details of the proof for embedded manifolds and leave it to the reader (as a straight forward exercise) to extend the proof to the intrinsic setting.

Definition 2.53. *Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold. Given a point $p \in M$ we say that M* **is geodesically complete at** *p if, for every $v \in T_pM$, there is a geodesic $\gamma : \mathbb{R} \to M$ (on the entire real axis) satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$ (or equivalently $V_p = T_pM$ where $V_p \subset T_pM$ is defined by (2.41)). M is called* **geodesically complete** *if it is geodesically complete at every point $p \in M$.*

Definition 2.54. *Let (M, d) be a metric space. A subset $A \subset M$ is called* **bounded** *if*

$$\sup_{p\in A} d(p, p_0) < \infty$$

for some (and hence every) point $p_0 \in M$.

Example 2.55. A manifold $M \subset \mathbb{R}^n$ can be contained in a bounded subset of $\mathbb{R}^n$ and still not be bounded with respect to the metric (2.3). An example is the 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ y = \sin(1/x)\}$.

Exercise 2.56. Let (M, d) be a metric space. Prove that every compact subset $K \subset M$ is closed and bounded. Find an example of a metric space that contains a closed and bounded subset that is not compact.

Theorem 2.57 (**Completeness**)**.** *Let $M \subset \mathbb{R}^n$ be a connected m-dimensional manifold and let $d : M \times M \to [0, \infty)$ be the distance function defined by (2.1), (2.2), and (2.3). Then the following are equivalent.*

(i) *There is a point $p \in M$ such that M is geodesically complete at p.*

(ii) *M is geodesically complete.*

(iii) *(M, d) is a complete metric space.*

(iv) *Every closed and bounded subset of M is compact.*

Theorem 2.58 (**Hopf–Rinow**)**.** *Let $M \subset \mathbb{R}^n$ be a connected m-manifold and let $p \in M$. Assume M is geodesically complete at p. Then, for every $q \in M$, there is a geodesic $\gamma : [0, 1] \to M$ such that*

$$\gamma(0) = p, \qquad \gamma(1) = q, \qquad L(\gamma) = d(p, q).$$

Lemma 2.59. *Let X be a vector field on M and $\gamma : (0, T) \to M$ be an integral curve of X such that the limit*

$$p_0 = \lim_{t\to 0} \gamma(t)$$

exists. Define $\gamma_0 : [0, T) \to M$ by

$$\gamma_0(t) := \begin{cases} p_0, & \text{for } t = 0, \\ \gamma(t), & \text{for } 0 < t < T. \end{cases}$$

Then γ_0 is differentiable at $t = 0$ and $\dot\gamma_0(0) = X(p_0)$.

Proof. Fix a constant $\varepsilon > 0$, choose $\rho > 0$ so small that

$$p \in M, \quad |p - p_0| \le \rho \qquad \Longrightarrow \qquad |X(p) - X(p_0)| \le \varepsilon,$$

and choose $\delta > 0$ so small that

$$0 < t \le \delta \qquad \Longrightarrow \qquad |\gamma(t) - p_0| \le \rho.$$

Then, for $0 < s < t \le \delta$, we have

$$\begin{aligned} |\gamma(t) - \gamma(s) - (t-s)X(p_0)| &= \left| \int_s^t (\dot\gamma(r) - X(p_0))\, dr \right| \\ &= \left| \int_s^t (X(\gamma(r)) - X(p_0))\, dr \right| \\ &\le \int_s^t |X(\gamma(r)) - X(p_0)|\, dr \\ &\le (t-s)\varepsilon \\ &\le t\varepsilon. \end{aligned}$$

Taking the limit $s \to 0$ we obtain

$$\left| \frac{\gamma(t) - p_0}{t} - X(p_0) \right| = \lim_{s \to 0} \frac{|\gamma(t) - \gamma(s) - (t-s)X(p_0)|}{t} \le \varepsilon$$

for $0 < t < \delta$. This proves the lemma. □

Theorem 2.58 implies Theorem 2.57. We prove that (iv) implies (iii). Thus we assume that every closed and bounded subset of M is compact and choose a Cauchy sequence $p_i \in M$. Choose i_0 such that, for all $i, j \in \mathbb{N}$, we have

$$i, j \ge i_0 \qquad \Longrightarrow \qquad d(p_i, p_j) \le 1,$$

and define

$$c := \max_{1 \le i \le i_0} d(p_1, p_i) + 1.$$

Then, for $i \ge i_0$, we have

$$d(p_1, p_i) \le d(p_1, p_{i_0}) + d(p_{i_0}, p_i) \le d(p_1, p_{i_0}) + 1 \le c.$$

Thus $d(p_1, p_i) \le c$ for every $i \in \mathbb{N}$. Hence the set $\{p_i \,|\, i \in I\}$ is bounded and so is its closure. By (iv) this implies that the sequence p_i has a convergent subsequence. Since p_i is a Cauchy sequence, this implies that p_i converges. Thus we have proved that (iv) implies (iii).

We prove that (iii) implies (ii). Assume, by contradiction, that M is not geodesically complete. Then there is a point $p \in M$ and a tangent vector $v \in T_pM$ such that the maximal existence interval $I_{p,v}$ is not equal to $\mathbb{R}$. Replacing v by $-v$, if necessary, we may assume that there is a $T > 0$ such that $[0, T) \subset I_{p,v}$ and $T \notin I_{p,v}$. Let $\gamma : I_{p,v} \to M$ be the geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then

$$|\dot{\gamma}(t)| = |v|$$

for every $t \in [0, T)$ and hence

$$d(\gamma(s), \gamma(t)) \le L(\gamma|_{[s,t]}) \le (t - s) \, |v|$$

for $0 < s < t < T$. By (iii) this implies that the limit

$$p_1 := \lim_{t \to T} \gamma(t)$$

exists in M. Now choose a compact neighborhood $K \subset M$ of p_1 and assume, without loss of generality, that $\gamma(t) \in K$ for $0 \le t < T$. Then there is a constant $c > 0$ such that

$$|h_q(w, w)| \le c \, |w|^2$$

for all $q \in K$ and $w \in T_qM$. Hence

$$|\ddot{\gamma}(t)| = \left| h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \right| \le c \, |\dot{\gamma}(t)|^2 = c \, |v|^2 , \qquad 0 \le t < T.$$

This in turn implies

$$|\dot{\gamma}(s) - \dot{\gamma}(t)| \le \int_s^t |\ddot{\gamma}(r)| \; dr \le (t - s) c \, |v|^2$$

for $0 < s < t < T$. It follows that the limit

$$v_1 := \lim_{t \to T} \dot{\gamma}(t)$$

exists in $\mathbb{R}^n$. Since $\gamma(t)$ converges to p_1 as t tends to T, we have

$$v_1 = \lim_{t \to T} \Pi(\gamma(t)) \dot{\gamma}(t) = \Pi(p_1) v_1 \in T_{p_1}M.$$

Thus $(\gamma(t), \dot{\gamma}(t))$ converges to $(p_1, v_1) \in TM$ as t tends to T. Since $(\gamma, \dot{\gamma})$ is an integral curve of a vector field on TM, it follows from Lemma 2.59 that γ extends to a geodesic on the interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$. This contradicts our assumption and proves that (iii) implies (ii). That (ii) implies (i) is obvious.

We prove that (i) implies (iv). Let $K \subset M$ be a closed and bounded subset. Then

$$r := \sup_{q \in K} d(p_0, q) < \infty.$$

Hence it follows from Theorem 2.58 that, for every $q \in K$ there is a tangent vector $v \in T_{p_0}M$ such that $|v| = d(p_0, q) \le r$ and $\exp_{p_0}(v) = q$. Thus

$$K \subset \exp_{p_0}(\overline{B}_r(p_0)), \qquad \overline{B}_r(p_0) = \{v \in T_{p_0}M \mid |v| \le r\}.$$

Consider the set

$$\widetilde{K} := \left\{v \in T_{p_0}M \mid |v| \le r, \, \exp_{p_0}(v) \in K\right\}.$$

This is a closed and bounded subset of the Euclidean space $T_{p_0}M$. Hence $\widetilde{K}$ is compact and K is its image under the exponential map. Since the exponential map $\exp_{p_0} : T_{p_0}M \to M$ is continuous it follows that K is compact. Thus (i) implies (iv) and this proves the theorem. $\square$

Lemma 2.60. *Let $M \subset \mathbb{R}^n$ be a connected m-manifold and $p \in M$. Suppose $\varepsilon > 0$ is smaller than the injectivity radius of M at p and denote*

$$\Sigma_1(p) := \{v \in T_pM \mid |v| = 1\}, \qquad S_\varepsilon(p) := \left\{p' \in M \mid d(p, p') = \varepsilon\right\}.$$

Then the map

$$\Sigma_1(p) \to S_\varepsilon(p) : v \mapsto \exp_p(\varepsilon v)$$

is a diffeomorphism and, for all $q \in M$, we have

$$d(p, q) > \varepsilon \qquad \Longrightarrow \qquad d(S_\varepsilon(p), q) = d(p, q) - \varepsilon.$$

Proof. By Theorem 2.44 we have $d(p, \exp_p(v)) = |v|$ for every $v \in T_pM$ with $|v| \le \varepsilon$ and $d(p, p') > \varepsilon$ for every $p' \in M \setminus \left\{\exp_p(v) \mid v \in T_pM, \, |v| \le \varepsilon\right\}$. This shows that $S_\varepsilon(p) = \exp_p(\varepsilon \Sigma_1(p))$ and, since ε is smaller than the injectivity radius, the map $\Sigma_1(p) \to S_\varepsilon(p) : v \mapsto \exp_p(\varepsilon v)$ is a diffeomorphism. To prove the second assertion let $q \in M$ such that

$$r := d(p, q) > \varepsilon.$$

Fix a constant $\delta > 0$ and choose a smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $L(\gamma) \le r + \delta$. Choose $t_0 > 0$ such that $\gamma(t_0)$ is the last point of the curve on $S_\varepsilon(p)$, i.e. $\gamma(t_0) \in S_\varepsilon(p)$ and $\gamma(t) \notin S_\varepsilon(p)$ for $t_0 < t \le 1$. Then

$$d(\gamma(t_0), q) \le L(\gamma|_{[t_0,1]}) = L(\gamma) - L(\gamma|_{[0,t_0]}) \le L(\gamma) - \varepsilon \le r + \delta - \varepsilon.$$

This shows that

$$d(S_\varepsilon(p), q) \le r + \delta - \varepsilon$$

for every $\delta > 0$ and therefore

$$d(S_\varepsilon(p), q) \le r - \varepsilon.$$

Moreover,

$$d(p', q) \ge d(p, q) - d(p, p') = r - \varepsilon$$

for every $p' \in S_\varepsilon(p)$. Hence $d(S_\varepsilon(p), q) = r - \varepsilon$ as claimed. □

Proof of Theorem 2.58. By assumption $M \subset \mathbb{R}^n$ is a connected submanifold, and $p \in M$ is given such that the exponential map

$$\exp_p : T_pM \to M$$

is defined on the entire tangent space at p. Fix a point $q \in M \setminus \{p\}$ so that

$$0 < r := d(p, q) < \infty.$$

Choose a constant $\varepsilon > 0$ smaller than the injectivity radius of M at p and smaller than r. Then, by Lemma 2.60, we have

$$d(S_\varepsilon(p), q) = r - \varepsilon.$$

Hence there is a tangent vector $v \in T_pM$ such that

$$d(\exp_p(\varepsilon v), q) = r - \varepsilon, \qquad |v| = 1.$$

Define the curve $\gamma : [0, r] \to M$ by

$$\gamma(t) := \exp_p(tv).$$

Claim. *For every $t \in [0, r]$ we have*

$$d(\gamma(t), q) = r - t.$$

In particular, $\gamma(r) = q$ and $L(\gamma) = r = d(p, q)$.

Consider the subset

$$I := \{t \in [0, r] \mid d(\gamma(t), q) = r - t\} \subset [0, r].$$

This set is nonempty, because $\varepsilon \in I$, it is obviously closed, and

$$t \in I \qquad \Longrightarrow \qquad [0, t] \subset I. \tag{2.45}$$

Namely, if $t \in I$ and $0 \leq s \leq t$ then

$$d(\gamma(s), q) \leq d(\gamma(s), \gamma(t)) + d(\gamma(t), q) \leq t - s + r - t = r - s$$

and

$$d(\gamma(s), q) \geq d(p, q) - d(p, \gamma(s)) \geq r - s.$$

Hence $d(\gamma(s), q) = r - s$ and hence $s \in I$. This proves (2.45).

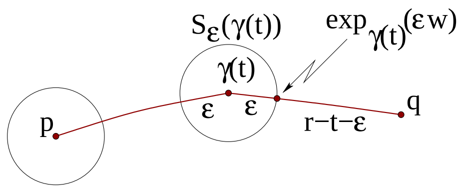


Figure 2.9: The proof of the Hopf–Rinow theorem.

We prove that I is open (in the relative topology of $[0, r]$). Let $t \in I$ be given with $t < r$. Choose a constant $\varepsilon > 0$ smaller than the injectivity radius of M at $\gamma(t)$ and smaller than $r - t$. Then, by Lemma 2.60 with p replaced by $\gamma(t)$, we have

$$d(S_\varepsilon(\gamma(t)), q) = r - t - \varepsilon.$$

Next we choose $w \in T_{\gamma(t)}M$ such that

$$|w| = 1, \qquad d(\exp_{\gamma(t)}(\varepsilon w), q) = r - t - \varepsilon.$$

Then

$$\begin{aligned} d(\gamma(t-\varepsilon), \exp_{\gamma(t)}(\varepsilon w)) &\geq d(\gamma(t-\varepsilon), q) - d(\exp_{\gamma(t)}(\varepsilon w), q) \\ &= (r - t + \varepsilon) - (r - t - \varepsilon) \\ &= 2\varepsilon. \end{aligned}$$

The converse inequality is obvious, because both points have distance ε to $\gamma(t)$ (see Figure 2.9). Thus we have proved that

$$d(\gamma(t-\varepsilon), \exp_{\gamma(t)}(\varepsilon w)) = 2\varepsilon.$$

Since $\gamma(t-\varepsilon) = \exp_{\gamma(t)}(-\varepsilon\dot{\gamma}(t))$ it follows from Lemma 2.61 below that $w = \dot{\gamma}(t)$. Hence $\exp_{\gamma(t)}(sw) = \gamma(t+s)$ and this implies that

$$d(\gamma(t+\varepsilon), q) = r - t - \varepsilon.$$

Thus $t + \varepsilon \in I$ and, by (2.45), we have $[0, t+\varepsilon] \in I$. Thus we have proved that I is open. In other words, I is a nonempty subset of $[0, r]$ which is both open and closed, and hence $I = [0, r]$. This proves the claim and the theorem. □

Lemma 2.61 (**Curve Shortening Lemma**). *Let $M \subset \mathbb{R}^n$ be an m-manifold and $p \in M$. Let $\varepsilon > 0$ be smaller than the injectivity radius of M at p. Then, for all $v, w \in T_pM$, we have*

$$|v| = |w| = \varepsilon, \quad d(\exp_p(v), \exp_p(w)) = 2\varepsilon \qquad \Longrightarrow \qquad v + w = 0.$$

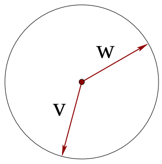


Figure 2.10: Two unit tangent vectors.

Proof. We will prove that, for all $v, w \in T_pM$, we have

$$\lim_{\delta \to 0} \frac{d(\exp_p(\delta v), \exp_p(\delta w))}{\delta} = |v - w| \,. \tag{2.46}$$

Assume this holds and suppose, by contradiction, that we have two tangent vectors $v, w \in T_pM$ satisfying

$$|v| = |w| = 1, \qquad v + w \neq 0, \qquad d(\exp_p(\varepsilon v), \exp_p(\varepsilon w)) = 2\varepsilon.$$

Then $|v - w| < 2$ (Figure 2.10). Thus by (2.46) there is a $\delta > 0$ such that

$$\delta < \varepsilon, \qquad d(\exp_p(\delta v), \exp_p(\delta w)) < 2\delta.$$

The triangle inequality gives

$$\begin{aligned} d(\exp_p(\varepsilon v), \exp_p(\varepsilon w)) &\leq d(\exp_p(\varepsilon v), \exp_p(\delta v)) \\ &\quad + d(\exp_p(\delta v), \exp_p(\delta w)) \\ &\quad + d(\exp_p(\delta w), \exp_p(\varepsilon w)) \\ &< \varepsilon - \delta + 2\delta + \varepsilon - \delta \\ &= 2\varepsilon, \end{aligned}$$

contradicting our assumption.

It remains to prove (2.46). For this we use the ambient space $\mathbb{R}^n$ and observe that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{d(\exp_p(\delta v),\exp_p(\delta w))}{\delta}\\
&=\lim_{\delta\to 0}\frac{d(\exp_p(\delta v),\exp_p(\delta w))}{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}\frac{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}{\delta}\\
&=\lim_{\delta\to 0}\frac{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}{\delta}\\
&=\lim_{\delta\to 0}\left|\frac{\exp_p(\delta v)-p}{\delta}-\frac{\exp_p(\delta w)-p}{\delta}\right|\\
&=\left|v-w\right|.
\end{aligned}$$

Here the second equality follows from Lemma 2.7. □

2.6 Riemannian metrics

We wish to carry over the fundamental notions of differential geometry to the intrinsic setting. First we need a notion of the length of a tangent vector to define the length of a curve via (2.1). Second we must introduce the covariant derivative of a vector field along a curve. With this understood all the definitions, theorems, and proofs in this chapter carry over in an almost word by word fashion to the intrinsic setting.

2.6.1 Riemannian metrics

We will always consider norms that are induced by inner products. But in general there is no ambient space that can induce an inner product on each tangent space. This leads to the following definition.

Definition 2.62. *Let M be a smooth m-manifold. A* **Riemannian metric** *on M is a collection of inner products*

$$T_pM\times T_pM\to\mathbb{R}:(v,w)\mapsto g_p(v,w),$$

one for every $p\in M$, such that the map

$$M\to\mathbb{R}:p\mapsto g_p(X(p),Y(p))$$

is smooth for every pair of vector fields $X,Y\in\mathrm{Vect}(M)$. We will also denote it by $\langle v,w\rangle_p$ and drop the subscript p if the base point is understood from the context. A smooth manifold equipped with a Riemannian metric is called a **Riemannian manifold**.

Example 2.63. If $M \subset \mathbb{R}^n$ is a smooth submanifold then a Riemannian metric on M is given by restricting the standard inner product on $\mathbb{R}^n$ to the tangent spaces $T_pM \subset \mathbb{R}^n$. This is the first fundamental form of an embedded manifold.

More generally, assume that M is a Riemannian m-manifold in the intrinsic sense of Definition 2.62 with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. Then the Riemannian metric g determines a collection of smooth functions

$$g_\alpha = (g_{\alpha,ij})_{i,j=1}^m : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m \times m},$$

one for each $\alpha \in A$, defined by

$$\xi^T g_\alpha(x)\eta := g_p(v,w), \quad \phi_\alpha(p) = x, \quad d\phi_\alpha(p)v = \xi, \quad d\phi_\alpha(p)w = \eta, \quad (2.47)$$

for $x \in \phi_\alpha(U_\alpha)$ and $\xi, \eta \in \mathbb{R}^m$. Each matrix $g_\alpha(x)$ is symmetrix and positive definite. Note that the tangent vectors v and w in (2.47) can also be written in the form $v = [\alpha, \xi]_p$ and $w = [\alpha, \eta]_p$. Choosing standard basis vectors $\xi = e_i$ and $\eta = e_j$ in $\mathbb{R}^m$ we obtain

$$[\alpha, e_i]_p = d\phi_\alpha(p)^{-1} e_i =: \frac{\partial}{\partial x^i}(p)$$

and hence

$$g_{\alpha,ij}(x) = \left\langle \frac{\partial}{\partial x^i}(\phi_\alpha^{-1}(x)), \frac{\partial}{\partial x^j}(\phi_\alpha^{-1}(x)) \right\rangle.$$

For different coordinate charts the maps g_α and g_β are related through the transition map $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ via

$$g_\alpha(x) = d\phi_{\beta\alpha}(x)^T g_\beta(\phi_{\beta\alpha}(x)) d\phi_{\beta\alpha}(x) \quad (2.48)$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Equation (2.48) can also be written in the shorthand notation $g_\alpha = \phi_{\beta\alpha}^* g_\beta$ for $\alpha, \beta \in A$.

Exercise 2.64. Every collection of smooth maps $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$ with values in the set of positive definite symmetric matrices that satisfies (2.48) for all $\alpha, \beta \in A$ determines a global Riemannian metric via (2.47).

In this intrinsic setting there is no canonical metric on M (such as the metric induced by $\mathbb{R}^n$ on an embedded manifold). In fact, it is not completely obvious that a manifold admits a Riemannian metric and this is the content of the next lemma.

Lemma 2.65. *Every paracompact Hausdorff manifold admits a Riemannian metric.*

Proof. Let m be the dimension of M and let $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ be an atlas on M. By Theorem 1.149 there is a partition of unity $\{\theta_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. Now there are two equivalent ways to construct a Riemannian metric on M.

The first method is to carry over the standard inner product on $\mathbb{R}^m$ to the tangent spaces T_pM for $p \in U_\alpha$ via the coordinate chart ϕ_α, multiply the resulting Riemannian metric on U_α by the compactly supported function θ_α, extend it by zero to all of M, and then take the sum over all α. This leads to the following formula. The inner product of two tangent vectors $v, w \in T_pM$ is defined by

$$\langle v, w\rangle_p := \sum_{p\in U_\alpha} \theta_\alpha(p)\langle d\phi_\alpha(p)v, d\phi_\alpha(p)w\rangle, \tag{2.49}$$

where the sum runs over all $\alpha \in A$ with $p \in U_\alpha$ and the inner product is the standard inner product on $\mathbb{R}^m$. Since $\mathrm{supp}(\theta_\alpha) \subset U_\alpha$ for each α and the sum is locally finite we find that the function $M \to \mathbb{R} : p \mapsto \langle X(p), Y(p)\rangle_p$ is smooth for every pair of vector fields $X, Y \in \mathrm{Vect}(M)$. Moreover, the right hand side of (2.49) is symmetric in v and w and is positive for $v = w \neq 0$ because each summand is nonnegative and each summand with $\theta_\alpha(p) > 0$ is positive. Thus equation (2.49) defines a Riemannian metric on M.

The second method is to define the functions $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$ by

$$g_\alpha(x) := \sum_{\gamma\in A} \theta_\gamma(\phi_\alpha^{-1}(x)) d\phi_{\gamma\alpha}(x)^T d\phi_{\gamma\alpha}(x) \tag{2.50}$$

for $x \in \phi_\alpha(U_\alpha)$ where each summand is defined on $\phi_\alpha(U_\alpha \cap U_\gamma)$ and is understood to be zero for $x \notin \phi_\alpha(U_\alpha \cap U_\gamma)$. We leave it to the reader to verify that these functions are smooth and satisfy the condition (2.48) for all $\alpha, \beta \in A$. Moreover, the formulas (2.49) and (2.50) determine the same Riemannian metric on M. (Prove this!) This proves the lemma. □

Once our manifold M is equipped with a Riemannian metric we can define the length of a curve $\gamma : [0, 1] \to M$ by the formula (2.1) and it is invariant under reparametrization as before (see Remark 2.2). Also the distance function $d : M \times M \to \mathbb{R}$ is defined by the same formula (2.3). We prove that it still defines a metric on M and that this metric induces the same topology as the given atlas, as in the case of embedded manifolds in Euclidean space.

Lemma 2.66. *Let M be a connected smooth Riemannian manifold and let $d : M \times M \to [0, \infty)$ be the function defined by* (2.1)*,* (2.2)*, and* (2.3)*. Then d is a metric and induces the same topology as the smooth structure.*

Proof. We prove the following.

Claim. *Fix a point $p_0 \in M$ and let $\phi : U \to \Omega$ be a coordinate chart onto an open subset $\Omega \subset \mathbb{R}^m$ such that $p_0 \in U$. Then there is an open neighborhood $V \subset U$ of p_0 and constants $\delta, r > 0$ such that*

$$\delta \left|\phi(p) - \phi(p_0)\right| \le d(p, p_0) \le \delta^{-1} \left|\phi(p) - \phi(p_0)\right| \tag{2.51}$$

for every $p \in V$ and $d(p, p_0) \ge \delta r$ for every $p \in M \setminus V$.

The claim shows that $d(p, p_0) > 0$ for every $p \in M \setminus \{p_0\}$ and hence d satisfies condition (i) in Lemma 2.4. The proofs of (ii) and (iii) remain unchanged in the intrinsic setting and this shows that d defines a metric on M. Moreover, it follows from the claim that a sequence $p_\nu \in M$ satisfies $\lim_{\nu\to\infty} d(p_\nu, p_0) = 0$ if and only if $p_\nu \in U$ for ν sufficiently large and $\lim_{\nu\to\infty} \left|\phi(p_\nu) - \phi(p_0)\right| = 0$. In other words, p_ν converges to p_0 with respect to the metric d if and only if p_ν converges to p_0 in the manifold topology. This implies that the topology induced by d coincides with the original topology of M. (See Exercise 2.68 below.)

To prove the claim we denote the inverse of the coordinate chart ϕ by $\psi := \phi^{-1} : \Omega \to M$ and define the map $g = (g_{ij})_{i,j=1}^m : \Omega \to \mathbb{R}^{m\times m}$ by

$$g_{ij}(x) := \left\langle \frac{\partial\psi}{\partial x^i}(x), \frac{\partial\psi}{\partial x^j}(x) \right\rangle_{\psi(x)}$$

for $x \in \Omega$. Then a smooth curve $\gamma : [0, 1] \to U$ has the length

$$L(\gamma) = \int_0^T \sqrt{\dot c(t)^T g(c(t)) \dot c(t)}\, dt, \qquad c(t) := \phi(\gamma(t)). \tag{2.52}$$

Let $x_0 := \phi(p_0) \in \Omega$ and choose $r > 0$ such that $\overline{B}_r(x_0) \subset \Omega$. Then there is a constant $\delta \in (0, 1]$ such that

$$\delta \left|\xi\right| \le \sqrt{\xi^T g(x) \xi} \le \delta^{-1} \left|\xi\right| \tag{2.53}$$

for all $x \in B_r(x_0)$ and $\xi, \eta \in \mathbb{R}^m$. Define the open set $V \subset U$ by

$$V := \phi^{-1}(B_r(x_0)).$$

Now let $p \in V$ and denote $x := \phi(p) \in B_r(x_0)$. Then, for every smooth curve $\gamma : [0,1] \to V$ with $\gamma(0) = p_0$ and $\gamma(1) = p$, the curve $c := \phi \circ \gamma$ takes values in $B_r(x_0)$ and satisfies $c(0) = x_0$ and $c(1) = x$. Hence, by (2.52) and (2.53), we have

$$L(\gamma) \geq \delta \int_0^1 |\dot{c}(t)| \, dt \geq \delta \left| \int_0^1 \dot{c}(t) \, dt \right| = \delta \, |x - x_0| \, .$$

If $\gamma : [0,1] \to M$ is a smooth curve with endpoints $\gamma(0) = p_0$ and $\gamma(1) = p$ whose image is not entirely contained in V then there is a time $T \in (0,1]$ such that $\gamma(t) \in V$ for $0 \leq t < T$ and $\gamma(T) \in \partial V$. Hence $c(t) := \phi(\gamma(t)) \in B_r(x_0)$ for $0 \leq t < T$ and $|c(T) - x_0| = r$. Hence the same argument shows that

$$L(\gamma) \geq \delta r.$$

This shows that $d(p_0, p) \geq \delta r$ for $p \in M \setminus V$ and $d(p_0, p) \geq \delta \, |\phi(p) - \phi(p_0)|$ for $p \in V$. If $p \in V$, $x := \phi(p)$, and $c(t) := x_0 + t(x - x_0)$ then $\gamma := \psi \circ c$ is a smooth curve in V with endpoints $\gamma(0) = p_0$ and $\gamma(1) = p$ and, by (2.52) and (2.53), we have

$$L(\gamma) \leq \delta^{-1} \int_0^1 |\dot{c}(t)| \, dt = \delta^{-1} \, |x - x_0| \, .$$

This proves the claim and the lemma. □

Exercise 2.67. Choose a coordinate chart $\phi : U \to \Omega$ with $\phi(p_0) = 0$ such that the metric in local coordinates satisfies

$$g_{ij}(0) = \delta_{ij}.$$

Refine the estimate (2.51) in the proof of Lemma 2.66 and show that

$$\lim_{p,q \to p_0} \frac{d(p,q)}{|\phi(p) - \phi(q)|} = 1.$$

This is the intrinsic analogue of Lemma 2.7. Use this to prove that equation (2.46) continues to hold for all Riemannian manifolds, i.e.

$$\lim_{\delta \to 0} \frac{d(\exp_p(\delta v), \exp_p(\delta w))}{\delta} = |v - w|$$

for $p \in M$ and $v, w \in T_pM$. Extend Lemma 2.61 to the intrinsic setting.

Exercise 2.68. Let M be a manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ and let $d : M \times M \to [0, \infty)$ be any metric on M. Prove that the following are equivalent.

(a) For every point $p_0 \in M$, every sequence $p_\nu \in M$, and every $\alpha \in M$ with $p_0 \in U_\alpha$ we have

$$\lim_{\nu \to \infty} d(p_\nu, p_0) = 0 \quad \iff \quad \begin{array}{l} p_\nu \in U_\alpha \text{ for } \nu \text{ sufficiently large and} \\ \lim_{\nu \to \infty} |\phi_\alpha(p_\nu) - \phi_\alpha(p_0)| = 0. \end{array}$$

(b) For every subset $W \subset M$ we have

$$W \text{ is } d\text{-open} \quad \iff \quad \phi_\alpha(U_\alpha \cap W) \text{ is open in } \mathbb{R}^m \text{ for all } \alpha \in A.$$

Condition (b) asserts that d induces the topology on M determined by the manifold structure via (1.53).

Example 2.69 (Fubini-Study metric). The complex projective space $\mathbb{C}P^n$ carries a natural Riemannian metric, defined as follows. Identify $\mathbb{C}P^n$ with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the diagonal action of the circle S^1:

$$\mathbb{C}P^n = S^{2n+1}/S^1.$$

Then the tangent space of the equivalence class

$$[z] = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$$

of a point $z = (z_0, \ldots, z_n) \in S^{2n+1}$ can be identified with the orthogonal complement of $\mathbb{C}z$ in $\mathbb{C}^{n+1}$. Now choose the inner product on $T_{[z]}\mathbb{C}P^n$ to be the one inherited from the standard inner product on $\mathbb{C}^{n+1}$ via this identification. The resulting metric on $\mathbb{C}P^n$ is called the **Fubini–Study metric**. **Exercise:** Prove that the action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ induces a transitive action of the quotient group

$$\mathrm{PSU}(n+1) := \mathrm{U}(n+1)/S^1$$

by isometries. If $z \in S^1$ prove that the unitary matrix

$$g := 2zz^* - \mathbb{1}$$

descends to an isometry ϕ on $\mathbb{C}\mathrm{P}^n$ with fixed point $p := [z]$ and $d\phi(p) = -\mathrm{id}$. Show that, in the case $n = 1$, the pullback of the Fubini–Study metric on $\mathbb{C}P^1$ under the stereographic projection

$$S^2 \setminus \{(0,0,1)\} \to \mathbb{C}P^1 \setminus \{[0:1]\} : (x_1, x_2, x_3) \mapsto \left[1 : \frac{x_1 + \mathbf{i}x_2}{1 - x_3}\right]$$

is one quarter of the standard metric on S^2.

Example 2.70. Think of the complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$ as a quotient of the space

$$\mathcal{F}_k(\mathbb{C}^n) := \left\{ D \in \mathbb{C}^{n\times k} \,|\, D^*D = \mathbb{1} \right\}$$

of unitary k-frames in $\mathbb{C}^n$ by the right action of the unitary group $\mathrm{U}(k)$. The space $\mathcal{F}_k(\mathbb{C}^n)$ inherits a Riemannian metric from the ambient Eucliden space $\mathbb{C}^{n\times k}$. Show that the tangent space of of $\mathrm{G}_k(\mathbb{C}^n)$ at a point $\Lambda = \mathrm{im} D$, with $D \in \mathcal{F}_k(\mathbb{C}^n)$ can be identified with the space

$$T_\Lambda \mathrm{G}_k(\mathbb{C}^n) = \left\{ \widehat{D} \in \mathbb{C}^{n\times k} \,|\, D^*\widehat{D} = 0 \right\}.$$

Define the inner product on this tangent space to be the restriction of the standard inner product on $\mathbb{C}^{n\times k}$ to this subspace. **Exercise:** Prove that the unitary group $\mathrm{U}(n)$ acts on $\mathrm{G}_k(\mathbb{C}^n)$ by isometries.

2.6.2 Covariant derivatives

A subtle point in this discussion is how to extend the notion of *covariant derivative* to general Riemannian manifolds. In this case the idea of projecting the derivative in the ambient space orthogonally onto the tangent space has no obvious analogue. Instead we shall see how the covariant derivatives of vector fields along curves can be characterized by several axioms and these can be used to define the covariant derivative in the intrinsic setting. An alternative, but somewhat less satisfactory, approach is to carry over the formula for the covariant derivative in local coordinates to the intrinsic setting and show that the result is independent of the choice of the coordinate chart. Of course, these approaches are equivalent and lead to the same result. We formulate them as a series of exercises. The details are straightforward and can be safely left to the reader.

Let M be a Riemannian m-manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ and suppose that the Riemannian metric is in local coordinates given by

$$g_\alpha = (g_{\alpha,ij})_{i,j=1}^m : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$$

for $\alpha \in A$. These functions satisfy (2.48) for all $\alpha, \beta \in A$.

Definition 2.71. *Let $f : N \to M$ be a smooth map between manifolds. A* **vector field along** f *is a collection of tangent vectors $X(q) \in T_{f(q)}M$, one for each $q \in N$, such that the map $N \to TM : q \mapsto (f(q), X(q))$ is smooth. The space of vector fields along f will be denoted by* $\mathrm{Vect}(f)$.

As before we will not distinguish in notation between the collection of tangent vectors $X(q) \in T_{f(q)}M$ and the associated map $N \to TM$ and denote them both by X.

Remark 2.72 (Levi-Civita connection). There is a unique collection of linear operators

$$\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$$

(called the **covariant derivative**), one for every smooth curve $\gamma : I \to M$ on an open interval $I \subset \mathbb{R}$, satisfying the following axioms.

(Leibnitz rule) For every smooth curve $\gamma : I \to M$, every smooth function $\lambda : I \to \mathbb{R}$, and every vector field $X \in \mathrm{Vect}(\gamma)$, we have

$$\nabla(\lambda X) = \dot{\lambda} X + \lambda \nabla X. \tag{2.54}$$

(Chain rule) Let $\Omega \subset \mathbb{R}^n$ be an open set, $c : I \to \Omega$ be a smooth curve, $\gamma : \Omega \to M$ be a smooth map, and X be a smooth vector field along γ. Denote by $\nabla_i X$ the covariant derivative of X along the curve $x^i \mapsto \gamma(x)$ (with the other coordinates fixed). Then $\nabla_i X$ is a smooth vector field along γ and the covariant derivative of the vector field $X \circ c \in \mathrm{Vect}(\gamma \circ c)$ is

$$\nabla(X \circ c) = \sum_{j=1}^{n} \dot{c}^j(t) \nabla_j X(c(t)). \tag{2.55}$$

(Riemannian) For any two vector fields $X, Y \in \mathrm{Vect}(\gamma)$ we have

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle. \tag{2.56}$$

(Torsion free) Let $I, J \subset \mathbb{R}$ be open intervals and $\gamma : I \times J \to M$ be a smooth map. Denote by ∇_s the covariant derivative along the curve $s \mapsto \gamma(s,t)$ (with t fixed) and by ∇_t the covariant derivative along the curve $t \mapsto \gamma(s,t)$ (with s fixed). Then

$$\nabla_s \partial_t \gamma = \nabla_t \partial_s \gamma. \tag{2.57}$$

Exercise 2.73. Prove that the covariant derivatives of vector fields along curves in an embedded manifold $M \subset \mathbb{R}^n$ satisfy the axioms of Remark 2.72. Prove the assertion of Remark 2.72. **Hint:** Use Theorem 2.27.

Exercise 2.74. The **Christoffel symbols** of the Riemannian metric are the functions

$$\Gamma^k_{\alpha,ij} : \phi_\alpha(U_\alpha) \to \mathbb{R}.$$

defined by

$$\Gamma^k_{\alpha,ij} := \sum_{\ell=1}^m g_\alpha^{k\ell} \frac{1}{2} \left(\frac{\partial g_{\alpha,\ell i}}{\partial x^j} + \frac{\partial g_{\alpha,\ell j}}{\partial x^i} - \frac{\partial g_{\alpha,ij}}{\partial x^\ell} \right)$$

(see Theorem 2.27). Prove that they are related by the equation

$$\sum_k \frac{\partial \phi^{k'}_{\beta\alpha}}{\partial x^k} \Gamma^k_{\alpha,ij} = \frac{\partial^2 \phi^{k'}_{\beta\alpha}}{\partial x^i \partial x^j} + \sum_{i',j'} \left(\Gamma^{k'}_{\beta,i'j'} \circ \phi_{\beta\alpha} \right) \frac{\partial \phi^{i'}_{\beta\alpha}}{\partial x^i} \frac{\partial \phi^{j'}_{\beta\alpha}}{\partial x^j}.$$

for all $\alpha, \beta \in A$.

Exercise 2.75. Denote $\psi_\alpha := \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to M$. Prove that the covariant derivative of a vector field

$$X(t) = \sum_{i=1}^m \xi^i_\alpha(t) \frac{\partial \psi_\alpha}{\partial x^i}(c_\alpha(t))$$

along $\gamma = \psi_\alpha \circ c_\alpha : I \to M$ is given by

$$\nabla X(t) = \sum_{k=1}^m \left(\dot{\xi}^k_\alpha(t) + \sum_{i,j=1}^m \Gamma^k_{\alpha,ij}(c(t)) \xi^i_\alpha(t) \dot{c}^j_\alpha(t) \right) \frac{\partial \psi_\alpha}{\partial x^k}(c_\alpha(t)). \tag{2.58}$$

Prove that ∇X is independent of the choice of the coordinate chart.

2.6.3 Geodesics

With the covariant derivative understood, we can again define geodesics on M as smooth curves $\gamma : I \to M$ that satisfy the equation $\nabla\dot{\gamma} = 0$. Then all the results we have proved about geodesics will translate word for word to general manifolds. In particular, geodesics are critical points of the energy functional on the space of paths with fixed endpoints, through each point and in each direction there is a unique geodesic on some time interval, and a manifold is complete as a metric space if and only if the geodesics through some (and hence every) point in M exist in each direction for all time.

Exercise 2.76. The real projective space $\mathbb{R}P^n$ inherits a Riemannian metric from S^n as it is a quotient of S^n by an isometric involution. Prove that each geodesic in S^n with its standard metric descends to a geodesic in $\mathbb{R}P^n$.

Exercise 2.77. Let $f : S^3 \to S^2$ be the **Hopf fibration** defined by

$$f(z,w) = \left(|z|^2 - |w|^2, 2\mathrm{Re}\,\bar{z}w, 2\mathrm{Im}\,\bar{z}w\right)$$

Prove that the image of a great circle in S^3 is a nonconstant geodesic in S^2 if and only if it is orthogonal to the fibers of f, which are also great circles. Here we identify S^3 with the unit sphere in $\mathbb{C}^2$. (See also Exercise 1.84.)

Exercise 2.78. Prove that a nonconstant geodesic $\gamma : \mathbb{R} \to S^{2n+1}$ descends to a nonconstant geodesic in $\mathbb{C}P^n$ with the Fubini–Study metric (see Example 2.69) if and only if $\dot{\gamma}(t) \perp \mathbb{C}\gamma(t)$ for every $t \in \mathbb{R}$.

Exercise 2.79. Consider the manifold

$$\mathcal{F}_k(\mathbb{R}^n) := \left\{D \in \mathbb{R}^{n\times k} \,|\, D^T D = \mathbb{1}\right\}$$

of orthonormal k-frames in $\mathbb{R}^n$, equipped with the Riemannian metric inherited from the standard inner product

$$\langle X, Y\rangle := \mathrm{trace}(X^T Y)$$

on the space of real $n \times k$-matrices.

(a) Prove that

$$T_D\mathcal{F}_k(\mathbb{R}^n) = \left\{X \in \mathbb{R}^{n\times k} \,|\, D^T X + X^T D = 0\right\},$$
$$T_D\mathcal{F}_k(\mathbb{R}^n)^\perp = \left\{DA \,|\, A = A^T \in \mathbb{R}^{k\times k}\right\}.$$

and that the orthogonal projection $\Pi(D) : \mathbb{R}^{n\times k} \to T_D\mathcal{F}_k(\mathbb{R}^n)$ is given by

$$\Pi(D)X = X - \frac{1}{2}D\left(D^T X + X^T D\right).$$

(b) Prove that the second fundamental form of $\mathcal{F}_k(\mathbb{R}^n)$ is given by

$$h_D(X)Y = -\frac{1}{2}D\left(X^T Y + Y^T X\right)$$

for $D \in \mathcal{F}_k(\mathbb{R}^n)$ and $X, Y \in T_D\mathcal{F}_k(\mathbb{R}^n)$.

(c) Prove that a smooth map $\mathbb{R} \to \mathcal{F}_k(\mathbb{R}^n) : t \mapsto D(t)$ is a geodesic if and only if it satisfies the differential equation

$$\ddot{D} = -D\dot{D}^T\dot{D}. \tag{2.59}$$

Prove that the function $D^T\dot{D}$ is constant for every geodesic in $\mathcal{F}_k(\mathbb{R}^n)$. Compare this with Example 2.42.

Exercise 2.80. Let $\mathrm{G}_k(\mathbb{R}^n) = \mathcal{F}_k(\mathbb{R}^n)/\mathrm{O}(k)$ be the real Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$, equipped with the Riemannian metric of Example 2.70, adapted to the real setting. Prove that a geodesics

$$\mathbb{R} \to \mathcal{F}_k(\mathbb{R}^n) : t \mapsto D(t)$$

descends to a nonconstant geodesic in $\mathrm{G}_k(\mathbb{R}^n)$ if and only if

$$D^T \dot{D} \equiv 0, \qquad \dot{D} \not\equiv 0.$$

Deduce that the exponential map on $\mathrm{G}_k(\mathbb{R}^n)$ is given by

$$\exp_\Lambda(\widehat{\Lambda}) = \mathrm{im}\left(D \cos\left(\left(\widehat{D}^T \widehat{D}\right)^{1/2} \right) + \widehat{D} \left(\widehat{D}^T \widehat{D}\right)^{-1/2} \sin\left(\left(\widehat{D}^T \widehat{D}\right)^{1/2} \right) \right)$$

for $\Lambda \in \mathcal{F}_k(\mathbb{R}^n)$ and $\widehat{\Lambda} \in T_\Lambda \mathcal{F}_k(\mathbb{R}^n) \setminus \{0\}$. Here we dentify the tangent space $T_\Lambda \mathcal{F}_k(\mathbb{R}^n)$ with the space of linear maps from Λ to $\Lambda^\perp$, and choose the matrices $D \in \mathcal{F}_k(\mathbb{R}^n)$ and $\widehat{D} \in \mathbb{R}^{n \times k}$ such that

$$\Lambda = \mathrm{im}\, D, \qquad D^T \widehat{D} = 0, \qquad \widehat{\Lambda} \circ D = \widehat{D} : \mathbb{R}^k \to \Lambda^\perp = \ker D^T.$$

Prove that the group $\mathrm{O}(n)$ acts on $\mathrm{G}_k(\mathbb{R}^n)$ by isometries. Which subgroup acts trivially?

Exercise 2.81. Carry over Exercises 2.79 and 2.80 to the complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$. Prove that the group $\mathrm{U}(n)$ acts on $\mathrm{G}_k(\mathbb{C}^n)$ by isometries. Which subgroup acts trivially?

Chapter 3

The Levi-Civita connection

The covariant derivative of a vector field along a curve was introduced in Section 2.2. For a submanifold of Euclidean space it is given by the orthogonal projection of the derivative in the ambient space onto the tangent space. In Section 2.3 we have seen that the covariant derivative is determined by the Christoffel symbols in local coordinates and thus can be carried over to the intrinsic setting. It takes the form of a family of linear operators

$$\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma),$$

one for every smooth curve $\gamma : I \to M$, and these operators are uniquely characterized by the axioms of Remark 2.72 in Section 2.6. This family of linear operators is the *Levi-Civita connection*. It can be extended in a natural manner to vector fields along any smooth map with values in M and, in particular, to vector fields along the identity map, i.e. to the space of vector fields on M. However, in this chapter we will focus attention on the covariant derivatives of vector fields along curves, show how they give rise to parallel transport, examine the frame bundle, discuss motions without "sliding, twisting, and wobbling", and prove the development theorem.

3.1 Parallel transport

Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. Then each tangent space of M is an m-dimensional real vector space and hence is isomorphic to $\mathbb{R}^m$. Thus any two tangent spaces T_pM and T_qM are of course isomorphic to each other. While there is no canonical isomorphism from T_pM to T_qM we shall see that every smooth curve γ in M connecting p to q induces an isomorphism between the tangent spaces via parallel transport of tangent vectors along γ.

Definition 3.1. *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. A vector field X along γ is called* **parallel** *if $\nabla X(t) = 0$ for every $t \in I$.*

Example 3.2. Assume $m = k$ so that $M \subset \mathbb{R}^m$ is an open set. Then a vector field along a smooth curve $\gamma : I \to M$ is a smooth map $X : I \to \mathbb{R}^m$. Its covariant derivative is equal to the ordinary derivative $\nabla X(t) = \dot{X}(t)$ and hence X is is parallel if and only it is constant.

In general, a vector field X along a smooth curve $\gamma : I \to M$ is parallel if and only if $\dot{X}(t)$ is orthogonal to $T_{\gamma(t)}M$ for every t and, by the Gauss–Weingarten formula (2.37), we have

$$\nabla X = 0 \qquad \iff \qquad \dot{X} = h_\gamma(\dot{\gamma}, X).$$

The next theorem shows that any given tangent vector $v_0 \in T_{\gamma(t_0)}M$ extends uniquely to a parallel vector field along γ.

Theorem 3.3. *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$ be given. Then there is a unique parallel vector field $X \in \mathrm{Vect}(\gamma)$ such that $X(t_0) = v_0$.*

Proof. Choose a basis $e_1, \dots, e_m$ of the tangent space $T_{\gamma(t_0)}M$ and let

$$X_1, \dots, X_m \in \mathrm{Vect}(\gamma)$$

be vector fields along γ such that

$$X_i(t_0) = e_i, \qquad i = 1, \dots, m.$$

(For example choose $X_i(t) := \Pi(\gamma(t))e_i$.) Then the vectors $X_i(t_0)$ are linearly independent. Since linear independence is an open condition there is a constant $\varepsilon > 0$ such that the vectors $X_1(t), \dots, X_m(t) \in T_{\gamma(t)}M$ are linearly independent for every $t \in I_0 := (t_0 - \varepsilon, t_0 + \varepsilon) \cap I$. Since $T_{\gamma(t)}M$ is an m-dimensional real vector space this implies that the vectors $X_i(t)$ form a basis of $T_{\gamma(t)}M$ for every $t \in I_0$. We express the vector $\nabla X_i(t) \in T_{\gamma(t)}M$ in this basis and denote the coefficients by $a_i^k(t)$ so that

$$\nabla X_i(t) = \sum_{k=1}^m a_i^k(t) X_k(t).$$

The resulting functions $a_i^k : I_0 \to \mathbb{R}$ are smooth. Likewise, if $X : I \to \mathbb{R}^n$ is any vector field along γ then there are smooth functions $\xi^i : I_0 \to \mathbb{R}$ such that

$$X(t) = \sum_{i=1}^m \xi^i(t) X_i(t) \qquad \text{for all } t \in I_0.$$

The derivative of X is given by

$$\dot{X}(t) = \sum_{i=1}^{m} \Big(\dot{\xi}^i(t) X_i(t) + \xi^i(t) \dot{X}_i(t) \Big)$$

and the covariant derivative by

$$\begin{aligned} \nabla X(t) &= \sum_{i=1}^{m} \Big(\dot{\xi}^i(t) X_i(t) + \xi^i(t) \nabla X_i(t) \Big) \\ &= \sum_{i=1}^{m} \dot{\xi}^i(t) X_i(t) + \sum_{i=1}^{m} \xi^i(t) \sum_{k=1}^{m} a_i^k(t) X_k(t) \\ &= \sum_{k=1}^{m} \left(\dot{\xi}^k(t) + \sum_{i=1}^{m} a_i^k(t) \xi^i(t) \right) X_k(t) \end{aligned}$$

for $t \in I_0$. Hence $\nabla X(t) = 0$ if and only if

$$\dot{\xi}(t) + A(t)\xi(t) = 0, \qquad A(t) := \begin{pmatrix} a_1^1(t) & \cdots & a_m^1(t) \\ \vdots & & \vdots \\ a_1^m(t) & \cdots & a_m^m(t) \end{pmatrix}.$$

Thus we have translated the equation $\nabla X = 0$ over the interval I_0 into a time dependent linear ordinary differential equation. By a theorem in Analysis II [12] this equation has a unique solution for any initial condition at any point in I_0. Thus we have proved that every $t_0 \in I$ is contained in an interval $I_0 \subset I$, open in the relative topology of I, such that, for every $t_1 \in I_0$ and every $v_1 \in T_{\gamma(t_1)}M$, there is a unique parallel vector field $X : I_0 \to \mathbb{R}^n$ along $\gamma|_{I_0}$ satisfying $X(t_1) = v_1$. We formulate this condition on the interval I_0 as a logical formula:

$$\forall\, t_1 \in I_0 \;\; \forall\, v_1 \in T_{\gamma(t_1)}M \;\; \exists!\, X \in \mathrm{Vect}(\gamma|_{I_0}) \ni \nabla X = 0,\; X(t_1) = v_1. \quad (3.1)$$

If two intervals $I_0, I_1 \subset I$ satisfy this condition and have nonempty intersection then their union $I_0 \cup I_1$ also satisfies (3.1). (Prove this!) Now define

$$J := \bigcup \{ I_0 \subset I \,|\, I_0 \text{ is an } I\text{-open interval, } I_0 \text{ satisfies (3.1), } t_0 \in I_0 \}.$$

This interval J satisfies (3.1). Moreover, it is nonempty and, by definition, it is open in the relative topology of I. We prove that it is also closed in the relative topology of I. Thus let $(t_\nu)_{\nu \in \mathbb{N}}$ be a sequence in J converging to a point $t^* \in I$. By what we have proved above, there is a constant $\varepsilon > 0$ such that the interval $I^* := (t^* - \varepsilon, t^* + \varepsilon) \cap I$ satisfies (3.1). Since the sequence $(t_\nu)_{\nu \in \mathbb{N}}$ converges to t^* there is a ν such that $t_\nu \in I^*$. Since $t_\nu \in J$ there is an interval $I_0 \subset I$, open in the relative topology of I, that contains t_0 and t_ν and satisfies (3.1). Hence the interval $I_0 \cup I^*$ is open in the relative topology of I, contains t_0 and t^*, and satisfies (3.1). This shows that $t^* \in J$. Thus we have proved that the interval J is nonempty and open and closed in the relative topology of I. Hence $J = I$ and this proves the theorem. □

Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. For $t_0, t \in I$ we define the map

$$\Phi_\gamma(t, t_0) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$$

by $\Phi_\gamma(t, t_0)v_0 := X(t)$ where $X \in \mathrm{Vect}(\gamma)$ is the unique parallel vector field along γ satisfying $X(t_0) = v_0$. The collection of maps $\Phi_\gamma(t, t_0)$ for $t, t_0 \in I$ is called **parallel transport along** γ. Recall the notation

$$\gamma^*TM = \left\{(s, v) \,|\, s \in I,\ v \in T_{\gamma(s)}M\right\}$$

for the pullback tangent bundle. This set is a smooth submanifold of $I \times \mathbb{R}^n$. (See Theorem 1.100 and Corollary 1.102.) The next theorem summarizes the properties of parallel transport. In particular, the last assertion shows that the covariant derivative can be recovered from the parallel transport maps.

Theorem 3.4 (Parallel Transport). *Let $\gamma : I \to M$ be a smooth curve on an interval $I \subset \mathbb{R}$.*

(i) *The map $\Phi_\gamma(t, s) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is linear for all $s, t \in I$.*

(ii) *For all $r, s, t \in I$ we have*

$$\Phi_\gamma(t, s) \circ \Phi_\gamma(s, r) = \Phi_\gamma(t, r), \qquad \Phi_\gamma(t, t) = 0.$$

(iii) *For all $s, t \in I$ and all $v, w \in T_{\gamma(s)}M$ we have*

$$\langle \Phi_\gamma(t, s)v, \Phi_\gamma(t, s)w \rangle = \langle v, w \rangle.$$

Thus $\Phi_\gamma(t, s) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is an orthogonal transformation.

(iv) *If $J \subset \mathbb{R}$ is an interval and $\alpha : J \to I$ is a smooth map then, for all $s, t \in J$, we have*

$$\Phi_{\gamma \circ \alpha}(t, s) = \Phi_\gamma(\alpha(t), \alpha(s)).$$

(v) *The map $I \times \gamma^*TM \to \gamma^*TM : (t, (s, v)) \mapsto (t, \Phi_\gamma(t, s)v)$ is smooth.*

(vi) *For all $X \in \mathrm{Vect}(\gamma)$ and $t, t_0 \in I$ we have*

$$\frac{d}{dt}\Phi_\gamma(t_0, t)X(t) = \Phi_\gamma(t_0, t)\nabla X(t).$$

Proof. Assertion (i) is obvious because the sum of two parallel vector fields along γ is again parallel and the product of a parallel vector field with a constant real number is again parallel. Assertion (ii) follows directly from the uniqueness statement in Theorem 3.3.

We prove (iii). Fix a number $s \in I$ and two tangent vectors

$$v.w \in T_{\gamma(s)}M.$$

Define the vector fields $X, Y \in \mathrm{Vect}(\gamma)$ along γ by

$$X(t) := \Phi_\gamma(t,s)v, \qquad Y(t) := \Phi_\gamma(t,s)w.$$

By definition of Φ_γ these vector fields are parallel. Hence

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle = 0.$$

Hence the function $I \to \mathbb{R} : t \mapsto \langle X(t), Y(t)\rangle$ is constant and this proves (iii).

We prove (iv). Fix an element $s \in J$ and a tangent vector $v \in T_{\gamma(\alpha(s))}M$. Define the vector field X along γ by

$$X(t) := \Phi_\gamma(t, \alpha(s))v$$

for $t \in I$. Thus X is the unique parallel vector field along γ that satisfies

$$X(\alpha(s)) = v.$$

Denote

$$\widetilde{\gamma} := \gamma \circ \alpha : J \to M, \qquad \widetilde{X} := X \circ \alpha : I \to \mathbb{R}^n$$

Then $\widetilde{X}$ is a vector field along $\widetilde{\gamma}$ and, by the chain rule, we have

$$\frac{d}{dt}\widetilde{X}(t) = \frac{d}{dt}X(\alpha(t)) = \dot{\alpha}(t)X(\alpha(t)).$$

Projecting orthogonally onto the tangent space $T_{\gamma(\alpha(t))}M$ we obtain

$$\nabla\widetilde{X}(t) = \dot{\alpha}(t)\nabla X(\alpha(t)) = 0$$

for every $t \in J$. Hence $\widetilde{X}$ is the unique parallel vector field along $\widetilde{\gamma}$ that satisfies $\widetilde{X}(s) = v$. Thus

$$\Phi_{\widetilde{\gamma}}(t,s)v = \widetilde{X}(t) = X(\alpha(t)) = \Phi_\gamma(\alpha(t), \alpha(s))v.$$

This proves (iv).

We prove (v). Fix a point $t_0 \in I$, choose an orthonormal basis $e_1, \dots, e_m$ of $T_{\gamma(t_0)}M$, and denote

$$X_i(t) := \Phi_\gamma(t, t_0)e_i$$

for $t \in I$ and $i = 1, \dots, m$. Thus $X_i \in \mathrm{Vect}(\gamma)$ is the unique parallel vector field along γ such that $X_i(t_0) = e_i$. Then by (iii) we have

$$\langle X_i(t), X_j(t)\rangle = \delta_{ij}$$

for all $i, j \in \{1, \dots, m\}$ and all $t \in I$. Hence the vectors $X_1(t), \dots, X_m(t)$ form an orthonormal basis of $T_{\gamma(t)}M$ for every $t \in I$. This implies that, for each $s \in I$ and each tangent vector $v \in T_{\gamma(s)}M$, we have

$$v = \sum_{i=1}^m \langle X_i(s), v\rangle X_i(s).$$

Since each vector field X_i is parallel it satisfies $X_i(t) = \Phi_\gamma(t, s)X_i(s)$. Hence

$$\Phi_\gamma(t, s)v = \sum_{i=1}^m \langle X_i(s), v\rangle X_i(t) \tag{3.2}$$

for all $s, t \in I$ and $v \in T_{\gamma(s)}M$. This proves (v)

We prove (vi). Let $X_1, \dots, X_m \in \mathrm{Vect}(\gamma)$ be as in the proof of (v). Thus every vector field X along γ can be written in the form

$$X(t) = \sum_{i=1}^m \xi^i(t)X_i(t), \qquad \xi^i(t) := \langle X_i(t), X(t)\rangle.$$

Since the vector fields X_i are parallel we have

$$\nabla X(t) = \sum_{i=1}^m \dot\xi^i(t)X_i(t)$$

for all $t \in I$. Hence

$$\Phi_\gamma(t_0, t)X(t) = \sum_{i=1}^m \xi^i(t)X_i(t_0), \qquad \Phi_\gamma(t_0, t)\nabla X(t) = \sum_{i=1}^m \dot\xi^i(t)X_i(t_0).$$

Evidently, the derivative of the first sum with respect to t is equal to the second sum. This proves (vi) and the theorem. □

Remark 3.5. For $s, t \in I$ we can think of the linear map

$$\Phi_\gamma(t,s)\Pi(\gamma(s)) : \mathbb{R}^n \to T_{\gamma(t)}M \subset \mathbb{R}^n$$

as a real $n \times n$ matrix. The formula (3.2) in the proof of (v) shows that this matrix can be expressed in the form

$$\Phi_\gamma(t,s)\Pi(\gamma(s)) = \sum_{i=1}^{m} X_i(t)X_i(s)^T \in \mathbb{R}^{n\times n}.$$

The right hand side defines a smooth matrix valued function on $I \times I$ and this is equivalent to the assertion in (v).

Remark 3.6. It follows from assertions (ii) and (iii) in Theorem 3.4 that

$$\Phi_\gamma(t,s)^{-1} = \Phi_\gamma(s,t) = \Phi_\gamma(t,s)^*$$

for all $s, t \in I$. Here the linear map $\Phi_\gamma(t,s)^* : T_{\gamma(t)}M \to T_{\gamma(s)}M$ is understood as the adjoint operator of $\Phi_\gamma(t,s) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ with respect to the inner products on the two subspaces of $\mathbb{R}^n$ inherited from the Euclidean inner product on the ambient space.

Exercise 3.7. Carry over the proofs of Theorems 3.3 and 3.4 to the intrinsic setting.

The two theorems in this section carry over verbatim to any smooth vector bundle $E \subset M \times \mathbb{R}^n$ over a manifold. As in the case of the tangent bundle one can define the covariant derivative of a section of E along γ as the orthogonal projection of the ordinary derivative in the ambient space $\mathbb{R}^n$ onto the fiber $E_{\gamma(t)}$. Instead of *parallel vector fields* one then speaks about *horizontal sections* and one proves as in Theorem 3.3 that there is a unique horizontal section along γ through any point in any of the fibers $E_{\gamma(t_0)}$. This gives parallel transport maps from $E_{\gamma(s)}$ to $E_{\gamma(t)}$ for any pair $s, t \in I$ and Theorem 3.4 carries over verbatim to all vector bundles $E \subset M \times \mathbb{R}^n$. We spell this out in more detail in the case where $E = TM^\perp \subset M \times \mathbb{R}^n$ is the normal bundle of M.

Let $\gamma : I \to M$ is a smooth curve in M. A **normal vector field along** γ is a smooth map $Y : I \to \mathbb{R}^n$ such that $Y(t) \perp T_{\gamma(t)}M$ for every $t \in M$. The set of normal vector fields along γ will be denoted by

$$\mathrm{Vect}^\perp(\gamma) := \left\{ Y : I \to \mathbb{R}^n \,|\, Y \text{ is smooth and } Y(t) \perp T_{\gamma(t)}M \text{ for all } t \in I \right\}.$$

This is again a real vector space. The **covariant derivative** of a normal vector field $Y \in \mathrm{Vect}^\perp(\gamma)$ at $t \in I$ is defined as the orthogonal projection of the ordinary derivative onto the orthogonal complement of $T_{\gamma(t)}M$ and will be denoted by

$$\nabla^{\perp}Y(t) := \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot{Y}(t). \tag{3.3}$$

Thus the covariant derivative defines a linear operator

$$\nabla^{\perp} : \mathrm{Vect}^{\perp}(\gamma) \to \mathrm{Vect}^{\perp}(\gamma).$$

There is a version of the Gauss–Weingarten formula for the covariant derivative of a normal vector field. This is the content of the next lemma.

Lemma 3.8. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. For $p \in M$ and $u \in T_pM$ define the linear map $h_p(u) : T_pM \to T_pM^{\perp}$ by*

$$h_p(u)v := h_p(u,v) = \big(d\Pi(p)u\big)v \tag{3.4}$$

for $v \in T_pM$. Then the following holds.
(i) *The adjoint operator $h_p(u)^* : T_pM^{\perp} \to T_pM$ is given by*

$$h_p(u)^*w = \big(d\Pi(p)u\big)w, \qquad w \in T_pM^{\perp}. \tag{3.5}$$

(ii) *If $I \subset \mathbb{R}$ is an interval, $\gamma : I \to M$ is a smooth curve, and $Y \in \mathrm{Vect}^{\perp}(\gamma)$ then the derivative of Y satisfies the* **Gauss–Weingarten formula**

$$\dot{Y}(t) = \nabla^{\perp}Y(t) - h_{\gamma(t)}(\dot{\gamma}(t))^*Y(t). \tag{3.6}$$

Proof. Since $\Pi(p) \in \mathbb{R}^{n\times n}$ is a symmetric matrix for every $p \in M$ so is the matrix $d\Pi(p)u$ for every $p \in M$ and every $u \in T_pM$. Hence

$$\langle v, h_p(u)^*w\rangle = \langle h_p(u)v, w\rangle = \big\langle \big(d\Pi(p)u\big)v, w\big\rangle = \big\langle v, \big(d\Pi(p)u\big)w\big\rangle$$

for every $v \in T_pM$ and every $w \in T_pM^{\perp}$. This proves (i).

To prove (ii) we observe that, for $Y \in \mathrm{Vect}^{\perp}(\gamma)$ and $t \in I$, we have

$$\Pi(\gamma(t))Y(t) = 0.$$

Differentiating this identity we obtain

$$\Pi(\gamma(t))\dot{Y}(t) + \big(d\Pi(\gamma(t))\dot{\gamma}(t)\big)Y(t) = 0$$

and hence

$$\begin{aligned} \dot{Y}(t) &= \dot{Y}(t) - \Pi(\gamma(t))\dot{Y}(t) - \big(d\Pi(\gamma(t))\dot{\gamma}(t)\big)Y(t) \\ &= \nabla^{\perp}Y(t) - h_{\gamma(t)}(\dot{\gamma}(t))^*Y(t) \end{aligned}$$

for $t \in I$. Here the last equation follows from (i) and the definition of $\nabla^{\perp}$. This proves the lemma. □

Theorem 3.3 and its proof carry over to the normal bundle $TM^\perp$. Thus, if $\gamma : I \to M$ is a smooth curve then, for all $s \in I$ and $w \in T_{\gamma(s)}M^\perp$, there is a unique normal vector field $Y \in \mathrm{Vect}^\perp(\gamma)$ such that $\nabla^\perp Y \equiv 0$ and $Y(s) = w$. This gives rise to parallel transport maps

$$\Phi_\gamma^\perp(t,s) : T_{\gamma(s)}M^\perp \to T_{\gamma(t)}M^\perp$$

defined by $\Phi_\gamma^\perp(t,s)w := Y(t)$ for $s,t \in I$ and $w \in T_{\gamma(s)}M^\perp$, where Y is the unique normal vector field along γ satisfying $\nabla^\perp Y \equiv 0$ and $Y(s) = w$. These parallel transport maps satisfy exactly the same conditions that have been spelled out in Theorem 3.4 for the tangent bundle and the proof carries over verbatim to the present setting.

3.2 The frame bundle

3.2.1 Frames of a vector space

Let V be an m-dimensional real vector space. A **frame of** V is a basis $e_1,\dots,e_m$ of V. It determines a vector space isomorphism $e : \mathbb{R}^m \to V$ via

$$e\xi := \sum_{i=1}^m \xi^i e_i, \qquad \xi = (\xi^1,\dots,\xi^m) \in \mathbb{R}^m.$$

Conversely, each isomorphism $e : \mathbb{R}^m \to V$ determines a basis $e_1,\dots,e_m$ of V via $e_i = e(0,\dots,0,1,0\dots,0)$, where the coordinate 1 appears in the ith place. The set of vector space isomorphisms from $\mathbb{R}^m$ to V will be denoted by

$$\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) := \{e : \mathbb{R}^m \to V \,|\, e \text{ is a vector space isomorphism}\}.$$

The general linear group $\mathrm{GL}(m) = \mathrm{GL}(m,\mathbb{R})$ (of nonsingular real $m \times m$-matrices) acts on this space by composition on the right via

$$\mathrm{GL}(m) \times \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) \to \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) : (a,e) \mapsto a^*e := e \circ a.$$

This is a **contravariant group action** in that

$$a^*b^*e = (ba)^*e, \qquad \mathbb{1}^*e = e$$

for $a,b \in \mathrm{GL}(m)$ and $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$. Moreover, the action is **free**, i.e. for all $a \in \mathrm{GL}(m)$ and $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$, we have

$$a^*e = e \qquad \Longleftrightarrow \qquad a = \mathbb{1}.$$

It is **transitive** in that for all $e, e' \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ there is a group element $a \in \mathrm{GL}(m)$ such that $e' = a^* e$. Thus we can identify the space $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ with the group $\mathrm{GL}(m)$ via the bijection

$$\mathrm{GL}(m) \to \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) : a \mapsto a^* e_0$$

induced by a fixed element $e_0 \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$. This identification is not canonical; it depends on the choice of e_0. The space $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ admits a bijection to a group but is not itself a group.

3.2.2 The frame bundle

Now let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold. The **frame bundle** of M is the set

$$\mathcal{F}(M) := \{(p, e) \,|\, p \in M,\; e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)\}\,. \tag{3.7}$$

We can think of a frame $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$ as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ whose image is T_pM and hence as a $n \times m$-matrix (of rank m). The basis of T_pM associated to a frame is given by the columns of the matrix $e \in \mathbb{R}^{n\times m}$. Thus the frame bundle of an embedded manifold $M \subset \mathbb{R}^n$ is a submanifold of the Euclidean space $\mathbb{R}^n \times \mathbb{R}^{n\times m}$.

The frame bundle $\mathcal{F}(M)$ is equipped with a natural projection

$$\pi : \mathcal{F}(M) \to M$$

which sends to each pair $(p, e) \in \mathcal{F}(M)$ to the base boint $\pi(p, e) := p$. The **fiber of** $\mathcal{F}(M)$ over $p \in M$ is the set

$$\mathcal{F}(M)_p := \left\{e \in \mathbb{R}^{n\times m} \,|\, (p, e) \in \mathcal{F}(M)\right\} = \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$$

The frame bundle admits a right action of the group $\mathrm{GL}(m)$ via

$$\mathrm{GL}(m) \times \mathcal{F}(M) \to \mathcal{F}(M) : (a, (p, e)) \mapsto a^*(p, e) := (p, a^* e).$$

This group action preserves the fibers of the frame bundle and its action on each fiber $\mathcal{F}(M)_p = \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$ is free and transitive. Thus each fiber of $\mathcal{F}(M)$ can be identified with the group $\mathrm{GL}(m)$ but the fibers of $\mathcal{F}(M)$ are not themselves groups.

Lemma 3.9. *$\mathcal{F}(M)$ is a smooth manifold of dimension $m + m^2$ and the projection $\pi : \mathcal{F}(M) \to M$ is a submersion.*

Proof. Let $p_0 \in M$, choose a local coordinate chart $\phi : U \to \Omega$ on an open neighborhood $U \subset M$ of p_0 with values in an open set $\Omega \subset \mathbb{R}^m$, and denote its inverse by $\psi := \phi^{-1} : \Omega \to U$. Define the open set $\widetilde{U} \subset \mathcal{F}(M)$ by

$$\widetilde{U} := \pi^{-1}(U) = \{(p, e) \in \mathcal{F}(M) \,|\, p \in U\} = (U \times \mathbb{R}^{n \times m}) \cap \mathcal{F}(M)$$

and the map $\widetilde{\psi} : \Omega \times \mathrm{GL}(m) \to \widetilde{U}$ by

$$\widetilde{\psi}(x, a) := \big(\psi(x), d\psi(x) \circ a\big)$$

for $x \in \Omega$ and $a \in \mathrm{GL}(m)$. Then $\widetilde{\psi}$ is a diffeomorphism from the open subset $\widetilde{\Omega} := \Omega \times \mathrm{GL}(m)$ of $\mathbb{R}^m \times \mathbb{R}^{m \times m}$ to the relatively open subset $\widetilde{U}$ of $\mathcal{F}(M)$. Its inverse $\widetilde{\phi} := \widetilde{\psi}^{-1} : \widetilde{U} \to \widetilde{\Omega}$ is given by

$$\widetilde{\phi}(p, e) = \big(\phi(p), d\phi(p) \circ e\big)$$

for $p \in U$ and $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$. It is indeed smooth and is the desired coordinate chart on $\mathcal{F}(M)$. Thus we have proved that $\mathcal{F}(M)$ is a smooth manifold of dimension $m + m^2$.

We prove that the projection $\pi : \mathcal{F}(M) \to M$ is a submersion. Let $(p_0, e_0) \in \mathcal{F}(M)$ and $v_0 \in T_{p_0}M$. Then there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p_0$ and $\dot{\gamma}(0) = v_0$. For $t \in \mathbb{R}$ define

$$e(t) := \Phi_\gamma(t, 0) \circ e_0 : \mathbb{R}^m \to T_{\gamma(t)}M.$$

Then $e(t) \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{\gamma(t)}M)$ and the map $\beta : \mathbb{R} \to \mathcal{F}(M)$ defined by $\beta(t) := (\gamma(t), e(t))$ is a smooth curve in $\mathcal{F}(M)$ satisfying $\pi(\beta(t)) = \gamma(t)$. Differentiating this equation at $t = 0$ we obtain

$$d\pi(p_0, e_0)(v_0, \dot{e}(0)) = v_0.$$

Thus we have proved that the linear map $d\pi(p_0, e_0) : T_{(p_0,e_0)}\mathcal{F}(M) \to T_{p_0}M$ is surjective for every pair $(p_0, e_0) \in \mathcal{F}(M)$. Hence π is a submersion, as claimed. This proves the lemma. □

The frame bundle $\mathcal{F}(M)$ is a **principal bundle** over M with **structure group** $\mathrm{GL}(m)$. More generally, a principal bundle over a manifold B with structure group G is a smooth manifold P equipped with a surjective submersion $\pi : P \to B$ and a smooth contravariant action $\mathrm{G} \times P \to P : (g, p) \mapsto pg$ by a Lie group G such that $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in \mathrm{G}$ and such that the group G acts freely and transitively on the fiber $P_b = \pi^{-1}(b)$ for each $b \in B$. In this manuscript we shall only be concerned with the frame bundle of a manifold M and the orthonormal frame bundle.

3.2.3 The orthonormal frame bundle

The **orthonormal frame bundle of** M is the set

$$\mathcal{O}(M) := \left\{(p,e) \in \mathbb{R}^n \times \mathbb{R}^{n\times m} \,|\, p \in M,\ \operatorname{im} e = T_pM,\ e^Te = \mathbb{1}_{m\times m}\right\}.$$

If we denote by

$$e_i := e(0,\dots,0,1,0,\dots,0)$$

the basis of T_pM induced by the isomorphism $e : \mathbb{R}^m \to T_pM$ then we have

$$e^Te = \mathbb{1} \quad\iff\quad \langle e_i, e_j\rangle = \delta_{ij} \quad\iff\quad \begin{array}{l} e_1,\dots,e_m \text{ is an} \\ \text{orthonormal basis.}\end{array}$$

Thus $\mathcal{O}(M)$ is the bundle of orthonormal frames of the tangent spaces T_pM or the bundle of orthogonal isomorphisms $e : \mathbb{R}^m \to T_pM$. It is a principal bundle over M with structure group $\mathrm{O}(m)$.

Exercise 3.10. Prove that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$ and that the obvious projection $\pi : \mathcal{O}(M) \to M$ is a submersion. Prove that the action of $\mathrm{GL}(m)$ on $\mathcal{F}(M)$ restricts to an action of the orthogonal group $\mathrm{O}(m)$ on $\mathcal{O}(M)$ whose orbits are the fibers

$$\mathcal{O}(M)_p := \left\{e \in \mathbb{R}^{n\times m} \,\middle|\, (p,e) \in \mathcal{O}(M)\right\} = \left\{e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM) \,\middle|\, e^Te = \mathbb{1}\right\}.$$

Hint: If $\phi : U \to \Omega$ is a coordinate chart on M with inverse $\psi : \Omega \to U$ then $e_x := d\psi(x)(d\psi(x)^T d\psi(x))^{-1/2} : \mathbb{R}^m \to T_{\psi(x)}M$ is an orthonormal frame of the tangent space $T_{\psi(x)}M$ for every $x \in \Omega$.

3.2.4 The tangent bundle of the frame bundle

We have seen in Lemma 3.9 that the frame bundle $\mathcal{F}(M)$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^{n\times m}$. Next we examine the tangent space of $\mathcal{F}(M)$ at a point $(p,e) \in \mathcal{F}(M)$. By Definition 1.21, this tangent space is given by

$$T_{(p,e)}\mathcal{F}(M) = \left\{ (\dot\gamma(0), \dot e(0)) \,\middle|\, \begin{array}{l} \mathbb{R} \to \mathcal{F}(M) : t \mapsto (\gamma(t), e(t)) \\ \text{is a smooth curve satisfying} \\ \gamma(0) = p \text{ and } e(0) = e \end{array} \right\}.$$

It is convenient to consider two kinds of curves in $\mathcal{F}(M)$, namely vertical curves with constant projections to M and horizontal lifts of curves in M. We denote by $\mathcal{L}(\mathbb{R}^m, T_pM)$ the space of linear maps from $\mathbb{R}^m$ to T_pM.

Definition 3.11. *Let $\gamma : \mathbb{R} \to M$ be a smooth curve. A curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ is called a* **lift of** γ *if*

$$\pi \circ \beta = \gamma.$$

Any such lift has the form $\beta(t) = (\gamma(t), e(t))$ with $e(t) \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{\gamma(t)}M)$. The associated curve of frames $e(t)$ of the tangent spaces $T_{\gamma(t)}M$ is called a **moving frame along** γ. *A curve $\beta(t) = (\gamma(t), e(t)) \in \mathcal{F}(M)$ is called* **horizontal** *or a* **horizontal lift of** γ *if the vector field $X(t) := e(t)\xi$ along γ is parallel for every $\xi \in \mathbb{R}^m$. Thus a horizontal lift of γ has the form*

$$\beta(t) = (\gamma(t), \Phi_\gamma(t,0)e)$$

for some $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{\gamma(0)}M)$.

Lemma 3.12. **(i)** *The tangent space of $\mathcal{F}(M)$ at $(p,e) \in \mathcal{F}(M)$ is the direct sum*

$$T_{(p,e)}\mathcal{F}(M) = H_{(p,e)} \oplus V_{(p,e)}$$

of the **horizontal space**

$$H_{(p,e)} := \left\{ (v, h_p(v)e) \,\middle|\, v \in T_pM \right\} \tag{3.8}$$

and the **vertical space**

$$V_{(p,e)} := \{0\} \times \mathcal{L}(\mathbb{R}^m, T_pM). \tag{3.9}$$

(ii) *The vertical space $V_{(p,e)}$ is the kernel of the linear map*

$$d\pi(p,e) : T_{(p,e)}\mathcal{F}(M) \to T_pM.$$

(iii) *A curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ is horizontal if and only if it is tangent to the horizontal spaces, i.e. $\dot\beta(t) \in H_{\beta(t)}$ for every $t \in \mathbb{R}$.*

(iv) *If $\beta : \mathbb{R} \to \mathcal{F}(M)$ is a horizontal curve so is $a^*\beta$ for every $a \in \mathrm{GL}(m)$.*

Proof. If $\beta : \mathbb{R} \to \mathcal{F}(M)$ is a vertical curve with $\pi\circ\beta \equiv p$ then $\beta(t) = (p, e(t))$ where $e(t) \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$. Hence $\dot e(0) \in \mathcal{L}(\mathbb{R}^m, T_pM)$. Conversely, for every $\hat e \in \mathcal{L}(\mathbb{R}^m, T_pM)$, the curve

$$\mathbb{R} \to \mathcal{L}(\mathbb{R}^m, T_pM) : t \mapsto e(t) := e + t\hat e$$

takes values in the open set $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$ for t sufficiently small and satisfies $\dot e(0) = \hat e$. This shows that $V_{(p,e)} \subset T_{(p,e)}\mathcal{F}(M)$.

Now fix a tangent vector $v \in T_pM$, let $\gamma : \mathbb{R} \to M$ be a smooth curve satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$, and let $\beta : \mathbb{R} \to \mathcal{F}(M)$ be the horizontal lift of γ with $\beta(0) = (p, e)$. Then

$$\beta(t) = (\gamma(t), e(t)), \qquad e(t) := \Phi_\gamma(t, 0)e.$$

Fix a vector $\xi \in \mathbb{R}^m$ and consider the vector field

$$X(t) := e(t)\xi = \Phi_\gamma(t, 0)e\xi$$

along γ. This vector field is parallel and hence, by the Gauss–Weingarten formula, it satisfies

$$\dot e(0)\xi = \dot X(0) = h_{\gamma(0)}(\dot\gamma(0), X(0)) = h_p(v)e\xi.$$

Here we have used (3.4). Thus we have obtained the formula

$$\dot\gamma(0) = v, \qquad \dot e(0) = h_p(v)e$$

for the derivative of the horizontal lift of γ. This shows that the horizontal space $H_{(p,e)}$ is contained in the tangent space $T_{(p,e)}\mathcal{F}(M)$. Since

$$\dim H_{(p,e)} = m, \qquad \dim V_{(p,e)} = m^2, \qquad \dim T_{(p,e)}\mathcal{F}(M) = m + m^2,$$

assertion (i) follows.

To prove (ii) we observe that

$$V_{(p,e)} \subset \ker\, d\pi(p, e)$$

and that $d\pi(p, e) : T_{(p,e)}\mathcal{F}(M) \to T_pM$ is surjective, by Lemma 3.9. Hence

$$\dim \ker d\pi(p, e) = \dim T_{(p,e)}\mathcal{F}(M) - \dim T_pM = m^2 = \dim V_{(p,e)}$$

and this proves (ii).

We have already seen that every horizontal curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ satisfies $\dot\beta(t) \in H_{\beta(t)}$ for every $t \in \mathbb{R}$. Conversely, if $\beta(t) = (\gamma(t), e(t))$ is a smooth curve in $\mathcal{F}(M)$ satisfying $\dot\beta(t) \in H_{\beta(t)}$ then

$$\dot e(t) = h_{\gamma(t)}(\dot\gamma(t))e(t)$$

for every $t \in \mathbb{R}$. By the Gauss–Weingarten formula this implies that the vector field $X(t) = e(t)\xi$ along γ is parallel for every $\xi \in \mathbb{R}^m$ and hence the curve β is horizontal. This proves (iii).

Assertion (iv) follows from (iii) and the fact that the horizontal tangent bundle $H \subset T\mathcal{F}(M)$ is invariant under the induced action of the group $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$. This proves the lemma. □

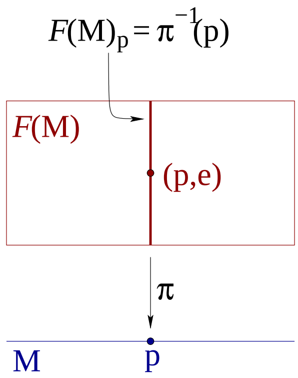
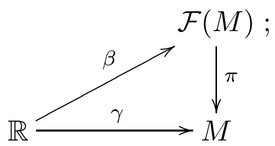


Figure 3.1: The frame bundle.

The reason for the terminology introduced in Definition 3.11 is that one draws the extremely crude picture of the frame bundle displayed in Figure 3.1. One thinks of $\mathcal{F}(M)$ as "lying over" M. One would then represent the equation $\gamma = \pi \circ \beta$ by the following commutative diagram:



hence the word "lift". The vertical space is tangent to the vertical line in Figure 3.1 while the horizontal space is transverse to the vertical space. This crude imagery can be extremely helpful.

Exercise 3.13. The group $\mathrm{GL}(m)$ acts on $\mathcal{F}(M)$ by diffeomorphisms. Thus for each $a \in \mathrm{GL}(m)$ the map

$$\mathcal{F}(M) \to \mathcal{F}(M) : (p, e) \mapsto a^*(p, e)$$

is a diffeomorphism of $\mathcal{F}(M)$. The derivative of this diffeomorphism is a diffeomorphism of the tangent bundle $T\mathcal{F}(M)$ and this is called the induced action of $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$. Prove that the horizontal and vertical subbundles are invariant under the induced action of $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$.

Exercise 3.14. Prove that $H_{(p,e)} \subset T_{(p,e)}\mathcal{O}(M)$ and that

$$T_{(p,e)}\mathcal{O}(M) = H_{(p,e)} \oplus V'_{(p,e)}, \qquad V'_{(p,e)} := V_{(p,e)} \cap T_{(p,e)}\mathcal{O}(M),$$

for every $(p, e) \in \mathcal{O}(M)$.

3.2.5 Basic vector fields

Every $\xi \in \mathbb{R}^m$ determines a vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ defined by

$$B_\xi(p,e) := \big(e\xi, h_p(e\xi)e\big) \tag{3.10}$$

for $(p,e) \in \mathcal{F}(M)$. This vector field is horizontal, i.e.

$$B_\xi(p,e) \in H_{(p,e)},$$

and projects to $e\xi$, i.e.

$$d\pi(p,e)B_\xi(p,e) = e\xi$$

for all $(p,e) \in \mathcal{F}(M)$. These two conditions determine the vector field B_ξ uniquely. It is called the **basic vector field** corresponding to ξ.

Theorem 3.15. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. Then the following are equivalent.*

(i) *M is geodesically complete.*

(ii) *The vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ is complete for every $\xi \in \mathbb{R}^m$.*

(iii) *For every interval $I \subset \mathbb{R}$, every smooth map $\xi : I \to \mathbb{R}^m$, every $t_0 \in I$, and every $(p_0, e_0) \in \mathcal{F}(M)$ there is a smooth curve $\beta : I \to \mathcal{F}(M)$ satisfying the differential equation*

$$\dot\beta(t) = B_{\xi(t)}(\beta(t)) \quad \forall\, t \in I, \qquad \beta(t_0) = (p_0, e_0). \tag{3.11}$$

Proof. That (iii) implies (ii) is obvious. Just take $I = \mathbb{R}$ and let $\xi : \mathbb{R} \to \mathbb{R}^m$ be a constant map.

We prove that (ii) implies (i). Let $p_0 \in M$ and $v_0 \in T_{p_0}M$ be given. Let $e_0 \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{p_0}M)$ be any isomorphism and choose $\xi \in \mathbb{R}^m$ such that

$$e_0\xi = v_0.$$

By (ii) the vector field B_ξ has a unique integral curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ with $\beta(0) = (p_0, e_0)$. Write β in the form $\beta(t) = (\gamma(t), e(t))$. Then

$$\dot\gamma(t) = e(t)\xi, \qquad \dot e(t) = h_{\gamma(t)}(e(t)\xi)e(t),$$

and hence

$$\ddot\gamma(t) = \dot e(t)\xi = h_{\gamma(t)}(e(t)\xi)e(t)\xi = h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)).$$

By the Gauss–Weingarten formula, this implies $\nabla\dot\gamma(t) = 0$ for every t and so $\gamma : \mathbb{R} \to M$ is a geodesic satisfying $\gamma(0) = p_0$ and $\dot\gamma(0) = e_0\xi = v_0$. This shows that M is geodesically complete.

We prove that (i) implies (iii). Thus we assume that $T > t_0$ is such that $T \in I$ and the solution of (3.11) exists on the interval $[t_0, T)$ but cannot be extended to $[t_0, T + \varepsilon)$ for any $\varepsilon > 0$. We shall use the metric completeness of M to prove the existence of the limit

$$(p_1, e_1) = \lim_{\substack{t \to T \\ t < T}} \beta(T) \in \mathcal{F}(M) \tag{3.12}$$

and this implies, by Lemma 2.59, that the solution can be extended to the interval $[t_0, T]$, a contradiction. To prove the existence of the limit (3.12) we write $\beta(t) =: (\gamma(t), e(t))$ so that γ and e satisfy the equations

$$\dot\gamma(t) = e(t)\xi(t), \quad \dot e(t) = h_{\gamma(t)}(\dot\gamma(t))e(t), \quad \gamma(0) = p_0, \quad e(0) = e_0. \tag{3.13}$$

on the half-open interval $[t_0, T)$. In particular, $\dot e(t)\eta$ is orthogonal to $T_{\gamma(t)}M$ and $e(t)\zeta \in T_{\gamma(t)}M$ for all $\eta, \zeta \in \mathbb{R}^m$ and $t \in [t_0, T)$. Hence

$$\frac{d}{dt}\langle e(t)\eta, e(t)\zeta\rangle = \langle \dot e(t)\eta, e(t)\zeta\rangle + \langle e(t)\eta, \dot e(t)\zeta\rangle = 0. \tag{3.14}$$

In particular, the function $t \mapsto |e(t)\eta|$ is constant. This implies

$$\|e(t)\| := \sup_{\eta\neq 0} \frac{|e(t)\eta|}{|\eta|} = \sup_{\eta \neq 0} \frac{|e_0\eta|}{|\eta|} = \|e_0\| \tag{3.15}$$

and thus

$$|\dot\gamma(t)| = |e(t)\xi(t)| \le \|e_0\| \, |\xi(t)| \le \|e_0\| \sup_{t_0 \le s \le T} |\xi(s)| =: c_T \tag{3.16}$$

for $t_0 \le t < T$. With the metric $d : M \times M \to [0, \infty)$ defined by (2.3) we obtain

$$d(\gamma(s), \gamma(t)) \le L(\gamma|_{[s,t]}) \le (t - s)c_T$$

for $t_0 < s < t < T$. Since (M, d) is a complete metric space it follows that $\gamma(t)$ converges to some point $p_1 \in M$ as t tends to T. Thus there is a constant $c > 0$ such that

$$\left\|h_{\gamma(t)}(v)\right\| := \sup_{0 \neq u \in T_pM} \frac{|h_p(v)u|}{|u|} \le c\,|v| \tag{3.17}$$

for $t_0 \le t < T$. Combining (3.13), (3.15), (3.16), and (3.17) we obtain

$$\begin{aligned}
\|\dot e(t)\| &= \left\|h_{\gamma(t)}(\dot\gamma(t))e(t)\right\| \\
&\le \left\|h_{\gamma(t)}(\dot\gamma(t))\right\| \|e(t)\| \\
&= \left\|h_{\gamma(t)}(\dot\gamma(t))\right\| \|e_0\| \\
&\le c\,|\dot\gamma(t)|\,\|e_0\| \\
&\le cc_T\,\|e_0\|.
\end{aligned}$$

This implies

$$\|e(t) - e(s)\| \le cc_T \|e_0\| (t - s)$$

for $t_0 < s < t < T$. Hence $e(t)$ converges to a matrix $e_1 \in \mathbb{R}^{n\times m}$ as t tends to T. The image of this limit matrix is contained in $T_{p_1}M$ because

$$\Pi(p_1)e_1 = \lim_{t\to T} \Pi(\gamma(t))e(t) = \lim_{t\to T} e(t) = e_1.$$

Moreover, the matrix e_1 is invertible because the function $t \mapsto e(t)^T e(t)$ is constant, by (3.14), and hence the matrix

$$e_1^T e_1 = \lim_{t\to T} e(t)^T e(t) = e_0^T e_0$$

is positive definite. Thus $\beta(t)$ converges to an element $(p_1, e_1) \in \mathcal{F}(M)$ as $t \in [t_0, T)$ tends to T and so, by Lemma 2.59, the solution β extends to the interval $[t_0, T + \varepsilon)$ for some $\varepsilon > 0$. A similar argument gives a contradiction in the case $T < t_0$ and this proves the theorem. □

Exercise 3.16 (Basic vector fields in the intrinsic setting). Let M be a Riemannian m-manifold with an atlas $\mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$. Prove that the frame bundle (3.7) admits the structure of a smooth manifold with the open cover $\widetilde{U}_\alpha := \pi^{-1}(U_\alpha)$ and coordinate charts

$$\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to \phi_\alpha(U_\alpha) \times \mathrm{GL}(m)$$

given by

$$\widetilde{\phi}_\alpha(p, e) := (\phi_\alpha(p), d\phi_\alpha(p)e).$$

Prove that the derivatives of the horizontal curves in Definition 3.11 form a horizontal subbundle $H \subset \mathcal{F}(M)$ whose fibers $H_{(p,e)}$ can in local coordinates be described as follows. Let

$$x := \phi_\alpha(p), \qquad a := d\phi_\alpha(p)e \in \mathrm{GL}(m).$$

and $(\hat{x}, \hat{a}) \in \mathbb{R}^m \times \mathbb{R}^{m\times m}$. This pair has the form

$$(\hat{x}, \hat{a}) = d\widetilde{\phi}_\alpha(p, e)(\hat{p}, \hat{e}), \qquad (\hat{p}, \hat{e}) \in H_{(p,e)},$$

if and only if

$$\hat{a}_\ell^k = -\sum_{i,j=1}^m \Gamma_{\alpha,ij}^k(x)\hat{x}^i a_\ell^j$$

for $k, \ell = 1, \dots, m$, where the functions $\Gamma_{\alpha,ij}^k : \phi_\alpha(U_\alpha) \to \mathbb{R}$ are the Christoffel symbols. Show that, for every $\xi \in \mathbb{R}^m$ there is a unique horizontal vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ such that $d\pi(p, e)B_\xi(p, e) = e\xi$ for $(p, e) \in \mathcal{F}(M)$.

Exercise 3.17. Extend Theorem 3.15 to the intrinsic setting.

3.3 Motions and developments

Our aim in this sections is to define motion without sliding, twisting, or wobbling. This is the motion that results when a heavy object is rolled, with a minimum of friction, along the floor. It is also the motion of the large snowball a child creates as it rolls it into the bottom part of a snowman.

We shall eventually justify mathematically the physical intuition that either of the curves of contact in such ideal rolling may be specified arbitrarily; the other is then determined uniquely. Thus for example the heavy object may be rolled along an arbitrary curve on the floor; if that curve is marked in wet ink another curve will be traced in the object. Conversely if a curve is marked in wet ink on the object, the object may be rolled so as to trace a curve on the floor. However, if both curves are prescribed, it will be necessary to slide the object as it is being rolled if one wants to keep the curves in contact.

We assume throughout this section that M and M' are two m-dimensional submanifolds of $\mathbb{R}^n$. Objects on M' will be denoted by the same letter as the corresponding objects on M with primes affixed. Thus for example $\Pi'(p') \in \mathbb{R}^{n\times n}$ denotes the orthogonal projection of $\mathbb{R}^n$ onto the tangent space $T_{p'}M'$, ∇' denotes the covariant derivative of a vector field along a curve in M', and $\Phi'_{\gamma'}$ denotes parallel transport along a curve in M'.

3.3.1 Motion

Definition 3.18. *A* **motion of M along M' (on an interval $I \subset \mathbb{R}$)** *is a triple (Ψ, γ, γ') of smooth maps*

$$\Psi : I \to \mathrm{O}(n), \qquad \gamma : I \to M, \qquad \gamma' : I \to M'$$

such that

$$\Psi(t)T_{\gamma(t)}M = T_{\gamma'(t)}M' \qquad \forall\, t \in I.$$

Note that a motion also matches normal vectors, i.e.

$$\Psi(t)T_{\gamma(t)}M^{\perp} = T_{\gamma'(t)}M'^{\perp} \qquad \forall\, t \in I.$$

Remark 3.19. Associated to a motion (Ψ, γ, γ') of M along M' is a family of (affine) isometries $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\psi_t(p) := \gamma'(t) + \Psi(t)\big(p - \gamma(t)\big) \tag{3.18}$$

for $t \in I$ and $p \in \mathbb{R}^n$. These isometries satisfy

$$\psi_t(\gamma(t)) = \gamma'(t), \qquad d\psi_t(\gamma(t))T_{\gamma(t)}M = T_{\gamma'(t)}M' \qquad \forall\, t \in I.$$

Remark 3.20. There are three operations on motions.

Reparametrization. If (Ψ, γ, γ') is a motion of M along M' on an interval $I \subset \mathbb{R}$ and $\alpha : J \to I$ is a smooth map between intervals then the triple $(\Psi \circ \alpha, \gamma \circ \alpha, \gamma' \circ \alpha)$ is a motion of M along M' on the interval J.

Inversion. If (Ψ, γ, γ') is a motion of M along M' then $(\Psi^{-1}, \gamma, \gamma)$ is a motion of M' along M.

Composition. If (Ψ, γ, γ') is a motion of M along M' on an interval I and $(\Psi', \gamma', \gamma'')$ is a motion of M' along M'' on the same interval, then $(\Psi'\Psi, \gamma, \gamma'')$ is a motion of M along M''.

We now give the three simplest examples of "bad" motions; i.e. motions which do not satisfy the concepts we are about to define. In all three of these examples, p is a point of M and M' is the affine tangent space to M at p:

$$M' := p + T_pM = \{p + v \,|\, v \in T_pM\}.$$

Example 3.21 (Pure sliding). Take a nonzero tangent vector $v \in T_pM$ and let

$$\gamma(t) := p, \qquad \gamma'(t) = p + tv, \qquad \Psi(t) := \mathbb{1}.$$

Then $\dot\gamma(t) = 0$, $\dot\gamma'(t) = v \neq 0$, and so $\Psi(t)\dot\gamma(t) \neq \dot\gamma'(t)$. (See Figure 3.2.)

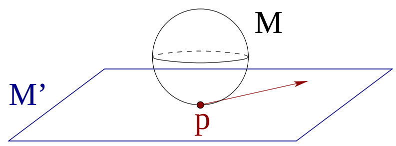


Figure 3.2: Pure sliding.

Example 3.22 (Pure twisting). Let γ and γ' be the constant curves $\gamma(t) = \gamma'(t) = p$ and take $\Psi(t)$ to be the identity on $T_pM^\perp$ and any curve of rotations on the tangent space T_pM. As a concrete example with $m = 2$ and $n = 3$ one can take M to be the sphere of radius one centered at the point $(0, 1, 0)$ and p to be the origin:

$$M := \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + (y-1)^2 + z^2 = 1\right\} p := (0, 0, 0).$$

Then M' is the (x, z)-plane and $A(t)$ is any curve of rotations in the (x, z)-plane, i.e. about the y-axis $T_pM^\perp$. (See Figure 3.3.)

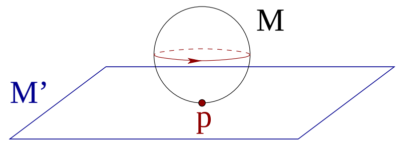


Figure 3.3: Pure twisting.

Example 3.23 (Pure wobbling). This is the same as pure twisting except that $\Psi(t)$ is the identity on T_pM and any curve of rotations on $T_pM^\perp$. As a concrete example with $m = 1$ and $n = 3$ one can take M to be the circle of radius one in the (x, y)-plane centered at the point $(0, 1, 0)$ and p to be the origin:

$$M := \left\{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + (y-1)^2 = 1\right\}, \qquad p := (0, 0, 0).$$

Then M' is the x-axis and $\Psi(t)$ is any curve of rotations in the (y, z)-plane, i.e. about the axis M'. (See Figure 3.4.)

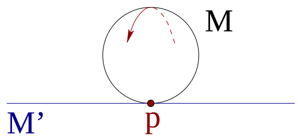


Figure 3.4: Pure wobbling.

3.3.2 Sliding

When a train slides on the track (e.g. in the process of stopping suddenly), there is a terrific screech. Since we usually do not hear a screech, this means that the wheel moves along without sliding. In other words the velocity of the point of contact in the train wheel M equals the velocity of the point of contact in the track M'. But the track is not moving; hence the point of contact in the wheel is not moving. One may explain the paradox this way: the train is moving forward and the wheel is rotating around the axle. The velocity of a point on the wheel is the sum of these two velocities. When the point is on the bottom of the wheel, the two velocities cancel.

Definition 3.24. *A motion* (Ψ, γ, γ') *is* **without sliding** *if*

$$\Psi(t)\dot{\gamma}(t) = \dot{\gamma}'(t)$$

for every t.

Here is the geometric picture of the no sliding condition. As explained in Remark 3.19 we can view a motion as a smooth family of isometries

$$\psi_t(p) := \gamma'(t) + \Psi(t)\big(p - \gamma(t)\big)$$

acting on the manifold M with $\gamma(t) \in M$ being the point of contact with M'. Differentiating the curve $t \mapsto \psi_t(p)$ which describes the motion of the point $p \in M$ in the space $\mathbb{R}^n$ we obtain

$$\frac{d}{dt}\psi_t(p) = \dot{\gamma}'(t) - \Psi(t)\dot{\gamma}(t) + \dot{\Psi}(t)\big(p - \gamma(t)\big).$$

Taking $p = \gamma(t_0)$ we find

$$\left.\frac{d}{dt}\right|_{t=t_0} \psi_t(\gamma(t_0)) = \dot{\gamma}'(t_0) - \Psi(t_0)\dot{\gamma}(t_0).$$

This expression vanishes under the no sliding condition. In general the curve $t \mapsto \psi_t(\gamma(t_0))$ will be non-constant, but (when the motion is without sliding) its velocity will vanish at the instant $t = t_0$; i.e. at the instant when it becomes the point of contact. In other words *the motion is without sliding if and only if the point of contact is motionless.*

We remark that if the motion is without sliding we have:

$$\left|\dot{\gamma}'(t)\right| = |\Psi(t)\dot{\gamma}(t)| = |\dot{\gamma}(t)|$$

so that the curves γ and γ' have the same arclength:

$$\int_{t_0}^{t_1} \left|\dot{\gamma}'(t)\right| \, dt = \int_{t_0}^{t_1} |\dot{\gamma}(t)| \, dt$$

on any interval $[t_0, t_1] \subset I$. Hence any motion with $\dot{\gamma} = 0$ and $\dot{\gamma}' \neq 0$ is not without sliding (such as the example of pure sliding above).

Exercise 3.25. Give an example of a motion where $|\dot{\gamma}(t)| = |\dot{\gamma}'(t)|$ for every t but which is not without sliding.

Example 3.26. We describe mathematically the motion of the train wheel. Let the center of the wheel move right parallel to the x-axis at height one and the wheel have radius one and make one revolution in 2π units of time. Then the track M' is the x-axis and we take

$$M := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + (y-1)^2 = 1\right\}.$$

Choose

$$\begin{aligned}\gamma(t) &= (\cos(t-\pi/2), 1+\sin(t-\pi/2)) \\ &= (\sin(t), 1-\cos(t)), \\ \gamma'(t) &= (t, 0),\end{aligned}$$

and define $A(t) \in \mathrm{GL}(2)$ by

$$\Psi(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

The reader can easily verify that this is a motion without sliding. A fixed point p_0 on M, say $p_0 = (0,0)$, sweeps out a cycloid with parametric equations

$$x = t - \sin(t), \qquad y = 1 - \cos(t).$$

(Check that $(\dot{x}, \dot{y}) = (0,0)$ when $y = 0$; i.e. for $t = 2n\pi$.)

Remark 3.27. These same formulas give a motion of a sphere M rolling without sliding along a straight line in a plane M'. Namely in coordinates (x, y, z) the sphere has equation

$$x^2 + (y-1)^2 + z^2 = 1,$$

the plane is $y = 0$ and the line is the x-axis. The z-coordinate of a point is unaffected by the motion. Note that the curve γ' traces out a straight line in the plane M' and the curve γ traces out a great circle on the sphere M.

Remark 3.28. The operations of reparametrization, inversion, and composition respect motion without sliding; i.e. if (Ψ, γ, γ') and $(\Psi', \gamma', \gamma'')$ are motions without sliding on an interval I and $\alpha : J \to I$ is a smooth map between intervals, then the motions $(\Psi \circ \alpha, \gamma \circ \alpha, \gamma' \circ \alpha)$, $(\Psi^{-1}, \gamma', \gamma)$, and $(\Psi'\Psi, \gamma, \gamma'')$ are also without sliding. The proof is immediate from the definition.

3.3.3 Twisting and wobbling

A motion (Ψ, γ, γ') on an intervall $I \subset \mathbb{R}$ transforms vector fields along γ into vector fields along γ' by the formula

$$X'(t) = (\Psi X)(t) := \Psi(t)X(t) \in T_{\gamma'(t)}M'$$

for $t \in I$ and $X \in \mathrm{Vect}(\gamma)$; so $X' \in \mathrm{Vect}(\gamma')$.

Lemma 3.29. *Let* (Ψ, γ, γ') *be a motion of* M *along* M' *on an interval* $I \subset \mathbb{R}$. *Then the following are equivalent.*

(i) *The instantaneous velocity of each tangent vector is normal, i.e. for* $t \in I$

$$\dot{\Psi}(t) T_{\gamma(t)} M \subset T_{\gamma'(t)} M'^{\perp}.$$

(ii) Ψ *intertwines covariant differentiation, i.e. for* $X \in \mathrm{Vect}(\gamma)$

$$\nabla'(\Psi X) = \Psi \nabla X.$$

(iii) Ψ *transforms parallel vector fields along* γ *into parallel vector fields along* γ', *i.e. for* $X \in \mathrm{Vect}(\gamma)$

$$\nabla X = 0 \qquad \Longrightarrow \qquad \nabla'(\Psi X) = 0.$$

(iv) Ψ *intertwines parallel transport, i.e. for* $s, t \in I$ *and* $v \in T_{\gamma(s)} M$

$$\Psi(t) \Phi_\gamma(t, s) v = \Phi'_{\gamma'}(t, s) \Psi(s) v.$$

A motion that satisfies these conditions is called **without twisting**.

Proof. We prove that (i) is equivalent to (ii). A motion satisfies the equation

$$\Psi(t) \Pi(\gamma(t)) = \Pi'(\gamma'(t)) \Psi(t)$$

for every $t \in I$. This restates the condition that $\Psi(t)$ maps tangent vectors of M to tangent vectors of M' and normal vectors of M to normal vectors of M'. Differentiating the equation $X'(t) = A(t) X(t)$ we obtain

$$\dot{X}'(t) = \Psi(t) \dot{X}(t) + \dot{\Psi}(t) X(t).$$

Applying $\Pi'(\gamma'(t))$ this gives

$$\nabla' X' = \Psi \nabla X + \Pi'(\gamma') \dot{\Psi} X.$$

Hence (ii) holds if and only if $\Pi'(\gamma'(t)) \dot{\Psi}(t) = 0$ for every $t \in I$. Thus we have proved that (i) is equivalent to (ii). That (ii) implies (iii) is obvious.

We prove that (iii) implies (iv). Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)} M$. Define $X \in \mathrm{Vect}(\gamma)$ by $X(t) := \Phi_\gamma(t, t_0) v_0$ and let $X' := \Psi X \in \mathrm{Vect}(\gamma')$ Then $\nabla X = 0$, hence by (iii) $\nabla' X' = 0$, and hence

$$X'(t) = \Phi'_{\gamma'}(t, s) X'(t_0) = \Phi'_{\gamma'}(t, s) \Psi(t_0) v_0.$$

Since $X'(t) = \Psi(t) X(t) = \Psi(t) \Phi_\gamma(t, t_0) v_0$, this implies (iv).

We prove that (iv) implies (ii). Let $X \in \mathrm{Vect}(\gamma)$ and $X' := \Psi X \in \mathrm{Vect}(\gamma')$. By (iv) we have

$$\Phi'_{\gamma'}(t_0, t)X'(t) = \Psi(t_0)\Phi_\gamma(t_0, t)X(t).$$

Differentiating this equation with respect to t at $t = t_0$ and using Theorem 3.4, we obtain $\nabla' X'(t_0) = \Psi(t_0)\nabla X(t_0)$. This proves the lemma. □

Lemma 3.30. *Let (Ψ, γ, γ') be a motion of M along M' on an interval $I \subset \mathbb{R}$. Then the following are equivalent.*

(i) *The instantaneous velocity of each normal vector is tangent, i.e. for $t \in I$*

$$\dot\Psi(t)T_{\gamma(t)}M^\perp \subset T_{\gamma'(t)}M'.$$

(ii) *Ψ intertwines normal covariant differentiation, i.e. for $Y \in \mathrm{Vect}^\perp(\gamma)$*

$$\nabla'^\perp(\Psi Y) = \Psi\nabla^\perp Y.$$

(iii) *Ψ transforms parallel normal vector fields along γ into parallel normal vector fields along γ', i.e. for $Y \in \mathrm{Vect}^\perp(\gamma)$*

$$\nabla^\perp Y = 0 \qquad \Longrightarrow \qquad \nabla'^\perp(\Psi Y) = 0.$$

(iv) *Ψ intertwines parallel transport of normal vector fields, i.e. for $s, t \in I$ and $w \in T_{\gamma(s)}M^\perp$*

$$\Psi(t)\Phi^\perp_\gamma(t, s)w = \Phi'^\perp_{\gamma'}(t, s)\Psi(s)w.$$

A motion that satisfies these conditions is called **without wobbling**.

The proof of Lemma 3.30 is analogous to that of Lemma 3.29 and will be omitted.

In summary a *motion is without twisting iff tangent vectors at the point of contact are rotating towards the normal space and it is without wobbling iff normal vectors at the point of contact are rotating towards the tangent space.* In case $m = 2$ and $n = 3$ motion without twisting means that the instantaneous axis of rotation is parallel to the tangent plane.

Remark 3.31. The operations of reparametrization, inversion, and composition respect motion without twisting, respectively without wobbling; i.e. if (Ψ, γ, γ') and $(\Psi', \gamma', \gamma'')$ are motions without twisting, respectively without wobbling, on an interval I and $\alpha : J \to I$ is a smooth map between intervals, then the motions $(\Psi \circ \alpha, \gamma \circ \alpha, \gamma' \circ \alpha)$, $(\Psi^{-1}, \gamma', \gamma)$, and $(\Psi'\Psi, \gamma, \gamma'')$ are also without twisting, respectively without wobbling.

Remark 3.32. Let $I \subset \mathbb{R}$ be an interval and $t_0 \in I$. Given curves $\gamma : I \to M$ and $\gamma' : I \to M'$ and an orthogonal matrix $\Psi_0 \in \mathrm{O}(n)$ such that

$$\Psi_0 T_{\gamma(t_0)} M = T_{\gamma'(t_0)} M'$$

there is a unique motion (Ψ, γ, γ') of M along M' (with the given γ and γ') **without twisting or wobbling** satisfying the initial condition:

$$\Psi(t_0) = \Psi_0.$$

Indeed, the path of matrices $\Psi : I \to \mathrm{O}(n)$ is uniquely determined by the conditions (iv) in Lemma 3.29 and Lemma 3.30. It is given by the explicit formula

$$\begin{aligned} \Psi(t)v = \Phi'_{\gamma'}(t, t_0)\Psi_0\Phi_\gamma(t_0, t)\Pi(\gamma(t))v \\ + \Phi'^{\perp}_{\gamma'}(t, t_0)\Psi_0\Phi^{\perp}_\gamma(t_0, t)\big(v - \Pi(\gamma(t))v\big) \end{aligned} \tag{3.19}$$

for $t \in I$ and $v \in \mathbb{R}^n$. We prove below a somewhat harder result where the motion is without twisting, wobbling, or sliding. It is in this situation that γ and γ' determine one another (up to an initial condition).

Remark 3.33. We can now give another interpretation of parallel transport. Given $\gamma : \mathbb{R} \to M$ and $v_0 \in T_{\gamma(t_0)} M$ take M' to be an affine subspace of the same dimension as M. Let (Ψ, γ, γ') be a motion of M along M' without twisting (and, if you like, without sliding or wobbling). Let $X' \in \mathrm{Vect}(\gamma')$ be the constant vector field along γ' (so that $\nabla' X' = 0$) with value

$$X'(t) = \Psi_0 v_0, \qquad \Psi_0 := \Psi(t_0).$$

Let $X \in \mathrm{Vect}(\gamma)$ be the corresponding vector field along γ so that

$$\Psi(t)X(t) = \Psi_0 v_0$$

Then $X(t) = \Phi_\gamma(t, t_0)v_0$. To put it another way, imagine that M is a ball. To define parallel transport along a given curve γ roll the ball (without sliding) along a plane M' keeping the curve γ in contact with the plane M'. Let γ' be the curve traced out in M'. If a constant vector field in the plane M' is drawn in wet ink along the curve γ' it will mark off a (covariant) parallel vector field along γ in M.

Exercise 3.34. Describe parallel transport along a great circle in a sphere.

3.3.4 Development

A development is an intrinsic version of motion without sliding or twisting.

Definition 3.35. *A* **development of** M **along** M' **(on an interval** I**)** *is a triple* (Φ, γ, γ') *where* $\gamma : I \to M$ *and* $\gamma' : I \to M'$ *are smooth paths and* Φ *is a family of orthogonal isomorphisms*

$$\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$$

such that Φ *intertwines parallel transport, i.e.*

$$\Phi(t)\Phi_\gamma(t, s) = \Phi'_{\gamma'}(t, s)\Phi(s)$$

for all $s, t \in I$*, and*

$$\Phi(t)\dot{\gamma}(t) = \dot{\gamma}'(t)$$

for all $t \in I$*. In particular, the family* Φ *of isomorphisms is smooth, i.e. if* X *is a smooth vector field along* γ *then the formula* $X'(t) := \Phi(t)X(t)$ *defines a smooth vector field along* γ'*.*

Lemma 3.36. *Let* $I \subset \mathbb{R}$ *be an interval,* $\gamma : I \to M$ *and* $\gamma' : I \to M'$ *be smooth curves, and* $\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ *be a family of orthogonal isomorphisms parametrized by* $t \in I$*. Then the following are equivalent.*

(i) (Φ, γ, γ') *is a development.*

(ii) *There exists a motion* (Ψ, γ, γ') *without sliding and twisting such that* $\Phi(t) = \Psi(t)|_{T_{\gamma(t)}M}$ *for every* $t \in I$*.*

(iii) *There exists a motion* (Ψ, γ, γ') *of* M *along* M' *without sliding, twisting, and wobbling such that* $\Phi(t) = \Psi(t)|_{T_{\gamma(t)}M}$ *for every* $t \in I$*.*

Proof. That (iii) implies (ii) and (ii) implies (i) is obvious. To prove that (i) implies (iii) choose any $t_0 \in I$ and any orthogonal matrix $\Psi_0 \in \mathrm{O}(n)$ such that $\Psi_0|_{T_{\gamma(t_0)}M} = \Phi(t_0)$ and define $\Psi(t) : \mathbb{R}^n \to \mathbb{R}^n$ by (3.19). □

Remark 3.37. The operations of reparametrization, inversion, and composition yield developments when applied to developments; i.e. if (Φ, γ, γ') is a development of M along M', on an interval I, $(\Phi', \gamma', \gamma'')$ is a development of M' along M'' on the same interval I, and $\alpha : J \to I$ is a smooth map of intervals, then the triples

$$(\Phi \circ \alpha, \gamma \circ \alpha, \gamma' \circ \alpha), \qquad (\Phi^{-1}, \gamma', \gamma)), \qquad (\Phi'\Phi, \gamma, \gamma'')$$

are all developments.

Theorem 3.38 (Development theorem). *Let $p_0 \in M$, $p_0' \in M'$ and*

$$\Phi_0 : T_{p_0}M \to T_{p_0'}M'$$

be an orthogonal isomorphism. Fix a smooth curve $\gamma' : \mathbb{R} \to M$ and a time $t_0 \in \mathbb{R}$ such that $\gamma'(t_0) = p_0'$. Then the following holds.

(i) *There exists a development $(\Phi, \gamma, \gamma'|_I)$ on some open interval $I \subset \mathbb{R}$ containing t_0 that satisfies the initial condition*

$$\gamma(t_0) = p_0, \qquad \Phi(t_0) = \Phi_0. \tag{3.20}$$

(ii) *Any two developments $(\Phi_1, \gamma_1, \gamma'|_{I_1})$ and $\Phi_2, \gamma_2, \gamma'|_{I_2})$ as in (i) on two intervals I_1 and I_2 agree on the intersection $I_1 \cap I_2$, i.e. $\gamma_1(t) = \gamma_2(t)$ and $\Phi_1(t) = \Phi_2(t)$ for every $t \in I_1 \cap I_2$.*

(iii) *If M is complete then (i) holds with $I = \mathbb{R}$.*

Proof. Let $\gamma : \mathbb{R} \to M$ be any smooth map such that $\gamma(t_0) = p_0$ and define $\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ by

$$\Phi(t) := \Phi'_{\gamma'}(t, t_0)\Phi_0\Phi_\gamma(t_0, t) \tag{3.21}$$

This is an orthogonal transformation for every t and it intertwines parallel transport. However, in general $\Phi(t)\dot\gamma(t)$ will not be equal to $\dot\gamma'(t)$. To construct a development that satisfies this condition, we choose an orthonormal frame $e_0 : \mathbb{R}^m \to T_{p_0}M$ and, for $t \in \mathbb{R}$, define $e(t) : \mathbb{R}^m \to T_{\gamma(t_0)}M$ by

$$e(t) := \Phi_\gamma(t, t_0)e_0. \tag{3.22}$$

We can think of $e(t)$ as a real $k\times m$-matrix and the map $\mathbb{R} \to \mathbb{R}^{n\times m} : t \mapsto e(t)$ is smooth. In fact, the map $t \mapsto (\gamma(t), \mathbf{e}(t))$ is a smooth path in the frame bundle $\mathcal{F}(M)$. Define the smooth map $\xi : \mathbb{R} \to \mathbb{R}^m$ by

$$\dot\gamma'(t) = \Phi'_{\gamma'}(t, t_0)\Phi_0 e_0 \xi(t). \tag{3.23}$$

We prove the following

Claim: *The triple (Φ, γ, γ') is a development on an interval $I \subset \mathbb{R}$ if and only if the path $t \mapsto (\gamma(t), e(t))$ satisfies the differential equation*

$$(\dot\gamma(t), \dot e(t)) = B_{\xi(t)}(\gamma(t), e(t)) \tag{3.24}$$

for every $t \in I$, where $B_{\xi(t)} \in \mathrm{Vect}(\mathcal{F}(M))$ denotes the basic vector field associated to $\xi(t) \in \mathbb{R}^m$ (see equation (3.10)*).*

The triple (Φ, γ, γ') is a development on I if and only if $\Phi(t)\dot{\gamma}(t) = \dot{\gamma}'(t)$ for every $t \in I$. By (3.21) and (3.23) this is equivalent to the condition

$$\Phi'_{\gamma'}(t, t_0)\Phi_0\Phi_\gamma(t_0, t)\dot{\gamma}(t) = \dot{\gamma}'(t) = \Phi'_{\gamma'}(t, t_0)\Phi_0 e_0 \xi(t),$$

hence to

$$\Phi_\gamma(t_0, t)\dot{\gamma}(t) = e_0\xi(t),$$

and hence to

$$\dot{\gamma}(t) = \Phi_\gamma(t, t_0)e(t)\xi(t) = e(t)\xi(t) \tag{3.25}$$

for every $t \in I$. By (3.22) and the Gauss–Weingarten formula, we have

$$\dot{e}(t) = h_{\gamma(t)}(\dot{\gamma}(t))e(t)$$

for every $t \in \mathbb{R}$. Hence it follows from (3.10) that (3.25) is equivalent to (3.24). This proves the claim.

Assertions (i) and (ii) follow immediately from the claim. Assertion (iii) follows from the claim and Theorem 3.15. This proves the theorem. □

Remark 3.39. As any two developments $(\Phi_1, \gamma_1, \gamma'|_{I_1})$ and $(\Phi_2, \gamma_2, \gamma'|_{I_2})$ on two intervals I_1 and I_2 satisfying the initial condition (3.20) agree on $I_1 \cap I_2$ there is a development defined on $I_1 \cup I_2$. Hence there is a unique *maximally defined development* $(\Phi, \gamma, \gamma'|_I)$, defined on a maximal interval I, associated to γ', p_0, Φ_0.

Remark 3.40. The statement of Theorem 3.38 is essentially symmetric in M and M' as the operation of inversion carries developments to developments. Hence given $\gamma : \mathbb{R} \to M$, $p_0' \in M'$, $t_0 \in \mathbb{R}$, and $\Phi_0 : T_{\gamma(t_0)}M \to T_{p_0'}M'$, we may speak of the development (Φ, γ, γ') corresponding to γ with initial conditions $\gamma'(t_0) = p_0'$ and $\Phi(t_0) = \Phi_0$.

Corollary 3.41. *Let $p_0 \in M$, $p_0' \in M'$ and $\Psi_0 \in \mathrm{O}(n)$ such that*

$$\Psi_0 T_{p_0}M = T_{p_0'}M'.$$

Fix a smooth curve $\gamma' : \mathbb{R} \to M$ and a time $t_0 \in \mathbb{R}$ such that $\gamma'(t_0) = p_0'$. Then the following holds.

(i) *There exists a motion $(\Psi, \gamma, \gamma'|_I)$ without sliding, twisting and wobbling on some open interval $I \subset \mathbb{R}$ containing t_0 that satisfies the initial condition $\gamma(t_0) = p_0$ and $\Psi(t_0) = \Psi_0$.*

(ii) *Any two motions as in (i) on two intervals I_1 and I_2 agree on the intersection $I_1 \cap I_2$.*

(iii) *If M is complete then (i) holds with $I = \mathbb{R}$.*

Proof. Theorem 3.38 and Remark 3.32. □

Chapter 4

Curvature

4.1 Isometries

Let M and M' be submanifolds of $\mathbb{R}^n$. An isometry is an isomorphism of the intrinsic geometries of M and M'. Recall the definition of the intrinsic distance function $d : M \times M \to [0, \infty)$ (in Section 2.1) by

$$d(p,q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma), \qquad L(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt$$

for $p, q \in M$. Let d' denote the intrinisic distance function on M'.

Theorem 4.1 (**Isometries**). *Let $\phi : M \to M'$ be a bijective map. Then the following are equivalent.*

(i) *ϕ intertwines the distance functions on M and M', i.e. for all $p, q \in M$*

$$d'(\phi(p), \phi(q)) = d(p,q)$$

(ii) *ϕ is a diffeomorphism and $d\phi(p) : T_pM \to T_{\phi(p)}M'$ is an orthogonal isomorphism for every $p \in M$.*

(iii) *ϕ is a diffeomorphism and $L(\phi \circ \gamma) = L(\gamma)$ for every smooth curve $\gamma : [a, b] \to M$.*

ϕ is called an **isometry** *if it satisfies these equivalent conditions. In the case $M = M'$ the isometries $\phi : M \to M$ form a group denoted by $\mathcal{I}(M)$ and called the* **isometry group** *of M.*

Lemma 4.2. *For every $p \in M$ there is a constant $\varepsilon > 0$ such that, for all $v, w \in T_pM$ with $0 < |w| < |v| < \varepsilon$, we have*

$$d(\exp_p(w), \exp_p(v)) = |v| - |w| \qquad \Longrightarrow \qquad w = \frac{|w|}{|v|} v. \tag{4.1}$$

Remark 4.3. It follows from the triangle inequality and Theorem 2.44 that

$$d(\exp_p(v), \exp_p(w)) \geq d(\exp_p(v), p) - d(\exp_p(w), p) = |v| - |w|$$

whenever $0 < |w| < |v| < \mathrm{inj}(p)$. Lemma 4.2 asserts that equality can only hold when w is a positive multiple of v or, to put it differently, that the distance between $\exp_p(v)$ and $\exp_p(w)$ must be strictly bigger that $|v| - |w|$ whenever w is not a positive multiple of v.

Proof of Lemma 4.2. As in Corollary 2.38 we denote

$$B_\varepsilon(p) := \{v \in T_pM \mid |v| < \varepsilon\}, \\ U_\varepsilon(p) := \{q \in M \mid d(p,q) < \varepsilon\}.$$

By Theorem 2.44 and the definition of the injectivity radius, the exponential map at p is a diffeomorphism $\exp_p : B_\varepsilon(p) \to U_\varepsilon(p)$ for $\varepsilon < \mathrm{inj}(p)$. Choose $0 < r < \mathrm{inj}(p)$. Then the closure of $U_r(p)$ is a compact subset of M. Hence there is a constant $\varepsilon > 0$ such that $\varepsilon < r$ and $\mathrm{inj}(p') < \varepsilon$ for every $p' \in \overline{U_r(p)}$ Since $\varepsilon < r$ we have

$$\varepsilon < \mathrm{inj}(p') \qquad \forall\, p' \in U_\varepsilon(p). \tag{4.2}$$

Thus $\varepsilon_{p'} : B_\varepsilon(p') \to U_\varepsilon(p')$ is a diffeomorphism for every $p' \in U_\varepsilon(p)$. Denote

$$p_1 := \exp_p(w), \qquad p_2 := \exp_p(v).$$

Then, by assumption, we have

$$d(p_1, p_2) = |v| - |w| < \varepsilon.$$

Since $p_1 \in U_\varepsilon(p)$ it follows from our choice of ε that $\varepsilon < \mathrm{inj}(p_1)$. Hence there is a unique tangent vector $v_1 \in T_{p_1}M$ such that

$$|v_1| = d(p_1, p_2) = |v| - |w|, \qquad \exp_{p_1}(v_1) = p_2.$$

Following first the shortest geodesic from p to p_1 and then the shortest geodesic from p_1 to p_2 we obtain (after suitable reparametrization) a smooth $\gamma : [0,2] \to M$ such that

$$\gamma(0) = p, \qquad \gamma(1) = p_1, \qquad \gamma(2) = p_2,$$

and

$$L(\gamma|_{[0,1]}) = d(p, p_1) = |w|, \qquad L(\gamma|_{[1,2]}) = d(p_1, p_2) = |v| - |w|.$$

Thus $L(\gamma) = |v| = d(p, p_2)$. Hence, by Theorem 2.44, there is a smooth function $\beta : [0, 2] \to [0, 1]$ satisfying

$$\beta(0) = 0, \qquad \beta(2) = 1, \qquad \dot{\beta}(t) \geq 0, \qquad \gamma(t) = \exp_p(\beta(t)v)$$

for every $t \in [0, 2]$. This implies

$$\exp_p(w) = p_1 = \gamma(1) = \exp_p(\beta(1)v), \qquad 0 \leq \beta(1) \leq 1.$$

Since w and $\beta(1)v$ are both elements of $B_\varepsilon(p)$ and $\exp_p$ is injective on $B_\varepsilon(p)$, this implies $w = \beta(1)v$. Since $\beta(1) \geq 0$ we have $\beta(1) = |w| \,/\, |v|$. This proves (4.1) and the lemma. □

Proof of Theorem 4.1. That (ii) implies (iii) follows from the definition of the length of a curve. Namely

$$\begin{aligned} L(\phi \circ \gamma) &= \int_a^b \left| \frac{d}{dt} \phi(\gamma(t)) \right| dt \\ &= \int_a^b |d\phi(\gamma(t))\dot{\gamma}(t)| \; dt \\ &= \int_a^b |\dot{\gamma}(t)| \; dt \\ &= L(\gamma). \end{aligned}$$

In the third equation we have used (ii). That (iii) implies (i) follows immediately from the definition of the intrinsic distance functions d and d'.

We prove that (i) implies (ii). Fix a point $p \in M$ and choose $\varepsilon > 0$ so small that $\varepsilon < \mathrm{inj}(p)$ and that the assertion of Lemma 4.2 holds for the point $p' := \phi(p) \in M'$. Then there is a unique homeomorphism

$$\Phi_p : B_\varepsilon(p) \to B_\varepsilon(\phi(p))$$

such that the following diagram commutes.

$$\begin{array}{ccccccc} T_pM & \supset & B_\varepsilon(p) & \xrightarrow{\Phi_p} & B_\varepsilon(\phi(p)) & \subset & T_{\phi(p)}M' . \\ & & \Big\downarrow{\scriptstyle \exp_p} & & \Big\downarrow{\scriptstyle \exp'_{\phi(p)}} & & \\ M & \supset & U_\varepsilon(p) & \xrightarrow{\phi} & U_\varepsilon(\phi(p)) & \subset & M' \end{array}$$

Here the vertical maps are diffeomorphisms and $\phi : U_\varepsilon(p) \to U_\varepsilon(\phi(p))$ is a homeomorphism by (i). Hence $\Phi_p : B_\varepsilon(p) \to B_\varepsilon(\phi(p))$ is a homeomorphism.

Claim 1. *The map* Φ_p *satisfies the following equations for every* $v \in B_\varepsilon(p)$ *and every* $t \in [0,1]$*:*

$$\exp'_{\phi(p)}(\Phi_p(v)) = \phi(\exp_p(v)), \tag{4.3}$$

$$|\Phi_p(v)| = |v|\,, \tag{4.4}$$

$$\Phi_p(tv) = t\Phi_p(v). \tag{4.5}$$

Equation (4.3) holds by definition. To prove (4.4) we observe that, by Theorem 2.44, we have

$$\begin{aligned} |\Phi_p(v)| &= d'(\phi(p), \exp'_{\phi(p)}(\Phi_p(v))) \\ &= d'(\phi(p), \phi(\exp_p(v))) \\ &= d(p, \exp_p(v)) \\ &= |v|\,. \end{aligned}$$

Here the second equation follows from (4.3) and the third equation from (i). Equation (4.5) holds for $t = 0$ because $\Phi_p(0) = 0$ and for $t = 1$ it is a tautology. Hence assume $0 < t < 1$. Then

$$\begin{aligned} d'(\exp'_{\phi(p)}(\Phi_p(tv)), \exp'_{\phi(p)}(\Phi_p(v))) &= d'(\phi(\exp_p(tv)), \phi(\exp_p(v))) \\ &= d(\exp_p(tv), \exp_p(v) \\ &= |v| - |tv| \\ &= |\Phi_p(v)| - |\Phi_p(tv)|\,. \end{aligned}$$

Here the first equation follows from (4.3), the second equation from (i), the third equation from Theorem 2.44 and the fact that $|v| < \mathrm{inj}(p)$, and the last equation follows from (4.4). Since $0 < |\Phi_p(tv)| < |\Phi_p(v)| < \varepsilon$ we can apply Lemma 4.2 and obtain

$$\Phi_p(tv) = \frac{|\Phi_p(tv)|}{|\Phi_p(v)|}\Phi_p(v) = t\Phi_p(v).$$

This proves Claim 1.

By Claim 1, Φ_p extends to a bijective map $\Phi_p : T_pM \to T_{\phi(p)}M'$ via

$$\Phi_p(v) := \frac{1}{\delta}\Phi_p(\delta v),$$

where $\delta > 0$ is chosen so small that $\delta\,|v| < \varepsilon$. The right hand side of this equation is independent of the choice of δ. Hence the extension is well defined. It is bijective because the original map Φ_p is a bijection from $B_\varepsilon(p)$ to $B_\varepsilon(\phi(p))$. The reader may verify that the extended map satisfies the conditions (4.4) and (4.5) for all $v \in T_pM$ and all $t \geq 0$.

Claim 2. *The extended map $\Phi_p : T_pM \to T_{\phi(p)}M'$ is linear and preserves the inner product.*

It follows from the equation (2.46) in the proof of Lemma 2.61 that

$$\begin{aligned}
|v-w| &= \lim_{t\to 0}\frac{d(\exp_p(tv),\exp_p(tw))}{t} \\
&= \lim_{t\to 0}\frac{d'(\phi(\exp_p(tv)),\phi(\exp_p(tw)))}{t} \\
&= \lim_{t\to 0}\frac{d'(\exp'_{\phi(p)}(\Phi_p(tv))),\exp'_{\phi(p)}(\Phi_p(tw))))}{t} \\
&= \lim_{t\to 0}\frac{d'(\exp'_{\phi(p)}(t\Phi_p(v))),\exp'_{\phi(p)}(t\Phi_p(w))))}{t} \\
&= |\Phi_p(v)-\Phi_p(w)|\,.
\end{aligned}$$

Here the second equation follows from (i), the third from (4.3), the fourth from (4.4), and the last equation follows again from (2.46). By polarization we obtain

$$\begin{aligned}
2\langle v,w\rangle &= |v|^2+|w|^2-|v-w|^2 \\
&= |\Phi_p(v)|^2+|\Phi_p(w)|^2-|\Phi_p(v)-\Phi_p(w)|^2 \\
&= 2\langle\Phi_p(v),\Phi_p(w)\rangle.
\end{aligned}$$

Thus Φ_p preserves the inner product. Hence, for all $v_1, v_2, w \in T_pM$, we have

$$\begin{aligned}
\langle\Phi_p(v_1+v_2),\Phi_p(w)\rangle &= \langle v_1+v_2,w\rangle \\
&= \langle v_1,w\rangle+\langle v_2,w\rangle \\
&= \langle\Phi_p(v_1),\Phi_p(w)\rangle+\langle\Phi_p(v_2),\Phi_p(w)\rangle \\
&= \langle\Phi_p(v_1)+\Phi_p(v_2),\Phi_p(w)\rangle.
\end{aligned}$$

Since Φ_p is surjective, this implies

$$\Phi_p(v_1+v_2)=\Phi_p(v_1)+\Phi_p(v_2)$$

for all $v_1, v_2 \in T_pM$. With $v_1 = v$ and $v_2 = -v$ we obtain

$$\Phi_p(-v)=-\Phi_p(v)$$

for every $v \in T_pM$ and by (4.5) this gives

$$\Phi_p(tv)=t\Phi_p(v)$$

for all $v \in T_pM$ and $t \in \mathbb{R}$. This proves Claim 2.

Claim 3. *ϕ is smooth and $d\phi(p) = \Phi_p$.*

By (4.3) we have

$$\phi = \exp'_{\phi(p)} \circ \Phi_p \circ \exp_p^{-1} : U_\varepsilon(p) \to U_\varepsilon(\phi(p)).$$

Since Φ_p is linear, this shows that the restriction of ϕ to the open set $U_\varepsilon(p)$ is smooth. Moreover, for every $v \in T_pM$ we have

$$d\phi(p)v = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp_p(tv)) = \left.\frac{d}{dt}\right|_{t=0} \exp'_{\phi(p)}(t\Phi_p(v)) = \Phi_p(v).$$

Here we have used equations (4.3) and (4.5) as well as Lemma 2.36. This proves Claim 3 and the theorem. □

Exercise 4.4. Prove that every isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map

$$\psi(p) = Ap + b$$

where $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. Thus ψ is a composition of translation and rotation. **Hint:** Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^m$. Prove that any two vectors $v, w \in \mathbb{R}^n$ that satisfy $|v| = |w|$ and $|v - e_i| = |w - e_i|$ for $i = 1, \dots, n$ must be equal.

Remark 4.5. If $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of the ambient Euclidean space with $\psi(M) = M'$ then certainly $\phi := \psi|_M$ is an isometry from M onto M'. On the other hand, if M is a plane manifold

$$M = \{(0, y, z) \in \mathbb{R}^3 \,|\, 0 < y < \pi/2\}$$

and M' is the cylindrical manifold

$$M' = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ x > 0,\ y > 0\}$$

Then the map $\phi : M \to M'$ defined by

$$\phi(0, y, z) := (\cos(y), \sin(y), z)$$

is an isometry which is *not* of the form $\phi = \psi|_M$. Indeed, an isometry of the form $\phi = \psi|_M$ necessarily preserves the second fundamental form (as well as the first) in the sense that

$$d\psi(p)h_p(v, w) = h'_{\psi(p)}(d\psi(p)v, d\psi(p)w)$$

for $v, w \in T_pM$ but in the example h vanishes identically while h' does not.

We may thus distinguish two fundamental question:

I. Given M and M' when are they extrinsically isomorphic, i.e. when is there an ambient isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(M) = M'$?

II. Given M and M' when are they intrinsically isomorphic, i.e. when is there an isometry $\phi : M \to M'$ from M onto M'?

As we have noted, both the first and second fundamental forms are preserved by extrinsic isomorphisms while only the first fundamental form need be preserved by an intrinsic isomorphism (i.e. an isometry).

A question which occurred to Gauss (who worked for a while as a cartographer) is this: Can one draw a perfectly accurate map of a portion of the earth? (i.e. a map for which the distance between points on the map is proportional to the distance between the corresponding points on the surface of the earth). We can now pose this question as follows: Is there an isometry from an open subset of a sphere to an open subset of a plane? Gauss answered this question negatively by associating an invariant, the Gaussian curvature $K : M \to \mathbb{R}$, to a surface $M \subset \mathbb{R}^3$. According to his *Theorema Egregium*

$$K' \circ \phi = K$$

for an isometry $\phi : M \to M'$. The sphere has positive curvature; the plane has zero curvature; hence the perfectly accurate map does not exist. Our aim is to explain these ideas.

We shall need a concept slightly more general than that of "isometry".

Definition 4.6. *A smooth map $\phi : M \to M'$ is called a* **local isometry** *if its derivative*

$$d\phi(p) : T_pM \to T_{\phi(p)}M'$$

is an orthogonal linear isomorphism for every $p \in M$.

Remark 4.7. Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{m'}$ be manifolds and $\phi : M \to M'$ be a map. The following are equivalent.

(i) ϕ is a local isometry.

(ii) For every $p \in M$ there are open neighborhoods $U \subset M$ and $U' \subset M'$ such that the restriction of ϕ to U is an isometry from U onto U'.

That (ii) implies (i) follows immediately from Theorem 4.1. On the other hand (i) implies that $d\phi(p)$ is invertible so that (ii) follows from the inverse function theorem.

Example 4.8. The map

$$\mathbb{R} \to S^1 : \theta \mapsto e^{i\theta}$$

is a local isometry but not an isometry.

Exercise 4.9. Let $M \subset \mathbb{R}^n$ be a compact connected 1-manifold. Prove that M is diffeomorphic to the circle S^1. Define the length of a compact connected Riemannian 1-manifold. Prove that two compact connected 1-manifolds $M, M' \subset \mathbb{R}^n$ are isometric if and only if they have the same length. **Hint:** Let $\gamma : \mathbb{R} \to M$ be a geodesic with $|\dot{\gamma}(t)| \equiv 1$. Show that γ is not injective; otherwise construct an open cover of M without finite subcover. If $t_0 < t_1$ with $\gamma(t_0) = \gamma(t_1)$ show that $\dot{\gamma}(t_0) = \dot{\gamma}(t_1)$; otherwise show that $\gamma(t_0 + t) = \gamma(t_1 - t)$ for all t and find a contradiction.

We close this section with a result which asserts that two local isometries that have the same value and the same derivative at a single point must agree everywhere, provided that the domain is connected.

Lemma 4.10. *Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ be smooth m-manifolds and assume that M is connected. Let $\phi : M \to M'$ and $\psi : M \to M'$ be local isometries and let $p_0 \in M$ such that*

$$\psi(p_0) = \psi(p_0) =: p_0', \qquad d\phi(p_0) = d\psi(p_0) : T_{p_0}M \to T_{p_0'}M'.$$

Then $\phi(p) = \psi(p)$ for every $p \in M$.

Proof. Define the set

$$M_0 := \{p \in M \,|\, \phi(p) = \psi(p),\, d\phi(p) = d\psi(p)\}\,.$$

This set is obviously closed. We prove that M_0 is open. Let $p \in M_0$ and choose $U \subset M$ and $U' \subset M'$ as in Remark 4.7 (ii). Denote

$$\Phi_p := d\phi(p) = d\psi(p) : T_pM \to T_{p'}M', \qquad p' := \phi(p) = \psi(p)$$

The proof of Theorem 4.1 shows that there is a constant $\varepsilon > 0$ such that $U_\varepsilon(p) \subset U$, $U_\varepsilon(p') \subset U'$, and

$$q \in U_\varepsilon(p) \qquad \Longrightarrow \qquad \phi(q) = \exp'_{p'} \circ \Phi_p \circ \exp_p^{-1}(q) = \psi(q).$$

Hence $U_\varepsilon(p) \subset M_0$. Thus M_0 is open, closed, and nonempty. Since M is connected it follows that $M_0 = M$ and this proves the lemma. □

4.2 The Riemann curvature tensor

4.2.1 Definition and Gauss–Codazzi

Let $M \subset \mathbb{R}^n$ be a smooth manifold and $\gamma : \mathbb{R}^2 \to M$ be a smooth map. Denote by (s,t) the coordinates on $\mathbb{R}^2$. Let $Z \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ, i.e. $Z : \mathbb{R}^2 \to \mathbb{R}^n$ is a smooth map such that $Z(s,t) \in T_{\gamma(s,t)}M$ for all s and t. The **covariant partial derivatives** of Z with respect to the variables s and t are defined by

$$\nabla_s Z := \Pi(\gamma)\frac{\partial Z}{\partial s}, \qquad \nabla_t Z := \Pi(\gamma)\frac{\partial Z}{\partial t}.$$

In particular $\partial_s\gamma = \partial\gamma/\partial s$ and $\partial_t\gamma = \partial\gamma/\partial t$ are vector fields along γ and we have

$$\nabla_s\partial_t\gamma - \nabla_t\partial_s\gamma = 0$$

as both terms on the left are equal to $\Pi(\gamma)\partial_s\partial_t\gamma$. Thus ordinary partial differentiation and covariant partial differentiation commute. The analogous formula (which results on replacing ∂ by ∇ and γ by Z) is in general false. Instead we have the following.

Definition 4.11. *The* **Riemann curvature tensor** *assigns to each $p \in M$ the bilinear map*

$$R_p : T_pM \times T_pM \to \mathcal{L}(T_pM, T_pM)$$

characterized by the equation

$$R_p(u,v)w = \big(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\big)(0,0) \tag{4.6}$$

for $u, v, w \in T_pM$ where $\gamma : \mathbb{R}^2 \to M$ is a smooth map and $Z \in \mathrm{Vect}(\gamma)$ is a smooth vector field along γ such that

$$\gamma(0,0) = p, \qquad \partial_s\gamma(0,0) = u, \qquad \partial_t\gamma(0,0) = v, \qquad Z(0,0) = w. \tag{4.7}$$

We must prove that R is well defined, i.e. that the right hand side of equation (4.6) is independent of the choice of γ and Z. This follows from the Gauss–Codazzi formula which we prove next. Recall that the second fundamental form can be viewed as a linear map $h_p : T_pM \to \mathcal{L}(T_pM, T_pM^\perp)$ and that, for $u \in T_pM$, the linear map $h_p(u) \in \mathcal{L}(T_pM, T_pM^\perp)$ and its dual $h_p(u)^* \in \mathcal{L}(T_pM^\perp, T_pM)$ are given by

$$h_p(u)v = \big(d\Pi(p)u\big)v, \qquad h_p(u)^*w = \big(d\Pi(p)u\big)w$$

for $v \in T_pM$ and $w \in T_pM^\perp$.

Theorem 4.12. *The Riemann curvature tensor is well defined and given by the* **Gauss–Codazzi formula**

$$R_p(u,v) = h_p(u)^* h_p(v) - h_p(v)^* h_p(u) \tag{4.8}$$

for $u, v \in T_pM$.

Proof. Let $u, v, w \in T_pM$ and choose a smooth map $\gamma : \mathbb{R}^2 \to M$ and a smooth vector field Z along γ such that (4.7) holds. Then, by the Gauss–Weingarten formula (2.37), we have

$$\begin{aligned} \nabla_t Z &= \partial_t Z - h_\gamma(\partial_t\gamma)Z \\ &= \partial_t Z - \bigl(d\Pi(\gamma)\partial_t\gamma\bigr)Z \\ &= \partial_t Z - \bigl(\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\, Z. \end{aligned}$$

Hence

$$\begin{aligned} \partial_s\nabla_t Z &= \partial_s\partial_t Z - \partial_s\Bigl(\bigl(\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\, Z\Bigr) \\ &= \partial_s\partial_t Z - \bigl(\partial_s\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\, Z - \bigl(\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\,\partial_s Z \\ &= \partial_s\partial_t Z - \bigl(\partial_s\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\, Z - \bigl(d\Pi(\gamma)\partial_t\gamma\bigr)\bigl(\nabla_s Z + h_\gamma(\partial_s\gamma)Z\bigr) \\ &= \partial_s\partial_t Z - \bigl(\partial_s\partial_t\bigl(\Pi\circ\gamma\bigr)\bigr)\, Z - h_\gamma(\partial_t\gamma)\nabla_s Z - h_\gamma(\partial_t\gamma)^* h_\gamma(\partial_s\gamma)Z. \end{aligned}$$

Interchanging s and t and taking the difference we obtain

$$\begin{aligned} \partial_s\nabla_t Z - \partial_t\nabla_s Z &= h_\gamma(\partial_s\gamma)^* h_\gamma(\partial_t\gamma)Z - h_\gamma(\partial_t\gamma)^* h_\gamma(\partial_s\gamma)Z \\ &\quad + h_\gamma(\partial_s\gamma)\nabla_t Z - h_\gamma(\partial_t\gamma)\nabla_s Z. \end{aligned}$$

Here the first two terms on the right are tangent to M and the last two terms on the right are orthogonal to $T_\gamma M$. Hence

$$\begin{aligned} \nabla_s\nabla_t Z - \nabla_t\nabla_s Z &= \Pi(\gamma)\bigl(\partial_s\nabla_t Z - \partial_t\nabla_s Z\bigr) \\ &= h_\gamma(\partial_s\gamma)^* h_\gamma(\partial_t\gamma)Z - h_\gamma(\partial_t\gamma)^* h_\gamma(\partial_s\gamma)Z. \end{aligned}$$

Evaluating the right hand side at $s = t = 0$ we find that

$$\bigl(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\bigr)(0,0) = h_p(u)^* h_p(v)w - h_p(v)^* h_p(u)w.$$

This proves the Gauss–Codazzi equation and shows that the left hand side is independent of the choice of γ and Z. This proves the theorem. □

4.2.2 The covariant derivative of a global vector field

So far we have only defined the covariant derivatives of vector fields along curves. The same method can be applied to global vector fields. This leads to the following definition.

Definition 4.13 (Covariant derivative). *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and X be a vector field on M. Fix a point $p \in M$ and a tangent vector $v \in T_pM$. The* **covariant derivative of X at p in the direction** v *is the tangent vector*

$$\nabla_v X(p) := \Pi(p) dX(p) v \in T_pM,$$

where $\Pi(p) \in \mathbb{R}^{n \times n}$ denotes the orthogonal projection onto T_pM.

Remark 4.14. If $\gamma : I \to M$ is a smooth curve on an interval $I \subset \mathbb{R}$ and $X \in \mathrm{Vect}(M)$ is a smooth vector field on M then $X \circ \gamma$ is a smooth vector field along γ. The covariant derivative of $X \circ \gamma$ is related to the covariant derivative of X by the formula

$$\nabla(X \circ \gamma)(t) = \nabla_{\dot{\gamma}(t)} X(\gamma(t)). \tag{4.9}$$

Remark 4.15 (Gauss–Weingarten formula). Differentiating the equation $X = \Pi X$ (understood as a function from M to $\mathbb{R}^n$) and using the notation $\partial_v X(p) := dX(p)v$ for the derivative of X at p in the direction v we obtain the **Gauss–Weingarten formula** for global vector fields:

$$\partial_v X(p) = \nabla_v X(p) + h_p(v) X(p). \tag{4.10}$$

Remark 4.16 (Levi-Civita connection). Differentiating a vector field Y on M in the direction of another vector field X we obtain a vector field $\nabla_X Y \in \mathrm{Vect}(M)$ defined by

$$(\nabla_X Y)(p) := \nabla_{X(p)} Y(p)$$

for $p \in M$. This gives rise to a family of linear operators

$$\nabla_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M),$$

one for every vector field $X \in \mathrm{Vect}(M)$, and the assignment

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto \nabla_X$$

is itself a linear operator. This operator is called the **Levi-Civita connection** on the tangent bundle TM. It satisfies the conditions

$$\nabla_{fX}(Y) = f\nabla_X Y, \tag{4.11}$$

$$\nabla_X(fY) = f\nabla_X Y + (\mathcal{L}_X f)Y \tag{4.12}$$

$$\mathcal{L}_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{4.13}$$

$$\nabla_Y X - \nabla_X Y = [X, Y], \tag{4.14}$$

for all $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in \mathscr{F}(M)$, where $\mathcal{L}_X f = df \circ X$ and $[X, Y] \in \mathrm{Vect}(M)$ denotes the Lie bracket of the vector fields X and Y. The next lemma asserts that the Levi-Civita connection is uniquely determined by (4.13) and (4.14).

Lemma 4.17 (Uniqueness Lemma). *There is a unique linear operator*

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto \nabla_X$$

satisfying equations (4.13) *and* (4.14) *for all* $X, Y, Z \in \mathrm{Vect}(M)$.

Proof. Existence follows from the properties of the Levi-Civita connection. We prove uniqueness. Let $X \mapsto D_X$ be any linear operator from $\mathrm{Vect}(M)$ to $\mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M))$ that satisfies (4.13) and (4.14). Then we have

$$\begin{aligned} \mathcal{L}_X\langle Y, Z\rangle &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle, \\ \mathcal{L}_Y\langle X, Z\rangle &= \langle D_Y X, Z\rangle + \langle X, D_Y Z\rangle, \\ -\mathcal{L}_Z\langle X, Y\rangle &= -\langle D_Z X, Y\rangle - \langle X, D_Z Y\rangle. \end{aligned}$$

Adding these three equations we find

$$\begin{aligned} &\mathcal{L}_X\langle Y, Z\rangle + \mathcal{L}_Y\langle Z, X\rangle - \mathcal{L}_Z\langle X, Y\rangle \\ &= 2\langle D_X Y, Z\rangle + \langle D_Y X - D_X Y, Z\rangle \\ &\quad + \langle X, D_Y Z - D_Z Y\rangle + \langle Y, D_X Z - D_X Z\rangle \\ &= 2\langle D_X Y, Z\rangle + \langle [X, Y], Z\rangle + \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle. \end{aligned}$$

The same equation holds for the Levi-Civita connection and hence

$$\langle D_X Y, Z\rangle = \langle \nabla_X Y, Z\rangle.$$

This implies $D_X Y = \nabla_X Y$ for all $X, Y \in \mathrm{Vect}(M)$. □

Remark 4.18 (The Levi-Civita connection in local coordinates). Let $\phi : U \to \Omega$ be a coordinate chart on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$. In such a coordinate chart a vector field $X \in \mathrm{Vect}(M)$ is represented by a smooth map

$$\xi = (\xi^1, \ldots, \xi^m) : \Omega \to \mathbb{R}^m$$

defined by

$$\xi(\phi(p)) = d\phi(p)X(p)$$

for $p \in U$. If $Y \in \mathrm{Vect}(M)$ is represented by η then $\nabla_X Y$ is represented by the function

$$(\nabla_\xi \eta)^k := \sum_{i=1}^m \frac{\partial \eta^k}{\partial x^i}\xi^i + \sum_{i,j=1}^m \Gamma_{ij}^k \xi^i \eta^j. \tag{4.15}$$

Here the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are the Christoffel symbols defined by

$$\Gamma_{ij}^k := \sum_{\ell=1}^m g^{k\ell} \frac{1}{2}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right), \tag{4.16}$$

where g_{ij} is the metric tensor and g^{ij} is the inverse matrix so that

$$\sum_j g_{ij} g^{jk} = \delta_i^k$$

(see Theorem 2.27). This formula can be used to prove the existence statement in Lemma 4.17 and hence define the Levi-Civita connection in the intrinsic setting.

Exercise 4.19. In the proof of Lemma 4.17 we did not actually use that the operator $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is linear nor that the operator $X \mapsto D_X$ is linear. Prove directly that if a map

$$D_X : \mathcal{L}(M) \to \mathcal{L}(M)$$

satisfies (4.13) for all $Y, Z \in \mathrm{Vect}(M)$ then D_X is linear. Prove that every map

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto D_X$$

that satisfies (4.14) is linear.

4.2.3 A global formula

Lemma 4.20. *For $X, Y, Z \in \mathrm{Vect}(M)$ we have*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z. \tag{4.17}$$

Proof. Fix a point $p \in M$. Then the right hand side of equation (4.17) at p remains unchanged if we multiply each of the vector fields X, Y, Z by a smooth function $f : M \to [0,1]$ that is equal to one near p. Choosing f with compact support we may therefore assume that the vector fields X and Y are complete. Let ϕ^s denote the flow of X and ψ^t the flow of Y. Define the map $\gamma : \mathbb{R}^2 \to M$ by

$$\gamma(s,t) := \psi^s \circ \psi^t(p), \qquad s, t \in \mathbb{R}.$$

Then

$$\partial_s \gamma = X(\gamma), \qquad \partial_t \gamma = (\phi^s_* Y)(\gamma).$$

Hence, by Remark 4.15, we have

$$\nabla_s (Z \circ \gamma) = (\nabla_X Z)(\gamma), \qquad \nabla_t (Z \circ \gamma) = \left(\nabla_{\phi^s_* Y} Z\right)(\gamma).$$

This implies

$$\nabla_s \nabla_t (Z \circ \gamma) = \left(\nabla_{\partial_s \gamma} \nabla_{\phi^s_* Y} Z\right)(\gamma) + \left(\nabla_{\partial_s \phi^s_* Y} Z\right)(\gamma).$$

Since

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \phi^s_* Y = [X, Y]$$

and $\partial_s \gamma = X(\gamma)$ we obtain

$$\begin{aligned}
\nabla_s \nabla_t (Z \circ \gamma)(0,0) &= \nabla_X \nabla_Y Z(p) + \nabla_{[X,Y]} Z(p), \\
\nabla_t \nabla_s (Z \circ \gamma)(0,0) &= \nabla_Y \nabla_X Z(p).
\end{aligned}$$

Hence

$$\begin{aligned}
R_p(X(p), Y(p))Z(p) &= \left(\nabla_s \nabla_t (Z \circ \gamma) - \nabla_t \nabla_s (Z \circ \gamma)\right)(0,0) \\
&= \nabla_X \nabla_Y Z(p) - \nabla_Y \nabla_X Z(p) + \nabla_{[X,Y]} Z(p).
\end{aligned}$$

This proves the lemma. □

Remark 4.21. Equation (4.17) can be written succinctly as

$$[\nabla_X, \nabla_Y] + \nabla_{[X,Y]} = R(X, Y). \tag{4.18}$$

This can be contrasted with the equation

$$[\mathcal{L}_X, \mathcal{L}_Y] + \mathcal{L}_{[X,Y]} = 0 \tag{4.19}$$

for the operator $\mathcal{L}_X$ on the space of real valued functions on M.

Remark 4.22. Equation (4.17) can be used to define the Riemann curvature tensor. To do this one must again prove that the right hand side of equation (4.17) at p depends only on the values $X(p), Y(p), Z(p)$ of the vector fields X, Y, Z at the point p. For this it suffices to prove that the map

$$\mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M) : (X, Y, Z) \mapsto R(X, Y)Z$$

is linear over the Ring $\mathscr{F}(M)$ of smooth real valued functions on M, i.e.

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)fZ = fR(X, Y)Z \tag{4.20}$$

for $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in \mathscr{F}(M)$. The formula (4.20) follows easily form the equations (4.11), (4.12), (4.19), and $[X, fY] = f[X, Y] - (\mathcal{L}_X f)Y$. It follows from (4.20) that the right hand side of (4.17) at p depends only on the vectors $X(p)$, $Y(p)$, $Z(p)$. The proof requires two steps. One first shows that if X vanishes near p then the right hand side of (4.17) vanishes at p (and similarly for Y and Z). Just multiply X by a smooth function equal to zero at p and equal to one on the support of X; then $fX = X$ and hence the vector field $R(X, Y)Z = R(fX, Y)Z = fR(X, Y)Z$ vanishes at p. Second, we choose a local frame $E_1, \ldots, E_m \in \mathrm{Vect}(M)$, i.e. vector fields that form a basis of T_pM for each p in some open set $U \subset M$. Then we may write

$$X = \sum_{i=1}^m \xi^i E_i, \qquad Y = \sum_{j=1}^m \eta^j E_j, \qquad Z = \sum_{k=1}^m \zeta^k E_k$$

in U. Using the first step and the $\mathscr{F}(M)$-multilinearity we obtain

$$R(X, Y)Z = \sum_{i,j,k=1}^m \xi^i \eta^j \zeta^k R(E_i, E_j)E_k$$

in U. If $X'(p) = X(p)$ then $\xi^i(p) = \xi'^i(p)$ so if $X(p) = X'(p)$, $Y(p) = Y'(p)$, $Z(p) = Z'(p)$ then $(R(X, Y)Z)(p) = (R(X', Y')Z')(p)$ as required.

4.2.4 Symmetries

Theorem 4.23. *The Riemann curvature tensor satisfies*

$$R(Y,X) = -R(X,Y) = R(X,Y)^*, \tag{4.21}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{4.22}$$

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle, \tag{4.23}$$

for $X,Y,Z,W \in \mathrm{Vect}(M)$. *Equation* (4.22) *is the* **first Bianchi identity**.

Proof. The first equation in (4.21) is obvious from the definition and the second equation follows immediately from the Gauss–Codazzi formula (4.8). Alternatively, we may choose a smooth map $\gamma : \mathbb{R}^2 \to M$ and two vector fields Z, W along γ. Then

$$\begin{aligned}
0 &= \partial_s\partial_t\langle Z,W\rangle - \partial_t\partial_s\langle Z,W\rangle \\
&= \partial_s\langle\nabla_t Z,W\rangle + \partial_s\langle Z,\nabla_t W\rangle - \partial_t\langle\nabla_s Z,W\rangle - \partial_t\langle Z,\nabla_s W\rangle \\
&= \langle\nabla_s\nabla_t Z,W\rangle + \langle Z,\nabla_s\nabla_t W\rangle - \langle\nabla_t\nabla_s Z,W\rangle - \langle Z,\nabla_t\nabla_s W\rangle \\
&= \langle R(\partial_s\gamma,\partial_t\gamma)Z,W\rangle - \langle Z,R(\partial_s\gamma,\partial_t\gamma)W\rangle.
\end{aligned}$$

This proof has the advantage that it carries over to the intrinsic setting. We prove the first Bianchi identity using (4.14) and (4.17):

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X + \nabla_{[Y,Z]}X \\
&\quad +\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y + \nabla_{[Z,X]}Y \\
&= \nabla_{[Y,Z]}X - \nabla_X[Y,Z] + \nabla_{[Z,X]}Y - \nabla_Y[Z,X] + \nabla_{[X,Y]}Z - \nabla_Z[X,Y] \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]].
\end{aligned}$$

The last term vanishes by the Jacobi identity. We prove (4.23) by combining the first Bianchi identity with (4.21):

$$\begin{aligned}
&\langle R(X,Y)Z,W\rangle - \langle R(Z,W)X,Y\rangle \\
&= -\langle R(Y,Z)X,W\rangle - \langle R(Z,X)Y,W\rangle - \langle R(Z,W)X,Y\rangle \\
&= \langle R(Y,Z)W,X\rangle + \langle R(Z,X)W,Y\rangle + \langle R(W,Z)X,Y\rangle \\
&= \langle R(Y,Z)W,X\rangle - \langle R(X,W)Z,Y\rangle \\
&= \langle R(Y,Z)W,X\rangle - \langle R(W,X)Y,Z\rangle.
\end{aligned}$$

Note that the first line is related to the last by a cyclic permutation. Repeating this argument we find

$$\langle R(Y,Z)W,X\rangle - \langle R(W,X)Y,Z\rangle = \langle R(Z,W)X,Y\rangle - \langle R(X,Y)Z,W\rangle.$$

combining these two identities we obtain (4.23). This proves the theorem □

Remark 4.24. We may think of a vector field X on M as a section of the tangent bundle. This is reflected in the alternative notation

$$\Omega^0(M,TM) := \mathrm{Vect}(M).$$

A 1**-form on** M **with values in the tangent bundle** is a collection of linear maps $A(p) : T_pM \to T_pM$, one for every $p \in M$, which is smooth in the sense that for every smooth vector field X on M the assignment $p \mapsto A(p)X(p)$ defines again a smooth vector field on M. We denote by

$$\Omega^1(M,TM)$$

the space of smooth 1-forms on M with values in TM. The covariant derivative of a vector field Y is such a 1-form with values in the tangent bundle which assigns to every $p \in M$ the linear map $T_pM \to T_pM : v \mapsto \nabla_v Y(p)$. Thus we can think of the covariant derivative as a linear operator

$$\nabla : \Omega^0(M,TM) \to \Omega^1(M,TM).$$

The equation (4.11) asserts that the operators $X \mapsto \nabla_X$ indeed determine a linear operator from $\Omega^0(M,TM)$ to $\Omega^1(M,TM)$. Equation (4.12) asserts that this linear operator ∇ is a **connection** on the tangent bundle of M. Equation (4.13) asserts that ∇ is a **Riemannian connection** and equation (4.14) asserts that ∇ is **torsion free**. Thus Lemma 4.17 can be restated as asserting that the **Levi-Civita connection** is the unique torsion free Riemannian connection on the tangent bundle.

Exercise 4.25. Extend the notion of a connection to a general vector bundle E, both as a collection of linear operators $\nabla_X : \Omega^0(M,E) \to \Omega^0(M,E)$, one for every vector field $X \in \mathrm{Vect}(M)$, and as a linear operator

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$$

satisfying the analogue of equation (4.12). Interpret this equation as a Leibnitz rule for the product of a function on M with a section of E. Show that $\nabla^\perp$ is a connection on $TM^\perp$. Extend the notion of curvature to connections on general vector bundles.

Exercise 4.26. Show that the field which assigns to each $p \in M$ the multilinear map $R_p^\perp : T_pM \times T_pM \to \mathcal{L}(T_pM^\perp, T_pM^\perp)$ characterized by

$$R^\perp(\partial_s\gamma, \partial_t\gamma)Y = \nabla_s^\perp\nabla_t^\perp Y - \nabla_t^\perp\nabla_s^\perp Y$$

for $\gamma : \mathbb{R}^2 \to M$ and $Y \in \mathrm{Vect}^\perp(\gamma)$ satisfies the equation

$$R_p^\perp(u,v) = h_p(u)h_p(v)^* - h_p(v)h_p(u)^*$$

for $p \in M$ and $u, v \in T_pM$.

4.2.5 Examples and exercises

Example 4.27. Let $\mathrm{G} \subset \mathrm{O}(n)$ be a **Lie subgroup**, i.e. a subgroup that is also a submanifold. Consider the Riemannian metric on G induced by the inner product

$$\langle v, w\rangle := \operatorname{trace}(v^T w) \tag{4.24}$$

on the ambient space $\mathfrak{gl}(n,\mathbb{R}) = \mathbb{R}^{n\times n}$. Let $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ be the Lie algebra of G. Then the Riemann curvature tensor on G can be expressed in terms of the Lie bracket (see item (d) below).

(a) The maps $g \mapsto ag$, $g \mapsto ga$, $g \mapsto g^{-1}$ are isometries of G for every $a \in \mathrm{G}$.

(b) A smooth map $\gamma : \mathbb{R} \to \mathrm{G}$ is a geodesic if and only if there exist matrices $g \in \mathrm{G}$ and $\xi \in \mathfrak{g}$ such that

$$\gamma(t) = g\exp(t\xi).$$

For $\mathrm{G} = \mathrm{O}(n)$ we have seen this in Example 2.42 and the proof in the general case is similar. Hence the exponential map $\exp : \mathfrak{g} \to \mathrm{G}$ defined by the exponential matrix (as in Section 1.6) agrees with the time-1-map of the geodesic flow (as in Section 2.4).

(c) Let $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth curve and $X \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ. Then the covariant derivative of X is given by

$$\gamma(t)^{-1}\nabla X(t) = \frac{d}{dt}\gamma(t)^{-1}X(t) + \frac{1}{2}\left[\gamma(t)^{-1}\dot\gamma(t), \gamma(t)^{-1}X(t)\right]. \tag{4.25}$$

(**Exercise:** Prove equation (4.25). **Hint:** Since $\mathfrak{g} \subset \mathfrak{o}(n)$ we have the identity $\operatorname{trace}((\xi\eta + \eta\xi)\zeta) = 0$ for all $\xi, \eta, \zeta \in \mathfrak{g}$.)

(d) The Riemann curvature tensor on G is given by

$$g^{-1}R_g(u,v)w = -\frac{1}{4}[[g^{-1}u, g^{-1}v], g^{-1}w]. \tag{4.26}$$

Note that the first Bianchi identity is equivalent to the Jacobi identity. (**Exercise:** Prove equation (4.26).)

Exercise 4.28. Prove that every Lie subgroup of $\mathrm{O}(n)$ is a closed subset and hence is compact. Show that the inner product (4.24) on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ of a Lie subgroup $\mathrm{G} \subset \mathrm{O}(n)$ is invariant under conjugation:

$$\langle \xi, \eta\rangle = \langle g\xi g^{-1}, g\eta g^{-1}\rangle$$

for all $g \in \mathrm{G}$ and all $\xi, \eta \in \mathfrak{g}$. Show that

$$\langle [\xi, \eta], \zeta\rangle = \langle \xi, [\eta, \zeta]\rangle$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$.

Example 4.29. Let $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ be any Lie subgroup, not necessarily contained in $\mathrm{O}(n)$, and let

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$$

be its Lie algebra. Fix any inner product on the Lie algebra $\mathfrak{g}$ (not necessarily invariant under conjugation) and consider the Riemannian metric on G defined by

$$\langle v, w\rangle_g := \langle vg^{-1}, wg^{-1}\rangle$$

for $v, w \in T_g\mathrm{G}$. This metric is called **right invariant**.

(a) Define the linear map $A : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$\langle A(\xi)\eta, \zeta\rangle = \frac{1}{2}\Big(\langle \xi, [\eta, \zeta]\rangle - \langle \eta, [\zeta, \xi]\rangle - \langle \zeta, [\xi, \eta]\rangle\Big)$$

for $\xi, \eta, \zeta \in \mathfrak{g}$. Then A is the unique linear map that satisfies

$$A(\xi) + A(\xi)^* = 0, \qquad A(\eta)\xi + A(\xi)\eta = [\xi, \eta]$$

for all $\xi, \eta \in \mathfrak{g}$. Here $A(\xi)^*$ denotes the adjoint operator with respect to the given inner product on $\mathfrak{g}$. Note that $A(\xi)\eta = -\frac{1}{2}[\xi, \eta]$ whenever the inner product on $\mathfrak{g}$ is invariant under conjugation.

(b) Let $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth curve and $X \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ. Then the covariant derivative of X is given by

$$\nabla X = \left(\frac{d}{dt}(X\gamma^{-1}) + A(\dot\gamma\gamma^{-1})X\gamma^{-1}\right)\gamma.$$

(**Exercise:** Prove this.) Hence a smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ is a geodesic if and only if it satisfies the equation

$$\frac{d}{dt}(\dot\gamma\gamma^{-1}) + A(\dot\gamma\gamma^{-1})\dot\gamma\gamma^{-1} = 0.$$

(c) The Riemann curvature tensor on G is given by

$$\big(R_g(u, v)w\big)g^{-1} = \Big(A\left(\left[ug^{-1}, vg^{-1}\right]\right) + \left[A\left(ug^{-1}\right), A\left(vg^{-1}\right)\right]\Big)wg^{-1}$$

for $g \in \mathrm{G}$ and $u, v, w \in T_g\mathrm{G}$. (**Exercise:** Prove this.)

4.3 Generalized Theorema Egregium

We will now show that geodesics, covariant differentiation, parallel transport, and the Riemann curvature tensor are all intrinsic, i.e. they are intertwined by isometries. In the extrinsic setting these results are somewhat surprising since these objects are all defined using the second fundamental form, whereas isometries need not preserve the second fundamental form in any sense but only the first fundamental form.

Below we shall give a formula expressing the Gaussian curvature of a surface M^2 in $\mathbb{R}^3$ in terms of the Riemann curvature tensor and the first fundamental form. It follows that the Gaussian curvature is also intrinsic. This fact was called by Gauss the "Theorema Egregium" which explains the title of this section.

4.3.1 Pushforward

We assume throughout this section that $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ are smooth submanifolds of the same dimension m. As in Section 4.1 we denote objects on M' by the same letters as objects in M with primes affixed. In particular, g' denotes the first fundamental form on M' and R' denotes the Riemann curvature tensor on M'.

Let $\phi : M \to M'$ be a diffeomorphism. Using ϕ we can move objects on M to M'. For example the pushforward of a smooth curve $\gamma : I \to M$ is the curve

$$\phi_*\gamma := \phi \circ \gamma : I \to M',$$

the pushforward of a smooth function $f : M \to \mathbb{R}$ is the function

$$\phi_* f := f \circ \phi^{-1} : M' \to \mathbb{R},$$

the pushforward of a vector field $X \in \mathrm{Vect}(\gamma)$ along a curve $\gamma : I \to M$ is the vector field $\phi_*X \in \mathrm{Vect}(\phi_*\gamma)$ defined by

$$(\phi_*X)(t) := d\phi(\gamma(t))X(t)$$

for $t \in I$, and the pushforward of a global vector field $X \in Vect(M)$ is the vector field $\phi_*X \in \mathrm{Vect}(M')$ defined by

$$(\phi_*X)(\phi(p)) := d\phi(p)X(p)$$

for $p \in M$. Recall that the first fundamental form on M is the Riemannian metric g defined as the restriction of the Euclidean inner product on the ambient space to each tangent space of M. It assigns to each $p \in M$ the bilinear map $g_p \in T_pM \times T_pM \to \mathbb{R}$ given by

$$g_p(u,v) = \langle u, v\rangle, \qquad u, v \in T_pM.$$

Its pushforward is the Riemannian metric which assigns to each $p' \in M'$ the inner product $(\phi_* g)_{p'} : T_{p'}M' \times T_{p'}M' \to \mathbb{R}$ defined by

$$(\phi_* g)_{\phi(p)}(d\phi(p)u, d\phi(p)v) := g_p(u,v)$$

for $p := \phi^{-1}(p') \in M$ and $u, v \in T_pM$. The pushforward of the Riemann curvature tensor is the tensor which assigns to each $p' \in M'$ the bilinear map $(\phi_* R)_{p'} : T_{p'}M' \times T_{p'}M' \to \mathcal{L}\left(T_{p'}M', T_{p'}M'\right)$, defined by

$$(\phi_* R)_{\phi(p)}(d\phi(p)u, d\phi(p)v) := d\phi(p) R_p(u,v)\, d\phi(p)^{-1}$$

for $p := \phi^{-1}(p') \in M$ and $u, v \in T_pM$.

4.3.2 Theorema Egregium

Theorem 4.30 (**Theorema Egregium**). *The first fundamental form, covariant differentiation, geodesics, parallel transport, and the Riemann curvature tensor are intrinsic. This means that for every isometry $\phi : M \to M'$ the following holds.*

(i) $\phi_* g = g'$.

(ii) *If $X \in \mathrm{Vect}(\gamma)$ is a vector field along a smooth curve $\gamma : I \to M$ then*

$$\nabla'(\phi_* X) = \phi_* \nabla X \tag{4.27}$$

and if $X, Y \in \mathrm{Vect}(M)$ are global vector fields then

$$\nabla'_{\phi_* X} \phi_* Y = \phi_*(\nabla_X Y). \tag{4.28}$$

(iii) *If $\gamma : I \to M$ is a geodesic then $\phi \circ \gamma : I \to M'$ is a geodesic.*

(iv) *If $\gamma : I \to M$ is a smooth curve then for all $s, t \in I$:*

$$\Phi'_{\phi\circ\gamma}(t,s) d\phi(\gamma(s)) = d\phi(\gamma(t)) \Phi_\gamma(t,s). \tag{4.29}$$

(v) $\phi_* R = R'$.

Proof. Assertion (i) is simply a restatement of Theorem 4.1. To prove (ii) we choose a local smooth parametrization $\psi : \Omega \to U$ of an open set $U \subset M$, defined on an open set $\Omega \subset \mathbb{R}^m$, so that $\psi^{-1} : U \to \Omega$ is a coordinate chart. Suppose without loss of generality that $\gamma(t) \in U$ for all $t \in I$ and define $c : I \to \Omega$ and $\xi : I \to \mathbb{R}^m$ by

$$\gamma(t) = \psi(c(t)), \qquad X(t) = \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)).$$

Recall from equations (2.20) and (2.21) that

$$\nabla X(t) = \sum_{k=1}^m \left(\dot\xi^k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(c(t)) \dot c^i(t) \xi^j(t) \right) \frac{\partial \psi}{\partial x^k}(c(t)),$$

where the Christoffel symbols $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are defined by

$$\Pi(\psi) \frac{\partial^2 \psi}{\partial x^i \partial x^j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial \psi}{\partial x^k}.$$

Now consider the same formula for $\phi^* X$ using the parametrization

$$\psi' := \phi \circ \psi : \Omega \to U' := \phi^{-1}(U) \subset M'.$$

The Christoffel symbols ${\Gamma'}_{ij}^k : \Omega \to \mathbb{R}$ associated to this parametrization of U' are defined by the same formula as the Γ_{ij}^k with ψ replaced by ψ'. But the metric tensor for ψ agrees with the metric tensor for ψ':

$$g_{ij} = \left\langle \frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle = \left\langle \frac{\partial \psi'}{\partial x^i}, \frac{\partial \psi'}{\partial x^j} \right\rangle.$$

Hence it follows from Theorem 2.27 that ${\Gamma'}_{ij}^k = \Gamma_{ij}^k$ for all i, j, k. This implies that the covariant derivative of $\phi_* X$ is given by

$$\begin{aligned} \nabla'(\phi_* X) &= \sum_{k=1}^m \left(\dot\xi^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c) \dot c^i \xi^j \right) \frac{\partial \psi'}{\partial x^k}(c) \\ &= d\phi(\psi(c)) \sum_{k=1}^m \left(\dot\xi^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c) \dot c^i \xi^j \right) \frac{\partial \psi}{\partial x^k}(c) \\ &= \phi^* \nabla X. \end{aligned}$$

This proves (4.27). Equation (4.28) follows immediately from (4.27) and Remark 4.14.

Here is a second proof of (ii). For every vector field $X \in \mathrm{Vect}(M)$ we define the operator $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ by

$$D_X Y := \phi^* \left(\nabla_{\phi_* X} \phi_* Y\right).$$

Then, for all $X, Y \in \mathrm{Vect}(M)$, we have

$$D_Y X - D_X Y = \phi^* \left(\nabla_{\phi_* Y} \phi_* X - \nabla_{\phi_* X} \phi_* Y\right) = \phi^*[\phi_* X, \phi_* Y] = [X, Y].$$

Moreover, it follows from (i) that

$$\begin{aligned}
\phi_* \mathcal{L}_X \langle Y, Z\rangle &= \mathcal{L}_{\phi_* X} \langle \phi_* Y, \phi_* Z\rangle \\
&= \langle \nabla_{\phi_* X} \phi_* Y, \phi_* Z\rangle + \langle \phi_* Y, \nabla_{\phi_* X} \phi_* Z\rangle \\
&= \langle \phi_* D_X Y, \phi_* Z\rangle + \langle \phi_* Y, \phi_* D_X Z\rangle \\
&= \phi_* \big(\langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle\big).
\end{aligned}$$

and hence $\mathcal{L}_X \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$ for all $X, Y, Z \in \mathrm{Vect}(M)$. Thus the operator $X \mapsto D_X$ satisfies (4.13) and (4.14) and, by Lemma 4.17, it follows that $D_X Y = \nabla_X Y$ for all $X, Y \in \mathrm{Vect}(M)$. This completes the second proof of (ii).

We prove (iii). Since ϕ preserves the first fundamental form it also preserves the energy of curves, namely

$$E(\phi \circ \gamma) = E(\gamma)$$

for every smooth map $\gamma : [0, 1] \to M$. Hence γ is a critical point of the energy functional if and only if $\phi \circ \gamma$ is a critical point of the energy functional. Alternatively it follows from (ii) that

$$\nabla' \left(\frac{d}{dt} \phi \circ \gamma\right) = \nabla' \phi_* \dot\gamma = \phi_* \nabla \dot\gamma$$

for every smooth curve $\gamma : I \to M$. If γ is a geodesic the last term vanishes and hence $\phi \circ \gamma$ is a geodesic as well. As a third proof we can deduce (iii) from the formula $\phi(\exp_p(v)) = \exp_{\phi(p)}(d\phi(p)v)$ in the proof of Theorem 4.1.

We prove (iv). For $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)} M$ define

$$X(t) := \Phi_\gamma(t, t_0) v_0, \qquad X'(t) := \Phi'_{\phi\circ\gamma}(t, t_0) d\phi(\gamma(t_0)) v_0.$$

By (ii) the vector fields X' and $\phi_* X$ along $\phi \circ \gamma$ are both parallel and they agree at $t = t_0$. Hence $X'(t) = \phi_* X(t)$ for all $t \in I$ and this proves (4.29).

We prove (v). Fix a smooth map $\gamma : \mathbb{R}^2 \to M$ and a smooth vector field Z along γ, and define $\gamma' = \phi \circ \gamma : \mathbb{R}^2 \to M'$ and $Z' := \phi_* Z \in \mathrm{Vect}(\gamma')$. Then it follows from (ii) that

$$\begin{aligned} R'(\partial_s\gamma', \partial_t\gamma')Z' &= \nabla'_s\nabla'_t Z' - \nabla'_t\nabla'_s Z' \\ &= \phi_* \left(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\right) \\ &= d\phi(\gamma)R(\partial_s\gamma, \partial_t\gamma)Z \\ &= (\phi_* R)(\partial_s\gamma', \partial_t\gamma')Z'. \end{aligned}$$

This proves (v) and the theorem. □

4.3.3 The Riemann curvature tensor in local coordinates

Given a local coordinate chart $\psi^{-1} : U \to \Omega$ on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$, we define the vector fields $E_1, \dots, E_m$ along ψ by

$$E_i(x) := \frac{\partial\psi}{\partial x^i}(x) \in T_{\psi(x)}M.$$

These vector fields form a basis of $T_{\psi(x)}M$ for every $x \in \Omega$. The coefficients $g_{ij} : \Omega \to \mathbb{R}$ of the first fundamental form are

$$g_{ij} = \langle E_i, E_j \rangle .$$

Recall from Theorem 2.27 that the Christoffel $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ are the coefficients of the Levi-Civita connection, defined by

$$\nabla_i E_j = \sum_{k=1}^m \Gamma^k_{ij} E_k$$

and that they are given by the formula

$$\Gamma^k_{ij} := \sum_{\ell=1}^m g^{k\ell}\frac{1}{2}\big(\partial_i g_{i\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\big).$$

Define the coefficients $R^\ell_{ijk} : \Omega \to \mathbb{R}$ of the curvature tensor by

$$R(E_i, E_j)E_k = \sum_{\ell=1}^m R^\ell_{ijk} E_\ell. \tag{4.30}$$

These coefficients are given by

$$R^\ell_{ijk} := \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \sum_{\nu=1}^m \left(\Gamma^\ell_{i\nu}\Gamma^\nu_{jk} - \Gamma^\ell_{j\nu}\Gamma^\nu_{ik}\right). \tag{4.31}$$

The coefficients of the Riemann curvature tensor have the symmetries

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}, \qquad R_{ijk\ell} := \sum_\nu R_{ijk}^\nu g_{\nu\ell}, \tag{4.32}$$

and the first Bianchi identity has the form

$$R_{ijk}^\ell + R_{jki}^\ell + R_{kij}^\ell = 0. \tag{4.33}$$

Warning: Care must be taken with the ordering of the indices. Some authors use the notation R_{kij}^ℓ for what we call R_{ijk}^ℓ and $R_{\ell kij}$ for what we call $R_{ijk\ell}$.

Exercise 4.31. Prove equations (4.31), (4.32), and (4.33). Use (4.31) to give an alternative proof of Theorem 4.30.

4.3.4 Gaussian curvature

As a special case we shall now consider a **hypersurface** $M \subset \mathbb{R}^{m+1}$, i.e. a smooth submanifold of codimension one. We assume that there is a smooth map $\nu : M \to \mathbb{R}^{m+1}$ such that, for every $p \in M$, we have $\nu(p) \perp T_pM$ and $|\nu(p)| = 1$. Such a map always exists locally (see Example 2.9). Note that $\nu(p)$ is an element of the unit sphere in $\mathbb{R}^{m+1}$ for every $p \in M$ and hence we can regard ν as a map from M to S^m:

$$\nu : M \to S^m.$$

Such a map is called a **Gauss map** for M. Note that if $\nu : M \to S^2$ is a Gauss map so is $-\nu$, but this is the only ambiguity when M is connected. Differentiating ν at $p \in M$ we obtain a linear map

$$d\nu(p) : T_pM \to T_{\nu(p)}S^m = T_pM$$

Here we use the fact that $T_{\nu(p)}S^m = \nu(p)^\perp$ and, by definition of the Gauss map ν, the tangent space of M at p is also equal to $\nu(p)^\perp$. Thus $d\nu(p)$ is linear map from the tangent space of M at p to itself.

Definition 4.32. *The* **Gaussian curvature** *of the hypersurface M is the real valued function $K : M \to \mathbb{R}$ defined by*

$$K(p) := \det\bigl(d\nu(p) : T_pM \to T_pM\bigr)$$

for $p \in M$. (Replacing ν by $-\nu$ has the effect of replacing K by $(-1)^m K$; so K is independent of the choice of the Gauss map when m is even.)

Remark 4.33. Given a subset $B \subset M$ the set $\nu(B) \subset S^m$ is often called the **spherical image** of B. If ν is a diffeomorphism on a neighborhood of B the change of variables formula for an integral gives

$$\int_{\nu(B)} \mu_S = \int_B |K| \mu_M$$

where μ_M and μ_S denote the volume elements on M and S^m, respectively. Introducing the notation $\mathrm{Area}_M(B) := \int_B \mu_M$ we obtain the formula

$$|K(p)| = \lim_{B \to p} \frac{\mathrm{Area}_S(\nu(B))}{\mathrm{Area}_M(B)}.$$

This says that the curvature at p is roughly the ratio of the (m-dimensional) area of the spherical image $\nu(B)$ to the area of B where B is a very small open neighborhood of p in M. The sign of $K(p)$ is positive when the linear map $d\nu(p) : T_pM \to T_pM$ preserves orientation and negative when it reverses orientation.

Remark 4.34. We see that the Gaussian curvature is a natural generalization of **Euler's curvature** for a plane curve. Indeed if $M \subset \mathbb{R}^2$ is a 1-manifold and $p \in M$ we can choose a curve $\gamma = (x, y) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $|\dot{\gamma}(s)| = 1$ for every s. This curve parametrizes M by the arclength and the unit normal vector pointing to the right with respect to the orientation of γ is $\nu(x, y) = (\dot{y}, -\dot{x})$. This is a local Gauss map and its derivative $(\ddot{y}, -\ddot{x})$ is tangent to the curve. The inner product of the latter with the unit tangent vector $\dot{\gamma} = (\dot{x}, \dot{y})$ is the Gaussian curvature. Thus

$$K := \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} = \frac{d\theta}{ds}$$

where s is the arclength parameter and θ is the angle made by the normal (or the tangent) with some constant line. With this convention K is positive at a left turn and negative at a right turn.

Exercise 4.35. The Gaussian curvature of a sphere of radius r is constant and has the value r^{-m}.

Exercise 4.36. Show that the Gaussian curvature of the surface $z = x^2 - y^2$ is -4 at the origin.

We now restrict to the case of **surfaces**, i.e. of 2-dimensional submanifolds of $\mathbb{R}^3$. Figure 4.1 illustrates the difference between positive and negative Gaussian curvature in dimension two.

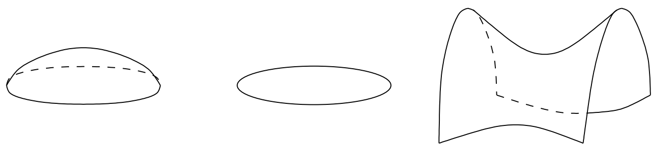


Figure 4.1: Positive and negative Gaussian curvature.

Theorem 4.37 (Gaussian curvature). *Let $M \subset \mathbb{R}^3$ be a surface and fix a point $p \in M$. If $u, v \in T_pM$ is a basis then*

$$K(p) = \frac{\langle R(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2}. \tag{4.34}$$

Moreover, for all $u, v, w \in T_pM$, we have

$$R(u,v)w = -K(p)\langle \nu(p), u \times v\rangle \nu(p) \times w. \tag{4.35}$$

Proof. The orthogonal projection of $\mathbb{R}^3$ onto the tangent space $T_pM = \nu(p)^\perp$ is given by the 3×3-matrix

$$\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^T.$$

Hence

$$d\Pi(p)u = -\nu(p)(d\nu(p)u)^T - \big(d\nu(p)u\big)\nu(p)^T.$$

Here the first summand is the second fundamental form, which maps T_pM to $T_pM^\perp$, and the second summand is its dual, which maps $T_pM^\perp$ to T_pM. Thus

$$\begin{aligned} h_p(v) &= \nu(p)\big(d\nu(p)v\big)^T : T_pM \to T_pM^\perp, \\ h_p(u)^* &= \big(d\nu(p)u\big)\nu(p)^T : T_pM^\perp \to T_pM. \end{aligned}$$

By the Gauss–Codazzi formula this implies

$$\begin{aligned} R_p(u,v)w &= h_p(u)^*h_p(v)w - h_p(v)^*h_p(u)w \\ &= \big(d\nu(p)u\big)\big(d\nu(p)v\big)^T w - \big(d\nu(p)v\big)\big(d\nu(p)u\big)^T w \\ &= \langle d\nu(p)v, w\rangle d\nu(p)u - \langle d\nu(p)u, w\rangle d\nu(p)v \end{aligned}$$

and hence

$$\langle R_p(u,v)w, z\rangle = \langle d\nu(p)u, z\rangle\langle d\nu(p)v, w\rangle - \langle d\nu(p)u, w\rangle\langle d\nu(p)v, z\rangle. \tag{4.36}$$

Now fix four tangent vectors $u, v, w, z \in T_pM$ and consider the composition

$$\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^3 \xrightarrow{C} \mathbb{R}^3$$

of the linear maps

$$\begin{aligned}
A\xi &:= \xi^1\nu(p) + \xi^2 u + \xi^3 v, \\
B\eta &:= \begin{cases} d\nu(p)\eta, & \text{if } \eta \perp \nu(p), \\ \eta, & \text{if } \eta \in \mathbb{R}\nu(p), \end{cases} \\
C\zeta &:= \begin{pmatrix} \langle \zeta, \nu(p)\rangle \\ \langle \zeta, z\rangle \\ \langle \zeta, w\rangle \end{pmatrix}.
\end{aligned}$$

This composition is represented by the matrix

$$CBA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle d\nu(p)u, z\rangle & \langle d\nu(p)v, z\rangle \\ 0 & \langle d\nu(p)u, w\rangle & \langle d\nu(p)v, w\rangle \end{pmatrix}.$$

Hence, by (4.36), we have

$$\begin{aligned}
\langle R_p(u,v)w, z\rangle &= \det(CBA) \\
&= \det(A)\det(B)\det(C) \\
&= \langle \nu(p), u \times v\rangle K(p) \langle \nu(p), z \times w\rangle \\
&= -K(p)\langle \nu(p), u \times v\rangle \langle \nu(p) \times w, z\rangle.
\end{aligned}$$

This implies (4.35) and

$$\begin{aligned}
\langle R_p(u,v)v, u\rangle &= K(p)\langle \nu(p), u \times v\rangle^2 \\
&= K(p)\,|u \times v|^2 \\
&= K(p)\left(|u|^2\,|v|^2 - \langle u, v\rangle^2\right).
\end{aligned}$$

This proves the theorem. □

Corollary 4.38 (Theorema Egregium of Gauss). *The Gaussian curvature is intrinsic, i.e. if $\phi : M \to M'$ is an isometry of surfaces in $\mathbb{R}^3$ then*

$$K = K' \circ \phi : M \to \mathbb{R}.$$

Proof. Theorem 4.30 and Theorem 4.37. □

Exercise 4.39. For $m = 1$ the Gaussian curvature is clearly *not* intrinsic as any two curves are locally isometric (parameterized by arclength). Show that the curvature $K(p)$ is intrinsic for even m while its absolute value $|K(p)|$ is intrinsic for odd $m \geq 3$. **Hint:** We still have the equation (4.36) which, for $z = u$ and $v = w$, can be written in the form

$$\langle R_p(u,v)v, u\rangle = \det \begin{pmatrix} \langle d\nu(p)u, u\rangle & \langle d\nu(p)u, v\rangle \\ \langle d\nu(p)v, u\rangle & \langle d\nu(p)v, v\rangle \end{pmatrix}.$$

Thus, for an orthonormal basis $v_1, \ldots, v_m$ of T_pM, the 2×2 minors of the matrix

$$(\langle d\nu(p)v_i, v_j\rangle)_{i,j=1,\ldots,m}$$

are intrinsic. Hence everything reduces to the following assertion.

Lemma. *The determinant of an $m \times m$ matrix is an expression in its 2×2 minors if m is even; the absolute value of the determinant is an expression in the 2×2 minors if m is odd and greater than or equal to 3.*

The lemma is proved by induction on m. For the absolute value, note the formula

$$\det(A)^m = \det(\det(A)I) = \det(AB) = \det(A)\det(B)$$

for an $m \times m$ matrix A where B is the transposed matrix of cofactors.

4.3.5 Gaussian curvature in local coordinates

If $M \subset \mathbb{R}^n$ is a 2-manifold (not necessarily embedded in $\mathbb{R}^3$) we can use equation (4.34) as the definition of the Gaussian curvature $K : M \to \mathbb{R}$. Let $\psi : \Omega \to U$ be a local parametrization of an open set $U \subset M$ defined on an open set $\Omega \subset \mathbb{R}^2$. Denote the coordinates in $\mathbb{R}^2$ by (x, y) and define the functions $E, F, G : \Omega \to \mathbb{R}$ by

$$E := |\partial_x \psi|^2, \qquad F := \langle \partial_x \psi, \partial_y \psi\rangle, \qquad G := |\partial_y \psi|^2.$$

We abbreviate

$$D := EG - F^2.$$

Then the composition of the Gaussian curvature $K : M \to \mathbb{R}$ with the parametrization ψ is given by the explicit formula

$$\begin{aligned}
K\circ\psi &= \frac{1}{D^2}\det\begin{pmatrix} E & F & \partial_y F - \frac{1}{2}\partial_x G \\ F & G & \frac{1}{2}\partial_y G \\ \frac{1}{2}\partial_x E & \partial_x F - \frac{1}{2}\partial_y E & -\frac{1}{2}\partial_y^2 E + \partial_x\partial_y F - \frac{1}{2}\partial_x^2 G \end{pmatrix} \\
&\quad -\frac{1}{D^2}\det\begin{pmatrix} E & F & \frac{1}{2}\partial_y E \\ F & G & \frac{1}{2}\partial_x G \\ \frac{1}{2}\partial_y E & \frac{1}{2}\partial_x G & 0 \end{pmatrix} \\
&= -\frac{1}{2\sqrt{D}}\frac{\partial}{\partial x}\left(\frac{E\partial_x G - F\partial_y E}{E\sqrt{D}}\right) \\
&\quad +\frac{1}{2\sqrt{D}}\frac{\partial}{\partial y}\left(\frac{2E\partial_x F - F\partial_x E - E\partial_y E}{E\sqrt{D}}\right).
\end{aligned}$$

This expression simplifies dramatically when $F = 0$ and we get

$$K\circ\psi = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial x}\frac{\partial_x G}{\sqrt{EG}} + \frac{\partial}{\partial y}\frac{\partial_y E}{\sqrt{EG}}\right) \tag{4.37}$$

Exercise 4.40. Prove that the Riemannian metric

$$E = G = \frac{4}{(1+x^2+y^2)^2}, \qquad F = 0,$$

on $\mathbb{R}^2$ has constant constant curvature $K = 1$ and the Riemannian metric

$$E = G = \frac{4}{(1-x^2-y^2)^2}, \qquad F = 0,$$

on the open unit disc has constant curvature $K = -1$.

4.4 The Cartan–Ambrose–Hicks theorem

In this section we address what might be called the “fundamental problem of intrinsic differential geometry”: when are two manifolds isometric? In some sense the theorem of this section answers that question (at least locally) although the equivalent conditions given there are probably more difficult to verify in most examples than the condition that there exist an isometry. We begin with the observation that, by Lemma 4.10, an isometry $\phi : M \to M'$ between connected manifolds, if it exists, is uniquely determined by its value and derivative at a single point. In other words, there cannot be too many isometries.

4.4.1 Homotopy

Definition 4.41. *Let M be a manifold and $I = [a, b]$ be a compact interval. A* **(smooth) homotopy** *of maps from I to M is a smooth map*

$$\gamma : [0, 1] \times I \to M.$$

We often write

$$\gamma_\lambda(t) = \gamma(\lambda, t)$$

for $\lambda \in [0, 1]$ and $t \in I$ and call γ a **(smooth) homotopy between** γ_0 **and** γ_1*. We say the homotopy has* **fixed endpoints** *if $\gamma_\lambda(a) = \gamma_0(a)$ and $\gamma_\lambda(b) = \gamma_0(b)$ for all $\lambda \in [0, 1]$. (See Figure 4.2.)*

We remark that a homotopy and a variation are essentially the same thing, namely a curve of maps (curves). The difference is pedagogical. We used the word "variation" to describe a curve of maps through a given map; when we use this word we are going to differentiate the curve to find a tangent vector (field) to the given map. The word "homotopy" is used to describe a curve joining two maps; it is a global rather than a local (infinitesimal) concept.

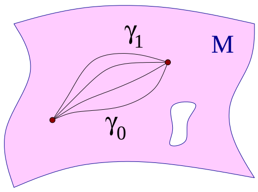


Figure 4.2: A homotopy with fixed endpoints.

Definition 4.42. *The manifold M is called* **simply connected** *if for any two curves $\gamma_0, \gamma_1 : [a, b] \to M$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$ there is a homotopy from γ_0 to γ_1 with endpoints fixed. (The idea is that the space $\Omega_{p,q}$ of curves from p to q is connected.)*

Remark 4.43. Two smooth maps $\gamma_0, \gamma_1 : [a, b] \to M$ with the same endpoints can be connected by a continuous homotopy if and only if hey can be connected by a smooth homotopy. This follows from the Weierstrass approximation theorem.

Remark 4.44. The topological space $\Omega_{p,q}$ of all smooth maps $\gamma : [0, 1] \to M$ with the endpoints p and q is connected for some pair of points $p, q \in M$ if and only if it is connected for every pair of points $p, q \in M$. (Prove this!)

Example 4.45. The Euclidean space $\mathbb{R}^m$ is simply connected, for any two curves $\gamma_0, \gamma_1 : [a,b] \to \mathbb{R}^m$ with the same endpoints can be joined by the homotopy

$$\gamma_\lambda(t) := \gamma_0(t) + \lambda(\gamma_1(t) - \gamma_0(t)).$$

The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected, for the curves

$$\gamma_n(t) := e^{2\pi \mathbf{i} nt}, \qquad 0 \le t \le 1,$$

are not homotopic with fixed endpoints for distinct n.

Exercise 4.46. Prove that the m-sphere S^m is simply connected for $m \neq 1$.

4.4.2 The global C-A-H theorem

Theorem 4.47 (Global C-A-H theorem). *Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ be connected, simply connected, complete m-manifolds. Let $p_0 \in M$, $p_0' \in M'$, and $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Then the following are equivalent.*

(i) *There is an isometry $\phi : M \to M'$ satisfying*

$$\phi(p_0) = p_0', \qquad d\phi(p_0) = \Phi_0. \tag{4.38}$$

(ii) *If (Φ, γ, γ') is a development satisfying the initial condition*

$$\gamma(0) = p_0, \qquad \gamma'(0) = p_0', \qquad \Phi(0) = \Phi_0 \tag{4.39}$$

then

$$\gamma(1) = p_0 \qquad \Longrightarrow \qquad \gamma'(1) = p_0', \quad \Phi(1) = \Phi_0$$

(iii) *If $(\Phi_0, \gamma_0, \gamma_0')$ and $(\Phi_1, \gamma_1, \gamma_1')$ are developments satisfying the initial condition (4.39) then*

$$\gamma_0(1) = \gamma_1(1) \qquad \Longrightarrow \qquad \gamma_0'(1) = \gamma_1'(1).$$

(iv) *If (Φ, γ, γ') is a development satisfying (4.39) then $\Phi_* R_\gamma = R'_{\gamma'}$.*

Lemma 4.48. *If $\phi : M \to M'$ is a local isometry satisfying (4.38) and (Φ, γ, γ') is a development satisfying the initial condition (4.39) then*

$$\gamma'(t) = \phi(\gamma(t)), \qquad \Phi(t) = d\phi(\gamma(t))$$

for every t.

Example 4.49. Before giving the proof let us interpret the conditions in case M and M' are two-dimensional spheres of radius r and r' respectively in three-dimensional Euclidean space $\mathbb{R}^3$. Imagine that the spheres are tangent at $p_0 = p_0'$. Clearly the spheres will be isometric exactly when $r = r'$. Condition (ii) says that if the spheres are rolled along one another without sliding or twisting then the endpoint $\gamma'(1)$ of one curve of contact depends only on the endpoint $\gamma(1)$ of the other and not on the intervening curve $\gamma(t)$. By Theorem 4.37 the Riemann curvature of a 2-manifold at p is determined by the Gaussian curvature $K(p)$; and for spheres we have $K(p) = 1/r$.

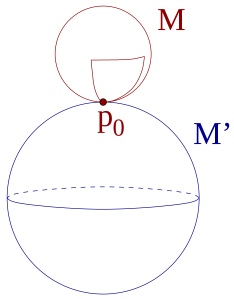


Figure 4.3: Diagram for Example 4.49.

Exercise 4.50. Let γ be the closed curve which bounds an octant as shown in the diagram for Example 4.49. Find γ'.

Exercise 4.51. Show that in case M is two-dimensional, the condition $\Phi(1) = \Phi_0$ in Theorem 4.47 may be dropped from (ii).

Proof of Lemma 4.48. Let $I \subset \mathbb{R}$ be an interval containing zero and let $\gamma : I \to M$ be a smooth curve such that $\gamma(0) = p_0$. Define

$$\gamma'(t) := \phi(\gamma(t)), \qquad \Phi(t) := d\phi(\gamma(t))$$

for $t \in I$. Then $\dot{\gamma}' = \Phi\dot{\gamma}$ by the chain rule and, for every vector field X along γ, we have

$$\Phi\nabla X = \nabla'(\Phi X)$$

by Theorem 4.30. Hence it follows from Lemma 3.29 and Lemma 3.36 that (Φ, γ, γ') is a development. Since (Φ, γ, γ') satisfies the initial condition (4.39) the assertion follows from the uniqueness result for developments in Theorem 3.38. This proves the lemma. □

Proof of Theorem 4.47. We first prove a slightly different theorem. Namely, we weaken condition (i) to assert that ϕ is a local isometry (i.e. not necessarily bijective), and prove that this weaker condition is equivalent to (ii), (iii), and (iv) whenever M is connected and simply connected and M' is complete. Thus we drop the hypotheses that M be complete and M' be connected and simply connected.

We prove that (i) implies (ii). Given a development as in (ii) we have, by Lemma 4.48,

$$\gamma'(1) = \phi(\gamma(1)) = \phi(p_0) = p_0', \qquad \Phi(1) = d\phi(\gamma(1)) = d\phi(p_0) = \Phi_0,$$

as required.

We prove that (ii) implies (iii) when M' is complete. Choose developments $(\Phi_i, \gamma_i, \gamma_i')$ for $i = 0, 1$ as in (iii). Define a curve $\gamma : [0,1] \to M$ by "composition"

$$\gamma(t) := \begin{cases} \gamma_0(2t), & 0 \le t \le 1/2, \\ \gamma_1(2-2t), & 1/2 \le t \le 1, \end{cases}$$

so that γ is continuous and piecewise smooth and $\gamma(1) = p_0$. By Theorem 3.38 there is a development (Φ, γ, γ') on the interval $[0,1]$ satisfying (4.39) (because M' is complete). Since $\gamma(1) = p_0$ it follows from (ii) that $\gamma'(1) = p_0'$ and $\Phi(1) = \Phi_0$. By the uniqueness of developments and the invariance under reparametrization, we have

$$(\Phi(t), \gamma(t), \gamma'(t)) = \begin{cases} (\Phi_0(2t), \gamma_0(2t), \gamma_0'(2t)), & 0 \le t \le 1/2, \\ (\Phi_1(2-2t), \gamma_1(2-2t), \gamma_1'(2-2t)), & 1/2 \le t \le 1. \end{cases}$$

Hence $\gamma_0'(1) = \gamma'(1/2) = \gamma_1'(1)$ as required.

We prove that (iii) implies (i) when M' is complete and M is connected. We define $\phi : M \to M'$ as follows. For $p \in M$ we choose a smooth curve $\gamma : [0,1] \to M$ with $\gamma(0) = p_0$ and $\gamma(1) = p$ (because M is connected); by Theorem 3.38 there is a development (Φ, γ, γ') with $\gamma'(0) = p_0'$ and $\Phi(0) = \Phi_0$ (because M' is complete); now define

$$\phi(p) := \gamma'(1).$$

According to (iii) the point $\gamma'(1)$ is independent of the choice of the curve γ; thus ϕ is well defined. Moreover, by definition of ϕ, we have

$$\phi(\gamma(t)) = \gamma'(t)$$

for $0 \le t \le 1$. (Thus ϕ is smooth: see Exercise 4.55 below.) Differentiating this equation we have

$$d\phi(\gamma(t))\dot\gamma(t) = \dot\gamma'(t) = \Phi(t)\dot\gamma(t).$$

But each $\Phi(t)$ is an orthogonal transformation so

$$|d\phi(\gamma(t))\dot{\gamma}(t)| = |\Phi(t)\dot{\gamma}(t)| = |\dot{\gamma}(t)|\,.$$

Given $p \in M$ and $v \in T_pM$ we may always choose γ such that $\gamma(1) = p$ and $\dot{\gamma}(1) = v$ so that

$$|d\phi(p)v| = |v|\,.$$

Hence ϕ is a local isometry as required.

We prove that (i) implies (iv). Given a development as in (ii) we have

$$\gamma'(t) = \phi(\gamma(t)), \qquad \Phi(t) = d\phi(\gamma(t))$$

for every t, by Lemma 4.48. Hence it follows from Theorem 4.30 that

$$\Phi(t)_* R_{\gamma(t)} = (\phi_* R)_{\gamma'(t)} = R'_{\gamma'(t)}$$

for every t as required.

We prove that (iv) implies (iii) when M' is complete and M is simply connected. Choose developments $(\Phi_i, \gamma_i, \gamma'_i)$ for $i = 0, 1$ as in (iii). Since M is simply connected there is a homotopy

$$[0,1] \times [0,1] \to M : (\lambda, t) \mapsto \gamma(\lambda, t) = \gamma_\lambda(t)$$

from γ_0 to γ_1 with endpoints fixed. By Theorem 3.38 there is, for each λ, a development $(\Phi_\lambda, \gamma_\lambda, \gamma'_\lambda)$ on the interval $[0, 1]$ with initial conditions

$$\gamma'_\lambda(0) = p'_0, \qquad \Phi_\lambda(0) = \Phi_0$$

(because M' is complete). The proof of Theorem 3.38 also shows that $\gamma_\lambda(t)$ and $\Phi_\lambda(t)$ depend smoothly on both t and λ. We must prove that

$$\gamma'_1(1) = \gamma'_0(1).$$

To see this we will show that, for each fixed t, the curve

$$\lambda \mapsto (\Phi_\lambda(t), \gamma_\lambda(t), \gamma'_\lambda(t))$$

is a development; then by the definition of development we have that the curve $\lambda \mapsto \gamma'_\lambda(1)$ is smooth and

$$\partial_\lambda \gamma'_\lambda(1) = \Phi_\lambda(1)\partial_\lambda \gamma_\lambda(1) = 0$$

as required.

First choose a basis $e_1, \dots, e_m$ for $T_{p_0}M$ and extend to get vector fields $E_i \in \mathrm{Vect}(\gamma)$ along the homotopy γ by imposing the conditions that the vector fields $t \mapsto E_i(\lambda, t)$ be parallel, i.e.

$$\nabla_t E_i(\lambda, t) = 0, \qquad E_i(\lambda, 0) = e_i. \tag{4.40}$$

Then the vectors $E_1(\lambda, t), \dots E_m(\lambda, t)$ form a basis of $T_{\gamma_\lambda(t)}M$ for all λ and t. Second, define the vector fields E_i' along γ' by

$$E_i'(\lambda, t) := \Phi_\lambda(t) E_i(\lambda, t) \tag{4.41}$$

so that

$$\nabla'_t E_i' = 0.$$

Third, define the functions $\xi^1, \dots, \xi^m : [0, 1]^2 \to \mathbb{R}$ by

$$\partial_t \gamma =: \sum_{i=1}^m \xi^i E_i, \qquad \partial_t \gamma' = \sum_{i=1}^m \xi^i E_i'. \tag{4.42}$$

Here the second equation follows from (4.41) and the fact that $\Phi_\lambda \partial_t \gamma = \partial_t \gamma'$.

Now consider the vector fields

$$X' := \partial_\lambda \gamma', \qquad Y_i' := \nabla'_\lambda E_i' \tag{4.43}$$

along γ'. They satisfy the equations

$$\begin{aligned} \nabla'_t X' &= \nabla'_t \partial_\lambda \gamma' \\ &= \nabla'_\lambda \partial_t \gamma' \\ &= \nabla'_\lambda \Big(\sum_{i=1}^m \xi^i E_i' \Big) \\ &= \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i' + \xi^i Y_i' \big) \end{aligned}$$

and

$$\nabla'_t Y_i' = \nabla'_t \nabla'_\lambda E_i' - \nabla'_\lambda \nabla'_t E_i' = R'(\partial_t \gamma', \partial_\lambda \gamma') E_i'.$$

To sum up we have $X'(\lambda, 0) = Y_i'(\lambda, 0) = 0$ and

$$\nabla'_t X' = \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i' + \xi^i Y_i' \big), \qquad \nabla'_t Y_i' = R'(\partial_t \gamma', \partial_\lambda \gamma') E_i'. \tag{4.44}$$

On the other hand, the vector fields

$$X' := \Phi_\lambda \partial_\lambda \gamma, \qquad Y_i' := \Phi_\lambda \nabla_\lambda E_i \tag{4.45}$$

along γ' satisfy the same equations, namely

$$\begin{aligned}
\nabla_t' X' &= \Phi_\lambda \nabla_t \partial_\lambda \gamma \\
&= \Phi_\lambda \nabla_\lambda \partial_t \gamma \\
&= \Phi_\lambda \nabla_\lambda \left(\sum_{i=1}^m \xi^i E_i \right) \\
&= \Phi_\lambda \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i + \xi^i \nabla_\lambda E_i \big) \\
&= \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i' + \xi^i Y_i' \big)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_t' Y_i' &= \Phi_\lambda \big(\nabla_t \nabla_\lambda E_i - \nabla_\lambda \nabla_t E_i \big) \\
&= \Phi_\lambda R(\partial_t \gamma, \partial_\lambda \gamma) E_i \\
&= R'(\Phi_\lambda \partial_t \gamma, \Phi_\lambda \partial_\lambda \gamma) \Phi_\lambda E_i \\
&= R'(\partial_t \gamma', X') E_i'.
\end{aligned}$$

Here the last but one equation follows from (iv).

Since the tuples (4.43) and (4.45) satisfy the same differential equation (4.44) and vanish at $t = 0$ they must agree. Hence

$$\partial_\lambda \gamma' = \Phi_\lambda \partial_\lambda \gamma, \qquad \nabla_\lambda' E_i' = \Phi_\lambda \nabla_\lambda E_i$$

for $i = 1, \ldots, m$. This says that $\lambda \mapsto (\Phi_\lambda(t), \gamma_\lambda(t), \gamma_\lambda'(t))$ is a development. For $t = 1$ we obtain $\partial_\lambda \gamma'(\lambda, 1) = 0$ as required.

Now the modified theorem (where ϕ is a local isometry) is proved. The original theorem follows immediately. Condition (iv) is symmetric in M and M'; hence if we assume (iv) we have local isometries $\phi : M \to M'$, $\psi : M' \to M$ with

$$\phi(p_0) = p_0', \qquad d\phi(p_0) = \Phi_0, \qquad \psi(p_0') = p_0, \qquad d\psi(p') = \Phi_0^{-1}.$$

But then $\psi \circ \phi$ is a local isometry with $\psi \circ \phi(p_0) = p_0$ and $d(\psi \circ \phi)(p_0) = \mathrm{id}$. Hence $\psi \circ \phi$ is the identity. Similarly $\phi \circ \psi$ is the identity so ϕ is bijective (and $\psi = \phi^{-1}$) as required. □

Remark 4.52. The proof of Theorem 4.47 shows that the various implications in the weak version of the theorem (where ϕ is only a local isometry) require the following conditions on M and M':

(i) always implies (ii), (iii), and (iv);

(ii) implies (iii) whenever M' is complete;

(iii) implies (i) whenever M' is complete and M is connected;

(iv) implies (iii) whenever M' is complete and M is simply connected.

4.4.3 The local C-A-H theorem

Theorem 4.53 (Local C-A-H Theorem). *Let M and M' be smooth m-manifolds, fix two points $p_0 \in M$ and $p_0' \in M'$, and let $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Let $r > 0$ is smaller than the injectvity radii of M at p_0 and of M' at p_0' and denote*

$$U_r := \{p \in M \mid d(p_0, p) < r\}, \qquad U_r' := \left\{p' \in M' \mid d'(p_0', p') < r\right\}.$$

Then the following are equivalent.

(i) *There is an isometry $\phi : U_r \to U_r'$ satisfying* (4.38).

(ii) *If (Φ, γ, γ') is a development on an interval $I \subset \mathbb{R}$ with $0 \in I$, satisfying the initial condition* (4.39) *as well as*

$$\gamma(I) \subset U_r, \qquad \gamma'(I) \subset U_r',$$

then

$$\gamma(1) = p_0 \qquad \Longrightarrow \qquad \gamma'(1) = p_0', \quad \Phi(1) = \Phi_0.$$

(iii) *If $(\Phi_0, \gamma_0, \gamma_0')$ and $(\Phi_1, \gamma_1, \gamma_1')$ are developments as in (ii) then*

$$\gamma_0(1) = \gamma_1(1) \qquad \Longrightarrow \qquad \gamma_0'(1) = \gamma_1'(1).$$

(iv) *If $v \in T_{p_0}M$ with $|v| < r$ and*

$$\gamma(t) := \exp_{p_0}(tv), \quad \gamma'(t) := \exp'_{p_0'}(t\Phi_0 v), \quad \Phi(t) := \Phi'_{\gamma'}(t, 0)\Phi_0\Phi_\gamma(0, t),$$

then $\Phi(t)_* R_{\gamma(t)} = R'_{\gamma'(t)}$ for $0 \le t \le 1$.

Moreover, when these equivalent conditions hold, ϕ is given by

$$\phi(\exp_{p_0}(v)) = \exp'_{p_0'}(\Phi_0 v).$$

for $v \in T_{p_0}M$ with $|v| < r$.

Lemma 4.54. *Let $p \in M$ and $v, w \in T_pM$ such that $|v| < \mathrm{inj}(p)$. For $0 \le t \le 1$ define*

$$\gamma(t) := \exp(tv), \qquad X(t) := \frac{\partial}{\partial\lambda}\bigg|_{\lambda=0} \exp_p\big(t(v+\lambda w)\big) \in T_{\gamma(t)}M.$$

Then

$$\nabla_t\nabla_t X = R(\dot\gamma, X)\dot\gamma, \qquad X(0) = 0, \qquad \nabla_t X(0) = w. \tag{4.46}$$

A vector field along γ satisfying the first equation in (4.46) *is called a* **Jacobi field along** γ.

Proof. Write $\gamma(\lambda, t) := \exp_p(t(v+\lambda w))$ and $X := \partial_\lambda\gamma$ to all λ and t. Since $\gamma(\lambda, 0) = p$ for all λ we have $X(\lambda, 0) = 0$ and

$$\nabla_t X(\lambda, 0) = \nabla_t\partial_\lambda\gamma(\lambda, 0) = \nabla_\lambda\partial_t\gamma(\lambda, 0) = \frac{d}{d\lambda}\big(v + \lambda w\big) = w.$$

Moreover, $\nabla_t\partial_t\gamma = 0$ and hence

$$\begin{aligned}
\nabla_t\nabla_t X &= \nabla_t\nabla_t\partial_\lambda\gamma \\
&= \nabla_t\nabla_\lambda\partial_t\gamma - \nabla_\lambda\nabla_t\partial_t\gamma \\
&= R(\partial_t\gamma, \partial_\lambda\gamma)\partial_t\gamma \\
&= R(\partial_t\gamma, X)\partial_t\gamma.
\end{aligned}$$

This proves the lemma. □

Proof of Theorem 4.53. The proofs (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv) are as before; the reader might note that when $L(\gamma) \le r$ we also have $L(\gamma') \le r$ for any development so that there are plenty of developments with $\gamma : [0,1] \to U_r$ and $\gamma' : [0,1] \to U'_r$. The proof that (iv) implies (i) is a little different since (iv) here is somewhat weaker than (iv) of the global theorem: the equation $\Phi_* R = R'$ is only assumed for certain developments.

Hence assume (iv) and define $\phi : U_r \to U'_r$ by

$$\phi := \exp'_{p'_0} \circ \Phi_0 \circ \exp_{p_0}^{-1} : U_r \to U'_r.$$

We must prove that ϕ is an isometry. Thus we fix a point $q \in U_r$ and a tangent vector $u \in T_qM$ and choose $v, w \in T_pM$ with $|v| < r$ such that

$$\exp_{p_0}(v) = q, \qquad d\exp_{p_0}(v)w = u. \tag{4.47}$$

Define $\gamma : [0,1] \to U_r$, $\gamma' : [0,1] \to U'_r$, $X \in \mathrm{Vect}(\gamma)$, and $X' \in \mathrm{Vect}(\gamma')$ by

$$\gamma(t) = \exp_{p_0}(tv), \qquad X(t) := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \exp_{p_0}(t(v+\lambda w))$$
$$\gamma'(t) = \exp'_{p'_0}(t\Phi_0 v), \qquad X'(t) := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \exp_{p_0}(t(\Phi_0 v+\lambda \Phi_0 w)).$$

Then, by definition of ϕ, we have

$$\gamma' := \phi\circ\gamma, \qquad d\phi(\gamma)X = X'. \tag{4.48}$$

Moreover, by Lemma 4.54, X is a solution of (4.46) and X' is a solution of

$$\nabla_t\nabla_t X' = R'(\partial_t\gamma', X')\partial_t\gamma', \qquad X(\lambda,0)=0, \qquad \nabla_t X(\lambda,0) = \Phi_0 w. \tag{4.49}$$

Now define $\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ by

$$\Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t).$$

Then Φ intertwines covariant differentiation. Since $\dot\gamma$ and $\dot\gamma'$ are parallel vector fields with $\dot\gamma'(0) = \Phi_0 v = \Phi(0)\dot\gamma(0)$, we have

$$\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$$

for every t. Moreover, it follows from (iv) that $\Phi_* R_\gamma = R'_{\gamma'}$. Combining this with (4.46) we obtain

$$\nabla'_t\nabla'_t(\Phi X) = \Phi\nabla_t\nabla_t X = R'(\Phi\dot\gamma, \Phi X)\Phi\dot\gamma = R'(\dot\gamma', \Phi X)\dot\gamma'.$$

Hence the vector field ΦX along γ' also satisfies the initial value problem (4.49) and thus

$$\Phi X = X' = d\phi(\gamma)X.$$

Here we have also used (4.48). Using (4.47) we find

$$\gamma(1) = \exp_{p_0}(v) = q, \qquad X(1) = d\exp_{p_0}(v)w = u,$$

and so

$$d\phi(q)u = d\phi(\gamma(1))X(1) = X'(1) = \Phi(1)u.$$

Since $\Phi(1) : T_{\gamma(1)}M \to T_{\gamma'(1)}M'$ is an orthogonal transformation this gives

$$|d\phi(q)u| = |\Phi(1)u| = |u|.$$

Hence ϕ is an isometry as claimed. □

Exercise 4.55. Let $\phi : M \to M'$ be a map between manifolds. Assume that $\phi\circ\gamma$ is smooth for every smooth curve $\gamma : [0,1] \to M$. Prove that ϕ is smooth.

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4.5 Flat spaces

Our aim in the next few sections is to give applictions of the Cartan-Ambrose-Hicks Theorem. It is clear that the hypothesis $\Phi_* R = R'$ for *all* developments will be difficult to verify without drastic hypotheses on the curvature. The most drastic such hypothesis is that the curvature vanishes identically.

Definition 4.56. *A Riemannian manifold M is called* **flat** *if the Riemann curvature tensor R vanishes identically.*

Theorem 4.57. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold.*

(i) *M is flat if and only if every point has a neighborhood which is isometric to an open subset of $\mathbb{R}^m$, i.e. at each point $p \in M$ there exist local coordinates $x^1, \dots, x^m$ such that the coordinate vectorfields $E_i = \partial/\partial x^i$ are orthonormal.*

(ii) *Assume M is connected, simply connected, and complete. Then M is flat if and only if there is an isometry $\phi : M \to \mathbb{R}^m$ onto Euclidean space.*

Proof. Assertion (i) follows immediately from Theorem 4.53 and (ii) follows immediately from Theorem 4.47. □

Exercise 4.58. Carry over the Cartan–Ambrose–Hicks theorem and Theorem 4.57 to the intrinsic setting.

Exercise 4.59. A one-dimensional manifold is always flat.

Exercise 4.60. If M_1 and M_2 are flat so is $M = M_1 \times M_2$.

Example 4.61. By Exercises 4.59 and 4.60 the standard torus

$$\mathbb{T}^m = \left\{ z = (z_1, \dots, z_m) \in \mathbb{C}^m \,\middle|\, |z_1| = \cdots = |z_m| = 1 \right\}$$

is flat.

Exercise 4.62. For $b \geq a > 0$ and $c \geq 0$ define $M \subset \mathbb{C}^3$ by

$$M := \left\{ (u, v, w) \in \mathbb{C}^3 \mid |u| = a,\ |v| = b,\ w = cuv \right\}.$$

Then M is diffeomorphic to a torus (a product of two circles) and M is flat. If M' is similarly defined from numbers $b' \geq a' > 0$ and $c' \geq 0$ then there is an isometry $\phi : M \to M'$ if and only if $(a, b, c) = (a', b', c')$, i.e. $M = M'$. (**Hint:** Each circle $u = u_0$ is a geodesic as well as each circle $v = v_0$; the numbers a, b, c can be computed from the length of the circle $u = u_0$, the length of the circle $v = v_0$, and the angle between them.)

Exercise 4.63 (Developable manifolds). Let $n = m+1$ and let $E(t)$ be a one-parameter family of hyperplanes in $\mathbb{R}^n$. Then there is a smooth map $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$E(t) = u(t)^\perp, \qquad |u(t)| = 1, \tag{4.50}$$

for every t. We assume that $\dot{u}(t) \neq 0$ for every t so that $u(t)$ and $\dot{u}(t)$ are linearly independent. Show that

$$L(t) := u(t)^\perp \cap \dot{u}(t)^\perp = \lim_{s \to t} E(t) \cap E(s). \tag{4.51}$$

Thus $L(t)$ is a linear subspace of dimension $m-1$. Now let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that

$$\langle \dot{\gamma}(t), u(t) \rangle = 0, \qquad \langle \dot{\gamma}(t), \dot{u}(t) \rangle \neq 0 \tag{4.52}$$

for all t. This means that $\dot{\gamma}(t) \in E(t)$ and $\dot{\gamma}(t) \notin L(t)$; thus $E(t)$ is spanned by $L(t)$ and $\dot{\gamma}(t)$. For $t \in \mathbb{R}$ and $\varepsilon > 0$ define

$$L(t)_\varepsilon := \{ v \in L(t) \mid |v| < \varepsilon \}.$$

Let $I \subset \mathbb{R}$ be a bounded open interval such that the restriction of γ to the closure of I is injective. Prove that, for $\varepsilon > 0$ sufficiently small, the set

$$M_0 := \bigcup_{t \in I} \Big(\gamma(t) + L(t)_\varepsilon \Big)$$

is a smooth manifold of dimension $m = n - 1$. A manifold which arises this way is called **developable**. Show that the tangent spaces of M_0 are the original subspaces $E(t)$, i.e.

$$T_p M_0 = E(t) \qquad \text{for} \qquad p \in \gamma(t) + L(t)_\varepsilon.$$

(One therefore calls M_0 the *"envelope"* of the hyperplanes $\gamma(t) + E(t)$.) Show that M_0 is flat (hint: use Gauss–Codazzi). If (Φ, γ, γ') is a development of M_0 along $\mathbb{R}^m$, show that the map $\phi : M_0 \to \mathbb{R}^m$, defined by

$$\phi(\gamma(t) + v) := \gamma'(t) + \Phi(t) v$$

for $v \in L(t)_\varepsilon$, is an isometry onto an open set $M_0' \subset \mathbb{R}^m$. Thus a development *"unrolls"* M_0 onto the Euclidean space $\mathbb{R}^m$. When $n = 3$ and $m = 2$ one can visualize M_0 as a twisted sheet of paper (see Figure 4.4).

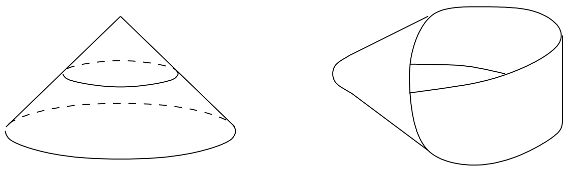

Figure 4.4: Developable surfaces.

Remark 4.64. Given a codimension-1 submanifold $M \subset \mathbb{R}^{m+1}$ and a curve $\gamma : \mathbb{R} \to M$ we may form the **osculating developable** M_0 to M along γ by taking

$$E(t) := T_{\gamma(t)}M.$$

This developable has common affine tangent spaces with M along γ as $T_{\gamma(t)}M_0 = E(t) = T_{\gamma(t)}M$ for every t. This gives a nice interpretation of parallel transport: M_0 may be unrolled onto a hyperplane where parallel transport has an obvious meaning and the identification of the tangent spaces thereby defines parallel transport in M. (See Remark 3.33.)

Exercise 4.65. Each of the following is a developable surface in $\mathbb{R}^3$.

(i) A cone on a plane curve $\Gamma \subset H$, i.e.

$$M = \{tp + (1-t)q \,|\, t > 0, q \in \Gamma\}$$

where $H \subset \mathbb{R}^3$ is an affine hyperplane, $p \in \mathbb{R}^3 \setminus H$, and $\Gamma \subset H$ is a 1-manifold.

(ii) A cylinder on a plane curve Γ, i.e.

$$M = \{q + tv \,|\, q \in \Gamma,\ t \in \mathbb{R}\}$$

where H and Γ are as in (i) and v is a fixed vector not parallel to H. (This is a cone with the cone point p at infinity.)

(iii) The tangent developable to a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, i.e.

$$M = \{\gamma(t) + s\dot{\gamma}(t) \,|\, |t - t_0| < \varepsilon,\ 0 < s < \varepsilon\},$$

where $\dot{\gamma}(t_0)$ and $\ddot{\gamma}(t_0)$ are linearly independent and $\varepsilon > 0$ is sufficiently small.

(iv) The normal developable to a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, i.e.

$$M = \{\gamma(t) + s\ddot{\gamma}(t) \,|\, |t - t_0| < \varepsilon,\ |s| < \varepsilon\}$$

where $|\dot{\gamma}(t)| = 1$ for all t, $\ddot{\gamma}(t_0) \neq 0$, and $\varepsilon > 0$ is sufficiently small.

Remark 4.66. A 2-dimensional submanifold $M \subset \mathbb{R}^3$ is called a **ruled surface** if there is a straight line in M through every point. Every developable surface is ruled, however, there are ruled surfaces that are not developable. An example is the **elliptic hyperboloid** of one sheet depicted in Figure 4.5:

$$M := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \right\}.$$

This manifold has negative Gaussian curvature and there are two straight lines through every point in M. (**Exercise:** Prove all this.)

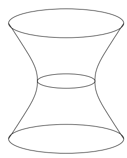

Figure 4.5: A circular one-sheeted hyperboloid.

4.6 Symmetric spaces

In the last section we applied the Cartan-Ambrose-Hicks Theorem in the flat case; the hypothesis $\Phi_* R = R'$ was easy to verify since both sides vanish. To find more general situations where we can verify this hypothesis note that for any development (Φ, γ, γ') satisfying the initial conditions $\gamma(0) = p_0$, $\gamma'(0) = p'_0$, and $\Phi(0) = \Phi_0$, we have

$$\Phi(t) = \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t)$$

so that the hypothesis $\Phi_* R = R'$ is certainly implied by the three hypotheses

$$\begin{aligned} \Phi_\gamma(t,0)_* R_{p_0} &= R_{\gamma(t)} \\ \Phi'_{\gamma'} t,0)_* R'_{p'_0} &= R'_{\gamma'(t)} \\ (\Phi_0)_* R_{p_0} &= R'_{p'_0}. \end{aligned}$$

The last hypothesis is a condition on the initial linear isomorphism

$$\Phi_0 : T_{p_0}M \to T_{p'_0}M'$$

while the former hypotheses are conditions on M and M' respectively, namely, that the Riemann curvature tensor is invariant by parallel transport. It is rather amazing that this condition is equivalent to a rather simple geometric condition as we now show.

4.6.1 Symmetric spaces

Definition 4.67. *A Riemannian manifold M is called* **symmetric about the point** $p \in M$ *if there is a (necessarily unique) isometry $\phi : M \to M$ satisfying*

$$\phi(p) = p, \qquad d\phi(p) = -\mathrm{id}. \tag{4.53}$$

M is called a **symmetric space** *if it is symmetric about each of its points. A Riemannian manifold M is called* **locally symmetric about the point** $p \in M$ *if, for $r > 0$ sufficiently small, there is an isometry*

$$\phi : U_r(p, M) \to U_r(p, M), \qquad U_r(p, M) := \{q \in M \mid d(p, q) < r\},$$

satisfying (4.53)*; M is called a* **locally symmetric space** *if it is locally symmetric about each of its points.*

Remark 4.68. The proof of Theorem 4.70 below will show that, if M is locally symmetric, the isometry $\phi : U_r(p, M) \to U_r(p, M)$ with $\phi(p) = p$ and $d\phi(p) = -\mathrm{id}$ exists whenever $0 < r \le \mathrm{inj}(p)$.

Exercise 4.69. Every symmetric space is complete. **Hint:** If $\gamma : I \to M$ is a geodesic and $\phi : M \to M$ is a symmetry about the point $\gamma(t_0)$ for $t_0 \in I$ then

$$\phi(\gamma(t_0 + t)) = \gamma(t_0 - t)$$

for all $t \in \mathbb{R}$ with $t_0 + t, t_0 - t \in I$.

Theorem 4.70. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Then the following are equivalent.*

(i) *M is locally symmetric.*

(ii) *The covariant derivative ∇R (defined below) vanishes identically, i.e.*

$$(\nabla_v R)_p(v_1, v_2)w = 0$$

for all $p \in M$ and $v, v_1, v_2, w \in T_pM$.

(iii) *The curvature tensor R is invariant under parallel transport, i.e.*

$$\Phi_\gamma(t, s)_* R_{\gamma(s)} = R_{\gamma(t)} \tag{4.54}$$

for every smooth curve $\gamma : \mathbb{R} \to M$ and all $s, t \in \mathbb{R}$.

Before proving Theorem 4.70 we note some immediate corollaries.

Corollary 4.71. *Let M and M' be locally symmetric spaces and fix two points $p_0 \in M$ and $p_0' \in M'$, and let $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Let $r > 0$ be less than the injectivity radius of M at p_0 and the injectivity radius of M' at p_0'. Then the following holds.*

(i) *There is an isometry $\phi : U_r(p_0, M) \to U_r(p_0', M')$ with $\phi(p_0) = p_0'$ and $d\phi(p_0) = \Phi_0$ if and only Φ_0 intertwines R and R':*

$$(\Phi_0)_* R_{p_0} = R'_{p_0'}. \tag{4.55}$$

(ii) *Assume M and M' are connected, simply connected, and complete. Then there is an isometry $\phi : M \to M'$ with $\phi(p_0) = p_0'$ and $d\phi(p_0) = \Phi_0$ if and only if Φ_0 satisfies* (4.55)*.*

Proof. In both (i) and (ii) the "only if" statement follows from Theorem 4.30 (Theorema Egregium) with $\Phi_0 := d\phi(p_0)$. To prove the "if" statement, let (Φ, γ, γ') be a development satisfying $\gamma(0) = p_0$, $\gamma'(p_0) = p_0'$, and $\Phi(0) = \Phi_0$. Since R and R' are invariant under parallel transport, by Theorem 4.70, it follows from the discussion in the beginning of this section that $\Phi_* R = R'$. Hence assertion (i) follows from the local C-A-H Theorem 4.53 and (ii) follows from the global C-A-H Theorem 4.47 □

Corollary 4.72. *A connected, simply connected, complete, locally symmetric space is symmetric.*

Proof. Corollary 4.71 (ii) with $M' = M$, $p_0' = p_0$, and $\Phi_0 = -\mathrm{id}$. □

Corollary 4.73. *A connected symmetric space M is* **homogeneous***; i.e. given $p, q \in M$ there exists an isometry $\phi : M \to M$ with $\phi(p) = q$.*

Proof. If M is simply connected the assertion follows from Corollary 4.71 (ii) with $M = M'$, $p_0 = p$, $p_0' = q$, and $\Phi_0 = \Phi_\gamma(1, 0) : T_pM \to T_qM$, where $\gamma : [0, 1] \to M$ is a curve from p to q. If M is not simply connected we can argue as follows. There is an equivalence relation on M defined by

$$p \sim q \quad :\Longleftrightarrow \quad \exists \text{ isometry } \phi : M \to M \ni \phi(p) = q.$$

Let $p, q \in M$ and suppose that $d(p, q) < \mathrm{inj}(p)$. By Theorem 2.44 there is a unique shortest geodesic $\gamma : [0, 1] \to M$ connecting p to q. Since M is symmetric there is an isometry $\phi : M \to M$ such that $\phi(\gamma(1/2)) = \gamma(1/2)$ and $d\phi(\gamma(1/2)) = -\mathrm{id}$. This isometry satisfies $\phi(\gamma(t)) = \gamma(1 - t)$ and hence $\phi(p) = q$. Thus $p \sim q$ whenever $d(p, q) < \mathrm{inj}(p)$. This shows that each equivalence class is open, hence each equivalence class is also closed, and hence there is only one equivalence class because M is connected. This proves the corollary. □

4.6.2 The covariant derivative of the curvature

For two vector spaces V, W and an integer $k \geq 1$ we denote by $\mathcal{L}^k(V,W)$ the vector space of multi-linear maps from $V^k = V \times \cdots \times V$ to W. Thus $\mathcal{L}^1(V,W) = \mathcal{L}(V,W)$ is the space of linear maps from V to W.

Definition 4.74. The **covariant derivative of the Riemann curvature tensor** assigns to every $p \in M$ a linear map

$$(\nabla R)_p : T_pM \to \mathcal{L}^2(T_pM, \mathcal{L}(T_pM, T_pM)$$

such that

$$\begin{aligned}(\nabla R)(X)(X_1,X_2)Y = \nabla_X\big(R(X_1,X_2)Y\big) - R(\nabla_X X_1, X_2)Y \\ - R(X_1, \nabla_X X_2)Y - R(X_1,X_2)\nabla_X Y\end{aligned} \tag{4.56}$$

for all $X, X_1, X_2, Y \in \mathrm{Vect}(M)$. We also use the notation

$$(\nabla_v R)_p := (\nabla R)_p(v)$$

for $p \in M$ and $v \in T_pM$ so that

$$(\nabla_X R)(X_1,X_2)Y := (\nabla R)(X)(X_1,X_2)Y$$

for all $X, X_1, X_2, Y \in \mathrm{Vect}(M)$.

Remark 4.75. One verifies easily that the map

$$\mathrm{Vect}(M)^4 \to \mathrm{Vect}(M) : (X, X_1, X_2, Y) \mapsto (\nabla_X R)(X_1,X_2)Y,$$

defined by the right hand side of equation (4.56), is multi-linear over the ring of functions $\mathscr{F}(M)$. Hence it follows as in Remark 4.22 that ∇R is well defined, i.e. that the right hand side of (4.56) at $p \in M$ depends only on the tangent vectors $X(p), X_1(p), X_2(p), Y(p)$.

Remark 4.76. Let $\gamma : I \to M$ be a smooth curve on an interval $I \subset \mathbb{R}$ and

$$X_1, X_2, Y \in \mathrm{Vect}(\gamma)$$

be smooth vector fields along γ. Then equation (4.56) continues to hold with X replaced by $\dot\gamma$ and each ∇_X on the right hand side replaced by the covariant derivative of the respective vector field along γ:

$$\begin{aligned}(\nabla_{\dot\gamma} R)(X_1,X_2)Y = \nabla(R(X_1,X_2)Y) - R(\nabla X_1, X_2)Y \\ - R(X_1, \nabla X_2)Y - R(X_1,X_2)\nabla Y.\end{aligned} \tag{4.57}$$

Theorem 4.77. **(i)** *If* $\gamma : \mathbb{R} \to M$ *is a smooth curve such that* $\gamma(0) = p$ *and* $\dot\gamma(0) = v$ *then*

$$(\nabla_v R)_p = \frac{d}{dt}\Big|_{t=0} \Phi_\gamma(0,t)_* R_{\gamma(t)} \tag{4.58}$$

(ii) *The covariant derivative of the Riemann curvature tensor satisfies the* **second Bianchi identity**

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0. \tag{4.59}$$

Proof. We prove (i). Let $v_1, v_2, w \in T_pM$ and choose parallel vector fields $X_1, X_2, Y \in \mathrm{Vect}(\gamma)$ along γ satisfying the initial conditions $X_1(0) = v_1$, $X_2(0) = v_2$, $Y(0) = w$. Thus

$$X_1(t) = \Phi_\gamma(t,0)v_1, \qquad X_2(t) = \Phi_\gamma(t,0)v_2, \qquad Y(t) = \Phi_\gamma(t,0)w.$$

Then the last three terms on the right vanish in equation (4.57) and hence

$$\begin{aligned}
(\nabla_v R)(v_1,v_2)w &= \nabla(R(X_1,X_2)Y)(0) \\
&= \frac{d}{dt}\Big|_{t=0} \Phi_{\gamma(0,t)} R_{\gamma(t)}(X_1(t), X_2(t))Y(t) \\
&= \frac{d}{dt}\Big|_{t=0} \Phi_{\gamma(0,t)} R_{\gamma(t)}(\Phi_\gamma(t,0)v_1, \Phi_\gamma(t,0)v_2)\Phi_\gamma(t,0)w \\
&= \frac{d}{dt}\Big|_{t=0} \left(\Phi_\gamma(0,t)_* R_{\gamma(t)}\right)(v_1,v_2)w.
\end{aligned}$$

Here the second equation follows from Theorem 3.4. This proves (i).

We prove (ii). Choose a smooth function $\gamma : \mathbb{R}^3 \to M$ and denote by (r,s,t) the coordinates on $\mathbb{R}^3$. If Y is a vector field along γ we have

$$\begin{aligned}
(\nabla_{\partial_r\gamma} R)(\partial_s\gamma, \partial_t\gamma)Y &= \nabla_r\big(R(\partial_s\gamma,\partial_t\gamma)Y\big) - R(\partial_s\gamma,\partial_t\gamma)\nabla_r Y \\
&\quad - R(\nabla_r\partial_s\gamma, \partial_t\gamma)Y - R(\partial_s\gamma, \nabla_r\partial_t\gamma)Y \\
&= \nabla_r\left(\nabla_s\nabla_t Y - \nabla_t\nabla_s Y\right) - \left(\nabla_s\nabla_t - \nabla_t\nabla_s\right)\nabla_r Y \\
&\quad + R(\partial_t\gamma, \nabla_r\partial_s\gamma)Y - R(\partial_s\gamma, \nabla_t\partial_r\gamma)Y.
\end{aligned}$$

Permuting the variables r, s, t cyclically and taking the sum of the resulting three equations we obtain

$$\begin{aligned}
&(\nabla_{\partial_r\gamma} R)(\partial_s\gamma,\partial_t\gamma)Y + (\nabla_{\partial_s\gamma} R)(\partial_t\gamma,\partial_r\gamma)Y + (\nabla_{\partial_t\gamma} R)(\partial_r\gamma,\partial_s\gamma)Y \\
&= \nabla_r\left(\nabla_s\nabla_t Y - \nabla_t\nabla_s Y\right) - \left(\nabla_s\nabla_t - \nabla_t\nabla_s\right)\nabla_r Y \\
&\quad + \nabla_s\left(\nabla_t\nabla_r Y - \nabla_r\nabla_t Y\right) - \left(\nabla_t\nabla_r - \nabla_r\nabla_t\right)\nabla_s Y \\
&\quad + \nabla_t\left(\nabla_r\nabla_s Y - \nabla_s\nabla_r Y\right) - \left(\nabla_r\nabla_s - \nabla_s\nabla_r\right)\nabla_t Y.
\end{aligned}$$

The terms on the right hand side cancel out. This proves the theorem. □

Proof of Theorem 4.70. We prove that (iii) implies (i). This follows from the local Cartan–Ambrose–Hicks Theorem 4.53 with

$$p_0' = p_0 = p, \qquad \Phi_0 = -\mathrm{id} : T_pM \to T_pM.$$

This isomorphism satisfies

$$(\Phi_0)_* R_p = R_p.$$

Hence it follows from the discussion in the beginning of this section that

$$\Phi_* R = R'$$

for every development (Φ, γ, γ') of M along itself satisfying

$$\gamma(0) = \gamma'(0) = p, \qquad \Phi(0) = -\mathrm{id}.$$

Hence, by the local C-A-H Theorem 4.53, there is an isometry

$$\phi : U_r(p, M) \to U_r(p, M)$$

satisfying

$$\phi(p) = p, \qquad d\phi(p) = -\mathrm{id}$$

whenever $0 < r < \mathrm{inj}(p, M)$.

We prove that (i) implies (ii). By Theorem 4.30 (Theorema Egregium), every isometry $\phi : M \to M'$ preserves the Riemann curvature tensor and covariant differentiation, and hence also the covariant derivative of the Riemann curvature tensor, i.e.

$$\phi_*(\nabla R) = \nabla' R'.$$

Applying this to the local isometry $\phi : U_r(p, M) \to U_r(p, M)$ we obtain

$$\left(\nabla_{d\phi(q)v} R\right)_{\phi(q)} (d\phi(q)v_1, d\phi(q)v_2) = d\phi(q) \left(\nabla_v R\right) (v_1, v_2) d\phi(q)^{-1}.$$

for all $v, v_1, v_2 \in T_pM$ Since

$$d\phi(p) = -\mathrm{id}$$

this shows that ∇R vanishes at p.

We prove that (ii) imlies (iii). If ∇R vanishes then then equation (4.58) in Theorem 4.77 shows that the function

$$s \mapsto \Phi_\gamma(t, s)_* R_{\gamma(s)} = \Phi_\gamma(t, 0)_* \Phi(0, s)_* R_{\gamma(s)}$$

is constant and hence is everywhere equal to $R_{\gamma(t)}$. This implies (4.54) and completes the proof of the theorem. □

The covariant derivative of the curvature in local coordinates

Let $\phi : U \to \Omega$ be a local coordinate chart on M with values in an open set $\Omega \subset \mathbb{R}^m$, denote its inverse by $\psi := \phi^{-1} : \Omega \to U$, and let

$$E_i(x) := \frac{\partial \psi}{\partial x^i}(x) \in T_{\psi(x)}M, \qquad x \in \Omega, \qquad i = 1, \dots, m,$$

be the local frame of the tangent bundle determined by this coordinate chart. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ denote the Christoffel symbols and $R_{ijk}^\ell : \Omega \to \mathbb{R}$ the coefficients of the Riemann curvature tensor so that

$$\nabla_i E_j = \sum_k \Gamma_{ij}^k E_k, \qquad R(E_i, E_j)E_k = \sum_\ell R_{ijk}^\ell E_\ell.$$

Given $i, j, k, \ell \in \{1, \dots, m\}$ we can express the vector field $(\nabla_{E_i} R)(E_j, E_k)E_\ell$ along ψ for each $x \in \Omega$ as a linear combination of the basis vectors $E_i(x)$. This gives rise to functions $\nabla_i R_{jk\ell}^\nu : \Omega \to \mathbb{R}$ defined by

$$(\nabla_{E_i} R)(E_j, E_k)E_\ell =: \sum_\nu \nabla_i R_{jk\ell}^\nu E_\nu. \tag{4.60}$$

These functions are given by

$$\begin{aligned} \nabla_i R_{jk\ell}^\nu &= \partial_i R_{jk\ell}^\nu + \sum_\mu \Gamma_{i\mu}^\nu R_{jk\ell}^\mu \\ &\quad - \sum_\mu \Gamma_{ij}^\mu R_{\mu k\ell}^\nu - \sum_\mu \Gamma_{ik}^\mu R_{j\mu\ell}^\nu - \sum_\mu \Gamma_{i\ell}^\mu R_{jk\mu}^\nu. \end{aligned} \tag{4.61}$$

The second Bianchi identity has the form

$$\nabla_i R_{jk\ell}^\nu + \nabla_j R_{ki\ell}^\nu + \nabla_k R_{ij\ell}^\nu = 0. \tag{4.62}$$

Exercise: Prove equations (4.61) and (4.62). **Warning:** As in Subsection 4.3.3, care must be taken with the ordering of the indices. Some authors use the notation $\nabla_i R_{\ell jk}^\nu$ for what we call $\nabla_i R_{jk\ell}^\nu$.

4.6.3 Examples and exercises

Example 4.78. A flat manifold is locally symmetric.

Example 4.79. If M_1 and M_2 are (locally) symmetric, so is $M = M_1 \times M_2$.

Example 4.80. $M = \mathbb{R}^m$ with the standard metric is a symmetric space. Recall that the isometry group $\mathcal{I}(\mathbb{R}^m)$ consists of all affine transformations of the form

$$\phi(x) = Ax + b, \qquad A \in \mathrm{O}(m), \qquad b \in \mathbb{R}^m.$$

(See Exercise 4.4.) The isometry with fixed point $p \in \mathbb{R}^m$ and $d\phi(p) = -\mathrm{id}$ is given by $\phi(x) = 2p - x$ for $x \in \mathbb{R}^m$.

Example 4.81. The flat tori of Exercise 4.62 in the previous section are symmetric (but not simply connected). This shows that the hypothesis of simply connectivity cannot be dropped in the Corollary 4.71 (ii).

Example 4.82. Below we define manifolds of constant curvature and show that they are locally symmetric. The simplest example, after a flat space, is the unit sphere $S^m = \left\{x \in \mathbb{R}^{m+1} \mid |x| = 1\right\}$. The symmetry ϕ of the sphere about a point $p \in M$ is given by

$$\phi(x) := -x + 2\langle p, x\rangle p$$

for $x \in S^m$. This extends to an orthogonal linear transformation of the ambient space. In fact the group of isometries of S^m is the group $\mathrm{O}(m+1)$ of orthogonal linear transformations of $\mathbb{R}^{m+1}$: see Example 4.101 below. In accordance with Corollary 4.73 this group acts transitively on S^m.

Example 4.83. A compact two-dimensional manifold of constant negative curvature is locally symmetric (as its universal cover is symmetric) but not homogeneous (as closed geodesics of a given period are isolated). Hence it is not symmetric. This shows that the hypothesis that M be simply connected cannot be dropped in the Corollary 4.72.

Example 4.84. The real projective space $\mathbb{R}P^n$ with the metric inherited from S^n is a symmetric space and the orthogonal group $\mathrm{O}(n+1)$ acts on it by isometries. The complex projective space $\mathbb{C}P^n$ with the Fubini–Study metric is a symmetric space and the unitary group $\mathrm{U}(n+1)$ acts on it by isometries: see Example 2.69. The complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ is a symmetric space and the unitary group $\mathrm{U}(n)$ acts on it by isometries: see Example 2.70. (**Exercise:** Prove this.)

Example 4.85. The simplest example of a symmetric space which is not of constant curvature is the orthogonal group $\mathrm{O}(n) = \left\{g \in \mathbb{R}^{n\times n} \mid g^T g = \mathbb{1}\right\}$ with the Riemannian metric (4.24) of Example 4.27. The symmetry ϕ about the point $a \in \mathrm{O}(n)$ is given by $\phi(g) = ag^{-1}a$. This discussion extends to every Lie subgroup $\mathrm{G} \subset \mathrm{O}(n)$. (**Exercise:** Prove this.)

4.7 Constant curvature

In the Section 4.3 we saw that the Gaussian curvature of a two-dimensional surface is intrinsic: we gave a formula for it in terms of the Riemann curvature tensor and the first fundamental form. We may use this formula to define the Gaussian curvature for *any* two-dimensional manifold (even if its codimension is greater than one). We make a slightly more general definition.

4.7.1 Sectional curvature

Definition 4.86. *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional manifold. Let $p \in M$ and $E \subset T_pM$ be a 2-dimensional linear subspace of the tangent space. The* **sectional curvature** *of M at (p, E) is the number*

$$K(p, E) = \frac{\langle R_p(u, v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2} \tag{4.63}$$

where $u, v \in E$ are linearly independent (and hence form a basis of E).

The right hand side of (4.63) remains unchanged if we multiply u or v by a nonzero real number or add to one of the vectors a real multiple of the other; hence it depends only on the linear subspace spanned by u ad v.

Example 4.87. If $M \subset \mathbb{R}^3$ is a 2-manifold then, by Theorem 4.37, the sectional curvature $K(p, T_pM) = K(p)$ is the Gaussian curvature of M at p. More generally, for any 2-manifold $M \subset \mathbb{R}^n$ (whether or not it has codimension one) we *define* the **Gaussian curvature** of M at p by

$$K(p) := K(p, T_pM).$$

Example 4.88. If $M \subset \mathbb{R}^{m+1}$ is a submanifold of codimension one and $\nu : M \to S^m$ is a Gauss map then the sectional curvature of a 2-dimensional subspace $E \subset T_pM$ spanned by two linearly independent tangent vectors $u, v \in T_pM$ is given by

$$K(p, E) = \frac{\langle u, d\nu(p)u\rangle\langle v, d\nu(p)v\rangle - \langle u, d\nu(p)v\rangle^2}{|u|^2|v|^2 - \langle u, v\rangle^2}. \tag{4.64}$$

This follows from equation (4.36) in the proof of Theorem 4.37 which holds in all dimensions. In particular, when $M = S^m$, we have $\nu(p) = p$ and hence $K(p, E) = 1$ for all p and E. For a sphere of radius r we have $\nu(p) = p/r$ and hence $K(p, E) = 1/r^2$.

Example 4.89. Let $\mathrm{G} \subset \mathrm{O}(n)$ be a Lie subgroup equipped with the Riemannian metric

$$\langle v, w \rangle := \mathrm{trace}(v^T w)$$

for $v, w \in T_g\mathrm{G} \subset \mathbb{R}^{n\times n}$. Then, by Example 4.27, the sectional curvature of G at the identity matrix $\mathbb{1}$ is given by

$$K(\mathbb{1}, E) = \frac{1}{4}\left|[\xi, \eta]\right|^2$$

for every 2-dimensional linear subspace $E \subset \mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ with an orthonormal basis ξ, η.

Exercise 4.90. Let $E \subset T_pM$ be a 2-dimensional linear subspace, let $r > 0$ be smaller than the injectivity radius of M at p, and let $N \subset M$ be the 2-dimensional submanifold given by

$$N := \exp_p\left(\{v \in E \mid |v| < r\}\right).$$

Show that the sectional curvature $K(p, E)$ of M at (p, E) agrees with the Gauss curvature of N at p.

Exercise 4.91. Let $p \in M \subset \mathbb{R}^n$ and let $E \subset T_pM$ be a 2-dimensional linear subspace. For $r > 0$ let L denote the ball of radius r in the $(n - m + 2)$ dimensional affine subspace of $\mathbb{R}^n$ through p and parallel to the vector subspace $E + T_pM^\perp$:

$$L = \left\{p + v + w \,|\, v \in E,\ w \in T_pM^\perp,\ |v|^2 + |w|^2 < r^2\right\}.$$

Show that, for r sufficiently small, $L \cap M$ is a 2-dimensional manifold with Gauss curvature $K_{L\cap M}(p)$ at p given by

$$K_{L\cap M}(p) = K(p, E).$$

4.7.2 Constant sectional curvature

Definition 4.92. *Let $k \in \mathbb{R}$ and $m \geq 2$ be an integer. An m-manifold $M \subset \mathbb{R}^n$ is said to have* **constant sectional curvature** *k if $K(p, E) = k$ for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$.*

Theorem 4.93. *An m-dimensional manifold $M \subset \mathbb{R}^n$ has constant sectional curvature k if and only if*

$$\langle R_p(v_1, v_2)v_3, v_4\rangle = k\Big(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\Big) \tag{4.65}$$

for all $p \in M$ and all $v_1, v_2, v_3, v_4 \in T_pM$.

Proof. The "only if" statement follows immediately from the definition with $v_1 = v_2 = u$ and $v_2 = v_3 = v$. To prove the converse, we assume that M has constant curvature k. Fix a point $p \in M$ and define the multi-linear map $Q : T_pM^4 \to \mathbb{R}$ by

$$Q(v_1, v_2, v_3, v_4) := \langle R_p(v_1, v_2)v_3, v_4\rangle - k\Big(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\Big).$$

Then Q satisfies the equations

$$Q(v_1, v_2, v_3, v_4) + Q(v_2, v_1, v_3, v_4) = 0, \tag{4.66}$$

$$Q(v_1, v_2, v_3, v_4) + Q(v_2, v_3, v_1, v_4) + Q(v_3, v_1, v_2, v_4) = 0, \tag{4.67}$$

$$Q(v_1, v_2, v_3, v_4) - Q(v_3, v_4, v_1, v_2) = 0, \tag{4.68}$$

$$Q(u, v, u, v) = 0 \tag{4.69}$$

for all $u, v, v_1, v_2, v_3, v_4 \in T_pM$. Here the first three equations follow from Therem 4.23 and the last follows from the definition of Q and the hypothesis that M have constant sectional curvature k.

We must prove that Q vanishes. Using (4.68) and (4.69) we find

$$\begin{aligned} 0 &= Q(u, v_1 + v_2, u, v_1 + v_2) \\ &= Q(u, v_1, u, v_2) + Q(u, v_2, uv_1) \\ &= 2Q(u, v_1, u, v_2). \end{aligned}$$

for all $u, v_1, v_2 \in T_pM$. This implies

$$\begin{aligned} 0 &= Q(u_1 + u_2, v_1, u_1 + u_2, v_2) \\ &= Q(u_1, v_1, u_2, v_2) + Q(u_2, v_1, u_1, v_2). \end{aligned}$$

for all $u_1, u_2, v_1, v_2 \in TpM$. Hence

$$\begin{aligned} Q(v_1, v_2, v_3, v_4) &= -Q(v_3, v_2, v_1, v_4) \\ &= Q(v_2, v_3, v_1, v_4) \\ &= -Q(v_3, v_1, v_2, v_4) - Q(v_1, v_3, v_3, v_4). \end{aligned}$$

Here the second equation follows from (4.66) and the last from (4.67). Thus

$$Q(v_1, v_2, v_3, v_4) = -\frac{1}{2}Q(v_3, v_1, v_2, v_4) = \frac{1}{2}Q(v_1, v_3, v_2, v_4)$$

for all $v_1, v_2, v_3, v_4 \in T_pM$ and, repeating this argument,

$$Q(v_1, v_2, v_3, v_4) = \frac{1}{4}Q(v_1, v_2, v_3, v_4).$$

Hence $Q \equiv 0$ as claimed. This proves the theorem. $\square$

Remark 4.94. The symmetric group S_4 on four symbols acts naturally on the space $\mathcal{L}^4(T_pM, \mathbb{R})$ of multi-linear maps from T_pM^4 to $\mathbb{R}$. The conditions (4.66), (4.67), (4.68), and (4.69) say that the four elements

$$\begin{aligned} a &= \mathrm{id} + (12) \\ c &= \mathrm{id} + (123) + (132) \\ b &= \mathrm{id} - (34) \\ d &= \mathrm{id} + (13) + (24) + (13)(24) \end{aligned}$$

of the group ring of S_4 annihilate Q. This suggests an alternate proof of Theorem 4.93. A representation of a finite group is completely reducible so one can prove that $Q = 0$ by showing that any vector in any irreducible representation of S_4 which is annihilated by the four elements a, b, c and d must necessarily be zero. This can be checked case by case for each irreducible representation. (The group S_4 has 5 irreducible representations: two of dimension 1, two of dimension 3, and one of dimension 2.)

If M and M' are two m-dimensional manifolds with constant curvature k then every orthogonal linear isomorphism $\Phi : T_pM \to T_{p'}M'$ intertwines the Riemann curvature tensors by Theorem 4.93. Hence by the appropriate version (local or global) of the C-A-H Theorem we have the following corollaries.

Corollary 4.95. *Every Riemannian manifold with constant sectional curvature is locally symmetric.*

Proof. Theorem 4.70 and Theorem 4.93. □

Corollary 4.96. *Let M and M' be m-dimensional Riemannian manifolds with constant curvature k and let $p \in M$ and $p' \in M'$. If $r > 0$ is smaller than the injectivity radii of M at p and of M' at p' then, for every orthogonal linear isomorphism $\Phi : T_pM \to T_{p'}M'$, there is an isometry*

$$\phi : U_r(p, M) \to U_r(p', M')$$

such that

$$\phi(p) = p', \qquad d\phi(p) = \Phi.$$

Proof. This follows from Corollary 4.71 and Corollary 4.95. Alternatively one can use Theorem 4.93 and the local C-A-H Theorem 4.53. □

Corollary 4.97. *Any two connected, simply connected, complete Riemannian manifolds with the same constant sectional curvature and the same dimension are isometric.*

Proof. Theorem 4.93 and the global C-A-H Theorem 4.47. □

Corollary 4.98. *Let $M \subset \mathbb{R}^n$ be a connected, simply connected, complete manifold. Then the following are equivalent.*

(i) *M has constant sectional curvature.*

(ii) *For every pair of points $p, q \in M$ and every orthogonal linear isomorphism $\Phi : T_pM \to T_qM$ there is an isometry $\phi : M \to M$ such that*

$$\phi(p) = q, \qquad d\phi(p) = \Phi.$$

Proof. That (i) implies (ii) follows immediately from Theorem 4.93 and the global C-A-H Theorem 4.47. Conversely assume (ii). Then, for every pair of points $p, q \in M$ and every orthogonal linear isomorphism $\Phi : T_pM \to T_qM$, it follows from Theorem 4.30 (Theorema Egregium) that $\Phi_* R_p = R_q$ and hence $K(p, E) = K(q, \Phi E)$ for every 2-dimensional linear subspace $E \subset T_pM$. Since, for every pair of points $p, q \in M$ and of 2-dimensional linear subspaces $E \subset T_pM$, $F \subset T_qM$, we can find an orthogonal linear isomorphism $\Phi : T_pM \to T_qM$ such that $\Phi E = F$, this implies (i). □

Corollary 4.98 asserts that a connected, simply connected, complete Riemannian m-manifold M has constant sectional curvature if and only if the isometry group $\mathcal{I}(M)$ acts transitively on its orthonormal frame bundle $\mathcal{O}(M)$. Note that, by Lemma 4.10, this group action is also free.

4.7.3 Examples and exercises

Example 4.99. Any flat Riemannian manifold has constant sectional curvature $k = 0$.

Example 4.100. The manifold $M = \mathbb{R}^m$ with its standard metric is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = 0$.

Example 4.101. For $m \geq 2$ the unit sphere $M = S^m$ with its standard metric is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = 1$. Hence, by Corollary 4.97, every connected simply connected, complete Riemannian manifold with positive sectional curvature $k = 1$ is compact. Moreover, by Corollary 4.98, the isometry group $\mathcal{I}(S^m)$ is isomorphic to the group $\mathrm{O}(m+1)$ of orthogonal linear transformations of $\mathbb{R}^{m+1}$. Thus, by Corollary 4.98, the orthonormal frame bundle $\mathcal{O}(S^m)$ is diffeomorphic to $\mathrm{O}(m+1)$. This follows also from the fact that, if $v_1, \dots, v_m$ is an orthonormal basis of $T_pS^m = p^\perp$ then $p, v_1, \dots, v_m$ is an orthonormal basis of $\mathbb{R}^{m+1}$.

Example 4.102. A product of spheres is *not* a space of constant sectional curvature, but it *is* a symmetric space. **Exercise:** Prove this.

Example 4.103. For $n \geq 4$ the orthogonal group $\mathrm{O}(n)$ is not a space of constant sectional curvature, but it is a symmetric space and has nonnegative sectional curvature (see Example 4.89).

4.7.4 Hyperbolic space

The **hyperbolic space** $\mathbb{H}^m$ is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = -1$. A model for $\mathbb{H}^m$ can be constructed as follows. A point in $\mathbb{R}^{m+1}$ will be denoted by

$$p = (x_0, x), \qquad x_0 \in \mathbb{R}, \qquad x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

Let $Q : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ denote the symmetric bilinear form given by

$$Q(p, q) := -x_0 y_0 + x_1 y_1 + \cdots + x_m y_m$$

for $p = (x_0, x), q = (y_0, y) \in \mathbb{R}^{m+1}$. Since Q is nondegenerate the space

$$\mathbb{H}^m := \left\{ p = (x_0, x) \in \mathbb{R}^{m+1} \,|\, Q(p, p) = -1,\, x_0 > 0 \right\}$$

is a smooth m-dimensional submanifold of $\mathbb{R}^{m+1}$ and the tangent space of $\mathbb{H}^m$ at p is given by

$$T_p\mathbb{H}^m = \left\{ v \in \mathbb{R}^{m+1} \,|\, Q(p, v) = 0 \right\}.$$

For $p = (x_0, x) \in \mathbb{R}^{m+1}$ and $v = (\xi_0, \xi) \in \mathbb{R}^{m+1}$ we have

$$p \in \mathbb{H}^m \quad \Longleftrightarrow \quad x_0 = \sqrt{1 + |x|^2},$$
$$v \in T_p\mathbb{H}^m \quad \Longleftrightarrow \quad \xi_0 = \frac{\langle \xi, x \rangle}{\sqrt{1 + |x|^2}}.$$

Now let us define a Riemannian metric on $\mathbb{H}^m$ by

$$g_p(v, w) := Q(v, w) = \langle \xi, \eta \rangle - \xi_0 \eta_0 = \langle \xi, \eta \rangle - \frac{\langle \xi, x \rangle \langle \eta, x \rangle}{1 + |x|^2} \tag{4.70}$$

for $v = (\xi_0, \xi) \in T_p\mathbb{H}^m$ and $w = (\eta_0, \eta) \in T_p\mathbb{H}^m$.

Theorem 4.104. *$\mathbb{H}^m$ is a connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = -1$.*

We remark that the manifold $\mathbb{H}^m$ does not quite fit into the extrinsic framework of most of this manuscript as it is not exhibited as a submanifold of Euclidean space but rather of "pseudo-Euclidean space": the positive definite inner product $\langle v, w\rangle$ of the ambient space $\mathbb{R}^{m+1}$ is replaced by a nondegenerate symmetric bilinear form $Q(v, w)$. However, all the theory developed thus far goes through (reading $Q(v, w)$ for $\langle v, w\rangle$) provided we make the additional hypothesis (true in the example $M = \mathbb{H}^m$) that the first fundamental form $g_p = Q|_{T_pM}$ is positive definite. For then $Q|_{T_pM}$ is nondegenerate and we may define the orthogonal projection $\Pi(p)$ onto T_pM as before. The next lemma summarizes the basic observations; the proof is an exercise in linear algebra.

Lemma 4.105. *Let Q be a symmetric bilinear form on a vector space V and for each subspace E of V define its orthogonal complement by*

$$E^{\perp_Q} := \{w \in V \,|\, Q(u, v) = 0\ \forall v \in E\}.$$

Assume Q is nondegenerate, i.e. $V^{\perp_Q} = \{0\}$. Then, for every linear subspace $E \subset V$, we have

$$V = E \oplus E^{\perp_Q} \qquad \Longleftrightarrow \qquad E \cap E^{\perp_Q} = \{0\},$$

i.e. $E^{\perp_Q}$ is a vector space complement of E if and only if the restriction of Q to E is nondegenerate.

Proof of Theorem 4.104. The proofs of the various properties of $\mathbb{H}^m$ are entirely analogous to the corresponding proofs for S^m. Thus the unit normal field to $\mathbb{H}^m$ is given by $\nu(p) = p$ for $p \in \mathbb{H}^m$ although the "square of its length" is $Q(p, p) = -1$.

For $p \in \mathbb{H}^m$ we introduce the Q-orthogonal projection $\Pi(p)$ of $\mathbb{R}^{m+1}$ onto $T_p\mathbb{H}^m$. It is characterized by the conditions

$$\Pi(p)^2 = \Pi(p), \qquad \ker\Pi(p) \perp_Q \mathrm{im}\Pi(p), \qquad \mathrm{im}\Pi(p) = T_p\mathbb{H}^m,$$

and is given by the explicit formula

$$\Pi(p)v = v + Q(v, p)p$$

for $v \in \mathbb{R}^{m+1}$. The covariant derivative of a vector field $X \in \mathrm{Vect}(\gamma)$ along a smooth curve $\gamma : \mathbb{R} \to \mathbb{H}^m$ is given by

$$\begin{aligned}\nabla X(t) &= \Pi(\gamma(t))\dot{X}(t) \\ &= \dot{X}(t) + Q(\dot{X}(t), \gamma(t))\gamma(t) \\ &= \dot{X}(t) - Q(X(t), \dot{\gamma}(t))\gamma(t).\end{aligned}$$

The last identity follows by differentiating the equation $Q(X, \gamma) \equiv 0$. This can be interpreted as the hyperbolic Gauss–Weingarten formula as follows. For $p \in \mathbb{H}^m$ and $u \in T_p\mathbb{H}^m$ we introduce, as before, the second fundamental form $h_p(u) : T_p\mathbb{H}^m \to (T_p\mathbb{H}^m)^{\perp_Q}$ via

$$h_p(u)v := \big(d\Pi(p)u\big)v$$

and denote its Q-adjoint by

$$h_p(u)^* : (T_p\mathbb{H}^m)^{\perp_Q} \to T_p\mathbb{H}^m.$$

For every $p \in \mathbb{R}^{m+1}$ we have

$$\Big(d\Pi(p)u\Big)v = \frac{d}{dt}\bigg|_{t=0} \big(v + Q(v, p + tu)(p + tu)\big) = Q(v, p)u + Q(v, u)p,$$

where the first summand on the right is tangent to $\mathbb{H}^m$ and the second summand is Q-orthogonal to $T_p\mathbb{H}^m$. Hence

$$h_p(u)v = Q(v, u)p, \qquad h_p(u)^*w = Q(w, p)u \tag{4.71}$$

for $v \in T_p\mathbb{H}^m$ and $w \in (T_p\mathbb{H}^m)^{\perp_Q}$.

With this understood, the Gauss-Weingarten formula

$$\dot{X} = \nabla X + h_\gamma(\dot\gamma)X$$

extends to the present setting. The reader may verify that the operators $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$ thus defined satisfy the axioms of Remark 2.72 and hence define the Levi-Civita connection on $\mathbb{H}^m$.

Now a smooth curve $\gamma : I \to \mathbb{H}^m$ is a geodesic if and only if it satisfies the equivalent conditions

$$\nabla\dot\gamma \equiv 0 \quad\iff\quad \ddot\gamma(t) \perp_Q T_{\gamma(t)}\mathbb{H}^m \;\; \forall\, t \in I \quad\iff\quad \ddot\gamma = Q(\dot\gamma, \dot\gamma)\gamma.$$

A geodesic must satisfy the equation

$$\frac{d}{dt}Q(\dot\gamma, \dot\gamma) = 2Q(\ddot\gamma, \dot\gamma) = 0$$

because $\ddot\gamma$ is a scalar multiple of γ, and so $Q(\dot\gamma, \dot\gamma)$ is constant. Let $p \in \mathbb{H}^m$ and $v \in T_p\mathbb{H}^m$ be given with

$$Q(v, v) = 1.$$

Then the geodesic $\gamma : \mathbb{R} \to \mathbb{H}^m$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ is given by

$$\gamma(t) = \cosh(t)p + \sinh(t)v, \tag{4.72}$$

where

$$\cosh(t) := \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) := \frac{e^t - e^{-t}}{2}.$$

In fact we have $\ddot\gamma(t) = \gamma(t) \perp_Q T_{\gamma(t)}\mathbb{H}^m$. It follows that the geodesics exist for all time and hence $\mathbb{H}^m$ is geodesically complete. Moreover, being diffeomorphic to Euclidean space, $\mathbb{H}^m$ is connected and simply connected.

It remains to prove that $\mathbb{H}^m$ has constant sectional curvature $k = -1$. To see this we use the Gauss–Codazzi formula in the hyperbolic setting, i.e.

$$R_p(u, v) = h_p(u)^* h_p(v) - h_p(v)^* h_p(u). \tag{4.73}$$

By equation (4.71), this gives

$$\begin{aligned}
\langle R_p(u, v)v, u\rangle &= Q(h_p(u)u, h_p(v)v) - Q(h_p(v)u, h_p(u)v) \\
&= Q(Q(u, u)p, Q(v, v)p) - Q(Q(u, v)p, Q(u, v)p) \\
&= -Q(u, u)Q(v, v) + Q(u, v)^2 \\
&= -g_p(u, u)g_p(v, v) + g_p(u, v)^2
\end{aligned}$$

for all $u, v \in T_p\mathbb{H}^m$. Hence

$$K(p, E) = \frac{\langle R_p(u, v)v, u\rangle}{g_p(u, u)g_p(v, v) - g_p(u, v)^2} = -1$$

for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$ with a basis u, v. This proves the theorem. □

Exercise 4.106. Prove that the pullback of the metric on $\mathbb{H}^m$ under the diffeomorphism

$$\mathbb{R}^m \to \mathbb{H}^m : x \mapsto \left(\sqrt{1 + |x|^2}, x\right)$$

is given by

$$|\xi|_x = \sqrt{|\xi|^2 - \frac{\langle x, \xi\rangle^2}{1 + |x|^2}}$$

or, equivalently, by the metric tensor,

$$g_{ij}(x) = \delta_{ij} - \frac{x_i x_j}{1 + |x|^2} \tag{4.74}$$

for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$.

Exercise 4.107. The **Poincaré model** of hyperbolic space is the open unit disc $\mathbb{D}^m \subset \mathbb{R}^m$ equipped with the **Poincaré metric**

$$|\eta|_y = \frac{2\,|\eta|}{1-|y|^2}$$

for $y \in \mathbb{D}^m$ and $\eta \in \mathbb{R}^m = T_y\mathbb{D}^m$. Thus the metric tensor is given by

$$g_{ij}(y) = \frac{4\delta_{ij}}{\left(1-|y|^2\right)^2}, \qquad y \in \mathbb{D}^m. \tag{4.75}$$

Prove that the diffeomorphism

$$\mathbb{D}^m \to \mathbb{H}^m : y \mapsto \left(\frac{1+|y|^2}{1-|y|^2}, \frac{2y}{1-|y|^2}\right)$$

is an isometry with inverse

$$\mathbb{H}^m \to \mathbb{D}^m : (x_0, x) \mapsto \frac{x}{1+x_0}.$$

Interpret this map as a stereographic projection from the *south pole* $(-1, 0)$.

Exercise 4.108. The composition of the isometries in Exercise 4.106 and Exercise 4.107 is the diffeomorphism $\mathbb{R}^m \to \mathbb{D}^m : x \mapsto y$ given by

$$y = \frac{x}{\sqrt{1+|x|^2}+1}, \qquad x = \frac{2y}{1-|y|^2}, \qquad \sqrt{1+|x|^2} = \frac{1+|y|^2}{1-|y|^2}.$$

Prove that this is an isometry intertwining the Riemannian metrics (4.74) and (4.75). Find a formula for the geodesics in the Poincaré disc $\mathbb{D}^m$. **Hint:** Use Exercise 4.110 below.

Exercise 4.109. Prove that the isometry group of $\mathbb{H}^m$ is the pseudo-orthogonal group

$$\mathcal{I}(\mathbb{H}^m) = \mathrm{O}(m,1) := \left\{ g \in \mathrm{GL}(m+1) \,\middle|\, \begin{array}{c} Q(gv, gw) = Q(v,w) \\ \text{for all } v, w \in \mathbb{R}^{m+1} \end{array} \right\}.$$

Thus, by Corollary 4.98, the orthonormal frame bundle $\mathcal{O}(\mathbb{H}^m)$ is diffeomorphic to $\mathrm{O}(m,1)$.

Exercise 4.110. Prove that the exponential map

$$\exp_p : T_p\mathbb{H}^m \to \mathbb{H}^m$$

is given by

$$\exp_p(v) = \cosh\left(\sqrt{Q(v,v)}\right) p + \frac{\sinh\left(\sqrt{Q(v,v)}\right)}{\sqrt{Q(v,v)}} v \tag{4.76}$$

for $v \in T_p\mathbb{H}^m = p^{\perp_Q}$. Prove that this map is a diffeomorphism for every $p \in \mathbb{H}^m$. Thus any two points in $\mathbb{H}^m$ are connected by a unique geodesic. Prove that the intrinsic distance function on hyperbolic space is given by

$$d(p,q) = \cosh^{-1}\left(Q(p,q)\right) \tag{4.77}$$

for $p, q \in \mathbb{H}^m$. Compare this with Example 2.23 and Example 2.41.

4.8 Nonpositive sectional curvature

In the previous section we have seen that any two points in a connected, simply connected, complete manifold M of constant negative curvature can be connected by a unique geodesic (see Exercise 4.110). Thus the entire manifold M is geodesically convex and its injectivity radius is infinity. This continues to hold in much greater generality for manifolds with nonpositive sectional curvature. It is convenient, at this point, to extend the discussion to Riemannian manifolds in the intrinsic setting. In particular, at some point in the proof of the main theorem of this section and in our main example, we shall work with a Riemannian metric that does not arise (in any obvious way) from an embedding.

Definition 4.111. *A Riemannian manifold M is said to have* **nonpositive sectional curvature** *if $K(p,E) \le 0$ for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$ or, equivalently,*

$$\langle R_p(u,v)v, u\rangle \le 0$$

for all $p \in M$ and all $u, v \in T_pM$.

4.8.1 The theorem of Hadamard and Cartan

The next theorem shows that every connected, simply connected, complete Riemannian manifold with nonpositive sectional curvature is diffeomorphic to Euclidean space and has infinite injectivity radius. This is in sharp contrast to positive curvature manifolds as the example $M = S^m$ shows.

Theorem 4.112 (Cartan–Hadamard). *Let M be a connected, simply connected, complete Riemannian manifold. Then the following are equivalent.*

(i) *M has nonpositive sectional curvature.*

(ii) *The derivative of each exponential map is length increasing, i.e.*

$$\left|d\exp_p(v)\hat{v}\right| \geq |\hat{v}|$$

for all $p \in M$ and all $v, \hat{v} \in T_pM$.

(iii) *Each exponential map is* **distance increasing**, *i.e.*

$$d(\exp_p(v_0), \exp_p(v_1)) \geq |v_0 - v_1|$$

for all $p \in M$ and all $v, w \in T_pM$.

Moreover, if these equivalent conditions are satisfied then the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. Thus any two points in M can be connected by a unique geodesic.

Lemma 4.113. *Let M and M' be connected, simply connected, complete Riemannian manifolds and $\phi : M \to M'$ be a local isometry. Then ϕ is bijective and hence is an isometry.*

Proof. This follows by combining the weak and strong versions of the global C-A-H Theorem 4.47. Fix a point $p_0 \in M$ and define

$$p_0' := \phi(p_0), \qquad \Phi_0 := d\phi(p_0).$$

Then the tuple M, M', p_0, p_0', Φ_0 satisfies condition (i) of the weak version of Theorem 4.47. Hence this tuple also satisfies condition (iv) of Theorem 4.47. Since M and M' are connected, simply connected, and complete we may apply the strong version of Theorem 4.47 to obtain an isometry $\psi : M \to M'$ satisfying

$$\psi(p_0) = p_0', \qquad d\psi(p_0) = \Phi_0.$$

Since every isometry is also a local isometry and M is connected it follows from Lemma 4.10 that $\phi(p) = \psi(p)$ for all $p \in M$. Hence ϕ is an isometry, as required. ∎

Remark 4.114. Refining the argument in the proof of Lemma 4.113 one can show that a local isometry $\phi : M \to M'$ must be surjective whenever M is complete and M' is connected. None of these assumptions can be removed. (Take an isometric embedding of a disc in the plane or an embedding of a complete space M into a space with two components, one of which is isometric to M.)

Likewise, one can show that a local isometry $\phi : M \to M'$ must be injective whenever M is complete and connected and M' is simply connected. Again none of these asumptions can be removed. (Take a covering $\mathbb{R} \to S^1$, or a covering of a disjoint union of two isometric complete simply connected spaces onto one copy of this space, or some noninjective immersion of a disc into the plane and choose the pullback metric on the disc.)

Exercise 4.115. Let $\xi : [0, \infty) \to \mathbb{R}^n$ be a smooth function such that

$$\xi(0) = 0, \qquad \dot{\xi}(0) \neq 0, \qquad \xi(t) \neq 0 \quad \forall\, t > 0.$$

Prove that the function $f : [0, \infty) \to \mathbb{R}$ given by

$$f(t) := |\xi(t)|$$

is smooth. **Hint:** The function $\eta : [0, \infty) \to \mathbb{R}^n$ defined by

$$\eta(t) := \begin{cases} t^{-1}\xi(t), & \text{for } t > 0, \\ \dot{\xi}(0), & \text{for } t = 0, \end{cases}$$

is smooth. Show that f is differentiable and

$$\dot{f} = \frac{\langle \eta, \dot{\xi} \rangle}{|\eta|}.$$

Exercise 4.116. Let $\xi : \mathbb{R} \to \mathbb{R}^n$ be a smooth function such that

$$\xi(0) = 0, \qquad \ddot{\xi}(0) = 0.$$

Prove that there are constant $\varepsilon > 0$ and $c > 0$ such that, for all $t \in \mathbb{R}$:

$$|t| < \varepsilon \qquad \Longrightarrow \qquad |\xi(t)|^2 \, |\dot{\xi}(t)|^2 - \langle \xi(t), \dot{\xi}(t) \rangle^2 \leq c\, |t|^6 .$$

Hint: Write $\xi(t) = tv + \eta(t)$ and $\dot{\xi}(t) = v + \dot{\eta}(t)$ with $\eta(t) = O(t^3)$ and $\dot{\eta}(t) = O(t^2)$. Show that the terms of order 2 and 4 cancel in the Taylor expansion at $t = 0$.

Proof of Theorem 4.112. We prove that (i) implies (ii). Let $p \in M$ and $v, \hat{v} \in T_pM$ be given. Assume without loss of generality that $\hat{v} \neq 0$ and define $\gamma : \mathbb{R} \to M$ and $X \in \mathrm{Vect}(\gamma)$ by

$$\gamma(t) := \exp_p(tv), \qquad X(t) := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \exp_p(t(v + \lambda\hat{v})) \in T_{\gamma(t)}M \tag{4.78}$$

for $t \in \mathbb{R}$. Then

$$X(0) = 0, \qquad \nabla X(0) = \hat{v} \neq 0, \qquad X(t) = d\exp_p(tv)t\hat{v}, \tag{4.79}$$

and, by Lemma 4.54, X is a Jacobi field along γ:

$$\nabla\nabla X = R(\dot{\gamma}, X)\dot{\gamma}. \tag{4.80}$$

It follows from Exercise 4.115 with $\xi(t) := \Phi_\gamma(0,t)X(t)$ that the function $[0,\infty) \to \mathbb{R} : t \mapsto |X(t)|$ is smooth and

$$\left.\frac{d}{dt}\right|_{t=0} |X(t)| = |\nabla X(0)| = |\hat{v}|\,.$$

Moreover, for $t > 0$, we have

$$\begin{aligned}
\frac{d^2}{dt^2}|X| &= \frac{d}{dt}\frac{\langle X, \nabla X\rangle}{|X|} \\
&= \frac{|\nabla X|^2 + \langle X, \nabla\nabla X\rangle}{|X|} - \frac{\langle X, \nabla X\rangle^2}{|X|^3} \\
&= \frac{|X|^2|\nabla X|^2 - \langle X, \nabla X\rangle^2}{|X|^3} + \frac{\langle X, R(\dot{\gamma}, X)\dot{\gamma}\rangle}{|X|} \\
&\geq 0.
\end{aligned} \tag{4.81}$$

Here the third equation follows from the fact that X is a Jacobi field along γ, and the last inequality follows from the nonpositive sectional curvature condition in (i) and from the Cauchy–Schwarz inequality. Thus the second derivative function $[0,\infty) \to \mathbb{R} : t \mapsto |X(t)| - t\,|\hat{v}|$ is nonnegative; so its first derivative is nondecreasing and it vanishes at $t = 0$; thus

$$|X(t)| - t\,|\hat{v}| \geq 0$$

for every $t \geq 0$. In particular, for $t = 1$ we obtain

$$\left|d\exp_p(v)\hat{v}\right| = |X(1)| \geq |\hat{v}|\,.$$

as claimed. Thus we have proved that (i) implies (ii).

We prove that (ii) implies (i). Assume, by contradiction, that (ii) holds but there is a point $p \in M$ and a pair of vectors $v, \hat{v} \in T_pM$ such that

$$\langle R_p(v, \hat{v})v, \hat{v} \rangle < 0. \tag{4.82}$$

Define $\gamma : \mathbb{R} \to M$ and $X \in \mathrm{Vect}(\gamma)$ by (4.78) so that (4.79) and (4.80) are satisfied. Thus X is a Jacobi field with

$$X(0) = 0, \qquad \nabla X(0) = \hat{v} \neq 0.$$

Hence it follows from Exercise 4.116 with

$$\xi(t) := \Phi_\gamma(0, t)X(t)$$

that there is a constant $c > 0$ such that, for $t > 0$ sufficiently small, we have the inequality

$$|X(t)|^2 \, |\nabla X(t)|^2 - \langle X(t), \nabla X(t) \rangle^2 \leq ct^6.$$

Moreover,

$$|X(t)| \geq \delta t, \qquad \langle X(t), R(\dot{\gamma}(t), X(t))\dot{\gamma}(t) \rangle \leq -\varepsilon t^2,$$

for t sufficiently small, where the second inequality follows from (4.82). Hence, by (4.81), we have

$$\frac{d^2}{dt^2} |X| = \frac{|X|^2 |\nabla X|^2 - \langle X, \nabla X \rangle^2}{|X|^3} + \frac{\langle X, R(\dot{\gamma}, X)\dot{\gamma} \rangle}{|X|} \leq \frac{ct^3}{\delta^3} - \frac{\varepsilon t}{\delta}.$$

Integrating this inequality over an interval $[0, t]$ with $ct^2 < \varepsilon\delta^2$ we get

$$\frac{d}{dt} |X(t)| < \left. \frac{d}{dt} \right|_{t=0} |X(t)| = |\nabla X(0)|$$

Integrating this inequality again gives

$$|X(t)| < t \, |\nabla X(0)|$$

for small t, and hence

$$\left| d\exp_p(tv)t\hat{v} \right| = |X(t)| < t \, |\nabla X(0)| = t \, |\hat{v}| \, .$$

This contradicts (ii).

We prove that (ii) implies that the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. By (ii) $\exp_p$ is a *local diffeomorphism*, i.e. its derivative $d\exp_p(v) : T_pM \to T_{\exp_p(v)}M$ is bijective for every $v \in T_pM$. Hence we can define a metric on $M' := T_pM$ by pulling back the metric on M under the exponential map. To make this more explicit we choose a basis $e_1, \dots, e_m$ of T_pM and define the map $\psi : \mathbb{R}^m \to M$ by

$$\psi(x) := \exp_p\left(\sum_{i=1}^m x^i e_i\right)$$

for $x = (x^1, \dots, x^m) \in \mathbb{R}^m$. Define the metric tensor by

$$g_{ij}(x) := \left\langle \frac{\partial\psi}{\partial x^i}(x), \frac{\partial\psi}{\partial x^j}(x)\right\rangle, \qquad i,j = 1,\dots,m.$$

Then $(\mathbb{R}^m, g)$ is a Riemannian manifold (covered by a single coordinate chart) and $\psi : (\mathbb{R}^m, g) \to M$ is a local isometry, by definition of g. The manifold $(\mathbb{R}^m, g)$ is clearly connected and simply connected. Moreover, for every tangent vector $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^m = T_0\mathbb{R}^m$, the curve $\mathbb{R} \to \mathbb{R}^m : t \mapsto t\xi$ is a geodesic with respect to g (because ψ is a local isometry and the image of the curve under ψ is a geodesic in M). Hence it follows from Theorem 2.57 that $(\mathbb{R}^m, g)$ is complete. Since both $(\mathbb{R}^m, g)$ and M are connected, simply connected, and complete, the local isometry ψ is bijective, by Lemma 4.113. Thus the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism as claimed. It follows that any two points in M are connected by a unique geodesic.

We prove that (ii) implies (iii). Fix a point $p \in M$ and two tangent vectors $v_0, v_1 \in T_pM$. Let $\gamma : [0,1] \to M$ be the geodesic with endpoints $\gamma(0) = \exp_p(v_0)$ and $\gamma(1) = \exp_p(v_1)$ and let $v : [0,1] \to T_pM$ be the unique curve satisfying $\exp_p(v(t)) = \gamma(t)$ for all t. Then $v(0) = v_0$, $v(1) = v_1$, and

$$\begin{aligned}
d(\exp_p(v_0), \exp_p(v_1)) &= L(\gamma) \\
&= \int_0^1 \left|d\exp_p(v(t))\dot v(t)\right| \, dt \\
&\geq \int_0^1 |\dot v(t)| \, dt \\
&\geq \left|\int_0^1 \dot v(t)\, dt\right| \\
&= |v_1 - v_0| .
\end{aligned}$$

Here the third inequality follows from (ii). This shows that (ii) implies (iii).

We prove that (iii) implies (ii). Fix a point $p \in M$ and a tangent vector $v \in T_pM$ and denote

$$q := \exp_p(v).$$

By (iii) the exponential map $\exp_q : T_qM \to M$ is injective and, since M is complete, it is bijective (see Theorem 2.58). Hence there is a unique geodesic from q to any other point in M and therefore, by Theorem 2.44, we have

$$|w| = d(q, \exp_q(w)) \tag{4.83}$$

for every $w \in T_qM$. Now define

$$\phi := \exp_q^{-1} \circ \exp_p : T_pM \to T_qM.$$

This map satisfies

$$\phi(v) = 0.$$

Moreover, it is differentiable in a neighborhood of v and, by the chain rule, we have

$$d\phi(v) = d\exp_p(v) : T_pM \to T_qM.$$

Now choose $w := \phi(v + \hat{v})$ in (4.83) with $\hat{v} \in T_pM$. Then

$$\exp_q(w) = \exp_q(\phi(v + \hat{v})) = \exp_p(v + \hat{v})$$

and hence

$$|\phi(v + \hat{v}))| = d(\exp_p(v), \exp_p(v + \hat{v})) \geq |\hat{v}|,$$

where the last inequality follows from (iii). This gives

$$\begin{aligned} \left|d\exp_p(v)\hat{v}\right| &= |d\phi(v)\hat{v}| \\ &= \lim_{t\to 0} \frac{|\phi(v + t\hat{v})|}{t} \\ &\geq \lim_{t\to 0} \frac{|t\hat{v}|}{t} \\ &= |\hat{v}|. \end{aligned}$$

Thus we have proved that (iii) implies (ii). This completes the proof of the theorem. □

4.8.2 Positive definite symmetric matrices

We close this manuscript with an example of a nonpositive sectional curvature manifold which plays a key role in Donaldson's beautiful paper on Lie algebra theory [3]. Let n be a positive integer and consider the space

$$\mathscr{P} := \left\{ P \in \mathbb{R}^{n\times n} \,|\, P^T = P > 0 \right\}$$

of positive definite symmetric $n\times n$-matrices. (The notation "$P > 0$" means $\langle x, Px\rangle > 0$ for every nonzero vector $x \in \mathbb{R}^n$.) Thus $\mathscr{P}$ is an open subset of the vector space

$$\mathscr{S} := \left\{ S \in \mathbb{R}^{n\times n} \,|\, S^T = S \right\}$$

of symmetric matrices and hence the tangent space of $\mathscr{P}$ is $T_P\mathscr{P} = \mathscr{S}$ for every $P \in \mathscr{P}$. However, we do not use the metric inherited from the inclusion into $\mathscr{S}$ but define a Riemannian metric by

$$\langle S_1, S_2\rangle_P := \mathrm{trace}\left(S_1P^{-1}S_2P^{-1}\right) \qquad (4.84)$$

for $P \in \mathscr{P}$ and $S_1, S_2 \in \mathscr{S} = T_P\mathscr{P}$.

Theorem 4.117. *The space $\mathscr{P}$ with the Riemannian metric* (4.84) *is a connected, simply connected, complete Riemannian manifold with nonpositive sectional curvature. Moreover, $\mathscr{P}$ is a symmetric space and the group* $\mathrm{GL}(n, \mathbb{R})$ *of nonsingular $n \times n$-matrices acts on $\mathscr{P}$ by isometries via*

$$g_*P := gPg^T \qquad (4.85)$$

for $g \in \mathrm{GL}(n, \mathbb{R})$ and $P \in \mathscr{P}$.

Remark 4.118. The paper [3] by Donaldson contains an elementary direct proof that the manifold $\mathscr{P}$ with the metric (4.84) satisfies the assertions of Theorem 4.112.

Remark 4.119. The submanifold

$$\mathscr{P}_0 := \{P \in \mathscr{P} \,|\, \det(P) = 1\}$$

of positive definite symmetric matrices with determinant one is totally geodesic (see Remark 4.120 below). Hence all the assertions of Theorem 4.117 (with $\mathrm{GL}(n, \mathbb{R})$ replaced by $\mathrm{SL}(n, \mathbb{R})$) remain valid for $\mathscr{P}_0$.

Remark 4.120. Let M be a Riemannian manifold and $L \subset M$ be a submanifold. Then the following are equivalent.
(i) If $\gamma : I \to M$ is a geodesic on an open interval I such that $0 \in I$ and

$$\gamma(0) \in L, \qquad \dot\gamma(0) \in T_{\gamma(0)}L,$$

then there is a constant $\varepsilon > 0$ such that $\gamma(t) \in L$ for $|t| < \varepsilon$.
(ii) If $\gamma : I \to L$ is a smooth curve on an open interval I and Φ_γ denotes parallel transport along γ in M then

$$\Phi_\gamma(t,s)T_{\gamma(s)}L = T_{\gamma(t)}L \qquad \forall\, s,t \in I.$$

(iii) If $\gamma : I \to L$ is a smooth curve on an open interval I and $X \in \mathrm{Vect}(\gamma)$ is a vector field along γ (with values in TM) then

$$X(t) \in T_{\gamma(t)}L \quad \forall\, t \in I \qquad \Longrightarrow \qquad \nabla X(t) \in T_{\gamma(t)}L \quad \forall\, t \in I.$$

A submanifold that satisfies these equivalent conditions is called **totally geodesic**

Exercise 4.121. Prove the equivalence of (i), (ii), and (iii) in Remark 4.120. **Hint:** Choose suitable coordinates and translate each of the three assertions into conditions on the Christoffel symbols.

Exercise 4.122. Prove that $\mathscr{P}_0$ is a totally geodesic submanifold of $\mathscr{P}$. Prove that, in the case $n = 2$, $\mathscr{P}_0$ is isometric to the hyperbolic space $\mathbb{H}^2$.

Proof of Theorem 4.117. The manifold $\mathscr{P}$ is obviously connected and simply connected as it is a convex open subset of a finite dimensional vector space. The remaining assertions will be proved in five steps.

Step 1. *Let $I \to \mathscr{P} : t \mapsto P(t)$ be a smoth path in $\mathscr{P}$ and $I \to \mathscr{S} : t \mapsto S(t)$ be a vector field along P. Then the covariant derivative of S is given by*

$$\nabla S = \dot S - \frac{1}{2}SP^{-1}\dot P - \frac{1}{2}\dot P P^{-1}S. \tag{4.86}$$

The formula (4.86) determines a family of linear operators on the spaces of vector fields along paths that satisfy the torsion free condition

$$\nabla_s \partial_t P = \nabla_t \partial_s P$$

for every smooth map $\mathbb{R}^2 \to \mathscr{P} : (s,t) \mapsto P(s,t)$ and the Leibnitz rule

$$\nabla \langle S_1, S_2 \rangle_P = \langle \nabla S_1, S_2 \rangle_P + \langle S_1, \nabla S_2 \rangle_P$$

for any two vector fields S_1 and S_2 along P. These two conditions determine the covariant derivative uniquely (see Theorem 2.27 and Remark 2.72).

Step 2. *The geodesics in $\mathscr{P}$ are given by*

$$\begin{aligned}\gamma(t) &= P\exp(tP^{-1}S) \\ &= \exp(tSP^{-1})P \\ &= P^{1/2}\exp(tP^{-1/2}SP^{-1/2})P^{1/2}\end{aligned} \tag{4.87}$$

for $P \in \mathscr{P}$, $S \in \mathscr{S} = T_P\mathscr{P}$, and $t \in \mathbb{R}$. In particular $\mathscr{P}$ is complete.

The curve $\gamma : \mathbb{R} \to \mathscr{P}$ defined by (4.87) satisfies

$$\dot\gamma(t) = S\exp(tP^{-1}S) = SP^{-1}\gamma(t).$$

Hence it follows from Step 1 that

$$\nabla\dot\gamma(t) = \ddot\gamma(t) - \dot\gamma(t)\gamma(t)^{-1}\dot\gamma(t) = \ddot\gamma(t) - SP^{-1}\dot\gamma(t) = 0$$

for every $t \in \mathbb{R}$. Hence γ is a geodesic. Note also that the curve $\gamma : \mathbb{R} \to \mathscr{P}$ in (4.87) satisfies $\gamma(0) = P$ and $\dot\gamma(0) = S$.

Step 3. *The curvature tensor on $\mathscr{P}$ is given by*

$$\begin{aligned}R_P(S,T)A = &-\frac{1}{4}SP^{-1}TP^{-1}A - \frac{1}{4}AP^{-1}TP^{-1}S \\ &+\frac{1}{4}TP^{-1}SP^{-1}A + \frac{1}{4}AP^{-1}SP^{-1}T\end{aligned} \tag{4.88}$$

for $P \in \mathscr{P}$ and $S, T, A \in \mathscr{S}$.

Choose smooth maps $P : \mathbb{R}^2 \to \mathscr{P}$ and $A : \mathbb{R}^2 \to \mathscr{S}$ (understood as a vector field along P) and denote $S := \partial_s P$ and $T := \partial_t P$. Then

$$R(S,T)A = \nabla_s\nabla_t A - \nabla_t\nabla_s A$$

and $\partial_s T = \partial_t S$. By Step 1 we have

$$\begin{aligned}\nabla_s A &= \partial_s A - \frac{1}{2}AP^{-1}S - \frac{1}{2}SP^{-1}A, \\ \nabla_t A &= \partial_t A - \frac{1}{2}AP^{-1}T - \frac{1}{2}TP^{-1}A,\end{aligned}$$

and hence

$$\begin{aligned}R(S,T)A \;=\; &\partial_s\nabla_t A - \frac{1}{2}(\nabla_t A)P^{-1}S - \frac{1}{2}SP^{-1}(\nabla_t A) \\ &-\partial_t\nabla_s A + \frac{1}{2}(\nabla_s A)P^{-1}T + \frac{1}{2}TP^{-1}(\nabla_s A).\end{aligned}$$

Now Step 3 follows by a direct calculation which we leave to the reader.

Step 4. *The manifold $\mathscr{P}$ has nonpositive sectional curvature.*

By Step 3 with $A = T$ and equation (4.84) we have

$$\begin{aligned}
\langle S, R_P(S,T)T\rangle_P &= \mathrm{trace}\left(SP^{-1}R_P(S,T)TP^{-1}\right) \\
&= -\frac{1}{4}\mathrm{trace}\left(SP^{-1}SP^{-1}TP^{-1}TP^{-1}\right) \\
&\quad -\frac{1}{4}\mathrm{trace}\left(SP^{-1}TP^{-1}TP^{-1}SP^{-1}\right) \\
&\quad +\frac{1}{4}\mathrm{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&\quad +\frac{1}{4}\mathrm{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&= -\frac{1}{2}\mathrm{trace}\left(SP^{-1}TP^{-1}TP^{-1}SP^{-1}\right) \\
&\quad +\frac{1}{2}\mathrm{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&= -\frac{1}{2}\mathrm{trace}\left(X^TX\right) + \frac{1}{2}\mathrm{trace}\left(X^2\right),
\end{aligned}$$

where $X := SP^{-1}TP^{-1}$. Write $X =: (x_{ij})_{i,j=1,\dots,n}$. Then, by the Cauchy–Schwarz inequality, we have

$$\mathrm{trace}(X^2) = \sum_{i,j} x_{ij}x_{ji} \le \sum_{i,j} x_{ij}^2 = \mathrm{trace}(X^TX)$$

for every matrix $X \in \mathbb{R}^{n\times n}$. Hence $\langle S, R_P(S,T)T\rangle_P \le 0$ for all $P \in \mathscr{P}$ and all $S, T \in \mathscr{S}$. This proves Step 4.

Step 5. *$\mathscr{P}$ is a symmetric space.*

Given $A \in \mathscr{P}$ define the map $\phi : \mathscr{P} \to \mathscr{P}$ by

$$\phi(P) := AP^{-1}A.$$

This map is a diffeomorphism, fixed the matrix $A = \phi(A)$, and satisfies

$$d\phi(P)S = -AP^{-1}SP^{-1}A$$

for $P \in \mathscr{P}$ and $S \in \mathscr{S}$. Hence $d\phi(A) = -\mathrm{id}$ and, for all $P \in \mathscr{P}$ and all $S \in \mathscr{S}$, we have

$$(d\phi(P)S)\phi(P)^{-1} = -AP^{-1}SA^{-1}$$

and therefore

$$|d\phi(P)S|^2_{\phi(P)} = \mathrm{trace}\left(\left(AP^{-1}SA^{-1}\right)^2\right) = \mathrm{trace}\left(\left(P^{-1}S\right)^2\right) = |S|^2_P\,.$$

Hence ϕ is an isometry and this proves the theorem. □

Remark 4.123. The space $\mathscr{P}$ can be identified with the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ via polar decomposition.

Remark 4.124. Theorem 4.117 carries over verbatim to the complex setting. Just replace $\mathscr{P}$ by the space $\mathscr{H}$ of positive definite Hermitian matrices

$$H = H^* > 0,$$

where H^* denotes the conjugate transposed matrix of $H \in \mathbb{C}^{n\times n}$. The inner product is then defined by the same formula as in the real case, namely

$$\left\langle \hat{H}_1, \hat{H}_2 \right\rangle_H := \operatorname{trace}\left(\hat{H}_1 H^{-1} \hat{H}_2 H^{-1}\right)$$

for $H \in \mathscr{H}$ and two Hermitian matrices $\hat{H}_1, \hat{H}_2 \in T_H\mathscr{H}$. The assertions of Theorem 4.117 remain valid with $\mathrm{GL}(n,\mathbb{R})$ replaced by $\mathrm{GL}(n,\mathbb{C})$. This space $\mathscr{H}$ can be identified with the quotient $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ and, likewise, the subspace $\mathscr{H}_0$ of positive definite Hermitian matrices with determinant one can be identified with the quotient $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$. This quotient (with nonpositive sectional curvature) can be viewed as a kind of dual of the Lie group $\mathrm{SU}(n)$ (with nonnegative sectional curvature). **Exercise:** Prove this! Show that, in the case $n = 2$, the space $\mathscr{H}_0$ is isometric to hyperbolic 3-space.

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