Symplectic Topology
Example Sheet 6

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Adjunction Inequality

Let (M, J) be an almost complex 4-manifold, (Σ, j) be a closed connected Riemann surface, and $u : \Sigma \to M$ be a simple J-holomorphic curve. Denote the set of critical points by $C(u) := \{z \in \Sigma \,|\, du(z) = 0\}$ and the set of self-intersections by $Z(u) := \{\{z_0, z_1\} \subset \Sigma \,|\, z_0 \neq z_1,\, u(z_0) = u(z_1)\}$. Then $C(u)$ and $Z(u)$ are finite sets. For $\{z_0, z_1\} \in Z(u)$ (respectively $z \in C(u)$) denote by $\iota(u; \{z_0, z_1\}) \in \mathbb{Z}$ (respectively $\iota(u; z) \in \mathbb{Z}$) the sum of the intersection indices of nearby self-intersections of an immersed perturbation of u with transverse self-intersections. These numbers are well defined by the Micallef–White theorem. Moreover, $\iota(u; \{z_0, z_1\}) \geq 1$ for every $\{z_0, z_1\} \in Z(u)$ (with equality iff the intersection is transverse), $\iota(u, z) \geq 1$ for every $z \in C(u)$, and

$$A \cdot A - c_1(A) + \chi(\Sigma) = 2 \left(\sum_{\{z_0,z_1\} \in Z(u)} \iota(u; \{z_0, z_1\}) + \sum_{z \in C(u)} \iota(u; z) \right), \quad (1)$$

where $A := [u] = u_*[\Sigma] \in H_2(M; \mathbb{Z})$, $A \cdot A := u \cdot u$ denotes the self-intersection number of u, and $c_1(A) := \langle c_1(TM, J), A \rangle$.

Exercise 6.1. Verify equation (1) for the holomorphic curve $u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ defined by $u([z_0 : z_1]) := [z_0^3 : z_0 z_1^2 : z_1^3]$.

Exercise 6.2. Prove the adjunction formula $g = \frac{(d-1)(d-2)}{2}$ for the genus of an embedded degree-d curve $C \subset \mathbb{C}\mathrm{P}^2$ by degenerating C to a union of d lines in general position. How many self-intersections does an immersed degree-d curve $u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ have?

Hirzebruch Signature Theorem

Let M be a closed oriented smooth 4-manifold. Then the formula

$$H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \to \mathbb{R} : (\omega, \tau) \mapsto \int_M \omega \wedge \tau \tag{2}$$

defines a nondegenerate symmetric bilinear form on the deRham cohomology group $H^2(M;\mathbb{R})$. The **signature of** M is defined as the signature of the quadratic form (2) and is denoted by

$$\sigma(M) := b^+(M) - b^-(M), \qquad b^+(M) + b^-(M) = \dim H^2(M;\mathbb{R}).$$

Here $b^\pm(M)$ denotes the dimension of a maximal positive (respectively negative) subspace of $H^2(M;\mathbb{R})$ with respect to (2).

The composition of (2) with the homomorphism $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$ is the quadratic form $H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z} : (a, b) \mapsto \langle a \cup b, [M] \rangle$ and is dual to the intersection pairing $Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$, given by $Q_M(A, B) := A \cdot B$ for $A, B \in H_2(M;\mathbb{Z})$.

Let J be an almost complex structure on M. The **Hirzebruch signature theorem** asserts that its first Chern class $c := c_1(TM, J) \in H^2(M;\mathbb{Z})$ satisfies

$$c^2 = 2\chi(M) + 3\sigma(M). \tag{3}$$

Here $c^2 := \langle c \cup c, [M] \rangle \in \mathbb{Z}$. A theorem of Wu asserts that a cohomology class $c \in H^2(M;\mathbb{Z})$ is the first Chern class of an almost complex structure on M if and only if it satisfies equation (3) and is an integral lift of the second Stiefel–Whitney class $\mathrm{w}_2(TM) \in H^2(M;\mathbb{Z}/2\mathbb{Z})$. (If $H^2(M;\mathbb{Z})$ has no 2-torsion then a cohomology class $c \in H^2(M;\mathbb{Z})$ is an integral lift of $\mathrm{w}_2(TM)$ if and only if the number $\langle c, A \rangle - A \cdot A$ is even for every $A \in H_2(M;\mathbb{Z})$.)

Exercise 6.3. Prove that the n-fold connected sum

$$n\mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \# \cdots \# \mathbb{C}\mathrm{P}^2$$

admits an almost complex structure if and only if n is odd.

Exercise 6.4. Let $X \subset \mathbb{C}\mathrm{P}^3$ be a degree-d hypersurface. Compute the Euler characteristic and signature of X and the Chern class of TX. **Hint:** Use the fact that X is simply connected.

Exercise 6.5. Let $X \subset \mathbb{C}\mathrm{P}^4$ be a degree-d hypersurface. Compute the Betti numbers of X and the Chern class of TX. **Hint:** Use the fact that X is simply connected and that $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$. Thus, by Poincaré duality, it remains to compute $b_3 = \dim H_3(X;\mathbb{R})$.

The Linearized Cauchy–Riemann Operator

Let (M, J) be an almost complex manifold, equipped with a Riemannian metric

$$g = \langle \cdot, \cdot \rangle$$

with respect to which J is skew-adjoint, denote by

$$\omega := \langle J\cdot, \cdot \rangle$$

the nondegenerate 2-form on M determined by g and J, and denote by ∇ the Levi-Civita connection of g. Let (Σ, j) be a closed connected Riemann surface and let $u : \Sigma \to M$ be a smooth map. The operator

$$D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}_J(\Sigma, u^*TM)$$

is defined by

$$D_u\xi := (\nabla\xi)^{0,1} - \tfrac{1}{2}J(u)\left(\nabla_\xi J(u)\right)\partial_J(u), \tag{4}$$

where

$$(\nabla\xi)^{0,1} := \tfrac{1}{2}(\nabla\xi + J(u)\nabla\xi \circ j), \qquad \partial_J(u) := \tfrac{1}{2}(du - J(u)du \circ j).$$

Exercise 6.6. Prove that, in local coordinates, the Christoffel symbols on the right hand side of equation (4) cancel whenever u is a J-holomorphic curve.

Exercise 6.7. Define the connection $\widetilde{\nabla}$ on TM by

$$\widetilde{\nabla}_Y X := \nabla_Y X - \tfrac{1}{2}J\left(\nabla_Y J\right)X \tag{5}$$

for $X, Y \in \mathrm{Vect}(M)$. Prove that $\widetilde{\nabla}$ is a Riemannian connection and $\widetilde{\nabla}J = 0$. Prove that, for every smooth map $u : \Sigma \to M$, the connection $\widetilde{\nabla}$ in (5) induces a unique differential operator

$$d^{\widetilde{\nabla}} : \Omega^1(\Sigma, u^*TM) \to \Omega^2(\Sigma, u^*TM)$$

that satisfies

$$d^{\widetilde{\nabla}}(\alpha\xi) = (d\alpha)\xi - \alpha \wedge \widetilde{\nabla}\xi$$

for every $\alpha \in \Omega^1(\Sigma)$ and every $\xi \in \Omega^0(\Sigma, u^*TM)$.

Exercise 6.8. Prove that

$$\begin{aligned} D_u\xi &= \tfrac{1}{2}\left(\widetilde{\nabla}\xi + J(u)\widetilde{\nabla}\xi\circ j\right) \\ &\quad + \tfrac{1}{4}N_J(\xi, \partial_J(u)) \\ &\quad + \tfrac{1}{4}\left(J\nabla_{\bar\partial_J(u)}J + \nabla_{J\bar\partial_j(u)}J\right)\xi \\ &\quad - \tfrac{1}{4}\left(J\nabla_\xi J + \nabla_{J\xi}J\right)\partial_J(u). \end{aligned}$$

Note that the first term on the right is a complex linear first order operator from $\Omega^0(\Sigma, u^*TM)$ to $\Omega^{0,1}_J(\Sigma, u^*TM)$, the second and third terms are complex anti-linear zeroth order operators from $\Omega^0(\Sigma, u^*TM)$ to $\Omega^{0,1}_J(\Sigma, u^*TM)$, and the last term is a complex linear zeroth order operator from $\Omega^0(\Sigma, u^*TM)$ to $\Omega^{0,1}_J(\Sigma, u^*TM)$. Moreover, the third term vanishes whenever u is a J-holomorphic curve, and the last two terms vanish whenever ω is closed.

Exercise 6.9. Assume ω is closed, fix a volume form $\mathrm{dvol}_\Sigma \in \Omega^2(\Sigma)$ compatible with the orientation, and consider the Riemannian metric

$$\langle\cdot,\cdot\rangle_\Sigma := \mathrm{dvol}_\Sigma(\cdot, j\cdot)$$

on Σ determined by dvol_Σ and j. Let $u : \Sigma \to M$ be a smooth map and define the linear operator

$$D_u^* : \Omega^{0,1}_J(\Sigma, u^*TM) \to \Omega^0(\Sigma, u^*TM)$$

by

$$D_u^*\eta := \frac{d^{\widetilde{\nabla}}(\eta\circ j)}{\mathrm{dvol}_\Sigma} + \frac{1}{4}\frac{(\nabla_\eta J)\wedge\partial_J(u)}{\mathrm{dvol}_\Sigma} \tag{6}$$

for $\eta \in \Omega^{0,1}_J(\Sigma, u^*TM)$. Prove that

$$\int_\Sigma \langle\eta, D_u\xi\rangle \mathrm{dvol}_\Sigma = \int_\Sigma \langle D_u^*\eta, \xi\rangle \mathrm{dvol}_\Sigma$$

for every $\xi \in \Omega^0(\Sigma, u^*TM)$ and every $\eta \in \Omega^{0,1}_J(\Sigma, u^*TM)$. Thus D_u^* is the formal adjoint operator of D_u with repect to the L^2 inner products on $\Omega^0(\Sigma, u^*TM)$ and $\Omega^{0,1}_J(\Sigma, u^*TM)$, determined by the Riemannian metrics on Σ and M.