Symplectic Topology
Example Sheet 8

Dietmar Salamon
ETH Zürich

19 April 2013

Exercise 8.1 (Isoperimetric Inequality). Let (V, ω) be a symplectic vector space and let $J \in \mathcal{J}(V, \omega)$ be an ω-compatible linear complex structure. Associated to a smooth loop $\gamma : \mathbb{R}/\mathbb{Z} \to M$ are the **symplectic action** $A(\gamma)$, the **energy** $E(\gamma)$, and the **length** $L(\gamma)$, defined by

$$A(\gamma) := \frac{1}{2}\int_0^1 \omega(\dot\gamma(t), \gamma(t))\,dt,$$
$$E(\gamma) := \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2\,dt,$$
$$L(\gamma) := \int_0^1 |\dot\gamma(t)|\,dt,$$

where $|v| := \sqrt{\omega(v, Jv)}$ for $v \in V$. Prove that

$$|A(\gamma)| \le \frac{1}{4\pi}L(\gamma)^2 \le \frac{1}{2\pi}E(\gamma). \tag{1}$$

If γ is nonconstant, prove that $|A(\gamma)| = (2\pi)^{-1}E(\gamma)$ if and only if the image of γ is a circle. **Hint:** Assume $(V, \omega, J) = (\mathbb{C}^n, \omega_0, \mathbf{i})$ and write γ as a Fourier series $\gamma(t) = \sum_{k=-\infty}^{\infty} v_k e^{2\pi \mathbf{i} kt}$ with $v_k \in \mathbb{C}^n$. Prove that

$$A(\gamma) = -\pi \sum_{k=-\infty}^{\infty} k\,|v_k|^2, \qquad E(\gamma) = 2\pi^2 \sum_{k=-\infty}^{\infty} k^2\,|v_k|^2.$$

Prove that $A(\gamma) \le (2\pi)^{-1}E(\gamma)$. Approximate γ by immersed loops and reparametrize by the arc length.

Exercise 8.2. Prove the isoperimetric inequality for the local symplectic action of sufficiently small loops in a compact symplectic manifold (M, ω) with an ω-compatible almost complex structure J for any constant $c > 1/4\pi$.
Hint: Reduce the problem to Exercise 8.1 via Darboux charts.

Exercise 8.3. Consider the family of Möbius trasformations $u_\varepsilon : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ given by

$$u_\varepsilon(z) := \varepsilon z$$

for $z \in \mathbb{C} \cup \{\infty\} \cong \mathbb{C}\mathrm{P}^1$. Compute the supremums-norm of du_ε with respect to the Fubini–Study metric on source and target. Prove that

$$\lim_{\varepsilon \to 0} \|du_\varepsilon\|_{L^\infty} = \infty.$$

Exercise 8.4. Consider the family of quadrics

$$Q_\varepsilon := \left\{ [x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, xy = \varepsilon z^2 \right\} = u_\varepsilon(\mathbb{C}\mathrm{P}^1),$$

where $u_\varepsilon : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ is the holomorphic curve, defined by

$$u_\varepsilon([w_0 : w_1]) := [w_0^2 : \varepsilon w_1^2 : w_0 w_1].$$

Prove that

$$\lim_{\varepsilon \to 0} \|du_\varepsilon\|_{L^\infty} = \infty.$$

Prove that u_ε converges to $u([w_0 : w_1]) := [w_0 : 0 : w_1]$ uniformly on every compact subset of $\mathbb{C}\mathrm{P}^1 \setminus \{[0 : 1]\}$. Prove that there exists a sequence of Möbius transformations $\phi_\varepsilon : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ such that $u_\varepsilon \circ \phi_\varepsilon$ converges to $v([w_0 : w_1]) := [0 : w_1 : w_0]$ uniformly on compact subsets of $\mathbb{C}\mathrm{P}^1 \setminus \{[1 : 0]\}$. Compute the homology classes $[u], [v], [u_\varepsilon] \in H_2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$. Show that

$$[u] + [v] = [u_\varepsilon].$$

Exercise 8.5. Let $\mathbb{D} \subset \mathbb{C}$ be the closed unit disc and $u : \mathbb{D} \setminus \{0\} \to M$ be a continuously differentiable function. Suppose that there exist constants $0 < \mu < 1$ and $c > 0$ such that

$$|du(z)| \leq \frac{c}{|z|^{1-\mu}}$$

for every $z \in \mathbb{D} \setminus \{0\}$. Prove that u is Hölder continuous with exponent μ, i.e. there exists a constant $C > 0$ such that

$$d(u(z), u(w)) \leq C|z - w|^\mu$$

for all $z, w \in \mathbb{D} \setminus \{0\}$. If M is compact, deduce that u extends to a Hölder continuous function from $\mathbb{D}$ to M.