# Phase Functions and Path Integrals

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This note is an introduction to our forthcoming paper [17]. There we show how to construct the metaplectic representation using Feynman path integrals. We were led to this by our attempts to understand Atiyah's explanation of topological quantum field theory in [2].

Like Feynman's original approach in [9] (see also [10]) an action integral plays the role of a phase function. Unlike Feynman, we use paths in phase space rather than configuration space and use the symplectic action integral rather than the (classical) Lagrangian integral. We eventually restrict to (inhomogeneous) quadratic Hamiltonians so that the finite dimensional approximation to the path integral is a Gaussian integral. In evaluating this Gaussian integral the signature of a quadratic form appears. This quadratic form is a discrete approximation to the second variation of the action integral.

For Lagrangians of the form kinetic energy minus potential energy, evaluated on curves in configuaration space, the index of the second variation is well-defined and, via the Morse Index Theorem,[1] related to the Maslov In-

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[1]See [8] for example.

dex of the corresponding linear Hamiltonian system. The second variation of the symplectic action has both infinite index and infinite coindex. However, this second variation does have a well-defined signature via the aforementioned discrete approximation. This signature can be expressed in terms of the Maslov index of the corresponding linear Hamiltonian system. This is a symplectic analog of the Morse Index Theorem.

Our treatment is motivated by the formal similarity between Feynman path integrals and the Fourier integral operators of Hörmander [13]. A key point of Hörmander's theory is that the phase function which appears in the expression for a Fourier integral operator can be replaced by another phase function which defines the same symplectic relation. This is how Feynman path integrals can be evaluated: one replaces the symplectic action by the generating function of the corresponding symplectic relation.

In sections 1 and 2 we review how to use phase functions to construct Lagrangian manifolds and symplectic relations. These generalities are motivated by the examples in section 3 where the phase function is the action integral.

Our topic has a vast literature. Our formula for the metaplectic representation appears in [16] where it is obtained by other arguments. Souriau [26] found an explicit solution for the quantum harmonic oscillator involving the Maslov index (thus correcting Feynman's original formula which is valid only for short times). Keller [14] first noticed the phase shift due to the Maslov index in Theorem 5.2 below and for this reason the Maslov index is sometimes called the *Keller-Maslov index*. Duistermaat‘s article [8] explains how to interpret the Morse index in terms of the Maslov index but in the situation studied here the Morse index is undefined. The article [1] explains how Feynman and Dirac [7] were motivated by using the method of stationary phase to obtain classical mechanics as the limit (as $\hbar \to 0$) of quantum mechanics. Daubechies and Klauder [5] (see also [6]) have formulated a theory of path integrals on phase space where the Hamiltonian function can be any polynomial. They remark that the 'time slicing' construction used by Feynman does not generalize. However, our Hamiltonians are at worst quadratic and Feynman's original method is adequate.

# 1 Lagrangian manifolds

A **variational family** is a pair

$$\pi : P \to X, \qquad \phi : P \to \mathbb{R}$$

consisting of a surjective submersion $\pi$ between manifolds $P$ and $X$, and a smooth function $\phi$ on $P$. Each choice of $x \in X$ determines a constrained variational problem

$$\textit{extremize } \phi(c) \quad \textit{subject to} \quad \pi(c) = x.$$

We call a critical point of $\phi|\pi^{-1}(x)$ a **fiber critical point** of $\phi$. Denote by $C(\pi, \phi) \subset P$ the set of all fiber critical points $c \in P$ of $\phi$. At a fiber critical point $c$ the differential $d\phi(c)$ vanishes on the vertical tangent space $\ker d\pi(c) = T_c\pi^{-1}(x)$. This means that there exists a **Lagrange multiplier** $y \in T_x^*X$ such that

$$d\phi(c)\gamma = \langle y, d\pi(c)\gamma \rangle \tag{1}$$

for every $\gamma \in T_cP$. The Lagrange multiplier $y$ is uniquely determined since $d\pi(c)$ is surjective. Consider the map

$$\lambda_{\pi,\phi} : C(\pi, \phi) \to T^*X$$

defined by $\lambda_{\pi,\phi}(c) = (x, y)$ where $x = \pi(c)$ and $y$ is given by (1). Denote its image by

$$\Lambda(\pi, \phi) = \{(x, y) \in T^*X : \exists c \in \pi^{-1}(x) \text{ such that } (1)\}.$$

If $\Lambda = \Lambda(\pi, \phi)$ is a set of this form then we say that $(\pi, \phi)$ *defines* $\Lambda$ and call $\phi$ a **phase function** for $\Lambda$ with respect to $\pi$. An extreme case is where $P = X$ and $\pi : P \to X$ is the identity so that $\phi$ is a function on $X$ and $\Lambda = \mathrm{Gr}(d\phi)$. In this case $\phi$ is called a **generating function** for $\Lambda$.

Let $N_\pi \subset T^*P$ denote the **fiber normal bundle**:

$$N_\pi = \{(c, b) \in T^*P : b \in \ker(d\pi(c))^\perp\}.$$

This is a co-isotropic submanifold of $T^*P$ and its symplectic quotient is $T^*X$. In the lingo of [28] $\Lambda(\pi, \phi)$ is the **symplectic reduction** of the Lagrangian manifold $\mathrm{Gr}(d\phi)$. Recall that two submanifolds $G$ and $N$ of a manifold $W$ are said to intersect

- **transversally** in $W$ iff $T_zW = T_zG + T_zN$, and

- **cleanly** in $W$ iff $T_z(G \cap N) = T_zG \cap T_zN$

for $z \in G \cap N$. (For clean interesection impose the condition that the intersection $G \cap N$ be a submanifold; for transverse intersections this follows. A transversal intersection is automatically clean.) We call the variational family $(\pi, \phi)$ **tranversal** (resp. **clean**) iff $\mathrm{Gr}(d\phi)$ intersects $N_\pi$ transversally (resp. cleanly) in $P$.

**Proposition 1.1** *If $(\pi, \phi)$ is a clean variational family, then $\Lambda(\pi, \phi)$ is an immersed Lagrangian manifold. If $(\pi, \phi)$ is a transversal variational family, then $\lambda_{\pi,\phi}$ is a Lagrangian immersion.*[2]

**Proof:** Localize and choose co-ordinates so that $P = X \times U$ where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^N$ and that $\pi : X \times U \to X$ is the projection

$$\pi(c) = x, \qquad c = (x, u).$$

Then $C(\pi, \phi)$ is defined by the equation $\partial_u\phi = 0$ and $\Lambda(\pi, \phi)$ is defined by eliminating $u$ from the equations

$$\partial_u\phi = 0, \qquad y = \partial_x\phi.$$

The family is transversal iff 0 is a regular value of $\partial_u\phi$ and clean iff $C(\pi, \phi)$ is a manifold and the tangent space at a point $c = (x, u) \in C(\pi, \phi)$ is given by

$$T_cC(\pi, \phi) = \{(\xi, \upsilon) \in \mathbb{R}^n \times \mathbb{R}^N : \partial_u\partial_x\phi(x, u)\xi + \partial_u\partial_u\phi(x, u)\upsilon = 0\}.$$

To prove Proposition 1.1 fix $c = (x, u) \in C(\pi, \phi)$ and apply the next lemma with $A = \partial_x\partial_x\phi(x, u)$, $B = \partial_x\partial_u\phi(x, u)$, $B^T = \partial_u\partial_x\phi(x, u)$, $C = \partial_u\partial_u\phi(x, u)$, $d\partial_u\phi(x, u) = (B^T, C)$, $T = T_cC(\pi, \phi)$, $\ell = d\lambda_{\pi,\phi}(c)$.

**Lemma 1.2** *Suppose that $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{N\times N}$ are symmetric and that $B \in \mathbb{R}^{n\times N}$. Let*

$$T = \{(\xi, \upsilon) : B^T\xi + C\upsilon = 0\} \subset \mathbb{R}^n \times \mathbb{R}^N$$

[2] In the transversal case this is due to Hörmander [13]. The clean case is folklore.

*and* $\ell : T \to \mathbb{R}^n \times \mathbb{R}^n$ *by*

$$\ell(\xi, \upsilon) = (\xi, A\xi + B\upsilon).$$

*Then* $\ell(T) \subset \mathbb{R}^n \times \mathbb{R}^n$ *is a Lagrangian subspace.*

**Proof:** Note that $(\xi, \eta) \in \ell(T)$ iff the inhomogeneous system

$$\begin{aligned} B\upsilon &= -A\xi + \eta \\ C\upsilon &= -B^T\xi \end{aligned}$$

has a solution $\upsilon$. Hence $(\xi, \eta) \in \ell(T)$ iff

$$B^T\xi' + C\upsilon' = 0 \Longrightarrow \langle \xi', -A\xi + \eta \rangle + \langle \upsilon', -B^T\xi \rangle = 0.$$

On the other hand $(\xi, \eta) \in \ell(T)^\omega$ iff

$$B^T\xi' + C\upsilon' = 0 \Longrightarrow \langle \xi, A\xi' + B\upsilon' \rangle - \langle \eta, \xi' \rangle = 0.$$

Hence $\ell(T) = \ell(T)^\omega$. □

At a critical point of a function, the Hessian is a well-defined quadratic form on the tangent space; at a fiber critical point $c$ the **vertical Hessian** is defined on the vertical tangent space. By Taylor's theorem the vertical Hessian $\Phi$ is characterized by the equation

$$\phi(c + \gamma) = \phi(c) + \tfrac{1}{2}\Phi(\gamma) + O(\|\gamma\|^3)$$

for $d\pi(c)\gamma = 0$. Here $c + \gamma = \exp_c(\gamma) \in P$ where exp is an exponential map which carries vertical tangent vectors to the fiber; $\Phi$ is independent of the choice.

**Proposition 1.3** *Assume that* $(\pi, \phi)$ *is a transversal variational family, and* $c \in C(\pi, \phi)$*. Then* $\Phi$ *is non-degenerate iff* $d\pi(c) : T_cC(\pi, \phi) \to T_xX$ *is an isomorphism.*

**Proof:** In local coordinates $c = (x, u)$ the tangent space $T_cC(\pi, \phi)$ is defined by the equation

$$\partial_u\partial_x\phi(c)\hat{x} + \partial_u\partial_u\phi(c)\hat{u} = 0.$$

Hence the projection $(\hat{x}, \hat{u}) \mapsto \hat{x}$ is an isomorphism on $T_cC(\pi, \phi)$ if and only if the Hessian matrix $\Phi = \partial_u\partial_u\phi(c)$ is invertible. □

**Definition 1.4** A fiber critical point $c \in C(\pi, \phi)$ is called **nondegenerate** if the fiber Hessian $\Phi$ is nondegenerate. This implies that

**(1)** $\mathrm{Gr}(d\phi)$ and $N_\pi$ intersect transversally at $c$,

**(2)** $d\pi(c) : T_c C(\pi, \phi) \to T_x X$ is invertible, and

**(3)** $T_x^* X \cap T_{(x,y)} \Lambda(\pi, \phi) = 0$.

In (3) $T_x^* X \subset T_{(x,y)} T^* X$ is the vertical tangent space of the cotangent bundle. The inverse

$$G = d\pi(c)^{-1} : T_x X \to T_c C(\pi, \phi)$$

of the projection in (2) is called the **Green's function** of $\phi$ at $c$. By the implicit function theorem $\pi | C(\pi, \phi)$ is a diffeomorphism in a neighborhood of $x = \pi(c)$: we denote the local inverse by $g$ and call it the **nonlinear Green's function**. Clearly

$$G = dg(x).$$

**Remark 1.5** If the projection $\pi : C(\pi, \phi) \to X$ is a diffeomorphism, there is a global nonlinear Green's function $g : X \to P$. Its image is the set $g(X) = C(\pi, \phi)$ of fiber critical points. In this case $\Lambda(\pi, \phi)$ admits a generating function $\phi \circ g : X \to \mathbb{R}$.

# 2 Symplectic relations

A **symplectic relation** from a symplectic manifold $M_0$ to a symplectic manifold $M_1$ is a Lagrangian submanifold of $\bar{M}_0 \times M_1$. The bar indicates that the sign of the symplectic form in the first factor has been reversed. We do not assume that $M_0$ and $M_1$ have the same dimension. A clean variational family $(\pi_{01}, \phi_{01})$ with $\pi_{01} : P_{01} \to X_0 \times X_1$ determine a symplectic relation from $T^* X_0$ to $T^* X_1$. The construction is as in the previous section but with a sign change. As before at a critical point $c$ there is a Lagrange multiplier $(-y_0, y_1) \in T_{x_0}^* X_0 \times T_{x_1}^* X_1$ such that

$$d\phi_{01}(c)\gamma = \langle y_1, \xi_1 \rangle - \langle y_0, \xi_0 \rangle \tag{2}$$

for every $\gamma \in T_c P$ where $d\pi(c)\gamma = (\xi_0, \xi_1)$. Denote by

$$R_{01} = \{(x_0, y_0, x_1, y_1) \in T^* X_0 \times T^* X_1 : \exists c \in \pi_{01}^{-1}(x_0, x_1) \text{ such that } (2)\}$$

the relation induced by $\phi_{01}$. We call $\phi_{01}$ a **phase function** for $R_{01}$ and call $R_{01}$ the relation defined by $(\pi_{01}, \phi_{01})$. If $(\pi_{01}, \phi_{01})$ is a clean variational family then $R_{01}$ is a symplectic relation. In the extreme case where $P_{01} = X_0 \times X_1$ and $\pi_{01}$ is the identity we call $\phi_{01}$ a **generating function** for $R_{01}$. If $\psi_{10} : T^*X_0 \to T^*X_1$ is a symplectomorphism then its graph $R_{01} = \mathrm{Gr}(\psi_{10})$ is a symplectic relation. In this case we call $\phi_{01}$ a phase function (or generating function) for $\psi_{10}$.

The **composition** of two relations $R_{01} \subset \bar{M}_0 \times M_1$ and $R_{12} \subset \bar{M}_1 \times M_2$ is the relation

$$R_{01} \# R_{12} = \{(z_0, z_2) \,:\, (z_0, z_1) \in R_{01},\, (z_1, z_2) \in R_{12}\}\,.$$

Note that by our conventions the graph operation is a contravariant functor:

$$\mathrm{Gr}(\psi_{21} \circ \psi_{10}) = \mathrm{Gr}(\psi_{10}) \# \mathrm{Gr}(\psi_{21}).$$

A Lagrangian manifold is a special case of a symplectic relation (take $M_0$ to be a point) and we have the formula

$$\psi_{21}(\Lambda_1) = \Lambda_1 \# \mathrm{Gr}(\psi_{21})$$

if $\Lambda_1 \subset M_1$ is Lagrangian.

Let $(\pi_{01}, \phi_{01})$ and $(\pi_{12}, \phi_{12})$ be variational families with

$$\pi_{01} : P_{01} \to X_0 \times X_1, \qquad \pi_{12} : P_{12} \to X_1 \times X_2.$$

Define another variational family $(\pi_{02}, \phi_{02})$ by

$$P_{02} = \{(c_{01}, c_{12}) \in P_{01} \times P_{12} \,:\, \pi_{01}(c_{01}) = (x_0, x_1),\, \pi_{12}(c_{12}) = (x_1, x_2)\}$$

with $\pi_{02} : P_{02} \to X_0 \times X_2$ given by

$$\pi_{02}(c_{01}, c_{12}) = (x_0, x_2)$$

and $\phi_{02} : P_{02} \to \mathbb{R}$ by

$$\phi_{02}(c_{01}, c_{12}) = \phi_{01}(c_{01}) + \phi_{12}(c_{12}).$$

Let $R_{jk} \subset T^*X_j \times T^*X_k$ be the relation defined by $(\pi_{jk}, \phi_{jk})$.

**Proposition 2.1** $R_{02} = R_{01} \# R_{12}$.

**Proof:** Fix $c_{02} = (c_{01}, c_{12}) \in P_{02}$ and

$$\gamma_{02} = (\gamma_{01}, \gamma_{12}) \in T_{c_{01}} P_{01} \times T_{c_{12}} P_{12}$$

and introduce the notations $(x_j, x_k) = \pi_{jk}(c_{jk})$ and

$$d\pi_{01}(c_{01})\gamma_{01} = (\xi_0, \xi_1), \quad d\pi_{12}(c_{12})\gamma_{12} = (\xi_1', \xi_2).$$

The tangent space to $P_{02}$ at is the set of all pairs $\gamma_{02}$ such that $\xi_1 = \xi_1'$. The tangent space to the fiber of $P_{02}$ is defined by the three constraints $\xi_0 = 0$, $\xi_1 = \xi_1'$, and $\xi_2 = 0$. Then $c_{02}$ is a fiber critical point iff there are Lagrange multipliers $y_0 \in T^*_{x_0}X_0$, $y_1 \in T^*_{x_1}X_1$, and $y_2 \in T^*_{x_2}X_2$ such that

$$d\phi_{01}(c_{01})\gamma_{01} + d\phi_{12}(c_{12})\gamma_{12} = \langle y_2, \xi_2\rangle + \langle y_1, \xi_1 - \xi_1'\rangle - \langle y_0, \xi_0\rangle$$

for *all* $(\gamma_{01}, \gamma_{12}) \in T_{c_{01}}P_{01} \times T_{c_{12}}P_{12}$. In particular

$$d\phi_{01}(c_{01})\gamma_{01} = \langle y_1, \xi_1\rangle - \langle y_0, \xi_0\rangle$$

(take $\gamma_{12} = 0$) and

$$d\phi_{12}(c_{12})\gamma_{12} = \langle y_2, \xi_2\rangle - \langle y_1, \xi_1'\rangle$$

(take $\gamma_{01} = 0$). This shows that $R_{02} \subset R_{01}\#R_{12}$. For the reverse inclusion argue backwards. □

The composition operation has the following interpretation. For $x \in X$ we identify $T^*_xX$ with the vertical tangent space

$$V = T^*_xX \subset T_{(x,y)}T^*X.$$

It is a Lagrangian submanifold of $T^*X$. Now fix $x_0 \in X_0$ and $x_2 \in X_2$. The goal is to find all pairs $y_0 \in T^*_{x_0}X_0$ and $y_2 \in T^*_{x_2}X_2$ such that

$$(x_0, y_0, x_2, y_2) \in R_{02}.$$

These points correspond to Lagrangian intersections of the image of $T^*_{x_0}X_0$ under $R_{01}$ with the preimage of $T^*_{x_2}X_2$ under $R_{12}$. For every point

$$(x_1, y_1) \in \left(T^*_{x_0}X_0\#R_{01}\right) \cap \left(R_{12}\#T^*_{x_2}X_2\right) \tag{3}$$

in this intersection there exist points $y_0 \in T^*_{x_0}X_0$ and $y_2 \in T^*_{x_2}X_2$ such that $(x_0, y_0, x_1, y_1) \in R_{01}$ and $(x_1, y_1, x_2, y_2) \in R_{12}$ and hence $(x_0, y_0, x_2, y_2) \in R_{02}$ as required. In the special case where $R_{jk}$ is the graph of a symplectomorphism each intersection point $(x_1, y_1)$ determines $y_0$ and $y_2$ uniquely. Thus given $x_0$ and $x_2$ there is a one-to-one correspondence of Lagrangian intersection points in $T^*X_1$ with points in $R_{02} \cap \left(T^*_{x_0}X_0 \times T^*_{x_2}X_2\right)$.

Now assume that $R_{jk}$ is a manifold and fix

$$(x_j, y_j, x_k, y_k) \in R_{jk}.$$

Let $c_{02} = (c_{01}, c_{12}) \in P_{02}$ be the corresponding fiber critical point so that

$$\pi_{jk}(c_{jk}) = (x_j, x_k)$$

With this notation we have (3). In the tangent space $T_{(x_1,y_1)}T^*X_1$ there are three interesting Lagrangian subspaces:

$$L_0 = T_{(x_1,y_1)}\left(T^*_{x_0}X_0 \# R_{01}\right), \qquad L_2 = T_{(x_1,y_1)}\left(R_{12} \# T^*_{x_2}X_2\right),$$

and the vertical space

$$V = T^*_{x_1}X_1.$$

We assume that $L_0$ and $L_2$ are transverse to $V$. Then the pair $(L_0, L_2)$ determines a quadratic form on the quotient space $T_{x_1}X_1 = T_{(x_1,y_1)}T^*X_1/T^*_{x_1}X_1$. To define it choose a Lagrangian complement $H$ of $V$:

$$T_{(x_1,y_1)}T^*X_1 = H \oplus V.$$

Identify $H$ with the dual space $V^*$ using the symplectic form on $T_{(x_1,y_1)}T^*X_1$. Since $L_0$ and $L_2$ are transverse to $V$ there exist quadratic forms $Q_j : H \to V = H^*$ such that

$$L_0 = \mathrm{Gr}(Q_0), \qquad L_2 = \mathrm{Gr}(Q_2).$$

There is a natural projection (isomorphism) $H \to T_{x_1}X_1$ and the difference

$$Q = Q_0 - Q_2 \tag{4}$$

descends to a quadratic form $T_{x_1}X_1 \to T^*_{x_1}X_1$. The result is independent of the choice of $H$. We abuse language and identify $Q$ with a form on $T_{x_1}X_1$. The form $Q$ is called the **composition form** of $(L_0, L_2)$. Denote by $\Phi_{jk}$ the fiber Hessian at $c_{jk}$. Assume that the fiber critical points $c_{jk} \in C_{\phi_{jk}}$ are nondegenerate and denote the Green's function by $G_{jk} : T_{x_j}X_j \times T_{x_k}X_k \to T_{c_{jk}}C_{\phi_{jk}}$. Define

$$G_0\xi_1 = G_{01}(0, \xi_1), \qquad G_2\xi_1 = G_{12}(\xi_1, 0).$$

**Proposition 2.2** *The linear map*

$$T_{c_{02}}\pi_{02}^{-1}(x_0,x_2) \to T_{c_{01}}\pi_{01}^{-1}(x_0,x_1) \times T_{x_1}X_1 \times T_{c_{12}}\pi_{12}^{-1}(x_1,x_2)$$

*given by*

$$\gamma_{02} = (\gamma_{01},\gamma_{12}) \mapsto (\gamma_{01} - G_0\xi_1, \xi_1, \gamma_{12} - G_2\xi_1)$$

*where* $\xi_1 = d\pi_1(c_{01})\gamma_{01}$ *is an isomorphism. Moreover,*

$$\Phi_{02}(\gamma_{02}) = \Phi_{01}(\gamma_{01} - G_0\xi_1) + \Phi_{12}(\gamma_{12} - G_2\xi_1) + \langle Q\xi_1, \xi_1\rangle. \tag{5}$$

**Proof:** In local coordinates we have

$$\phi_{02}(x_0,u_{01},x_1,u_{12},x_2) = \phi_{01}(x_0,u_{01},x_1) + \phi_{12}(x_1,u_{12},x_2).$$

The relation $R_{01}$ is defined by eliminating $u_{01}$ from the nonlinear system

$$\begin{aligned} -y_0 &= \partial_{x_0}\phi_{01}(x_0,u_{01},x_1) \\ y_1 &= \partial_{x_1}\phi_{01}(x_0,u_{01},x_1) \\ 0 &= \partial_{u_{01}}\phi_{01}(x_0,u_{01},x_1). \end{aligned}$$

The last equation defines the set $C_{\phi_{01}}$. The Lagrangian manifold $T^*_{x_0}X_0\#R_{01}$ is defined by fixing $x_0$ and eliminating $u_{01}$ from the last two. The tangent space $T_{c_{01}}R_{01}$ is defined by eliminating $\upsilon_{01}$ from

$$\begin{aligned} -\eta_0 &= (\partial_{x_0}\partial_{x_0}\phi_{01})\xi_0 + (\partial_{x_0}\partial_{x_1}\phi_{01})\xi_1 + (\partial_{x_0}\partial_{u_{01}}\phi_{01})\upsilon_{01} \\ \eta_1 &= (\partial_{x_1}\partial_{x_0}\phi_{01})\xi_0 + (\partial_{x_1}\partial_{x_1}\phi_{01})\xi_1 + (\partial_{x_1}\partial_{u_{01}}\phi_{01})\upsilon_{01} \\ 0 &= (\partial_{u_{01}}\partial_{x_0}\phi_{01})\xi_0 + (\partial_{u_{01}}\partial_{x_1}\phi_{01})\xi_1 + (\partial^2_{u_{01}}\phi_{01})\upsilon_{01}. \end{aligned}$$

The last equation defines the tangent space to $C_{\phi_{01}}$ and the Green's function $G_{01}$ is given by solving for $\upsilon_{01}$. Thus

$$G_0\xi_1 = (0, \Gamma_0\xi_1, \xi_1), \qquad \Gamma_0 = -\left(\partial^2_{u_{01}}\phi_{01}\right)^{-1}\partial_{u_{01}}\partial_{x_1}\phi_{01}.$$

To define the Lagrangian subspace $L_0 = T_{(x_1,y_1)}\left(T^*_{x_0}X_0\#R_{01}\right)$, set $\xi_0 = 0$ and eliminate $\upsilon_{01}$ in the last two equations. Hence $L_0$ is the graph of the symmetric matrix

$$Q_0 = (\partial^2_{x_1}\phi_{01}) + (\partial_{x_1}\partial_{u_{01}}\phi_{01})\Gamma_0.$$

Similarly, $L_2$ is the graph of $Q_2$ where

$$-Q_2 = (\partial^2_{x_1}\phi_{12}) + (\partial_{x_1}\partial_{u_{12}}\phi_{12})\Gamma_2, \qquad \Gamma_2 = -\left(\partial^2_{u_{12}}\phi_{12}\right)^{-1}\partial_{u_{12}}\partial_{x_1}\phi_{12}.$$

Now the tangent vector $\gamma_{02}$ is in local co-ordinates given by

$$\gamma_{02} = (0, \upsilon_{01}, \xi_1, \upsilon_{12}, 0)$$

with $\gamma_{01} = (0, \upsilon_{01}, \xi_1)$ and $\gamma_{12} = (\xi_1, \upsilon_{12}, 0)$. Hence

$$\gamma_{01} - G_0\xi_1 = (0, \upsilon_{01} - \Gamma_0\xi_1, 0)$$

and

$$\begin{aligned}\Phi_{01}(\gamma_{01} - G_0\xi_1) &= \langle(\partial^2_{u_{01}}\phi_{01})\upsilon_{01}, \upsilon_{01}\rangle + 2\langle(\partial_{u_{01}}\partial_{x_1}\phi_{01})\xi_1, \upsilon_{01}\rangle \\ &\quad -\langle\xi_1, (\partial_{x_1}\partial_{u_{01}}\phi_{01})\Gamma_0\xi_1\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\Phi_{12}(\gamma_{12} - G_2\xi_1) &= \langle(\partial^2_{u_{12}}\phi_{12})\upsilon_{12}, \upsilon_{12}\rangle + 2\langle(\partial_{u_{12}}\partial_{x_1}\phi_{12})\xi_1, \upsilon_{12}\rangle \\ &\quad -\langle\xi_1, (\partial_{x_1}\partial_{u_{12}}\phi_{12})\Gamma_2\xi_1\rangle.\end{aligned}$$

Subtract these two identities from the Hessian

$$\begin{aligned}\Phi_{02}(\gamma_{02}) &= \langle(\partial^2_{u_{01}}\phi_{01})\upsilon_{01}, \upsilon_{01}\rangle + \langle(\partial^2_{u_{12}}\phi_{12})\upsilon_{12}, \upsilon_{12}\rangle \\ &\quad +2\langle(\partial_{u_{01}\partial_{x_1}}\phi_{01})\xi_1, \upsilon_{01}\rangle + 2\langle(\partial_{u_{12}\partial_{x_1}}\phi_{12})\xi_1, \upsilon_{12}\rangle \\ &\quad +\langle(\partial^2_{x_1}\phi_{01})\xi_1, \xi_1\rangle + \langle(\partial^2_{x_1}\phi_{12})\xi_1, \xi_1\rangle.\end{aligned}$$

and use the above formulae for $Q_0$ and $Q_2$ to prove the proposition. $\square$

# 3 Examples

In our examples, except for the last one, the space $P$ is a space of paths and the work of the previous section can be interpreted formally. Alternatively one can introduce Hilbert manifold structures and generalize the previous work to the infinite dimensional case.

**Example 3.1** Let $X$ be a manifold and $L : \mathbb{R} \times TX \to \mathbb{R}$ be a function. Fix $t_0, t_1 \in \mathbb{R}$ and take $P_{01} = P(t_0, t_1)$ to be the space of all paths $c : [t_0, t_1] \to X$. Take $X_0 = X_1 = X$ and define the projection $\pi = \pi_{01}$ by evaluation at the endpoints:

$$\pi(c) = (c(t_0), c(t_1)).$$

The phase function $\phi = \phi_{01}$ is the Lagrangian action integral

$$\phi(c) = \int_{t_0}^{t_1} L(t, c(t), \dot{c}(t))\, dt$$

from the calculus of variations. A tangent vector $\gamma \in T_c P_{01}$ is a vectorfield along $c$ and it is vertical iff it vanishes at the endpoints. A curve $c$ is a fiber critical point iff it satisfies the Euler-Lagrange equations

$$\dot{y} = \partial_x L, \qquad y = \partial_{\dot{x}} L$$

where $L = L(t, x, \dot{x})$. The right side of Equation (2) consists of the boundary terms which result from the integration by parts in the derivation of the Euler-Lagrange equations. If $t_2$ is replaced by $t$ and allowed to vary between $t_1$ and $t_2$, the restriction $c|[t_0, t]$ is a fiber critical point of the new problem on $P(t_0, t)$ and the Lagrange mutiplier $y(t)$ is the $y(t)$ which appears in the Euler-Lagrange equations. When the Legendre transformation

$$TX \to T^*X : (x, \dot{x}) \mapsto (x, y), \qquad y = \partial_{\dot{x}} L$$

is a diffeomorphism, the Euler-Lagrange equations take the form of Hamilton's equations

$$\dot{x} = \partial_y H, \qquad \dot{y} = -\partial_x H$$

where $H : T^*X \to \mathbb{R}$ is defined by eliminating $\dot{x}$ (via the Legendre transformation) from

$$H(t, x, y) = \langle y, \dot{x} \rangle - L(t, x, \dot{x}).$$

**Example 3.2** Specialize the previous example by taking $X$ a Riemannian manifold with energy function $L(x, \dot{x}) = \frac{1}{2}|\dot{x}|_x^2$. Then the fiber critical points are the *geodesics*. The Hessian $\Phi = \Phi_{01}$ is defined by

$$\Phi(\gamma) = \int_{t_0}^{t_1} \langle (W\gamma)(t), \gamma(t) \rangle_{c(t)}\, dt$$

where the operator $W$ is given by

$$W\gamma = \frac{D^2\gamma}{dt^2} + R(\dot{c}, \gamma)\dot{c}.$$

Here $D/dt$ denotes the covariant derivative along $c$ and $\gamma(t) \in T_{c(t)}X$. The linear second order differential equation $W\gamma = 0$ is the *Jacobi equation.* The geodesic is non-degenerate when its end points are not *conjugate* and the Green's function $G$ has its usual interpretation of assigning to the boundary conditions $\xi_0 \in T_{x(t_0)}X$ and $\xi_1 \in T_{c(t_1)}X$ the unique solution $\gamma = G(\xi_0, \xi_1)$ of the boundary value problem

$$W\gamma = 0, \qquad \gamma(t_0) = \xi_0, \qquad \gamma(t_1) = \xi_1.$$

**Example 3.3** Again take $X_0 = X_1 = X$ but now take $\mathcal{P}(t_0, t_1)$ the space of all curves

$$c = (x, y) : [t_0, t_1] \to T^*X$$

with projection $\pi = \pi_{t_0 t_1} : \mathcal{P}(t_0, t_1) \to X \times X$ given by

$$\pi(c) = (x(t_0), x(t_1)).$$

A time-dependent Hamiltonian $H : \mathbb{R} \times T^*X \to \mathbb{R}$ determines a one-form $\sigma_H$ on $\mathbb{R} \times T^*M$ via

$$\sigma_H = \langle y, dx \rangle - H\, dt$$

called the **action form** of $H$. Define the phase function $\phi = \phi_{t_0 t_1}$ to be the integral

$$\phi(c) = \int_c \sigma_H$$

of the action form along $c$ is called the **action integral**. A more explicit formula is

$$\phi(c) = \int_{t_0}^{t_1} \Big( \langle y, \dot{x} \rangle - H(t, x, y) \Big)\, dt.$$

where $c(t) = (x(t), y(t))$. As before the vertical critical points of $(\pi, \phi)$ are the solutions of the Euler-Lagrange equations of this functional. They are Hamilton's equations

$$\dot{x} = \partial_y H, \qquad \dot{y} = -\partial_x H.$$

The Lagrange multipliers in Equation (2) are given by $y_0 = y(t_0)$ and $y_1 = y(t_1)$. Assume for simplicity that these differential equations are complete. Then the symplectic relation determined by $(\pi_{t_0t_1}, \phi_{t_0t_1})$ is $\mathrm{Gr}(\psi_{t_0}^{t_1})$ where $t \mapsto \psi_{t_0}^{t}(x,y)$ is the solution of Hamilton's equations satisfying the initial condition $\psi_{t_0}^{t_0}(x,y) = (x,y)$. These symplectomorphisms define an **evolution system** meaning that

$$\psi_{t_1}^{t_2} \circ \psi_{t_0}^{t_1} = \psi_{t_0}^{t_2}, \qquad \psi_{t_0}^{t_0} = \mathbb{1}.$$

**Remark 3.4** Assume that $c_0$ is a nondegenerate fiber critical point in the previous example. Then there is a local nonlinear Green's function which assigns to every point $(x_0, x_1)$ near $\pi(c_0)$ the unique solution $c(t) = (x(t), y(t))$ near $c_0$ of Hamilton's equation which satisfies $x(t_0) = x_0$ and $x(t_1) = x_1$. Let

$$S_{t_0t_1}(x_0, x_1) = \int_{t_0}^{t_1} \Big( \langle y, \dot{x} \rangle - H(t, x, y) \Big)\, dt$$

denote the action integral of this solution. This is a local generating function of the symplectomorphism $\psi_{t_0}^{t_1}$.

**Remark 3.5** Now fix $t_0$ and $x_0$. Then the generating function $S(t,x) = S_{t_0t}(x_0, x)$ satisfies the **Hamilton-Jacobi equation**

$$\partial_t S + H(t, x, \partial_x S) = 0.$$

To prove this differentiate the identity

$$S(t, x(t)) = \int_{t_0}^{t} \Big( \langle y(s), \dot{x}(s) \rangle - H(s, x(s), y(s)) \Big)\, ds$$

with respect to $t$ and use $y = \partial S / \partial x$.

**Example 3.6** By a **partition** of $\mathbb{R}$ we mean an infinite discrete subset $\mathcal{T} \subset \mathbb{R}$ extending to infinity in both directions. Every $t \in \mathcal{T}$ has a unique **successor** $t^+ \in \mathcal{T}$ and **predecessor** $t^- \in \mathcal{T}$ defined by

$$t^- = \sup \mathcal{T} \cap (-\infty, t), \qquad t^+ = \inf \mathcal{T} \cap (t, \infty).$$

Denote the **mesh** of $\mathcal{T}$ by

$$|\mathcal{T}| = \sup_{t \in \mathcal{T}} |t^+ - t|.$$

Now take $X = \mathbb{R}^n$, $T^*X = \mathbb{R}^n \times \mathbb{R}^n$, and fix a time dependent Hamiltonian $H(t, x, y)$ and a partition $\mathcal{T}$. Let $t_0, t_1 \in \mathcal{T}$ with $t_0 < t_1$. Define the space

$$\mathcal{P}^{\mathcal{T}}(t_0, t_1) = \{c = (x, y) \,:\, x : \mathcal{T} \cap [t_0, t_1] \to \mathbb{R}^n,\, y : \mathcal{T} \cap [t_0, t_1) \to \mathbb{R}^n\}$$

of discrete paths in $\mathbb{R}^{2n}$. These discrete paths are finite sequences of length $N$ and $N - 1$ where $N$ is the cardinality of the finite set $\mathcal{T} \cap [t_0, t_1]$. The projection $\pi = \pi_{t_0 t_1} : \mathcal{P}^{\mathcal{T}}(t_0, t_1) \to \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$\pi(c) = (x(t_0), x(t_1)).$$

The **discrete action functional** $\phi^{\mathcal{T}} : \mathcal{P}^{\mathcal{T}}(t_0, t_1) \to \mathbb{R}$ is defined by

$$\phi^{\mathcal{T}}(c) = \sum_{\substack{t \in \mathcal{T} \\ t_0 \le t < t_1}} \Big( \langle y(t), x(t^+) - x(t) \rangle - H(t, x(t^+), y(t))(t^+ - t) \Big).$$

The vertical critical points of $(\pi^{\mathcal{T}}, \phi^{\mathcal{T}})$ are the solutions of the **discrete Hamiltonian equations**

$$\begin{array}{rcl} x(t^+) - x(t) & = & \partial_y H(t, x(t^+), y(t))(t^+ - t) \\ y(t^+) - y(t) & = & -\partial_x H(t, x(t^+), y(t))(t^+ - t). \end{array} \tag{6}$$

These equations define $(x(t^+), y(t^+))$ implicitly in terms of $(x(t), y(t))$. Let $(y_0, y_1)$ be the Lagrange multipliers in equation (2). Then $y_0 = y(t_0)$ and $y_1 = y(t_1)$ is defined by equation (6).

**Remark 3.7** Assume that $c_0^{\mathcal{T}}$ is a nondegenerate fiber critical point in the previous example and define a discrete generating function $S^{\mathcal{T}}_{t_0 t_1}(x_0, x_1)$ as in the continuous case. Now fix a time interval $[t_0, t_1]$, let the mesh $|\mathcal{T}|$ of the partition go to zero, and let $c_0^{\mathcal{T}}$ converge to a nondegenerate fiber critical point of the continuous variational problem. Then we have a limit

$$\lim_{|\mathcal{T}| \to 0} S^{\mathcal{T}} = S.$$

This follows from standard arguments in the discretization of ordinary differential equations.

# 4 Hessians

We now compare the fiber Hessians in Examples 3.3 and 3.6. We take $X = \mathbb{R}^n$ and $T^*X = \mathbb{R}^n \times \mathbb{R}^n$. To simplify the notation we assume that for each $t$ the Hamiltonian $H(t, x, y)$ is homogeneous quadratic in $(x, y)$:

$$H(t, x, y) = \tfrac{1}{2}\langle H_{xx}(t)x, x\rangle + \langle H_{yx}(t)x, y\rangle + \tfrac{1}{2}\langle H_{yy}(t)y, y\rangle$$

where $H_{xx}(t)$, $H_{yx}(t)$, $H_{yy}(t)$ are $n \times n$ matrices with $H_{xx}$ and $H_{yy}$ symmetric. We abbreviate $H_{xy} = H_{yx}^T$.

## Continuous time

In the continuous time case the fiber Hessian $\Phi = \Phi_{t_0t_1}$ is given by

$$\Phi(\gamma) = \langle W\gamma, \gamma\rangle$$

for $\gamma = (\xi, \eta) : [t_0, t_1] \to \mathbb{R}^n \times \mathbb{R}^n$ with $\xi(t_0) = \xi(t_1) = 0$. The inner product on the right is the $L^2$ inner product and the fiber Hessian $W = W_{t_0t_1}$ is the self-adjoint operator on $L^2([t_0, t_1], \mathbb{R}^n \times \mathbb{R}^n)$ with dense domain

$$\mathcal{W}(t_0, t_1) = H_0^1([t_0, t_1], \mathbb{R}^n) \times H^1([t_0, t_1], \mathbb{R}^n)$$

given by $W(\xi, \eta) = (u, v)$ where

$$u = -\dot{\eta} - H_{xx}\xi - H_{xy}\eta, \qquad v = \dot{\xi} - H_{yx}\xi - H_{yy}\eta,$$

We call $W = W_{t_0t_1}$ the **second variation** from $t_0$ to $t_1$. By Proposition 1.3 the Hessian is nondegenerate if and only if the symplectomorphism $\psi_{t_0}^{t_1}$ generated by $H$ admits a generating function.

## Discrete time

In discrete time we do the analogous thing. For $t_0, t_1 \in \mathcal{T}$ with $t_0 < t_1$ define

$$\mathcal{W}^{\mathcal{T}}(t_0, t_1) = \left\{\gamma = (\xi, \eta) \in \mathcal{P}^{\mathcal{T}}(t_0, t_1) : \xi(t_0) = \xi(t_1) = 0\right\}.$$

This is a Hilbert space with the approximate $L^2$-norm

$$\|\gamma\|_{\mathcal{T}}^2 = \sum_{t_0 \le t < t_1} \left(|\xi(t^+)|^2 + |\eta(t)|^2\right)(t^+ - t).$$

In this case the fiber Hessian is the (finite dimensional) symmetric operator $W^{\mathcal{T}} = W^{\mathcal{T}}_{t_0t_1} : \mathcal{W}^{\mathcal{T}}(t_0, t_1) \to \mathcal{W}^{\mathcal{T}}(t_0, t_1)$ given by $W^{\mathcal{T}}(\xi, \eta) = (u, v)$ where

$$u(t) = -\frac{\eta(t) - \eta(t^-)}{t - t^-} - H_{xx}(t^-)\xi(t) - H_{xy}(t^-)\eta(t^-),$$

$$v(t) = \frac{\xi(t^+) - \xi(t)}{t^+ - t} - H_{yx}(t)\xi(t^+) - H_{yy}(t)\eta(t).$$

We call $W^{\mathcal{T}} = W^{\mathcal{T}}_{t_0t_1}$ the **discrete second variation** from $t_0$ to $t_1$. By Proposition 1.3 the Hessian is nondegenerate if and only if the affine symplectomorphism $\phi^{t_1}_{t_0}$ generated by the discrete Hamiltonian equations admits a generating function.

## Signature

The operator $W^{\mathcal{T}}$ is defined on a finite dimensional space and hence has a well defined index (number of negative eigenvalues), coindex (number of positive eigenvalues), signature (coindex minus index), and nullity. For the operator $W$ the index and coindex are both infinite and hence the signature is undefined. However, the signature of $W^{\mathcal{T}}$ stabilizes when the mesh of the partition gets sufficiently small. It is related to the Maslov index $\mu(t_0, t_1, H)$ of the Hamiltonian flow (defined below) as follows.

**Theorem 4.1** *Assume that $W_{t_0t_1}$ is non-degenerate. If the mesh $|\mathcal{T}|$ is sufficiently small then*

$$\operatorname{sign} W^{\mathcal{T}}_{t_0t_1} = 2\mu(t_0, t_1, H).$$

Here is the definition of the Maslov index. Let $\mathrm{Sp}(2n)$ denote the symplectic group and $\widetilde{\mathrm{Sp}}(2n)$ its universal cover. Think of an element of $\widetilde{\mathrm{Sp}}(2n)$ covering $\Psi$ as a homotopy class of paths starting at $\mathbb{1}$ and ending at $\Psi$. Define the **Maslov cycle**

$$\Sigma = \{\Psi \in \mathrm{Sp}(2n) : \Psi\,(0 \times \mathbb{R}^n) \cap (0 \times \mathbb{R}^n) \neq \{0\}\},$$

and its complement

$$\mathrm{Sp}_0(2n) = \mathrm{Sp}(2n) \setminus \Sigma$$

and denote by $\widetilde{\Sigma}$ and $\widetilde{\mathrm{Sp}}_0(2n)$ the preimages under the covering map. For $\Psi_{10}, \Psi_{21} \in \mathrm{Sp}_0(2n)$ let $Q(\Psi_{21}, \Psi_{10})$ denote the composition form of the pair

$(\Psi_{10}(0 \times \mathbb{R}^n), \Psi_{21}^{-1}(0 \times \mathbb{R}^n))$. If the matrices $\Psi_{kj}$ are written in block matrix notation

$$\Psi_{kj} = \begin{pmatrix} A_{kj} & B_{kj} \\ C_{kj} & D_{kj} \end{pmatrix} \tag{7}$$

then the composition form is given by

$$Q = B_{21}^{-1} B_{20} B_{10}^{-1}.$$

**Theorem 4.2** *There is a unique locally constant map* $\mu : \widetilde{\mathrm{Sp}}_0(2n) \to n/2 + \mathbb{Z}$ *such that*

$$\mu(\widetilde{\Psi}_{20}) = \mu(\widetilde{\Psi}_{21}) + \mu(\widetilde{\Psi}_{10}) + \tfrac{1}{2}\mathrm{sign}\, Q(\Psi_{21}, \Psi_{10})$$

*whenever* $\widetilde{\Psi}_{20} = \widetilde{\Psi}_{21}\widetilde{\Psi}_{10}$ *and* $\widetilde{\Psi}_{kj}$ *covers* $\Psi_{kj}$. *This is called the* **Maslov index**.

The number $\mu(\widetilde{\Psi})$ of Theorem 4.2 is essentially the intersection number of $\widetilde{\Psi}$ with the Maslov cycle. The definition is modified to adjust for the fact that the curve $\widetilde{\Psi}$ begins at the identity (which is an element of the Maslov cycle). For details see [19]. The number $\mu(t_0, t_1, H)$ of Theorem 4.1 is the Maslov index of the curve $[t_0, t_1] \to \mathrm{Sp}(2n) : t \mapsto \Psi_{t_0}^t$ defined by the evolution system generated by $H$.

**Remark 4.3** Suppose that the evolution system generated by the Hamiltonian $H$ is a symplectic shear

$$\Psi_{t_0}^{t_1} = \begin{pmatrix} \mathbb{1} & B(t_1, t_0) \\ 0 & \mathbb{1} \end{pmatrix}.$$

Then $B(t_2, t_0) = B(t_2, t_1) + B(t_1, t_0)$ and $B(t_0, t_0) = \mathbb{1}$. The Maslov index is given by

$$\mu(t_0, t_1, H) = -\tfrac{1}{2}\mathrm{sign}\, B(t_1, t_0).$$

For any two symmetric matrices $A$, $B$ such that $A$, $B$, $A+B$, and $A^{-1}+B^{-1}$ are nonsingular we have the signature identity

$$\mathrm{sign}(A) + \mathrm{sign}(B) = \mathrm{sign}(A + B) + \mathrm{sign}(A^{-1} + B^{-1}).$$

This proves that the Maslov index as defined by intersection numbers satisfies the composition formula of Theorem 4.2 in the case of symplectic shears. The signature identity is obvious if the matrices are simultaneously diagonalizable. The general case can be proved with a homotopy argument.

Assume that $t_0, t_1, t_2 \in \mathcal{T}$ are such that the Hessians $W_{t_j t_k}$ are nondegenerate and denote by $Q^{\mathcal{T}}$ the corresponding composition form as in equation (4). The composition forms $Q^{\mathcal{T}}$ converge to the composition form $Q$ of the continuous time problem as the mesh $|\mathcal{T}|$ tends to zero. If the mesh is sufficiently small then, by Proposition 2.2,

$$\operatorname{sign} W^{\mathcal{T}}_{t_0 t_2} = \operatorname{sign} W^{\mathcal{T}}_{t_0 t_1} + \operatorname{sign} W^{\mathcal{T}}_{t_1 t_2} + \operatorname{sign} Q^{\mathcal{T}}.$$

Thus the signature of the discrete second variation $W^{\mathcal{T}}$ satisfies the composition formula of Theorem 4.2 and this can be used to prove Theorem 4.1. Alternatively one can prove Theorem 4.1 first in the special case of a symplectic shear and then use a homotopy argument.

**Remark 4.4** Theorem 4.2 is essentially due to Leray [16]. Leray's index $m(\widetilde{\Psi})$ is related to ours by the formula

$$m(\widetilde{\Psi}) = \mu(\widetilde{\Psi}) + \frac{n}{2}.$$

# 5 Feynman path integrals

Heuristically a variational family $(\pi, \phi)$ together with some sort of measure on the fibers determines a distribution on the base

$$f(x) = \int_{\substack{c \in P \\ \pi(c)=x}} e^{i\phi(c)/\hbar} \mathcal{D}c. \tag{8}$$

If the base is a product $X = X_0 \times X_1$ the distribution may be interpreted as an integral kernel

$$K(x_1, x_0) = \int_{\substack{c \in P \\ \pi(c)=(x_0,x_1)}} e^{i\phi(c)/\hbar} \mathcal{D}c$$

of an operator from a space of functions on $X_0$ to a space of functions on $X_1$:

$$Uf(x_1) = \int_{X_0} K(x_1, x_0) f(x_0) dx_0.$$

Formally the Feynman path integral is an example of this. The composition formula of Proposition 2.1 should correspond to the composition of operators.

Consider a time dependent quadratic Hamiltonian

$$\begin{aligned} H(t,x,y) &= H_0(t) + \langle H_x(t), x\rangle + \langle H_y(t), y\rangle \\ &\quad + \tfrac{1}{2}\langle H_{xx}(t)x, x\rangle + \langle H_{yx}(t)x, y\rangle + \tfrac{1}{2}\langle H_{yy}(t)y, y\rangle \end{aligned}$$

where $H_{xx}(t)$, $H_{yx}(t)$, $H_{yy}(t)$ are as before, $H_x(t), H_y(t) \in \mathbb{R}^n$, and $H_0(t) \in \mathbb{R}$. Let $\phi(c)$ denote the action integral. The Feynman path integral associated to $H$ is the formal expression

$$\mathcal{U}(t_1, t_0, H)f(x_1) = \int\limits_{\substack{c\in\mathcal{P}(t_0,t_1)\\ x(t_1)=x_1}} e^{i\phi(c)/\hbar} f(x(t_0))\mathcal{D}c.$$

where $c = (x, y)$. Feynman was led to integrals of this type by physical considerations. He assigned a phase $e^{i\phi(c)/\hbar}$ to each classical path $c$ and summed over all paths $c$. Our goal is to interpret this integral as a limit in the same way Feynman did. The discrete analogue of the path integral is the expression

$$\mathcal{U}^{\mathcal{T}}(t_1, t_0, H)f(x_1) = \int\limits_{\substack{c\in\mathcal{P}^{\mathcal{T}}(t_0,t_1)\\ x(t_1)=x_1}} e^{i\phi^{\mathcal{T}}(c)/\hbar} f(x(t_0))\mathcal{D}c$$

where

$$\mathcal{D}\,c = \prod_{t_0\le t<t_1} (2\pi\hbar)^{-n} \det\left(\mathbb{1} - (t^+ - t)H_{xy}\right)^{1/2} dx(t)dy(t).$$

The order of integration is the time-order, i.e. first $dx(t_0)$, then $dy(t_0)$, then $dx(t_0^+)$ etc. The notation $\mathcal{D}\,c$ hides the normalization which makes the Feynman product a unitary operator. The integral does not converge absolutely as an integral in all its variables. Interchanging the order of integration requires justification.

**Theorem 5.1** *The limit*

$$\mathcal{U}(t_1, t_0, H) = \lim_{|\mathcal{T}|\to 0} \mathcal{U}^{\mathcal{T}}(t_1, t_0, H)$$

*exists in the strong operator topology. It is a unitary operator on* $L^2(\mathbb{R}^n)$*. Here the partitions partition the interval* $[t_0, t_1]$.

We now give an explicit formula for the operator $\mathcal{U}(t_1, t_0, H)$. According to the philosophy of Fourier integral operators it should be possible to replace $\phi$ by any other phase function defining the same symplectic relation provided that $\mathcal{D}c$ is modified appropriately. In the case at hand the symplectic relation is the graph of the evolution system $\psi_{t_0}^{t_1}$ (see Example 3.3) so it is natural to seek a formula in terms of the generating function $S(x_0, x_1)$ from $t_0$ to $t_1$. Let $\Psi_{t_0}^{t_1}$ denote the linear part of $\psi_{t_0}^{t_1}$, $\mu = \mu(t_0, t_1, H)$ denote the Maslov index of $[t_0, t_1] \to \mathrm{Sp}(2n) : t \mapsto \Psi_{t_0}^{t}$, and $B = B(t_1, t_0)$ denote the right upper block in the block decomposition (7) of $\Psi_{t_0}^{t_1}$.

**Theorem 5.2** *If $\psi_{t_0}^{t_1}$ admits a generating function then $\mathcal{U}(t_1, t_0, H)$ is given by*

$$\mathcal{U}(t_1, t_0, H)f(x_1) = \frac{(2\pi\hbar)^{-n/2}}{|\det B|^{1/2}} e^{i\pi\mu/2} \int_{\mathbb{R}^n} e^{i\,S(x_0,x_1)/\hbar} f(x_0)\, dx_0.$$

The formula is first proved in the case of discrete time and then convergence as well as the continuous time formula are obvious. To prove the analogous formula in discrete time note that Taylor's formula

$$\phi^{\mathcal{T}}(c) = S^{\mathcal{T}}(x_0, x_1) + \tfrac{1}{2}\langle W_{t_0 t_1}^{\mathcal{T}}\gamma, \gamma\rangle$$

is exact (since the action is quadratic). Here $c = c_0 + \gamma$. $c_0$ is a fiber critical point with $\pi(c_0) = (x_0, x_1)$, so $S^{\mathcal{T}}(x_0, x_1) = \phi^{\mathcal{T}}(c_0)$. Now integrate over $\gamma$. Then the Maslov index appears as the signature of $W_{t_0 t_1}^{\mathcal{T}}$ according to Theorem 4.1.

Associated to the Hamiltonian $H(t, x, y)$ is a second order differential operator $H(t, Q, P)$ where $Q_j$ and $P_j$ denote the self-adjoint operators

$$(P_j f)(x) = -i\hbar\partial_j f(x), \qquad (Q_j f)(x) = x_j f(x),$$

and $H(t, Q, P)$ results from $H(t, x, y)$ by making the following substitutions

$$x_j \mapsto Q_j, \qquad y_j \mapsto P_j,$$

$$x_j x_k \mapsto Q_j Q_k, \qquad y_j y_k \mapsto P_j P_k, \qquad x_k y_j \mapsto Q_k P_j - \frac{i\hbar}{2}\delta_{jk}\mathbb{1}.$$

Pay attention to the mixed term: $Q_j$ and $P_j$ do not commute. If the Hamiltonian has the form $H = \frac{1}{2}|y^2| + V(x)$ the equation in the next theorem is the Schrödinger equation.

**Theorem 5.3** *The operators $\mathcal{U}(t, t_0, H)$ are the evolution operators of the time-dependent partial differential equation*

$$i\hbar \frac{\partial u}{\partial t} = H(t, Q, P)u.$$

**Proof:** Assume that $\psi_{t_0}^t$ admits a generating function and let $S(t, x, x_0)$ be given by the action. Let $B(t)$ denote the right upper block in the block decomposition of $\Psi_{t_0}^t = d\psi_{t_0}^t$ and abbreviate $\lambda = e^{i\pi\mu(t,t_0,H)/2}\,(2\pi\hbar)^{-n/2}$. Then

$$u(t, x) = \mathcal{U}(t, t_0, H)f(x) = \lambda |\det B(t)|^{-1/2} \int_{\mathbb{R}^n} e^{iS(t,x,x_0)/\hbar} f(x_0)\, dx_0.$$

Differentiating with respect to $x$ gives

$$P_j u = \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} \frac{\partial S}{\partial x_j} e^{iS/\hbar} f$$

and

$$P_j P_k u = -i\hbar \frac{\partial^2 S}{\partial x_j \partial x_k} u + \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_k} e^{iS/\hbar} f$$

Hence the right hand side of the equation is

$$\begin{aligned} H(t, Q, P)u &= -i\hbar \tfrac{1}{2} \mathrm{tr}\,(H_{yx} + H_{yy} D B^{-1})\, u \\ &\quad + \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} H(t, x, \partial_x S) e^{iS/\hbar} f. \end{aligned}$$

Here we have used the identity $\partial^2 S/\partial x^2 = DB^{-1}$ where $D = D(t)$ is the lower right block in the block decomposition (7) of $\Psi_{t_0}^t$. Now

$$\begin{aligned} \frac{d}{dt} |\det\, B|^{-1/2} &= -\tfrac{1}{2} \mathrm{tr}\,(\dot{B} B^{-1}) |\det\, B|^{-1/2} \\ &= -\tfrac{1}{2} \mathrm{tr}\,(H_{yx} + H_{yy} D B^{-1}) |\det\, B|^{-1/2} \end{aligned}$$

and hence

$$i\hbar \frac{\partial u}{\partial t} = -i\hbar \tfrac{1}{2} \mathrm{tr}\,(H_{yx} + H_{yy} D B^{-1})\, u - \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} \frac{\partial S}{\partial t} e^{iS/\hbar} f.$$

Since $S$ satisfies the Hamilton-Jacobi equation $\partial_t S + H(t, x, \partial_x S) = 0$ this proves the statement whenever $\psi_{t_0}^t$ admits a generating function. The general case follows since both sides of the equation depend continuously on $H$. □

# 6 Geometric Quantization

A time dependent Hamiltonian $H$ on $\mathbb{R}^{2n}$ determines an evolution system on $W = \mathbb{R}^{2n} \times \mathrm{U}(1)$ via the formula

$$g_{t_0}^{t_1}(z_0, u_0) = \left(\psi_{t_0}^{t_1}(z_0), u_0 e^{i\phi(c)/\hbar}\right)$$

for $(z_0, u_0) \in W = \mathbb{R}^{2n} \times \mathrm{U}(1)$ where $\psi_{t_0}^{t_1}$ is the evolution system generated by $H$, $\phi(c)$ is the symplectic action integral evaluated at the curve $c(t) = \psi_{t_0}^{t}(z_0)$. If the generating function $S = \phi(c)$ of $\psi_{t_0}^{t_1}$ is defined then

$$g_{t_0}^{t_1}(z_0, u_0) = \left(\psi_{t_0}^{t_1}(z_0), u_0 e^{iS(x_0,x_1)/\hbar}\right) \tag{9}$$

where $z_j = (x_j, y_j)$, $z_1 = \psi_{t_0}^{t_1}(z_0)$. The group $\mathrm{ESp}(W, \hbar)$ of all diffeomorphisms of $W$ of form $g_{t_0}^{t_1}$ where $H$ runs over the time dependent (inhomogeneous) quadratic Hamiltonians $\mathbb{R} \to \mathcal{F}_2$ is called the **extended symplectic group**. The various groups $\mathrm{ESp}(W, \hbar)$ depend set-theoretically on $\hbar$ but are isomorphic as abstract groups. There is a central extension

$$1 \to \mathrm{U}(1) \to \mathrm{ESp}(W, \hbar) \to \mathrm{ASp}(\mathbb{R}^{2n}) \to 1$$

where $\mathrm{ASp}(\mathbb{R}^{2n})$ denotes the **affine symplectic group**; the projection is given by $g_{t_0}^{t_1} \mapsto \psi_{t_0}^{t_1}$ and the $\mathrm{U}(1)$ subgroup consists of those $g_{t_0}^{t_1}$ where $H$ is constant.

If the Hamiltonian $H$ is time independent then the corresponding evolution systems $\psi_{t_0}^{t_1}$ and $g_{t_0}^{t_1}$ are flows: denote by $X_H$ and $Y_H$ the vector fields generating these flows. Then $X_H$ is the Hamiltonian vector field of $H$, and $Y_H$ is a lift of $X_H$ to $L$. The Lie algebra to $\mathrm{ASp}(\mathbb{R}^{2n})$ is the image of quadratic Hamiltonians under the representation $H \mapsto X_H$ but this representation is not faithful as the constant Hamiltonians map to zero. However the representation $H \mapsto Y_H$ *is* faithful. Differentiating gives the following

**Proposition 6.1** *The vector field $Y_H$ on $W$ is given by*

$$Y_H(z, u) = (X_H(z), uis_H/\hbar), \qquad s_H = \langle y, \partial_y H\rangle - H.$$

Souriau [25] and Kostant [15] describe the extended symplectic group as the group of bundle automorphisms of the $U(1)$ bundle $W \to \mathbb{R}^{2n}$ which cover affine symplectic transformations and preserve the connection form

$$\alpha = -\frac{i}{\hbar}\langle y, dx\rangle + u^{-1} du.$$

# 7 Representations

The group $\mathrm{EMp}(2n)$ of all unitary operators of the form

$$U = \mathcal{U}(t_1, t_0, H) \tag{10}$$

where $H$ runs over the time dependent quadratic Hamiltonians and $t_1, t_0$ range over the real numbers form a finite dimensional group called the **extended metaplectic goup**. The subgroup $\mathrm{Mp}(2n)$ obtained by taking only homogeneous quadratic Hamiltonians $H$ in (10) is called the **metaplectic group**. The subgroup $\mathrm{HG}(2n)$ obtained by taking only affine Hamiltonians $H$ in (10) is called the **Heisenberg group**. By Theorem 5.2 the map

$$\mathrm{EMp}(2n) \to \mathrm{ESp}(W, \hbar) : \mathcal{U}(t_1, t_0, H) \mapsto \mathfrak{g}_{t_0}^{t_1}(H)$$

is a well-defined double cover (which depends on $\hbar$). This repesentation of the double cover of the symplectic group is called *Siegel-Shale-Weil representation* or the *metaplectic representation*. The restriction of the double cover to the Heisenberg group is injective and the resulting representation is called the **Heisenberg representation**.

Here is a more explicit description of the Heisenberg representation. If $H$ is an affine Hamiltonian with constant coefficients then

$$\mathcal{U}(t, t_0, H) = \mathcal{T}((t - t_0)H)$$

where

$$\mathcal{T}(H) = e^{-iH_0/\hbar - i\langle H_x, x\rangle/\hbar + i\langle H_x, H_y\rangle/2\hbar} f(x - H_y).$$

If $\Psi$ is a symplectic matrix then the map $H \mapsto \mathcal{T}(H \circ \Psi)$ is another such representation corresponding to the same value of Planck's constant $\hbar$. By the Stone-von Neumann theorem these representations are unitarily isomorphic. In other words there exists a unitary operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, unique up to multiplication by a complex number of modulus 1, such that

$$\mathcal{T}(H \circ \Psi) = U^{-1} \circ \mathcal{T}(H) \circ U.$$

Such an intertwining operator $U$ may be taken as a lift of $\Psi$ to the metaplectic group. This is apparently how the metaplectic representation was discovered (see [23]). The elements of the metaplectic representation are thus viewed as intertwining operators of various incarnations of the Heisenberg representation. See [20] for an exposition in terms of co-adjoint orbits and polarizations.

# 8 Quantum field theory

By generalizing from affine symplectomorphisms to affine symplectic relations it should be possible to generalize the extended metaplectic representation to the *extended metaplectic functor*. An *extended Lagrangian subspace* is a Legendrian submanifold of $W$ which covers an affine Lagrangian subspace of $\mathbb{R}^{2n}$. A quadratic function $S : \mathbb{R}^n \to \mathbb{R}$ determines an extended Lagrangian subspace $L(S)$ via

$$L(S) = \{(x, y, u) \in W : y = \partial_x S(x),\ u = e^{iS(x)/\hbar}\}.$$

An element of the extended symplectic group can be interpreted as an extended Lagrangian subspace of the external tensor product $W^* \otimes W$ over $\bar{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$. (The bar indicates that the sign of the symplectic form in the first factor is reversed.) More generally given circle bundles $W_0 \to \mathbb{R}^{2n_0}$ and $W_1 \to \mathbb{R}^{2n_1}$ as in Section 6 let $W_{01} \to \bar{\mathbb{R}}^{2n_0} \times \mathbb{R}^{2n_1}$ be endowed with the connection form $\alpha_1 - \alpha_0$. Then an *extended symplectic relation* is an extended Lagrangian subspace of $\bar{\mathbb{R}}^{2n_0} \times \mathbb{R}^{2n_1}$. Extended Lagrangian subspaces appear as the special case $n_0 = 0$. The extended metaplectic functor assigns to each extended symplectic relation a distribution on $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1}$, determined by the relation only up to a sign, and respecting the operation of composition defined in section 2. In the case of an extended symplectomorphism $g_{t_0}^{t_1}(H)$ the distribution is the distribution kernel of $\mathcal{U}(t_0, t_1, H)$. For a quadratic generating function $S(x)$ the distribution is $e^{iS(x)/\hbar}$ multiplied by a normalizing factor. Composition of extended symplectic relations corresponds to composition of distribution kernels; there should be a formula like

$$\mathcal{U}(R_{01} \# R_{12}) = \mathrm{tr}(\mathcal{U}(R_{01}) \otimes \mathcal{U}(R_{12})).$$

The extended metaplectic functor should give a simple model of Segal's axioms for topological quantum field theory. Taking the homology of a Riemann surface as the underlying symplectic vector space should lead to a $(2+1)$-dimensional theory. This is what Atiyah calls the Abelian case (without the lattice).

# References


[1] S. Albevario and R. Hoegh-Krohn, Feynman path integrals and the corresponding method of stationary phase, in *Feynman Path Integrals*, ed.

S. Albevario et al, Springer Lecture Notes in Physics **106** (1978) 3-57.

[2] M.F. Atiyah, *The Geometry and Physics of Knots*, Cambridge University Press, 1990.

[3] C. Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold, *Invent. Math.* **73** (1983), 33–49.

[4] C.C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions of Hamiltonian equations, *Commun. Pure Appl. Math.* **37** (1984), 207–253.

[5] I. Daubechies and J.R. Klauder, Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians, *Phys. Rev Letters* **52** (1984) 1161; *J. Math. Physics* **26** (1985) 2239-2256.

[6] I. Daubechies, J.R. Klauder, and T. Paul, Wiener measures for path integrals with affine kinematic variables, *J. Math. Physics* **28** (1987) 85-102.

[7] P.A.M. Dirac, The Lagrangian in quantum mechanics, Phys. Zeitschr. d. Sovyetunion **3** (1933) 64-72.

[8] J.J. Duistermaat, On the Morse index in variational calculus, *Advances in Mathematics* **21** (1976), 173–195.

[9] R.P. Feynman, Space-time approach to non-relativistic quantum mechanics, *Rev. Mod. Phisics* **20** (1948), 367–387.

[10] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw-Hill, 1965.

[11] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, AMS Math Surveys **14**, 1977.

[12] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, 1984.

[13] L. Hörmander, Fourier integral operators I, *Acta Math.* **127** (1971), 79–183.

[14] J.B. Keller, Corrected Bohr-Sommerfeld quantum conditions for non-separable systems, *Annals of Physics* **4** (1958), 180–188.

[15] B. Kostant, Quantization and unitary representations, in *Modern Analysis and its Applications* Springer Lecture Notes in Math **170** (1970) 87-207.

[16] J. Leray, *Lagrangian Analysis and Quantum Mechanics*, MIT press, 1981.

[17] J.W. Robbin and D.A. Salamon, Path integrals on phase space and the metaplectic representation, Preprint, 1992.

[18] J.W. Robbin, and D.A. Salamon, The spectral flow and the Maslov index, Preprint 1992.

[19] J.W. Robbin, and D.A. Salamon, A Maslov index for paths, Preprint 1992.

[20] P.L. Robinson and J.H. Rawnsley, *The metaplectic representation, $Mp^C$ structures and geometric quantization* , Memoirs of the American Mathematical Society **81**, 1989.

[21] D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, to appear in Comm. Pure Appl. Math.

[22] I.E. Segal, Foundations of the theory of dynamical systems of infinitely many degrees of freedom (I), *Mat. Fys. Medd. Danske Vid. Selsk.* **31** (1959) 1-39.

[23] I.E. Segal, Transforms for operators and symplectic automorphisms over a locally compact abelian group, *Math. Scand.* **13** (1963) 31-43.

[24] D. Shale, Linear symmetries of free boson fields, *Trans. AMS* **103** (1962), 149–167.

[25] J.M. Souriau, *Structures des Systemes Dynamiques*, Dunod, Paris, 1970.

[26] J.M. Souriau, Construction explicite de l'indice de Maslov. *Group Theoretical Methods in Physics* Springer Lecture Notes in Physics **50** (1975), 117–148.

[27] A. Weil, Sur certaine groupes d'operateurs unitaires, *Acta math* **111** (1964), 143–211.

[28] A. Weinstein, *Lectures on Symplectic Manifolds*, AMS Reg. Conf. Ser. Math. **29**, 1977.