ERRATA FOR *INTRODUCTION TO SYMPLECTIC TOPOLOGY*

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Abstract. These notes correct a few typos and errors in *Introduction to Symplectic Topology* (2nd edition, OUP 1998, reprinted 2005). We also include some additional clarifying material. We thank everyone who pointed out errors.

p 45/46: Proposition 2.22 should read:

The unitary group $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible. Moreover, every compact subgroup* $\mathrm{G} \subset \mathrm{Sp}(2n)$ *is conjugate to a subgroup of* $\mathrm{U}(n)$.

The proof contains a mistake on page 46: the matrix P obtained by averaging the matrices $\Psi^T\Psi$ over all $\Psi \in \mathrm{G}$ using the Haar measure need not be symplectic. To obtain a symplectic positive definite symmetric matrix that defines a G-invariant inner product on $\mathbb{R}^{2n}$ one can use the following argument.

Let G be a compact subgroup of $\mathrm{Sp}(2n)$ and $S = S^T \in \mathbb{R}^{2n\times 2n}$ be a positive definite symmetric matrix satisfying

$$\Psi^T S \Psi = S \qquad \forall\ \Psi \in \mathrm{G}.$$

Such a matrix can be obtained by averaging the matrices $\Psi^T\Psi$ over $\Psi \in \mathrm{G}$ using the Haar measure $C(\mathrm{G}, \mathbb{R}) \to \mathbb{R}$ for a compact Lie group. Define

$$A := S^{-1} J_0.$$

This matrix is nonsingular and S-skew-adjoint (i.e. it is skew-adjoint with respect to the inner product determined by S). Moreover, A commutes with each element of G. Hence there is a unique S-self-adjoint positive definite square root Q of $-A^2$:

$$Q^T S = SQ > 0, \qquad Q^2 = -A^2.$$

This matrix also commutes with each element of G as well as with A. (To see this use the S-orthogonal eigenspace decomposition of A.) Now define

$$P := -J_0 Q^{-1} A = (J_0)^T (SQ)^{-1} J_0.$$

This matrix is symmetric and positive definite. Moreover it is symplectic, i.e.

$$P^T J_0 P = P J_0 P = -J_0 Q^{-1} A Q^{-1} A = J_0,$$

and, since $Q^{-1}A$ commutes with every element of G, it satisfies

$$\Psi^T P \Psi = P \qquad \forall\ \Psi \in G.$$

Hence $P^{1/2}\Psi P^{-1/2}$ is an orthogonal symplectic matrix for every $\Psi \in \mathrm{G}$ and so $P^{1/2}\mathrm{G}P^{-1/2}$ is a subgroup of $\mathrm{U}(n)$.

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p 48: In Theorem 2.29 the Maslov index μ is best thought of as a function not a functor.

p 52/53: In Theorem 2.35 the last sentence in the product axiom reads: "In particular, a constant loop $\Lambda(t) \equiv \Lambda_0$ has Maslov index zero." However, this assertion does not follow from the product axiom and therefore this sentence should be deleted. Instead this property should be added as the zero axiom:

(zero) A constant loop $\Lambda(T) \equiv \Lambda_0$ has Maslov index zero.

This axiom should be inserted after the direct sum axiom and before the normalization axiom. The beginning of Theorem 2.35 should now be rephrased as follows.

There is a function μ, called the Maslov index, which assigns an integer $\mu(\Lambda)$ to every loop $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$ of Lagrangian subspaces and satisfies the following axioms:

After the formulation of the axioms the theorem should end with the following sentence:

The Maslov index is uniquely determined by the homotopy, product, direct sum, and zero axioms.

In the proof of Theorem 2.35 on page 53, line 17, the condition $U_j(1) = \pm\mathbb{1}$ should be replaced by the displayed formula

$$U_j(1) = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The last paragraph of the proof should be replaced by the following two paragraphs.

Thus we have proved that our Maslov index, as defined above, satisfies the homotopy axiom. That it also satisfies the product, direct sum, zero, and normalization axioms is obvious. This proves the existence statement of the theorem.

We prove that the normalization axiom follows from the direct sum, product, and zero axioms. The loop $\Lambda(t) := e^{\pi i t}\mathbb{R} \subset \mathbb{C}$ satisfies

$$\Lambda(t) \oplus \Lambda(t) = \Psi(t)\Lambda_0,$$

where $\Lambda_0 = \mathbb{R}^2 \subset \mathbb{C}^2$ is the constant loop and $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{U}(2)$ is the loop of unitary matrices given by

$$\Psi(t) := e^{\pi i t} \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}, \qquad 0 \le t \le 1.$$

The loop Ψ has Maslov index $\mu(\Psi) = 1$. Hence it follows from the direct sum, product, and zero axioms that

$$2\mu(\Lambda) = \mu(\Lambda \oplus \Lambda) = \mu(\Lambda_0) + 2\mu(\Psi) = 2.$$

Hence $\mu(\Lambda) = 1$ as claimed. The proof that all five axioms determine the Maslov index uniquely is left to the reader.

p 63-65: In the book we prove that, given a symplectic form ω on a manifold M, the space of ω-compatible, respectively ω-tame, almost complex structures on M is contractible. In fact, more is true: the space of nondegenerate 2-forms is homotopy equivalent to the space of almost complex structures. To see this one can fix a Riemannian metric on M and first show that the inclusion of the space of skew-adjoint almost complex structures on M into the space of all almost complex structures is a homotopy equivalence. Second, one can show that the obvious map from the space of skew-adjoint almost complex structures to the space of nondegenerate 2-forms on M is a homotopy equivalence. The proof requires the following strengthening of the analogous result for vector spaces: Proposition 2.50 on page 63.

Let V be an even dimensional vector space, denote by $\mathfrak{Met}(V)$ the space of inner products $g : V \times V \to \mathbb{R}$, by $\mathfrak{Symp}(V)$ the space of nondegenerate skew-symmetric bilinear forms $\omega : V \times V \to \mathbb{R}$, and by $\mathcal{J}(V)$ the space of automorphisms $J : V \to V$ that satisfy $J^2 = -1\!\!1$. Given $\omega \in \mathfrak{Symp}(V)$ and $g \in \mathfrak{Met}(V)$ we denote by

$$\begin{aligned} \mathcal{J}(V, \omega) &:= \{J \in \mathcal{J}(V) \,|\, J^*\omega = \omega \text{ and } \omega(v, Jv) > 0 \text{ for } 0 \neq v \in V\}, \\ \mathcal{J}(V, g) &:= \{J \in \mathcal{J}(V) \,|\, J^*g = g\} \end{aligned}$$

the space of complex structures on V that are compatible with ω, respectively g.

Proposition 2.50. **(i)** *For every $\omega \in \mathfrak{Symp}(V)$ the space $\mathcal{J}(V, \omega)$ is homeomorphic to the space $\mathcal{P}$ of positive definite symplectic matrices.*

(ii) *For every $g \in \mathfrak{Met}(V)$ the inclusion $\mathcal{J}(V, g) \hookrightarrow \mathcal{J}(V)$ is a homotopy equivalence.*

(iii) *For every $g \in \mathfrak{Met}(V)$ the map $\mathcal{J}(V, g) \to \mathfrak{Symp}(V) : J \mapsto g(J\cdot, \cdot)$ is a homotopy equivalence.*

(iv) *For every $\omega \in \mathfrak{Symp}(V)$ the map $\mathcal{J}(V, \omega) \to \mathfrak{Met}(V) : J \mapsto \omega(\cdot, J\cdot)$ is a homotopy equivalence.*

(v) *There is a $\mathrm{GL}(V)$-equivariant smooth map*

$$\mathfrak{Met}(V) \times \mathfrak{Symp}(V) \to \mathcal{J}(V) : (g, \omega) \mapsto J_{g,\omega} \tag{1}$$

satisfying the following conditions. First, $J_{g,\omega}$ is compatible with both g and ω. Second, for all $g \in \mathfrak{Met}(V)$, $\omega \in \mathfrak{Symp}(V)$, and $J \in \mathcal{J}(V)$, we have

$$g = \omega(\cdot, J\cdot) \qquad \Longrightarrow \qquad J_{g,\omega} = J.$$

Third, for each $g \in \mathfrak{Met}(V)$ the map

$$\mathfrak{Symp}(V) \to \mathcal{J}(V, g) : \omega \mapsto J_{g,\omega}$$

is a homotopy inverse of the map in (iii). Fourth, for each $\omega \in \mathfrak{Symp}(V)$ the map

$$\mathfrak{Met}(V) \to \mathcal{J}(V, \omega) : g \mapsto J_{g,\omega}$$

is a homotopy inverse of the map in (iv).

Corollary. *$\mathcal{J}(V, \omega)$ is contractible and, for every $g \in \mathfrak{Met}(V)$, the map*

$$\mathfrak{Symp}(V) \to \mathcal{J}(V) : \omega \mapsto J_{g,\omega}$$

is a homotopy equivalence.

Proof. The first assertion follows from Proposition 2.50 (i) and Lemma 2.21. It also follows from Proposition 2.50 (iv). The second assertion follows from (ii) and (iii) in Proposition 2.50. □

Proof of Proposition 2.50. We prove (i). By Theorem 2.3 we may assume $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$. A matrix $J \in \mathbb{R}^{2n\times 2n}$ is an ω_0-compatible complex structure if and only if

$$J^2 = -1\!\!1, \qquad J^T J_0 J = J_0, \qquad \langle v, -J_0 J v\rangle > 0 \;\; \forall\, v \neq 0.$$

The first two identities imply that

$$(J_0 J)^T = -J^T J_0 = J^T J_0 J^2 = J_0 J.$$

Hence $P := -J_0 J$ is symmetric, positive definite, and symplectic. Conversely, if P has these properties then $J = J_0 P \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$. This proves (i).

We prove (ii). It suffices to assume $V = \mathbb{R}^{2n}$ with the standard inner product g_0. Denote by $\mathrm{GL}^+(2n, \mathbb{R})$ the group of $2n \times 2n$ matrices with positive determinant and consider the composition

$$\mathcal{J}(\mathbb{R}^{2n}, g_0) \cong \frac{\mathrm{SO}(2n)}{\mathrm{U}(n)} \hookrightarrow \frac{\mathrm{GL}^+(2n,\mathbb{R})}{\mathrm{U}(n)} \to \frac{\mathrm{GL}^+(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})} \cong \mathcal{J}(\mathbb{R}^{2n}).$$

Here the last homeomorphism is induced by the transitive action of $\mathrm{GL}^+(2n,\mathbb{R})$ on $\mathcal{J}(\mathbb{R}^{2n})$ with stabilizer $\mathrm{GL}(n,\mathbb{C})$ at J_0. Similarly, the first homeomorphism is induced by the transitive action of $\mathrm{SO}(2n)$ on $\mathcal{J}(\mathbb{R}^{2n}, g_0)$ with stabilizer $\mathrm{U}(n)$ at J_0. The inclusion $\mathrm{SO}(2n)/\mathrm{U}(n) \hookrightarrow \mathrm{GL}^+(2n,\mathbb{R})/\mathrm{U}(n)$ is a homotopy equivalence whose homotopy inverse is induced by polar decomposition:

$$\mathrm{GL}^+(2n,\mathbb{R}) \to \mathrm{SO}(2n) : \Psi \mapsto \Psi(\Psi^T\Psi)^{-1/2}.$$

The projection $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{U}(n) \to \mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ is a fibration with contractible fibers $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ (diffeomorphic to the space of positive definite Hermitian matrices). Thus all four maps are homotopy equivalences and this proves (ii).

We construct the map (1) in (v). Given $g \in \mathfrak{Met}(V)$ and $\omega \in \mathfrak{Symp}(V)$ define the automorphism $A : V \to V$ by

$$\omega(v, w) = g(Av, w).$$

The identity $\omega(v,w) = -\omega(w,v)$ is equivalent to $g(Av,w) = -g(v,Aw)$. Therefore A is g-skew-adjoint. Hence, writing A^* for the g-adjoint of A, we find that the automorphism $P := A^*A = -A^2$ is g-positive definite. Hence there is a unique automorphism $Q : V \to V$ which is g-self-adjoint, g-positive definite, and satisfies

$$Q^2 = P = -A^2.$$

(To see this, use the fact that P can be represented as a diagonal matrix with positive diagonal entries in a suitable g-orthonormal basis of V. Details are given in Exercise 2.52 below.) Then Q commutes with A and it follows easily that

$$J_{g,\omega} := Q^{-1}A$$

is a complex structure compatible with ω and g. Conversely, if g and ω are related by $g = \omega(\cdot, J\cdot)$ for some $J \in \mathcal{J}(V)$ then $A = J$ and $Q = 1\!\!1$ and hence $J_{g,\omega} = J$. Now fix an element $\Phi \in \mathrm{GL}(V)$. If g is replaced by $\Phi^*g = g(\Phi\cdot, \Phi\cdot)$ and ω is replaced by $\Phi^*\omega$, then the automorphism A in the above construction is replaced by $\Phi^{-1}A\Phi$ and Q by $\Phi^{-1}Q\Phi$. Hence

$$J_{\Phi^*g,\Phi^*\omega} = \Phi^{-1}J_{g,\omega}\Phi.$$

This shows that the map (1) is $\mathrm{GL}(V)$-equivariant. Smoothness of the map (1) is proved in Exercise 2.52.

We prove (iii). Fix an inner product $g \in \mathfrak{Met}(V)$. Then the composition of the map

$$\mathcal{J}(V, g) \to \mathrm{Symp}(V) : J \mapsto \omega_J := g(J\cdot, \cdot)$$

with the map $\mathfrak{Symp}(V) \to \mathcal{J}(V, g) : \omega \mapsto J_{g,\omega}$ is the identity. The converse composition is the map

$$\mathfrak{Symp}(V) \to \mathfrak{Symp}(V) : \omega \mapsto \omega_{J_{g,\omega}} = g\left(\left(-A_{g,\omega}^2\right)^{-1/2} A_{g,\omega}\cdot, \cdot\right),$$

where $A_{g,\omega} : V \to V$ is the automorphism associated to g and ω via $\omega = g(A_{g,\omega}\cdot, \cdot)$. This map is homotopic to the identity via

$$(t, \omega) \mapsto g\left(\left(-A_{g,\omega}^2\right)^{-t/2} A_{g,\omega}\cdot, \cdot\right), \qquad 0 \le t \le 1.$$

This proves (iii).

We prove (iv). Fix a nondegenerate 2-form $\omega \in \mathfrak{Symp}(V)$ and denote by $g_J := \omega(\cdot, J\cdot) \in \mathfrak{Met}(V)$ the inner product associated to an ω-compatible complex structure $J \in \mathcal{J}(M, \omega)$. Then $J_{g_J,\omega} = J$ for every $J \in \mathcal{J}(V, \omega)$ and so the composition of the maps $J \mapsto g_J$ and $g \mapsto J_{g,\omega}$ is the identity. The converse composition $\mathfrak{Met}(V) \to \mathfrak{Met}(V) : g \mapsto g_{J_{g,\omega}}$ is homotopic to the identity because the set $\mathfrak{Met}(V)$ is convex. This proves the proposition. $\square$

p 67: Rephrase part (i) in Exercise 2.52 as follows.

(i) Prove that the map $(g, \omega) \mapsto J_{g,\omega}$ in Proposition 2.50 is smooth. Assume $V = \mathbb{R}^{2n}$, let $\omega \in \mathfrak{Symp}(\mathbb{R}^{2n})$ be represented by a skew-symmetric matrix

$$B = -B^T \in \mathbb{R}^{2n\times 2n}, \qquad \omega(v, w) = (Bv)^T w,$$

and let $g \in \mathfrak{Met}(\mathbb{R}^{2n})$ be represented by a positive definite symmetric matrix

$$S = S^T \in \mathbb{R}^{2n\times 2n}, \qquad g(v, w) = v^T S w$$

The formula $\omega(v, w) = g(Av, w)$ determines the matrix $A = S^{-1}B$. Prove that the g-adjoint of A is represented by the matrix $A^* = S^{-1}A^T S = -A$. Prove that the g-square root Q of the matrix

$$P := A^*A = -A^2 = S^{-1}B^T S^{-1}B$$

is given by

$$Q = S^{-1/2}\left(S^{-1/2}B^T S^{-1} B S^{-1/2}\right)^{1/2} S^{1/2}.$$

Deduce that the map $(S, B) \mapsto J = Q^{-1}S^{-1}B$ is smooth.

p 70: The following assertion can be added to Proposition 2.63:

(iii) *Fix an inner product g on E. Then the map $\omega \mapsto J_{g,\omega}$ in Proposition 2.50 induces a homotopy equivalence from the space of symplectic bilinear forms on E to the space of complex structures on E.*

This follows immediately from Proposition 2.50 as formulated above.

p 118: The following assertion can be added to Proposition 4.1.

(iii) *Fix a Riemannian metric g on M. Then the map $\omega \mapsto J_{g,\omega}$ in Proposition 2.50 induces a homotopy equivalence from the space of nondegenerate 2-forms on M to the space of almost complex structures on M.*

p 209: The proof of Theorem 6.17 can be simplified as follows.

Proof of Theorem 6.17. Denote by $\mathfrak{g} \to \mathrm{Vect}(F) : \xi \mapsto X_\xi$ the infinitesimal action of the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and let $\mu : F \to \mathfrak{g}^*$ be a moment map for the action. Choose a connection 1-form $A \in \mathcal{A}(P)$ and define $\pi_A : TP \times TF \to TF$ by

$$\pi_A(v, \widehat{x}) := \widehat{x} - X_{A_p(v)}(x)$$

for $v \in T_pP$ and $\widehat{x} \in T_xF$. Then the 2-form $\tau_A \in \Omega^2(P \times F)$, defined by

$$\tau_A := \sigma - d\langle \mu, A \rangle = \pi_A^* \sigma - \langle \mu, F_A \rangle, \tag{2}$$

is closed, G-invariant, and horizontal. (Here we slightly abuse notation and identify objects on P and F with their pullbacks to the product $P \times F$.) Hence it descends to a closed 2-form on $M = P \times_{\mathrm{G}} F$. The explicit formula

$$\tau_A\left((v_1, \widehat{x}_1), (v_2, \widehat{x}_2)\right) = \sigma\left(\widehat{x}_1 - X_{A_p(v_1)}(x), \widehat{x}_2 - X_{A_p(v_2)}(x)\right) - \langle \mu(x), F_A(v_1, v_2) \rangle$$

for $v_1, v_2 \in T_pP$ and $\widehat{x}_1, \widehat{x}_2 \in T_xF$ shows that it restricts to the symplectic form on each fiber. This proves the theorem. □

Exercise. Prove the second equation in (2).

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