The problem is that, in general, it may be exceedingly difficult, or impossible, to find a closed-form solution to the linear, time-variable state equation when A_0 and A_1 (or α) are functions of time. But to carry our present idea a little further, we may discuss a fuzzy state-transition matrix $\Phi_f(t,t_0)$ such that

$$\frac{d}{dt}\Phi_f(t,t_0)=[A_0+\alpha A_1](t)\Phi_f(t,t_0)$$

and

$$\Phi_f(t_0,t_0)=I, \text{ the identity matrix.}$$

The unique solution to (12) is given by X in terms of $\Phi_f(t,t_0)$ in the form

$$X(t)=\Phi_f(t,t_0)X(0),$$

and the average solution of $X(t)$ in terms of the FEV is given by

$$\text{FEV}\{X(t)\}=\text{FEV}\{\Phi_f(t,t_0)X(0)\}.$$

The average of a matrix can be defined as the average of its terms, and thus the FEV of the matrix $\Phi_f(t,t_0)X(0)$ is nothing more than the FEV of each term of the matrix. It should be noted that the practical computation of FEV $\{\Phi_f(t,t_0)X(0)\}$ depends upon the monotonic properties of the elements involved in the matrix $\Phi_f(t,t_0)$. It is also clear that if $\alpha \ll 1$, perturbation theory can be used in order to evaluate FEV$\{X(t)\}$.

IV. Conclusions

In this paper we investigate the applicability of fuzzy processes and fuzzy statistics to the problem of modeling of fuzzy systems. The fuzzy systems discussed in this paper are represented by fuzzy differential equations. Our study will hopefully be one of the foundations that will enable us to use fuzzy models in more productive applications. The aim of this paper is to show how fuzzy set theory can be applied in an imprecise modeling structure where some behavior of the system or some data are not precisely known.

Clearly, both the merits and defects of our method originate mainly from the fact that we have tried to cover problems usually considered as belonging to disciplines which require a long list of *a priori* assumptions and/or expensive testing before some mathematical evaluations can be performed.

We feel that our techniques can be easily applied to a sufficiently complex class of problems.

We have investigated, throughout the preliminary course of this research, a variety of additional applications of the framework developed in this paper. These include the evaluation of fuzzy phenomena in pattern recognition and classification, weather prediction and forecast evaluation, hydrological forecasting, quality control, the evaluation of eigenvalue spectrum for a particle in a fuzzy medium, and the topic of the measurement problem in quantum mechanics. A paper discussing these applications is under preparation.

Preliminary results of our investigations into these fields are very promising, and it is our hope that future studies on the analytical properties of fuzzy set theory and their applications will concentrate on these and many other areas. This will enable better applications and a clear understanding of the role of fuzzy statistics in many engineering and scientific models.

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Observers and Duality Between Observation and State Feedback for Time Delay Systems

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***Abstract*—This paper deals with the problem of observer construction for systems with general time-delays in state and output. The spectrum of the infinitesimal generator of the observer semigroup is determined explicitly, and the observer equation is decomposed into a finite- and an infinite-dimensional component.**

For systems with delays appearing in the state variables only duality results are obtained and the observer is described in the more concrete form of a differential delay equation.

I. Introduction

In [2] Bhat and Koivo developed an observer theory for systems with a single time delay in state. Their method is based on Hale's spectral decomposition [8, ch. 7] and they got the sufficient condition for the existence of a stable observer that the reduced finite-dimensional system, obtained by spectral projection, is observable. Moreover, in [3] they derived an observability criterion for the reduced system in terms of the original system parameters.

In this paper it is shown that a modified version of the observability criterion in [3] holds for systems with general state- and output-delays. For the same class of systems an observer theory is developed analogously to that in [2] and the observer is described as an abstract evolution equation in the Banach space $X=C([-h,0];\mathbb{R}^n)$. The observer semigroup $T_K(t)$ is obtained by finite-dimensional perturbation of the original semigroup $T(t)$ and does not directly correspond to a delay equation. Hence, the results of Hale [8, ch. 7] are not directly applicable. In this paper $T_K(t)$ is studied in detail. In particular, the spectrum of its infinitesimal generator is determined explicitly. The sufficient condition in [2] for the existence of a stable observer is generalized and shown to be necessary also. Based on the spectral decomposition of the original system the observer is decomposed into a finite and an infinite-dimensional part.

In the special case that there are no delays in the output mapping of the system, it is shown that $T_K(t)$ is isomorphic to a semigroup which corresponds to a delay equation. This representation leads to a more concrete description of the observer. Moreover, in this situation duality is investigated between the observer semigroup $T_K(t)$ and the state feedback concept of Pandolfi [13].

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II. Notation and Preliminaries

Consider the following time-invariant linear system with general delays in state and output:

$$\dot{x}(t)=L(x_t)+Bu(t)$$
$$y(t)=\Gamma(x_t) \tag{1}$$

where $u\in\mathbf{R}^l$, $x\in\mathbf{R}^n$, $y\in\mathbf{R}^m$, and

$$L(\varphi)=\int_{-h}^{0}d\eta(\tau)\varphi(\tau)$$
$$\Gamma(\varphi)=\int_{-h}^{0}d\gamma(\tau)\varphi(\tau)$$

for $\varphi\in X=C([-h,0];\mathbf{R}^n)$. Here $\eta(\cdot)$ and $\gamma(\cdot)$ are matrix-valued functions of bounded variation from the interval $[-h,0]$ into $\mathbf{R}^{nn}$, respectively, $\mathbf{R}^{mn}$. If $x(t)$, $t\geq -h$ is a solution of (1), then the function segment

$$x_t(\tau)=x(t+\tau),\quad -h\leqslant\tau\leqslant 0,$$

is the state of the system (1) at instant t. Let $T(t)$, $t\geq 0$, be the C_0-semigroup on X which corresponds to the unforced motions of (1), i.e., $T(t)\varphi=x_t$, where $x(t)$ solves (1) with $u(\cdot)\equiv 0$ and initial state $x_0=\varphi\in X$. The infinitesimal generator A of $T(t)$ is given by

$$\mathfrak{D}(A)=\{\varphi\in X|\varphi\in C^1,\dot{\varphi}(0)=L(\varphi)\}$$
$$A\varphi=\dot{\varphi}$$

(Hale [8, Section 7.1]). The semigroup approach leads to the representation of the solutions of (1) by the variation-of-constants formula

$$x_t=T(t)\varphi+\int_0^t T(t-s)X_0Bu(s)\,ds \tag{2}$$

where $X_0:[-h,0]\to\mathbf{R}^{nn}$ is defined by $X_0(\tau)=0$ for $\tau<0$, $X_0(0)=I$, and $T(t)X_0$ denotes the restriction X_t of the fundamental solution of (1) to the interval $[t-h,t]$ (see Hale [8, p. 146]).

The spectrum of A is given by the zeros of $\det\Delta(\lambda)$, where

$$\Delta(\lambda)=\lambda I-L(e^{\lambda\cdot})=\lambda I-\int_{-h}^{0}e^{\lambda\tau}d\eta(\tau). \tag{3}$$

A spectral decomposition of the state space X can be obtained through the homogeneous formal adjoint equation of (1),

$$\dot{x}(t)=L^T(x_t)=\int_{-h}^{0}d\eta^T(\tau)x(t+\tau).$$

The infinitesimal generator A^* of the corresponding semigroup $T^*(t)$, $t\geq 0$, is dual to A via the bilinear form

$$\langle\psi,\varphi\rangle=\psi^T(0)\varphi(0)+\int_{-h}^{0}\int_{\tau}^{0}\psi^T(\tau-\sigma)d\eta(\tau)\varphi(\sigma)d\sigma \tag{4}$$

on X (Hale [8, Sections 7.3 and 7.4]).[1]

Now define $Z=C([-h,0];\mathbf{C}^n)$ and let $\mathfrak{D}_Z(A)=\{\varphi\in Z|\dot{\varphi}\in Z,\dot{\varphi}(0)=L(\varphi)\}$ be the domain of A on Z. For any finite symmetric subset Λ of $\sigma(A)=\sigma(A^*)$ let X_Λ, respectively Z_Λ, be the corresponding real respectively complex, generalized eigenspace of A. There exist complementary subspaces X^Λ and Z^Λ such that

$$X=X_\Lambda\oplus X^\Lambda,\ Z=Z_\Lambda\oplus Z^\Lambda. \tag{5}$$

If X_Λ^* is defined analogously, then there exist bases $\{\varphi_1,\cdots,\varphi_N\}$ of X_Λ and $\{\psi_1,\cdots,\psi_N\}$ of X_Λ^* such that $\Phi=[\varphi_1,\cdots,\varphi_N]$ and $\Psi=[\psi_1,\cdots,\psi_N]$ $\in C([-h,0];\mathbf{R}^{nN})$ satisfy

$$\langle\Psi,\Phi\rangle=\Psi^T(0)\Phi(0)+\int_{-h}^{0}\int_{\tau}^{0}\Psi^T(\tau-\sigma)d\eta(\tau)\Phi(\sigma)d\sigma=I.$$

[1] There is a slight difference in the notation. Hale describes the formal adjoint equation as a backward equation in the state space $C([0,h];\mathbf{R}^{n*})$. Correspondingly our bilinear form results from [8, eq. (2.5) p. 169], by defining $\psi(\tau)=\alpha^T(-\tau)$, $-h\leqslant\tau\leqslant 0$.

Now the projection P_Λ of X onto X_Λ along X^Λ can be described by

$$P_\Lambda\varphi=\Phi\langle\Psi,\varphi\rangle,\quad\varphi\in X$$

where $\langle\Psi,\varphi\rangle$ denotes the column vector with components $\langle\psi_j,\varphi\rangle$, $j=1,\cdots,N$ (Hale [8, Section 7.3]). Finally, there exists a real $N\times N$ matrix A_Λ such that $A\Phi=\Phi A_\Lambda$.

Now let $x(t)$, $t\geq -h$, solve (1) with initial state $x_0=\varphi\in X$. Then the projection of x_t into X_Λ is given by $P_\Lambda x_t=\Phi x_\Lambda(t)$, where $x_\Lambda(t)=\langle\Psi,x_t\rangle\in\mathbf{R}^N$ satisfies

$$\dot{x}_\Lambda(t)=A_\Lambda x_\Lambda(t)+\Psi^T(0)Bu(t) \tag{6}$$
$$x_\Lambda(0)=\langle\Psi,\varphi\rangle.$$

The output corresponding to $P_\Lambda x_t$ is given by

$$y_\Lambda(t)=\Gamma P_\Lambda x_t=\Gamma\Phi x_\Lambda(t). \tag{7}$$

The following observability result for the reduced finite-dimensional system (6), (7) has been proved by Bhat and Koivo [3] for systems with a single time delay in state. Our proof shows that a generalization of their criterion holds for systems with general time delays in state and output.

Proposition 1: The system (6), (7) is observable iff for all $\lambda\in\Lambda$

$$\operatorname{rank}\begin{bmatrix}\Delta(\lambda)\\ \Gamma(e^{\lambda\cdot})\end{bmatrix}=n. \tag{8}$$

Proof: We show first that—given $\lambda\in\Lambda$ and $y\in\mathbf{C}^n$—we have $\Delta(\lambda)y=0$ iff there exists an $x\in\mathbf{C}^N$ such that

$$y=\Phi(0)x,(\lambda I-A_\Lambda)x=0.$$

Necessity: Let $\Delta(\lambda)y=0$ and $\varphi(\tau)=e^{\lambda\tau}y$, $-h\leqslant\tau\leqslant 0$. Then $\varphi\in\ker(\lambda I-A)$ (Hale [8, p. 169]) and thus $\varphi\in Z_\Lambda$, since $\lambda\in\Lambda$. It follows that $\varphi=\Phi x$ for some $x\in\mathbf{C}^N$. Hence, $y=\varphi(0)=\Phi(0)x$ and

$$\lambda\Phi x=\lambda\varphi=A\varphi=A\Phi x=\Phi A_\Lambda x$$

which implies $\lambda x=A_\Lambda x$.

Sufficiency: Let $\lambda x=A_\Lambda x$, $y=\Phi(0)x$, and $\varphi:=\Phi x$. Then

$$\lambda\varphi=\lambda\Phi x=\Phi A_\Lambda x=A\Phi x=A\varphi$$

and hence $y=\varphi(0)\in\ker\Delta(\lambda)$ (Hale [8, p. 169]).

Now by the condition of Hautus the reduced system (6), (7) is observable iff for all $\lambda\in\Lambda=\sigma(A_\Lambda)$

$$\ker\begin{bmatrix}\lambda I-A_\Lambda\\ \Gamma\Phi\end{bmatrix}=\{0\}.$$

We show that this is equivalent to (8).

First let $0\neq x\in\mathbf{C}^N$ with $(\lambda I-A_\Lambda)x=0$ and $\Gamma\Phi x=0$. Then with $y:=\Phi(0)x$ it follows

$$\Phi(\tau)x=\Phi(0)e^{A_\Lambda\tau}x=e^{\lambda\tau}\Phi(0)x=e^{\lambda\tau}y$$

for $-h\leqslant\tau\leqslant 0$. Thus, $y\neq 0$ and

$$\Gamma(e^{\lambda\cdot})y=\Gamma\Phi x=0.$$

Moreover, the first part of the proof implies $\Delta(\lambda)y=0$.

Conversely, let $0\neq y\in\mathbf{C}^n$ such that $\Delta(\lambda)y=0$ and $\Gamma(e^{\lambda\cdot})y=0$. Then there exists some $x\in\mathbf{C}^N$ such that

$$y=\Phi(0)x,(\lambda I-A_\Lambda)x=0.$$

Again we conclude $\Phi x=e^{\lambda\cdot}y$ and thus

$$\Gamma\Phi x=\Gamma(e^{\lambda\cdot})y=0$$

Clearly, x must be nonzero, since y is. Q.E.D.

III. The Perturbed Semigroup

In this section we study the C_0-semigroup $T_K(t)$, $t \geq 0$, generated by the closed linear operator $A_K = A + \mathcal{K}\Gamma$, where $K \in \mathbf{R}^{Nm}$ and $\mathcal{K}: \mathbf{R}^m \to X$ is the bounded linear operator given by

$$\mathcal{K}y = \Phi K y, \qquad y \in \mathbf{R}^m.$$

$T_K(t)$ satisfies the following perturbation formula:

$$T_K(t)\varphi = T(t)\varphi + \int_0^t T(t-s)\mathcal{K}\Gamma T_K(s)\varphi\, ds$$
$$= T(t)\varphi + \int_0^t T_K(t-s)\mathcal{K}\Gamma T(s)\varphi\, ds \tag{9}$$

(see Curtain and Pritchard [4, Theorem 2.31]).[2] The above semigroup was introduced by Bhat and Koivo in [2] for the construction of an observer for systems with a single time delay in state.

Note that the semigroup $T_K(t)$ does not directly arise from a delay system of the form (1), because the infinitesimal generators of such semigroups always map φ into $\dot{\varphi}$. In fact, $T_K(t)$ is not a translation semigroup; for $\varphi \in X$ we have to treat $z(t,\tau) = (T_K(t)\varphi)(\tau)$ as a function of two variables ($t \geq 0, -h \leq \tau \leq 0$). Nevertheless, $T_K(t)$ is related to a differential delay equation if there are no delays in the output mapping, i.e., $\Gamma: X \to \mathbf{R}^m$ is given by

$$\Gamma\varphi = C\varphi(0) = CP\varphi \tag{10}$$

where C is a real $m \times n$ matrix and $P: X \to \mathbf{R}^n$ maps φ into $\varphi(0)$. In this case some considerations analogous to those done by Bhat and Koivo [2, Appendix] show that

$$z(t) = z(t,0) = PT_K(t)\varphi$$

satisfies the following hereditary differential equation:

$$\dot{z}(t) = L(z_t) + \Phi(0)KCz(t) + \int_{-h}^0 d\eta(\tau)\int_\tau^0 \Phi(\tau-\sigma)KCz(t+\sigma)\, d\sigma. \tag{11}$$

If, moreover, $\mathfrak{T}: X \to X$ is the linear operator defined by

$$(\mathfrak{T}\varphi)(\tau) = \varphi(\tau) + \int_\tau^0 \Phi(\tau-\sigma)KC\varphi(\sigma)\, d\sigma$$

and $S(t)$ is the semigroup which corresponds to (11), then we have the following result.

Theorem 1: If Γ is given by (10), then

$$T_K(t) = \mathfrak{T}S(t)\mathfrak{T}^{-1}.$$

Proof: Let $\tilde{A}$ be the infinitesimal generator of $S(t)$ and $\psi \in \mathfrak{D}(\tilde{A})$. Then ψ is continuously differentiable and

$$\dot{\psi}(0) = L(\mathfrak{T}\psi) + \Phi(0)KC\psi(0). \tag{12}$$

Hence, $\varphi = \mathfrak{T}\psi$ is also continuously differentiable and

$$\dot{\varphi}(\tau) = \dot{\psi}(\tau) - \Phi(\tau)KC\psi(0) + \int_\tau^0 \Phi(\tau-\sigma)KC\dot{\psi}(\sigma)\, d\sigma.$$

By (12) this implies

$$\dot{\varphi}(0) = \dot{\psi}(0) - \Phi(0)KC\psi(0) = L(\mathfrak{T}\psi) = L(\varphi)$$

and thus $\varphi \in \mathfrak{D}(A)$. Moreover, we have

$$A_K\varphi = \dot{\varphi} + \Phi KC\varphi(0) = \dot{\varphi} + \Phi KC\psi(0) = \mathfrak{T}\dot{\psi} = \mathfrak{T}\tilde{A}\psi.$$

Applying a general result of semigroup theory (Bernier and Manitius [1, Lemma 5.3]), we obtain $T_K(t)\mathfrak{T} = \mathfrak{T}S(t)$ and hence the statement follows from invertibility of $\mathfrak{T}$. Q.E.D.

[2] The second equality in (9) follows from interchanging the roles of $T(t)$ and $T_K(t)$.

The previous theorem shows that $z(t) = PT_K(t)\varphi = PS(t)\mathfrak{T}^{-1}\varphi$ in fact satisfies (11) with initial condition $z_0 = \psi = \mathfrak{T}^{-1}\varphi$. Moreover it follows from Theorem 1 that the spectrum of A_K coincides with the spectrum of $\tilde{A}$ which is determined by the characteristic matrix function of (11):

$$\Delta_K(\lambda) = \Delta(\lambda) - \langle e^{\lambda\cdot}, \Phi\rangle KC. \tag{13}$$

Corollary 1:

$$\sigma(A + \Phi KCP) = \{\lambda \in \mathbb{C} \mid \det \Delta_K(\lambda) = 0\}.$$

Now let us return to the general case that there are delays in the output mapping also. In this situation it is more difficult to find a representation of $T_K(t)$ analogous to (11). The interested reader is referred to [14]. Here we will restrict ourselves to prove some properties of $T_K(t)$.

1). The infinitesimal generator A_K has a compact resolvent operator, since A has and since for $\lambda \in \rho(A_K) \cap \rho(A)$ the following resolvent formula holds:

$$(\lambda I - A_K)^{-1} = (\lambda I - A)^{-1}\left[I + \mathcal{K}\Gamma(\lambda I - A_K)^{-1}\right].$$

2). $T_K(t)$ is a compact operator for $t \geq h$. This follows from the first equation in (9), since $T(t)$ is compact for $t \geq h$ and the integral term is a bounded operator whose range is contained in the finite-dimensional subspace X_Λ.

The first property implies that A_K has a pure point spectrum (see Hille and Phillips [10, Theorem 5.14.2]) and from the second property it follows that for every $t \geq h$

$$\sigma(T_K(t)) \setminus \{0\} \subset P\sigma(T_K(t)). \tag{14}$$

The semigroup $T_K(t)$ is said to be of exponential growth $\omega_0 \in \mathbb{R}$ if for every $\epsilon > 0$ there exists some $M < \infty$ such that for all $t \geq 0$

$$\|T_K(t)\| \leq Me^{(\omega_0+\epsilon)t}$$

and if ω_0 is the smallest number with this property. From semigroup theory it is known that

$$\omega_0 = \lim_{t\to\infty} t^{-1}\log\|T_K(t)\| \geq \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A_K)\} \tag{15}$$

(see, e.g., Dunford and Schwartz [5, ch. VIII]). Now (14) allows us to apply a result of Zabczyk [16, Lemma 1] and we have equality in (15). Hence, we have proved the following.

Proposition 2: Let $\omega \in \mathbf{R}$. Then the following statements are equivalent:

i) There exists an $M < \infty$ such that $\|T_K(t)\| \leq Me^{\omega t}$.

ii) $\mathrm{Re}\lambda < \omega - \epsilon$ for every eigenvalue λ of A_K and some $\epsilon > 0$.

The above proposition shows the importance of determining the spectrum of A_K. This is done by the theorem below.

Theorem 2:

$$\sigma(A_K) = (\sigma(A) \setminus \Lambda) \cup \sigma(A_\Lambda + K\Gamma\Phi).$$

Proof: First let $\lambda \in \sigma(A_\Lambda + K\Gamma\Phi)$. Then there exists some nonzero $x \in \mathbb{C}^N$ such that $(A_\Lambda + K\Gamma\Phi)x = \lambda x$. Hence, $\Phi x \neq 0$ and

$$A_K\Phi x = A\Phi x + \mathcal{K}\Gamma\Phi x = \Phi(A_\Lambda x + K\Gamma\Phi x) = \lambda\Phi x$$

which implies $\lambda \in P\sigma(A_K)$.

Now let $\lambda \in \sigma(A) \setminus \Lambda$. Then $\mathfrak{R}(\lambda I - A)$ is a proper subset of Z (see Hale [8, p. 168, Lemma 2.1]). We prove that $\mathfrak{R}(\lambda I - A_K) \subset \mathfrak{R}(\lambda I - A)$ and hence $\lambda \in \sigma(A_K)$. Indeed, since $\lambda \notin \Lambda = \sigma(A_\Lambda)$, we have

$$(\lambda I - A)\Phi(\lambda I - A_\Lambda)^{-1} = \Phi.$$

Hence, for all $\varphi \in \mathfrak{D}_Z(A)$

$$(\lambda I - A_K)\varphi = (\lambda I - A)\varphi - \Phi K\Gamma(\varphi)$$
$$= (\lambda I - A)\left[\varphi - \Phi(\lambda I - A_\Lambda)^{-1}K\Gamma(\varphi)\right],$$

and the desired inclusion follows.

Finally, let $\lambda \in \sigma(A_K)$ and $\lambda \notin \sigma(A) \setminus \Lambda$. Then there exists some nonzero $\varphi \in \mathcal{D}_Z(A)$ such that

$$\begin{aligned}\Phi K \Gamma(\varphi) &= (\lambda I - A)\varphi \\ &= \Phi(\lambda I - A_\Lambda)\langle \Psi, \varphi\rangle + (\lambda I - A)(I - P_\Lambda)\varphi. \qquad (16)\end{aligned}$$

It follows from (5) that $(\lambda I - A)(I - P_\Lambda)\varphi = 0$. Since $\lambda \notin \sigma(A) \setminus \Lambda, \lambda I - A$ is one-to-one on Z^Λ and thus $(I - P_\Lambda)\varphi = 0$, i.e., $\varphi = \Phi\langle \Psi, \varphi\rangle$. Hence, $\langle \Psi, \varphi\rangle$ is nonzero and

$$K\Gamma\Phi\langle \Psi, \varphi\rangle = K\Gamma(\varphi) = (\lambda I - A_\Lambda)\langle \Psi, \varphi\rangle$$

where the latter equality follows from (16) and the fact that $(I - P_\Lambda)\varphi = 0$. We conclude that $\lambda \in \sigma(A_\Lambda + K\Gamma\Phi)$ and the theorem is proved. Q.E.D.

From finite-dimensional observer theory (see, e.g., Wonham [15, ch. 3]) it is well known that the spectrum of $A_\Lambda + K\Gamma\Phi$ can be placed arbitrarily symmetric to the real axis by choice of K if the reduced system (6), (7) is observable. Hence, if (8) holds for all $\lambda \in \Lambda$, then for every symmetric set Λ' of N complex numbers there exists some real $N \times m$ matrix K such that

$$\sigma(A + \Phi K \Gamma) = (\sigma(A) \setminus \Lambda) \cup \Lambda'$$

(Proposition 1 and Theorem 2).

We close this section with a decomposition result for the perturbed semigroup $T_K(t)$ with respect to (5).

Proposition 3:

i) $(I - P_\Lambda)T_K(t) = T(t)(I - P_\Lambda) = (I - P_\Lambda)T(t)$.

ii) For $\varphi \in X$ *the function* $z_\Lambda(t) = \langle \Psi, T_K(t)\varphi\rangle \in \mathbf{R}^N$ *satisfies the differential equation*

$$\begin{aligned}\dot{z}_\Lambda(t) &= (A_\Lambda + K\Gamma\Phi)z_\Lambda(t) + K\Gamma(I - P_\Lambda)T_K(t)\varphi \\ z_\Lambda(0) &= \langle \Psi, \varphi\rangle.\end{aligned}$$

iii) $T_K(t)\Phi = \Phi e^{(A_\Lambda + K\Gamma\Phi)t}$.

Proof:

i) Since $X_\Lambda \subset \mathcal{D}(A)$, we have $(I - P_\Lambda)\varphi \in \mathcal{D}(A)$ for every $\varphi \in \mathcal{D}(A_K) = \mathcal{D}(A)$. Moreover,

$$(I - P_\Lambda)A_K\varphi = (I - P_\Lambda)A\varphi = A(I - P_\Lambda)\varphi.$$

Hence, i) follows from Bernier and Manitius [1, Lemma 5.3] and the fact that $T(t)$ commutes with P_Λ.

ii) Applying (9) we obtain

$$\begin{aligned}z_\Lambda(t) &= \langle \Psi, T(t)\varphi + \int_0^t T(t-s)\Phi K \Gamma T_K(s)\varphi\, ds\rangle \\ &= e^{A_\Lambda t}\langle \Psi, \varphi\rangle + \int_0^t e^{A_\Lambda(t-s)} K\Gamma T_K(s)\varphi\, ds.\end{aligned}$$

Hence, $z_\Lambda(t)$ satisfies the differential equation

$$\begin{aligned}\dot{z}_\Lambda(t) &= A_\Lambda z_\Lambda(t) + K\Gamma T_K(t)\varphi \\ &= A_\Lambda z_\Lambda(t) + K\Gamma\Phi\langle \Psi, T_K(t)\varphi\rangle + K\Gamma(I - P_\Lambda)T_K(t)\varphi \\ &= (A_\Lambda + K\Gamma\Phi)z_\Lambda(t) + K\Gamma(I - P_\Lambda)T_K(t)\varphi.\end{aligned}$$

iii) By i) we have $(I - P_\Lambda)T_K(t)\Phi = 0$ and thus

$$T_K(t)\Phi = P_\Lambda T_K(t)\Phi = \Phi\langle \Psi, T_K(t)\Phi\rangle.$$

Now iii) follows from ii). Q.E.D

IV. Duality

Within this section we consider the special case that there are no delays in the output mapping, i.e., Γ is given by (10). Then the dual system of (1) is described by

$$\begin{aligned}\dot{x}(t) &= L^T(x_t) + C^T u(t) \\ y(t) &= B^T x(t). \qquad (17)\end{aligned}$$

For this system Pandolfi's feedback concept via the feedback matrix K^T leads to the equation

$$\begin{aligned}\dot{x}(t) &= L^T(x_t) + C^T K^T \langle x_t, \Phi\rangle^T \\ &= L^T(x_t) + C^T K^T \Phi^T(0)x(t) \\ &\quad + C^T K^T \int_{-h}^0 \int_\tau^0 \Phi^T(\tau - \sigma)\, d\eta^T(\tau) x(t+\sigma)\, d\sigma \qquad (18)\end{aligned}$$

(see [13]). Note that (18) results from (11) by transposition of matrices and hence has the characteristic matrix function $\Delta_K^T(\lambda)$. Thus, it follows from Corollary 1 and Theorem 2 that Pandolfi's pole shifting result [13, Theorem 2] also holds without the restrictive assumption that the new eigenvalues are not contained in the spectrum of A.

Now let $T_K^*(t)$ denote the semigroup corresponding to (18) and A_K^* its infinitesimal generator. Then we obtain the following duality results between the perturbed semigroup $T_K(t)$ and $T_K^*(t)$ via the bilinearform (4). These are analogous to Hale's duality results for the original semigroup $T(t)$ [8, Section 7.3].

Theorem 3:

i) For all $\psi \in \mathcal{D}(A_K^*)$ *and* $\varphi \in \mathcal{D}(A_K) = \mathcal{D}(A)$

$$\langle A_K^*\psi, \varphi\rangle = \langle \psi, A_K\varphi\rangle.$$

ii) For all $\varphi, \psi \in X$ *and all* $t \geq 0$

$$\langle T_K(t)\psi, \varphi\rangle = \langle \psi, T_K(t)\varphi\rangle.$$

Proof:

i) Note that $\psi \in \mathcal{D}(A_K^*)$ iff ψ is continuously differentiable and

$$\dot{\psi}(0) = L^T(\psi) + C^T K^T \langle \psi, \Phi\rangle^T.$$

Hence, for $\psi \in \mathcal{D}(A_K^*)$ and $\varphi \in \mathcal{D}(A)$ we have

$$\begin{aligned}\langle A_K^*\psi, \varphi\rangle &= \dot{\psi}^T(0)\varphi(0) + \int_{-h}^0\int_\tau^0 \dot{\psi}^T(\tau - \sigma)\, d\eta(\tau)\varphi(\sigma)\, d\sigma \\ &= [L^T(\psi) + C^T K^T\langle \psi, \Phi\rangle^T]^T \varphi(0) \\ &\quad - \int_{-h}^0 \Big\{ [\psi^T(\tau - \sigma)\, d\eta(\tau)\varphi(\sigma)]_{\sigma=\tau}^{\sigma=0} - \int_\tau^0 \psi^T(\tau - \sigma)\, d\eta(\tau)\dot{\varphi}(\sigma)\, d\sigma \Big\} \\ &= \langle \psi, \Phi\rangle K C\varphi(0) + \psi^T(0)L(\varphi) \\ &\quad + \int_{-h}^0\int_\tau^0 \psi^T(\tau - \sigma)\, d\eta(\tau)\dot{\varphi}(\sigma)\, d\sigma \\ &= \psi^T(0)[\dot{\varphi}(0) + \Phi(0)KC\varphi(0)] \\ &\quad + \int_{-h}^0\int_\tau^0 \psi^T(\tau - \sigma)\, d\eta(\tau)[\dot{\varphi}(\sigma) + \Phi(\sigma)KC\varphi(0)]\, d\sigma \\ &= \langle \psi, A_K\varphi\rangle.\end{aligned}$$

ii) First let $\psi \in \mathcal{D}(A_K^*)$ and $\varphi \in \mathcal{D}(A)$. Then by i) we have

$$\begin{aligned}\frac{d}{ds}&\langle T_K^*(t-s)\psi, T_K(s)\varphi\rangle \\ &= -\langle A_K^* T_K^*(t-s)\psi, T_K(s)\varphi\rangle + \langle T_K^*(t-s)\psi, A_K T_K(s)\varphi\rangle \\ &= 0\end{aligned}$$

for $0 < s < t$ and hence ii) holds in this case. Since A and A_K^* are densely defined, the statement follows. Q.E.D.

V. The Dynamic Observer

The representation (2) of the solutions of (1) leads to the following description of the observer as an abstract evolution equation in the state space X

$$z(t,\cdot) = T_K(t)\psi + \int_0^t T_K(t-s)[X_0 Bu(s) - \mathcal{K}y(s)]\, ds \qquad (19)$$

(compare Bhat and Koivo [2] or Gressang and Lamont [7]). We have to be careful in the interpretation of the integral term in (19) because X_0 is not continuous. First, $T_K(t)X_0$ has to be defined explicitly. Motivated by (9) this is done as follows:

$$T_K(t)X_0 := T(t)X_0 + \int_0^t T_K(t-s)\mathcal{K}\Gamma T(s)X_0\,ds \tag{20}$$

where

$$\Gamma T(t)X_0 = \Gamma X_t = \int_{\max\{-t,-h\}}^{0} d\gamma(\tau)X(t+\tau) \in \mathbf{R}^{mn}$$

is measurable and bounded on $[0,h]$ and continuous for $t>h$. With this definition $T_K(t)X_0$—as a function from $[-h,0]$ into $\mathbf{R}^{nn}$—is not continuous if $t<h$. Hence, the first integral in (19) has to be taken in $L^\infty([-h,0];\mathbf{R}^n)$ instead of X or must be interpreted pointwise. Both possibilities lead to the same value of the integral which is in fact in X.

Now let $u(s)$ in (19) coincide with the input of (1), respectively (2), and let $y(s)=\Gamma(x_s)$. Then for the error $e(t,\cdot)=z(t,\cdot)-x_t$ of the observer one obtains, by the use of (20),

$$\begin{aligned} e(t,\cdot) &= T_K(t)\psi + \int_0^t\Big[T(t-s)X_0 \\ &\quad + \int_0^{t-s} T_K(t-s-\sigma)\mathcal{K}\Gamma T(\sigma)X_0\,d\sigma\Big]Bu(s)\,ds \\ &\quad - \int_0^t T_K(t-s)\mathcal{K}\Gamma\Big[T(s)\varphi + \int_0^s T(s-\sigma)X_0Bu(\sigma)\,d\sigma\Big]ds \\ &\quad - T(t)\varphi - \int_0^t T(t-s)X_0Bu(s)\,ds \\ &= T_K(t)\psi - T(t)\varphi - \int_0^t T_K(t-s)\mathcal{K}\Gamma T(s)\varphi\,ds \\ &\quad + \int_0^t\int_s^t T_K(t-\sigma)\mathcal{K}\Gamma T(\sigma-s)X_0Bu(s)\,d\sigma\,ds \\ &\quad - \int_0^t\int_0^s T_K(t-s)\mathcal{K}\Gamma T(s-\sigma)X_0Bu(\sigma)\,d\sigma\,ds \end{aligned}$$

and hence by Fubini's theorem and (9)

$$e(t,\cdot) = T_K(t)(\psi-\varphi). \tag{21}$$

Definition 1: Let $\omega<0$. Then the observer (19) for the system (1) [respectively, (2)] is said to be ω-stable if for all initial states $\varphi,\psi\in X$ and any admissable control $u(\cdot)$ the error $e(t,\cdot)=z(t,\cdot)-x_t$ of the observer goes to zero with exponential decay rate ω, i.e.,

$$\lim_{t\to\infty} e^{-\omega t}\|e(t,\cdot)\| = 0.$$

Remark: By (21) and Proposition 2 the observer (19) is ω-stable if and only if $\mathrm{Re}\lambda<\omega$ for all $\lambda\in\sigma(A_K)$.[3]

Remark: In [11] Olbrot calls the system (1) *ω-detectable*, if $y(\cdot)\equiv 0$ implies $\lim_{t\to\infty} e^{-\omega t}x(t)=0$. Moreover, he shows that equivalently (8) holds for all $\lambda\in\sigma(A)$ with $\mathrm{Re}\lambda\geq\omega$.

Summarizing our results we get the following.

Theorem 4: Let $\omega<0$ and $\Lambda=\{\lambda\in\sigma(A)|\mathrm{Re}\lambda\geq\omega\}$. Then the following statements are equivalent.

i) There exists an ω-stable observer of the form (19).

ii) There exists a bounded linear operator $\mathcal{K}:\mathbf{R}^m\to X$ such that $\mathrm{Re}\lambda<\omega$ for all $\lambda\in\sigma(A+\mathcal{K}\Gamma)$.

iii) Equation (8) holds for all $\lambda\in\Lambda$.

iv) System (1) is ω-detectable.

v) System (6), (7) is observable.

Proof: "i)⇔ii)" and "iii)⇔iv)" follow from the above remarks, "iii)⇔v)" has been proved in Proposition 1 and the implication "v)⇒ii)" is a consequence of Theorem 2. Hence, it remains to show that ii) implies iii). Suppose that there exists some $\lambda\in\Lambda$ and some nonzero $y\in\mathbf{C}^n$ such that

$$\Delta(\lambda)y=0,\ \Gamma(e^{\lambda\cdot})y=0.$$

Then $\varphi=e^{\lambda\cdot}y\in\ker(\lambda I-A)$ (Hale [8, p. 169]) and $\Gamma(\varphi)=0$. Hence, for every linear $\mathcal{K}:\mathbf{C}^m\to Z$ we have

$$(A+\mathcal{K}\Gamma)\varphi=\lambda\varphi$$

which implies $\lambda\in\sigma(A+\mathcal{K}\Gamma)$. Q.E.D.

So far the observer has only been described as an abstract evolution equation in the Banach space X. It will be helpful for the treatment of this equation to study the decomposition which is obtained by projecting its solutions to the subspaces X_Λ and X^Λ. For this sake let $z(t,\cdot)$ be given by (19). Then it follows easily from Proposition 3ii) that $P_\Lambda z(t,\cdot)=\Phi z_\Lambda(t)$ where $z_\Lambda(t)=\langle\Psi,z(t,\cdot)\rangle\in\mathbf{R}^N$ satisfies the differential equation

$$\begin{aligned} \dot z_\Lambda(t) &= (A_\Lambda+K\Gamma\Phi)z_\Lambda(t)-Ky(t)+\Psi^T(0)Bu(t)+K\Gamma(z^\Lambda(t)) \\ z_\Lambda(0) &= \langle\Psi,\psi\rangle. \end{aligned} \tag{22}$$

Here $z^\Lambda(t)=(I-P_\Lambda)z(t,\cdot)\in X^\Lambda$ denotes the second infinite dimensional component of the observer. By Proposition 3i) $z^\Lambda(t)$ is given by

$$\begin{aligned} z^\Lambda(t) &= T(t)(I-P_\Lambda)\psi+\int_0^t T(t-s)(I-P_\Lambda)X_0Bu(s)\,ds \\ &= (I-P_\Lambda)w_t \end{aligned} \tag{23}$$

where $w(t)$, $t\geq -h$ is the solution of the original system equation (1) with initial condition $w_0=\psi$:

$$\begin{aligned} \dot w(t) &= L(w_t)+Bu(t) \\ w(\tau) &= \psi(\tau), \qquad -h\leq\tau\leq 0. \end{aligned}$$

Note that in its essential part the finite-dimensional component (22) of the observer is just the observer equation corresponding to the reduced system (6), (7) with an additional term $K\Gamma(z^\Lambda(t))$ resulting from the second component (23). Moreover, this second component is nothing else than the projection of the original system (1) on the complementary subspace X^Λ. On this subspace the system (1) is stable with exponential decay rate $\omega<0$, if $\Lambda=\{\lambda\in\sigma(A)|\mathrm{Re}\lambda\geq\omega\}$ (see Hale [8, Section 7.4]).

Pandolfi's control law for the system (1) is of the form

$$u(t)=F\langle\Psi,x_t\rangle \tag{24}$$

where F is a stabilizing matrix for the reduced system (6) (see [13]). Hence, in the closed loop—which means that x_t in (24) is replaced by the state $z(t,\cdot)$ of the observer (19) —the above decomposition (22), (23) of the observer has the advantage that $z_\Lambda(t)=\langle\Psi,z(t,\cdot)\rangle$ is already computed and can be used directly for the calculation of the input $u(t)=Fz_\Lambda(t)$ of (1). Since in this situation we only have to observe the "unstable part" $\langle\Psi,x_t\rangle$ of the system (1), there arises the question under which conditions the infinite-dimensional part (23) of the (full-order) observer (19) can be omitted. This is only possible, if the output $y_\Lambda(t)=\Gamma\Phi\langle\Psi,x_t\rangle$ of the reduced finite-dimensional system (6), (7) can be calculated from the output $y(t)=\Gamma(x_t)$ of the original system, i.e., if there exists some real $m\times m$ matrix D such that

$$D\Gamma(\varphi)=\Gamma\Phi\langle\Psi,\varphi\rangle, \qquad \varphi\in X. \tag{25}$$

If such matrix D exists, then the system

$$\dot z_\Lambda(t)=(A_\Lambda+K\Gamma\Phi)z_\Lambda(t)-KDy(t)+\Psi^T(0)Bu(t) \tag{26}$$

is a detector for the finite-dimensional component $\langle\Psi,x_t\rangle$ of the system (1). Indeed it is easy to see that the error $e_\Lambda(t)=z_\Lambda(t)-\langle\Psi,x_t\rangle$ satisfies the differential equation

$$\dot e_\Lambda(t)=(A_\Lambda+K\Gamma\Phi)e_\Lambda(t).$$

Now let us return to the special case of a system with delays appearing only in the state variable which means that Γ is given by (10). Then (11) leads to another more concrete concept of the observer, described by a differential delay equation

$$\begin{aligned} \dot z(t) &= L(z_t)+Bu(t)+\Phi(0)K[Cz(t)-y(t)] \\ &\quad + \int_{-h}^0 d\eta(\tau)\int_\tau^0 \Phi(\tau-\sigma)K[Cz(t+\sigma)-y(t+\sigma)]\,d\sigma. \end{aligned} \tag{27}$$

[3]Note that by Theorem 2 the spectrum of A_K is either finite or consists of a sequence of eigenvalues with real parts tending to $-\infty$. Hence ii) in Proposition 2 is equivalent to $\mathrm{Re}\lambda<\omega$ for all $\lambda\in\sigma(A_K)$.

In fact, if $x(t)$ is a solution of (1) and $z(t)$ a corresponding solution of (27), then the error $e(t)=z(t)-x(t)$ of this observer satisfies (11). Hence by Corollary 1 (27) is an ω-stable observer if and only if (19) is. The relation between both descriptions of the observer is analyzed precisely in the more general setting of the author's paper [14]. In particular, it can be shown that $z(t,0)$ satisfies (27) if $z(t,\cdot)$ is defined by (19). This follows also from solving the partial differential equation

$$\frac{\partial}{\partial t}z(t,\tau)=\frac{\partial}{\partial \tau}z(t,\tau)+\Phi(\tau)K[Cz(t,0)-y(t)]+X_0(\tau)Bu(t)$$
$$\frac{\partial}{\partial \tau}z(t,0)=L(z(t,\cdot)) \quad (28)$$

which is associated formally with the observer equation (19). Finally note that, in the special case of a single time delay in the state variable, (27) coincides with the final description of the observer obtained by Bhat and Koivo [2, eq. (20)].

Example 1: Consider the differential delay system

$$\begin{aligned}\dot{x}_1(t)&=-x_1(t-1)+x_2(t)\\ \dot{x}_2(t)&=\int_{-1}^{0}x_2(t+\tau)\,d\tau \\ y(t)&=x_1(t)\end{aligned} \quad (29)$$

which can be written in the form (1), (10) with

$$L(\varphi)=\begin{bmatrix}0&1\\0&0\end{bmatrix}\varphi(0)+\begin{bmatrix}-1&0\\0&0\end{bmatrix}\varphi(-1)+\int_{-1}^{0}\begin{bmatrix}0&0\\0&1\end{bmatrix}\varphi(\tau)\,d\tau$$
$$C=[1\quad 0].$$

Equation (29) is not in the class of delay systems for which observer concepts have been developed yet in the literature [2], [6], [9], [12]. The characteristic matrix function

$$\Delta(\lambda)=\begin{bmatrix}\lambda+e^{-\lambda}&-1\\0&\lambda-\dfrac{1-e^{-\lambda}}{\lambda}\end{bmatrix}$$

of (29) has a single root $\lambda_0=0.714556\cdots$ of multiplicity 1 in the closed right complex half-plane. Now define $\alpha_0:=\lambda_0+e^{-\lambda_0}$ and let $\mu_0<0$ be the new preassigned eigenvalue. Then we have

$$\Phi(0)=\begin{bmatrix}1\\ \alpha_0\end{bmatrix},\ A_\Lambda=\lambda_0,\ C\Phi(0)=1,\ K=\mu_0-\lambda_0.$$

Hence, in this case the observer equation (27) is given by

$$\dot{z}(t)=L(z_t)+(\mu_0-\lambda_0)\begin{bmatrix}1\\ \alpha_0\end{bmatrix}[z_1(t)-y(t)]$$
$$+(\mu_0-\lambda_0)\int_{-1}^{0}\begin{bmatrix}-e^{-\lambda_0(1+\sigma)}\\ \alpha_0\lambda_0^{-1}(1-e^{-\lambda_0(1+\sigma)})\end{bmatrix}[z_1(t+\sigma)-y(t+\sigma)]\,d\sigma. \quad (30)$$

By Theorem 2 and Corollary 1 the observer (30) has the same spectrum as (29) except λ_0 which is replaced by μ_0. This follows also directly from the fact that the characteristic matrix function

$$\Delta_K(\lambda)=\begin{bmatrix}\lambda+e^{-\lambda}+\dfrac{\mu_0-\lambda_0}{\lambda-\lambda_0}(\alpha_0-\lambda-e^{-\lambda})&-1\\ -\alpha_0\dfrac{\mu_0-\lambda_0}{\lambda-\lambda_0}\left(\lambda-\dfrac{1-e^{-\lambda}}{\lambda}\right)&\lambda-\dfrac{1-e^{-\lambda}}{\lambda}\end{bmatrix}$$

of (30) satisfies

$$\det\Delta_K(\lambda)\cdot(\lambda-\lambda_0)=\det\Delta(\lambda)\cdot(\lambda-\mu_0).$$

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Determination of Generic Dimensions of Controllable Subspaces and Its Application

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Abstract—**The controllable subspace and its dimension of a structured linear system vary as a function of the free parameters. However, the dimension is stable in the sense that it takes, for almost any system parameters, some maximal constant which is the generic rank of the controllability matrix. In this paper, this maximal constant is called the generic dimension of the controllable subspace. Two simple methods for determining generic dimensions of controllable subspaces are derived. As an application, the results are applied to the determination of system types of linear multivariable unity feedback systems.**

I. Introduction

Since Lin introduced the concept of structural controllability [1], there appeared many papers on the subject extending Lin's single-input results to the multiinput case or giving more elegant proofs to the structural controllability theorems [2]–[5]. At this point, however, considerations in these researches are mainly directed to determining whether or not a given system is structurally controllable. In order for the concept to be utilized in synthesis, e.g., robust synthesis of linear feedback systems, more work is needed.

This paper aims at extending some of the previous results to include the case where a system is structurally uncontrollable. In such a case, the controllable subspace and its dimension vary as a function of the system parameters. The dimension, however, is stable in the sense that it takes, for almost any system parameters, its maximal constant value which equals the generic rank of the controllability matrix. In this paper, this maximal constant value is called the generic dimension of the controlla-

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