

PROPAGATION IN HAMILTONIAN DYNAMICS AND RELATIVE SYMPLECTIC HOMOLOGY

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1. INTRODUCTION

1.1. **Stable propagation.** Let $M := U^*\mathbb{T}^n$ be the open unit cotangent bundle of the n -dimensional Euclidean torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We express the elements of M in terms of the canonical coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, where $q_i \equiv q_i + 1$. Thus we identify M with $\mathbb{T}^n \times \mathbb{D}^n$, where $\mathbb{D}^n = \{|p| < 1\}$ is the open unit ball in \mathbb{R}^n and $|v|$ stands for the Euclidean norm of a vector $v \in \mathbb{R}^n$. Consider the space \mathcal{H} of all smooth compactly supported functions on $[0, 1] \times M$. Every function $H \in \mathcal{H}$ gives rise to the Hamiltonian system on M

$$(1) \quad \dot{q} = \frac{\partial H}{\partial p}(t, q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(t, q, p).$$

The flow h_t which sends any initial condition $x(0) = (q(0), p(0))$ to the solution $x(t) = (q(t), p(t))$ at time t is called the **Hamiltonian flow** (or **isotopy**) generated by H , and the time-one-map h_1 is called the **Hamiltonian diffeomorphism** generated by H . The set of all Hamiltonian diffeomorphisms of M form a group denoted by \mathcal{D} .

Let $f_* = \{f_k\}_{k=1,2,\dots}$ be an infinite sequence of Hamiltonian diffeomorphisms. We consider f_* as a dynamical system as follows. Put $f^{(k)} = f_k \cdots f_1$. This sequence is called the **evolution** of f_* . The orbit $\{x_k\}_{k \in \mathbb{N}}$ of a point $x \in M$ is defined by $x_k = f^{(k)}x$. Classical dynamical systems (iterations of a single map f) correspond to constant sequences $f_k \equiv f$. Sequential systems arise naturally as perturbations of the classical ones. In a number of interesting situations stability has been observed in the sense that these perturbations inherit dynamical properties of the original system (cf. [PR]). In this paper we present a new stability phenomenon of this type – stable propagating behaviour – which we are going to describe next.

Every Hamiltonian diffeomorphism h has a *canonical* lift \tilde{h} to the universal cover $\widetilde{M} = \mathbb{R}^n \times \mathbb{D}^n$ of M . Throughout we denote

$$Q := \left\{ (q, p) \in \widetilde{M} \mid |p| < 1, \max_i |q_i| \leq 1/2 \right\};$$

this is a fundamental domain of the covering.

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Definition. A sequential system f_* **propagates to infinity with speed** (at least) c if for every vector $v \in \mathbb{R}^n$ with $|v| \leq c$ there exists a sequence of points $\tilde{x}_k \in Q$ such that the sequence $(p_k, q_k) := \tilde{f}^{(k)}(\tilde{x}_k)$ satisfies

$$\lim_{k \rightarrow \infty} \frac{q_k}{k} = v.$$

In other words, the projection to \mathbb{R}^n of $\tilde{f}^{(k)}(Q)$ covers the Euclidean ball of radius kc up to an error which is small with respect to k .

Let us illustrate this notion in the following simple example. Consider a Hamiltonian function $H \in \mathcal{H}$ which depends on momenta variables only: $H = H(p)$. The lift of the corresponding Hamiltonian flow to the universal cover is given by $\tilde{h}^t(q, p) = (q + t\nabla H(p), p)$. Therefore the projection of $\tilde{h}^k(Q)$ to \mathbb{R}^n coincides up to a bounded error with the set kI , where I denotes the image of the gradient map $p \mapsto \nabla H(p)$. Assume now that $H(0) > c$ for some $c > 0$. We claim that I contains the Euclidean ball of radius c centered at zero. Indeed, for every $v \in \mathbb{R}^n$ with $|v| \leq c$ the function $F(p) := H(p) - pv$ satisfies $F(0) > c$ and $F \leq c$ near $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Therefore F attains its maximum at a point $p_0 \in \mathbb{D}^n$. Hence $\nabla H(p_0) = v$ and the claim follows. This shows that in our example we have propagation with speed c .

The group \mathcal{D} carries a remarkable *biinvariant* metric ρ called *Hofer's metric* (see [H]). The corresponding geometry provides us with a suitable language for the study of various stable phenomena in Hamiltonian dynamics. Given a diffeomorphism $f \in \mathcal{D}$, write $\rho(\text{id}, f)$ for $\inf(\max F - \min F)$ where the infimum is taken over all Hamiltonians $F \in \mathcal{H}$ generating f . Define Hofer's distance $\rho(f, g)$ between two elements $f, g \in \mathcal{D}$ as $\rho(\text{id}, fg^{-1})$.

Theorem A (Stable propagation). *Let $0 < a < c$ and suppose that h is a Hamiltonian diffeomorphism generated by a Hamiltonian $H \in \mathcal{H}$ such that $H(t, q, 0) \geq c$ for all t and q . Let f_* be any sequential system such that $\rho(f_i, h) < a$ for all $i \in \mathbb{N}$. Then f_* propagates to infinity with speed $c - a$.*

Theorem A will be proved in Section 2.3 below. Interestingly enough, such a propagating behaviour may be completely destroyed by an appropriate arbitrarily C^∞ -small dissipative perturbation even in the framework of classical dynamics. In Section 2.4 below we elaborate this in the case $n = 1$. We show that every Hamiltonian diffeomorphism h generated by the Hamiltonian $H = H(p)$ admits an arbitrarily small smooth perturbation f such that the images $\tilde{f}^k(Q)$, $k \in \mathbb{N}$, of the set Q under the iterates of \tilde{f} remain in a compact part of \tilde{M} .

Note that Theorem A does not provide any information about propagation of *individual* trajectories on the universal cover (and we doubt that such information is available at

all in this generality). The situation improves when one considers sequences $f_* = \{f_i\}$ which roughly speaking are uniformly distributed with respect to some “nice” measure on \mathcal{D} whose support is close to h in the sense of Hofer’s metric. It turns out that such sequential systems have trajectories which propagate to infinity with constant velocity. We refer the reader to Section 2.3 for the details.

1.2. Noncontractible closed orbits. The main tool for studying stable propagation as described in the previous section is an existence result for noncontractible periodic solutions of compactly supported Hamiltonian systems under quite robust assumptions on the Hamiltonian functions. This result is based on Floer homology filtered by the symplectic action and its proof is quite involved. A solution $x(t) = (q(t), p(t))$ of a Hamiltonian system generated by a function $H \in \mathcal{H}$ is called **periodic** if $p(1) = p(0)$ and $q(1) = q(0) + e$ for some integer vector $e \in \mathbb{Z}^n$. The lattice \mathbb{Z}^n is identified in a natural way with the fundamental group of M , hence we refer to e as the **homotopy class** of the solution. An important quantity associated to a periodic solution is its **action**

$$\mathcal{A}_H(x) = \int_0^1 \left(H(t, q(t), p(t)) - \sum_{i=1}^n p_i(t) \dot{q}_i(t) \right) dt.$$

Denote by $Z \subset M$ the zero section $\{p = 0\}$.

Theorem B. *For every compactly supported smooth Hamiltonian function $H \in \mathcal{H}$ and every $e \in \mathbb{Z}^n$ such that*

$$(2) \quad |e| \leq c := \inf_{[0,1] \times Z} H,$$

the Hamiltonian system (1) has a periodic solution $x(t)$ in the homotopy class e with action $\mathcal{A}_H(x) \geq c$.

The result above is *sharp* in the following sense. Firstly, the inequalities in Theorem B which guarantee the existence of periodic solutions in the class e cannot be improved. For instance, for every $\varepsilon > 0$ it is easy to produce a Hamiltonian of the form $H = H(|p|)$ whose restriction to $[0, 1] \times Z$ equals $1 - \varepsilon$ and such that all periodic solutions are contractible (such a function $H(|p|)$ can be obtained from $1 - |p|$ by an appropriate smoothing). Secondly, the zero section Z in the inequality (2) cannot be replaced by an arbitrary smooth section. Consider, for instance an arbitrary C^∞ -small perturbation $S = \{p = u(q)\}$ of Z such that the 1-form $u(q) dq$ on \mathbb{T}^n is **not** closed (in symplectic terms this means that the section S is non-Lagrangian; this is only possible when $n \geq 2$).

Theorem C. *Given any $c > 0$, there exists a Hamiltonian function $H \in \mathcal{H}$ such that $H(t, x) \geq c$ for every $t \in [0, 1]$ and every $x \in S$ and such that every 1-periodic solution of (1) is contractible.*

If one takes S as the graph of an exact 1-form on \mathbb{T}^n , the assertion of Theorem B remains valid with Z replaced by S . Thus the existence mechanism for noncontractible periodic solutions described in Theorem B is quite sensitive to the choice of a subset where the Hamiltonian is large enough. This phenomenon will be studied below in terms of a *relative symplectic capacity* (see Section 3).

On the other hand, the statement of Theorem B is *robust* from the following viewpoint: if the restriction of a Hamiltonian H to $[0, 1] \times Z$ is bigger than a certain positive number, (most of) the periodic solutions guaranteed by the theorem persist under C^0 -perturbations of the Hamiltonian. This robustness plays a crucial role in the study of stable propagation.

In Section 3 below we prove Theorem B and its generalization where the torus is replaced by a hyperbolic manifold. The proof uses Floer homology for the action functional on the space of noncontractible loops. The main difficulty we have to go round is as follows. Since we deal with compactly supported Hamiltonians, noncontractible solutions are “non-essential” from the viewpoint of Floer homology – they may not persist under deformations of the Hamiltonian. In brief, the idea of the proof can be described as follows. One can squeeze the Hamiltonian function H between two more or less standard functions, $H_- \leq H \leq H_+$ where H_- and H_+ depend only on $|p|$. The filtered Floer homologies of H_- and H_+ as well as the natural morphism between them can be computed explicitly. This morphism turns out to be nontrivial, and since it factors through the Floer homology of H , we get nontrivial information about the noncontractible periodic solutions corresponding to H . The calculations are quite involved, hence we introduce a convenient algebraic tool – relative symplectic homology – which helps us perform them in a more organized way.

Noncontractible orbits of Hamiltonian systems were studied earlier in a number of interesting situations. In a beautiful paper [GL] Gatiien and Lalonde considered the following setting. Let L_0 and L_1 be two disjoint closed Lagrangian submanifolds of a symplectic manifold. Assume that H is an autonomous Hamiltonian function which is “small” on L_0 and “large” on L_1 . It turns out that under certain additional assumptions of a topological nature one can prove the existence of noncontractible periodic orbits for the Hamiltonian flow generated by H . This result was the starting point for Theorem B. Another important idea of a symplectic capacity which feels the fundamental group is contained in Schwarz’s work [Sc]. We will develop it further in Section 3 below. Noncontractible orbits of autonomous Hamiltonians on cotangent bundles whose levels are starshaped were considered by a number of authors (see e.g. [C-1]). In contrast to our case, these closed orbits

are homologically essential in the sense of Floer homology. Let us mention finally that the interest in noncontractible periodic orbits on cotangent bundles comes from classical mechanics where one considers Hamiltonians $H(t, q, p)$ which are convex with respect to the momenta variable p . In this case the existence of closed orbits in given homotopy classes can be derived with methods from the classical calculus of variations.

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2. STABLE PROPAGATION IN SEQUENTIAL HAMILTONIAN DYNAMICS

In this section we study stable propagation along the lines mentioned in the Introduction. The main tool is Theorem B.

2.1. Preliminaries on Hamiltonian diffeomorphisms. Let M be an open manifold (i.e. M is connected, noncompact and has no boundary) and denote by

$$\pi : \widetilde{M} \rightarrow M$$

its universal cover. Let $\text{Diff}_0(M)$ denote the group of compactly supported diffeomorphisms of M that are isotopic to the identity by isotopies with compact support in $[0, 1] \times M$. Then every $h \in \text{Diff}_0(M)$ has a canonical lift $\widetilde{h} : \widetilde{M} \rightarrow \widetilde{M}$ to the universal cover. To see this, choose a compactly supported isotopy $[0, 1] \rightarrow \text{Diff}_0(M) : [0, 1] \mapsto h_t$ from $h_0 = \text{id}$ to $h_1 = h$. Given $\widetilde{x}_0 \in \widetilde{M}$, lift the path $[0, 1] \rightarrow M : t \mapsto h_t(\pi(\widetilde{x}_0))$ to a path $[0, 1] \rightarrow \widetilde{M} : t \mapsto \widetilde{x}(t)$ such that $\widetilde{x}(0) = \widetilde{x}_0$ and define $\widetilde{h}(\widetilde{x}_0) := \widetilde{x}(1)$. The following remark shows that this definition is independent of the choice of the isotopy $t \mapsto h_t$.

Remark 2.1.1. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ be two compactly supported isotopies such that $\varphi_1 = \psi_1 = h \in \text{Diff}_0(M)$. Given a point $x \in M$, consider the paths $t \mapsto \varphi_t(x)$ and $t \mapsto \psi_t(x)$ connecting x to $h(x)$. We claim that they are homotopic with fixed endpoints. To see this choose a path $[0, 1] \rightarrow M : s \mapsto \gamma(s)$ such that $\gamma(0) = x$ and $\gamma(1)$ lies outside the support of the isotopies. Looking at $\varphi_t(\gamma(s))$ and $\psi_t(\gamma(s))$ it is easy to see that both paths $t \mapsto \varphi_t(x)$ and $t \mapsto \psi_t(x)$ are homotopic with fixed endpoints to the path Γ obtained by going first from x to $\gamma(1)$ along γ and then from $\gamma(1)$ to $h(x)$ along $h(\gamma^{-1})$. Here γ^{-1} stands for the reverse of γ .

Now suppose that M is equipped with a symplectic form ω and denote by $\mathcal{D} \subset \text{Diff}_0(M)$ the group of Hamiltonian diffeomorphisms that are generated by compactly supported Hamiltonian functions on $[0, 1] \times M$. Given a compact subset $A \subset M$ and a real number c let us denote by $\mathcal{D}_c = \mathcal{D}_c(M, A) \subset \mathcal{D}$ the subset of Hamiltonian diffeomorphisms that

are generated by compactly supported Hamiltonian functions $H \in C_0^\infty([0, 1] \times M)$ with $\inf_{[0,1] \times A} H \geq c$. As before denote by ρ the Hofer metric on \mathcal{D} .

Proposition 2.1.2. *Let $c > a$ be positive numbers, and let $f, g \in \mathcal{D}$ be Hamiltonian diffeomorphisms with $f \in \mathcal{D}_c$ and $\rho(f, g) < a$. Then $g \in \mathcal{D}_{c-a}$.*

Proof. This is an immediate consequence of the following product formula. If the functions Φ_t and Ψ_t generate Hamiltonian flows φ_t and ψ_t , respectively, then the product flow $\varphi_t \circ \psi_t$ is generated by the Hamiltonian $\Phi_t + \Psi_t \circ \varphi_t^{-1}$. \square

In what follows we shall deal also with time periodic Hamiltonians, namely functions H satisfying $H(t, \cdot) = H(t + 1, \cdot)$. It is useful to think of these Hamiltonians as smooth functions $H : S^1 \times M \rightarrow \mathbb{R}$, where we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$. The following proposition shows that every $h \in \mathcal{D}_c$ can be generated by a periodic Hamiltonian bounded below by c on $S^1 \times A$.

Proposition 2.1.3. (Periodic Hamiltonians) *Let M be an open symplectic manifold and $A \subset M$ be a compact subset. Let h be a Hamiltonian diffeomorphism of M generated by a Hamiltonian $H \in C_0^\infty([0, 1] \times M)$ with $\inf_{[0,1] \times A} H \geq c$. Then there exists a Hamiltonian $\bar{H} \in C_0^\infty(S^1 \times M)$ with $\inf_{S^1 \times A} \bar{H} \geq c$ that generates h .*

Proof. Let h_t denote the Hamiltonian isotopy generated by H and f_t denote the Hamiltonian flow generated by a time independent compactly supported function $F : M \rightarrow \mathbb{R}$. Let $\tau : [0, 1] \rightarrow [0, 1]$ be a smooth nondecreasing function which equals 0 near $t = 0$ and equals 1 near $t = 1$. Then the Hamiltonian isotopy

$$\bar{h}_t := f_{t-\tau(t)} \circ h_{\tau(t)}$$

is generated by the Hamiltonian functions

$$\bar{H}_t := F + \tau'(t)(H_{\tau(t)} - F) \circ f_{\tau(t)-t}.$$

The function \bar{H}_t equals F near $t = 0$ and $t = 1$ and hence defines a smooth Hamiltonian on $S^1 \times M$. Moreover, $\bar{h}_1 = h_1 = h$. If F is chosen to be equal to c in a neighbourhood of A , then f_t is equal to the identity on A and $H_{\tau(t)} - F$ is nonnegative on A , and hence $\inf_A \bar{H}_t \geq c$ for every t . \square

In the remainder of this section we assume that $M = U^*\mathbb{T}^n$ is the open unit cotangent bundle and $A = Z = \mathbb{T}^n \subset U^*\mathbb{T}^n$ is the zero section. Given $e \in \mathbb{Z}^n$, we denote by T_e the deck transformation $(q, p) \rightarrow (q + e, p)$ of the covering. Note that 1-periodic solutions of the Hamiltonian system associated to H are in one-to-one correspondence with the fixed points of h . The homotopy class of the periodic solution $x(t)$ corresponding to the fixed point $x \in M$ (or, in brief, *the homotopy class of the fixed point x*) can be

determined as follows. Pick a lift y of x . Then $\tilde{h}(y) = T_e(y)$ for some $e \in \mathbb{Z}^n$ and this e is the homotopy class in question. In this terminology Theorem B asserts that every diffeomorphism $h \in \mathcal{D}_c = \mathcal{D}_c(U^*\mathbb{T}^n, \mathbb{T}^n)$ has a fixed point in the homotopy classes e for every e such that $|e| \leq c$. It turns out that the same result allows us to get information on the fixed points of the iterates h^k for $k \in \mathbb{N}$.

Proposition 2.1.4. *If $h \in \mathcal{D}_c$ then $h^k \in \mathcal{D}_{kc}$ for every $c > 0$ and every $k \in \mathbb{N}$. In particular, h^k has a fixed point in every homotopy class $e \in \mathbb{Z}^n$ such that $|e| \leq kc$.*

Proof. We have seen in Proposition 2.1.3 that, for every $h \in \mathcal{D}_c$, one can find a function $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ which is 1-periodic in time, satisfies $\inf_{\mathbb{R} \times A} H \geq c$, and generates h as the time-1-map. It follows that the k th iterate h^k is generated by $kH(kt, x)$, hence $h^k \in \mathcal{D}_{kc}$, and hence the result follows from Theorem B. \square

2.2. Preliminaries from ergodic theory. Let X be a compact metrizable topological space and $f : X \rightarrow X$ be a homeomorphism. Let \mathcal{M} denote the set of Borel probability measures on X and $\mathcal{M}(f) \subset \mathcal{M}$ denote the subset of f -invariant Borel probability measures. Both sets are convex and compact with respect to the weak-* topology ($\mu_i \rightarrow \mu$ iff $\int u d\mu_i \rightarrow \int u d\mu$ for every continuous function $u : X \rightarrow \mathbb{R}$). An f -invariant measure $\mu \in \mathcal{M}(f)$ is called **ergodic** if every Borel set $\Lambda \subset X$ such that $f(\Lambda) = \Lambda$ has measure $\mu(\Lambda) \in \{0, 1\}$.

Theorem 2.2.1. (Birkhoff's Ergodic Theorem [CFS]) *If $\mu \in \mathcal{M}(f)$ is an ergodic f -invariant Borel probability measure and $u \in C^0(X)$, then there exists a Borel set $Y \subset X$ such that $\mu(Y) = 1$ and*

$$\int u d\mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(f^k(x))$$

for every $x \in Y$.

Remark 2.2.2. Let \mathcal{X} be a Banach space and $\mathcal{K} \subset \mathcal{X}^*$ be a weak-* compact convex subset of its dual space. A point $x^* \in \mathcal{K}$ is called **extremal** if, for all $y^*, z^* \in \mathcal{K}$ and all real numbers t ,

$$x^* = (1-t)y^* + tz^*, \quad 0 < t < 1 \quad \implies \quad y^* = z^*.$$

The Krein–Milman theorem [R, p. 242] asserts that every weak-* compact convex subset of \mathcal{X}^* is equal to the weak-* closure of the convex hull of its extremal points. In particular, every compact convex subset $\mathcal{K} \subset \mathbb{R}^n$ with nonempty interior has at least $n + 1$ extremal points. Namely, if there were less than or equal to n extremal points then \mathcal{K} would be contained in an affine subspace of dimension less than n and so have empty interior.

Remark 2.2.3. Every extremal point of $\mathcal{M}(f)$ is ergodic. To see this suppose that $\mu \in \mathcal{M}(f)$ is not ergodic. Then there exists an invariant Borel set $\Lambda = f(\Lambda)$ such that $0 < \mu(\Lambda) < 1$. Let $t := \mu(\Lambda)$. Then $\mu = t\mu_\Lambda + (1-t)\mu_{X \setminus \Lambda}$ where $\mu_\Lambda \in \mathcal{M}(f)$ is defined by $\mu_\Lambda(A) := \mu(\Lambda \cap A)/\mu(\Lambda)$ and similarly for $\mu_{X \setminus \Lambda}$. Hence μ is not an extremal point of $\mathcal{M}(f)$.

Remark 2.2.4. A homeomorphism $f : X \rightarrow X$ is called **uniquely ergodic** if $\mathcal{M}(f)$ consists of a single point, i.e. there exists a unique f -invariant Borel probability measure μ on X . In this case, by Remark 2.2.3, the invariant measure μ is necessarily ergodic.

Example 2.2.5. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ be a vector with rationally independent components. Then the homeomorphism $\varphi : \mathbb{T}^d \rightarrow \mathbb{T}^d$ induced by the map $\mathbb{R}^d \rightarrow \mathbb{R}^d : y \mapsto y + \alpha$ is uniquely ergodic and the unique φ -invariant measure is the Lebesgue measure (cf. [CFS]).

Lemma 2.2.6. *Let $f : X \rightarrow X$ be a homeomorphism of a compact metric space and $u : X \rightarrow \mathbb{R}^n$ be a continuous function. Consider the linear projection*

$$\mathcal{M}(f) \rightarrow \mathbb{R}^n : \mu \mapsto R(\mu) := \int u d\mu.$$

Let r be an extremal point of the compact convex set $K := R(\mathcal{M}(f)) \subset \mathbb{R}^n$. Then there exists an ergodic f -invariant Borel probability measure $\mu \in \mathcal{M}(f)$ such that $r = \int u d\mu$.

Proof. The set $\mathcal{M}_r(f) := \{\mu \in \mathcal{M}(f) \mid R(\mu) = r\}$ is a nonempty weak-* compact convex subset of the dual space of $C^0(X)$. Hence, by the Krein–Milman theorem, it has an extremal point μ (Remark 2.2.2). We prove that μ is an extremal point of $\mathcal{M}(f)$. To see this, suppose that $\mu = (1-t)\mu_0 + t\mu_1$ such that $\mu_0, \mu_1 \in \mathcal{M}(f)$ and $0 < t < 1$. We must prove that $\mu_0 = \mu_1$. To see this, let $r_0 := R(\mu_0)$ and $r_1 := R(\mu_1)$. Then $r = (1-t)r_0 + tr_1$ and $r_0, r_1 \in K$. Since r is an extremal point of K it follows that $r_0 = r_1 = r$ and hence $\mu_0, \mu_1 \in \mathcal{M}_r(f)$. Since μ is an extremal point of $\mathcal{M}_r(f)$ this implies that $\mu_0 = \mu_1$. Thus we have proved that μ is an extremal point of $\mathcal{M}(f)$ and hence, by Remark 2.2.3, it is an ergodic f -invariant Borel probability measure. \square

2.3. Stable propagation revisited. Let us now return to the case where $M = U^*\mathbb{T}^n$ is the open unit cotangent bundle of the n -torus and \mathcal{D} is the group of compactly supported Hamiltonian diffeomorphisms of M . For $a > 0$ and $h \in \mathcal{D}$ denote by

$$B(h, a) := \{f \in \mathcal{D} \mid \rho(f, h) < a\}$$

the open ball of radius a in Hofer’s metric. Recall that Theorem A states the following. Let $c > a > 0$ be real numbers, $h \in \mathcal{D}_c$ be a Hamiltonian diffeomorphism, and f_* be a sequential system such that $f_i \in B(h, a)$ for every $i \in \mathbb{N}$. Then f_* propagates to infinity with speed $c - a$. We now prove Theorem A, assuming Theorem B.

Proof of Theorem A. Write $f_i = h\psi_i$ with $\rho(\text{id}, \psi_i) < a$. Then the evolution of f_* can be written as follows:

$$f^{(k)} = (h\psi_k h^{-1})(h^2\psi_{k-1}h^{-2}) \cdots (h^k\psi_1 h^{-k})h^k.$$

Combining the fact that ρ is biinvariant with the triangle inequality we get

$$\rho(f^{(k)}, h^k) < ka.$$

Now let $w \in \mathbb{R}^n$ be any vector such that $|w| \leq c - a$ and choose a sequence $v_k \in \mathbb{Z}^n$ such that

$$w = \lim_{k \rightarrow \infty} \frac{v_k}{k}, \quad |v_k| < k(c - a).$$

Since $h^k \in \mathcal{D}_{kc}$ (see Proposition 2.1.4) we obtain from Proposition 2.1.2 that $f^{(k)} \in \mathcal{D}_{k(c-a)}$. Hence, by Theorem B, there exists a point $\tilde{x}_k = (q_k, p_k) \in Q$ so that $\tilde{f}^{(k)}(\tilde{x}_k) = (q_k + v_k, p_k)$. Since the sequence q_k is bounded, we have $\lim_{k \rightarrow \infty} (q_k + v_k)/k = w$, and hence f_* propagates to infinity with speed $c - a$. \square

Now we turn to the study of the long time behaviour of *individual trajectories*. We start with the following general definition. Consider the trajectory $x_k = f^{(k)}(x)$ of a point $x \in M$ under a sequential system f_* . Let $\tilde{x} \in \widetilde{M}$ be a lift of x and consider the lifted trajectory $\tilde{x}_k = \tilde{f}^{(k)}(\tilde{x})$ in the universal cover.

Definition. A trajectory x_k of a sequential system with a lift $\tilde{x}_k = (q_k, p_k) \in \widetilde{M}$ is said to **propagate with velocity vector** $r \in \mathbb{R}^n$ if

$$\lim_{k \rightarrow \infty} \frac{q_k}{k} = r.$$

The Euclidean norm of r is called the **speed** of the trajectory x_k .

As an illustration, let us mention that every k -periodic orbit of a Hamiltonian diffeomorphism h in a nonzero homotopy class $e \in \mathbb{Z}^n$ propagates with velocity vector e/k . Note also that propagation in the universal cover \widetilde{M} corresponds to “rotation” in M . In classical dynamics the velocity vector of a propagating orbit is called the rotation vector.

Consider now the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Fix a vector $\alpha = (\alpha_1, \dots, \alpha_d)$ whose coordinates are independent over the rationals. Let $g : \mathbb{T}^d \rightarrow \mathcal{D}$ be a map which is continuous with respect to the C^0 -topology on \mathcal{D} . For a point $y \in \mathbb{T}^d$ define a sequential system

$$f_*(y) := \{g(y + k\alpha)\}_{k \in \mathbb{N}}.$$

Roughly speaking, the sequence $f_*(y)$ is “random” – it is uniformly distributed on the subset $g(\mathbb{T}^d)$. The next result is an improvement of Theorem A for (almost all) such sequences.

Theorem 2.3.1. *Let $0 < a < c$ be real numbers and $h \in \mathcal{D}_c$ be a Hamiltonian diffeomorphism. Let $g : \mathbb{T}^d \rightarrow \mathcal{D}$ be a continuous map whose image is contained in $B(h, a)$. Then, for set of Lebesgue measure one of points $y \in \mathbb{T}^d$, the sequential system $f_*(y)$ has at least $n + 1$ trajectories which propagate with speed at least $c - a$.*

In fact there exists a compact convex set $K \subset \mathbb{R}^n$ containing the closed ball of radius $c - a$ in \mathbb{R}^n centered at the origin so that the following holds: for every extremal point $r \in K$ there exists a subset $Y_r \subset \mathbb{T}^d$ of Lebesgue measure one such that, for every $y \in Y_r$, the system $f_(y)$ has a trajectory which propagates with velocity vector r .*

The basic feature of the irrational shift of the torus which is crucial for our purposes is unique ergodicity (see Example 2.2.5). The assertion of Theorem 2.3.1 continues to hold, and the proof remains the same, if one replaces the torus \mathbb{T}^d by an arbitrary compact metric space Y , the Lebesgue measure by any Borel probability measure σ , and the irrational shift by any σ -preserving uniquely ergodic homeomorphism of Y .

Proof of Theorem 2.3.1. Let \overline{M} denote the closed unit cotangent bundle and \widetilde{M} its universal cover. Write $Y := \mathbb{T}^d$ and $\varphi(y) := y + \alpha$ and denote by σ the Lebesgue measure on Y . Consider the skew-product map

$$\overline{M} \times Y \rightarrow \overline{M} \times Y : (x, y) \mapsto S(x, y) := (g(y)(x), \varphi(y))$$

and its canonical lift \widetilde{S} to $\widetilde{M} \times Y$. There exist functions $u, v : M \times Y \rightarrow \mathbb{R}^n$ so that

$$\widetilde{S}(q, p, y) = (q + u(q, p, y), v(q, p, y), \varphi(y)).$$

Here we slightly abuse notation and write $u(q, p, y)$ instead of $u(x, y)$, whenever $\tilde{x} = (q, p)$. The function u which measures displacement along the q -plane on \widetilde{M} will play an important role below. Note that u has compact support in $M \times Y$.

Let $\mathcal{M}(S)$ denote the set of all S -invariant Borel probability measures on $\overline{M} \times Y$. This space is convex and weak- $*$ compact (see Section 2.2). For a measure $\mu \in \mathcal{M}$ define its **rotation vector** $R(\mu) \in \mathbb{R}^n$ by

$$R(\mu) := \int u \, d\mu \in \mathbb{R}^n.$$

The map $R : \mathcal{M}(S) \rightarrow \mathbb{R}^n$ is affine and continuous. Hence its image $K := R(\mathcal{M}(S))$ is a compact convex subset of \mathbb{R}^n . This set has all the properties formulated in the theorem.

We prove that K contains a ball of radius $c - a$. Pick a vector $v \in \mathbb{R}^n$ with $|v| \leq c - a$ and a point $y \in Y$. Then, by Theorem A, there exists a sequence $x_k \in M$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} u(S^i(x_k, y)) = v.$$

Denote by $\delta_{i,k}$ the Dirac measure on $\overline{M} \times Y$ concentrated at $S^i(x_k, y)$ and consider the sequence of measures μ_k on $\overline{M} \times Y$ defined by

$$\mu_k := \frac{1}{k} \sum_{i=0}^{k-1} \delta_{i,k}.$$

By Alaoglu's theorem, this sequence has a limit point $\mu = \lim_{\nu \rightarrow \infty} \mu_{k_\nu}$ with respect to the weak-* topology. Since

$$\lim_{k \rightarrow \infty} \left(\int F \circ S d\mu_k - \int F d\mu_k \right) = \lim_{k \rightarrow \infty} \frac{F(S^k(x_k, y)) - F(x_k, y)}{k} = 0$$

for every continuous function $F : \overline{M} \times Y \rightarrow \mathbb{R}$ the limit point μ is S -invariant. Moreover, it satisfies

$$R(\mu) = \int u d\mu = \lim_{\nu \rightarrow \infty} \int u d\mu_{k_\nu} = \lim_{\nu \rightarrow \infty} \frac{1}{k_\nu} \sum_{i=0}^{k_\nu-1} u(S^i(x_{k_\nu}, y)) = v.$$

Hence $v \in K$.

We prove that the extremal points of K satisfy the requirements of the theorem. Let r be an extremal point of K . Then, by Lemma 2.2.6, there exists an ergodic measure $\mu_r \in \mathcal{M}(S)$ such that $R(\mu_r) = r$. By Birkhoff's Ergodic Theorem 2.2.1, there exists a Borel set $Z_r \subset \overline{M} \times Y$ such that $\mu_r(Z_r) = 1$ and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} u(S^i(z)) = \int u d\mu_r = r$$

for every $z \in Z_r$. Note that $Z_r \subset M \times Y$. Let $Y_r \subset Y$ be the image of Z_r under the obvious projection $\overline{M} \times Y \rightarrow Y$ and let σ_r be the pushforward of the measure μ_r (i.e. $\sigma_r(B) := \mu_r(\overline{M} \times B)$ for every Borel set $B \subset Y$). Then σ_r is a φ -invariant Borel probability measure on Y and so, by unique ergodicity, equals σ . Hence

$$\sigma(Y_r) = \sigma_r(Y_r) = \mu_r(\overline{M} \times Y_r) \geq \mu_r(Z_r) = 1.$$

Moreover, for every $y \in Y_r$ there exists a point $x \in M$ such that $(x, y) \in Z_r$. By definition of the set Z_r , the trajectory of the point x under the system $f_*(y)$ propagates with velocity vector r . \square

It is instructive to compare the proof above with Aubry–Mather theory for Hamiltonian diffeomorphisms of the annulus generated by a Hamiltonian which is convex with respect to the momentum p . Aubry and Mather constructed remarkable orbits which lie on special Cantor-type sets called cantori. These orbits propagate in our sense. Roughly speaking, these orbits are obtained as the limits of periodic ones. A crucial point for such a limiting procedure is the fact that, due to convexity, one can control the relative positions of closed

orbits in phase space. In our setting the Hamiltonians are not assumed to be convex, and we have no information on the positions of the closed orbits. We went round this difficulty by using an idea which goes back to Mather [M] – to look at limits of invariant measures which sit on periodic orbits provided by Theorem B.

2.4. A dissipative counter-example. Assume $n = 1$ so $M = S^1 \times (-1, 1)$ is the annulus. Let $H : (-1, 1) \rightarrow \mathbb{R}$ be any compactly supported function (which depends on the momenta variable only and has an arbitrarily large value at $p = 0$). Then the corresponding Hamiltonian diffeomorphism h has the form

$$h(q, p) = (q + H'(p), p),$$

where H' denotes the derivative of H . Recall that Q denotes the fundamental domain $Q := [-1/2, 1/2] \times (-1, 1)$ in the universal cover $\widetilde{M} = \mathbb{R} \times (-1, 1)$. We present an example of an arbitrarily small smooth perturbation f of h such that the images $\widetilde{f}^k(Q)$ under the iterates of \widetilde{f} remain in a compact part of \widetilde{M} , thus the propagating behaviour disappears. Assume without loss of generality that the support of $H = H(p)$ is contained in the open interval $(-1/2, 1/2)$ and let $\gamma := \max_p |H'(p)|$.

Let $u : [-1, 1] \rightarrow [-1, 1]$ be an orientation preserving diffeomorphism such that $u(s) > s$ for $-3/4 < s < 3/4$ and $u(s) = s$ for $|s| \geq 3/4$. Note that u can be chosen arbitrarily close to the identity. Choose $N \in \mathbb{N}$ such that $u^N(-2/3) > 2/3$.

Define a map $\varphi : M \rightarrow M$ by $\varphi(q, p) := (q, u(p))$ and let $f := \varphi h$. For a point $x = (q_0, p_0) \in Q$ consider its orbit $(q_i, p_i) := \widetilde{f}^i(x)$. We claim that $|q_i - q_0| \leq (N + 1)\gamma$ for all $x \in Q$ and $i \in \mathbb{Z}$. This universal estimate means the absence of propagation.

To prove this, note that h preserves the p -coordinate, and φ preserves the q -coordinate of each point. Moreover, the sequence p_i is nondecreasing. If $|p_0| \geq 3/4$ then $(p_i, q_i) = (p_0, q_0)$ for all $i \in \mathbb{Z}$. If $|p_0| < 3/4$ then $\lim_{i \rightarrow \pm\infty} p_i = \pm 3/4$. In this case let $j_0 \in \mathbb{Z}$ be the largest integer such that $p_{j_0} < -2/3$ and $j_1 > j_0$ be the smallest integer such that $p_{j_1} > 2/3$. Then $|j_1 - j_0| \leq N + 2$ and $q_i = q_{j_0}$ for $i \leq j_0$ and $q_i = q_{j_1}$ for $i \geq j_1$. For every $i \in [j_0, j_1 - 1]$ we have $|q_{i+1} - q_i| = |H'(p_i)| \leq \gamma$. Hence, for all $j, j' \in \mathbb{Z}$ such that $j' > j$,

$$|q_{j'} - q_j| \leq \sum_{i=j}^{j'-1} |q_{i+1} - q_i| \leq \sum_{i=j_0}^{j_1-1} |q_{i+1} - q_i| \leq (N + 1)\gamma.$$

3. RELATIVE SYMPLECTIC TOPOLOGY

Below we discuss existence and nonexistence results for noncontractible closed orbits in a more general topological framework. The main player in this section is a relative symplectic capacity – a symplectic invariant which provides a convenient language for

thinking about these results. Using this language, we prove Theorems B and C as stated in the Introduction.

3.1. Symplectic action. This is a preparatory section in which we set notation. Let (M, ω) be an open symplectic manifold. We assume throughout that the symplectic form ω is exact and fix a 1-form $\lambda \in \Omega^1(M)$ such that

$$d\lambda = \omega.$$

Let $S^1 := \mathbb{R}/\mathbb{Z}$ and denote the free loop space of M by $LM := C^\infty(S^1, M)$. We denote the set of free homotopy classes of loops in M by $\tilde{\pi}_1(M)$ and for $x \in LM$ we write $[x] \in \tilde{\pi}_1(M)$ for its free homotopy class. Given a subset $\alpha \subset \tilde{\pi}_1(M)$, we write

$$L_\alpha M := \{x \in LM \mid [x] \in \alpha\}.$$

We shall mostly consider single elements $\alpha \in \tilde{\pi}_1(M)$, however, for some of our applications it is useful to consider more general subsets of $\tilde{\pi}_1(M)$. We denote the space of compactly supported time dependent 1-periodic Hamiltonian functions on M by $\mathcal{H}(M) := C_0^\infty(S^1 \times M)$. We do not distinguish in notation between the function $H \in \mathcal{H}(M)$ and its lift $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ and, for $H \in \mathcal{H}(M)$ and $t \in \mathbb{R}$, we define $H_t : M \rightarrow \mathbb{R}$ by $H_t(x) := H(t, x)$. Every Hamiltonian function $H \in \mathcal{H}(M)$ determines a 1-periodic family of Hamiltonian vector fields $X_t = X_{t+1} \in \text{Vect}(M, \omega)$ via $\iota(X_t)\omega = -dH_t$. The space of 1-periodic solutions of the corresponding Hamiltonian differential equation representing a class in the set $\alpha \subset \tilde{\pi}_1(M)$ will be denoted by

$$(3) \quad \mathcal{P}(H; \alpha) := \{x \in L_\alpha M \mid \dot{x}(t) = X_t(x(t))\}.$$

The elements of $\mathcal{P}(H; \alpha)$ are the critical points of the symplectic action $\mathcal{A}_H : L_\alpha M \rightarrow \mathbb{R}$, defined by

$$(4) \quad \mathcal{A}_H(x) := \int_0^1 (H_t(x(t)) - \lambda(\dot{x}(t))) dt$$

for $x \in L_\alpha M$. The sole purpose of the 1-form λ is to fix a normalization of the symplectic action (which otherwise is only well defined up to an additive constant). Some of the invariants discussed in this paper do not depend on this normalization and this will be pointed out at the appropriate places. However, we do not indicate the dependence of \mathcal{A}_H on λ in the notation.

Remark 3.1.1. The notation in this section differs from the one used in Section 1.2, where the Hamiltonian functions are not required to be periodic in the t -variable. Proposition 2.1.3 shows that this does not effect the class of Hamiltonian diffeomorphisms to which the theory applies. It also does not effect the value of the symplectic action at

the periodic orbits. To see this, fix a compactly supported, but not necessarily periodic, Hamiltonian function $H \in C_0^\infty([0, 1] \times M)$ and let $[0, 1] \times M \rightarrow M : (t, x) \mapsto h_t(x)$ be the Hamiltonian isotopy generated by H . Fix any point $x_0 \in M$ and consider the path $x : [0, 1] \rightarrow M$ given by

$$x(t) := h_t(x_0).$$

Let $\gamma : [0, 1] \rightarrow M$ be a smooth path such that $\gamma(0)$ lies outside of the support of H and $\gamma(1) = x_0$. Differentiating the function $s \mapsto \mathcal{A}_H(x_s)$, where $x_s(t) := h_t(\gamma(s))$, we find

$$\mathcal{A}_H(x) = \int_0^1 (h^* \lambda - \lambda)(\dot{\gamma}(s)) ds.$$

Thus the action of x depends only on x_0 , h , and λ .

3.2. A relative symplectic capacity. Let M be an open symplectic manifold with symplectic form $\omega = d\lambda$ and $A \subset M$ be a compact subset. In this section we define a relative symplectic capacity

$$C(M, A) : 2^{\tilde{\pi}_1(M)} \times [-\infty, \infty) \rightarrow [0, \infty]$$

for the pair (M, A) . This capacity has the following feature: given $\alpha \subset \tilde{\pi}_1(M)$ and $a \in \mathbb{R} \cup \{-\infty\}$ such that $C(M, A; \alpha, a) < \infty$, every $H \in \mathcal{H}(M)$ with $\inf_{S^1 \times A} H > C(M, A; \alpha, a)$ must have a 1-periodic orbit representing one of the homotopy classes in α with symplectic action $\mathcal{A}_H(x) \geq a$. Moreover, $C(M, A; \alpha, a)$ is the optimal bound for the existence of such an orbit. More precisely, for $c > 0$ let

$$\mathcal{H}_c(M, A) := \left\{ H \in \mathcal{H}(M) \mid \inf_{S^1 \times A} H \geq c \right\}.$$

For a subset $\alpha \subset \tilde{\pi}_1(M)$ and a number $a \geq -\infty$ we define the **relative symplectic capacity** $C(M, A; \alpha, a) \geq 0$ by

$$C(M, A; \alpha, a) := \inf \left\{ c > 0 \mid \forall H \in \mathcal{H}_c(M, A) \exists x \in \mathcal{P}(H; \alpha) \text{ such that } \mathcal{A}_H(x) \geq a \right\}.$$

We use the convention that $\inf \emptyset = \infty$. The infimum in the definition of $C(M, A; \alpha, a)$ is always achieved (see Proposition 3.3.4 below).

In order to relax the notation we identify the single element subset $\{\alpha\} \subset \tilde{\pi}_1(M)$ with $\alpha \in \tilde{\pi}_1(M)$. Also, for $a = -\infty$ we abbreviate

$$C(M, A; \alpha) := C(M, A; \alpha, -\infty) = \inf \left\{ c > 0 \mid \mathcal{P}(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_c(M, A) \right\}.$$

Note that the invariant $C(M, A; \alpha)$ is independent of the choice of λ . It turns out that $C(M, A; \alpha)$ is *finite* in some interesting cases. To begin with let us consider the case

$A = \text{pt}$. Then the invariant $C(M, A; \alpha)$, with $\alpha = \tilde{\pi}_1(M)$, is analogous to the Hofer-Zehnder capacity [HZ]. The difference is that Hofer and Zehnder considered nonnegative and time-independent Hamiltonian functions. Lalonde [L] suggested to consider more general subsets A .

Here are some examples in which our capacity can be computed. Let X be a closed (i.e. compact without boundary) connected Riemannian manifold. Denote by

$$U^*X := \{v \in T^*X \mid |v| < 1\} \subset T^*X$$

the open unit cotangent bundle. This manifold is equipped with the canonical symplectic form $\omega_{\text{can}} = d\lambda_{\text{can}}$ of T^*X . We identify X with the zero section of U^*X and $\tilde{\pi}_1(X)$ with $\tilde{\pi}_1(U^*X)$. Moreover, in the case of the standard torus $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ there is a natural isomorphism

$$\tilde{\pi}_1(U^*\mathbb{T}^n) \cong \tilde{\pi}_1(\mathbb{T}^n) \cong \mathbb{Z}^n.$$

We denote the Euclidean norm of an integer vector $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ by

$$|k| := \sqrt{\sum_{j=1}^n k_j^2}.$$

Theorem 3.2.1. (i) *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the flat equilateral torus with the metric induced by the standard metric on \mathbb{R}^n . Then, for every $k \in \mathbb{Z}^n \cong \tilde{\pi}_1(U^*\mathbb{T}^n)$ and every $a \in \mathbb{R}$,*

$$C(U^*\mathbb{T}^n, \mathbb{T}^n; k, a) = \max\{|k|, a\}.$$

(ii) *Let X be a closed Riemannian manifold with negative sectional curvature. Then, for every $\alpha \in \tilde{\pi}_1(X) \cong \tilde{\pi}_1(U^*X)$ and every $a \in \mathbb{R}$,*

$$C(U^*X, X; \alpha, a) = \max\{\text{length}(\gamma_\alpha), a\},$$

where $\gamma_\alpha : S^1 \rightarrow X$ is the (unique up to time shift) closed geodesic that represents the homotopy class α .

3.3. Properties. In this section we establish some basic properties of the relative capacity and prove Theorems B and C, assuming Theorem 3.2.1 (i).

Proposition 3.3.1. (Monotonicity) *Let $A_2 \subset A_1 \subset M_1 \subset M_2$, where M_1 is an open subset of M_2 . Let $\iota_* : \tilde{\pi}_1(M_1) \rightarrow \tilde{\pi}_1(M_2)$ denote the map induced by the inclusion $\iota : M_1 \subset M_2$. Then, for $\alpha_i \in \tilde{\pi}_1(M_i)$ and $a_i \geq -\infty$, we have*

$$\iota_*^{-1}(\alpha_2) \subset \alpha_1, \quad a_1 \leq a_2 \quad \implies \quad C(M_1, A_1; \alpha_1, a_1) \leq C(M_2, A_2; \alpha_2, a_2).$$

In particular, if ι_* is injective then for every $\alpha \in \tilde{\pi}_1(M_1)$ and $a_1 \leq a_2$ we have

$$C(M_1, A_1; \alpha, a_1) \leq C(M_2, A_2; \iota_*(\alpha), a_2).$$

Proof. Let $c > C(M_2, A_2; \alpha_2, a_2)$. Then every Hamiltonian function $H \in \mathcal{H}_c(M_2, A_2)$ has a 1-periodic orbit representing a class in α_2 with action at least a_2 . In particular, for every $H \in \mathcal{H}_c(M_1, A_1)$ there exists a 1-periodic orbit $x : S^1 \rightarrow M_1$ such that $\iota_*[x] \in \alpha_2$ (and hence $[x] \in \alpha_1$) and $\mathcal{A}_H(x) \geq a_2 \geq a_1$. This shows that $c \geq C(M_1, A_1; \alpha_1, a_1)$. \square

Our next result provides a useful condition which guarantees that the relative capacity is trivial.

Proposition 3.3.2. (Displacement) *Suppose there exists a compactly supported Hamiltonian isotopy $f_t : M \rightarrow M$, $0 \leq t \leq 1$, such that $f_1(A) \cap A = \emptyset$ and all 1-periodic orbits of f_t are contractible. Then, for every constant $c > 0$, there exists a Hamiltonian $H \in \mathcal{H}_c(M, A)$ such that $\mathcal{P}(H; \alpha) = \emptyset$ for every nontrivial class $\alpha \in \tilde{\pi}_1(M)$. In particular, $C(M, A; \alpha) = \infty$ for every nontrivial class α .*

Proof. Let $F \in \mathcal{H}$ be the Hamiltonian function generating f_t and suppose that U is a neighbourhood of A such that $f_1(U) \cap U = \emptyset$. Choose a smooth function $G : M \rightarrow \mathbb{R}$ such that

$$\text{supp } G \subset U, \quad \inf_A G + \inf_{[0,1] \times M} F \geq c.$$

Let g_t denote the Hamiltonian flow of G and consider the Hamiltonian flow $h_t := f_t g_t$. Since $f_1(U) \cap U = \emptyset$, the Hamiltonian diffeomorphisms h_1 and f_1 have the same fixed points. Hence all 1-periodic orbits of the flow h_t are contractible: they have the form $x(t) = x(t+1) = h_t(x_0) = f_t(x_0)$ for some $x_0 \in M \setminus U$. The same holds for the conjugate isotopy $\varphi_t := f_1^{-1} h_t f_1$. Moreover, $\varphi_1 = g_1 f_1 = \psi_1$, where $\psi_t = g_t f_t$. Hence, by Remark 2.1.1, all 1-periodic orbits of the isotopy ψ_t are contractible as well. Observe now that the flow ψ_t is generated by the Hamiltonian functions $\Psi_t := G + F_t \circ g_t^{-1}$ and they satisfy $\inf_A \Psi_t \geq c$ for $x \in A$. By Proposition 2.1.3, there is a periodic Hamiltonian in $\mathcal{H}_c(M, A)$ with the same time-1-map as Ψ_t and, by Remark 2.1.1, the 1-periodic orbits of this periodic Hamiltonian are also contractible. \square

Let us now look at the relative capacity from the geometric viewpoint. The group \mathcal{D} of all compactly supported Hamiltonian diffeomorphisms of M carries Hofer's metric (see Section 1.1 above). Denote by $\mathcal{D}_c \subset \mathcal{D}$ the subset consisting of all Hamiltonian diffeomorphisms generated by functions from $\mathcal{H}_c(M, A)$. Recall that every $h \in \mathcal{D}$ has a canonical lift to the universal cover and hence the homotopy class (in $\tilde{\pi}_1(M)$) of a fixed point is independent of the choice of the compactly supported Hamiltonian generating h (see Remark 2.1.1).

Proposition 3.3.3. (Stability) (i) *Suppose that $C(M, A; \alpha) \leq c < \infty$ for some nontrivial class $\alpha \in \tilde{\pi}_1(M)$. Let $f \in \mathcal{D}_{c+a}$ be a Hamiltonian diffeomorphism, where $a > 0$.*

Then every compactly supported Hamiltonian diffeomorphism $h \in \mathcal{D}$ with $\rho(f, h) < a$ has a fixed point in the class α . In particular, $\rho(f, \text{id}) \geq a$.

(ii) Suppose that

$$\lim_{\substack{a > 0 \\ a \rightarrow 0}} C(M, A; 0, a) = 0$$

and let $c > 0$. Then $\rho(f, \text{id}) \geq c$ for every $f \in \mathcal{D}_c$.

Proof. We prove (i). Let $f \in \mathcal{D}_{c+a}$ and $h \in \mathcal{D}$ such that $\rho(f, h) < a$. Then, by Proposition 2.1.2, $h \in \mathcal{D}_c$ and hence there exists a Hamiltonian isotopy h_t with $h_1 = h$ whose Hamiltonian function belongs to $\mathcal{H}_c(M, A)$. By definition of the relative capacity, this flow has a 1-periodic orbit in the class α . In particular $h \neq \text{id}$. This proves (i).

We prove (ii). Suppose, by contradiction, that $\rho(f, \text{id}) < c$ for some $f \in \mathcal{D}_c$. Then, by Propositions 2.1.2 and 2.1.3, there exists a Hamiltonian function $H \in \mathcal{H}(M)$ that generates the identity and satisfies $\inf_{S^1 \times A} H > 0$. By assumption, there exists a constant $a > 0$ such that $C(M, A; 0, a) < \inf_{S^1 \times A} H$. Hence there exists a contractible periodic orbit $x \in \mathcal{P}(H; 0)$ with action $\mathcal{A}_H(x) \geq a$. On the other hand, since H generates the identity, every orbit is a contractible periodic orbit with action zero. This contradiction proves (ii). \square

Proposition 3.3.4. *Let $\alpha \subset \tilde{\pi}_1(M)$ and $a \geq -\infty$. Then every Hamiltonian $H \in \mathcal{H}(M)$ with $\inf_{S^1 \times A} H \geq C(M, A; \alpha, a)$ must have a 1-periodic orbit representing one of the homotopy classes in α with symplectic action $\mathcal{A}_H(x) \geq a$. In other words, the set $\{c > 0 \mid \forall H \in \mathcal{H}_c(M, A) \exists x \in \mathcal{P}(H; \alpha) \text{ such that } \mathcal{A}_H(x) \geq a\}$ is either empty or has a minimum.*

Proof. Without loss of generality we may assume that either $0 \notin \alpha$ or $a > 0$, since otherwise there is nothing to prove (all points x outside the support of H are constant periodic orbits in the class 0 and with action 0). Choose a compact subset $K \subset M$ such that $A \subset \text{Int } K$ and $S^1 \times K \supset \text{supp } H$. Next, let $\sigma : M \rightarrow \mathbb{R}$ be a Hamiltonian function supported in K and such that $\sigma|_A > 0$. Consider the sequence of Hamiltonians $H_n := H + \frac{1}{n}\sigma$. Clearly $\inf_{S^1 \times A} H_n > C(M, A; \alpha, a)$, hence for every n , H_n has a 1-periodic orbit x_n with $[x_n] \in \alpha$ and $\mathcal{A}_{H_n}(x_n) \geq a$. Note that $x_n \subset K$ for every n due to the assumption that either $0 \notin \alpha$ or $a > 0$.

Now H_n converges to H in the C^∞ topology and K is compact. Hence, replacing H_n by a suitable subsequence if necessary, we obtain a sequence of periodic orbits x_n , that converges to a periodic orbit x of H . It follows that, for n sufficiently large, the loops x_n all represent the same homotopy class. Hence $[x] = \lim_{n \rightarrow \infty} [x_n] \in \alpha$ and $\mathcal{A}_H(x) = \lim_{n \rightarrow \infty} \mathcal{A}_{H_n}(x_n) \geq a$. \square

Proof of Theorem B. Due to Proposition 2.1.3 and the discussion on homotopy classes following its proof we may assume our Hamiltonians to be defined on $S^1 \times U^*\mathbb{T}^n$ rather than on $[0, 1] \times U^*\mathbb{T}^n$. Let $H : S^1 \times U^*\mathbb{T}^n \rightarrow \mathbb{R}$ be a compactly supported Hamiltonian function and $k \in \mathbb{Z}^n$ an integer vector such that

$$|k| \leq c := \inf_{S^1 \times \mathbb{T}^n} H.$$

By Theorem 3.2.1 (i), $C(U^*\mathbb{T}^n, \mathbb{T}^n; k, c) = c$. It follows from the definition of our capacity and Proposition 3.3.4 that there exists a periodic solution $x \in \mathcal{P}(H; k)$ of (1) such that x represents the homotopy class k and $\mathcal{A}_H(x) \geq c$. This proves the assertion of Theorem B for periodic Hamiltonian functions. The assertion in the non-periodic case follows from the periodic case, Proposition 2.1.3, and Remark 3.1.1. \square

Proof of Theorem C. Let $S \subset U^*\mathbb{T}^n$ be a non-Lagrangian section and $c > 0$ be any real number. It is shown in [Po1, LS] that there exists a Hamiltonian function $H : U^*\mathbb{T}^n \rightarrow \mathbb{R}$ such that the vector field X_H is nowhere tangent to S :

$$x \in S \quad \implies \quad X_H(x) \notin T_x S.$$

We may assume without loss of generality that H has compact support. Now let $\varphi_t : U^*\mathbb{T}^n \rightarrow U^*\mathbb{T}^n$ denote the flow generated by X_H . Then there exists an $\varepsilon > 0$ such that

$$0 < t < \varepsilon \quad \implies \quad \varphi_t(S) \cap S = \emptyset.$$

If δ is sufficiently small then the only 1-periodic solutions of the Hamiltonian flow $t \mapsto \varphi_{\delta t}$ are constant and $\varphi_{\delta}(S) \cap S = \emptyset$. Hence, by Proposition 3.3.2, there exists, for every $c > 0$, a Hamiltonian function $F \in \mathcal{H}_c(U^*\mathbb{T}^n, S)$ such that $\mathcal{P}(F; k) = \emptyset$ for every nonzero integer vector $k \in \mathbb{Z}^n$. \square

3.4. Existence of closed orbits on hypersurfaces. As a by-product of our study of the relative capacity we obtain existence of closed orbits on hypersurfaces in various situations.

Theorem 3.4.1. *Let X be either \mathbb{T}^n or a closed negatively curved manifold and $H : T^*X \rightarrow \mathbb{R}$ a proper and bounded below Hamiltonian. Suppose that the sublevel set $\{H < c\}$ contains the zero section. Then for every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(T^*X)$ there exists a dense subset $S_\alpha \subset (c, \infty)$ with the property that for every $s \in S_\alpha$ the level set $\{H = s\}$ contains a closed orbit x_s in the class α and with $\int_{x_s} \lambda_{\text{can}} > 0$.*

Proof. Fix a nontrivial homotopy class $\alpha \in \tilde{\pi}_1(T^*X)$. To prove the statement of the theorem we shall show that for every $b > a > c$ there exists $s \in (a, b)$ such that the level set $\{H = s\}$ carries a closed orbit x in the class α and with $\int_x \lambda_{\text{can}} > 0$.

Pick a Riemannian metric g on X which in case $X = \mathbb{T}^n$ is the flat equilateral metric, or in the other case has negative curvature. Denote by $U^*X \subset T^*X$ the unit cotangent bundle of (X, g) endowed with the canonical symplectic structure.

Let $b > a > c$. Due to a rescaling argument we may assume without loss of generality that the sublevel set $\{H \leq b\}$ is contained in U^*X . Next, put $C_\alpha = C(U^*X, X; \alpha^{-1})$. Note that by Theorem 3.2.1 the number C_α is finite (and in fact equals the length of a geodesic in the class α^{-1}). Now choose a smooth function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- $\sigma(r) = C_\alpha$ for $r \leq a$.
- $\sigma(r) = 0$ for $r \geq b$.
- $\sigma'(r) < 0$ for $a < r < b$.

Consider the compactly supported Hamiltonian $F : U^*X \rightarrow \mathbb{R}$ defined by $F = \sigma \circ H$. As $F|_X = C_\alpha$ it follows from the definition of C_α that F has a 1-periodic orbit, say y , in the class α^{-1} . Note that y is nonconstant (because α is nontrivial). Therefore y must lie on one of the level sets $\{F = \rho\}$ where $0 < \rho < C_\alpha$. Since σ takes the interval (a, b) injectively onto $(0, C_\alpha)$ there exists $s \in (a, b)$ such that $\{F = \rho\} = \{H = s\}$, hence y lies on the level set $\{H = s\}$. Applying to y a suitable *orientation reversing* reparametrization will give us a closed orbit x_s of H in the class α . This follows easily from the fact that $X_F = (\sigma' \circ H)X_H$ and $\sigma'(s) < 0$.

Finally note that by Theorem 3.2.1, the orbit y has action $\mathcal{A}_F(y) \geq C_\alpha$, hence

$$C_\alpha \leq \int_0^1 (F(y(t)) - \lambda_{\text{can}}(\dot{y}(t))) dt = \rho + \int_{x_s} \lambda_{\text{can}} < C_\alpha + \int_{x_s} \lambda_{\text{can}}.$$

Consequently $\int_{x_s} \lambda_{\text{can}} > 0$. □

3.4.1. Symplectically convex boundaries. Let (M, ω) be a symplectic manifold and $U \subset M$ a relatively compact domain with smooth **convex** boundary $Q = \partial\bar{U}$. Recall that this means, by definition, that there exists a Liouville vector field Y (namely $\mathcal{L}_Y\omega = \omega$), defined on a neighbourhood of Q in M , such that Y points outside of U along Q . Note that in this case Q is a hypersurface of contact type since the vector field Y gives rise to a contact form $\lambda_Q = (\iota(Y)\omega)|_{TQ}$ on Q with $d\lambda_Q = \omega|_{TQ}$.

Denote by $\mathcal{L}_Q = \ker(\omega|_{TQ}) \subset TQ$ the characteristic line field of Q . The Reeb vector field R of λ_Q is a nonvanishing section of \mathcal{L}_Q and so defines an orientation on \mathcal{L}_Q . We call this the **canonical orientation** of \mathcal{L}_Q . It induces an orientation on each leaf of the characteristic foliation of Q (namely the foliation corresponding to \mathcal{L}_Q).

Corollary 3.4.2. *Let X be either \mathbb{T}^n or a closed negatively curved manifold. Let $U \subset T^*X$ be a relatively compact domain containing the zero section, and with smooth convex*

boundary $Q = \partial\bar{U}$. Let \mathcal{L}_Q be equipped with its canonical orientation. Then for every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(X)$ the characteristic foliation of Q has a closed leaf $x \subset Q$ with $j_*[x] = \alpha$, where $j_* : \tilde{\pi}_1(Q) \rightarrow \tilde{\pi}_1(X)$ is the map induced by the composition $j : Q \subset T^*X \xrightarrow{\pi} X$.

Remark. We were informed by C. Viterbo that the above corollary should follow also from the variational techniques of [HV].

Proof of Corollary 3.4.2. Denote by ω the (canonical) symplectic form on T^*X and let λ_Q be the contact form on Q induced by a local Liouville vector field Y as described above. Since U has convex boundary there exist a neighbourhood W of $Q = \partial\bar{U}$ in M and a diffeomorphism $\Phi : (-\varepsilon, \varepsilon) \times Q \rightarrow W$ such that (see [MS]):

- $\Phi(0, q) = q$ for every $q \in Q$.
- $\Phi((-\varepsilon, 0) \times Q) \subset U$, and $\Phi((0, \varepsilon) \times Q) \subset W \setminus \bar{U}$.
- $\Phi^*\omega = d(e^\tau \lambda_Q)$, where τ is the coordinate on $(-\varepsilon, \varepsilon)$.

Choose a smooth function $h : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ with the following properties:

- $h(\tau) = 0$ for $\tau \leq -\varepsilon/2$, and $h(\tau) = 1$ for $\tau \geq \varepsilon/2$.
- $h'(\tau) > 0$ for $-\varepsilon/2 < \tau < \varepsilon/2$.

Define now $H : W \rightarrow \mathbb{R}$ by $H(\Phi(\tau, q)) = h(\tau)$. Next, extend H to a smooth function $H : T^*X \rightarrow \mathbb{R}$ in such a way that:

- $H \equiv 0$ on $U \setminus W$.
- $H \geq 1$ on $T^*X \setminus (U \cup W)$.
- H is proper.

Pick a, b with $0 < a < b < 1$. By Theorem 3.4.1, there exists an $s_0 \in (a, b)$ such that the level set $\{H = s_0\}$ carries a closed orbit of H , say y , with $\pi_*[y] = \alpha$.

Since $0 < s_0 < 1$, there exists $\tau_0 \in (-\varepsilon/2, \varepsilon/2)$ such that $\Phi^{-1}(\{H = s_0\}) = \tau_0 \times Q$. Thus $\Phi^{-1}(y) = (\tau_0, x)$ where $x \subset Q$ is a closed curve. On the other hand $\Phi^{-1}(y)$ is a closed orbit for the Hamiltonian $h(\tau, q) = h(\tau)$ on $((-\varepsilon, \varepsilon) \times Q, d(e^\tau \lambda_Q))$. A simple computation shows that the Hamiltonian vector field X_h of h is given along $\tau_0 \times Q$ by $X_h|_{\tau_0 \times Q} = h'(\tau_0)e^{-\tau_0}R$, where R is the Reeb vector field of λ_Q . This proves that $x \subset Q$ is a closed leaf of the characteristic foliation of Q . The orientation of x agrees with the one of the characteristic foliation because $h'(\tau_0) > 0$.

Finally, the map $\Phi(\tau_0, \cdot) : Q \rightarrow T^*X$ is homotopic to the inclusion $Q \subset X$, hence $j_*[x] = \pi_*[y] = \alpha$. \square

The main point in the proofs of both Theorem 3.4.1 and Corollary 3.4.2 is that when X is either \mathbb{T}^n or a negatively curved manifold we have $C(U^*X, X; \alpha) < \infty$ for every $\alpha \in \tilde{\pi}_1(X)$. This suggests the following definition.

Definition 3.4.3. Let X be a smooth closed manifold. We say that a nontrivial free homotopy class $\alpha \in \tilde{\pi}_1(X)$ is **symplectically essential** if there exists a domain $U \subset T^*X$ containing the zero-section such that $C(U, X; \iota_*(\alpha)) < \infty$. Here $\iota_* : \tilde{\pi}_1(X) \rightarrow \tilde{\pi}_1(U)$ is the map induced by the inclusion $\iota : X \rightarrow U$ of the zero section into U .

Example. Let X be either \mathbb{T}^n or a closed negatively curved manifold. It follows from Theorem 3.2.1 that every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(X)$ is symplectically essential.

In this language, Theorem 3.4.1 has the following obvious generalization.

Theorem 3.4.4. *Let X be a smooth closed manifold and $H : T^*X \rightarrow \mathbb{R}$ be a proper and bounded below Hamiltonian. Suppose that the sublevel set $\{H < c\}$ contains the zero section. Then for every nontrivial symplectically essential class $\alpha \in \tilde{\pi}_1(X)$ there exists a dense subset $S_\alpha \subset (c, \infty)$ with the property that for every $s \in S_\alpha$ the level set $\{H = s\}$ contains a closed orbit whose projection to the zero section is in the class α^{-1} .*

The proof is an obvious modification of the one of Theorem 3.4.1. Similarly, the proof of the next result is analogous to that of Corollary 3.4.2.

Corollary 3.4.5. *Let X be a closed manifold and $U \subset T^*X$ a relatively compact domain containing the zero section, and with smooth convex boundary $Q = \partial\bar{U}$. Let \mathcal{L}_Q be equipped with its canonical orientation. Then for every nontrivial symplectically essential homotopy class $\alpha \in \tilde{\pi}_1(X)$ the characteristic foliation of Q has a closed leaf $x \subset Q$ with $j_*[x] = \alpha$, where $j_* : \tilde{\pi}_1(Q) \rightarrow \tilde{\pi}_1(X)$ is the map induced by the composition $j : Q \subset T^*X \xrightarrow{\pi} X$.*

3.5. Topological applications.

Definition (Manifolds of type \mathcal{F}). Let X be a smooth closed manifold. We say that X is of **type \mathcal{F}** if there exist two *nontrivial* free homotopy classes $\alpha_1, \alpha_2 \in \tilde{\pi}_1(X)$ such that:

- (i) α_1 and α_2 are not *positively proportional*, namely there exist no $k_1, k_2 \in \mathbb{N}$ with $\alpha_1^{k_1} = \alpha_2^{k_2}$.
- (ii) α_1 and α_2 are both symplectically essential. It is easy to see that this is equivalent to existence of one domain $U \subset T^*X$ containing the zero section and such that both $C(U, X; \iota_*(\alpha_1))$ and $C(U, X; \iota_*(\alpha_2))$ are finite. Here ι_* is the map induced by the inclusion of zero section X into U .

Examples. **(i)** Let X be a closed negatively curved manifold. Then X is of type \mathcal{F} . To see this note first that X is not simply connected because the universal cover is diffeomorphic to Euclidean space. Now let $\alpha \in \tilde{\pi}_1(X)$ be a nontrivial free homotopy class and denote by γ the unique (up to parametrization) closed geodesic representing the class α . We claim that α and α^{-1} are not positively proportional. Suppose otherwise that there exist

positive integers k and ℓ such that $\alpha^k = \alpha^{-\ell}$. Then the k 'th and ℓ 'th iterates γ^k and $\gamma^{-\ell}$ of γ and γ^{-1} , respectively, are two different geodesics representing the same free homotopy class. This is impossible for negatively curved manifolds because every geodesic has Morse index zero.

Now let $\alpha_1, \alpha_2 \in \tilde{\pi}_1(X)$ be two nontrivial positively non-proportional classes. By Theorem 3.2.1 (ii), $C(U^*X, X; \alpha_1)$ and $C(U^*X, X; \alpha_2)$ are finite, hence X is of type \mathcal{F} .

(ii) It follows from Theorem 3.2.1(i) that \mathbb{T}^n is also of type \mathcal{F} .

3.5.1. Applications to Hamiltonian circle actions.

Theorem 3.5.1. *Let (M, ω) be a compact symplectic manifold (possibly with boundary) with $\dim_{\mathbb{R}} M \geq 4$ and $L \subset \text{Int}(M)$ a compact connected Lagrangian submanifold. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \rightarrow [r, R)$ that is proper onto its image (R may possibly be ∞). Then L cannot be of type \mathcal{F} . In particular, L cannot be diffeomorphic to \mathbb{T}^n , $n \geq 2$, or to any negatively curved manifold.*

Note that the Theorem, as stated, is not true when $\dim_{\mathbb{R}} M = 2$. For example take $M = U^*S^1$ and let $L = S^1$ be the zero section. Then $U^*S^1 \setminus S^1$ is diffeomorphic to $S^1 \times ((-1, 0) \cup (0, 1))$ and has a circle action with moment map $\mu(q, p) = -|p|$ which satisfies all the assumptions of the theorem.

The crucial difference between dimension two and higher ones is that in dimension two, $M \setminus L$ might be disconnected whereas in higher dimension this never happens. Indeed the following 2-dimensional version of Theorem 3.5.1 holds:

Theorem 3.5.2. *Let (M, ω) be a compact symplectic 2-manifold (possibly with boundary) and $S^1 \approx L \subset \text{Int}(M)$ an embedded circle. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \rightarrow [r, R)$ that is proper onto its image (R may possibly be ∞). Then $M \setminus L$ is disconnected.*

Before we turn to the proofs of Theorems 3.5.1 and 3.5.2 let us remark that in both Theorems it is impossible to drop the assumption that μ is proper onto its image and that its image is a half open interval. Indeed take $M = \mathbb{T}^2 \approx S^1 \times S^1$ and $L = \text{pt} \times S^1$. Then $M \setminus L$ is diffeomorphic to $S^1 \times (0, 1)$ and so is connected. It has an obvious circle action whose moment map $\mu : S^1 \times (0, 1) \rightarrow (0, 1)$ is the projection onto the second factor. However, there is no Hamiltonian circle action on $S^1 \times (0, 1)$ such that the image of the moment map is a half open interval and the moment map is proper. One can easily produce higher dimension examples as well, e.g. multiplying M by another \mathbb{T}^2 factor and L by S^1 .

Proof of Theorem 3.5.1. By Darboux' theorem there exist an open relatively compact neighbourhood U_0 of L in T^*L , an open neighbourhood W_0 of L in M and a symplectomorphism $f : (U_0, \omega_{\text{can}}) \rightarrow (W_0, \omega)$ taking $L \subset U_0$ identically to $L \subset W_0$.

Note that $M \setminus U_0$ is compact and let

$$R_0 := \max_{M \setminus W_0} \mu, \quad U := f^{-1}(M \setminus \{\mu \leq R_0\}) \subset U_0.$$

Then $U \setminus L$ carries a circle action as well, with moment map $\mu \circ f : U \setminus L \rightarrow (R_0, R)$ which is proper onto its image. By reducing U_0 if necessary we may assume that U is connected. Note that $U \setminus L$ is also connected because the codimension of L in U is at least two. It follows that any two S^1 -orbits in $U \setminus L$ represent positively proportional homotopy classes in $\tilde{\pi}_1(U)$.

Now let $R_0 < R_1 < R_2 < R$ and choose a smooth function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- $\sigma(r) = 0$ for every $r \leq R_1$.
- $\sigma(r) = c$ for every $r \geq R_2$, where $c > 0$ is a constant that will be determined later.
- $\sigma'(r) > 0$ for every $R_1 < r < R_2$.

Define now a compactly supported Hamiltonian $H : U \rightarrow \mathbb{R}$ by putting $H := \sigma \circ \mu \circ f$ on $U \setminus L$, and extending H to be c on L . Note that the vector field X_H is everywhere tangent to the orbits of the circle action on $U \setminus L$ and moreover along $\mu^{-1}(R_1, R_2)$, X_H points in the direction induced by the circle action (because $\sigma' > 0$ there). Since any two orbits of the circle action represent positively proportional homotopy classes in $\tilde{\pi}_1(U)$ it follows that any two nonconstant closed orbits of X_H must also be positively proportional in $\tilde{\pi}_1(U)$.

Suppose now that L is of type \mathcal{F} . We shall get a contradiction by showing that once the constant c from the definition of σ is chosen to be large enough, X_H must carry two closed orbits whose classes are positively non-proportional. Indeed, if L is of type \mathcal{F} then there exist two positively non-proportional classes $\alpha_1, \alpha_2 \in \tilde{\pi}_1(L)$ and a domain U' containing the zero section such that both capacities $C(U', L; \iota'_*(\alpha_1))$ and $C(U', L; \iota'_*(\alpha_2))$ are finite, where ι'_* is the map induced by the inclusion of the zero section L into U' . By reducing the size of the Darboux neighbourhood W_0 if necessary we may assume that $U \subset U'$. Denote by $\iota : L \rightarrow U$ and $j : U \rightarrow U'$ the obvious inclusions and by $\pi : U \rightarrow L$ the obvious projection. Then $\pi_* \iota_* = \text{id}$ and hence $j_*^{-1}(\iota'_*(\alpha_i)) \subset \pi_*^{-1}(\alpha_i)$ for $i = 1, 2$. Hence, by monotonicity (Proposition 3.3.1), we have

$$\begin{aligned} C(U, L; \pi_*^{-1}(\alpha_1)) &\leq C(U_1, L; \iota'_*(\alpha_1)) < \infty \\ C(U, L; \pi_*^{-1}(\alpha_2)) &\leq C(U_2, L; \iota'_*(\alpha_2)) < \infty. \end{aligned}$$

Now choose the constant c in the definition of σ to be larger than both of $C(U, L; \pi_*^{-1}(\alpha_1))$ and $C(U, L; \pi_*^{-1}(\alpha_2))$. Since $H|_L = c$, the Hamiltonian H must have two periodic orbits $x_1, x_2 \subset U$ with $\pi_*[x_1] = \alpha_1$ and $\pi_*[x_2] = \alpha_2$. As α_1 and α_2 are not positively proportional, neither are the classes $[x_1]$ and $[x_2]$. This contradicts the fact, established above, that any two periodic orbits of X_H represent proportional homotopy classes in $\tilde{\pi}_1(U)$. \square

Proof of Theorem 3.5.2. Let W_0 be a Darboux neighbourhood of L in M as in the proof of Theorem 3.5.1. Choose an orientation on $L \approx S^1$ and denote by $\gamma_0 \in \tilde{\pi}_1(W_0)$ the homotopy class represented by the oriented circle L . By Theorem 3.2.1 both $C(W_0, L; \gamma_0)$ and $C(W_0, L; \gamma_0^{-1})$ are finite.

Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined in the proof of Theorem 3.5.1, where the constant c used in its definition is now taken to be larger than both of $C(W_0, L; \gamma_0)$ and $C(W_0, L; \gamma_0^{-1})$. Define now a compactly supported Hamiltonian $H : W_0 \rightarrow \mathbb{R}$ by putting $H := \sigma \circ \mu$ on $W_0 \setminus L$ and extending H to be c on L . By the choice of the constant c , H must have two periodic orbits $x_1, x_2 \subset W_0 \setminus L$ with $[x_1] = \gamma_0$ and $[x_2] = \gamma_0^{-1}$.

As in the proof of Theorem 3.5.1, every nonconstant periodic orbit of H represents a class that is positively proportional in $\tilde{\pi}_1(W_0)$ to a class represented by an orbit of the S^1 action. Thus both γ_0 and γ_0^{-1} are positively proportional to classes represented by orbits of the S^1 action.

Suppose, by contradiction, that $M \setminus L$ is connected. Then all the classes represented by orbits of the S^1 action are positively proportional in $\tilde{\pi}_1(M \setminus L)$, in particular also in $\tilde{\pi}_1(M)$. It follows that $\gamma_0 = [x_1]$ and $\gamma_0^{-1} = [x_2]$ become positively proportional when viewed as classes in $\tilde{\pi}_1(M)$, namely the classes $\iota_*(\gamma_0), (\iota_*(\gamma_0))^{-1} \in \tilde{\pi}_1(M)$ are positively proportional, where $\iota : W_0 \rightarrow M$ is the inclusion. Hence there exists $k > 0$ such that $(\iota_*(\gamma_0))^k$ is the trivial class in $\tilde{\pi}_1(M)$. Passing to homology we get that $k[L] = 0 \in H_1(M; \mathbb{Z})$, where $[L] \in H_1(M; \mathbb{Z})$ is the homology class represented by L . But $H_1(M; \mathbb{Z})$ has no torsion, hence $[L] = 0$. Finally, note that if an *embedded* circle L in an orientable surface M is zero in homology then $M \setminus L$ must be disconnected. Contradiction. \square

3.5.2. Applications to Stein manifolds. Let (W, J) be a Stein manifold. Recall that a smooth function $\varphi : W \rightarrow \mathbb{R}$ is called **plurisubharmonic** if the 2-form

$$\omega_\varphi := -d(d\varphi \circ J)$$

is a J -positive symplectic form, i.e. $\omega_\varphi(v, Jv) > 0$ for every nonzero tangent vector $v \in TW$. We denote by

$$g_\varphi(\cdot, \cdot) := \omega_\varphi(\cdot, J\cdot)$$

the associated Kähler metric. Let $\varphi : W \rightarrow \mathbb{R}$ be an **exhausting plurisubharmonic function**, namely in addition to being plurisubharmonic φ is also proper, bounded from

below and has no critical points outside some compact subset of W . Let

$$X_\varphi := \text{grad}_{g_\varphi} \varphi$$

be the gradient vector field of φ with respect to the metric g_φ . Then

$$\mathcal{L}_{X_\varphi} \omega_\varphi = dt(X_\varphi)\omega_\varphi = -d(g_\varphi(X_\varphi, J\cdot)) = -d(d\varphi \circ J) = \omega_\varphi,$$

and hence the flow $X_\varphi^t : W \rightarrow W$ of X_φ satisfies $(X_\varphi^t)^* \omega_\varphi = e^t \omega_\varphi$. Denote by Δ_φ the union of all the stable submanifold of the flow of X_φ :

$$(5) \quad \Delta_\varphi := \bigcup_{p \in \text{Crit}(\varphi)} W_p^s(X_\varphi).$$

Note that $\Delta_\varphi \subset W$ is the maximal compact invariant subset for the flow of X_φ . We call Δ_φ the **associated skeleton** of φ . When φ is Morse, each stable submanifold in the union (5) is isotropic with respect to ω_φ (see [EG]), in particular $\dim \Delta_\varphi \leq \frac{1}{2} \dim_{\mathbb{R}} W$. We remark that, even if φ is Morse, Δ_φ may have a quite “wild” structure. However, for a generic exhausting plurisubharmonic function φ , Δ_φ has the structure of an isotropic CW-complex (see [Bi]). In rare situations it may even happen that some φ 's (Morse or not) have smooth skeletons.

We now turn to a special class of Stein manifolds, namely those obtained from removing an ample divisor from a smooth algebraic variety. More precisely, let (M, J) be a closed algebraic manifold and $\Sigma \subset M$ a smooth ample divisor. It is well known that $(M \setminus \Sigma, J)$ is an affine variety, in particular Stein. The following theorem deals with topological restrictions on the possible smooth manifolds that may arise as skeletons of Stein manifolds of the type just mentioned.

Theorem 3.5.3. *Let (M, J) be a closed algebraic manifold and $\Sigma \subset M$ a smooth and reduced ample divisor. If the Stein manifold $(M \setminus \Sigma, J)$ admits an exhausting plurisubharmonic function $\varphi : M \setminus \Sigma \rightarrow \mathbb{R}$ whose skeleton Δ_φ is a smooth connected Lagrangian submanifold, then Δ_φ cannot be of type \mathcal{F} . In particular, Δ_φ cannot be diffeomorphic to \mathbb{T}^n , $n \geq 2$, or to any closed negatively curved manifold.*

Proof. Put $W := M \setminus \Sigma$ and endow W with the complex structure J . The proof has two steps.

Step 1. *For every exhausting plurisubharmonic function $\varphi : W \rightarrow \mathbb{R}$ there exists an open relatively compact domain $W_0 \subset W$ with the following properties:*

- (i) $W_0 \supset \Delta_\varphi$.
- (ii) The boundary $P := \partial \overline{W}_0$ is smooth, connected, and convex with respect to ω_φ .
- (iii) The leaves of the characteristic foliation (with respect to ω_φ) on P are all orbits of a free circle action.

The idea of the proof is the following. Since W is the complement of smooth ample divisor it is possible to endow W with an exhausting plurisubharmonic function φ_0 for which the above statement holds. In order to pass from φ_0 to any given plurisubharmonic function φ we modify φ_0 and φ at infinity to obtain new plurisubharmonic functions $\overline{\varphi}_0$ and $\overline{\varphi}$ for which the vector fields $X_{\overline{\varphi}_0}$ and $X_{\overline{\varphi}}$ are complete. Then by the theory of Eliashberg-Gromov [EG] the symplectic forms $\omega_{\overline{\varphi}_0}$ and $\omega_{\overline{\varphi}}$ are diffeomorphic. Hence the statement being true for φ_0 is true also for φ .

Here are the precise details. We first adjust φ at infinity so that the vector field X_φ becomes complete. This is a standard procedure ([EG], see also [Bi-Ci] Lemma 3.1). More precisely, let $\overline{\varphi} : W \rightarrow \mathbb{R}$ be an exhausting plurisubharmonic function with the following properties:

- $\overline{\varphi} = \varphi$ on a relatively compact domain $W' \subset W$ that contains Δ_φ .
- $\Delta_{\overline{\varphi}} = \Delta_\varphi$.
- The vector field $X_{\overline{\varphi}}$ is complete.

Next we endow W with another plurisubharmonic function φ_0 that arises from W being the complement of an ample divisor. For this purpose, denote by $\mathcal{L} = \mathcal{O}_M(\Sigma) \rightarrow M$ the holomorphic line bundle defined by Σ , and let $s : M \rightarrow \mathcal{L}$ be a holomorphic section with $\Sigma = \{s = 0\}$. Since Σ is ample there exists a hermitian metric $\|\cdot\|$ on \mathcal{L} for which the associated metric connection ∇ has positive curvature R^∇ . Then the (real) 2-form $\omega = \frac{1}{2\pi i} R^\nabla$ is a J -compatible symplectic form. Let $\varphi_0 : W \rightarrow \mathbb{R}$ be the function defined by $\varphi_0 = -\frac{1}{4\pi} \log \|s\|^2$. Then on W we have $\omega = -d(d\varphi_0 \circ J) = \omega_{\varphi_0}$, hence φ_0 is plurisubharmonic. Note that φ_0 is exhausting. Indeed φ_0 is proper, bounded below, and, since Σ is smooth and reduced, φ_0 has no critical points near Σ .

Let $E(\rho) := \{v \in \mathcal{L}|_\Sigma \mid \|v\| < \rho\}$ and $P(\rho) := \{v \in \mathcal{L}|_\Sigma \mid \|v\| = \rho\}$ be the radius- ρ disc and circle subbundles of $\mathcal{L}|_\Sigma$ and denote by $\pi : E(\rho) \rightarrow \Sigma$ the projection. Pick a connection 1-form γ on $P(1)$ such that $d\gamma = -\pi^*(\omega|_\Sigma)$ and consider the symplectic form $\omega_0 = \pi^*(\omega|_\Sigma) + d(r^2\gamma)$ where r is the radial coordinate on the fibres induced by the hermitian metric. A simple computation shows that the vector field $Z := \frac{r^2-1}{2r} \frac{\partial}{\partial r}$ defined on $E(1) \setminus \Sigma$ satisfies $\mathcal{L}_Z \omega_0 = \omega_0$. Moreover for every $0 < \rho < 1$, Z is transverse to $P(\rho)$ and points towards the inside of $E(\rho)$. Finally note that the leaves of characteristic foliation on $P(\rho)$ are all orbits of a free circle action, in fact they all coincide with the fibres of the principal circle bundle $P(\rho) \rightarrow \Sigma$. Since Σ is connected so is $P(\rho)$.

By the symplectic tubular neighbourhood theorem there exist a neighbourhood B of Σ in M , an $\varepsilon > 0$, and a symplectomorphism $f : (E(\varepsilon), \omega_0) \rightarrow (B, \omega)$ that sends $\Sigma \subset M$ identically onto $\Sigma \subset E(\varepsilon)$. Pick any $0 < \varepsilon_0 < \varepsilon$ and put $B_0 := f(E(\varepsilon_0))$ and $\mathcal{U}_0 := M \setminus \overline{B}_0$. Then $\mathcal{U}_0 \subset W$ is a relatively compact domain with convex boundary $\partial \overline{\mathcal{U}}_0$ on which the

characteristic foliation coincides with the orbits of a free circle action. Taking ε_0 to be smaller if necessary we may assume that \mathcal{U}_0 contains Δ_{φ_0} .

Similarly to φ we adjust φ_0 at infinity so that the vector field X_{φ_0} becomes complete. More precisely, let $\bar{\varphi}_0 : W \rightarrow \mathbb{R}$ be an exhausting plurisubharmonic function with the following properties:

- $\bar{\varphi}_0 = \varphi_0$ on an open subset containing $\bar{\mathcal{U}}_0$.
- $\Delta_{\bar{\varphi}_0} = \Delta_{\varphi_0}$.
- The vector field $X_{\bar{\varphi}_0}$ is complete.

By [EG] there exists a symplectomorphism $F : (W, \omega_{\bar{\varphi}_0}) \rightarrow (W, \omega_{\bar{\varphi}})$. Denote by $X_{\bar{\varphi}_0}^t$ the flow of $X_{\bar{\varphi}_0}$. Then we have $\bigcup_{t \geq 0} X_{\bar{\varphi}_0}^t(\mathcal{U}_0) = W$, hence for $t_0 > 0$ large enough $F(X_{\bar{\varphi}_0}^{t_0}(\mathcal{U}_0)) \supset \Delta_{\bar{\varphi}}$. Clearly, the domain $F(X_{\bar{\varphi}_0}^{t_0}(\mathcal{U}_0)) \subset (W, \omega_{\bar{\varphi}})$ has a smooth convex connected boundary whose characteristic foliation has leaves which are orbits of a free circle action. Since $(X_{\bar{\varphi}}^{-t})^* \omega_{\bar{\varphi}} = e^{-t} \omega_{\bar{\varphi}}$ the same holds also for each of the domains $X_{\bar{\varphi}}^{-t} \circ F \circ X_{\bar{\varphi}_0}^{t_0}(\mathcal{U}_0)$. Pick now $t_1 > 0$ large enough so that $W_0 := X_{\bar{\varphi}}^{-t_1} \circ F \circ X_{\bar{\varphi}_0}^{t_0}(\mathcal{U}_0) \subset W'$. Recall that on W' we have $\omega_{\bar{\varphi}} = \omega_{\varphi}$. Hence the domain W_0 satisfies all three conditions claimed in Step 1.

Step 2. *We prove the theorem.*

Let $\varphi : W \rightarrow \mathbb{R}$ be an exhausting plurisubharmonic function with skeleton Δ_{φ} which is a connected smooth manifold of type \mathcal{F} .

By Darboux' theorem there exist a neighbourhood $V(\Delta_{\varphi})$ of Δ_{φ} and a symplectic embedding $g : (V(\Delta_{\varphi}), \omega_{\varphi}) \rightarrow (T^*\Delta_{\varphi}, \omega_{\text{can}})$ which takes $\Delta_{\varphi} \subset V(\Delta_{\varphi})$ identically onto the zero section $\Delta_{\varphi} \subset T^*\Delta_{\varphi}$.

Let $X_{\varphi} := \text{grad}_{g_{\varphi}} \varphi$, denote by X_{φ}^t the flow of X_{φ} , and recall that $(X_{\varphi}^t)^* \omega_{\varphi} = e^t \omega_{\varphi}$. Let $W_0 \subset W$ be the domain defined by Step 1 and let $W_1 \subset W$ be a relatively compact domain containing \bar{W}_0 . From the definition of Δ_{φ} it follows that for $T > 0$ large enough we have $\Delta_{\varphi} \subset X_{\varphi}^{-T}(W_1) \subset V(\Delta_{\varphi})$.

Denote by $Y = p \frac{\partial}{\partial p}$ the standard Liouville vector field on $T^*\Delta_{\varphi}$. Then its flow Y^t satisfies $(Y^t)^* \omega_{\text{can}} = e^t \omega_{\text{can}}$. Hence $Y^T \circ g \circ X_{\varphi}^{-T} : (W_1, \omega_{\varphi}) \rightarrow (T^*\Delta_{\varphi}, \omega_{\text{can}})$ is a symplectic embedding. It follows that $U := Y^T \circ g \circ X_{\varphi}^{-T}(W_0) \subset T^*\Delta_{\varphi}$ is an open relatively compact domain containing the zero section and with convex boundary $Q := Y^T \circ g \circ X_{\varphi}^{-T}(P)$. Moreover, by condition (iii) in Step 1, the leaves of the characteristic foliation on Q coincide with the orbits of a free circle action on Q . Since Q is connected, it follows that any two leaves of the characteristic foliation on Q represent positively proportional homotopy classes. On the other hand, Δ_{φ} is of type \mathcal{F} and so, by Corollary 3.4.5, the characteristic foliation of Q contains two closed leaves whose homotopy classes are positively non-proportional. Contradiction. \square

4. A HOMOLOGICAL CAPACITY

In order to study and compute the relative capacity $C(M, A; \alpha)$ we shall define another quantity $\widehat{C}(M, A; \alpha) \geq C(M, A; \alpha)$ which captures the existence of *homologically essential* periodic orbits in given homotopy classes. Here the term “homologically essential” refers to Floer homology. The homological capacity $\widehat{C}(M, A; \alpha)$ will be defined in purely Floer-homological terms. It is easier to compute and enjoys some nice functorial properties. We begin with a brief discussion of convex boundaries.

4.1. Convex boundaries. Let (\overline{M}, ω) be a compact connected symplectic manifold with convex boundary and denote $M := \overline{M} \setminus \partial\overline{M}$. Recall ([EG], see also Section 3.4 above) that the boundary is called **convex** if there exist a vector field $X \in \text{Vect}(\overline{M})$ and a neighbourhood U of $\partial\overline{M}$ such that X points out on the boundary and is dilating on U , namely $\mathcal{L}_X\omega = \omega$ on U . Let φ_t denote the flow of X , suppose that $U = \{\varphi_t(x) \mid x \in \partial M, -\varepsilon < t \leq 0\}$, and denote by $\xi := \ker(\iota(X)\omega|_{T\partial\overline{M}})$ the contact structure on the boundary determined by X and ω . Under these hypotheses (the existence of X and U) there is an ω -compatible almost complex structure J on \overline{M} such that

$$J\xi = \xi, \quad \omega(X(x), J(x)X(x)) = 1, \quad D\varphi_t(x)J(x) = J(\varphi_t(x))D\varphi_t(x)$$

for all $x \in \partial\overline{M}$ and $t \in (-\varepsilon, 0]$. Such an almost complex structure is called **convex near the boundary**.

Consider the function $f : U \rightarrow \mathbb{R}$ given by

$$f(\varphi_t(x)) := e^t$$

for $x \in \partial\overline{M}$ and $-\varepsilon < t \leq 0$. A simple computation shows that its gradient with respect to the metric $\omega(\cdot, J\cdot)$ is X and hence $-d(df \circ J) = \omega$. This means that f is plurisubharmonic on U , or in other words the 2-form $-d(df \circ J)$ is positive on every J -complex tangent line in TU . Let $u : \Omega \rightarrow U$ be a nonconstant J -holomorphic curve, defined on a connected open subset $\Omega \subset \mathbb{C}$. Then the function $f \circ u : \Omega \rightarrow \mathbb{R}$ is subharmonic and, by the mean value inequality, cannot have a strict interior maximum. Hence a nonconstant J -holomorphic curve in \overline{M} cannot intersect $\partial\overline{M}$ at an interior point of its domain Ω .

Remark 4.1.1. Since J is invariant under the flow of X (near $\partial\overline{M}$) we have $0 = (\mathcal{L}_X J)v = (\nabla_X J)v + J\nabla_v X - \nabla_{Jv} X$, where ∇ denotes the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. Now consider the 1-form $\alpha := -df \circ J = \iota(X)\omega = \langle JX, \cdot \rangle$. A simple calculation, using $\mathcal{L}_X J = 0$, shows that

$$d\alpha(v, w) = \langle \nabla_v(JX), w \rangle - \langle \nabla_w(JX), v \rangle = 2\langle J\nabla_v X, w \rangle.$$

Hence the identity $d\alpha = \omega$ is equivalent to $\nabla_v X = v/2$. Since X is the gradient of f it follows that the Laplacian of $f \circ u$ is given by $\Delta(f \circ u) = |du|^2/2$ for every J -holomorphic curve $u : \Omega \rightarrow \overline{M}$.

Remark 4.1.2. The space of almost complex structures on \overline{M} that are convex near the boundary is connected. To see this fix first the dilating vector field X on a neighbourhood of $\partial\overline{M}$. Then the space of ω -compatible almost complex structures on the symplectic bundle $\xi = \ker(\iota(X)\omega|_{T\partial\overline{M}})$ is connected (see [MS]), hence the space of ω -compatible almost complex structures J satisfying (6) is also connected. Finally, we may allow X to vary since the space of vector fields that are dilating near $\partial\overline{M}$ and point out on $\partial\overline{M}$ is convex hence connected.

4.2. The setting. From now on our standing hypotheses are that (\overline{M}, ω) is a compact connected symplectic manifold with convex boundary $\partial\overline{M}$ and $A \subset M := \overline{M} \setminus \partial\overline{M}$ is a compact subset. We assume that the symplectic form is exact and fix a 1-form $\lambda \in \Omega^1(\overline{M})$ such that $d\lambda = \omega$. We call λ an ω -**primitive**.

As in section 2.1, we denote by $\mathcal{H} := \mathcal{H}(M) := C_0^\infty(S^1 \times M)$ the space of smooth compactly supported smooth Hamiltonian functions on $S^1 \times M$ and by $\mathcal{D} \subset \text{Diff}_0(M)$ the group of Hamiltonian diffeomorphisms of M generated by functions from \mathcal{H} . For $c > 0$ we denote by $\mathcal{H}_c = \mathcal{H}_c(M, A)$ the subspace of all Hamiltonian functions $H \in \mathcal{H}$ that satisfy $\inf_{S^1 \times A} h \geq c$ and by $\mathcal{D}_c = \mathcal{D}_c(M, A)$ the set of all Hamiltonian diffeomorphisms that are generated by functions from \mathcal{H}_c .

4.3. Floer homology. Floer homology is an essential ingredient in the definition of our invariants. The purpose of this section is to summarize the main building blocks of this theory needed for our applications. The reader is referred to [F4, FH2, CFH, V] for a detailed foundation of the subject (see also [Sa] for a general exposition).

Fix a *nontrivial* free homotopy class $\alpha \in \tilde{\pi}_1(M)$ and recall from Section 2.1 that $\mathcal{P}(H; \alpha) \subset L_\alpha M$ denotes the set of periodic solutions of the Hamiltonian system associated to $H \in \mathcal{H}$ and that these periodic solutions are the critical points of the symplectic action functional $\mathcal{A}_H : L_\alpha M \rightarrow \mathbb{R}$ defined by (4). The set of critical values of \mathcal{A}_H is called the **action spectrum** and will be denoted by

$$\mathcal{S}(H; \alpha) := \mathcal{A}_H(\mathcal{P}(H; \alpha)) = \{\mathcal{A}_H(x) \mid x \in L_\alpha M, \dot{x}(t) = X_{H_t}(x(t))\}.$$

Here $X_{H_t} \in \text{Vect}(M)$ is given by $\iota(X_{H_t})\omega = -dH_t$. Now let $-\infty \leq a < b \leq \infty$ and denote by $\mathcal{P}^{[a,b]}(H; \alpha)$ the set of 1-periodic solutions of the Hamiltonian system associated to H that represent the class α and whose action lies in the interval $[a, b]$:

$$\mathcal{P}^{[a,b]}(H; \alpha) := \mathcal{P}^b(H; \alpha) \setminus \mathcal{P}^a(H; \alpha), \quad \mathcal{P}^a(H; \alpha) := \{x \in \mathcal{P}(H; \alpha) \mid \mathcal{A}_H(x) < a\}.$$

Suppose that $H \in \mathcal{H}$ is a Hamiltonian function that satisfies the following hypothesis:

(H) $a, b \notin \mathcal{S}(H; \alpha)$ and every 1-periodic orbit $x \in \mathcal{P}(H; \alpha)$ is nondegenerate.

Then the Floer homology group $\mathrm{HF}^{[a,b]}(H; \alpha)$ is defined as the homology of a chain complex over \mathbb{Z}_2 generated by the 1-periodic orbits in $\mathcal{P}^{[a,b]}(H; \alpha)$.¹ It is useful to think of this chain complex as the quotient

$$\mathrm{CF}^{[a,b]}(H; \alpha) := \mathrm{CF}^b(H; \alpha) / \mathrm{CF}^a(H; \alpha), \quad \mathrm{CF}^a(H; \alpha) := \bigoplus_{x \in \mathcal{P}^a(H; \alpha)} \mathbb{Z}_2 x.$$

The Floer boundary operator is defined as follows. Let $J_t = J_{t+1} \in \mathcal{J}(M, \omega)$ be a t -dependent smooth family of ω -compatible almost complex structures on \overline{M} such that J_t is convex and independent of t near the boundary $\partial \overline{M}$ (see Section 4.1). Consider the Floer differential equation

$$(7) \quad \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0.$$

For a smooth solution $u : \mathbb{R} \times S^1 \rightarrow M$ of (7) define its energy to be

$$E(u) := \int_0^1 \int_{-\infty}^{\infty} |\partial_s u|^2 ds dt.$$

Then if $u : \mathbb{R} \times S^1 \rightarrow M$ is a smooth solution of (7) with finite energy then the limits

$$(8) \quad \lim_{s \rightarrow \pm\infty} u(s, t) = x^\pm(t), \quad \lim_{s \rightarrow \pm\infty} \partial_s u(s, t) = 0$$

exist and are uniform in the t -variable. Moreover, $x^\pm \in \mathcal{P}(H)$ and we have

$$E(u) = \mathcal{A}_H(x^-) - \mathcal{A}_H(x^+).$$

The following observations allow us to define Floer homology groups in the present situation.

- (i) Since every periodic solution $x \in \mathcal{P}(M; \alpha)$ is nonconstant (the class α is nontrivial) and J is convex near the boundary, there exists an open set $U \subset M$ such that $M \setminus U$ is compact and $u(\mathbb{R} \times S^1) \cap U = \emptyset$ for every finite energy solution of (7).
- (ii) By (i) and the energy identity, the space of finite energy solutions of (7) is compact with respect to C^∞ -convergence on compact sets, i.e. only the splitting into a finite sequence of adjacent Floer connecting orbits can occur in the limit.
- (iii) For a generic family of almost complex structures $J = \{J_t\}$ (that are convex and independent of t on U) the linearized operator for equation (7) is surjective for every finite energy solution of (7) in the homotopy class α (see [FHS]). Such a family of almost complex structures is called **regular** and the space of regular families of almost complex structures will be denoted by $\mathcal{J}_{\mathrm{reg}}(H; \alpha)$.

¹We use the convention that the complex generated by the empty set is 0.

For every $J \in \mathcal{J}_{\text{reg}}(H; \alpha)$ and every pair $x^\pm \in \mathcal{P}(H; \alpha)$ the space $\mathcal{M}(x^-, x^+; H, J)$ of solutions of (7) and (8) is a smooth manifold whose dimension near a solution u of (7) and (8) is given by the difference of the Conley–Zehnder indices (see [SZ]) of x^- and x^+ (relative to u). The subspace of solutions of index one will be denoted by $\mathcal{M}^1(x^-, x^+; H, J)$. For $J \in \mathcal{J}_{\text{reg}}(H; \alpha)$ it follows from (i) and (ii) that the quotient $\mathcal{M}^1(x^-, x^+; H, J)/\mathbb{R}$ (modulo time shift) is a finite set for every pair $x^\pm \in \mathcal{P}(H; \alpha)$. The Floer boundary operator $\partial^{H,J}$ on $\text{CF}^b(H; \alpha)$ is defined by

$$\partial^{H,J} x := \sum_{y \in \mathcal{P}^b(H; \alpha)} \#(\mathcal{M}^1(x, y; H, J)/\mathbb{R}) y$$

for every $x \in \mathcal{P}^b(H; \alpha)$. That this is indeed a boundary operator is proved as in Floer’s original work [F4]. The energy identity shows that $\text{CF}^a(H; \alpha)$ is a subcomplex, namely it is invariant under the Floer boundary operator. We thus get an induced boundary operator $[\partial^{H,J}]$ on the quotient $\text{CF}^{[a,b]}(H; \alpha)$. We denote the homology of the quotient complex by

$$\text{HF}^{[a,b]}(H, J; \alpha) := \frac{\ker([\partial^{H,J}] : \text{CF}^{[a,b]}(H; \alpha) \rightarrow \text{CF}^{[a,b]}(H; \alpha))}{\text{im}([\partial^{H,J}] : \text{CF}^{[a,b]}(H; \alpha) \rightarrow \text{CF}^{[a,b]}(H; \alpha))}.$$

These Floer homology groups are independent of the choice of the almost complex structure $J = \{J_t\}_{t \in S^1}$ in the sense that for any two almost complex structures $J_0, J_1 \in \mathcal{J}_{\text{reg}}(H; \alpha)$ there is a natural isomorphism

$$\tau_{J_1 J_0} : \text{HF}^{[a,b]}(H, J_0; \alpha) \rightarrow \text{HF}^{[a,b]}(H, J_1; \alpha).$$

If the two almost complex structure agree near the boundary then this follows from the standard arguments as in Floer’s original paper [F4] (choose a homotopy of almost complex structures $\{J_{s,t}\}$ from J_0 to J_1 , independent of s and t near the boundary, and use the solutions of equation (9) below with $H_{s,t} = H_t$ to construct the isomorphism between the two Floer homology groups; see also [Sa, SZ]). To show that the Floer homology groups are also independent of the choice of the *convex* almost complex structure near the boundary one can use the fact that the space of convex almost complex structures near the boundary is connected (Remark 4.1.2) and that the Floer chain complex associated to a regular almost complex structure remains unchanged under sufficiently small perturbations of J . The upshot is that the Floer homology groups are independent of J up to natural isomorphisms. For this reason we shall sometimes drop the argument J and refer to $\text{HF}^{[a,b]}(H; \alpha) := \text{HF}^{[a,b]}(H, J; \alpha)$ as the *Floer homology associated to H* .

4.4. Homotopy invariance. Following the work of Floer–Hofer [FH2], Cieliebak–Floer–Hofer [CFH], and Viterbo [V] we describe the local isomorphisms of Floer homology in a

given interval of the action spectrum. Consider the space

$$\mathcal{H}^{a,b}(M; \alpha) := \{H \in \mathcal{H}(M) \mid a, b \notin \mathcal{S}(H; \alpha)\}$$

of all Hamiltonians $H \in \mathcal{H}$ that do not contain a and b in their action spectrum. We consider the space \mathcal{H} with the strong Whitney C^∞ -topology. Note that the action spectrum $\mathcal{S}(H; \alpha)$ is compact for every H and is a lower semicontinuous function of H (i.e. for every open neighbourhood $V \subset \mathbb{R}$ of $\mathcal{S}(H; \alpha)$ there exists a neighbourhood $\mathcal{U} \subset \mathcal{H}$ of H such that $\mathcal{S}(H'; \alpha) \subset V$ for every $H' \in \mathcal{U}$). Hence the set $\mathcal{H}^{a,b}(M; \alpha)$ is open in \mathcal{H} . We now explain why the Floer homology groups $\mathrm{HF}^{[a,b]}(H; \alpha)$ are independent of H in every component of $\mathcal{H}^{a,b}(M; \alpha)$.

Fix a Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$ and choose a (convex) neighbourhood \mathcal{U} of H such that $\mathcal{U} \subset \mathcal{H}^{a,b}(M; \alpha)$. Now suppose that $H^+, H^- \in \mathcal{U}$ satisfy (H), i.e. all periodic solutions $x \in \mathcal{P}(H^\pm; \alpha)$ are nondegenerate. Connect H^- and H^+ by a smooth homotopy $\mathbb{R} \mapsto \mathcal{U} : s \mapsto H_s = \{H_{s,t}\}$ such that $H_{s,t} = H_t^-$ for $s \leq -T$ and $H_{s,t} = H_t^+$ for $s \geq T$. Consider the equation

$$(9) \quad \partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{s,t}}(u)) = 0,$$

where $s \mapsto \{J_{s,t}\}$ is a **regular homotopy** of families of almost complex structures. This means that $J_{s,t}$ satisfies the following conditions.

- $J_{s,t}$ is convex and independent of s and t near the boundary of \overline{M} .
- $J_{s,t} = J_t^-$ is regular for H_t^- for $s \leq -T$.
- $J_{s,t} = J_t^+$ is regular for H_t^+ for $s \geq T$.
- The finite energy solutions of (9) are transverse (i.e. the associated Fredholm operators are surjective) and hence form finite dimensional moduli spaces.

The key observation is the energy identity

$$(10) \quad E(u) = \mathcal{A}_{H^-}(x^-) - \mathcal{A}_{H^+}(x^+) + \int_0^1 \int_{-\infty}^{\infty} (\partial_s H)(s, t, u(s, t)) ds dt$$

for every solution of (9) and (8). It follows from (10) that

$$\mathcal{A}_{H^+}(x^+) \leq \mathcal{A}_{H^-}(x^-) + \int_{-\infty}^{\infty} \max_{S^1 \times M} \partial_s H_s ds.$$

In particular, if the homotopy has the form $H_{s,t} := H_{0,t} + \beta(s)(H_t^+ - H_t^-)$ for a nondecreasing function $\beta : \mathbb{R} \rightarrow [0, 1]$ we obtain $\partial_s H_s = \dot{\beta}(s)(H^+ - H^-)$ and hence

$$(11) \quad \mathcal{A}_{H^+}(x^+) \leq \mathcal{A}_{H^-}(x^-) + \max_{S^1 \times M} (H^+ - H^-).$$

Now choose $\varepsilon > 0$ such that

$$\mathcal{S}(H^\pm; \alpha) \cap [a - 4\varepsilon, a + 4\varepsilon] = \emptyset, \quad \mathcal{S}(H^\pm; \alpha) \cap [b - 4\varepsilon, b + 4\varepsilon] = \emptyset,$$

and suppose that $\sup_{S^1 \times M} |H^\pm - H| \leq \varepsilon$. Then $\sup_{S^1 \times M} |H^+ - H^-| \leq 2\varepsilon$ and hence, by (11), the Floer chain map (see [F4, FH2, CFH, V, Sa, SZ]) from $\text{CF}(H^-; \alpha)$ to $\text{CF}(H^+; \alpha)$ defined by the solutions of (9) preserves the subcomplexes CF^a and CF^b . The same applies to the Floer chain map from $\text{CF}(H^+; \alpha)$ to $\text{CF}(H^-; \alpha)$ and to the chain homotopy equivalence associated to a suitable homotopy of homotopies. Hence the solutions of (9) define a homomorphism $\text{CF}^{[a,b]}(H^-; \alpha) \rightarrow \text{CF}^{[a,b]}(H^+; \alpha)$ which induces an isomorphism of Floer homology, whenever H^\pm are sufficiently close to a given Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$.

Remark 4.4.1. (Local isomorphisms) The above discussion shows that every Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$ has a neighbourhood \mathcal{U} such that the Floer homology groups $\text{HF}^{[a,b]}(H', J'; \alpha)$, for every $H' \in \mathcal{U}$ that satisfies (H) and every regular almost complex structure $J' \in \mathcal{J}_{\text{reg}}(H'; \alpha)$, are naturally isomorphic. We can use these local isomorphisms to define the Floer homology groups $\text{HF}^{[a,b]}(H; \alpha)$ for every Hamiltonian $H \in \mathcal{H}^{a,b}(M; \alpha)$, whether or not it satisfies (H).

Remark 4.4.2. (Contractible loops) When $\alpha \in \tilde{\pi}_1(M)$ is the homotopy class of the constant loops we are not allowed to work with intervals $[a, b]$ that contain zero, since the Hamiltonians we work with always have degenerate periodic orbits with action zero as they vanish at infinity. In this case we are forced to work with either $0 < a < b \leq \infty$ or $-\infty \leq a < b < 0$.

Remark 4.4.3. (Composition) We emphasize that the canonical isomorphism

$$\text{HF}^{[a,b]}(H^-, J^-; \alpha) \rightarrow \text{HF}^{[a,b]}(H^+, J^+; \alpha)$$

only exists locally, when H^\pm are sufficiently close to a given Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$. It is easy to construct Hamiltonian functions $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ such that $\text{HF}^{[a,b]}(H_0; \alpha)$ is not isomorphic to $\text{HF}^{[a,b]}(H_1; \alpha)$. If H_0 and H_1 belong to the same component of $\mathcal{H}^{a,b}(M; \alpha)$ then there is a smooth path $[0, 1] \rightarrow \mathcal{H}^{a,b}(M; \alpha) : s \mapsto H_s$ connecting H_0 to H_1 . Hence in this case $\text{HF}^{[a,b]}(H_0; \alpha)$ is isomorphic to $\text{HF}^{[a,b]}(H_1; \alpha)$. However, in general the isomorphism cannot be defined directly in terms of the solutions of (9). It can only be constructed as a composition of isomorphisms

$$\text{HF}^{[a,b]}(H_{s_i}; \alpha) \rightarrow \text{HF}^{[a,b]}(H_{s_{i+1}}; \alpha)$$

for a regular homotopy, where each of these isomorphisms is defined in terms of the solutions of (9). Moreover, it is an open question if this composition along a loop $s \mapsto H_s$ with $H_0 = H_1$ is always the identity.

4.5. Monotone homotopies. Suppose that $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ satisfy

$$H_0(t, x) \geq H_1(t, x)$$

for all $(t, x) \in \mathbb{R} \times M$ as well as (H) . Then there exists a homotopy $s \mapsto H_s$ from H_0 to H_1 such that $\partial_s H_s \leq 0$. We call such a homotopy of Hamiltonian functions **monotone**. In the monotone case it follows from (10) that the Floer chain map $\text{CF}(H_0; \alpha) \rightarrow \text{CF}(H_1; \alpha)$, defined in terms of the solutions of (9) preserves the subcomplexes CF^a and CF^b . Hence every monotone homotopy $s \mapsto H_s$ induces a natural homomorphism

$$\sigma_{H_1 H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \rightarrow \text{HF}^{[a,b]}(H_1; \alpha).$$

We call such a homomorphism **monotone**. The standard arguments in Floer homology [F4, FH2, CFH, V, Sa, SZ] show that this homomorphism is independent of the choice of the monotone homotopy of Hamiltonians, used to define it, and that

$$\sigma_{H_2 H_1} \circ \sigma_{H_1 H_0} = \sigma_{H_2 H_0},$$

whenever $H_0, H_1, H_2 \in \mathcal{H}^{a,b}(M; \alpha)$ satisfy $H_0 \geq H_1 \geq H_2$, and $\sigma_{HH} = \text{id}$ for every $H \in \mathcal{H}^{a,b}(M; \alpha)$.

The homomorphism $\sigma_{H_1 H_0}$ is in general neither injective nor surjective. For example, it may happen that during the homotopy the action of some periodic orbit of H_s leaves or enters the interval $[a, b)$. It turns out that this is the only possible reason for $\sigma_{H_1 H_0}$ not to be an isomorphism. More precisely, we have the following proposition which is an easy consequence of the theory developed in [FH1, CFH] (as outlined above) and appears in an explicit form in [V].

Proposition 4.5.1. *Let $-\infty \leq a < b \leq \infty$, $\alpha \in \tilde{\pi}_1(M)$ be a nontrivial homotopy class, and $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ be such that $H_0 \geq H_1$. Suppose that there exists a monotone homotopy $\{H_s\}_{0 \leq s \leq 1}$ from H_0 to H_1 such that $H_s \in \mathcal{H}^{a,b}(M; \alpha)$ for every $s \in [0, 1]$. Then $\sigma_{H_1 H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \rightarrow \text{HF}^{[a,b]}(H_1; \alpha)$ is an isomorphism. This continues to hold for the trivial homotopy class $\alpha = 0$ provided that $0 \notin [a, b]$.*

Proof. The monotone homomorphism $\sigma_{H_{s_1} H_{s_0}}$ agrees with the local isomorphism of Section 4.4 whenever s_0 and s_1 are both sufficiently close to a number $s \in [0, 1]$ such that $\mathcal{H}_s \in \mathcal{H}^{a,b}(H_s; \alpha)$. By assumption, we have $H_s \in \mathcal{H}^{a,b}(H_s; \alpha)$ for every $s \in [0, 1]$. Hence we can write $\sigma_{H_1 H_0}$ as a composition of finitely many isomorphisms of the form $\sigma_{H_{s_{i+1}} H_{s_i}}$, where $0 = s_0 < s_1 < \dots < s_{N-1} < s_N = 1$. \square

4.6. Direct and inverse limits. The next step towards defining the relative capacity is to define two kinds of *symplectic homologies* especially suited for our purposes. The definitions of these invariants require the algebraic notions of direct and inverse limits. In

this subsection we recall the basic definitions (for more details see [GM], but note that below we use somewhat different conventions than theirs).

Let (I, \preceq) be a partially ordered set. Think of I as a category with precisely one morphism from i to j whenever $i \preceq j$. Let R be a commutative ring. A **partially ordered system** of R -modules over I is a functor from (I, \preceq) into the category of R -modules. We write this functor as a pair (G, σ) where G assigns to each $i \in I$ an R -module G_i and σ assigns to each pair $i, j \in I$ with $i \preceq j$ a homomorphism $\sigma_{ji} : G_i \rightarrow G_j$ such that $\sigma_{kj} \circ \sigma_{ji} = \sigma_{ki}$ for $i \preceq j \preceq k$ and $\sigma_{ii} = \text{id}$ is the identity map on G_i .

The partially ordered set (I, \preceq) is called **upward directed** if for every pair $i, j \in I$ there exists an $\ell \in I$ such that $i \preceq \ell$ and $j \preceq \ell$. In this case the functor (G, σ) is called a **directed system of R -modules**. The **direct limit** of such a directed system is defined as the quotient

$$\varinjlim G := \varinjlim G_i := \{(i, x) \mid i \in I, x \in G_i\} / \sim$$

where $(i, x) \sim (j, y)$ iff there exists an $\ell \in I$ such that $i \preceq \ell$, $j \preceq \ell$ and $\sigma_{\ell i}(x) = \sigma_{\ell j}(y)$. Since I is upward directed, this is an equivalence relation. The direct limit is an R -module with the operations $[i, x] + [j, y] := [\ell, \sigma_{\ell i}(x) + \sigma_{\ell j}(y)]$ for $\ell \in I$ such that $i \preceq \ell$ and $j \preceq \ell$ and $r[i, x] := [i, rx]$ for every $r \in R$. For $i \in I$ we denote by $\iota_i : G_i \rightarrow \varinjlim G$ the homomorphism given by $\iota_i(x) := [i, x]$. Then $\iota_i = \iota_j \circ \sigma_{ji}$ for $i \preceq j$. Despite the notation, ι_i need not be injective. Up to isomorphism the direct limit is characterized by the following universal property. If H is any R -module and $\tau_i : G_i \rightarrow H$ is a family of homomorphisms, indexed by $i \in I$, such that $\tau_i = \tau_j \circ \sigma_{ji}$ whenever $i \preceq j$, then there exists a unique homomorphism $\tau : \varinjlim G \rightarrow H$ such that $\tau_i = \tau \circ \iota_i$ for every $i \in I$. (The homomorphism τ is given by $[i, x] \mapsto \tau_i(x)$.)

The partially ordered set (I, \preceq) is called **downward directed** if for every pair $i, j \in I$ there exists a $k \in I$ such that $k \preceq i$ and $k \preceq j$. In this case the functor (G, σ) is called an **inverse system of R -modules**. The **inverse limit** of such an inverse system is defined as

$$\varprojlim G := \varprojlim G_i := \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} G_i \mid i \preceq j \implies \sigma_{ji}(x_i) = x_j \right\}.$$

For $j \in I$ we denote by $\pi_j : \varprojlim G \rightarrow G_j$ the obvious projection to the j th component. Then $\pi_j = \sigma_{ji} \circ \pi_i$ for $i \preceq j$. Despite the notation, π_j need not be surjective. Up to isomorphism the inverse limit is characterized by the following universal property. If H is any R -module and $\tau_j : H \rightarrow G_j$ is a family of homomorphisms, indexed by $j \in I$, such that $\tau_j = \sigma_{ji} \circ \tau_i$ whenever $i \preceq j$, then there exists a unique homomorphism $\tau : H \rightarrow \varprojlim G$ such that $\tau_j = \pi_j \circ \tau$ for every $j \in I$. (The homomorphism τ is given by $y \mapsto \{\tau_i(y)\}_{i \in I}$.)

Remark. Note that inverse and direct limits are related via the following duality. Let (G, σ) be a directed system of R -modules and H be any R -module. Denote by (I^*, \preceq^*) the oppositely partially ordered set, namely $I^* := I$ and $i \preceq^* j$ iff $i \succeq j$. Then there exists a canonical isomorphism

$$\mathrm{Hom}_R\left(\varprojlim_{i \in I} G_i, H\right) \cong \varinjlim_{i \in I^*} \mathrm{Hom}_R(G_i, H).$$

In particular, if R is a field and the G_i are vector spaces over R then $(\varprojlim_{i \in I} G_i)^* \cong \varinjlim_{i \in I^*} G_i^*$.

In most of our applications the partially ordered set (I, \preceq) will be **bidirected**, i.e. both upward and downward directed. In this case we call the functor (G, σ) a **bidirected system of R -modules**. The next lemma follows directly from the definitions.

Lemma 4.6.1. *Let (I, \preceq) be a downward directed partially ordered set and $I' \subset I$ be an upward directed subset (with respect to the restriction of the partial order \preceq to I'). Let (G, σ) be a partially ordered system of R -modules over I . Then there exists a unique homomorphism*

$$T : \varprojlim_{i \in I} G_i \rightarrow \varinjlim_{i' \in I'} G_{i'}$$

such that the following diagram commutes for all $j', k' \in I'$ with $j' \preceq k'$:

$$\begin{array}{ccc} \varprojlim_{i \in I} G_i & \xrightarrow{T} & \varinjlim_{i' \in I'} G_{i'} \\ \pi_{j'} \downarrow & & \uparrow \iota_{k'} \\ G_{j'} & \xrightarrow{\sigma_{k'j'}} & G_{k'} \end{array}$$

Proof. The map T is given by $\{x_i\}_{i \in I} \mapsto [i', x_{i'}]$ for $i' \in I'$. By definition of direct and inverse limits, this map is independent of the choice of i' . \square

In Lemma 4.6.1 we **do not** assume that for every $i \in I$ there exists an $i' \in I'$ such that $i \preceq i'$ (and indeed this condition is not satisfied in our application). If this holds then the map T factors through π_i for every $i \in I$ and not just for $i \in I'$. However, there are examples where $G_i = \{0\}$ for some $i \in I$ and $T \neq 0$.

4.7. Exhausting sequences. To compute direct and inverse limits we introduce the notion of exhausting sequences. Let (G, σ) be a partially ordered system of R -modules over (I, \preceq) and denote $\mathbb{Z}^\pm := \{\nu \in \mathbb{Z} \mid \pm \nu > 0\}$. A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^+}$ is called **upward exhausting** for (G, σ) iff the following holds

- For every $\nu \in \mathbb{Z}^+$ we have $i_\nu \preceq i_{\nu+1}$ and $\sigma_{i_{\nu+1}i_\nu} : G_{i_\nu} \rightarrow G_{i_{\nu+1}}$ is an isomorphism.
- For every $i \in I$ there exists a $\nu \in \mathbb{Z}^+$ such that $i \preceq i_\nu$.

A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^-}$ is called **downward exhausting** for (G, σ) iff the following holds

- For every $\nu \in \mathbb{Z}^-$ we have $i_{\nu-1} \preceq i_\nu$ and $\sigma_{i_\nu i_{\nu-1}} : G_{i_{\nu-1}} \rightarrow G_{i_\nu}$ is an isomorphism.
- For every $i \in I$ there exists a $\nu \in \mathbb{Z}^-$ such that $i_\nu \preceq i$.

The importance of such sequences is that they can be used to simplify computations of direct and inverse limits.

Lemma 4.7.1. *Let (G, σ) be a partially ordered system of R -modules over (I, \preceq) .*

(i) *If $\{i_\nu\}_{\nu \in \mathbb{Z}^+}$ is an upward exhausting sequence for (G, σ) then (I, \preceq) is upward directed and the homomorphism $\iota_{i_\nu} : G_{i_\nu} \rightarrow \varinjlim G$ is an isomorphism for every $\nu \in \mathbb{Z}^+$.*

(ii) *If $\{i_\nu\}_{\nu \in \mathbb{Z}^-}$ is a downward exhausting sequence for (G, σ) then (I, \preceq) is downward directed and the homomorphism $\pi_{i_\nu} : \varprojlim G \rightarrow G_{i_\nu}$ is an isomorphism for every $\nu \in \mathbb{Z}^-$.*

Proof. To prove (i) we fix an integer $\nu \in \mathbb{Z}^+$. Let $x \in G_{i_\nu}$ and suppose that $\iota_{i_\nu}(x) = 0$. Then there exists an $i \in I$ such that $i_\nu \preceq i$ and $\sigma_{ii_\nu}(x) = 0$. Choose an integer $\nu' \geq \nu$ such that $i \preceq i_{\nu'}$. Then $\sigma_{i_{\nu'} i_\nu}(x) = \sigma_{i_{\nu'} i} \circ \sigma_{ii_\nu}(x) = 0$ and hence $x = 0$. Hence ι_{i_ν} is injective. Now let $y \in G_j$ and choose an integer $\nu' \geq \nu$ such that $j \preceq i_{\nu'}$. Since $\sigma_{i_{\nu'} i_\nu}$ is surjective there exists an $x \in G_{i_\nu}$ such that $\sigma_{i_{\nu'} i_\nu}(x) = \sigma_{i_{\nu'} j}(y)$. Hence $(j, y) \sim (i_\nu, x)$. This shows that ι_{i_ν} is surjective

To prove (ii) we fix an integer $\nu \in \mathbb{Z}^-$. Let $\{x_i\}_{i \in I} \in \varprojlim G$ such that $x_{i_\nu} = 0$. Given $i \in I$ choose an integer $\nu' \leq \nu$ such that $i_{\nu'} \preceq i$. Then $\sigma_{i_{\nu'} i_\nu}(x_{i_{\nu'}}) = x_{i_\nu} = 0$, hence $x_{i_{\nu'}} = 0$, and hence $x_i = \sigma_{ii_{\nu'}}(x_{i_{\nu'}}) = 0$. This shows that π_{i_ν} is injective. Now let $x \in G_{i_\nu}$. Given $i \in I$, choose an integer $\nu' \leq \nu$ such that $i_{\nu'} \preceq i$ and define $x_i \in G_i$ by

$$x_i := \sigma_{ii_{\nu'}}(x_{i_{\nu'}}), \quad \sigma_{i_{\nu'} i_\nu}(x_{i_{\nu'}}) := x.$$

Since $\sigma_{i_{\nu'} i_\nu}$ is surjective the element $x_i \in G_i$ exists. Since $\sigma_{i_{\nu'} i_\nu}$ is injective, the element x_i is unique, and it is independent of the choice of ν' . We prove that $\sigma_{ji}(x_i) = x_j$ whenever $i \preceq j$. To see this choose $\nu' \leq \nu$ such that $i_{\nu'} \preceq i \preceq j$. Then

$$x_j = \sigma_{ji_{\nu'}}(x_{i_{\nu'}}) = \sigma_{ji} \circ \sigma_{ii_{\nu'}}(x_{i_{\nu'}}) = \sigma_{ji}(x_i).$$

Hence $\{x_i\}_{i \in I} \in \varprojlim G$ and $\pi_{i_\nu}(\{x_i\}_{i \in I}) = x_{i_\nu} = x$. This shows that π_{i_ν} is surjective. \square

4.8. Symplectic homology. The set $\mathcal{H}(M)$ of Hamiltonian functions on $S^1 \times M$ with compact support is partially ordered by

$$H_0 \preceq H_1 \quad \iff \quad H_0(t, x) \geq H_1(t, x) \quad \forall (t, x) \in S^1 \times M.$$

This defines a bidirected partial order on $\mathcal{H}(M)$. Let $\alpha \in \tilde{\pi}_1(M)$ be a nontrivial homotopy class and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$. As in Section 4.4 we denote by $\mathcal{H}^{a,b}(M; \alpha)$ the

subset of all Hamiltonian functions $H \in \mathcal{H}(M)$ such that $a, b \notin \mathcal{S}(H; \alpha)$. In Subsection 4.5 we have seen that there is a natural homomorphism

$$\sigma_{H_1 H_0} : \mathrm{HF}^{[a,b]}(H_0; \alpha) \rightarrow \mathrm{HF}^{[a,b]}(H_1; \alpha)$$

whenever $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ satisfy $H_0 \preceq H_1$. These homomorphisms define an inverse (in fact bidirected) system of Floer homology groups over $(\mathcal{H}^{a,b}(M; \alpha), \preceq)$. The inverse limit of this system is called the **symplectic homology** of M in the homotopy class α for the action interval $[a, b]$. A version of this homology group was introduced in [FH2, CFH] for the homotopy class of contractible loops and later on for general homotopy classes in [C-2]. We denote it by

$$\mathrm{SH}^{[a,b]}(M; \alpha) := \varprojlim_{H \in \mathcal{H}^{a,b}(M; \alpha)} \mathrm{HF}^{[a,b]}(H; \alpha).$$

Now fix a compact subset $A \subset M$ and a constant $c \in \mathbb{R}$. Consider the set $\mathcal{H}_c^{a,b}(M, A; \alpha)$ of all Hamiltonian functions $H \in \mathcal{H}^{a,b}(M; \alpha)$ that satisfy $H > c$ on $S^1 \times A$, namely

$$\mathcal{H}_c^{a,b}(M, A; \alpha) := \left\{ H \in \mathcal{H}^{a,b}(M; \alpha) \mid \inf_{S^1 \times A} H > c \right\}.$$

This gives rise to a directed (in fact bidirected) system of Floer homology groups over $(\mathcal{H}_c^{a,b}(M, A; \alpha), \preceq)$. The direct limit of this system is called the **relative symplectic homology** of the pair (M, A) at the **level** c in the homotopy class α for the action interval $[a, b]$. We denote it by

$$\mathrm{SH}^{[a,b];c}(M, A; \alpha) := \varinjlim_{H \in \mathcal{H}_c^{a,b}(M, A; \alpha)} \mathrm{HF}^{[a,b]}(H; \alpha).$$

Remark. Since we have chosen to work with \mathbb{Z}_2 -coefficients all the Floer homology groups $\mathrm{HF}^{[a,b]}(H; \alpha)$ are in fact \mathbb{Z}_2 -vector spaces. Consequently also the symplectic homologies $\mathrm{SH}^{[a,b]}(M; \alpha)$ and $\mathrm{SH}^{[a,b];c}(M, A; \alpha)$ have the structure of \mathbb{Z}_2 -vector spaces.

An important feature of absolute and relative symplectic homologies is the existence of a homomorphism between them which factors through Floer homology.

Proposition 4.8.1. *Let $\alpha \in \tilde{\pi}_1(M)$ be a nontrivial homotopy class and suppose that $-\infty \leq a < b \leq \infty$. Then, for every $c \in \mathbb{R}$, there exists a unique homomorphism*

$$T_\alpha^{[a,b];c} : \mathrm{SH}^{[a,b]}(M; \alpha) \rightarrow \mathrm{SH}^{[a,b];c}(M, A; \alpha)$$

such that for any two Hamiltonian functions $H_0, H_1 \in \mathcal{H}_c^{a,b}(M, A; \alpha)$ with $H_0 \geq H_1$ the following diagram commutes:

$$\begin{array}{ccc} \overleftarrow{\text{SH}}^{[a,b]}(M; \alpha) & \xrightarrow{T_\alpha^{[a,b];c}} & \overrightarrow{\text{SH}}^{[a,b];c}(M, A; \alpha) \\ \pi_{H_0} \downarrow & & \uparrow \iota_{H_1} \\ \text{HF}^{[a,b]}(H_0; \alpha) & \xrightarrow{\sigma_{H_1, H_0}} & \text{HF}^{[a,b]}(H_1; \alpha) \end{array}$$

Here $\pi_{H_0} : \overleftarrow{\text{SH}}^{[a,b]}(M; \alpha) \rightarrow \text{HF}^{[a,b]}(H_0; \alpha)$ and $\iota_{H_1} : \text{HF}^{[a,b]}(H_1; \alpha) \rightarrow \overrightarrow{\text{SH}}^{[a,b];c}(M, A; \alpha)$ are the canonical homomorphisms introduced in Section 4.6. In particular, since $\sigma_{HH} = \text{id}$ for every $H \in \mathcal{H}_c^{a,b}(M, A; \alpha)$, we have

$$\begin{array}{ccc} \overleftarrow{\text{SH}}^{[a,b]}(M; \alpha) & \xrightarrow{T_\alpha^{[a,b];c}} & \overrightarrow{\text{SH}}^{[a,b];c}(M, A; \alpha) \\ & \searrow \pi_H & \nearrow \iota_H \\ & \text{HF}^{[a,b]}(H; \alpha) & \end{array}$$

The statements above continue to hold also for the trivial class $\alpha = 0$, provided that $0 \notin [a, b]$.

Proof. The proof follows at once from Lemma 4.6.1. \square

4.9. The homological relative capacity. For every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(M)$ and every real number $c > 0$ we define the set

$$\mathcal{A}_c(M, A; \alpha) := \left\{ a \in \mathbb{R} \mid \text{The homomorphism } T_\alpha^{[a, \infty);c} \text{ does not vanish} \right\},$$

where $T_\alpha^{[a, \infty);c} : \overleftarrow{\text{SH}}^{[a, \infty)}(M; \alpha) \rightarrow \overrightarrow{\text{SH}}^{[a, \infty);c}(M, A; \alpha)$ is the homomorphism from Proposition 4.8.1.

For the trivial homotopy class $\alpha = 0 \in \tilde{\pi}_1(M)$ we define $\mathcal{A}_c(M, A; 0)$ by the same formula except that we only consider real numbers $a > 0$ (for which $T_0^{[a, \infty);c} \neq 0$). The **homological relative capacity** of the pair (M, A) is the function

$$\widehat{C}(M, A) : \tilde{\pi}_1(M) \times [-\infty, \infty) \rightarrow [0, \infty]$$

which assigns to the class $\alpha \in \tilde{\pi}_1(M)$ and the number $a \geq -\infty$ the number

$$\widehat{C}(M, A; \alpha, a) := \inf \{ c > 0 \mid \sup \mathcal{A}_c(M, A; \alpha) > a \}.$$

Here we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. For $a = -\infty$ we abbreviate

$$\widehat{C}(M, A; \alpha) := \widehat{C}(M, A; \alpha, -\infty) = \inf \{ c > 0 \mid \mathcal{A}_c(M, A; \alpha) \neq \emptyset \}.$$

The latter quantity is independent of the ω -primitive λ while $\widehat{C}(M, A; \alpha, a)$ does depend on this choice: the set $\mathcal{A}_c(M, A; \alpha) =: \mathcal{A}_c^\lambda(M, A; \alpha)$ depends on λ , but for two ω -primitives λ, λ' we have $\mathcal{A}_c^{\lambda'}(M, A; \alpha) = \mathcal{A}_c^\lambda(M, A; \alpha) - \int_\alpha (\lambda' - \lambda)$.

Proposition 4.9.1. *Let $\alpha \in \widetilde{\pi}_1(M)$ and $a \in \mathbb{R}$. If $\widehat{C}(M, A; \alpha, a) < \infty$ then every compactly supported Hamiltonian H on $S^1 \times M$ with $H|_{S^1 \times A} \geq \widehat{C}(M, A; \alpha, a)$ has a 1-periodic orbit in the homotopy class α with action $\mathcal{A}_H(x) \geq a$. In particular,*

$$\widehat{C}(M, A; \alpha, a) \geq C(M, A; \alpha, a).$$

Proof. Assume first that $\inf_{S^1 \times A} H > \widehat{C}(M, A; \alpha, a)$. Then, by definition of $\widehat{C}(M, A; \alpha, a)$, there exist two real numbers b and c such that

$$0 < c < \inf_{S^1 \times A} H, \quad a < b, \quad b \in \mathcal{A}_c(M, A; \alpha).$$

Hence, by definition of the set $\mathcal{A}_c(M, A; \alpha)$, the homomorphism

$$T_\alpha^{[b, \infty); c} : \underset{\leftarrow}{\text{SH}}^{[b, \infty)}(M; \alpha) \rightarrow \underset{\rightarrow}{\text{SH}}^{[b, \infty); c}(M, A; \alpha)$$

is nonzero. Now choose a sequence of Hamiltonian functions $H_i \in \mathcal{H}(M)$ such that H_i converges to H in the C^∞ -topology, $b \notin \mathcal{S}(H_i; \alpha)$, and $\inf_{S^1 \times A} H_i > c$ for every i . Then $H_i \in \mathcal{H}_c^{[b, \infty)}(M, A; \alpha)$ and so, by Proposition 4.8.1, the nonzero homomorphism $T_\alpha^{[b, \infty); c}$ factors through the Floer homology group $\text{HF}^{[b, \infty)}(H_i; \alpha)$ for every i . Hence there exists a sequence of periodic orbits $x_i \in \mathcal{P}(H_i; \alpha)$ such that $\mathcal{A}_{H_i}(x_i) > b$. Passing to a converging subsequence we get a periodic orbit $x \in \mathcal{P}(H; \alpha)$ with $\mathcal{A}_H(x) \geq b > a$. This proves the assertion in the case $\inf_{S^1 \times A} H > \widehat{C}(M, A; \alpha, a)$. If $\inf_{S^1 \times A} H = \widehat{C}(M, A; \alpha, a)$ the result follows by another approximation argument. \square

Remark 4.9.2. Both relative capacities have the following rescaling property. Let $c > 0$ and replace the symplectic form ω by $c\omega$ and the ω -primitive λ by $c\lambda$. Then

$$\widehat{C}(M, A, c\lambda; \alpha, ca) = c\widehat{C}(M, A, \lambda; \alpha, a), \quad C(M, A, c\lambda; \alpha, ca) = cC(M, A, \lambda; \alpha, a).$$

To see this, note that the Hamiltonian function $\widetilde{H} := cH$ has the same Hamiltonian vector field with respect to $\widetilde{\omega} := c\omega$ as the Hamiltonian function H with respect to ω , and that the symplectic action of a periodic orbit $x \in \mathcal{P}(\widetilde{H}, \widetilde{\omega}; \alpha) = \mathcal{P}(H, \omega; \alpha)$ with respect to \widetilde{H} and $\widetilde{\lambda} := c\lambda$ is equal to c times the action with respect to H and λ .

5. COMPUTATION OF THE CAPACITIES

We are now in a position to compute in certain cases the relative symplectic homology of the unit cotangent bundle U^*X of a compact connected Riemannian manifold X without boundary. We shall always work with the Liouville form λ_{can} as a primitive of the canonical symplectic form ω_{can} . We consider the following two cases.

(T) $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the flat torus.

(N) X has negative sectional curvature.

In either case we identify $\tilde{\pi}_1(U^*X)$ with $\tilde{\pi}_1(X)$ and in the case of the torus we identify $\tilde{\pi}_1(\mathbb{T}^n)$ with \mathbb{Z}^n . More precisely, we identify $k \in \mathbb{Z}^n$ with the homotopy class of the loop $[0, 1] \rightarrow \mathbb{T}^n : t \mapsto tk + \mathbb{Z}^n$.

5.1. The main results. In this section we state the main results about the (relative) symplectic homology of open subsets of cotangent bundles and show how they can be used to establish Theorem 3.2.1 (i) and (ii). The subsequent sections are devoted to their proofs. The first result concerns the symplectic homology in the trivial homotopy class.

Theorem 5.1.1. *Assume (T) or (N) and consider the trivial class $\alpha = 0 \in \tilde{\pi}_1(X)$. Then, for $a, c > 0$, we have*

$$\underset{\leftarrow}{\text{SH}}^{[a, \infty)}(U^*X; 0) \cong H_*(X; \mathbb{Z}_2),$$

and

$$\underset{\rightarrow}{\text{SH}}^{[a, \infty); c}(U^*X, X; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2), & \text{if } a \leq c, \\ 0, & \text{if } c < a. \end{cases}$$

Moreover, the homomorphism $T_0^{[a, \infty); c} : \underset{\leftarrow}{\text{SH}}^{[a, \infty)}(U^*X; 0) \rightarrow \underset{\rightarrow}{\text{SH}}^{[a, \infty); c}(U^*X, X; 0)$ is an isomorphism for $0 < a \leq c$. In particular, for every $a \in \mathbb{R}$

$$\widehat{C}(U^*X, X; 0, a) = \max\{0, a\}.$$

Theorem 5.1.2. *Assume (T) or (N) and consider a nontrivial homotopy class $0 \neq \alpha \in \tilde{\pi}_1(X)$. Denote by $\ell > 0$ the (unique) length of the geodesics in the class α . Let $P := \mathbb{T}^n$ in the case (T) and $P := S^1$ in the case (N). Then, for every $a \in \mathbb{R}$ and $c > 0$ we have*

$$\underset{\leftarrow}{\text{SH}}^{[a, \infty)}(U^*X; \alpha) \cong \begin{cases} 0, & \text{if } a < \ell, \\ H_*(P; \mathbb{Z}_2), & \text{if } a \geq \ell. \end{cases}$$

and

$$\underset{\rightarrow}{\text{SH}}^{[a, \infty); c}(U^*X, X; \alpha) \cong \begin{cases} H_*(P; \mathbb{Z}_2), & \text{if } 0 < a \leq c, \\ 0, & \text{if } a > c. \end{cases}$$

Moreover, the homomorphism $T_\alpha^{[a, \infty); c} : \underset{\leftarrow}{\text{SH}}^{[a, \infty)}(U^*X; \alpha) \rightarrow \underset{\rightarrow}{\text{SH}}^{[a, \infty); c}(U^*X, X; \alpha)$ is an isomorphism for $\ell \leq a \leq c$. In particular, for every $a \in \mathbb{R}$

$$\widehat{C}(U^*X, X; \alpha, a) = \max\{\ell, a\}.$$

We are now in position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1 (i) and (ii). Assume that X satisfies (T) or (N). If $\alpha = 0$ we must prove that $C(U^*X, X; 0, a) = \max\{0, a\}$. To see this, note that every compactly supported Hamiltonian function $H \in \mathcal{H}(U^*X)$ has a contractible periodic orbit x with action $\mathcal{A}_H(x) = 0$ and hence $C(U^*X, X; 0, a) = 0$ whenever $a \leq 0$. If $a > 0$ then Theorem 5.1.1 asserts that $\widehat{C}(U^*X, X; 0, a) = a$ and hence

$$a = \widehat{C}(U^*X, X; 0, a) \geq C(U^*X, X; 0, a) \geq a.$$

Here the middle inequality follows from Proposition 4.9.1. To prove the right-hand inequality let $0 < \delta < a$ and choose any Hamiltonian function $H = H(p)$ that depends only on the momenta variables and satisfies $\max H = a - \delta$. Then every contractible periodic orbit $x \in \mathcal{P}(H; 0)$ is (up to parametrization) a contractible geodesic and hence, since X satisfies (T) or (N), is constant and has action $\mathcal{A}_H(x) = H(x) = a - \delta$. This shows that $C(U^*X, X; 0, a) \geq a - \delta$ for every $\delta > 0$. Thus we have proved that $C(U^*X, X; 0, a) = \max\{0, a\}$ as claimed.

Now assume $\alpha \neq 0$ and abbreviate $\ell := \ell(\gamma_\alpha)$ in the case (N) and $\ell := |k|$ in the case (T) with $\alpha = k \in \mathbb{Z}^n$. Then Theorem 5.1.2 asserts that $\widehat{C}(U^*X, X; \alpha, a) = \max\{\ell, a\}$ and hence

$$\max\{\ell, a\} = \widehat{C}(U^*X, X; \alpha, a) \geq C(U^*X, X; \alpha, a) \geq \max\{\ell, a\}$$

for every real number a . Again the middle inequality follows from Proposition 4.9.1 and the rightmost inequality from an explicit construction of a Hamiltonian function. Namely, for any $\delta > 0$ choose a compactly supported function $f : [0, 1) \rightarrow \mathbb{R}$ such that

$$f(r) = \begin{cases} m - \delta, & \text{for } r \text{ near } 0, \\ 0, & \text{for } r \text{ near } 1, \end{cases}$$

where $m := \max\{\ell, a\}$, and

$$f(r) < (1 - r)m, \quad -m < f'(r) \leq 0$$

for every r . Now consider the compactly supported Hamiltonian function $H := f(|p|)$ on U^*X . Its 1-periodic solutions are reparametrized closed geodesics. The sphere bundle $|p| = r$ contains a periodic orbit x in the class α if and only if $f'(r) = -\ell$ and the action of this periodic orbit is

$$\mathcal{A}_H(x) = f(r) - rf'(r) < f(r) + rm < m.$$

(See Lemma 5.3.2 below.) If $a \leq \ell$ then $f'(r) > -\ell$ for all r , hence there is no 1-periodic solution of length ℓ , and hence none in the class α . If $\ell \leq a$ then every 1-periodic solution has action $\mathcal{A}_H(x) < a$. In either case there is no 1-periodic orbit in the class α with action

at least a , and hence $C(U^*X, X; 0, a) \geq m - \delta$. Since this holds for every $\delta > 0$ we obtain $C(U^*X, X; 0, a) \geq m$ as claimed. \square

5.2. Morse–Bott theory in Floer homology. Let us return to the general setting of Section 4.2 where (\overline{M}, ω) is a compact connected symplectic manifold with convex boundary, $\omega = d\lambda$ is an exact symplectic form, $M = \overline{M} \setminus \partial\overline{M}$, $\mathcal{H} = \mathcal{H}(M)$ denotes the space of compactly supported functions on $S^1 \times M$, and \mathcal{J} denotes the space of 1-periodic ω -compatible almost complex structures $J_t = J_{t+1}$ on M .

A subset $P \subset \mathcal{P}(H)$ is called a **Morse–Bott manifold of periodic orbits** if the set $C_0 := \{x(0) \mid x \in P\}$ is a compact submanifold of M and $T_{x_0}C_0 = \ker(D\psi_1(x_0) - \mathbb{1})$ for every $x_0 \in C_0$.

Remark 5.2.1. The Morse–Bott condition can be reformulated as follows. Firstly, a subset $P \subset \mathcal{P}(H)$ is a compact submanifold of the loop space LM if and only if the set $C_0 = \{x(0) \mid x \in P\}$ is a compact submanifold of M . Secondly, for every $x \in P$ the kernel of the linear map $D\psi_1(x(0)) - \mathbb{1}$ on $T_{x(0)}M$ is isomorphic to the space of periodic solutions of the following *linearized Hamiltonian differential equation* for vector fields $\xi(t) \in T_{x(t)}M$ along x :

$$(12) \quad \nabla_{\dot{x}}\xi = \nabla_{\xi}X_H(x), \quad \xi(t+1) = \xi(t),$$

where ∇ stands for the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. To see this just note that ξ satisfies (12) if and only if $\xi(t) = D\psi_t(x(0))\xi(0)$ for all t . Note that every tangent vector of P is a solution of (12). The Morse–Bott condition can now be expressed in the form that P is a compact submanifold of LM and

$$T_xP = \{\xi \in C^\infty(S^1, x^*TM) \mid \xi \text{ satisfies (12)}\}.$$

We emphasize that the Hessian of the symplectic action functional $\mathcal{A}_H : LM \rightarrow \mathbb{R}$ at a critical point x is the linear operator $\xi \mapsto \nabla_{\xi}(\text{grad}H) - (\nabla_{\xi}J)\dot{x} - J\nabla_{\dot{x}}\xi = J(\nabla_{\xi}X_H - \nabla_{\dot{x}}\xi)$ on $C^\infty(S^1, x^*TM)$ (equipped with the L^2 -inner product). Hence the space of solutions of (12) is the kernel of the Hessian of \mathcal{A}_H at x , and the Morse–Bott condition asserts that the kernel of the Hessian agrees with the tangent space of the critical manifold P .

Theorem 5.2.2. *Let $-\infty \leq a < b \leq \infty$, $\alpha \in \tilde{\pi}_1(M)$, and $H \in \mathcal{H}^{a,b}(M; \alpha)$. Suppose that the set $P := \{x \in \mathcal{P}(H; \alpha) \mid a < \mathcal{A}_H(x) < b\}$ is a connected Morse–Bott manifold of periodic orbits. Then $\text{HF}^{[a,b]}(H; \alpha) \cong H_*(P; \mathbb{Z}_2)$.*

This is a version of Pozniak’s theorem [P] which was originally proved in the context of Floer homology for Lagrangian intersections. In this section we explain the reduction of this theorem to Pozniak’s original one.

In order to reformulate Floer homology in the Lagrangian setting let us consider the symplectic manifold

$$\widetilde{M} := M \times M, \quad \widetilde{\omega} := \omega \oplus (-\omega) = d\widetilde{\lambda}, \quad \widetilde{\lambda} := \lambda \oplus (-\lambda).$$

Since $\widetilde{\omega}$ is exact there are no nonconstant holomorphic spheres in \widetilde{M} , for any $\widetilde{\omega}$ -compatible almost complex structure $\widetilde{J} \in \mathcal{J}(\widetilde{M}, \widetilde{\omega})$. Since $\widetilde{\lambda}$ vanishes on the diagonal $\Delta \subset M \times M$, there are also no nonconstant holomorphic disks with boundary in Δ . Hence the standard theory of Floer homology for Lagrangian intersections applies as in Floer's original work [F1, F2, F3]. Given $H \in \mathcal{H}(M)$ define $\widetilde{H}_t : \widetilde{M} \rightarrow \mathbb{R}$ by

$$\widetilde{H}_t(x_0, x_1) := H_t(x_0) + H_{1-t}(x_1).$$

The Hamiltonian isotopy generated by \widetilde{H}_t with respect to $\widetilde{\omega}$ is given by $\widetilde{\psi}_t(x_0, x_1) = (\psi_t(x_0), \psi_{1-t} \circ \psi_1^{-1}(x_1))$. Given $J \in \mathcal{J}(M)$ define $\widetilde{J}_t \in \mathcal{J}(\widetilde{M}, \widetilde{\omega})$ by

$$(13) \quad \widetilde{J}_t := \psi_t^* J_t \times (-\psi_{1-t} \circ \psi_1^{-1})^* J_{1-t} = \left(\widetilde{\psi}_t \right)^* (J_t \times (-J_{1-t}))$$

for $0 \leq t \leq 1/2$. Given $u : \mathbb{R} \times S^1 \rightarrow M$ define $\widetilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \widetilde{M}$ by

$$\widetilde{u}(s, t) := (\psi_t^{-1}(u(s, t)), \psi_1 \circ \psi_{1-t}^{-1}(u(s, 1-t))) = \left(\widetilde{\psi}_t \right)^{-1} (u(s, t), u(s, 1-t)).$$

Then u satisfies (7) if and only if \widetilde{u} satisfies the Lagrangian boundary value problem

$$(14) \quad \partial_s \widetilde{u} + \widetilde{J}_t(\widetilde{u}) \partial_t \widetilde{u} = 0, \quad \widetilde{u}(s, 0) \in \Delta, \quad \widetilde{u}(s, 1/2) \in \text{graph}(\psi_1).$$

It satisfies (8) if and only if \widetilde{u} satisfies

$$(15) \quad \lim_{s \rightarrow \pm\infty} \widetilde{u}(s, t) = \widetilde{x}^\pm, \quad \lim_{s \rightarrow \pm\infty} \partial_s \widetilde{u}(s, t) = 0,$$

where $\widetilde{x}^\pm := (x^\pm(0), x^\pm(0)) \in \Delta \cap \text{graph}(\psi_1)$. The solutions of (14) can be interpreted as the gradient flow lines of the action functional

$$\widetilde{\mathcal{A}}(\widetilde{x}) := - \int_0^{1/2} \widetilde{\lambda}(\dot{\widetilde{x}}(t)) dt$$

on the space $\widetilde{\mathcal{P}}$ of paths $\widetilde{x} : [0, 1/2] \rightarrow \widetilde{M}$ with endpoints $\widetilde{x}(0) \in \Delta$ and $\widetilde{x}(1/2) \in \text{graph}(\psi_1)$ (with respect to the L^2 metric determined by \widetilde{J}). Note that the composition of $\widetilde{\mathcal{A}}$ with the map $LM \rightarrow \widetilde{\mathcal{P}} : x \mapsto \widetilde{x}$, given by $\widetilde{x}(t) := (\psi_t^{-1}(x(t)), \psi_1 \circ \psi_{1-t}^{-1}(x(1-t)))$, agrees with \mathcal{A}_H . Note also that this map induces a bijection

$$\widetilde{\pi}_1(M) = \pi_0(LM) \rightarrow \pi_0(\widetilde{\mathcal{P}}) : \alpha \mapsto \widetilde{\alpha}.$$

Hence the solutions of (14) can be used to define the Floer homology groups of the pair $(\Delta, \text{graph}(\psi_1))$ of Lagrangian submanifolds of $(\widetilde{M}, \widetilde{\omega})$. Moreover, the Floer homology groups defined by the solutions of (14) are independent of the choice of the (regular)

almost complex structure \tilde{J}_t used to define them. More precisely, denote by $\tilde{\mathcal{J}}$ the space of smooth functions $[0, 1/2] \rightarrow \mathcal{J}(\tilde{M}, \tilde{\omega}) : t \mapsto \tilde{J}_t$ such that $\tilde{J}_t = J \times (-J)$ near the boundary of \tilde{M} , where $J \in \mathcal{J}(M, \omega)$ is convex. Given a Hamiltonian $H \in \mathcal{H}$ that satisfies (H) denote by $\tilde{\mathcal{J}}_{\text{reg}}(H)$ the set of all almost complex structures $\tilde{J} \in \tilde{\mathcal{J}}$ such that every finite energy solution \tilde{u} of (14) is regular in the sense that the linearized operator along \tilde{u} is surjective. Then the solutions of (14) give rise to Lagrangian Floer homology groups $\text{HF}^{[a,b]}(\Delta, \text{graph}(\psi_1); \tilde{J}, \tilde{\alpha})$. Moreover, it follows as in [F1, F2, F3] (and as outlined above) that these Floer homology groups are independent of the almost complex structure $\tilde{J} \in \tilde{\mathcal{J}}_{\text{reg}}(H)$ used to define them. Note that if $J \in \mathcal{J}_{\text{reg}}(H)$ and \tilde{J} is given by (13) then $\tilde{J} \in \tilde{\mathcal{J}}_{\text{reg}}(H)$. Hence there is a natural isomorphism

$$\text{HF}^{[a,b]}(H; \alpha) \cong \text{HF}^{[a,b]}(\Delta, \text{graph}(\psi_1); \tilde{\alpha})$$

for every $\alpha \in \tilde{\pi}_1(M)$ and every Hamiltonian $H \in \mathcal{H}^{a,b}(M; \alpha)$, where $\tilde{\alpha}$ is the image of α under the above homomorphism $\tilde{\pi}_1(M) = \pi_0(LM) \rightarrow \pi_0(\tilde{\mathcal{P}})$. The advantage of the Lagrangian approach in the present context is that we can use any (regular) family of $\tilde{\omega}$ -compatible almost complex structures $\{\tilde{J}_t\}_{0 \leq t \leq 1/2}$ to define the Floer homology groups, and are not restricted to those arising from periodic families of almost complex structures on M via (13). Hence we can apply the results of Pozniak.

Let $\tilde{J} \in \tilde{\mathcal{J}}$ and $H \in \mathcal{H}$ and denote by $\mathbb{R} \rightarrow \text{Diff}(M, \omega) : t \mapsto \psi_t$ the Hamiltonian isotopy generated by H . The **graph** of a loop $x \in LM$ is the set

$$\Gamma(x) := \{(t, \psi_t^{-1}(x(t)), \psi_1 \circ \psi_{1-t}^{-1}(x(1-t))) \mid 0 \leq t \leq 1/2\} \subset [0, 1/2] \times \tilde{M}.$$

For a subset $P \subset LM$ we write $\Gamma(P) := \bigcup_{x \in P} \Gamma(x)$, and for a map $\tilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M}$ we write

$$\Gamma(\tilde{u}) := \{(t, \tilde{u}(s, t)) \mid s \in \mathbb{R}, 0 \leq t \leq 1/2\}.$$

A subset $P \subset \mathcal{P}(H)$ is called a **\tilde{J} -isolated periodic set** if there exists an open neighbourhood $U \subset [0, 1/2] \times \tilde{M}$ of $\Gamma(P)$ such that the following holds:

- (P1) The closure \bar{U} is a compact subset of $[0, 1/2] \times \tilde{M}$.
- (P2) If $\tilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M}$ is a finite energy solution of (14) with $\Gamma(\tilde{u}) \subset \bar{U}$ then there exists an $x \in P$ such that $u(s, t) = x(t)$ for every $(s, t) \in \mathbb{R}^2$.

An open neighbourhood $U \subset [0, 1/2] \times \tilde{M}$ of $\Gamma(P)$ that satisfies (P1) and (P2) is called **\tilde{J} -isolating**. Note that every \tilde{J} -isolated periodic set is compact.

Lemma 5.2.3. *Let $H \in \mathcal{H}(M)$. Then every Morse–Bott manifold $P \subset \mathcal{P}(H)$ of periodic orbits is a \tilde{J} -isolated periodic set for every almost complex structure $\tilde{J} \in \tilde{\mathcal{J}}$.*

Proof. We may assume without loss of generality that P is connected. Let \tilde{d}_t denote the distance function of the metric $\langle \cdot, \cdot \rangle_t := \tilde{\omega}(\cdot, \tilde{J}_t \cdot)$ and consider the open set

$$U := \left\{ (t, \tilde{x}) \mid 0 \leq t \leq 1/2, \tilde{x} \in \tilde{M}, \sup_{y \in P} \tilde{d}_t(\tilde{x}, (y(1/2 - t), y(1/2 + t))) < \varepsilon \right\} \subset [0, 1/2] \times \tilde{M}.$$

Let $\varepsilon > 0$ be sufficiently small. Then since C_0 is an isolated fixed point set for ψ_1 it follows that every $x \in \mathcal{P}(H)$ with $\Gamma(x) \subset \bar{U}$ is an element of P . Now the set U satisfies (P2) because every finite energy solution $\tilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M}$ of (14) with $\Gamma(\tilde{u}) \subset \bar{U}$ is asymptotic to the set P as $s \rightarrow \pm\infty$. Since $\mathcal{A}_{\tilde{H}} = \mathcal{A}_H$ is constant along P it follows that every such solution \tilde{u} has energy $E(\tilde{u}) = 0$ and hence has the form $\tilde{u}(s, t) = \tilde{x}(t) = (x(1/2 - t), x(1/2 + t))$ for some $x \in \mathcal{P}(H)$. \square

Lemma 5.2.4. *Let $H \in \mathcal{H}$ and $J \in \tilde{\mathcal{J}}$. Suppose that $P \subset \mathcal{P}(H)$ is a \tilde{J} -isolated periodic set and $U \subset [0, 1/2] \times \tilde{M}$ is a \tilde{J} -isolating neighbourhood of $\Gamma(P)$. Then there exist a compact neighbourhood $V \subset U$ of $\Gamma(P)$ and a constant $\delta > 0$ such that the following holds. If $\mathbb{R} \rightarrow \mathcal{H}(M) : s \mapsto H_s$ and $\mathbb{R} \rightarrow \mathcal{J}(\tilde{M}) : s \mapsto \tilde{J}_s$ are smooth homotopies such that*

$$\|H_s - H\|_{C^2} + \|\tilde{J}_s - \tilde{J}\|_{C^1} + \|\partial_s H_s\|_{C^2} + \|\partial_s \tilde{J}_s\|_{C^1} < \delta$$

and $\partial_s H_s = 0$ and $\partial_s \tilde{J}_s = 0$ for $|s| \geq 1$ then every finite energy solution $\tilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M}$ of (14) with (\tilde{H}, \tilde{J}) replaced by $(\tilde{H}_s, \tilde{J}_s)$ satisfies

$$\Gamma(\tilde{u}) \subset \bar{U} \quad \implies \quad \Gamma(\tilde{u}) \subset V.$$

Proof. Suppose, by contradiction, that there exist sequences

$$\mathbb{R} \rightarrow \mathcal{H} : s \mapsto H_s^\nu, \quad \mathbb{R} \rightarrow \tilde{\mathcal{J}} : s \mapsto \tilde{J}_s^\nu, \quad \tilde{u}^\nu : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M},$$

and $(s^\nu, t^\nu) \in \mathbb{R} \times [0, 1/2]$ such that the following holds:

- (i) $\lim_{\nu \rightarrow \infty} \sup_{s \in \mathbb{R}} \left(\|H_s^\nu - H_s\|_{C^2} + \|\partial_s H_s^\nu\|_{C^2} + \|\tilde{J}_s^\nu - \tilde{J}\|_{C^1} + \|\partial_s \tilde{J}_s^\nu\|_{C^1} \right) = 0.$
- (ii) $\partial_s H_s^\nu = 0$ and $\partial_s \tilde{J}_s^\nu = 0$ for $|s| \geq 1$.
- (iii) \tilde{u}^ν is a finite energy solution of (14) with (\tilde{H}, \tilde{J}) replaced by $(\tilde{H}_s^\nu, \tilde{J}_s^\nu)$.
- (iv) $\Gamma(\tilde{u}^\nu) \subset \bar{U}$ and $\lim_{\nu \rightarrow \infty} \tilde{u}^\nu(s^\nu, t^\nu) \in \partial U$.

Since there are no nonconstant \tilde{J}_t -holomorphic spheres in \tilde{M} and no nonconstant \tilde{J}_t -holomorphic disks with boundary in Δ , the first derivatives of the functions \tilde{u}_ν are uniformly bounded. Hence, by Floer–Gromov compactness [F4, G, MS, Sa], there exists a subsequence, still denoted by \tilde{u}^ν , such that the shifted sequence $\tilde{u}^\nu(s^\nu + s, t)$ converges in the C^1 -topology on compact sets to a finite energy solution $\tilde{u} : \mathbb{R} \times [0, 1/2] \rightarrow \tilde{M}$ of (14) such that $\Gamma(\tilde{u}) \subset \bar{U}$. By taking a further subsequence we may assume that $t^\nu \rightarrow t$ and hence $\tilde{u}(0, t) = \lim_{\nu \rightarrow \infty} \tilde{u}^\nu(s^\nu, t^\nu)$ satisfies $(t, \tilde{u}(0, t)) \in \partial U \subset [0, 1/2] \times (M \setminus U)$. This contradicts (P2). \square

Lemma 5.2.4 enables us to define the **local Floer homology** $\mathrm{HF}^{\mathrm{loc}}(H; P)$ of a \tilde{J} -isolated periodic set $P \subset \mathcal{P}(H)$ as follows. Choose a \tilde{J} -isolating neighbourhood $U \subset S^1 \times M$ of $\Gamma(P)$, let $\delta > 0$ be as in Lemma 5.2.4, choose a Hamiltonian function H' such that all periodic solutions $x \in \mathcal{P}(H')$ are nondegenerate and $\|H' - H\|_{C^2} < \delta/4$, and choose a regular almost complex structure $\tilde{J}' \in \tilde{\mathcal{J}}_{\mathrm{reg}}(H')$ such that $\|\tilde{J}' - \tilde{J}\|_{C^1} < \delta/4$. Then, by Lemma 5.2.4, all the Floer connecting orbits of (\tilde{H}', \tilde{J}') (i.e. solutions \tilde{u}' of (14) with (\tilde{H}, \tilde{J}) replaced by (\tilde{H}', \tilde{J}')) in \bar{U} are actually contained in U . Denote the set of local periodic orbits of H' near P by

$$\mathcal{P}(H'; U) := \{x' \in \mathcal{P}(H') \mid \Gamma(x') \subset U\}$$

and consider the local Floer chain complex

$$\mathrm{CF}^{\mathrm{loc}}(H'; U) := \bigoplus_{x' \in \mathcal{P}(H'; U)} \mathbb{Z}_2 x'.$$

The boundary operator $\partial^{H', \tilde{J}'; U} : \mathrm{CF}^{\mathrm{loc}}(H'; U) \rightarrow \mathrm{CF}^{\mathrm{loc}}(H'; U)$ is defined by counting the index-1 solutions \tilde{u}' of (14), with (\tilde{H}, \tilde{J}) replaced by (\tilde{H}', \tilde{J}') , such that $\Gamma(\tilde{u}') \subset U$. Since these solutions can never converge to the boundary of U it follows that $\partial^{H', \tilde{J}'; U}$ is indeed a boundary operator and the local Floer homology is defined by

$$\mathrm{HF}^{\mathrm{loc}}(H', \tilde{J}'; U) := H_*(\mathrm{CF}^{\mathrm{loc}}(H'; U), \partial^{H', \tilde{J}'; U}).$$

The same arguments as in Floer's original theory [F1, F2, F3] now show that this local Floer homology is independent (up to natural isomorphisms) of the isolating neighbourhood U , and of the perturbations H' and \tilde{J}' used to define it. We write

$$\mathrm{HF}^{\mathrm{loc}}(H; P) := \mathrm{HF}^{\mathrm{loc}}(H', \tilde{J}'; U)$$

for the local Floer homology in a \tilde{J} -isolating neighbourhood U of $\Gamma(P)$. Strictly speaking, this is a *connected simple system* in the sense of Conley, namely a small category whose objects are the triples $(H', \tilde{J}'; U)$ of local perturbations and whose morphisms are the canonical (unique) isomorphisms between the local Floer homologies $\mathrm{HF}^{\mathrm{loc}}(H'_0, \tilde{J}'_0, U_0)$ and $\mathrm{HF}^{\mathrm{loc}}(H'_1, \tilde{J}'_1, U_1)$. The details of this construction were carried out by Pozniak [P] in the context of Lagrangian intersections.

Theorem 5.2.5. (Pozniak [P]) *Let $H \in \mathcal{H}(M)$ and suppose that $P \subset \mathcal{P}(H)$ is a connected Morse–Bott manifold of periodic orbits. Then $\mathrm{HF}^{\mathrm{loc}}(H; P) \cong H_*(P; \mathbb{Z}_2)$.*

Proof. The local Floer homology of H near P is isomorphic to the local Floer homology of the pair of Lagrangian submanifolds $L_0 := \Delta \subset M \times M$ and $L_1 := \mathrm{graph}(\psi_1) \subset M \times M$ near their clean intersection $\Lambda := \{(x(0), x(0)) \mid x \in P\}$. Hence, by [P, Theorem 3.4.11], it is isomorphic to $H_*(\Lambda; \mathbb{Z}_2) \cong H_*(P; \mathbb{Z}_2)$. \square

Proof of Theorem 5.2.2. Fix a 1-periodic almost complex structure $J \in \mathcal{J}$ and let $\tilde{J} \in \tilde{\mathcal{J}}$ be given by (13). Then, by Lemma 5.2.3, P is a \tilde{J} -isolated periodic set. Let U be a \tilde{J} -isolating neighbourhood of $\Gamma(P)$ and choose a sequence of regular perturbations (H^ν, J^ν) that agree with (H, J) in some neighbourhood of $\partial\overline{M}$ and converge to (H, J) in the C^2 -norm. We claim that, for ν sufficiently large, all the Floer connecting orbits (i.e solutions of (7)) for the pair (H^ν, J^ν) in the homotopy class α and the action interval $[a, b]$ are contained in U . Otherwise, there has to be a sequence u^ν of such connecting orbits passing through $M \setminus U$ and we can argue as in the proof of Lemma 5.2.4 that, in the limit $\nu \rightarrow \infty$, there must be a finite energy solution of (7) for the pair (H, J) in the homotopy class α and the action interval $[a, b]$ that passes through $M \setminus U$. However, every such connecting orbit has the form $u(s, t) = x(t)$ for some $x \in P$ and so is contained in U . This contradiction proves the claim. Hence $\text{HF}^{[a,b]}(H; \alpha) \cong \text{HF}^{\text{loc}}(H; P)$, and hence the result follows from Theorem 5.2.5. \square

5.3. The main example. In this section we consider the case where $M = U^*X$ is the open unit cotangent bundle of a compact connected Riemannian n -manifold X without boundary that satisfies either (T) (i.e. X is a flat torus) or (N) (i.e. X has negative sectional curvature). We shall use the metric to identify the tangent bundle with the cotangent bundle and denote a point in U^*X by $x = (q, p)$ where $q \in X$ and $p \in T_q^*X$. Let $H : U^*X \rightarrow \mathbb{R}$ be a compactly supported Hamiltonian function of the form

$$H(q, p) = f(|p|),$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $f(r) = f(-r)$. The corresponding Hamiltonian differential equation has the form

$$(16) \quad \dot{q} = \frac{f'(|p|)}{|p|}p, \quad \nabla_{\dot{q}}p = 0.$$

Here ∇ denotes the Levi-Civita connection. Since $|p|$ is constant it follows that q is a geodesic for every solution (q, p) of (16). Moreover, since $f'(0) = 0$, the zero section $\{p = 0\}$ consists of constant solutions. There are other constant solutions $x(t) \equiv (q, p)$ whenever $f'(|p|) = 0$ but these will not be relevant in the context of the present paper.

Lemma 5.3.1. *The set $P_0 := \{x = (p, q) : S^1 \rightarrow T^*X \mid \dot{q} \equiv 0, p \equiv 0\}$ is a Morse-Bott manifold of periodic orbits for H if and only if $f''(0) \neq 0$.*

Proof. Since f is even there exists a smooth function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(r) = h(r^2)/2$. Then $h'(r^2) = f'(r)/r$ and $h'(0) = f''(0)$. So equation (16) reads

$$\dot{q} = h'(|p|^2)p, \quad \nabla_{\dot{q}}p = 0.$$

By Remark 5.2.1, P_0 is a Morse–Bott manifold if and only if the space of periodic solutions of the linearized equation is equal to the tangent space of P_0 for every $x = (q, p) \in P_0$. This means that the space of periodic solutions of the linearized equation has the same dimension as P_0 . Now the linearized equation at a constant solution with $p \equiv 0$ has the form

$$\frac{d}{dt}\hat{q} = h'(0)\hat{p} = f''(0)\hat{p}, \quad \frac{d}{dt}\hat{p} = 0.$$

If $f''(0) \neq 0$ then the space of periodic solutions of this equation has dimension $n = \dim P_0$ and if $f''(0) = 0$ it has dimension $2n$. \square

Lemma 5.3.1 continues to hold for any compact Riemannian manifold X . However, in the case (T) or (N), every nonconstant closed geodesic is not contractible. Let us now consider a nonzero homotopy class $0 \neq \alpha \in \tilde{\pi}_1(X)$ and denote by ℓ the (unique) length of the closed geodesics in the class α . The space of solutions to equation (16) that represent the class α consists of $\{(q(t), p(t))\}$, where

$$(17) \quad \begin{cases} q(t) \text{ is a geodesic in the class } \alpha, \text{ parametrized so that } |\dot{q}| \equiv \ell. \\ p(t) = \pm \frac{r}{\ell} \dot{q}(t), \text{ where } r > 0 \text{ is such that } f'(r) = \pm \ell. \end{cases}$$

Given $r > 0$ with $f'(r) = \pm \ell$ we denote

$$P^\pm(r, \alpha) := \left\{ x = (q, p) : S^1 \rightarrow U^*X \mid p(t), q(t) \text{ satisfy (17)} \right\}.$$

In the case (T) the space $P^\pm(r, \alpha)$ is diffeomorphic to X and in the case (N) it is diffeomorphic to S^1 .

Lemma 5.3.2. *Assume X satisfies (T) or (N) and let $\alpha \neq 0$, ℓ , $r > 0$ with $f'(r) = \pm \ell$, and $P^\pm(r, \alpha)$ be as above. Then $P^\pm(r, \alpha)$ is a Morse–Bott manifold of periodic orbits for H if and only if $f''(r) \neq 0$. Moreover, $\mathcal{A}_H(x) = f(r) - rf'(r) = f(r) \mp r\ell$ for every $x \in P^\pm(r, \alpha)$.*

Proof. As in the proof of Lemma 5.3.1, it follows from Remark 5.2.1 that $P^\pm(r, \alpha)$ is a Morse–Bott manifold if and only if the space of periodic solutions of the linearized equation has the same dimension as $P^\pm(r, \alpha)$, namely n in the case (T) and 1 in the case (N). We begin by linearizing equation (16). Given a path $x : \mathbb{R} \rightarrow LU^*X : s \mapsto x_s = (q_s, p_s)$, we represent a variation of x by a pair $\hat{x} = (\hat{q}, \hat{p})$ of periodic vector fields along q via $\hat{q} := \partial_s q_s$ and $\hat{p} := \nabla_s p_s$. Since $\partial_s |p| = |p|^{-1} \langle p, \nabla_s p \rangle$ the linearized equation has the form

$$(18) \quad \nabla_t \hat{q} = \frac{f'(r)}{r} \left(\hat{p} - \left\langle \frac{p}{r}, \hat{p} \right\rangle \frac{p}{r} \right) + f''(r) \left\langle \frac{p}{r}, \hat{p} \right\rangle \frac{p}{r}, \quad \nabla_t \hat{p} + \frac{f'(r)}{r} R(\hat{q}, p)p = 0.$$

Here $R \in \Omega^2(X, \text{End}(TX))$ denotes the Riemann curvature tensor. Note that in the case $f(r) = r^2/2$ we have $\nabla_t \hat{q} = \hat{p}$ and so (18) is equivalent to the standard Jacobi equation $\nabla_t \nabla_t \hat{q} + R(\hat{q}, p)q = 0$.

The periodic solutions of equation (18) form the kernel of the Hessian of the symplectic action (see Remark 5.2.1). Taking the pointwise inner product of the second equation in (18) with p and using $\nabla_t p = 0$ we find that $\langle p, \hat{p} \rangle$ is constant. Hence, taking the L^2 -inner product of the first equation in (18) with p and using the fact that $|p| = r$, we find that every periodic solution of (18) satisfies

$$f''(r) \langle p, \hat{p} \rangle \equiv 0$$

Moreover, taking the L^2 -inner product of the second equation in (18) with \hat{q} and using the first equation we find that every periodic solution of (18) satisfies

$$\int_0^1 \left(|\hat{p}|^2 - \left\langle \frac{p}{r}, \hat{p} \right\rangle^2 - \langle R(\hat{q}, p)p, \hat{q} \rangle \right) dt = 0.$$

Now suppose that X has nonpositive sectional curvature and that $f''(r) \neq 0$. Then $\hat{p} = 0$ and $\nabla_t \hat{q} = 0$ for every periodic solution of (18). Hence the space of periodic solutions of (18) has dimension n in the torus case (namely \hat{q} is uniquely determined by $\hat{q}(0) \in T_{q(0)}X$) and has dimension one in the negative curvature case (namely, \hat{q} is a scalar multiple of p). In both cases it follows that the kernel of the Hessian of the symplectic action at every point $x \in P^\pm(r, \alpha)$ has the same dimension as $P^\pm(r, \alpha)$ and hence is equal to the tangent space of $P^\pm(r, \alpha)$ at x . This is equivalent to the Morse–Bott nondegeneracy condition (Remark 5.2.1). If, on the other hand, $f''(0) = 0$ then the dimension of the space of periodic solutions of (18) is $n + 1$ in the torus case and is 2 in the negative curvature case. \square

5.4. Proof of Theorem 5.1.1. Fix a real number $c > 0$ and choose a smooth family of real functions $\{f_s(r)\}_{s \in \mathbb{R}}$, defined for $r \in \mathbb{R}$, with the following properties (see Figure 1):

(i) $f_s(-r) = f_s(r)$ for all s and r .

(ii) For every $s \in \mathbb{R}$

$$f_s(0) > c, \quad f_s''(0) < 0, \quad \lim_{s \rightarrow -\infty} f_s(0) = c, \quad \lim_{s \rightarrow \infty} f_s(0) = \infty.$$

(iii) For all s and r we have $\partial_s f_s(r) \geq 0$.

(iv) $f_s'(r) \leq 0$ for $r \geq 0$ and, for $s \geq 1$,

$$f_s(r) = \begin{cases} f_s(0)(1 - r^2), & \text{if } 0 \leq r \leq 1 - 1/4s, \\ 0, & \text{if } r \geq 1 - 1/8s. \end{cases}$$

(v) $f'_s(r) \leq 0$ for $r \leq 1/2$, $f'_s(r) \geq 0$ for $r \geq 1/2$, and, for $s \leq -1$,

$$f_s(r) = \begin{cases} f_s(0)(1 - r^2), & \text{if } 0 \leq r \leq 1/8|s|, \\ s, & \text{if } 1/4|s| \leq r \leq 1 - 1/4|s|, \\ 0, & \text{if } r \geq 1 - 1/8|s|. \end{cases}$$

(vi) For every s the only critical point r of f_s with $f_s(r) > 0$ is $r = 0$.

It is not hard to prove that such a family $\{f_s(r)\}_{s \in \mathbb{R}}$ indeed exists.

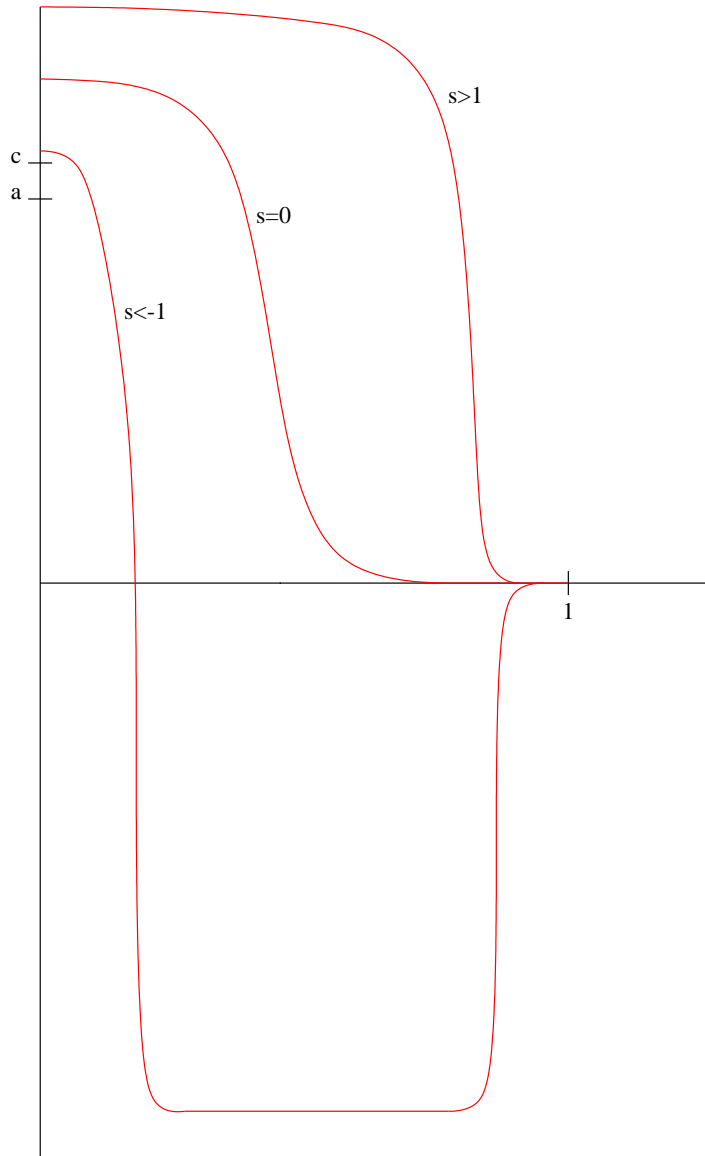


FIGURE 1. A family of profile functions.

Now define $H_s : U^*X \rightarrow \mathbb{R}$ by

$$H_s(q, p) := f_s(|p|).$$

Since X satisfies (T) or (N), every contractible closed geodesic is constant. Moreover, the constant periodic orbits $x \in \mathcal{P}(H_s)$ have the form $x \equiv (q, p)$ where $f'_s(|p|) = 0$, and the symplectic action of such a constant solution is $\mathcal{A}_{H_s}(x) = f_s(|p|)$. Hence, by (vi), the contractible periodic solutions of H_s with positive action have the form $x = (q, 0)$ with

$$\mathcal{A}_{H_s}(x) = f_s(0) > c.$$

By Lemma 5.3.1, these solutions form a Morse–Bott manifold of periodic orbits for H_s . Hence, by Theorem 5.2.2, we have

$$\mathrm{HF}^{[a, \infty)}(H_s; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2), & \text{if } 0 < a < f_s(0), \\ 0, & \text{if } 0 < f_s(0) < a, \end{cases}$$

for every $s \in \mathbb{R}$. By Proposition 4.5.1, the monotone homomorphism

$$\sigma_{H_{s_1} H_{s_0}} : \mathrm{HF}^{[a, \infty)}(H_{s_0}; \alpha) \rightarrow \mathrm{HF}^{[a, \infty)}(H_{s_1}; \alpha)$$

is an isomorphism whenever $s_1 \leq s_0$ and $a \notin [f_{s_1}(0), f_{s_0}(0)]$. Now, for every $H \in \mathcal{H}(U^*X)$ there exists an $s \in \mathbb{R}$ such that $H \leq H_s$. Hence, by Lemma 4.7.1 (ii), the homomorphism

$$\pi_s : \underset{\longleftarrow}{\mathrm{SH}}^{[a, \infty)}(U^*X; 0) \rightarrow \mathrm{HF}^{[a, \infty)}(H_s; 0)$$

is an isomorphism for every $s \in \mathbb{R}$ such that $f_s(0) > a$. Hence

$$\underset{\longleftarrow}{\mathrm{SH}}^{[a, \infty)}(U^*X; 0) \cong H_*(X; \mathbb{Z}_2).$$

Moreover, for every $H \in \mathcal{H}_c^{a, \infty}(U^*X, X; 0)$ there exists an $s \in \mathbb{R}$ such that $H_s \leq H$. Hence, by Lemma 4.7.1 (i), the homomorphism

$$\iota_s : \mathrm{HF}^{[a, \infty)}(H_s; 0) \rightarrow \underset{\longrightarrow}{\mathrm{SH}}^{[a, \infty); c}(U^*X, X; 0)$$

is an isomorphism for every $s \in \mathbb{R}$ in the case $a \leq c$, and for every $s \in \mathbb{R}$ with $f_s(0) < a$ in the case $a > c$. Hence

$$\underset{\longrightarrow}{\mathrm{SH}}^{[a, \infty); c}(U^*X, X; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2), & \text{if } a \leq c, \\ 0, & \text{if } a > c. \end{cases}$$

In the case $a > c$ it follows that $T_0^{[a, \infty); c} = 0$. In the case $a \leq c$ it follows from Proposition 4.8.1, that the map $T_0^{[a, \infty); c}$ can be expressed as the composition

$$T_0^{[a, \infty); c} = \iota_s \circ \pi_s : \underset{\longleftarrow}{\mathrm{SH}}^{[a, \infty)}(U^*X; 0) \rightarrow \underset{\longrightarrow}{\mathrm{SH}}^{[a, \infty); c}(U^*X, X; 0)$$

for every $s \in \mathbb{R}$. Hence, in this case, $T_0^{[a, \infty); c}$ is an isomorphism.

It remains to prove the statement on $\widehat{C}(U^*X, X; 0, a)$. Indeed, it follows from what we have proved above that $\mathcal{A}_c(U^*X, X; 0) = (0, c]$ for every $c > 0$. Therefore, for every $a \in \mathbb{R}$ we have:

$$\widehat{C}(U^*X, X; 0, a) = \inf \{c > 0 \mid \sup \mathcal{A}_c(U^*X, X; \alpha) > a\} = \max\{0, a\}.$$

The proof of Theorem 5.1.1 is complete. \square

5.5. Proof of Theorem 5.1.2. Fix a nontrivial homotopy class $\alpha \in \widetilde{\pi}_1(X)$ and let ℓ denote the length of the geodesics in this class. Moreover, fix a real number $c > 0$ and choose a smooth family of real functions $\{f_s(r)\}_{s \in \mathbb{R}}$, defined for $r \in \mathbb{R}$, with the following properties (see Figure 2):

- (i) $f_s(-r) = f_s(r)$ for all s and r .
- (ii) For every $s \in \mathbb{R}$

$$f_s(0) > c, \quad \lim_{s \rightarrow -\infty} f_s(0) = c, \quad \lim_{s \rightarrow \infty} f_s(0) = \infty.$$

- (iii) For all s and r we have $\partial_s f_s(r) \geq 0$.
- (iv) If $s \geq 1$ then

$$f_s(r) = \begin{cases} f_s(0), & \text{if } 0 \leq r \leq 1 - 3/8s, \\ 0, & \text{if } r \geq 1 - 1/8s, \end{cases}$$

$f'_s(r) \leq 0$ for all $r \geq 0$, and

$$f''_s(r) = \begin{cases} < 0, & \text{if } 1 - 3/8s < r < 1 - 2/8s, \\ > 0, & \text{if } 1 - 2/8s < r < 1 - 1/8s, \end{cases}$$

- (v) If $s \leq -1$ then

$$f_s(r) = \begin{cases} f_s(0), & \text{if } 0 \leq r \leq 1/8|s|, \\ s, & \text{if } 3/8|s| \leq r \leq 1 - 3/8|s|, \\ 0, & \text{if } r \geq 1 - 1/8|s|, \end{cases}$$

$f'_s(r) \leq 0$ for $r \leq 1/2$, $f'_s(r) \geq 0$ for $r \geq 1/2$, and

$$f''_s(r) = \begin{cases} < 0, & \text{if } 1/8|s| < r < 2/8|s|, \\ > 0, & \text{if } 2/8|s| < r < 3/8|s|. \end{cases}$$

- (vi) For every $s \in \mathbb{R}$ such that $f_s(0) > \ell$ there exist real numbers $r'_s > r_s > 0$ such that

$$f'_s(r_s) = f'_s(r'_s) = -\ell, \quad f''_s(r_s) < 0, \quad f''_s(r'_s) > 0,$$

and $f'_s(r) \neq -\ell$ for every $r \in [0, \infty) \setminus \{r_s, r'_s\}$.

- (vii) For every $s \in \mathbb{R}$, the only possible points $r > 0$ with $f'_s(r) = \ell$ must satisfy $f_s(r) < 0$.

It is not hard to show that such a family of functions $\{f_s(r)\}_{s \in \mathbb{R}}$ indeed exists. Now define

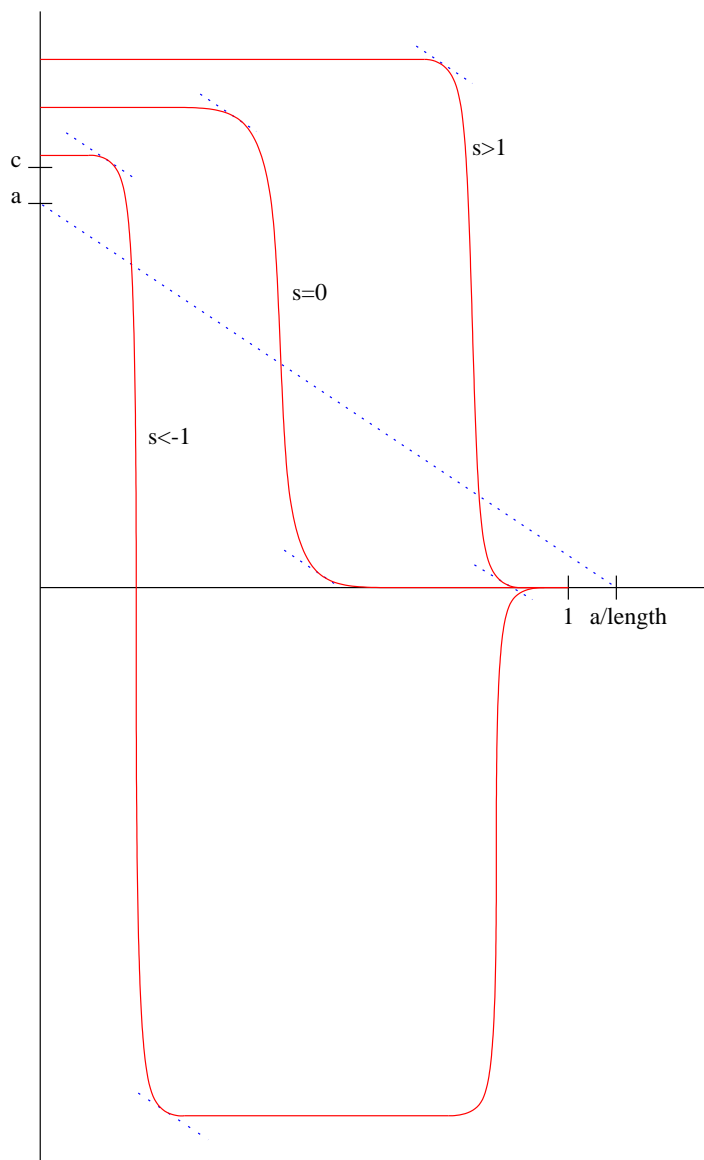


FIGURE 2. Another family of profile functions.

$H_s : U^*X \rightarrow \mathbb{R}$ by

$$H_s(q, p) := f_s(|p|).$$

Consider first periodic orbits of H_s that belong to one of the sets $P^+(r, \alpha)$, ($r > 0$), as defined by (17). We claim that the corresponding action is negative. Indeed, at such a value of $r > 0$ we have $f'_s(r) = \ell$, and the action is $f_s(r) - \ell r$ which is negative due to (vii).

Next, denote by $P_s := P^-(r_s, \alpha)$ and $P'_s := P^-(r'_s, \alpha)$ the other two components of the set of periodic solutions in the class α as defined by (17). Then P_s and P'_s are both diffeomorphic to \mathbb{T}^n in the case (T) and to S^1 in the case (N). Moreover, by (vi) and Lemma 5.3.2, they are Morse–Bott manifolds of periodic orbits for H_s for every $s \in \mathbb{R}$ and the values of the symplectic action functional on these two critical manifolds are

$$\mathcal{A}_{H_s}(P_s) = f_s(r_s) + r_s \ell, \quad \mathcal{A}_{H_s}(P'_s) = f_s(r'_s) + r'_s \ell.$$

Fix a real number a and denote $P := \mathbb{T}^n$ in the case (T) and $P := S^1$ in the case (N). We prove Theorem 5.1.2 in five steps.

Step 1. *If $a < \ell$ then $\text{SH}^{\leftarrow[a, \infty)}(U^*X; \alpha) = 0$.*

By (iv), $f'_s(r) \leq 0$ for every $s \geq 1, r > 0$ hence for $s \geq 1$ there are no periodic orbits of the type $P^+(r, \alpha)$. Thus for $s \geq 1$ the only families of periodic orbits are P_s and P'_s . Since both r_s and r'_s converge to 1 as $s \rightarrow \infty$ it follows that the sets P_s and P'_s both have action bigger than a for s sufficiently large. Hence

$$\text{HF}^{[a, \infty)}(H_s; \alpha) \cong \text{HF}^{[-\infty, \infty)}(H_s; \alpha) = 0$$

for s sufficiently large. The last equation holds because $\alpha \neq 0$, so $\text{HF}^{[-\infty, \infty)}(H; \alpha)$ is independent of H , and there is a Hamiltonian function with only contractible 1-periodic orbits. Now Step 1 follows from Lemma 4.7.1 (ii).

Step 2. *If $a \geq \ell$ then $\text{SH}^{\leftarrow[a, \infty)}(U^*X; \alpha) \cong H_*(P; \mathbb{Z}_2)$. Moreover, the homomorphism*

$$\pi_s : \text{SH}^{\leftarrow[a, \infty)}(U^*X; \alpha) \rightarrow \text{HF}^{[a, \infty)}(H_s; \alpha)$$

is an isomorphism whenever $f_s(0) > a$.

As $a > \ell > 0$ we may ignore all periodic orbits of the type $P^+(r, \alpha)$ and consider only the families P_s, P'_s . The numbers r_s and r'_s are the critical points of the function $f_{s, \ell} : [0, 1] \rightarrow \mathbb{R}$ given by

$$(19) \quad f_{s, \ell}(r) := f_s(r) + r \ell.$$

By (vi), the point r_s is a strict local maximum and the point $r'_s > r_s$ is a strict local minimum. Suppose that $f_s(0) > a$. Then $f_{s, \ell}(0) = f_s(0) > a$ and $f_{s, \ell}(1) = \ell \leq a$, hence it follows that $f_{s, \ell}(r_s) > a$ and $f_{s, \ell}(r'_s) < a$. This means that

$$\mathcal{A}_{H_s}(P_s) > a, \quad \mathcal{A}_{H_s}(P'_s) < a.$$

Hence, by Theorem 5.2.2, $\mathrm{HF}^{[a,\infty)}(H_s; \alpha) \cong H_*(P; \mathbb{Z}_2)$ and, by Proposition 4.5.1, the monotone homomorphism $\sigma_{H_{s_1} H_{s_0}} : \mathrm{HF}^{[a,\infty)}(H_{s_0}; \alpha) \rightarrow \mathrm{HF}^{[a,\infty)}(H_{s_1}; \alpha)$ is an isomorphism whenever $f_{s_i}(0) > a$ for $i = 0, 1$ and $s_1 \leq s_0$ and Step 2 follows from Lemma 4.7.1 (ii).

Step 3. *If $a > c > 0$ then $\mathrm{SH}^{[a,\infty);c}(U^*X, X; \alpha) = 0$.*

As $a > 0$ we can again ignore all orbits of type $P^+(r, \alpha)$. Since both r_s and r'_s converge to 0 as $s \rightarrow -\infty$ it follows that the sets P_s and P'_s both have action less than a for $-s$ sufficiently large. Hence $\mathrm{HF}^{[a,\infty)}(H_s; \alpha) = 0$ for $-s$ sufficiently large. Hence Step 3 follows from Lemma 4.7.1 (i).

Step 4. *If $0 < a \leq c$ then $\mathrm{SH}^{[a,\infty);c}(U^*X, X; \alpha) \cong H_*(P; \mathbb{Z}_2)$. Moreover, the homomorphism*

$$\iota_s : \mathrm{HF}^{[a,\infty)}(H_s; \alpha) \rightarrow \mathrm{SH}^{[a,\infty);c}(U^*X, X; \alpha)$$

is an isomorphism for $s \ll -1$.

Since $a > 0$, we may ignore as in previous steps orbits of type $P^+(r, \alpha)$. Let $f_{s,\ell} : [0, 1] \rightarrow \mathbb{R}$ be given by (19). Then, by (ii), $f_{s,\ell}(0) = f_s(0) > c \geq a$ and hence

$$\mathcal{A}_{H_s}(P_s) = f_{s,\ell}(r_s) > f_{s,\ell}(0) > a.$$

If $s < \min\{-1, a - \ell/2\}$ then $f_{s,\ell}(1/2) = s + \ell/2 < a$ and hence

$$\mathcal{A}_{H_s}(P'_s) = f_{s,\ell}(r'_s) < a.$$

By Theorem 5.2.2, $\mathrm{HF}^{[a,\infty)}(H_s; \alpha) \cong H_*(P_s; \mathbb{Z}_2)$ for $s < \min\{-1, a - \ell/2\}$. By Proposition 4.5.1, the monotone homomorphism $\sigma_{H_{s_1} H_{s_0}} : \mathrm{HF}^{[a,\infty)}(H_{s_0}; 0) \rightarrow \mathrm{HF}^{[a,\infty)}(H_{s_1}; 0)$ is an isomorphism for $s_1 < s_0 < \min\{-1, a - \ell/2\}$. Step 4 follows now from Lemma 4.7.1 (i).

Step 5. *If $\ell \leq a \leq c$ then the homomorphism*

$$T_\alpha^{[a,\infty);c} : \mathrm{SH}^{[a,\infty)}(U^*X; \alpha) \rightarrow \mathrm{SH}^{[a,\infty);c}(U^*X, X; \alpha)$$

is an isomorphism.

By (ii), $f_s(0) > c \geq a$ for every s . Hence, by Step 2, π_s is an isomorphism for every $s \in \mathbb{R}$. Moreover, by Step 4, ι_s is an isomorphism for $s \ll -1$. By Proposition 4.8.1, $T_\alpha^{[a,\infty);c} = \iota_s \circ \pi_s$ for every s . Hence $T_\alpha^{[a,\infty);c}$ is an isomorphism.

It remains to prove the statement on $\widehat{C}(U^*X, X; \alpha, a)$. Indeed, it follows from what we have proved above that $\mathcal{A}_c(U^*X, X; \alpha) = [\ell, c]$ for every $c > 0$. Therefore, for every $a \in \mathbb{R}$ we have:

$$\widehat{C}(U^*X, X; \alpha, a) = \inf \{c > 0 \mid \sup \mathcal{A}_c(U^*X, X; \alpha) > a\} = \max\{\ell, a\}.$$

The proof of Theorem 5.1.2 is complete. \square

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