The Maslov index for paths

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Maslov's famous index for a loop of Lagrangian subspaces was interpreted by Arnold [1] as an intersection number with an algebraic variety known as the Maslov cycle. Arnold's general position arguments apply equally well to the case of a path of Lagrangian subspaces whose endpoints lie in the complement of the Maslov cycle. Our aim in this paper is to define a Maslov index for any path regardless of where its endpoints lie. Our index is invariant under homotopy with fixed endpoints and is additive for catenations. Duistermaat [4] has proposed a Maslov index for paths which is not additive for catenations but is independent of the choice of the Lagrangian subspace used to define the Maslov cycle. By contrast our Maslov index depends on this choice.

We have been motivated by two applications in [10] and [12] as well as the index introduced by Conley and Zehnder in [2] and [3]. In [12] we show how to define a signature for a certain class of one dimensional first order differential operators whose index and coindex are infinite. In [10] we relate the Maslov index to Cauchy Riemann operators such as those that arise in

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Floer theory. Our index formula in [10] generalizes the one in [5] and the one in [13].

We use our Maslov index for paths of Lagrangian subspaces to define a Maslov index for paths of symplectic matrices. We characterize this latter index axiomatically. This leads us to a stratification of the symplectic group $\mathrm{Sp}(2n)$ where the connected strata are characterized by pairs (k, ν) with $k = 0, 1, \ldots, n$ and $\nu \in \mathbb{Z}_2$. Our Maslov index is invariant under homotopies where the endpoints are allowed to vary in a stratum. Our index μ satisfies the identity

$$\mu + \frac{k_a - k_b}{2} \equiv \nu_a - \nu_b \pmod{2}$$

where (k_a, ν_a) and (k_b, ν_b) characterize the strata at the left and right endpoints, respectively. The number on the left is an integer.

Arnold defined the intersection number for a closed loop by transversality arguments. For paths with fixed endpoints this approach does not work when an endpoint lies in a stratum of the Maslov cycle of codimension bigger than 1. To surmount this difficulty we introduce the notion of simple and regular crossings of the Maslov cycle. A simple crossing is a transverse crossing in the usual sense. It can only occur in a codimension-1 stratum. A regular crossing can occur at any point of the Maslov cycle and it has a well defined crossing index (an integer of modulus less than or equal to n). Regular crossings are isolated. For a curve with only regular crossings our Maslov index is the sum of the crossing indices with the endpoints contributing half. A crossing in a codimension-1 stratum is simple if and only if it is regular and for such crossings the crossing index is ± 1 in agreement with Arnold's definition.

1 Generalities

The standard symplectic structure on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is defined by

$$\omega(z_1, z_2) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle$$

for $z_k = (x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^n$. A subspace Λ is called **Lagrangian** iff it has dimension n and $\omega(z_1, z_2) = 0$ for all $z_1, z_2 \in \Lambda$. A **Lagrangian frame** for a Lagrangian subspace Λ is an injective linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ whose image is Λ. Such a frame has the form

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{1}$$

where X, Y are $n \times n$-matrices and

$$Y^T X = X^T Y.$$

The last n (or first n) columns of a symplectic matrix Ψ form a Lagrangian frame.

A quadratic form on a vector space V can be viewed as a map from V to the dual space V^*. If $\mathbb{R}^{2n} = V \oplus W$ is a Lagrangian splitting then W can be identified with V^* via the symplectic form. In this situation every Lagrangian subspace transverse to W is the graph of a quadratic form $A : V \to V^* = W$. For each Lagrangian subspace $\Lambda \in \mathcal{L}(n)$ we will define a canonical isomorphism

$$T_\Lambda \mathcal{L}(n) \to S^2(\Lambda) : (\Lambda, \hat{\Lambda}) \mapsto Q = Q(\Lambda, \hat{\Lambda})$$

between the tangent space at Λ and the space of quadratic forms on Λ.

Theorem 1.1 *Let $\Lambda(t) \in \mathcal{L}(n)$ be a curve of Lagrangian subspaces with $\Lambda(0) = \Lambda$ and $\dot{\Lambda}(0) = \hat{\Lambda}$.*

(1) *Let W be a fixed Lagrangian complement of Λ and for $v \in \Lambda$ and small t define $w(t) \in W$ by $v + w(t) \in \Lambda(t)$. Then the form*

$$Q(v) = \left.\frac{d}{dt}\right|_{t=0} \omega(v, w(t))$$

is independent of the choice of W.

(2) *If $Z(t) = (X(t), Y(t))$ is a frame for $\Lambda(t)$ as in (1) then*

$$Q(v) = \langle X(0)u, \dot{Y}(0)u \rangle - \langle Y(0)u, \dot{X}(0)u \rangle$$

where $v = Z(0)u$.

(3) *The form Q is natural in the sense that*

$$Q(\Psi\Lambda, \Psi\hat{\Lambda}) \circ \Psi = Q(\Lambda, \hat{\Lambda})$$

for a symplectic matrix Ψ.

Proof: Choose co-ordinates so that $\Lambda(0) = \mathbb{R}^n \times 0$. Then any Lagrangian complement of $\Lambda(0)$ is the graph of a symmetric matrix $B \in \mathbb{R}^{n\times n}$:

$$W = \{(By, y) \, : \, y \in \mathbb{R}^n\}$$

and for small t the Lagrangian subspace $\Lambda(t)$ is the graph of a symmetric matrix $A(t) \in \mathbb{R}^{n\times n}$:

$$\Lambda(t) = \{(x, A(t)) \, : \, x \in \mathbb{R}^n\}.$$

Hence $v = (x, 0)$, $w(t) = (By(t), y(t))$, and $y = A(t)(x + By(t))$. Hence $\omega(v, w(t)) = \langle x, y(t)\rangle$ and

$$Q(v) = \langle x, \dot{y}(0)\rangle = \langle x, \dot{A}(0)x\rangle.$$

The result is independent of B and this proves (1).

To prove (2) assume that $W = 0 \times \mathbb{R}^n$ is the vertical and choose a frame $Z(t) = (X(t), Y(t))$ for $\Lambda(t)$. Then $v = (X(0)u, Y(0)u)$ and $w(t) = (0, y(t))$ where $Y(0)u + y(t) = Y(t)X(t)^{-1}X(0)u$. Hence $\omega(v, w(t)) = \langle X(0)u, y(t)\rangle$ and

$$\begin{aligned} Q(v) &= \langle X(0)u, \dot{y}(0)\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle X(0)u, Y(0)X(0)^{-1}\dot{X}(0)u\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle \dot{X}(0)u, Y(0)u\rangle \end{aligned}$$

The last equation follows from the identity $X^TY = Y^TX$. This proves (2). Statement (3) is an obvious consequence of the definition. □

Remark 1.2 The previous work can be defined in the language of differential geometry as follows. Denote by $\mathcal{F}(n)$ the manifold of Lagrangian frames. Then we have a principal bundle

$$\mathrm{GL}(n; \mathbb{R}) \hookrightarrow \mathcal{F}(n) \to \mathcal{L}(n).$$

For $\Lambda \in \mathcal{L}(n)$ denote by $\mathcal{F}_\Lambda$ the fibre of $\mathcal{F}(n)$ over Λ. It is the set of Lagrangian frames for Λ. The tangent space to $\mathcal{F}(n)$ at $Z = (X, Y)$ is the space of all pairs $\zeta = (\xi, \eta)$ of $n \times n$-matrices such that the matrix

$$S(X, Y, \xi, \eta) = \xi^T Y - \eta^T X$$

is symmetric. The tangent space to the fibre over Λ is the subspace determined by $S(Z, \zeta) = 0$. The tangent space to $\mathcal{L}(n)$ at Λ can be identified with the quotient

$$T_\Lambda \mathcal{L}(n) = T_Z \mathcal{F}(n)/T_Z \mathcal{F}_\Lambda.$$

By Theorem 1.1 (2) the matrix $S(Z, \dot{Z})$ determines the quadratic form Q.

Remark 1.3 A **unitary Lagrangian frame** is one whose columns are orthonormal in $\mathbb{R}^{2n}$. The space of unitary frames is naturally diffeomorphic to the unitary group $\mathrm{U}(n)$ via $Z \mapsto X + iY$. Hence there is a principal bundle

$$\mathrm{O}(n) \hookrightarrow \mathrm{U}(n) \to \mathcal{L}(n).$$

Remark 1.4 Another principal bundle is

$$\mathrm{St}(2n) \hookrightarrow \mathrm{Sp}(2n) \to \mathcal{L}(n).$$

Here $\mathrm{Sp}(2n)$ denotes the symplectic group and $\mathrm{St}(2n)$ denotes the **stabilizer subgroup** of all symplectic matrices Ψ such that $\Psi(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$. The symplectic matrices have the block form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \Psi^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \tag{2}$$

and the stabilizer subgroup is defined by $B = 0$. In this bundle the projection $\mathrm{Sp}(2n) \to \mathcal{L}(n)$ sends Ψ to $\Psi(0 \times \mathbb{R}^n)$.

Remark 1.5 The graph

$$\mathrm{Gr}(G) = \{(x, Gx) \,:\, x \in \mathbb{R}^n\}$$

of a matrix $G \in \mathbb{R}^{n \times n}$ is Lagrangian iff G is symmetric. If Ψ is a symplectic matrix with $B = C = 0$ then

$$\Psi \mathrm{Gr}(G) = \mathrm{Gr}(DGD^T).$$

If Λ is a Lagrangian subspace transverse to the vertical $0 \times \mathbb{R}^n$ with frame $Z = (X, Y)$ then X is invertible and Λ is the graph of the symmetric matrix

$$G = YX^{-1}.$$

2 Lagrangian paths

Every Lagrangian subspace V determines a decomposition of the space of Lagrangian subspaces as a disjoint union

$$\mathcal{L}(n) = \bigcup_{k=0}^{n} \Sigma_k(V)$$

where $\Sigma_k(V)$ is the submanifold of those Lagrangian subspaces which intersect V in a subspace of dimension k. The codimension of $\Sigma_k(V)$ is $k(k+1)/2$. We will prove in Theorem 4.2 below that each stratum $\Sigma_k(V)$ is connected. The **Maslov cycle** determined by V is the algebraic variety

$$\Sigma(V) = \overline{\Sigma_1(V)} = \bigcup_{k=1}^{n} \Sigma_k(V).$$

The tangent space to $\Sigma_k(V)$ at a point $\Lambda \in \Sigma_k(V)$ is given by

$$T_\Lambda \Sigma_k(V) = \left\{ \hat{\Lambda} \in T_\Lambda \mathcal{L}(n) : Q(\Lambda, \hat{\Lambda})|_{\Lambda \cap V} = 0 \right\}.$$

Let $\Lambda : [a, b] \to \mathcal{L}(n)$ be smooth curve of Lagrangian subspaces. A **crossing** for Λ is a number $t \in [a, b]$ for which $\Lambda(t)$ intersects V nontrivially, i.e. for which $\Lambda(t) \in \Sigma(V)$. The set of crossings is compact. At each crossing time $t \in [a, b]$ we define the **crossing form**

$$\Gamma(\Lambda, V, t) = Q(\Lambda(t), \dot{\Lambda}(t))|_{\Lambda(t) \cap V}.$$

By Theorem 1.1 (3), the crossing form is natural in the sense that

$$\Gamma(\Psi\Lambda, \Psi V, t) \circ \Psi = \Gamma(\Lambda, V, t) \tag{3}$$

for every symplectic matrix Ψ. If $V = \mathbb{R}^n \times 0$ and $\Lambda(t) = \mathrm{Gr}(A(t))$ for a path of symmetric matrices $A(t)$ then at each crossing

$$\Gamma(\Lambda, V, t)(v) = \langle x, \dot{A}(t)x \rangle$$

for $v = (x, 0)$ with $x \in \ker A(t)$.

A curve $\Lambda : [a, b] \to \mathcal{L}(n)$ is tangent to $\Sigma_k(V)$ at a crossing t if and only if $\Lambda(t) \in \Sigma_k(V)$ and the crossing form $\Gamma(\Lambda, V, t) = 0$. A crossing t is called

regular if the crossing form $\Gamma(\Lambda, V, t)$ is nonsingular. It is called **simple** if it is regular and in addition $\Lambda(t) \in \Sigma_1(V)$. A curve has only simple crossings if and only if it is transverse to every $\Sigma_k(V)$. Intuitively, a curve has only regular crossings if and only if it is transverse to the algebraic variety $\Sigma(V)$. For a curve $\Lambda : [a, b] \to \mathcal{L}(n)$ with only regular crossings we define the **Maslov index**

$$\mu(\Lambda, V) = \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, V, a) + \sum_{a<t<b} \mathrm{sign}\,\Gamma(\Lambda, V, t) + \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, V, b)$$

where the summation runs over all crossings t. (It is easy to show that regular crossings are isolated. See Theorem 2.3 below.)

Lemma 2.1 *Suppose* $\Lambda_0, \Lambda_1 : [a, b] \to \mathcal{L}(n)$ *with* $\Lambda_0(a) = \Lambda_1(a)$ *and* $\Lambda_0(b) = \Lambda_1(b)$ *have only regular crossings. If* Λ_0 *and* Λ_1 *are homotopic with fixed endpoints then they have the same Maslov index.*

Lemma 2.2 *Every Lagrangian path* $\Lambda : [a, b] \to \mathcal{L}(n)$ *is homotopic with fixed endpoints to one having only regular crossings.*

These lemmata are proved below. Together they enable us to define the Maslov index for every continuous path. The Maslov index then has the following properties.

Theorem 2.3 **(Naturality)** *For* $\Psi \in \mathrm{Sp}(2n)$

$$\mu(\Psi\Lambda, \Psi V) = \mu(\Lambda, V).$$

(Catenation) *For* $a < c < b$

$$\mu(\Lambda, V) = \mu(\Lambda|_{[a,c]}, V) + \mu(\Lambda|_{[c,b]}, V).$$

(Product) *If* $n' + n'' = n$ *identify* $\mathcal{L}(n') \times \mathcal{L}(n'')$ *as a submanifold of* $\mathcal{L}(n)$ *in the obvious way. Then*

$$\mu(\Lambda' \oplus \Lambda'', V' \oplus V'') = \mu(\Lambda', V') + \mu(\Lambda'', V'').$$

(Localization) *If $V = \mathbb{R}^n \times 0$ and $\Lambda(t) = \mathrm{Gr}(A(t))$ then the Maslov index of Λ is given by the spectral flow*[1]

$$\mu(\Lambda, V) = \tfrac{1}{2}\mathrm{sign} A(b) - \tfrac{1}{2}\mathrm{sign} A(a). \tag{4}$$

(Homotopy) *Two paths $\Lambda_0, \Lambda_1 : [a, b] \to \mathcal{L}(n)$ with $\Lambda_0(a) = \Lambda_1(a)$ and $\Lambda_0(b) = \Lambda_1(b)$ are homotopic with fixed endpoints if and only if they have the same Maslov index.*

(Zero) *Every path $\Lambda : [a, b] \to \Sigma_k(V)$ has Maslov index $\mu(\Lambda, V) = 0$.*

Proof: We prove this theorem and the previous lemmata in eight steps.

Step 1: *Suppose $A(t) \in \mathbb{R}^{n\times n}$ is a smooth path of symmetric matrices with* $\operatorname{rank} A(0) = n - k$. *Then there exists a matrix $\hat{\alpha} \in \mathbb{R}^{k\times k}$, and smooth curves $D(t) \in \mathrm{GL}(n)$ and $\gamma(t) \in \mathrm{GL}(n-k)$ such that*

$$D(t)A(t)D(t)^T = \begin{pmatrix} \hat{\alpha}t + O(t^2) & 0 \\ 0 & \gamma(t) \end{pmatrix}.$$

for t near 0.

Without loss of generality assume that A has the block form

$$A(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t)^T & \gamma(t) \end{pmatrix}$$

where α and β vanish at 0 and γ is invertible. Now factor A as

$$A(t) = \begin{pmatrix} \mathbb{1} & \beta\gamma^{-1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \alpha - \beta\gamma^{-1}\beta^T & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \gamma^{-1}\beta(t)^T & \mathbb{1} \end{pmatrix}$$

Now take $D(t)$ to be the inverse of the first matrix on the right and $\hat{\alpha} = \dot{\alpha}(0)$.

Step 2: *Every Lagrangian path Λ may be approximated by a path Λ' with the same endpoints having regular crossings at the ends and simple crossings in the interior.*

By naturality of the crossing operator (3) we may assume that $V = \mathbb{R}^n \times 0$ and $\Lambda(a)$ and $\Lambda(b)$ are graphs. Now use step 1 at to obtain regular crossings at

[1] A special case of this formula appears in Duistermaat [4] Lemma 2.5.

the endpoints by a small perturbation. Then $\Lambda(t) \in \Sigma_0(V)$ for $a < t \leq a + \varepsilon$ and $b - \varepsilon \leq t < b$. Now use transversality.

Step 3: *Assume $V = \mathbb{R}^n \times 0$ and $\Lambda(t)$ is the graph of a symmetric matrix $A(t) \in \mathbb{R}^{n \times n}$ with a regular crossing at $t = t_0$. Then*

$$\operatorname{sign} A(t_0 \pm \varepsilon) = \operatorname{sign} A(t_0) \pm \operatorname{sign} \Gamma(\Lambda, V, t_0).$$

Hence the localization axiom holds for paths with regular crossings.

This follows from Kato's selection theorem [7] about the eigenvalues of a smooth one parameter family of symmetric matrices. Alternatively, note that the crossing operator $\Gamma(\Lambda, V, t_0)$ is represented by the symmetric matrix $\hat{\alpha}$ of step 1. To prove the localization axiom in the case of regular crossings note that both sides of (4) satisfy the catenation axiom.

Step 4: *If the path Λ in step 2 has only regular crossings then it has the same Maslov index as Λ'.*

The crossing index at the ends is unchanged. By naturality of the crossing operator (3) and step 3 we have

$$\operatorname{sign} \Gamma(\Lambda, V, t_0) = \sum_{|t - t_0| < \varepsilon} \operatorname{sign} \Gamma(\Lambda', V, t).$$

at each interior crossing $t_0 \in (a, b)$.

Step 5: *Two paths which have only simple crossings, are homotopic with fixed endpoints, and have endpoints in $\Sigma_0(V)$ have the same Maslov index.*

For $k \geq 2$ the codimension of $\Sigma_k(V)$ is bigger than 2 and hence a generic homotopy misses $\Sigma_k(V)$. By the arguments of [9] the intersection number is well-defined and depends only on the homotopy class.

Step 6: *We prove Lemma 2.1*

By naturality of the crossing operator (3) we may assume that $V = \mathbb{R}^n \times 0$ and $\Lambda_0(a) = \Lambda_1(a)$ and $\Lambda_0(b) = \Lambda_1(b)$ are graphs. By step 4 we may assume without loss of generality that Λ_0 and Λ_1 have only simple crossings in the interior. Choose $\varepsilon > 0$ so small that $\Lambda_j(t) \in \Sigma_0(V)$ and $\Lambda_s(t)$ is a graph for $a < t \leq a + \varepsilon$ and $b - \varepsilon \leq t < b$. Now perturb the homotopy if necessary such that the paths $s \mapsto \Lambda_s(a + \varepsilon)$ and $s \mapsto \Lambda_s(b - \varepsilon)$ have only simple crossings. By the localization axiom (step 3) these paths have Maslov index

$$\mu(s \mapsto \Lambda_s(a + \varepsilon)) = \tfrac{1}{2}\operatorname{sign} \Gamma(\Lambda_1, V, a) - \tfrac{1}{2}\operatorname{sign} \Gamma(\Lambda_0, V, a)$$

$$\mu(s \mapsto \Lambda_s(b-\varepsilon)) = \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda_0, V, b) - \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda_1, V, b).$$

Introduce the intermediate path Λ' which is the catenation of the paths $\Lambda_0|_{[a,a+\varepsilon]}$, $\Lambda_s(a+\varepsilon)$, $\Lambda_1|_{[a+\varepsilon,b-\varepsilon]}$, $\Lambda_{1-s}(b-\varepsilon)$, and $\Lambda_0|_{[b-\varepsilon,b]}$. By Step 5 all three paths Λ_0, Λ', and Λ_1 have the same Maslov index.

We have now proved Lemma 2.1 and Lemma 2.2. Hence the Maslov index is now well defined for all paths.

Step 7: *Two paths Λ_0 and Λ_1 with the same endpoints and the same Maslov index are homotopic.*

By step 6 we may assume without loss of generality that $\Lambda_0(t) = \Lambda_1(t)$ for $a \le t \le a+\varepsilon$ and $b-\varepsilon \le t \le b$. By step 4 we may assume that both paths have only simple crossings in the interior. Hence it suffices to prove that every loop $\gamma : S^1 \to \mathcal{L}(n)$ of Lagrangian subspaces which has only simple crossings and has Maslov index zero is contractible. Since the Maslov index is zero there is at least one pair of adjacent crossings of opposite sign. Since $\Sigma_1(V)$ is connected[2] homotop to a curve where these crossings occur at the same point of $\Sigma_1(V)$. Since $\Sigma_0(V)$ is contractible remove this pair of crossings.

We have now proved the homotopy axiom. The localization axiom was proved in step 3 for paths with only regular crossings. Hence it holds always. The naturality axiom, the catenation axiom, and the product axiom are all obvious for curves with regular crossings and hence hold generally. The final step is

Step 8: *The zero axiom holds.*

For any smooth path $\Lambda : [a,b] \to \Sigma_k(V)$ define the cone

$$K(t) = \left\{ \hat{\Lambda} \in T_{\Lambda(t)}\mathcal{L}(n) \,:\, Q(\Lambda(t), \hat{\Lambda})(v) > 0 \text{ for } v \in \Lambda(t) \cap V \right\}.$$

Choose a smooth section $\hat{\Lambda}(t) \in K(t)$ and choose a smooth deformation $[0,1] \times [a,b] \to \mathcal{L}(n) : (s,t) \mapsto \Lambda_s(t)$ such that

$$\Lambda_0 = \Lambda, \qquad \left.\frac{\partial \Lambda_s}{\partial s}\right|_{s=0} = \hat{\Lambda}.$$

[2]This is proved in Corollary 4.4 below.

Then $\Lambda_s(t) \in \Sigma_0(V)$ for $0 < s \leq \varepsilon$ with $\varepsilon > 0$ sufficiently small. The path Λ is homotopic to the catenation of the three paths $s \mapsto \Lambda_s(a)$, $t \mapsto \Lambda_\varepsilon(t)$, and $s \mapsto \Lambda_{\varepsilon-s}(b)$. The first has Maslov index k, the second has Maslov index 0 and the last has Maslov index $-k$. □

A Lagrangian homotopy $\Lambda_s : [a,b] \to \mathcal{L}(n)$ is called a **stratum homotopy** with respect to V if $\Lambda_s(a)$ and $\Lambda_s(b)$ each remain in the same stratum. This means that there are integers $k_a = k_a(\Lambda_s, V)$ and $k_b = k_b(\Lambda_s, V)$ such that

$$\Lambda_s(a) \in \Sigma_{k_a}(V), \qquad \Lambda_s(b) \in \Sigma_{k_b}(V)$$

for all s. The following theorem is a corollary of Theorem 4.7 below.

Theorem 2.4 *Two Lagrangian paths are stratum homotopic with respect to V if and only if they have the same invariants μ, k_a, k_b. These invariants are related by*

$$\mu + \frac{k_a - k_b}{2} \in \mathbb{Z}.$$

In particular, the Maslov index of a loop is an integer.

Remark 2.5 In applications it is often required to compute the the crossing form explicitly in terms of a Lagrangian frame. If $V = \mathbb{R}^n \times 0$ and $Z(t) = (X(t), Y(t))$ is a frame for $\Lambda(t)$ then at each crossing

$$\Gamma(\Lambda, V, t)(v) = \langle X(t)u, \dot{Y}(t)u \rangle$$

for $v = (X(t)u, 0)$ with $Y(t)u = 0$. If $V = 0 \times \mathbb{R}^n$ then

$$\Gamma(\Lambda, V, t)(v) = -\langle Y(t)u, \dot{X}(t)u \rangle$$

for $v = (0, Y(t)u)$ with $X(t)u = 0$. (See Theorem 1.1 (2).) This corresponds to the geometric picture for $n = 1$. If $\Lambda(t)$ crosses the x-axis and rotates towards the y-achsis then the rotation is counter-clockwise so the Maslov index is positive. Conversely, if $\Lambda(t)$ crosses the y-axis and rotates towards the x-achsis then the rotation is clockwise so the Maslov index is negative.

Remark 2.6 Let $\Lambda(t) = \Lambda(t+1)$ be a loop of Lagrangian subspaces. Choose a lift $Z(t) = (X(t), Y(t))$ of unitary frames. Then for any Lagrangian subspace V

$$\mu(\Lambda, V) = \frac{\alpha(1) - \alpha(0)}{\pi}, \qquad \det(X(t) + iY(t)) = e^{i\alpha(t)}.$$

To see this note that both sides are homotopy invariants, are additive for direct sums, and agree in the case $n = 1$. In the case $n = 1$ a unitary frame has the form $X(t) = \cos\alpha(t)$ and $Y(t) = \sin\alpha(t)$. A crossing occurs where $\alpha(t_0) \in \pi\mathbb{Z}$ and the crossing index is the sign of $\dot{\alpha}(t_0)$. (See previous remark.)

3 Lagrangian pairs

Now consider a pair of curves $\Lambda, \Lambda' : [a, b] \to \mathcal{L}(n)$. Define the **relative crossing form** $\Gamma(\Lambda, \Lambda', t)$ on $\Lambda(t) \cap \Lambda'(t)$ by

$$\Gamma(\Lambda, \Lambda', t) = \Gamma(\Lambda, \Lambda'(t), t) - \Gamma(\Lambda', \Lambda(t), t).$$

and call the crossing regular if this fom is nondegenerate. For a pair with only regular crossings define the **relative Maslov index** by

$$\mu(\Lambda, \Lambda') = \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, \Lambda', a) + \sum_{a<t<b} \mathrm{sign}\,\Gamma(\Lambda, \Lambda', t) + \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, \Lambda', b).$$

In the case $\Lambda'(t) \equiv V$ this agrees with the previous definition. Theorem 3.2 below can be used to extend the definition to continuous pairs. The relative Maslov index for pairs of Lagrangian paths was used by Viterbo [15] in the case of transverse endpoints.

Theorem 3.1 *The Maslov index is natural in the sense that*

$$\mu(\Psi\Lambda, \Psi\Lambda') = \mu(\Lambda, \Lambda')$$

for a path of symplectic matrices $\Psi : [a, b] \to \mathrm{Sp}(2n)$.

Proof: The crossing form of the pair $(\Psi\Lambda, \Psi\Lambda')$ at is given by

$$\Gamma(\Psi\Lambda, \Psi\Lambda', t) \circ \Psi(t) = \Gamma(\Lambda, \Lambda', t). \tag{5}$$

To prove this note that the forms $Q(\Psi\Lambda, \dot{\Psi}\Lambda, t)$ and $Q(\Psi\Lambda', \dot{\Psi}\Lambda', t)$ agree on the intersection $\Psi(t)(\Lambda(t) \cap \Lambda'(t))$. The equation (5) shows that if the pair (Λ, Λ') has only regular crossings then so does the pair $(\Psi\Lambda, \Psi\Lambda')$ and in this case the Maslov indices agree. Hence they agree always. □

Theorem 3.2 *Consider the symplectic vector space* $(\mathbb{R}^{2n}\times\mathbb{R}^{2n},(-\omega)\times\omega)$. *Then*[3]

$$\mu(\Psi\Lambda,\Lambda')=\mu(\mathrm{Gr}(\Psi),\Lambda\times\Lambda').$$

In particular, when $\Psi(t)\equiv\mathbb{1}$ *we have*

$$\mu(\Lambda,\Lambda')=\mu(\Delta,\Lambda\times\Lambda') \tag{6}$$

where $\Delta\subset\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ *is the diagonal.*

Proof: We first prove (6). Denote $\bar{\omega}=(-\omega)\times\omega$ and $\overline{\Lambda}(t)=\Lambda(t)\times\Lambda'(t)$. Then $\Delta\cap\overline{\Lambda}(t)$ is the set of pairs $\bar{v}=(v,v)$ with $v\in\Lambda(t)\cap\Lambda'(t)$. With this notation we shall prove that the crossing form is given by

$$\Gamma(\Delta,\Lambda\times\Lambda',t)(\bar{v})=\Gamma(\Lambda,\Lambda'(t),t)(v)-\Gamma(\Lambda',\Lambda(t),t)(v). \tag{7}$$

To see this choose a Lagrangian subspace $\overline{W}=W\times W'$ such that W is transverse to $\Lambda(t)$ and W' is transverse to $\Lambda'(t)$. Given $v\in\mathbb{R}^n$ and s near t choose $w(s)\in W$ and $w'(s)$ in W' such that

$$v+w(s)\in\Lambda(s),\qquad v+w'(s)\in\Lambda'(s).$$

Then $\bar{v}\in\Delta$, $\bar{w}(s)=(w(s),w'(s))\in\overline{W}$, $\bar{v}+\bar{w}(s)\in\overline{\Lambda}(s)$, and

$$-\bar{\omega}(\bar{v},\bar{w}(s))=\omega(v,w(s))-\omega(v,w'(s)).$$

Differentiate this identity with respect to s at $s=t$ to obtain (7). This proves the theorem in the case $\Psi(t)\equiv\mathbb{1}$.

To prove the result in the general case define the symplectomorphism $\overline{\Psi}(t)$ of $(R^{2n}\times\mathbb{R}^{2n},\bar{\omega})$ by $\overline{\Psi}(t)(z,z')=(z,\Psi(t)z')$. Then $\overline{\Psi}\Delta=\mathrm{Gr}(\Psi)$ and $\overline{\Psi}(\Lambda\times\Psi^{-1}\Lambda')=\Lambda\times\Lambda'$. Hence

$$\begin{aligned}\mu(\mathrm{Gr}(\Psi),\Lambda\times\Lambda') &= \mu(\Delta,\Lambda\times\Psi^{-1}\Lambda')\\ &= \mu(\Lambda,\Psi^{-1}\Lambda')\\ &= \mu(\Psi\Lambda,\Lambda').\end{aligned}$$

The second equality follows from (6) and the last equality from Theorem 3.1.
□

[3]Below we define the Maslov index of a path of symplectic matrices as the special case $\Lambda(t)=\Lambda'(t)=0\times\mathbb{R}^n$.

Care must be taken when considering homotopies of Lagrangian pairs. For example if (Λ_0, Λ_0') and (Λ_1, Λ_1') are Lagrangian pairs with the same endpoints and the same Maslov index they need not be homotopic with fixed endpoints. Only $\Lambda_0 \oplus \Lambda_0'$ and $\Lambda_1 \oplus \Lambda_1'$ are homotopic with fixed endpoints as Lagrangian paths in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega) \times \omega)$. However we have the following

Corollary 3.3 *The number $\mu(\Lambda, \Lambda')$ is a homotopy invariant in the case*

$$\Lambda(a) \cap \Lambda'(a) = 0, \qquad \Lambda(b) \cap \Lambda'(b) = 0.$$

The homotopy (Λ_s, Λ_s'), $0 \le s \le 1$, is required to preserve the condition on the endpoints. Two pairs (Λ_0, Λ_0') and (Λ_1, Λ_1') are homotopic in this sense if and only if they have the same Maslov index.

Proof: If two pairs are homotopic then, by Theorem 3.2, they have the same Maslov index. Conversely, suppose that $\Lambda_j(a) \cap \Lambda_j'(a) = 0 = \Lambda_j(b) \cap \Lambda_j'(b)$ for $j = 0, 1$ and $\mu(\Lambda_0, \Lambda_0') = \mu(\Lambda_1, \Lambda_1')$. Choose any smooth map $[0, 1] \times [a, b] \to \mathrm{Sp}(2n) : (s, t) \mapsto \Psi_s(t)$ such that

$$\Psi_0(t)\Lambda_0'(t) = V, \qquad \Psi_1(t)\Lambda_1'(t) = V.$$

Then $\Psi_j(a)\Lambda_j(a)$ and $\Psi_j(b)\Lambda_j(b)$ are transverse to V for $j = 0, 1$ and, by Theorem 3.1,

$$\mu(\Psi_0\Lambda_0, V) = \mu(\Psi_1\Lambda_1, V).$$

By Theorem 2.4 the paths $\Psi_0\Lambda_0$ and and $\Psi_1\Lambda_1$ are stratum homotopic with respect to V. Hence there exists a smooth homotopy $[0, 1] \times [a, b] \to \mathcal{L}(n) : (s, t) \mapsto \Lambda_s(t)$ from Λ_0 to Λ_1 such that

$$\Psi_s(a)\Lambda_s(a) \cap V = 0, \qquad \Psi_s(b)\Lambda_s(b) \cap V = 0$$

for all s. The required homotopy is the pair (Λ_s, Λ_s') where $\Lambda_s'(t) = \Psi_s(t)^{-1}V$.
□

Remark 3.4 The Maslov index for pairs gives rise to an alternative proof of the zero axiom for the Maslov index of paths. By the naturality Theorem 3.1 the axiom can be reduced to the case of a constant path: choose a path of symplectic matrices $\Psi(t)$ such that both $\Psi(t)V$ and $\Psi(t)\Lambda(t)$ are constant.

Theorem 3.5 *Let V_0, V_1, Λ_0, Λ_1 be any four Lagrangian subspaces.*

(1) *The half integer*

$$s(V_0, V_1; \Lambda_0, \Lambda_1) = \mu(\Lambda, V_1) - \mu(\Lambda, V_0)$$

is independent of the choice of the path $\Lambda : [a, b] \to \mathcal{L}(n)$ joining $\Lambda_0 = \Lambda(a)$ with $\Lambda_1 = \Lambda(b)$. It is called the **Hörmander index** *[6], [4].*

(2) $s(\Lambda_0, \Lambda_1; V_0, V_1) = -s(V_0, V_1; \Lambda_0, \Lambda_1)$.

(3) *If $V_j = \mathrm{Gr}(A_j)$ and $\Lambda_j = \mathrm{Gr}(B_j)$ for symmetric matrices A_j and B_j then*

$$\begin{aligned} s(V_0, V_1; \Lambda_0, \Lambda_1) &= \tfrac{1}{2}\mathrm{sign}\,(B_1 - A_1) - \tfrac{1}{2}\mathrm{sign}\,(B_0 - A_1) \\ &\quad -\tfrac{1}{2}\mathrm{sign}\,(B_1 - A_0) + \tfrac{1}{2}\mathrm{sign}\,(B_0 - A_0). \end{aligned}$$

Proof: Choose a path of symplectic matrices $\Psi : [0, 1] \to \mathrm{Sp}(2n)$ such that $\Psi(0) = \mathbb{1}$ and $\Psi(1)V_0 = V_1$. Consider the smooth map

$$[0, 1] \times [a, b] \to \mathcal{L}(n) : (s, t) \mapsto \Lambda(s, t) = \Psi(s)^{-1}\Lambda(t).$$

The Maslov index around the boundary (relative to the Lagrangian subspace V_0) is zero. Hence

$$\mu(\Lambda_1, \Psi V_0) - \mu(\Lambda_0, \Psi V_0) = \mu(\Lambda, V_1) - \mu(\Lambda, V_0).$$

The number on the left depends only on the endpoints of the path Λ while the number on the right is independent of the choice of Ψ. This proves (1) and (2). To prove (3) choose $\Lambda(t) = \mathrm{Gr}(B(t))$ where $B(t) = (1-t)B_0 + tB_1$. □

4 Symplectic paths

For a path of symplectic matrices $\Psi : [a, b] \to \mathrm{Sp}(2n)$ and a Lagrangian subspace V define the **Maslov index**

$$\mu(\Psi) = \mu(\Psi V, V), \qquad V = 0 \times \mathbb{R}^n.$$

The stratum

$$\mathrm{Sp}_k(2n) = \{\Psi \in \mathrm{Sp}(2n) : \dim(\Psi V \cap V) = k\}$$

is the preimage of $\Sigma_k(V)$ under the fibration $\mathrm{Sp}(2n) \to \mathcal{L}(n) : \Psi \mapsto \Psi V$ of Remark 1.4. Thus $\Psi \in \mathrm{Sp}_k(2n)$ iff $\operatorname{rank} B = n - k$ and $\Psi \in \mathrm{Sp}_0(2n)$ iff $\det(B) \neq 0$ in the block decomposition (2). The stratum $\mathrm{Sp}_k(2n)$ is a submanifold of $\mathrm{Sp}(2n)$ of codimension $k(k+1)/2$.

The Maslov index $\mu(\Psi)$ can be viewed as the intersection number of the path Ψ with the **Maslov cycle**

$$\overline{\mathrm{Sp}}_1(2n) = \mathrm{Sp}(2n) \setminus \mathrm{Sp}_0(2n) = \bigcup_{k=1}^{n} \mathrm{Sp}_k(2n).$$

The Maslov index is a half-integer and is an integer if the endpoints $\Psi(t_0)$ and $\Psi(t_1)$ lie in $\mathrm{Sp}_0(2n)$. The set $\mathrm{Sp}_0(2n)$ consists of those symplectic matrices Ψ which admit a generating function $S = S(x_0, x_1)$ in the sense that

$$(x_1, y_1) = \Psi(x_0, y_0) \quad \Longleftrightarrow \quad y_0 = -\frac{\partial S}{\partial x_0}, \;\; y_1 = \frac{\partial S}{\partial x_1}.$$

When we speak of crossings, regular crossings, etc for a curve of symplectic matrices Ψ we mean the corresponding concept for the curve of Lagrangian subspaces ΨV relative to the Maslov cycle $\Sigma(V)$. Since the last n columns of Ψ form a Lagrangian frame for ΨV the crossing form $\Gamma(\Psi, t) : \ker B(t) \to \mathbb{R}$ is given by

$$\Gamma(\Psi, t)(y) = -\langle D(t)y, \dot{B}(t)y\rangle$$

where $B(t)$ and $D(t)$ are as in (2).

Theorem 4.1 *The Maslov index is characterized by the following axioms.*

(Homotopy) *Two paths which begin at Ψ_0 and end at Ψ_1 are homotopic with end points fixed if and only if they have the same Maslov index.*

(Zero) *For each k every path in $\mathrm{Sp}_k(2n)$ has Maslov index zero.*

(Catenation) *If $\Psi : [a, b] \to \mathrm{Sp}(2n)$ and $a < c < b$ then*

$$\mu(\Psi) = \mu(\Psi|_{[a,c]}) + \mu(\Psi|_{[c,b]}).$$

(Product) *If* $n'+n''=n$ *identify* $\mathrm{Sp}(2n')\times\mathrm{Sp}(2n'')$ *as a subgroup of* $\mathrm{Sp}(2n)$ *in the obvious way. Then*

$$\mu(\Psi'\oplus\Psi'')=\mu(\Psi')+\mu(\Psi'').$$

(Normalization) *For a* **symplectic shear**

$$\Psi(t)=\begin{pmatrix}\mathbb{1} & B(t)\\ 0 & \mathbb{1}\end{pmatrix}$$

on the interval $[a,b]$ *the Maslov index is given by*

$$\mu(\Psi)=\tfrac{1}{2}\mathrm{sign}\,B(a)-\tfrac{1}{2}\mathrm{sign}\,B(b).$$

Proof: The homotopy axiom follows from the homotopy lifting property for the fibration $\mathrm{St}(2n)\hookrightarrow\mathrm{Sp}(2n)\to\mathcal{L}(n)$ of Remark 1.4. The other axioms are obvious. The proof that the axioms characterize the Maslov index requires the following

Theorem 4.2 *Each stratum* $Sp_k(2n)$ *has two components.*

Proof: Denote by $\mathcal{F}_k(n)$ the space of Lagrangian frames $Z=(X,Y)$ such that $\mathrm{rank}\,X=n-k$. Then there is a fibration

$$F\to\mathrm{Sp}_k(2n)\to\mathcal{F}_k(n)$$

whose fiber F is the space of symmetric $n\times n$-matrices. We prove that the space $\mathcal{F}_k(n)$ has two components. To see this we construct a map

$$\nu:\mathcal{F}_k(n)\to\mathbb{Z}_2$$

called the **parity**. A Lagrangian frame $Z=(X,Y)\in\mathcal{F}_k(n)$ is said to be in **normal form** if

$$X=\begin{pmatrix}X_1 & 0\\ 0 & 0\end{pmatrix},\qquad Y=\begin{pmatrix}Y_1 & 0\\ Y_3 & Y_4\end{pmatrix},$$

where $X_1\in\mathbb{R}^{(n-k)\times(n-k)}$ is nonsingular. The condition that the frame be Lagrangian is that the matrix $X_1^TY_1\in\mathbb{R}^{(n-k)\times(n-k)}$ be symmetric. For any Lagrangian frame in normal form define the parity $\nu(Z)$ by

$$(-1)^{\nu(Z)}=\mathrm{sign}\,\det(X_1)\det(Y_4).$$

Two Lagrangian frames $Z = (X, Y) \in \mathcal{F}_k(n)$ and $Z' = (X', Y') \in \mathcal{F}_k(n)$ are called **equivalent** if there exist matrices $L, M \in \mathbb{R}^{n\times n}$ such that

$$X' = L^T XM, \qquad Y' = L^{-1} YM, \qquad \det(L) > 0, \qquad \det(M) > 0. \tag{8}$$

Every Lagrangian frame is equivalent to one in normal form: first choose L and M such that $X' = L^T XM$ is as required; then the condition on $Y' = L^{-1}YM$ is satisfied automatically. If Z and Z' are equivalent normal forms related by (8) then

$$M = \begin{pmatrix} M_1 & 0 \\ M_3 & M_4 \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 & L_2 \\ 0 & L_4 \end{pmatrix},$$

hence

$$X_1' = L_1^T X_1 M_1, \qquad Y_4' = L_4^{-1} Y_4 M_4,$$

and hence $\nu(Z) = \nu(Z')$. This shows that the map ν extends to a unique map $\mathcal{F}_k(n) \to \mathbb{Z}_2$ which is invariant under the above equivalence relation. Now any two equivalent Lagrangian frames lie in the same component of $\mathcal{F}_k(n)$. Moreover, for $k < n$ every normal form can be connected to one where $Y = \mathbb{1}$ and either $X_1 = \mathbb{1}$ or $X_1 = \mathrm{diag}(1, \ldots, 1, -1)$. If $k = n$ then $X = 0$ and $(-1)^{\nu(Z)} = \mathrm{sign} \det Y$. Hence $\nu(Z)$ characterizes the components of $\mathcal{F}_k(n)$. □

Remark 4.3 If $Z = (B, \mathbb{1})$ is a Lagrangian frame of a graph (of x as a function of y) then the parity $\nu(B, \mathbb{1})$ is the number of negative eigenvalues of the symmetric matrix B mod 2.

Corollary 4.4 *Each stratum $\Sigma_k(V)$ is connected.*

Proof: Each Lagrangian subspace Λ admits two Lagrangian frames of distinct parity. □

Remark 4.5 For $\Psi \in \mathrm{Sp}_k(2n)$ define

$$\nu(\Psi) = \nu(B, D)$$

where B and D are the matrices in the block decomposition (2). For a symplectic shear ($A = D = \mathbb{1}$ and $C = 0$) the parity $\nu(\Psi)$ is the index of B modulo 2. Notice that

$$\Psi \in \mathrm{Sp}_0(2n) \quad \Longrightarrow \quad (-1)^{\nu(\Psi)} = \mathrm{sign}\det B,$$

$$\Psi \in \mathrm{Sp}_n(2n) \quad \Longrightarrow \quad (-1)^{\nu(\Psi)} = \mathrm{sign}\det D.$$

Remark 4.6 The open stratum $\mathrm{Sp}_0(2n)$ is homotopy equivalent to $\mathrm{O}(n)$. This follows from the exact sequence of the fibration

$$\mathrm{St}(2n) \to \mathrm{Sp}_0(2n) \to \Sigma_0(V).$$

The base $\Sigma_0(V)$ is contractible and hence $\mathrm{Sp}_0(2n)$ is homotopy equivalent to $\mathrm{St}(2n)$. Now $\mathrm{St}(2n)$ deformation retracts onto $\mathrm{GL}(n)$ (the set of all matrices $\Psi \in \mathrm{Sp}(2n)$ with $B = C = 0$) and $\mathrm{GL}(n)$ deformation retracts onto $\mathrm{O}(n)$. The two components of $\mathrm{Sp}_0(2n)$ are distinguished by the sign of the determinant of B in the block decomposition (2).

A homotopy $\Psi_s : [a, b] \to \mathrm{Sp}(2n)$ of symplectic paths is called a **stratum homotopy** if the ranks of the matrices $B_s(a)$ and $B_s(b)$ in the block decomposition (2) are independent of s. Call a path of symplectic shears in $\mathrm{Sp}(2n)$ **neutral** if it has Maslov index zero, its endpoints are in the open stratum $\mathrm{Sp}_0(2n)$, and $n \in 4\mathbb{Z}$. Two paths Ψ_0 and Ψ_1 are called **stably stratum homotopic** if there exist two neutral paths Ψ_0' and Ψ_1' such that $\Psi_0 \oplus \Psi_0'$ and $\Psi_1 \oplus \Psi_1'$ are stratum homotopic. This is an equivalence relation since a product of neutral paths is neutral. For any symplectic path $\Psi : [a, b] \to \mathrm{Sp}(2n)$ define $k_a = k_a(\Psi)$ and $k_b = k_b(\Psi)$ by

$$\Psi(a) \in \mathrm{Sp}_{k_a}(2n), \qquad \Psi(b) \in \mathrm{Sp}_{k_b}(2n)$$

and denote

$$\nu_a = \nu_a(\Psi) = \nu(\Psi(a)), \qquad \nu_b = \nu_b(\Psi) = \nu(\Psi(b)).$$

These four numbers are invariant under stable stratum homotopy. By the homotopy, zero, and catenation axioms the Maslov index is invariant under stratum homotopy. By the product axiom it is invariant under stable stratum homotopy.

Theorem 4.7 *Two paths in* $\mathrm{Sp}(2n)$ *are stratum homotopic if and only they have the same invariants* μ, k_a, k_b, ν_a, ν_b. *These invariants are related by*

$$\mu + \frac{k_a - k_b}{2} \in \mathbb{Z} \tag{9}$$

and

$$\mu + \frac{k_a - k_b}{2} \equiv \nu_a - \nu_b \,(\mathrm{mod}\ 2). \tag{10}$$

Theorem 4.8 *Two symplectic paths* $\Psi_0 : [a,b] \to \mathrm{Sp}(2n_0)$ *and* $\Psi_1 : [a,b] \to \mathrm{Sp}(2n_1)$ *are stably stratum homotopic if and only if* $n_1 - n_0 \in 4\mathbb{Z}$ *and they have the same invariants* μ, k_a, k_b, ν_a, ν_b.

Theorem 4.9 *Every symplectic path is stably stratum homotopic to a symplectic shear.*

Proof: We prove these three theorems in six steps.

Step 1: *A symplectic shear satisfies (9), (10)*

Assume first that $\Psi(t)$ is a symplectic shear with right upper block $B(t)$. Let n_b^- be the number of negative eigenvalues of $B(b)$ and $n_b^+ = n - n_b^- - k_b$ be the number of positive eigenvalues. Similarly for $B(a)$. Hence $\nu_a \equiv n_a^- (\mathrm{mod}\ 2)$ and $\nu_b \equiv n_b^- (\mathrm{mod}\ 2)$. Hence

$$\begin{aligned} \mu &= \tfrac{1}{2}\mathrm{sign}\, B(a) - \tfrac{1}{2}\mathrm{sign}\, B(b) \\ &= \frac{n_a^+ - n_a^-}{2} - \frac{n_b^+ - n_b^-}{2} \\ &= \frac{n - 2n_a^- - k_a}{2} - \frac{n - 2n_b^- - k_b}{2} \\ &= \frac{k_b - k_a}{2} + n_b^- - n_a^- . \end{aligned}$$

Step 2: *If* $n, k_a, k_b \in \mathbb{Z}$, $\nu_a, \nu_b \in \mathbb{Z}_2$, *and* $\mu \in \mathbb{R}$ *satisfy (9), (10), and*

$$0 \le k_a, k_b \le n, \qquad |\mu| < n - 1 - \frac{k_a + k_b}{2} \tag{11}$$

then there exists a symplectic shear in $\mathrm{Sp}(2n)$ *with these given invariants.*

Take $B(a)$ and $B(b)$ to be diagonal matrices with diagonal entries 0, 1, -1 such that $B(a)$ has k_a zeros and n_a^- minus signs with $n_a^- \equiv \nu_a (\mathrm{mod}\ 2)$ and similarly for $B(b)$. Then the proof of step 1 shows that the path of symplectic shears has Maslov index $\mu = (k_b - k_a)/2 + n_b^- - n_a^-$. Every number μ which satisfies (9), (10), and (11) can be obtained this way.

Step 3: *Two paths in* $\mathrm{Sp}(2n)$ *are stratum homotopic if and only if they have the same invariants* μ, k_a, k_b, ν_a, ν_b.

Suppose two paths have the same invariants. By Theorem 4.2 we may assume they have the same endpoints. Now use the homotopy axiom.

Step 4: *We prove Theorem 4.8.*

The indices are obviously invariant under stable stratum homotopy. Conversely, assume that the paths $\Psi_0 : [a, b] \to \mathrm{Sp}(2n_0)$ and $\Psi_1 : [a, b] \to \mathrm{Sp}(2n_1)$ have the same invariants and $n_1 - n_0 \in 4\mathbb{Z}$. Choose neutral extensions Ψ_0' and Ψ_1' such that $n_0 + n_0' = n_1 + n_1'$. Then the paths $\Psi_0 \oplus \Psi_0'$ and $\Psi_1 \oplus \Psi_1'$ have the same invariants. By step 3 they are stratum homotopic.

Step 5: *Every symplectic path* Ψ *is stably stratum homotopic to a symplectic shear.*

Choose a neutral path $\Psi' : [a, b] \to \mathrm{Sp}(2n')$ with $n' > 2\mu$. Now use step 2 to find a symplectic shear in $\mathrm{Sp}(2n + 2n')$ with the same invariants as Ψ. By step 3 this path is stratum homotopic to $\Psi \oplus \Psi'$.

Step 6: *Every symplectic path satisfies (9), (10)*

Use step 5 and step 1. □

Proof of Theorem 4.1 continued: The axioms assert that every putative Maslov index is a stable stratum homotopy invariant. By Theorem 4.9 every symplectic path is stably stratum homotopic to a symplectic shear. Hence the normalization axiom shows that the putative Maslov index and the Maslov index agree. □

Proof of Theorem 2.4: By naturality, assume without loss of generality that $V = 0 \times \mathbb{R}^n$. Then any lift $\Psi : [a, b] \to \mathrm{Sp}(2n)$ of a Lagrangian path $\Lambda : [a, b] \to \mathcal{L}(n)$ under the projection $\Psi \mapsto \Lambda = \Psi V$ has the same invariants μ, k_a, k_b. By Theorem 4.7 they satisfy (9). The numbers k_a and k_b are obviously stratum homotopy invariants. The Maslov index μ is by the homotopy, zero, and catenation axioms. Now assume that the Lagrangian paths Λ_0 and Λ_1 have the same invariants μ, k_a, k_b. By Corollary 4.4 we may assume that they have the same endpoints. Now use the homotopy axiom. □

Remark 4.10 Equation (9) shows that the Maslov index of a symplectic loop is an even integer. This follows also from the exact sequence of Remark 1.3.

Corollary 4.11 (Homotopy with a free end) *Two curves Ψ_0 and Ψ_1 in $\mathrm{Sp}(2n)$ with $\Psi_j(a) = \mathbb{1}$ and $\Psi_j(b) \in \mathrm{Sp}_0(2n)$ are homotopic within this class if and only if they have the same Maslov index.*

Proof: $k_a = 0$, $\nu_a = 0$, $k_b = n$. Hence (9) shows that that $\nu_b(\Psi_0) = \nu_b(\Psi_1)$ whenever $\mu(\Psi_0) = \mu(\Psi_1)$. □

Corollary 4.12 *If $\Psi(a) = \mathbb{1}$ and $\Psi(b) \in \mathrm{Sp}_0(2n)$ then $\mu(\Psi) + n/2 \in \mathbb{Z}$ and*

$$(-1)^{\mu(\Psi)+n/2} = \mathrm{sign}\, B$$

where B is as in (2).

Proof: Theorem 4.7 and Remark 4.5. □

Let $\Lambda_0, \Lambda_1 : \mathbb{R} \to \mathcal{L}(n)$ be Lagrangian paths with transverse endpoints. Choose orientations of Λ_0 and Λ_1 and define the **relative intersection number** $\varepsilon(\Lambda_0, \Lambda_1) = 1$ if the orientations $\mathbb{R}^{2n}$ induced by the two splittings $\Lambda_0(a) \oplus \Lambda_1(a)$ and $\Lambda_0(b) \oplus \Lambda_1(b)$ agree; otherwise define $\varepsilon(\Lambda_0, \Lambda_1) = -1$.

Corollary 4.13 *For Λ_0 and Λ_1 as above*

$$(-1)^{\mu(\Lambda_0,\Lambda_1)} = \varepsilon(\Lambda_0, \Lambda_1).$$

Proof: Choose a symplectic path $\Psi : [a, b] \to \mathrm{Sp}(2n)$ such that $\Psi(t)\Lambda_0(t) = \Lambda_1(t)$. By Theorem 4.7

$$\begin{aligned}
(-1)^{\mu(\Lambda_0,\Lambda_1)} &= \varepsilon(\mu(\Psi)) \\
&= (-1)^{\nu(\Psi(b))-\nu(\Psi(a))} \\
&= \mathrm{sign}\det\, B(a) \cdot \mathrm{sign}\det\, B(b) \\
&= \varepsilon(\Lambda_0, \Lambda_1).
\end{aligned}$$

Here $B(t)$ is as in (2) and the last but one equality follows from Remark 4.5. To prove the last equality assume $\Lambda_0(t) = 0 \times \mathbb{R}^n$ so that the matrix $Z(t) = (B(t), D(t))$ is a Lagrangian frame of $\Lambda_1(t)$ where D is also as in (2). □

5 Other Maslov indices

For two matrices Ψ_{21} and Ψ_{10} in $\mathrm{Sp}_0(2n)$ the define the **composition form**

$$Q(\Psi_{21}, \Psi_{10}) = B_{21}^{-1} B_{20} B_{10}^{-1}$$

where $\Psi_{20} = \Psi_{21}\Psi_{10}$ and B_{kj} is the right upper block in the decomposition (2) of Ψ_{kj}. In [11] the composition form is related to the boundary terms that arise in the calculus of variations. An alternative formula for the composition form is

$$Q(\Psi_{21}, \Psi_{10}) = B_{21}^{-1} A_{21} + D_{10} B_{10}^{-1}. \tag{12}$$

Let $\widetilde{\mathrm{Sp}}(2n)$ denote the universal cover of $\mathrm{Sp}(2n)$. Think of an element of $\widetilde{\mathrm{Sp}}(2n)$ covering Ψ as a homotopy class (fixed endpoints) of paths $\widetilde{\Psi} : [0,1] \to \mathrm{Sp}(2n)$ with $\widetilde{\Psi}(0) = \mathbb{1}$ and $\widetilde{\Psi}(1) = \Psi$. Denote by $\widetilde{\mathrm{Sp}}_k(2n)$ the preimage of $\mathrm{Sp}_k(2n)$ under the covering map. The following theorem is essentially due to Leray [8], p.52.

Theorem 5.1 *The restriction of the Maslov index to* $\widetilde{\mathrm{Sp}}_0(2n)$ *is the unique locally constant map* $\mu : \widetilde{\mathrm{Sp}}_0(2n) \to n/2 + \mathbb{Z}$ *such that*

$$\mu(\widetilde{\Psi}_{20}) = \mu(\widetilde{\Psi}_{21}) + \mu(\widetilde{\Psi}_{10}) + \tfrac{1}{2}\mathrm{sign}\, Q(\Psi_{21}, \Psi_{10}) \tag{13}$$

whenever $\widetilde{\Psi}_{20} = \widetilde{\Psi}_{21}\widetilde{\Psi}_{10}$ *and* $\widetilde{\Psi}_{kj}$ *covers* Ψ_{kj}.

Proof: We first prove that the Maslov index satisfies (13). Assume without loss of generality that $\widetilde{\Psi}_{10}(t)$ is constant for $t \geq 1/2$ and $\widetilde{\Psi}_{21}(t) = \mathbb{1}$ for $t \leq 1/2$. Denote $\mu_{kj} = \mu(\widetilde{\Psi}_{kj})$. Then

$$\begin{aligned}
\mu_{20} &= \mu_{10} + \mu(\widetilde{\Psi}_{21}\Psi_{10}V, V) \\
&= \mu_{10} + \mu(\Psi_{10}V, \widetilde{\Psi}_{21}^{-1}V) \\
&= \mu_{10} + \mu_{21} - \mu(V, \widetilde{\Psi}_{21}^{-1}V) + \mu(\Psi_{10}V, \widetilde{\Psi}_{21}^{-1}V) \\
&= \mu_{10} + \mu_{21} + \mu(\widetilde{\Psi}_{21}^{-1}V, V) - \mu(\widetilde{\Psi}_{21}^{-1}V, \Psi_{10}V) \\
&= \mu_{10} + \mu_{21} + s(\Psi_{10}V, V; V, \Psi_{21}^{-1}V).
\end{aligned}$$

The last equality follows from Theorem 3.5. If the matrices Ψ_{kj} are written in block form as in (2) then

$$s(\Psi_{10}V, V; V, \Psi_{21}^{-1}V) = \tfrac{1}{2}\mathrm{sign}\, Q, \qquad Q = B_{21}^{-1} B_{20} B_{10}^{-1}.$$

To see this take the formula of Theorem 3.5 (3) with $A_0 = -B_{10}D_{10}^{-1}$, $A_1 = B_0 = 0$, $B_1 = A_{21}^{-1}B_{21}$ and use the signature identity of Lemma 5.2 below with $A = A_0$ and $B = -B_1$. In these expressions for A_j and B_j the signs are reversed because the Lagrangian subspaces are graphs of x over y.

To prove uniqueness let $\rho : \mathrm{Sp}_0(2n) \to \mathbb{Z}$ be the difference of the Maslov index and a putative Maslov index. Then ρ is a homomorphism wherever defined. Take a one parameter subgroup $t \mapsto \Psi(t)$ of $\mathrm{Sp}(2n)$ which passes through every component of $\widetilde{\mathrm{Sp}}_0(2n)$ and hits the Maslov cycle when t is an integer. Then $\rho \circ \Psi$ defines a 'homomorphism' from $\mathbb{R} \setminus \mathbb{Z} \to \mathbb{Z}$. Any such homomorphism vanishes identically. $\square$

Lemma 5.2 (The signature identity) *Suppose A and B are real symmetric matrices and A, B, $A + B$ are nonsingular. Then*

$$\operatorname{sign} A + \operatorname{sign} B = \operatorname{sign}(A + B) + \operatorname{sign}(A^{-1} + B^{-1}).$$

Proof: The matrices A, B and $A + B$ are nonsingular if and only if the path $A(t) = A + tB$ of symmetric matrices has only regular crossings. Hence the spectral flow of this path is given by

$$\tfrac{1}{2}\operatorname{sign}(A + B) - \tfrac{1}{2}\operatorname{sign} A = \sum_t \operatorname{sign} P_t B P_t.$$

Here the sum is over all numbers $t \in (0, 1)$ such that $\det(A + tB) = 0$ and P_t denotes the orthogonal projection onto $\ker(A + tB)$. The analogous formula for the path $A^{-1} + t^{-1}B^{-1}$ on the interval $\varepsilon \le t \le 1$ is

$$\tfrac{1}{2}\operatorname{sign} B - \tfrac{1}{2}\operatorname{sign}(A^{-1} + B^{-1}) = \sum_t \operatorname{sign} Q_t B^{-1} Q_t.$$

where Q_t is the orthogonal projection onto $\ker(A^{-1} + t^{-1}B^{-1})$. Since $\ker(A^{-1} + t^{-1}B^{-1}) = B\ker(A + tB)$ the right hand sides agree.

Here is an alternative proof. The symmetric matrices

$$M = \begin{pmatrix} A & \mathbb{1} \\ \mathbb{1} & D \end{pmatrix}, \qquad N = \begin{pmatrix} D & \mathbb{1} \\ \mathbb{1} & A \end{pmatrix}$$

have the same signature. (They are similar.) The identity

$$PMP^T = \begin{pmatrix} A & 0 \\ 0 & D - A^{-1} \end{pmatrix}, \qquad P = \begin{pmatrix} \mathbb{1} & 0 \\ -A^{-1} & \mathbb{1} \end{pmatrix}$$

shows that the signature of M is $\operatorname{sign} A + \operatorname{sign}(D - A^{-1})$. Interchange D and A and use $\operatorname{sign} N = \operatorname{sign} M$ to obtain

$$\operatorname{sign} A + \operatorname{sign}(D - A^{-1}) = \operatorname{sign} D + \operatorname{sign}(A - D^{-1}).$$

Now replace D by $-B^{-1}$. □

Remark 5.3 Let $\Psi(t) = \Psi(t+1)$ be a loop of symplectic matrices. Then the Maslov index $\mu(\Psi)$ agrees with the usual definition:

$$\mu(\Psi) = \frac{\alpha(1) - \alpha(0)}{\pi}$$

where

$$\alpha(t) = \det(X(t) + iY(t)), \qquad \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = (\Psi\Psi^T)^{-1/2}\Psi.$$

This can be proved as in Remark 5.4 below. It also follows from Remark 2.6 above.

Remark 5.4 Consider the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \bar{\omega})$ with $\bar{\omega} = (-\omega) \times \omega$. If $\Psi(a) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(b)) \neq 0$ then the index

$$\mu_{CZ}(\Psi) = \mu(\mathrm{Gr}(\Psi), \Delta)$$

is called the **Conley-Zehnder index** [2], [13]. This index is an integer and satisfies

$$(-1)^{\mu(\Psi)-n} = \operatorname{sign}\det(\mathbb{1} - \Psi(b)).$$

This number is the parity of the Lagrangian frame $(\mathbb{1}, \Psi(b))$ for the graph of $\Psi(b)$. That our definition agrees with the one in [2] and [13] follows from the homotopy invariance, the product formula, and by examining the case $n = 1$. The path

$$\Psi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \qquad 0 \le t \le \varepsilon,$$

has Conley-Zehnder index $\mu_{CZ}(\Psi) = 1$ when $\varepsilon > 0$ is small. To see this with our definition choose a Lagrangian splitting $\mathbb{R}^2 \times \mathbb{R}^2 = \overline{V} \oplus \overline{W}$ where

$\overline{V} = \Delta$ and $\overline{W} = 0 \times \mathbb{R} \times \mathbb{R} \times 0$. (Warning: The sign of the symplectic form on the first factor $\mathbb{R}^2 \times 0$ is reversed.) Given $\bar{v} = (x_0, y_0, x_0, y_0) \in \overline{V}$ choose $\bar{w}(t) = (0, \eta(t), \xi(t), 0) \in \overline{W}$ such that $\bar{v} + \bar{w}(t) \in \overline{\Lambda}(t) = \mathrm{Gr}(\Psi(t))$. This means that

$$x_0 + \xi(t) = x_0 \cos t - (y_0 + \eta(t)) \sin t, \qquad y_0 = x_0 \sin t + (y_0 + \eta(t)) \cos t.$$

Differentiate this identity at $t = 0$ and use $\xi(0) = \eta(0) = 0$ to obtain $\dot{\eta}(0) = -x_0$ and $\dot{\xi}(0) = -y_0$. Hence the crossing form at $t = 0$ is given by

$$\begin{aligned} \Gamma(\mathrm{Gr}(\Psi), \Delta, 0)(\bar{v}) &= \left.\frac{d}{dt}\right|_{t=0} \bar{\omega}(\bar{v}, \bar{w}(t)) \\ &= -x_0 \dot{\eta}(0) - \dot{\xi}(0) y_0 \\ &= x_0^2 + y_0^2. \end{aligned}$$

This quadratic form has signature 2.

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