Symplectic Topology
Example Sheet 11

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Exercise 11.1. Define $f : \mathbb{R}^{2n} \to \mathbb{R}$ by $f(z) := \frac{1}{4}|z|^2$. Prove that

$$\omega_f := -d(df \circ J_0) = \omega_0.$$

Exercise 11.2. Let L be a Riemannian manifold. Define $f : T^*L \to \mathbb{R}$ by

$$f(q, v^*) := \tfrac{1}{2}|v^*|^2$$

for $v^* \in T_q^*L$. Let J be the almost complex structure on T^*L induced by the Riemannian metric. Prove that J is compatible with ω_{can} and

$$df \circ J = \lambda_{\mathrm{can}}, \qquad \omega_f := -d(df \circ J) = \omega_{\mathrm{can}}.$$

Hint 1: Choose standard coordinates $x^1, \dots, x^n, y_1, \dots y_n$ on T^*L, so that f is given by $f(x, y) = \frac{1}{2}\sum_{i,j} y_i g^{ij}(x) y_j$. Denote the coordinates on $T_{(q,v^*)}T^*L$ by $\xi^1, \dots, \xi^n, \eta_1, \dots, \eta_n$. Show that the Riemannian metric determines a splitting $T_{(q,v^*)}T^*L \cong T_qL \oplus T_q^*L$, which in local coordinates is given by

$$(\xi^i, \eta_j) \mapsto \left(\xi^i, \eta_j - \sum_{k,\ell} \Gamma^\ell_{jk}(x) y_\ell \xi^k\right).$$

Hint 2: Show that J is given by $(\xi, \eta) \mapsto (\widehat{\xi}, \widehat{\eta})$, where

$$\begin{aligned} \widehat{\xi}^i &:= -\sum_j g^{ij}(x)\left(\eta_j - \sum_{k,\ell} \Gamma^\ell_{jk}(x) y_\ell \xi^k\right), \\ \widehat{\eta}_j &:= \sum_{k,\ell} \Gamma^\ell_{jk}(x) y_\ell \widehat{\xi}^k + \sum_k g_{jk}(x)\xi^k. \end{aligned} \tag{1}$$

Exercise 11.3. Let (M,ω) be symplectically aspherical and consider the product $\widetilde{M} := M \times M$ with the symplectic form $\widetilde{\omega} := \mathrm{pr}_2^*\omega - \mathrm{pr}_1^*\omega$. Prove that

$$\int_{\mathbb{D}} v^*\widetilde{\omega} = 0$$

for every smooth map $v : \mathbb{D} \to \widetilde{M}$ (on the closed unit disc $\mathbb{D} \subset \mathbb{C}$) with boundary values in the diagonal $\Delta := \{(p,p)\,|\,p \in M\} \subset \widetilde{M}$.

Exercise 11.4. Let (M,ω) be a symplectic manifold without boundary, let $[0,1] \times M \to \mathbb{R} : (t,p) \mapsto H_t(p)$ be a compactly supported time-dependent smooth Hamiltonian function, let $[0,1] \times M \to M : (t,p) \mapsto \psi_t(p)$ be the Hamiltonian isotopy generated by H_t via

$$\partial_t\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}, \qquad \iota(X_t)\omega = dH_t, \tag{2}$$

and let $L \subset M$ be a Lagrangian submanifold. Prove that $\psi_t(L) = L$ for all t if and only if $H_t|_L$ is constant for every t.

Exercise 11.5. Let L be a closed manifold and let $\Lambda \subset T^*L$ be a compact exact Lagrangian submanifold and choose a compactly supported smooth function $H : T^*L \to \mathbb{R}$ such that

$$(\lambda_{\mathrm{can}} + dH)|_\Lambda = 0.$$

For $t \in \mathbb{R}$ define $\Lambda_t \subset T^*L$ and $H_t : T^*L \to \mathbb{R}$ by

$$\Lambda_t := \left\{(q, e^t v^*)\,|\,(q,v^*) \in \Lambda\right\}, \qquad H_t(q,v^*) := e^t H(q, e^{-t}v^*).$$

Let $\{\psi_t\}_{t\in\mathbb{R}}$ be the Hamiltonian isotopy generated by H_t via (2). Prove that $\psi_t(\Lambda) = \Lambda_t$ for every $t \in \mathbb{R}$.

Exercise 11.6. Show that the formula

$$\phi(s+\mathbf{i}t) := \frac{e^{\pi(s+it} - \mathbf{i}}{e^{\pi(s+it} + \mathbf{i}}$$

defines a holomorphic diffeomorphism from the strip $\mathbb{S} := \mathbb{R} + \mathbf{i}[0,1]$ to the twice punctured disc $\mathbb{D} \setminus \{\pm 1\}$.

Exercise 11.7. Let (M, ω) be a symplectic manifold, let $L \subset M$ be a Lagrangian submanifold, and let $F_{s,t}ds + G_{s,t}dt \in \Omega^1(\mathbb{D}, \Omega^0(M))$. Define

$$\widetilde{M} := \mathbb{D} \times M, \qquad \widetilde{L} := S^1 \times L,$$

and

$$\widetilde{\omega} := \omega - d^{\widetilde{M}}(Fds - Gdt) + cds \wedge dt.$$

Prove that $\widetilde{L}$ is a Lagrangian submanifold of $(\widetilde{M}, \widetilde{\omega})$ if and only if the function

$$\cos(\theta)G_{e^{i\theta}} - \sin(\theta)F_{e^{i\theta}} : M \to \mathbb{R}$$

is constant on L for every $\theta \in \mathbb{R}$.

Exercise 11.8. Let $\{X_t\}_{0 \le t \le 1}$ be a smooth family of vector fields on a compact Riemannian manifold M and let $x_\nu : [0,1] \to M$ be a sequence of smooth functions such that

$$\lim_{\nu \to \infty} \int_0^1 |\dot{x}_\nu(t) - X_t(x_\nu(t))|^2 \, dt = 0.$$

Prove that there exists a subsequence x_{ν_i} which converges uniformly, and weakly in thw $W^{1,2}$-topology, to a solution $x : [0,1] \to M$ of the differential equation

$$\dot{x}(t) = X_t(x(t)).$$

Hint: Embed M into some Euclidean space and use Arzéla–Ascoli and Banach–Alaoglu.

Exercise 11.9. Assume the moduli space $\mathcal{M}(A, J)/\mathrm{G}$ (of all J-holomorphic spheres with values in a closed almost complex manifold (M, J), representing the homology class $A \in H_2(M; \mathbb{Z})$, modulo the action of $\mathrm{G} := \mathrm{PSL}(2, \mathbb{C})$) is compact and that every element of $\mathcal{M}(A, J)$ is injective. Define

$$\widetilde{\mathcal{M}}_{0,k}(A, J) := \left\{ (u, z_1, \ldots, z_k) \in \mathcal{M}(A, J) \times (S^2)^k \,|\, z_i \neq z_j \text{ for } i \neq j \right\}.$$

The reparametrization group $\mathrm{G} = \mathrm{PSL}(2, \mathbb{C})$ acts on the space $\widetilde{\mathcal{M}}_{0,k}(A, J)$ by $\phi^*(u, z_1, \ldots, z_k) := (u \circ \phi, \phi^{-1}(z_1), \ldots, \phi^{-1}(z_k))$ for $\phi \in \mathrm{G}$. Denote the quotient space by $\mathcal{M}_{0,k}(A, J) := \widetilde{\mathcal{M}}_{0,k}(A, J)/\mathrm{G}$ and consider the evaluation map $\mathrm{ev}_J : \mathcal{M}_{0,k}(A, J) \to M^k \setminus \Delta$ given by $\mathrm{ev}_J([u, z_1, \ldots, z_k]) := (u(z_1), \ldots, u(z_k))$, where $\Delta := \left\{ (p_1, \ldots, p_k) \in M^k \,|\, p_i \neq p_j \text{ for } i \neq j \right\}$ denotes the *fat diagonal.* Prove that ev_J is proper, i.e. the preimage of a compact subset of $M^k \setminus \Delta$ is compact. Deduce that the image of ev_J is a closed subset of $M^k \setminus \Delta$ (in the relative topology).

Exercise 11.10. Consider the tautological line bundle

$$L := \widetilde{\mathbb{C}}^2 := \left\{([w_1 : w_2], (z_1, z_2)) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2 \,|\, w_1 z_2 = w_2 z_1\right\} \subset \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2$$

over $\mathbb{C}\mathrm{P}^1$. Denote the zero section of L by $Z := \mathbb{C}\mathrm{P}^1 \times \{0\}$. Prove that Z has self-intersection number $Z \cdot Z = -1$ in L. Find an orientation reversing diffeomorphism $f : L \to \mathbb{C}\mathrm{P}^2 \setminus \{[1:0:0]\}$. Is there an orientation preserving diffeomorphism from L to the complement of a point in $\mathbb{C}\mathrm{P}^2$?

Exercise 11.11. Let $X \subset \mathbb{C}\mathrm{P}^3$ be a smooth quadric (i.e. the zero set of a homogeneous polynomial of degree two in four variables z_0, z_1, z_2, z_3, with zero as a regular value of its restriction to $\mathbb{C}^4 \setminus \{0\}$, divided by $\mathbb{C}^*$). Let $H \subset \mathbb{C}\mathrm{P}^3$ be a hyperplane tangent to X. Prove that $X \cap H$ is a union of two lines intersecting in precisely one point. Deduce that X is diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. Find an explicit formula for such a diffeomorphism in the cases

$$X := \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,|\, z_0 z_1 = z_2 z_3\right\},$$
$$Y := \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,|\, z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\right\}.$$

Exercise 11.12. Let Σ be a closed oriented 2-manifold and let $\mathrm{dvol}_\Sigma \in \Omega^2(\Sigma)$ be an area form. Denote $\mathrm{G} := \mathrm{SO}(3)$ and identify its Lie algebra $\mathfrak{g} := \mathfrak{so}(3)$ with $\mathbb{R}^3$ (i.e. a vector $\xi \in \mathbb{R}^3$ determines a skew-symmetric matrix $R_\xi \in \mathfrak{so}(3)$ via the cross product $R_\xi \eta := \xi \times \eta$, so that $R_{\xi\times\eta} = [R_\xi, R_\eta]$ and $g R_\xi g^{-1} = R_{g\xi}$ for $\xi, \eta \in \mathbb{R}^3$ and $g \in \mathrm{SO}(3)$). Let $\pi : P \to \Sigma$ be a principal G-bundle. Denote the (right) action of G on P by $P \times \mathrm{G} \to P : (p, g) \mapsto pg$, denote the induced action of $g \in \mathrm{G}$ on the tangent bundle by $T_pP \to T_{pg}P : v \mapsto vg$, and denote the infinitesimal action of $\xi \in \mathbb{R}^3$ by

$$p\xi := \left.\frac{d}{dt}\right|_{t=0} p \exp(tR_\xi) \in T_pP$$

for $p \in P$. Let $A \in \Omega^1(P, \mathbb{R}^3)$ be a **connection 1-form** on P, i.e. it satisfies $A_{pg}(vg) = g^{-1}A_p(v)$ and $A_p(p\xi) = \xi$ for $p \in P$, $v \in T_pP$, $g \in \mathrm{G}$, $\xi \in \mathbb{R}^3$. Define the 1-form $\alpha \in \Omega^1(P \times S^2)$ by

$$\alpha_{(p,x)}(v, \widehat{x}) := \langle x, A_p(v) \rangle,$$

let $\mathrm{dvol}_{S^2} := x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$ be the standard SO(3)-invariant volume form on S^2, and define $\omega_{A,c} \in \Omega^2(P \times S^2)$ by

$$\omega_{A,c} := \mathrm{pr}_{S^2}^* \mathrm{dvol}_{S^2} - d\alpha + c \cdot (\pi \circ \mathrm{pr}_P)^* \mathrm{dvol}_\Sigma.$$

Prove that, for $c > 0$ sufficiently large, $\omega_{A,c}$ descends to a symplectic form on $M := P \times_{\mathrm{G}} S^2$, where $[p, x] \equiv [pg, g^{-1}x]$ for $p \in P$, $x \in S^2$, $g \in \mathrm{SO}(3)$.