Transversality in elliptic Morse theory for the symplectic action

Andreas Floer*, Helmut Hofer[†] and Dietmar Salamon[‡]
ETH Zürich[†] and University of Warwick[‡]

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Abstract

Our goal in this paper is to settle some transversality question for the perturbed nonlinear Cauchy-Riemann equations on the cylinder. These results play a central role in the definition of symplectic Floer homology and hence in the proof of the Arnold conjecture. There is currently no other reference to these transversality results in the open literature. Our approach does not require Aronszajn's theorem. Instead we derive the unique continuation theorem from a generalization of the Carleman similarity principle.

*A. Floer died on May 15th, 1991

1 Introduction

Let (M, ω) be a compact symplectic manifold and consider the differential equation

$$\dot{x}(t) = X_t(x(t)) \tag{1}$$

where $X_t = X_{t+1} : M \to TM$ is a smooth family of symplectic vector fields, i.e. the 1-forms $\iota(X_t)\omega$ are closed. The periodic solutions $x(t) = x(t+1)$ of (1) are the zeros of the closed 1-form Ψ_X on the loop space $\mathcal{L}$ of M defined by

$$\Psi_X(x; \xi) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), \xi(t))\, dt$$

for $\xi \in T_x\mathcal{L} = C^\infty(x^*TM)$. On the universal cover of $\mathcal{L}$ this 1-form is the differential $\Psi_X = d\mathcal{A}_X$ of the symplectic action functional $\mathcal{A}_X : \widetilde{\mathcal{L}} \to \mathbb{R}$. We shall assume throughout that the periodic solutions of (1) are all nondegenerate. This is equivalent to the condition that $\mathcal{A}_X$ is a Morse function.

The Floer homology groups of X can roughly be described as the Novikov homology of the closed 1-form Ψ_X on the loop space of M. The precise definition involves an infinite dimensional analogue of Witten's approach to Morse theory. Thus one considers the chain complex generated by the zeros of the 1-form Ψ_X and constructs a boundary operator by counting those gradient flow lines which connect two critical points of relative Morse index 1. In the finite dimensional context this construction requires the *Morse-Smale* transversality condition, namely that the stable and unstable manifolds of any two critical points intersect transversally. The infinite dimensional analogue of this condition is the subject of the present paper.

To be more precise, we must study the gradient flow lines of Ψ_X with respect to some L^2-metric which is induced by a t-dependent family of ω-compatible almost complex structures $J_t = J_{t+1} : TM \to TM$. These gradient flow lines are solutions $u : \mathbb{R}^2 \to M$ of the perturbed nonlinear Cauchy-Riemann equations

$$\partial_s u + J_t(u)(\partial_t u - X_t(u)) = 0 \tag{2}$$

which satisfy the periodicity condition $u(s, t+1) = u(s, t)$ and have limits

$$\lim_{s \to \pm\infty} u(s, t) = x^\pm(t) \tag{3}$$

which are periodic solutions of (1). The infinite dimensional analogue of the Morse-Smale condition asserts that the space $\mathcal{M}(x^-, x^+, X, J)$ of all smooth solutions of (2) and (3) is a smooth manifold of local dimension

$$\dim_u \mathcal{M}(x^-, x^+, X, J) = \mu(u)$$

where $\mu(u)$ is the Fredholm index of the operator obtained by linearizing (2). We shall prove in Theorem 5.1 that this condition is satisfied for a generic family of almost complex structures $J_t = J_{t+1}$ or for a generic family of symplectic vector fields $X_t = X_{t+1}$. The proof requires the next four sections. It is based on a unique continuation theorem (Proposition 3.1) which we prove with the help of the Carleman Similarity Principle (Theorem 2.2). Another key ingredient in the proof is the existence of an injective point for every solution of (2) (Theorem 4.3).

In the second part of the paper (Section 6-8) we focus on the case where the symplectic vector field $X_t = X$ and the almost complex structure $J_t = J$ are independent of t. More abstractly, this can be interpreted as the case where the action functional $\mathcal{A}_X : \widetilde{\mathcal{L}} \to \mathbb{R}$ and the L^2-metric on the loop space are invariant under the natural S^1-action. As a matter of fact, the loop space can be regarded as an infinite dimensional symplectic manifold and the action functional $\mathcal{A}_0$ (with $X = 0$) as a Hamiltonian function which generates the S^1-action. If $X_t = X$ is independent of t then $\mathcal{A}_X$ is an equivariant perturbation of $\mathcal{A}_0$. Now we are interested in such perturbations whose critical points are all fixed points of the S^1-action and in those connecting orbits on which S^1 acts freely. This means we consider solutions u of (2) and (3) with $X_t = X$ and $J_t = J$ such that the limits $x^\pm(t) = x^\pm$ are zeros of X and which are **simple** in the sense that $u(s, t + 1/m) \not\equiv u(s, t)$ for all integers $m > 1$. We shall prove in Theorem 7.4 that the space of such simple solutions is a smooth manifold of dimension $\mu(u)$. The proof is based on a technical transversality result for symmetric matrices (Theorem 6.1). In the case $\mu(u) \le 1$ we deduce in Theorem 8.1 that the solutions $u(s,t)$ of (2) and (3) with $X_t = X$ and $J_t = J$ must be independent of t, i.e. they must lie in the fixed point set of the S^1 action on the loop space. This result is used in [7] and [10] (in the case where $X = X_H$ is a Hamiltonian vector field) to prove that the Floer homology groups $HF_*(H, J)$ are naturally isomorphic to the ordinary homology of M.

2 The Carleman similarity principle

Let V be a finite dimensional complex vector space and denote by $S^2 = \mathbb{C} \cup \{\infty\}$ the Riemannian sphere. Consider the vector bundle $\Lambda^{0,1}T^*S^2 \otimes V$ over S^2 whose fibre over $z \in S^2$ is the space of complex anti-linear maps $T_zS^2 \to V$. The space $C^\infty(S^2, \Lambda^{0,1}T^*S^2 \otimes V) = \Omega^{0,1}(S^2, V)$ of smooth sections of this bundle is, of course, the space of complex anti-linear 1-forms on S^2 with values in V. The Cauchy-Riemann operator $\overline{\partial} : C^\infty(S^2, V) \to \Omega^{0,1}(S^2, V)$ is defined by

$$\overline{\partial}u = du + i \circ du \circ i.$$

For $p > 1$ this operator can be extended to the Sobolev space $W^{1,p}(S^2, V)$ of V-valued functions whose first derivatives are p-integrable. It then takes values in the space $L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes V)$ of L^p-sections of the bundle $\Lambda^{0,1}T^*S^2 \otimes V$. For later reference we state here a special case of the Riemann-Roch theorem.

Theorem 2.1 *For every $p > 1$ the operator*

$$\bar{\partial} : W^{1,p}(S^2, V) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes V)$$

is a Fredholm operator. Its index as a complex operator is given by

$$\operatorname{index} \overline{\partial} = \dim_{\mathbb{C}} V$$

Moreover, this operator is onto and its kernel consists of the constant functions.

That $\overline{\partial}$ is a Fredholm operator can be proved by the usual L^p estimates for elliptic operators. In our case this is just the Calderon-Zygmund inequality. The index formula follows from the explicit statements about the kernel and the cokernel. That the kernel consists of the constant maps is just the assertion of Liouville's theorem. That the operator is onto follows from the fact that holomorphic vector bundles over Riemann surfaces with negative Chern number do not have nonzero holomorphic sections. In the special case of Theorem 2.1 surjectivity can in fact be proved by constructing an explicit right inverse T of $\overline{\partial}$ given by the formula

$$(Tv)(z) = \lim_{\varepsilon \to 0} \left(\frac{-1}{2\pi} \int_{|\zeta| \geq \varepsilon} \frac{v(z+\zeta)}{\zeta} \right)$$

for $v \in C_0^\infty(\mathbb{C}, V)$ where $\zeta = s + it \in \mathbb{C}$. Then

$$\overline{\partial} \circ T(v) = v \mathrm{d}\bar{z}$$

for $v \in C_0^\infty(\mathbb{C}, V)$. Combining this with a change of coordinates and a similar result for complex anti-linear 1-forms which are supported in a neighbourhood of $\infty \in S^2$ we obtain that $\overline{\partial}$ has a dense range and is therefor onto. The proof that T actually extends to an operator from $L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes V)$ to $W^{1,p}(S^2, V)$ is, of course, equivalent to the Calderon-Zygmund inequality.

Let $\mathbb{F}$ denote either the real or complex numbers. For a complex vectorspace V we denote by $\mathcal{L}_{\mathbb{F}}(V)$ the $\mathbb{F}$-vectorspace of $\mathbb{F}$-linear maps. Likewise, we denote by $\mathrm{GL}_{\mathbb{F}}(V) \subset \mathcal{L}_{\mathbb{F}}(V)$ the open subset of invertible $\mathbb{F}$-linear maps.

Write $z = s + it$ and consider the first order elliptic system

$$\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0 \tag{4}$$

where $u : B_\varepsilon = \{z \in \mathbb{C} \,|\, |z| < \varepsilon\} \to \mathbb{C}^n$. We assume that the map $z \mapsto J(z)$ belongs to the Sobolev space $W^{1,p}\big(B_\varepsilon, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)\big)$ for some $p > 2$ and that $J(z) : \mathbb{C}^n \to \mathbb{C}^n$ is a complex structure for every z, i.e.

$$J(z)^2 = -\mathbb{1}.$$

Moreover, we assume the map $z \mapsto C(z)$ belongs to $L^p\big(B_\varepsilon, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)\big)$. The following result is a higher dimensional version of the **Carleman-Similarity principle** (cf.[24]). It says roughly that solutions of (4) behave like holomorphic maps.

Theorem 2.2 *Let $u \in W^{1,p}\big(B_\varepsilon, \mathbb{C}^n\big)$ be a solution of (4) with $u(0) = 0$. Then there exist a constant $0 < \delta < \varepsilon$, a map $\Phi \in W^{1,p}\big(B_\delta, \mathrm{GL}_\mathbb{R}(\mathbb{C}^n)\big)$, and a holomorphic map and $\sigma : B_\delta \to \mathbb{C}^n$ such that*

$$u(z) = \Phi(z)\sigma(z), \qquad \sigma(0) = 0, \qquad J(z)\Phi(z) = \Phi(z)i$$

for $z \in B_\delta$.

Proof: First choose a map $\Psi \in W^{1,p}\big(B_\delta, \mathrm{GL}_\mathbb{R}(\mathbb{C}^n)\big)$ such that

$$J(z)\Psi(z) = \Psi(z)i$$

for $z \in B_\delta$ and define $v \in W^{1,p}\big(B_\delta, \mathbb{C}^n\big)$ by $u(z) = \Psi(z)v(z)$. Then

$$\begin{aligned} 0 &= \partial_s u + J\partial_t u + Cu \\ &= (\partial_s \Psi)v + \Psi\partial_s v + J(\partial_t\Psi)v + J\Psi\partial_t v + C\Psi v \\ &= \Psi\left(\partial_s v + i\partial_t v + \widetilde{C}v\right) \end{aligned}$$

where $\widetilde{C} = \Psi^{-1}(\partial_s\Psi + J\partial_t\Psi + C\Psi) \in L^p\big(B_\varepsilon, \mathcal{L}_\mathbb{R}(\mathbb{C}^n)\big)$. With respect to the complex structure i we decompose $\widetilde{C}$ into the linear and anti-linear part

$$\widetilde{C}(z) = \widetilde{C}^+(z) + \widetilde{C}^-(z), \qquad \widetilde{C}^\pm(z) = \tfrac{1}{2}\left(\widetilde{C}(z) \mp i\widetilde{C}(z)i\right).$$

Now choose a map $B_\varepsilon \to \mathcal{L}_\mathbb{R}(\mathbb{C}^n) : z \mapsto D(z)$ such that

(a) $D(z)v(z) = v(z)$ for $z \in B_\varepsilon$,

(b) $D(z)$ is complex anti-linear for every $z \in B_\varepsilon$,

(c) $D \in L^\infty\big(B_\varepsilon, \mathcal{L}_\mathbb{R}(\mathbb{C}^n)\big)$.

For example, define $D(z)\zeta = |v(z)|^{-2}v(z)v(z)^T\bar{\zeta}$ whenever $v(z) \neq 0$ and $D(z) = 0$ otherwise. Then the linear map

$$A(z) = \widetilde{C}^+(z) + \widetilde{C}^-(z)D(z)$$

is complex linear and satisfies

$$A(z)v(z) = \widetilde{C}(z)v(z).$$

Moreover, $A \in L^p\big(B_\varepsilon, \mathcal{L}_\mathbb{C}(\mathbb{C}^n)\big)$. For $0 < \delta < \varepsilon$ define

$$A_\delta \in L^p\big(S^2, \mathcal{L}_\mathbb{C}(\mathbb{C}^n)\big)$$

by $A_\delta(z) = A(z)$ for $z \in B_\delta$ and $A_\delta(z) = 0$ otherwise. Now let $V = \mathcal{L}_\mathbb{C}(\mathbb{C}^n)$ and denote by $D_\delta : W^{1,p}(S^2, V) \to L^p(\Lambda^{0,1}T^*S^2 \otimes V)$ the perturbed Cauchy-Riemann operator

$$D_\delta\Theta = \bar{\partial}\Theta + A_\delta\Theta \mathrm{d}\bar{z}.$$

for $\Theta \in W^{1,p}(S^2, V)$. By Theorem 2.1 the operator $\Theta \mapsto (\bar{\partial}\Theta, \Theta(0))$ is bijective. Since

$$\lim_{\delta \to 0} \|A_\delta\|_{L^p} = 0$$

it follows that the operator $\Theta \mapsto (D_\delta \Theta, \Theta(0))$ is bijective for $\delta > 0$ sufficiently small. Hence, for $\delta > 0$ sufficiently small, there exists a unique map $\Theta_\delta \in W^{1,p}(S^2, V)$ such that

$$\mathrm{D}_\delta \Theta_\delta = 0, \qquad \Theta_\delta(0) = \mathbb{1}.$$

In particular, $\partial_s \Theta_\delta + i\partial_t \Theta_\delta + A\Theta_\delta = 0$ in B_δ. Since Θ_δ converges to the constant map $\Theta_0(z) = \mathbb{1}$ in the $W^{1,p}$-norm as $\delta \to 0$ we may choose δ so small that the complex linear map $\Theta_\delta(z) \in V = \mathbb{C}^{n\times n}$ is invertible for every $z \in S^2$.

Now we drop the subscript δ, denote $\Theta(z) = \Theta_\delta(z)$, and define

$$\Phi(z) = \Psi(z)\Theta(z), \qquad \sigma(z) = \Theta(z)^{-1} v(z).$$

Then obviously $\Psi \in W^{1,p}(S^2, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ and $\Psi(z)\sigma(z) = \Phi(z)v(z) = u(z)$. Moreover, in B_δ we have

$$\begin{aligned} 0 &= \partial_s v + i\partial_t v + \widetilde{C} v \\ &= \partial_s v + i\partial_t v + Av \\ &= \Theta\partial_s \sigma + i\Theta\partial_t \sigma + (\partial_s \Theta + i\partial_t \Theta + A\Theta)\,\sigma \\ &= \Theta(\partial_s \sigma + i\partial_t \sigma). \end{aligned}$$

Hence σ is holomorphic in B_δ. Moreover, by construction,

$$J(z)\Phi(z) = J(z)\Psi(z)\Theta(z) = \Psi(z)i\Theta(z) = \Psi(z)\Theta(z)i = \Phi(z)i$$

This proves the theorem. □

Here is an immediate consequence of the Carleman similarity principle.

Corollary 2.3 *Let $\ell \geq 2$ and $p > 2$. Let $J \in W^{\ell,p}(B_\varepsilon, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ and $C \in W^{\ell-1,p}(B_\varepsilon, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ with $J(z)^2 = -\mathbb{1}$ for every $z \in B_\varepsilon$. Let $u : B_\varepsilon \to \mathbb{C}^n$ be a nonconstant $W^{\ell,p}$-solution of (4) with $u(0) = 0$.*

(i) *There exists a constant $0 < \delta < \varepsilon$ such that $u(z) \neq 0$ for $0 < |z| < \delta$.*

(ii) *If $C = 0$ then there exists a constant $0 < \delta < \varepsilon$ such that $du(z) \neq 0$ for $0 < |z| < \delta$.*

Proof: In view of Theorem 2.2 we have

$$u(z) = \Phi(z)\sigma(z)$$

for $|z| < \delta$ where $\sigma : B_\delta \to \mathbb{C}^n$ is holomorphic and $\Phi(z) \in \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n)$ for $|z| < \delta$. Hence for $|z| < \delta$

$$u(z) = 0 \quad \Longleftrightarrow \quad \sigma(z) = 0$$

Since σ is holomorphic we have either $\sigma \equiv 0$ on a neighborhood of 0 or $\sigma(z) \neq 0$ on a punctured neighborhood of zero. This proves (i).

Assertion (ii) is obvious in the case $du(0) \neq 0$. Hence assume $du(0) = 0$. Differentiating the identity $\partial_s u + J(z)\partial_t u = 0$ with respect to s we obtain that the function $v = \partial_s u$ satisfies

$$\partial_s v + J(z)\partial_t v + (\partial_s J)(z)J(z)v = 0$$

Here we have used the identity $\partial_t u = J(z)\partial_s u = J(z)v$. It follows again from Theorem 2.2 that $v(z) = \partial_s u(z) = \Phi(z)\sigma(z)$ with Φ and σ as above and this proves the corollary. □

Remark 2.4 Standard elliptic regularity theory asserts that if $J \in W^{\ell,p}$ and $C \in W^{\ell-1,p}$ with $\ell \geq 2$ and $p > 2$ then every $W^{1,p}$-solution u of (4) is necessarily of class $W^{\ell,p}$. To see this just apply the operator $\partial_s - J\partial_t$ to the left hand side of (4) and use the local L^p-regularity theorem for the Laplace operator.

3 Unique continuation

In this section we show how the Carleman similarity principle can be used to prove a unique continuation theorem for J-holomorphic curves in almost complex manifolds. More precisely, consider the perturbed nonlinear Cauchy Riemann equations

$$\partial_s u + J(z,u)\partial_t u + Y(z,u) = 0. \tag{5}$$

Here we assume that the map $J : \mathbb{C} \times \mathbb{C}^n \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)$ is of class $W^{1,p}$ with $p > 2$ and

$$J(z,w)^2 = -1\!\!1$$

and the vector field $Y : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ is of class $W^{1,p}$. A $W^{1,p}$-function $u : \mathbb{C} \to \mathbb{C}^n$ is said to **vanish to infinite order** at a point $z_0 \in \mathbb{C}$ if

$$\lim_{r\to 0} \frac{\sup_{|z-z_0|\leq r} |u(z)|}{r^k} = 0$$

for all $k \geq 0$. Of course, a smooth function u vanishes to infinite order at a point z_0 if and only if all derivatives of u vanish at that point. In particular, for a smooth function the set of points at which it vanishes to infinite order is closed and for a holomorphic function it is open and closed. The next proposition asserts that this property of holomorphic functions persists for the solutions of (5).

Proposition 3.1 (Unique continuation) *Let $u, v : \Omega \to \mathbb{C}^n$ be two $W^{1,p}$-solutions of (5) defined in some open set $\Omega \subset \mathbb{C}$. Then the sets of points $z \in \Omega$ where $u - v$ vanishes to infinite order is open and closed. In particular, if Ω is connected and $u = v$ on some nonempty open subset of Ω then $u(z) = v(z)$ for all $z \in \Omega$.*

Proof: Define $w = u - v$. Then

$$\begin{aligned}
\partial_s w + J(z,u)\partial_t w &= J(z,u)\partial_t w + J(z,v)\partial_t v + Y(z,v) \\
&\quad - J(z,u)\partial_t u - Y(z,u) \\
&= \big(J(z,v) - J(z,u)\big)\partial_t v + Y(z,v) - Y(z,u) \\
&= \left(\int_0^1 \frac{d}{d\tau} J\big(z, u + \tau(v-u)\big)\, d\tau\right) \partial_t v \\
&\quad + \int_0^1 \frac{d}{d\tau} Y\big(z, u + \tau(v-u)\big)\, d\tau \\
&= -C(z)w
\end{aligned}$$

where $C : \Omega \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)$ is locally p-integrable. Hence $w(z) = u(z) - v(z)$ satisfies

$$\partial_s w + \widetilde{J}(z)\partial_t w + C(z)w = 0$$

where $\widetilde{J}(z) = J(z, u(z))$ is locally of class $W^{1,p}$. By Theorem 2.2 every point $z_0 \in \Omega$ admits a neighbourhood $B_\delta(z_0)$ in which w can be written in the form $w(z) = \Phi(z)\sigma(z)$ where σ is holomorphic and $\Phi(z)$ is invertible. Hence w vanishes to infinite order at $z \in B_\varepsilon(z_0)$ if and only if σ vanishes to infinite order at z. Hence the sets of such points is open and closed in $B_\delta(z_0)$. This proves the proposition. □

We will use the previous proposition for functions which are defined on all of $\mathbb{C} = \mathbb{R}^2$ and take values in a manifold. In this case Proposition 3.1 asserts that two solutions which agree to infinite order at a point must agree globally.

4 Injective points

In [12] Dusa McDuff proved that a J-holomorphic curve $u : \Sigma \to M$ in an almost complex manifold is either multiply covered or admits a point $z \in \Sigma$ such that

$$du(z) \neq 0, \qquad \{z\} = u^{-1}(u(z)).$$

(See also [13].) Such a point is called an **injective point** and the existence of such points plays a crucial role in the transversality theory for J-holomorphic curves. The purpose of this section is to prove an analogue of this result for the perturbed equation (5) in the case where both the almost complex structure $J(z,w)$ and the perturbation $Y(z,w)$ are independent of the variable $s = \mathrm{Re}\, z$. Hence consider the equation

$$\partial_s u + J(t,u)(\partial_t u - X(t,u)) = 0. \tag{6}$$

where $J : \mathbb{R} \times \mathbb{C}^n \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)$ and $X : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ are of class C^ℓ with $\ell \geq 2$ and $J(t,w)^2 = -\mathbb{1}$ for all $t \in \mathbb{R}$ and $w \in \mathbb{C}^n$. Thus every $W^{1,p}$-solution of (6) with $p > 2$ is necessarily of class $W^{\ell+1,p}$ and hence of class C^ℓ. We begin with an analogue of Corollary 2.3 (ii) for the nonlinear equation (6).

Lemma 4.1 *Let $u : B_\varepsilon \to \mathbb{C}^n$ be a C^ℓ-solution of (6) and assume that $\partial_s u \not\equiv 0$. Then the set of points $(s,t) \in B_\varepsilon$ with $\partial_s u(s,t) = 0$ is discrete.*

Proof: Let $\psi_t : \Omega_t \to \mathbb{C}^n$ be the local diffeomorphisms generated by the vector fields $X_t(w) = X(t,w)$ via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}. \tag{7}$$

It suffices to prove the lemma locally and hence we may assume that $u(s,t) \in \psi_t(\Omega_t)$ for $(s,t) \in B_\varepsilon$. Then the function $v(s,t) = \psi_t^{-1}(u(s,t))$ satisfies

$$\partial_s u = d\psi_t(v)\partial_s v, \qquad \partial_t u - X_t(u) = d\psi_t(v)\partial_s v.$$

Hence $d\psi_t(v)\partial_s v + J_t(u)d\psi_t(v)\partial_t v = 0$ where $J_t(w) = J(t,w)$ and this means that

$$\partial_s v + \psi_t^* J_t(v)\partial_t v = 0.$$

Moreover, v is nonconstant since otherwise $\partial_s u \equiv 0$. Hence it follows from Corollary 2.3 (ii) that the set of critical points of v is discrete. Since $dv(z) = 0$ if and only if $\partial_s u(z) = 0$ the lemma is proved. □

Lemma 4.2 *Let $u, v : B_\varepsilon \to \mathbb{C}^n$ be C^ℓ-solutions of (6) with $X = 0$ such that*

$$u(0) = v(0), \qquad du(0) \neq 0, \qquad dv(0) \neq 0.$$

Moreover, assume that there exists a constant $0 < \delta < \varepsilon$ such that for every $(s,t) \in B_\delta$ there exists an $s' \in \mathbb{R}$ such that $(s',t) \in B_\varepsilon$ and $u(s,t) = v(s',t)$. Then $v(z) = u(z)$ for $|z| < \varepsilon$.

Proof: Choose $\varepsilon > 0$ so small that $\Sigma = v(B_\varepsilon)$ is a submanifold of $\mathbb{C}^n$. By the implicit function theorem $v^{-1} : \Sigma \to B_\varepsilon$ extends to a C^ℓ-map defined on a neighbourhood of Σ. By assumption, $u(B_\delta) \subset \Sigma$ and hence the map $v^{-1} \circ u : B_\delta \to B_\varepsilon$ is of class C^ℓ. Moreover, our assumptions assert that this map takes the form $v^{-1} \circ u(s,t) = (\phi(s,t), t)$. Differentiating the formula $u(s,t) = v(\phi(s,t),t)$. we obtain

$$\begin{aligned} 0 &= \partial_s u(s,t) + J(t,u)\partial_t u(s,t) \\ &= \partial_s v(\phi,t)\partial_s\phi + J(t,v(\phi,t))\big(\partial_s v(\phi,t)\partial_t\phi + \partial_t v(\phi,t)\big) \\ &= \partial_s v(\phi,t)\partial_s\phi + \partial_t v(\phi,t)\partial_t\phi - \partial_s v(\phi,t) \\ &= \partial_s v(\phi,t)(\partial_s\phi - 1) + \partial_t v(\phi,t)\partial_t\phi \end{aligned}$$

Since $\partial_s v(\phi,t)$ and $\partial_t v(\phi,t)$ are linearly independent we deduce that $\partial_s\phi = 1$ and $\partial_t\phi = 0$. Hence $\phi(s,t) = s + s_0$ for some $s_0 \in \mathbb{R}$. Since $0 = u(0) = v(s_0, 0)$ we obtain $s_0 = 0$ and hence $\phi(s,t) = s$. This implies that u and v agree in a neighbourhood of 0. By unique continuation it follows that $u = v$ on B_ε (see Proposition 3.1). □

The next theorem is a global result in an almost complex manifold M. More precisely, let M be a manifold (without boundary) of real dimension $2n$ and fix a compactly supported C^ℓ-diffeomorphism $\phi : M \to M$ with $\ell \geq 2$. (Here compactly supported means that $\phi(x) = x$ outside a compact set.) Moreover, let $\mathbb{R} \to \mathrm{End}(TM) : t \mapsto J_t$ be a C^ℓ-family of almost complex structures on M and $\mathbb{R} \to \mathcal{X}(M) : t \mapsto X_t$ be a C^ℓ-family of vector fields such that

$$\phi^* J_{t+1} = J_t, \qquad \phi^* X_{t+1} = X_t.$$

Sometimes we write $X(t,p) = X_t(p)$ and $J(t,p) = J_t(p)$. Now let $u : \mathbb{R}^2 \to M$ be a C^ℓ-solution of (6) such that

$$u(s,t+1) = \phi(u(s,t)), \tag{8}$$

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t), \qquad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0. \tag{9}$$

Here the convergence is uniform in t. It follows that the limit curves $x^\pm(t)$ are solutions of the ordinary differential equation

$$\dot{x}(t) = X_t(x(t)), \qquad x(t+1) = \phi(x(t)). \tag{10}$$

These correspond to fixed points of the C^ℓ-diffeomorphism $\phi_X = \psi_1^{-1} \circ \phi$ where ψ_1 denotes the time-1-map generated by the vector fields X_t via (7). In principle we should be more careful with the domain of definition of the diffeomorphism ϕ_X. However, since the set $u(\mathbb{R} \times S^1) \cup x^-(S^1) \cup x^+(S^1)$ is compact we may assume without loss of generality that the vector fields X_t vanish outside a compact set and are therefore complete.

Now let $u : \mathbb{R}^2 \to M$ be a C^ℓ-solution of (6), (8) and (9). A point $(s,t) \in \mathbb{R}^2$ is called **regular** for u if

$$\partial_s u(s,t) \neq 0, \qquad u(s,t) \neq x^\pm(t), \qquad u(s,t) \notin u(\mathbb{R} - \{s\}, t).$$

We denote by $R(u)$ the set of regular points of u. In particular, these conditions mean that the map $s' \mapsto u(s',t)$ is an immersion near $s' = s$ and meets the point $u(s,t)$ only once. This notion is analogous to that of an injective point for J-holomorphic curves mentioned above. The next theorem is the main result of this section.

Theorem 4.3 *Let $u : \mathbb{R}^2 \to M$ be a C^ℓ-solution of (6), (8), and (9) such that $\partial_s u \not\equiv 0$. Then the set $R(u)$ of regular points for u is open and dense in $\mathbb{R}^2$.*

Proof: We first reduce the theorem to the case $X_t = 0$. Denote by $\psi_t : M \to M$ the time dependent flow generated by the vector fields X_t via (7). Since X_t is compactly supported so is ψ_t for every t. Recall from the proof of Lemma 4.1 that the functions $v(s,t) = \psi_t^{-1}(u(s,t))$ satisfy the partial differential equation

$$\partial_s v + \psi_t^* J_t(v) \partial_t v = 0$$

and note that

$$v(s,t+1) = \phi_X(v(s,t)), \qquad \lim_{s\to\pm\infty} v(s,t) = x^\pm(0), \qquad \phi_X = \psi_1^{-1} \circ \phi.$$

Since $R(u) = R(v)$ we may assume from now on that $X_t = 0$ for all t. In particular, this implies that $x^\pm(t)$ is independent of t and we denote $x^\pm = x^\pm(t)$.

We prove that $R(u)$ is open. Assume otherwise that there exists a point $(s,t) \in R(u)$ which can be approximated by a sequence $(s_\nu, t_\nu) \notin R(u)$. Then $\partial_s u(s_\nu, t_\nu) \neq 0$ and $u(s_\nu, t_\nu) \neq x^\pm$ for ν sufficiently large. Since $(s_\nu, t_\nu) \notin R(u)$ it follows that there exists a sequence $s'_\nu \in \mathbb{R}$ such that

$$u(s_\nu, t_\nu) = u(s'_\nu, t_\nu), \qquad s'_\nu \neq s_\nu.$$

If the sequence s'_ν is unbounded then, passing to a subsequence if necessary, we may assume that $s'_\nu \to \pm\infty$ and hence, by (9), $u(s'_\nu, t_\nu) \to x^\pm$. This implies $u(s,t) = x^\pm$ in contradiction to $(s,t) \in R(u)$. Hence the sequence s'_ν is bounded and we may assume without loss of generality that $s'_\nu \to s'$. Then $u(s,t) = u(s',t)$ and since $(s,t) \in R(u)$ we must have $s' = s$. Hence s'_ν and s_ν both converge to s and this contradicts the fact that $\partial_s u(s,t) \neq 0$. This proves that the set $R(u)$ is open.

We prove that $R(u)$ is dense. To see this recall from Lemma 4.1 that the set $C(u)$ of all points $(s,t) \in \mathbb{R}^2$ with $\partial_s u(s,t) = 0$ is discrete. Hence it suffices to prove that every point in $\mathbb{R}^2 - C(u)$ can be approximated by a sequence in $R(u)$. Now a point $(s,t) \notin C(u)$ can obviously be approximated by a sequence $(s_\nu, t) \in \mathbb{R}^2 - C(u)$ with $u(s_\nu, t) \neq x^\pm(t)$. In fact any sequence s_ν with $s_\nu \neq s$ will do. Hence we must prove that every point $(s_0, t_0) \in \mathbb{R} \times [0,1]$ with

$$\partial_s u(s_0, t_0) \neq 0, \qquad u(s_0, t_0) \neq x^\pm(t_0)$$

can be approximated by a sequence in $R(u)$. Assume otherwise that

$$B_\varepsilon(s_0, t_0) \cap R(u) = \emptyset$$

for some $\varepsilon > 0$. Choose ε so small and $T > 0$ so large that the following holds

(i) $u(s,t) \notin u(B_\varepsilon(s_0, t_0))$ for $|s| \geq T$ and $|t - t_0| \leq \varepsilon$.

(ii) If $|t - t_0| \leq \varepsilon$ then the map $[s_0 - \varepsilon, s_0 + \varepsilon] \to M : s \mapsto u(s,t)$ is an immersion.

Now, by Lemma 4.1, the set $C(u) \cap [-T, T] \times [0,1]$ is finite. Moving the point (s_0, t_0) slightly, if necessary, we may assume that $u(s_0, t_0) \neq u(s,t)$ whenever $(s,t) \in C(u) \cap [-T,T] \times [0,1]$. We may then shrink $\varepsilon > 0$ to obtain

(iii) $u(B_\varepsilon(s_0, t_0)) \cap u(C(u) \cap [-T,T] \times [0,1]) = \emptyset$.

This modification will not affect the conditions (i) and (ii) above.

Now it follows from (i) and (ii) that $\partial_s u(s,t) \neq 0$ and $u(s,t) \neq x^\pm(t)$ for all $(s,t) \in B_\varepsilon(s_0, t_0)$. Hence the condition $u(B_\varepsilon(s_0, t_0)) \cap R(u) = \emptyset$ means that for all $(s,t) \in B_\varepsilon(s_0, t_0)$ there exists an $s' \in \mathbb{R}$ such that $u(s,t) = u(s',t)$ and $s' \neq s$. In view of (iii) we have $\partial_s u(s',t) \neq 0$ for any such point s' and in view of (i) we have $|s'| \leq T$. Hence there can only be finitely many such points s' for each pair (s,t). (Otherwise there would be an accumulation point at which $\partial_s u = 0$ and we have just seen that this is impossible.) Hence let $s_1, \ldots, s_N \in [-T, T]$ be the points with

$$u(s_0, t_0) = u(s_1, t_0) = \cdots = u(s_N, t_0).$$

We claim that for every constant $r > 0$ there exists a $\delta > 0$ such that

$$u(B_{2\delta}(s_0, t_0)) \subset \bigcup_{j=1}^{N} u(B_r(s_j, t_0)).$$

Otherwise there would exist a sequence $(s_\nu, t_\nu) \to (s_0, t_0)$ such that

$$u(s_\nu, t_\nu) \notin B_\rho(s_j, t_0)$$

for every $j \geq 1$. But there exists a sequence $s'_\nu \neq s_\nu$ such that $u(s_\nu, t_\nu) = u(s'_\nu, t_\nu)$. By assumption $(s'_\nu, t_\nu) \notin B_\rho(s_j, t_0)$ and, by (ii), we have $|s_\nu - s'_\nu| \geq \varepsilon$. By (i), we have $|s'_\nu| \leq T$. Hence the sequence s'_ν has an accumulation point s' which must be distinct from all the points $s_0, \ldots, s_N$ but satisfies $u(s', t_0) = u(s_0, t_0)$. This contradiction proves the claim.

Now define

$$\Sigma_j = \{(s,t) \in \mathrm{cl}(B_\delta(s_0, t_0)) \,|\, u(s,t) \in \mathrm{cl}(u(B_r(s_j, t_0)))\}$$

for $j = 1, \ldots, N$. These sets are closed and

$$\mathrm{cl}(B_\delta(s_0, t_0)) = \Sigma_1 \cup \ldots \cup \Sigma_k$$

Hence at least one of the sets Σ_j has a nonempty interior. Assume without loss of generality that $\mathrm{int}(\Sigma_1) \neq \emptyset$ and $0 \in \mathrm{int}(\Sigma_1)$. Choose $\rho > 0$ so small that $B_\rho(0) \subset \Sigma_1 \subset B_\varepsilon(s_0, t_0)$ and note that

$$B_\rho(0) \cap B_r(s_1, t_0) = \emptyset$$

provided that $r > 0$ was chosen sufficiently small. On the other hand it follows from the definition of Σ_1 that for every $(s,t) \in B_\rho(0)$ there exists an $s' \in \mathbb{R}$ such that $(s', t) \in B_r(s_1, t_0)$ and $u(s,t) = u(s', t)$. Since the point s' is uniquely determined by s we may assume that $t_0 = 0$ and $u(0,0) = u(s_1, 0)$. This means that the functions $u(s,t)$ and $v(s,t) = u(s + s_1, t)$ satisfy the assumptions of Lemma 4.2 with $\varepsilon = r$ and $\delta = \rho$. Hence it follows from Lemma 4.2 that

$$u(s,t) = u(s + s_1, t)$$

in a neighbourhood of zero and hence, by unique continuation this equation holds on all of $\mathbb{R}^2$ (see Proposition 3.1). But this implies

$$u(s,t) = \lim_{k \to \pm\infty} u(s + k s_1, t) = x^{\pm}$$

for all s and t. Hence u is constant in contradiction to our assumption that $\partial_s u \not\equiv 0$. This proves the theorem. □

Remark 4.4 Assume $z_0 = (s_0, t_0) \in R(u)$. Then for any smooth function $\rho : R^2 \to \mathbb{R}$ which is supported in a sufficiently small neighbourhood of z_0 there exists a smooth cutoff function $\mathbb{R} \times M \to [0,1] : (t,p) \mapsto \beta_t(p)$ such that

$$\beta_t(u(s,t)) = \rho(s,t), \qquad \beta_{t+1} \circ \phi = \beta_t.$$

We leave the proof of this elementary fact as an exercise.

5 Transversality

In this section we specialize to the case where (M, ω) is a compact symplectic manifold, $\phi : M \to M$ is a symplectomorphism, and the almost complex structure $J_t : TM \to TM$ is compatible with ω. This means that the formula

$$\langle v, w \rangle_t = \omega(v, J_t w) \tag{11}$$

defines a Riemannian metric on M for every t. We also assume that X_t is a Hamiltonian vector field for every t. This means that there exists a smooth time dependent Hamiltonian function $\mathbb{R} \times M \to \mathbb{R} : (t,p) \mapsto H(t,p) = H_t(p)$ such that

$$\iota(X_t)\omega = dH_t.$$

We now assume that H and J satisfy the periodicity condition

$$H_t = H_{t+1} \circ \phi, \qquad J_t = \phi^* J_{t+1}.$$

We shall denote by $\mathcal{J}_\phi(M, \omega)$ the space of all smooth t-dependent almost complex structures $\mathbb{R} \to \mathrm{End}(TM) : t \mapsto J_t$ which are compatible with ω and satisfy $J_t = \phi^* J_{t+1}$. Likewise, denote by $C^\infty_\phi(M)$ the space of all smooth t-dependent Hamiltonian functions $\mathbb{R} \to C^\infty(M) : t \mapsto H_t$ which satisfy $H_t = H_{t+1} \circ \phi$.

We point out that the requirement on the vector field X_t to be Hamiltonian rather than symplectic (i.e. $\iota(X_t)\omega$ is exact rather than closed) poses no restriction at all. The proof of Theorem 4.3 shows that the vector field X can be removed from (6) at the expense of altering ϕ and J. So if we perturb J we can simply consider the case $X = 0$. However, if we perturb X it is essential to know that this can be done within the class of Hamiltonian vector fields.

Now the partial differential equation (6) can be written in the form

$$\overline{\partial}_{J,H}(u) = \partial_s u + J_t(u)\partial_t u - \nabla H_t(u) = 0 \tag{12}$$

where $\nabla H_t = J_t X_t$ denotes the gradient with respect to the t-dependent metric (11). As before we also assume that u satisfies

$$u(s, t+1) = \phi(u(s,t)), \tag{13}$$

$$\lim_{s \to \pm\infty} u(s,t) = \psi_t(x^\pm), \qquad \phi_H(x^\pm) = x^\pm \tag{14}$$

where the convergence is uniform in t and $\partial_s u(s,t)$ converges to zero, also uniformly in t, as s tends to $\pm\infty$. Here $\psi_t : M \to M$ denotes the family of symplectomorphism generated by the Hamiltonian vector fields $X_t = X_{H_t}$ via (7) and

$$\phi_H = \psi_1^{-1} \circ \phi$$

Hence, again as before, the limit curves $x^\pm(t) = \psi_t(x^\pm)$ are solutions of the ordinary differential equation $\dot{x}(t) = X_t(x(t))$ and the equation

$$\psi_{t+1} \circ \phi_H = \phi \circ \psi_t$$

shows that $x(t+1) = \phi(x(t))$. As in [10] one can prove that for a generic Hamiltonian function H the fixed points of ϕ_H are all nondegenerate. (In [10] this was proved for the case $\phi = \mathbb{1}$.)

If the fixed points of ϕ_H are all nondegenerate then for solutions of (12) and (13) the existence of the limits (14) is equivalent to the finiteness of the energy

$$E(u) = \tfrac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(|\partial_s u|^2 + |\partial_t u - X_t(u)|^2 \right) dtds < \infty.$$

If u satisfies (12), (13), and (14) then the energy of u is given by

$$E(u) = \int_{-\infty}^{\infty} \int_0^1 \omega(\partial_s u, \partial_t u)\, dtds + \int_0^1 H_t(x^+(t))\, dt - \int_0^1 H_t(x^-(t))\, dt. \quad (15)$$

In fact these solutions minimize the energy $E(u)$ among all smooth functions u which satisfy (13) and (14). These observations follows from standard arguments as in [5], [7], [19], [20].

A key theorem in Floer homology asserts that for a generic H or a generic J the space

$$\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+, \phi, H, J)$$

of all solutions of (12), (13) and (14) is a finite dimensional manifold. The proof is based on Fredholm theory and Thom-Smale transversality and we shall carry out the details in this section.

The first step is to linearize the partial differential equation (12). This leads to the first order differential operator

$$D_u : W^{1,p}_\phi(u^*TM) \to L^p_\phi(u^*TM)$$

defined by

$$D_u\xi = \nabla_s \xi + J_t(u)\nabla_s \xi + \nabla_\xi J_t(u)\partial_t u - \nabla_\xi \nabla H_t(u)$$

where ∇ denotes the Levi-Civita connection with respect to the t-dependent metric (11). For every inter $k \geq 0$, we denote by $W^{k,p}_\phi(u^*TM)$ the completion with respect to the $W^{k,p}$-norm of the space of smooth vector fields $\xi(s,t) \in T_{u(s,t)}M$ along u which satisfy $\xi(s,t+1) = d\phi(u(s,t))\xi(s,t)$ and have compact support in $\mathbb{R} \times S^1$. For $k = 0$ we denote $L^p_\phi(u^*TM) = W^{0,p}_\phi(u^*TM)$. If $x^\pm$ are nondegenerate fixed points of ϕ_H and u satisfies (13) and (14) then the operator D_u is Fredholm and its index

$$\mathrm{index} D_u = \mu(u)$$

is the Maslov class of u. That D_u is Fredholm was proved for the case $\phi = \mathbb{1}$ by Floer [7] and the index formula was proved by Salamon and Zehnder in [20] (see also [18]). The case of general ϕ is treated in [3]. The Maslov class $\mu(u)$ is invariant under homotopy, additive for catenations, and satisfies

$$\mu(u \# v) = \mu(u) + 2c_1(v^*TM)$$

for any sphere $v : S^2 \to M$. It is related to the Morse index as follows. If $\phi = \mathrm{id}$ and $H_t = H : M \to \mathbb{R}$ is a Morse function with sufficiently small second derivatives then the fixed points $x^\pm$ of ϕ_H are critical points of H and

$$\mu(u) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-)$$

whenever $u(s,t) = u(s)$ is independent of t. If $\phi = \mathrm{id}$ then these properties determine the Maslov class uniquely. In general one can choose a trivialization of the vector bundle $u^*TM \to \mathbb{R} \times S^1$ and express the Maslov class as the difference of the Conley-Zehnder indices corresponding to the ends. The Conley-Zehnder index is a version of the Maslov index for symplectic paths and was introduced in [1]. For details about the above assertions about the Fredholm index and the Maslov class we refer to [3], [17], [18], and [20].

Now if the operator D_u is onto then it follows from an infinite dimensional implicit function theorem that the space $\mathcal{M}(x^-, x^+)$ is a finite dimensional manifold of local dimension $\dim_u \mathcal{M}(x^-, x^+) = \mu(u)$ near u. Hence we denote by

$$\mathcal{HJ}_{\mathrm{reg}} = \mathcal{HJ}_{\mathrm{reg}}(M, \omega, \phi)$$

the space of all pairs $(H, J) \in C^\infty_\phi(M) \times \mathcal{J}_\phi(M, \omega)$ such that the fixed points of ϕ_H are all nondegenerate and the operator D_u is onto for all *connecting orbits* $u \in \mathcal{M}(x^-, x^+, \phi, H, J)$ and all fixed points $x^\pm \in \mathrm{Fix}(\phi_H)$.

We are now in a position to state the main theorem of this section. Recall that a subset of a complete metric space is said to be of the **second category** if it contains a countable intersection of open and dense sets. By Baire's category theorem, every set of the second category is dense. Recall also that the spaces $\mathcal{J}_\phi(M, \omega)$ and $C^\infty_\phi(M)$ with their C^∞-topology admit the structure of a complete metric spaces. Now fix a Hamiltonian function $H_0 \in C^\infty_\phi(M)$ such that the fixed points of ϕ_{H_0} are all nondegenerate. Denote by

$$C^\infty_\phi(M, H_0)$$

the subset of all $H \in C^\infty_\phi(M)$ which agree with H_0 up to second order on the solutions of $\dot{x}(t) = X_{H_0}(x(t), t)$ which satisfy $x(t+1) = \phi(x(t))$.

Theorem 5.1 *Let (M, ω) be a compact symplectic manifold and $\phi : M \to M$ be a symplectomorphism.*

(i) *Let $H \in C^\infty_\phi(M)$ and assume that the fixed points of ϕ_H are all nondegenerate. Then the set*

$$\begin{aligned} \mathcal{J}_{\mathrm{reg}} &= \mathcal{J}_{\mathrm{reg}}(M, \omega, \phi, H) \\ &= \{J \in \mathcal{J}_\phi(M, \omega) \,|\, (H, J) \in \mathcal{HJ}_{\mathrm{reg}}\} \end{aligned}$$

is of the second category in $\mathcal{J}_\phi(M, \omega)$.

(ii) *Let $J \in \mathcal{J}_\phi(M, \omega)$ and $H_0 \in C^\infty_\phi(M)$ and assume that the fixed points of ϕ_{H_0} are all nondegenerate. Then the set*

$$\begin{aligned} \mathcal{H}_{\mathrm{reg}} &= \mathcal{H}_{\mathrm{reg}}(M, \omega, \phi, H_0) \\ &= \left\{H \in C^\infty_\phi(M, H_0) \,|\, (H, J) \in \mathcal{HJ}_{\mathrm{reg}}\right\} \end{aligned}$$

is of the second category in $C^\infty_\phi(M, H_0)$.

Remark 5.2 The proof of Theorem 5.1 shows that it suffices to perturb the almost complex structure J_t outside a neighbourhood U of the points $\psi_t(x)$ where x runs through the (finitely many) fixed points of ϕ_H. This is because any connecting orbit which is independent of s must have positive energy and in the case $x^+ = x^-$ it follows from the energy identity (15) that $\int u^*\omega > 0$. So even in the case $x^- = x^+$ a nontrivial connecting orbit cannot stay close to the curves $\psi_t(x)$ where $x = \phi_H(x)$. Thus we may replace the space $\mathcal{J}_\phi(M, \omega)$ by the subspace of those almost complex structures which agree with a given structure J_t in U.

Unfortunately, however, we were not able to prove such a statement in the case of the Hamiltonian functions H_t. Here our proof requires possible perturbations arbitrarily close to the limit curves $x^\pm(t)$. The result should remain valid for Hamiltonian functions with support outside a fixed neighbourhood U of the curves $\psi_t(x)$ with $x \in \mathrm{Fix}(\phi_H)$ but this requires a modification of our argument below which we could not quite see how to do.

Remark 5.3 It is easy to see that instead of ω-compatible almost complex structures we can consider all structures J_t which are tamed by ω in the sense that $\omega(v, J_t v) > 0$ whenever $v \neq 0$. In this case the induced metric is given by $\langle v, w\rangle_t = \frac{1}{2}(\omega(v, J_t w) + \omega(w, J_t v))$ rather than (11).

Proof of Theorem 5.1: Fix a number $p > 2$ and two nondegenerate fixed points $x^\pm = \phi_H(x^\pm)$, denote $x^\pm(t) = \psi_t(x^\pm)$, and choose trivializations $\Phi^\pm(t) : \mathbb{R}^{2n} \to T_{x^\pm(t)}M$ with $\Phi^\pm(t+1) = d\phi(x^\pm(t))\Phi^\pm(t)$. Denote by

$$\mathcal{B} = \mathcal{B}^{1,p}(x^-, x^+, \phi)$$

the space of continuous maps $u : \mathbb{R}^2 \to M$ which satisfy (14), are locally of class $W^{1,p}$ and satisfy $\xi^+ \in W^{1,p}([T, \infty) \times [0, 1])$ and $\xi^- \in W^{1,p}((-\infty, -T] \times [0, 1])$ where $\xi^\pm(s, t) = \xi^\pm(s, t + 1) \in \mathbb{R}^{2n}$ is defined by $\exp_{x^{\pm}(t)}(\Phi^{\pm}(t)\xi^{\pm}(s,t)) = u(s,t)$ for $\pm s \geq T$ with T sufficiently large. The space $\mathcal{B}^{1,p}$ is an infinite dimensional Banach manifold with tangent space

$$T_u\mathcal{B} = W^{1,p}_{\phi}(u^*TM)$$

Consider the vector bundle $\mathcal{E} \to \mathcal{B}$ whose fiber over $u \in \mathcal{B}$ is the space

$$\mathcal{E}_u = L^p_{\phi}(u^*TM).$$

The left hand side of the equation (12) defines a section

$$\overline{\partial}_{J,H} : \mathcal{B} \to \mathcal{E}$$

of this bundle and the moduli space $\mathcal{M}(x^-, x^+, J)$ of connecting orbits is the zero set of this section. To prove that this moduli space is a manifold we must shows that $\overline{\partial}_{J,H}$ is transversal to the zero section. Now the tangent space of $\mathcal{E}$ splits at the zero section as $T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u$ and the composition of the differential $d\overline{\partial}_{H,J} : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$ with the projection $\pi_u : T_{(u,0)}\mathcal{E} \to \mathcal{E}_u$ is precisely the operator $D_u : T_u\mathcal{B} \to \mathcal{E}_u$ introduced above. Hence $\overline{\partial}_{H,J}$ is transversal to the zero section if and only if the operator D_u is onto for every $u \in \mathcal{M}(x^-, x^+)$ and this means that $(H, J) \in \mathcal{HJ}_{\mathrm{reg}}$.

Now denote by $\mathcal{J}^\ell = \mathcal{J}^\ell_\phi(M, \omega)$ the completion of $\mathcal{J}_\phi(M, \omega)$ with respect to the C^ℓ-topology. This space is a Banach manifold. Its tangent space is the space $T_J\mathcal{J}^\ell$ of C^ℓ-maps $\mathbb{R} \times TM \to TM : (t, p, v) \mapsto Y_t(p)v$ which satisfy

$$J_tY_t + Y_tJ_t = 0, \qquad \omega(Y_tv, w) + \omega(v, Y_tw) = 0, \qquad \phi^*Y_{t+1} = Y_t.$$

The first two conditions can be summarized as $Y_t \in C^\ell(\mathrm{End}(TM, J_t, \omega))$. The map $(u, J) \mapsto \overline{\partial}_{H,J}(u)$ defines a section of the bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{J}^\ell$ with fiber $\mathcal{E}_{(u,J)} = \mathcal{E}_u$ which we denote by

$$\mathcal{F} : \mathcal{B} \times \mathcal{J}^\ell \to \mathcal{E}, \qquad \mathcal{F}(u, J) = \overline{\partial}_{H,J}(u).$$

The zero set of this section is the universal moduli space

$$\mathcal{M}(x^-, x^+, \mathcal{J}^\ell) = \left\{(u, J) \in \mathcal{B} \times \mathcal{J}^\ell \,|\, \overline{\partial}_{H,J}(u) = 0\right\}.$$

We shall prove that the section $\mathcal{F}$ is transverse to the zero section and hence $\mathcal{M}(x^-, x^+, \mathcal{J}^\ell)$ is a Banach manifold. To see this note that the differential $D\mathcal{F}(u, J) = \pi_u \circ d\mathcal{F}(u, J) : T_u\mathcal{B} \times T_J\mathcal{J}^\ell \to \mathcal{E}_u$ at a point (u, J) with $\mathcal{F}(u, J) = 0$ is given by

$$D\mathcal{F}(u, J)(\xi, Y) = D_u\xi + Y_t(u)(\partial_t u - X_t(u))$$

for $\xi \in T_u\mathcal{B} = W^{1,p}_\phi(u^*TM)$ and $Y \in T_J\mathcal{J}^\ell$. We must prove that the operator $D\mathcal{F}(u, J)$ is onto for every $(u, J) \in \mathcal{M}(x^-, x^+, \mathcal{J}^\ell)$.

Since the operator D_u is Fredholm it suffices to prove that $D\mathcal{F}(u, J)$ has a dense range. Hence choose $q > 1$ such that $1/p + 1/q = 1$ and suppose that $\eta \in L^q_\phi(u^*TM)$ is in the annihilator of the range of $D\mathcal{F}(u,J)$. This means that

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta, D_u\xi\rangle\, dtds = 0 \tag{16}$$

for all $\xi \in W^{1,p}_\phi(u^*TM)$ and

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta, Y_t(u)\partial_t u\rangle\, dtds = 0 \tag{17}$$

for all $Y \in T_J\mathcal{J}^\ell$. The first equation asserts that η is a weak solution of $D_u^*\eta = 0$ where D_u^* is the formal adjoint operator of D_u which can be obtained from D_u by replacing ∇_s with $-\nabla_s$. Since D_u^* is an elliptic first order operator with coefficients of class C^ℓ it follows that η is of class C^ℓ and is a strong solution of $D_u^*\eta = 0$. In local coordinates the operator D_u is of the form (4) and hence the unique continuation result (Proposition 3.1) shows that it suffices to prove that η vanishes on some open set. We shall in fact prove that $\eta(s,t) = 0$ for $(s,t) \in R(u)$ where $R(u) \subset \mathbb{R}^2$ is defined as in Section 4. In view of Theorem 4.3 this set is open and dense in $\mathbb{R}^2$.

Assume, by contradiction, that there is a point $z_0 = (s_0, t_0) \in R(u)$ with $\eta(z_0) \neq 0$. Then it is easy to see that there exists a

$$Y_0 \in \mathrm{End}(T_{u(z_0)}M, J_{t_0}, \omega)$$

such that

$$\langle \eta(z_0), Y_0\partial_s u(z_0)\rangle > 0.$$

(See for example [20], p. 1346.) Now choose any $Y \in T_J\mathcal{J}^\ell$ such that $Y_{t_0}(z_0) = Y_0$. Multiply Y_t by a cutoff function β_t as in Remark 4.4 to obtain a section $\widetilde{Y}_t = \beta_t Y_t$ for which the left hand side of (17) does no vanish. This contradiction shows that $\eta(z) = 0$ for all $z \in R(u)$ and hence $\eta = 0$. Thus we have proved that the operator $D\mathcal{F}(u,J)$ has a dense range and is therefore onto whenever $\overline{\partial}_{H,J}(u) = 0$. This implies that the space $\mathcal{M}(x^-, x^+, \mathcal{J}^\ell)$ is a Banach manifold.

Now consider the projection

$$\mathcal{M}(x^-, x^+, \mathcal{J}^\ell) \to \mathcal{J}^\ell : (u,J) \mapsto J.$$

This is a Fredholm map between Banach manifolds and hence it follows from the Sard Smale theorem [22] that the set

$$\mathcal{J}^\ell_{\mathrm{reg}}(x^-, x^+) \subset \mathcal{J}^\ell$$

of regular values of this projection is of the second category in $\mathcal{J}^\ell$. Now it follows from the usual argument in Thom-Smale transversality that the regular values of the above projection are precisely those almost complex structures $J \in \mathcal{J}^\ell$ for which the operator D_u is surjective whenever $u \in \mathcal{M}(x^-, x^+, J)$.

Thus we have proved statement (i) in the C^ℓ category. Although this would suffice for most applications it is more elegant to work with the full statement in the C^∞ category. This can be reduced to the C^ℓ-statement via the following argument which is due to Taubes [23] and was also used in [13].

Consider the space

$$\mathcal{J}_{\mathrm{reg,K}} \subset \mathcal{J}$$

of all smooth almost complex structures such that D_u is onto for all $x^\pm$ and all $u \in \mathcal{M}(x^-, x^+, J)$ which satisfy $|\partial_s u(s,t)| \le K$ for all s and t. Then

$$\mathcal{J}_{\mathrm{reg}} = \bigcap_{K>0} \mathcal{J}_{\mathrm{reg,K}}$$

and we prove that the set $\mathcal{J}_{\mathrm{reg,K}}$ is open and dense in $\mathcal{J}$ with respect to the C^∞-topology. We prove first that the complement of $\mathcal{J}_{\mathrm{reg,K}}$ is closed. Thus let $J_\nu \notin \mathcal{J}_{\mathrm{reg,K}}$ converge to $J \in \mathcal{J}$ in the C^∞-topology. Then there exists a sequence u_ν of connecting orbits with respect to J_ν such that D_{u_ν} is not onto. We may assume without loss of generality that $u_\nu \in \mathcal{M}(x^-, x^+, J_\nu)$ for some fixed pair $x^\pm = \phi_H(x^\pm)$. Since the first derivatives of u_ν are uniformly bounded we may also assume without loss of generality that u_ν converges *weakly* to a finite collection of connecting orbits $v_j \in \mathcal{M}(x_{j-1}, x_j, J)$ with $x_0 = x^-$ and $x_N = x^+$. If all the D_{v_j} were onto then, by the usual gluing argument, it would follow that also D_{u_ν} is onto for ν sufficiently large. Hence one of the limit operators D_{v_j} is not onto and this shows that $J \notin \mathcal{J}_{\mathrm{reg,K}}$. Thus we have proved that $\mathcal{J}_{\mathrm{reg,K}}$ is open in $\mathcal{J}$ with respect to the C^∞-topology.

Now we shall prove that $\mathcal{J}_{\mathrm{reg,K}}$ is dense in $\mathcal{J}$ in the C^∞-topology. To see this note first that

$$\mathcal{J}_{\mathrm{reg,K}} = \mathcal{J}^\ell_{\mathrm{reg,K}} \cap \mathcal{J}$$

where $\mathcal{J}^\ell_{\mathrm{reg,K}}$ is defined as $\mathcal{J}_{\mathrm{reg,K}}$ but with the C^∞-topology replaced by the C^ℓ topology. But we have proved above that $\mathcal{J}^\ell_{\mathrm{reg,K}}$ is dense in $\mathcal{J}^\ell$ with respect to the C^ℓ-topology. Hence the set $\mathcal{J}^\ell_{\mathrm{reg,K}}$ is both open and dense in $\mathcal{J}^\ell$ with respect to the C^ℓ-topology. This implies that $\mathcal{J}_{\mathrm{reg,K}}$ is dense in $\mathcal{J}$ with respect to the C^ℓ-topology. (Take $J \in \mathcal{J}$, approximate it in the C^ℓ topology by an element $J' \in \mathcal{J}^\ell_{\mathrm{reg,K}}$, and then approximate J' by an element $J'' \in \mathcal{J}_{\mathrm{reg,K}} = \mathcal{J}^\ell_{\mathrm{reg,K}} \cap \mathcal{J}$ in the C^ℓ-topology.) Thus we have proved that the set $\mathcal{J}_{\mathrm{reg,K}}$ is dense in $\mathcal{J}$ with respect to the C^ℓ topology for every ℓ. But this implies that $\mathcal{J}_{\mathrm{reg,K}}$ is dense in $\mathcal{J}$ with respect to the C^∞ topology. (Given $J \in \mathcal{J}$ and $\nu \in \mathbb{N}$ choose $J_\nu \in \mathcal{J}_{\mathrm{reg,K}}$ such that $\|J - J_\nu\|_{C^\nu} \le 2^{-\nu}$. Then J_ν converges to J in the C^∞-topology.) Hence we have represented $\mathcal{J}_{\mathrm{reg}}$ as a countable intersection of open and dense sets and this proves statement (i).

The proof of (ii) is essentially the same as that of (i). The only point of difference is that we now consider the bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{H}^\ell$ where $\mathcal{H}^\ell = C^\ell_\phi(M, H_0)$. The fibers of the bundle $\mathcal{E}$ are given by $\mathcal{E}_{(u,H)} = L^p_\phi(u^*TM)$ as before. The section $\mathcal{F} : \mathcal{B} \times \mathcal{H}^\ell \to \mathcal{E}$ is given by $\mathcal{F}(u, H) = \overline{\partial}_{J,H}(u)$ and its differential $D\mathcal{F}(u, H) : T_u\mathcal{B} \times \mathcal{H} \to \mathcal{E}_u$ is of the form

$$D\mathcal{F}(u, H)(\xi, h) = D_u\xi - \nabla h_t(u).$$

In this case a section $\eta \in L^q_\phi(u^*TM)$ is in the annihilator of the image of $D\mathcal{F}(u,H)$ if and only if it is of class C^ℓ wit $D_u^*\eta = 0$ and

$$\int_{-\infty}^{\infty}\int_0^1 dh_t(u)\eta\, dtds = 0 \tag{18}$$

for all $h \in \mathcal{H}^\ell$. Again we must prove that any such η vanishes on some open set. The details were carried out in [20], pp. 1349–1351, and we reproduce the argument here.

We first prove that $\eta(s,t)$ and $\partial_s u(s,t)$ are linearly dependent for all $(s,t) \in \mathbb{R}^2$. Suppose otherwise that $\partial_s u$ and η are linearly independent at some point (s_0,t_0). We may assume without loss of generality that $0 < t_0 < 1$ and, by Theorem 4.3, we may also asume without loss of generality that $(s_0,t_0) \in R(u)$. Then there exists a neighbourhood $U_0 \subset (0,1)\times M$ of $(t_0, u(s_0,t_0))$ such that

$$V_0 = \left\{(s,t) \in \mathbb{R}^2 \,|\, (t, u(s,t)) \in U_0\right\}$$

is a small neighbourhood of (s_0,t_0). Now for $\varepsilon > 0$ sufficiently small and t sufficiently close to t_0 there is an embedding $g_t : B_\varepsilon(0,s_0) \to U_0$ defined by $g_t(r,s) = \exp_{u(s,t)}(r\eta(s,t))$. This embedding satisfies

$$g_t(0,s) = u(s,t), \qquad \partial_r g_t(0,s) = \eta(s,t).$$

Since g_t is an embedding there exists a Hamiltonian function $h_t : M \to \mathbb{R}$ such that the map $(0,1)\times M \to \mathbb{R} : (t,x) \mapsto h_t(x)$ is supported in U_0 and

$$h_t(g_t(r,s)) = r\beta(r)\beta(s-s_0)\beta(t-t_0)$$

where $\beta : \mathbb{R} \to [0,1]$ is a cutoff function which is equal to 1 near 0. Differentiating this identity with respect to r at $r = 0$ we obtain

$$dh_t(u(s,t))\eta(s,t) = \beta(s-s_0)\beta(t-t_0)$$

for all s and t. Moreover h_t vanishes for t near 0 or 1 and hence we can extend h_t to all $t \in \mathbb{R}$ such that $h_t = h_{t+1}\circ\phi$. Thus we have found a function $h \in \mathcal{H}^\ell$ such that the left hand side of (18) does not vanish. This contradiction proves that $\eta(s,t)$ and $\partial_s u(s,t)$ must be linearly dependent for all $(s,t) \in \mathbb{R}^2$.

Now recall that $C(u)$ is the set of points (s,t) with $\partial_s u = 0$. By what we have just proved there is a unique function $\lambda : \mathbb{R}^2 - C(u) \to \mathbb{R}$ such that

$$\eta(s,t) = \lambda(s,t)\partial_s u(s,t)$$

for $(s,t) \in \mathbb{R}^2 - C(u)$. We prove that

$$\partial_s\lambda(s,t) = 0$$

for all $(s,t) \in \mathbb{R}^2 - C(u)$. Assume otherwise that there exists a point $(s_0,t_0) \in \mathbb{R}^2 - C(u)$ such that $\partial_s\lambda(s_0,t_0) \neq 0$. Since $R(u)$ is dense in $\mathbb{R}^2$ we may assume without loss of generality that $(s_0,t_0) \in R(u)$. Choose a smooth function $\rho : \mathbb{R}^2 \to [0,1]$ with support in a neighbourhood V_0 of (s_0, t_0) as above such that $\int \rho \partial_s \lambda \neq 0$ and hence

$$\int_{V_0} \lambda \partial_s \rho \neq 0.$$

Now choose $h_t : M \to \mathbb{R}$ such that $h_t(u(s,t)) = \rho(s,t)$ (see Remark 4.4). Then $dh_t(u)\eta = \lambda \partial_s \rho$ and it follows again that the left hand side of (18) is nonzero. Thus we have proved that $\partial_s \lambda(s,t) = 0$ for all $(s,t) \in \mathbb{R}^2 - C(u)$. Since $C(u)$ is a discrete set it follows that $\lambda(s,t)$ extends to a C^ℓ-function on $\mathbb{R}^2$ which is independent of s. Hence $\lambda(s,t) = \lambda(t)$ is defined for all $t \in \mathbb{R}$ and

$$\eta(s,t) = \lambda(t)\partial_s u(s,t)$$

for all $(s,t) \in \mathbb{R}^2$. Now assume $\eta \neq 0$. Then it follows from Corollary 2.3 that the set of points where $\eta(s,t)$ vanishes is discrete. Hence $\lambda(t)$ must be nonzero for all t. We assume, without loss of generality, that

$$\lambda(t) > 0$$

for all t. (Otherwise replace η by $-\eta$.) Now use

$$D_u \partial_s u = 0, \qquad D_u^* \eta = 0$$

to obtain

$$\begin{aligned}
\frac{d}{ds}\int_0^1 \langle \eta, \partial_s u\rangle\, dt &= \int_0^1 \left(\langle \eta, \nabla_s \partial_s u\rangle + \langle \nabla_s \eta, \partial_s u\rangle\right) dt \\
&= \int_0^1 \left(\langle \eta, D_u \partial_s u\rangle - \langle D_u^* \eta, \partial_s u\rangle\right) dt \\
&= 0.
\end{aligned}$$

On the other hand

$$\int_0^1 \langle \eta, \partial_s u\rangle = \int_0^1 \lambda(t)|\partial_s u(s,t)|^2\, dt > 0$$

and so the total integral over $\mathbb{R} \times [0,1]$ would be infinite. But since $\eta \in L^q_\phi(u^*TM)$ and $\partial_s u \in L^p_\phi(u^*TM)$ with $1/p + 1/q = 1$ this integral has to be finite. This contradiction shows that $\eta = 0$.

Thus we have proved that the section $(u,H) \mapsto \overline{\partial}_{J,H}(u)$ of the bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{H}^\ell$ is transverse to the zero section. The remainder of the proof of (ii) is precisely the same as that of (i) and can be safely left to the reader. □

Remark 5.4 Theorem 5.1 generalizes under suitable assumptions to non-compact manifolds. For example in [9] the first and the second auther considered Hamiltonian systems in $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard complex structure J_0 and the standard symplectic form ω_0. These are given by

$$\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j, \qquad J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

in the coordinates $(z_1, \dots, z_n)$ with $z_j = x_j + iy_j$. They considered smooth Hamiltonian functions $H_t = H_{t+1} : \mathbb{C}^n \to \mathbb{R}$ which satisfy the conditions

$$|\partial_t \nabla H_t(z)| \le c(1 + |z|), \qquad \left|\nabla^2 H_t(z)\right| \le c$$

for a suitable constant $c > 0$. Moreover H is required to satisfy the asymptotic condition

$$\lim_{|z|\to\infty} \frac{|\nabla H_t(z) - Az|}{|z|} = 0$$

for a symmetric matrix A whose spectrum does not intersect the lattice $2\pi\mathbb{Z}$. This means that the equation

$$J_0 \dot{z}(t) = Az(t), \qquad z(0) = z(1)$$

has only the trivial solution. In this case the transversality theorems of this section remain valid for almost complex structures $J = J(t, z)$ which are compatible with ω_0 and agree with J_0 for large $|z|$.

Our next goal is to study the transversality problem for the partial differential equation (6) in the case where $\phi = \mathbb{1}$ and J and X are independent of t. This is a severe restriction and the techniques of this section will break down in this case. In the time independent case Theorem 4.3 is useless and transversality can only be expected for *simple solutions*. Moreover we need an additional technical result about symmetric $(2n \times 2n)$-matrices which will be discussed in the next section.

6 Symmetric matrices

Consider $\mathbb{R}^{2n}$ with the standard complex structure

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

and the standard inner product. Denote the vector space of symmetric $(2n \times 2n)$-matrices by $\mathcal{S} = \left\{ S \in \mathbb{R}^{2n\times 2n} \,|\, S^T = S \right\}$. Now consider the subset

$$\mathcal{S}_{\mathrm{reg}} \subset \mathcal{S}$$

of all those matrices $S = S^T \in \mathbb{R}^{2n\times 2n}$ such that for any four real numbers a, b, α, β there is no nonzero solution $\zeta \in \mathbb{R}^{2n}$ of the equations

$$(SJ_0 - J_0 S - a - bJ_0)\zeta = 0 \tag{19}$$

and

$$(SJ_0 - J_0 S - a - bJ_0)S\zeta - \alpha\zeta - \beta J_0 \zeta = 0. \tag{20}$$

Our goal in this section is to prove the following theorem.

Theorem 6.1 *Assume $n \ge 2$. Then the set $\mathcal{S}_{\mathrm{reg}}$ is open and dense in $\mathcal{S}$. Moreover, if $S \in \mathcal{S}_{\mathrm{reg}}$ then $\tau\Phi^T S\Phi \in \mathcal{S}_{\mathrm{reg}}$ for every real number $\tau \ne 0$ and any unitary matrix $\Phi \in \mathrm{U}(n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n)$.*

Remark 6.2 If we identify $\mathbb{R}^{2n} = \mathbb{C}^n$ in the usual way by $z = x + iy$ with $x, y \in \mathbb{R}^n$ then a real linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ can be written in the form

$$T(\zeta) = A\zeta + B\bar{\zeta}$$

for $\zeta \in \mathbb{C}^n$ where $A, B \in \mathbb{C}^{n\times n}$ represent the complex linear and the complex anti-linear part of T. In real notation T is represented by multiplication with a symmetric matrix if and only if

$$A = A^*, \qquad B = B^T$$

where B^T denotes the transposed matrix and $A^* = \bar{A}^T$ the conjugate transpose. With $\lambda = \frac{i}{2}(a + ib)$ and $\mu = \frac{i}{2}(\alpha + i\beta)$ the equations (19) and (20) can be written in the form

$$B\bar{\zeta} - \lambda\zeta = 0, \qquad B\overline{(A\zeta + B\bar{\zeta})} - \lambda(A\zeta + B\bar{\zeta}) - \mu\zeta = 0.$$

Now assume $n = 1$. Then B is just a complex number and A is a real number. Hence the first equation has a solution $\lambda = \bar{B}$, $\zeta = B$ and the second equation can obviously be solved for μ. Hence in this case $\mathcal{S}_{\text{reg}}$ is the empty set.

Proof of Theorem 6.1: $\mathcal{S}_{\text{reg}}$ is open: Let S_ν be a sequence in $\mathcal{S} - \mathcal{S}_{\text{reg}}$ and assume that S_ν converges to $S \in \mathcal{S}$. Then there exist corresponding sequences a_ν, b_ν, α_ν, β_ν such that the equations (19) and (20) have a solution $\zeta_\nu \in \mathbb{R}^{2n}$ with $|\zeta_\nu| = 1$. Since ζ_ν and $J_0\zeta_\nu$ are orthogonal it follows that

$$a_\nu^2 + b_\nu^2 \le 4\,\|S_\nu\|^2 .$$

Similarly, the sequences α_ν and β_ν are bounded and so we may assume without loss of generality that the sequences a_ν, b_ν, α_ν, β_ν, and ζ_ν all converge. Hence in the limit we obtain a nonzero solution of (19) and (20) and this shows that $S \notin \mathcal{S}_{\text{reg}}$. □

The proof that $\mathcal{S}_{\text{reg}}$ is dense will occupy the rest of this section. We shall begin by examining (19). Denote by

$$\Sigma \subset \mathcal{S} \times \mathbb{R}^2 \times \mathbb{R}^{2n}$$

the set of all quadruples (S, a, b, ζ) with $S = S^T \in \mathbb{R}^{2n\times 2n}$, $a, b \in \mathbb{R}$, $\zeta \in \mathbb{R}^{2n}$ such that

$$(SJ_0 - J_0S - a - bJ_0)\zeta = 0$$

and $\zeta \ne 0$.

Lemma 6.3 *Σ is a smooth submanifold of $\mathcal{S} \times \mathbb{R}^2 \times \mathbb{R}^{2n}$.*

Proof: We must prove that 0 is a regular value of the map

$$f : \mathcal{S} \times \mathbb{R}^2 \times (\mathbb{R}^{2n} - \{0\}) \to \mathbb{R}^{2n}$$

defined by

$$f(S, a, b, \zeta) = (SJ_0 - J_0S - a - bJ_0)\zeta.$$

We shall prove in fact that whenever $\zeta \neq 0$ then the differential of this map with respect to S is onto. This means that for every $\eta \in \mathbb{R}^{2n}$ and every nonzero vector $\zeta \in \mathbb{R}^{2n}$ there exists a symmetric matrix $\hat{S} \in \mathbb{R}^{2n\times 2n}$ such that $(\hat{S}J_0 - J_0\hat{S})\zeta = \eta$. Denote $A = \hat{S} + J_0\hat{S}J_0$ and $\xi = J_0\zeta$. Then we must prove that the equation

$$A\xi = \eta, \qquad A^T = A = J_0 A J_0$$

has a solution $A \in \mathbb{R}^{2n\times 2n}$ for all $\xi, \eta \in \mathbb{R}^{2n}$ with $\xi \neq 0$. Such a matrix is given by the explicit formula

$$\begin{aligned} A &= \frac{1}{|\xi|^2}\left(\eta\xi^T + \xi\eta^T\right) + \frac{1}{|\xi|^2} J_0 \left(\eta\xi^T + \xi\eta^T\right) J_0 \\ &\quad - \frac{\langle \eta, \xi\rangle}{|\xi|^4}\left(\xi\xi^T + J_0\xi\xi^T J_0\right) - \frac{\langle \eta, J_0\xi\rangle}{|\xi|^4}\left(J_0\xi\xi^T - \xi\xi^T J_0\right). \end{aligned}$$

This proves the lemma. □

Now denote by $\mathcal{S}_1 \subset \mathcal{S}$ the set of regular values of the projection

$$\Sigma \to \mathcal{S} : (S, a, b, \zeta) \mapsto S.$$

in view of Sard's theorem this set is of the second category in $\mathcal{S}$ and, in particular, it is dense.

Lemma 6.4 *The set $\mathcal{S}_1$ is open and dense in $\mathcal{S}$. Moreover, $S \in \mathcal{S}_1$ if and only if* 0 *is a regular value of the map $f_S : \mathbb{R}^2 \times (\mathbb{R}^{2n} - \{0\}) \to \mathbb{R}^{2n}$ defined by*

$$f_S(a, b, \zeta) = (SJ_0 - J_0 S - a - bJ_0)\zeta.$$

Proof: The second assertion follows from standard Thom-Smale transversality theory for paramerized smooth maps $\Lambda \times X \to Y : (\lambda, x) \mapsto f(\lambda, x) = f_\lambda(x)$. Let $y \in Y$ be a regular value of f and denote $\Sigma = f^{-1}(y)$. Then y is a regular value of $f_\lambda : X \to Y$ if and only if λ is a regular value of the projection $\Sigma \to \Lambda : (\lambda, x) \mapsto \lambda$. Apply this to the above map f with $\Lambda = \mathcal{S}$, $X = \mathbb{R}^2 \times (\mathbb{R}^{2n} - \{0\})$, $Y = \mathbb{R}^{2n}$, and $y = 0$ to obtain the required characterization of the set $\mathcal{S}_1$.

Now, by Sard's theorem, the set $\mathcal{S}_1$ is dense in $\mathcal{S}$. That $\mathcal{S}_1$ is open follows from the fact that for every S the set

$$\{(a, b, \zeta) \,|\, f_S(a, b, \zeta) = 0,\ |\zeta| = 1\}$$

is compact and that $df_S(a, b, \zeta)$ is onto if and only if $df_S(a, b, t\zeta)$ is onto for any $t \in \mathbb{R} - \{0\}$. □

Now denote by

$$\Gamma \subset \mathcal{S}_1 \times \mathbb{R}^4 \times (\mathbb{R}^{2n} - \{0\})$$

the set of all sixtuples $(S, a, b, \alpha, \beta, \zeta)$ which satisfy (19) and (20).

Lemma 6.5 Γ *is a smooth submanifold of $\mathcal{S}_1 \times \mathbb{R}^4 \times \mathbb{R}^{2n}$ of dimension*

$$\dim \Gamma = \dim \mathcal{S} + 4 - 2n.$$

Proof: We must prove that 0 is a regular value of the map

$$\mathcal{F} : \mathcal{S}_1 \times \mathbb{R}^4 \times (\mathbb{R}^{2n} - \{0\}) \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$$

defined by

$$\mathcal{F}(S, a, b, \alpha, \beta, \zeta) = (f(S, a, b, \zeta), f(S, a, b, S\zeta) - \alpha\zeta - \beta J_0\zeta).$$

In fact, it suffices to differentiate $\mathcal{F}$ with respect to S, α and β. The differential of $\mathcal{F}$ in these directions is a linear operator $L = (L_1, L_2) : \mathcal{S} \times \mathbb{R}^2 \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ given by

$$\begin{aligned} L_1(\hat{S}, \hat{\alpha}, \hat{\beta}) &= (\hat{S}J_0 - J_0\hat{S})\zeta, \\ L_2(\hat{S}, \hat{\alpha}, \hat{\beta}) &= (\hat{S}J_0 - J_0\hat{S})S\zeta \\ &\quad + (SJ_0 - J_0S - a - bJ_0)\hat{S}\zeta \\ &\quad - (\hat{\alpha} + \hat{\beta}J_0)\zeta. \end{aligned}$$

The strategy is now as follows. Given $\eta_1, \eta_2 \in \mathbb{R}^{2n}$ first choose $\hat{S}_1$ such that

$$(\hat{S}_1 J_0 - J_0\hat{S}_1)\zeta = \eta_1.$$

That this is possible was shown in the proof of Lemma 6.3. Secondly, use the fact that $S \in \mathcal{S}_1$ and hence, by Lemma 6.4, 0 is a regular value of f_S. This implies that there exist $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$ and $\xi \in \mathbb{R}^{2n}$ such that

$$(SJ_0 - J_0S - a - bJ_0)\xi - (\hat{\alpha} + \hat{\beta}J_0)\zeta = \eta_2 - (\hat{S}J_0 - J_0\hat{S})S\zeta.$$

The final step is to find a matrix $A \in \mathbb{R}^{2n \times 2n}$ such that

$$A\zeta = \xi, \qquad A^T = A, \qquad AJ_0 = J_0A.$$

An explicit formula for A is given by

$$\begin{aligned} A &= \frac{1}{|\zeta|^2}\left(\xi\zeta^T + \zeta\xi^T\right) - \frac{1}{|\zeta|^2}J_0\left(\xi\zeta^T + \zeta\xi^T\right)J_0 \\ &\quad - \frac{\langle \xi, \zeta \rangle}{|\zeta|^4}\left(\zeta\zeta^T - J_0\zeta\zeta^T J_0\right) + \frac{\langle \xi, J_0\zeta \rangle}{|\zeta|^4}\left(J_0\zeta\zeta^T + \zeta\zeta^T J_0\right). \end{aligned}$$

Now it follows from the previous three equations that

$$L_1(\hat{S}, \hat{\alpha}, \hat{\beta}) = \eta_1, \qquad L_2(\hat{S}, \hat{\alpha}, \hat{\beta}) = \eta_2$$

with $\hat{S} = \hat{S}_1 + A$. Here we have used the fact that, since A is complex linear, $\hat{S}J_0 - J_0\hat{S} = \hat{S}_1 J_0 - J_0\hat{S}_1$. □

Proof of Theorem 6.1: $\mathcal{S}_{\mathrm{reg}}$ is dense: Denote by $\mathcal{S}_2 \subset \mathcal{S}_1$ the set of regular values of the projection

$$\Gamma \to \mathcal{S}_1 : (S, a, b, \alpha, \beta, \zeta) \mapsto S.$$

Then for every $S \in \mathcal{S}_2$ the set

$$\Gamma_S = \{(a, b, \alpha, \beta, \zeta) \,|\, (S, a, b, \alpha, \beta, \zeta) \in \Gamma\}$$

is a manifold of dimension

$$\dim \Gamma_S = \dim \Gamma - \dim \mathcal{S} = 4 - 2n.$$

This set consists precisely of the quintuples $(a, b, \alpha, \beta, \zeta)$ which satisfy (19) and (20) and $\zeta \neq 0$. Moreover, if Γ_S is nonempty then it is at least 1-dimensional because it is invariant under the action of $\mathbb{R} - \{0\}$ given by $(a, b, \alpha, \beta, \zeta) \mapsto (a, b, \alpha, \beta, t\zeta)$. Hence in the case $n \geq 2$ we conclude that $\Gamma_S = \emptyset$ for every $S \in \mathcal{S}_2$ and this implies $\mathcal{S}_2 \subset \mathcal{S}_{\text{reg}}$. Hence, by Sard's theorem, $\mathcal{S}_{\text{reg}}$ is dense in $\mathcal{S}_1$ and hence in $\mathcal{S}$. □

7 Transversality for simple curves

In this section we examine the solutions of the equation (6) in the case where both the almost complex structure J and the symplectic vector field X are independent of t and, moreover, the symplectomorphism ϕ is the identity. Hence consider the partial differential equation

$$\partial_s u + J(u)(\partial_t u - X(u)) = 0 \tag{21}$$

with boundary condition $u(s, t + 1) = u(s, t)$ and limit condition

$$\lim_{s \to \pm\infty} u(s, t) = x^{\pm} \tag{22}$$

where $x^{\pm}$ are zeros of the vector field X. We shall assume that both zeros are nondegenerate as 1-periodic solutions of $\dot{x} = X(x)$. This means that the eigenvalues of $dX(x^{\pm})$ are not integer multiples of $2\pi i$. If X is sufficiently small in the C^1-topology then this simply means that $dX(x^{\pm})$ is nonsingular.

Since X is a symplectic vector field (in the sense that $\iota(X)\omega$ is closed) it follows that JX is locally the gradient of a smooth function and our assumptions imply that $x^{\pm}$ are nondegenerate critical points of this function and are therefore hyperbolic zeros of the (gradient) vector field $Y = JX : M \to TM$. If X is a Hamiltonian vector field then

$$\iota(X)\omega = dH$$

for some function $H : M \to \mathbb{R}$ and the vector field $JX = \nabla H$ is the gradient field of H. In this case it is interesting from the point of view of Morse theory to study the space of gradient flow lines $\gamma : \mathbb{R} \to M$ which run from x^- to x^+:

$$\dot{\gamma} = J(\gamma)X(\gamma), \qquad \lim_{s \to \pm\infty} \gamma(s) = x^{\pm}. \tag{23}$$

These gradient lines form special solutions of (21), namely those which are independent of t. Of course, this remains valid if X is only a symplectic vector field. In this case the Hamiltonian function H is to be replaced by the closed 1-form

$$\alpha = \iota(X)\omega$$

and JX can be interpreted as the *gradient vector field* of α. So in this case the solutions of (23) could be thought of as the gradient flow lines of the closed 1-form α. Now these special flow lines give rise to a chain complex which in the Hamiltonian case generates the homology of M (cf. [19], [21], [25]) and in the general case generates the Novikov homology of the 1-form α (cf. [11], [14], [16])

The important question is now if the solutions of (21) are all independent of t and thus degenerate to gradient flow lines of the form (23). An elementary example on $M = S^2$ shows that this can in general not be expected (cf. [10]). However, in this example the solutions which depend on t in a nontrivial way all have Maslov class $\mu(u) \geq 2$. In Theorem 8.1 we shall prove that this holds in general. In order to formulate this more precisely we first recall that under our assumptions the Maslov class of a smooth map $u : \mathbb{R} \times S^1 \to M$ which satisfies (21) and (22) is given by

$$\mu(u) = \mathrm{ind}_\alpha(x^+) - \mathrm{ind}_\alpha(x^-) + 2\int u^* c_1 \tag{24}$$

provided that X is sufficiently small in the C^1-topology. Here $\mathrm{ind}_\alpha(x)$ denotes denotes the Morse index, i.e. the dimension of the negative part of the Hessian of the closed 1-form α at the critical point x.

A function $u : \mathbb{R}^2 \to M$ which satisfies $u(s, t+1) = u(s,t)$ is called **simple** if for every integer $m > 1$ there exists a point $(s,t) \in \mathbb{R}^2$ such that $u(s, t + 1/m) \neq u(s,t)$. For any two zeros $x^\pm$ of X we denote by

$$\mathcal{M}^*(x^-, x^+, X, J)$$

the space of simple solutions of (21) and (22). Our goal is to prove that for a generic choice of H and J this space is a finite dimensional manifold of dimension $\mu(u)$ near u. Then it follows that the space $\mathcal{M}^*(x^-, x^+, X, J)$ cannot contain any t-dependent solutions unless $\mu(u) \geq 2$ because these always come in (at least) 2-dimensional families. To state the result more precisely we make the following definition. We denote by

$$\psi_X : M \to M$$

the time-1-map of the symplectic differential equation $\dot{x} = X(x)$.

Definition 7.1 *A symplectic vector field $X \in \mathcal{X}(M, \omega)$ is called* **admissable** *if the following holds.*

(i) *Every zero p of X is a nondegenerate fixed point of ψ_X, i.e.* $\det(\mathbb{1} - d\psi_X(p)) \neq 0$. *Equivalently, the spectrum of the linear transformation $dX(p) : T_pM \to T_pM$ does not intersect the set $2\pi i\mathbb{Z}$.*

(ii) *There exists an almost complex structure $J \in \mathcal{J}(M, \omega)$ such that for each zero p_j of X and each unitary frame $\Phi_j : \mathbb{R}^{2n} \to T_{p_j}M$ (i.e. $\Phi_j J_0 = J(p_j)\Phi_j$ and $\Phi_j^*\omega = \omega_0$) we have*

$$S_j = J_0 \Phi_j^{-1} dX(p_j) \Phi_j \in \mathcal{S}_{\mathrm{reg}}.$$

Denote by

$$\mathcal{X}_{\text{ad}}(M,\omega)$$

the set of admissable symplectic vector fields. Given a cohomology class $a \in H^1(M,\mathbb{R})$ we denote by $\mathcal{X}(M,\omega,a)$ the set of vector fields $X \in \mathcal{X}(M,\omega)$ such that the 1-form $\iota(X)\omega$ represents the class a and

$$\mathcal{X}_{\text{ad}}(M,\omega,a) = \mathcal{X}_{\text{ad}}(M,\omega) \cap \mathcal{X}(M,\omega,a).$$

Given a vector field $X \in \mathcal{X}_{\text{ad}}(M,\omega)$ we denote by

$$\mathcal{J}_{\text{ad}}(M,\omega)$$

the set of all almost complex structures $J \in \mathcal{J}(M,\omega)$ which satisfy (ii) above.

Lemma 7.2 *Assume* $\dim M = 2n \geq 4$.

(i) *For every $a \in H^1(M,\mathbb{R})$ the set $\mathcal{X}_{\text{ad}}(M,\omega,a)$ is dense in $\mathcal{X}(M,\omega,a)$ with respect to the C^∞-topology.*

(ii) *The set $\mathcal{X}_{\text{ad}}(M,\omega)$ is open in $\mathcal{X}(M,\omega)$ with respect to the C^1-topology.*

(iii) *For every $X \in \mathcal{X}(M,\omega)$ the set $\mathcal{J}_{\text{ad}}(M,\omega,X)$ is open in $\mathcal{J}(M,\omega)$ with respect to the C^0-topology.*

(iv) *If the pair (X,J) satisfies the conditions of Definition 7.1 then so does the pair $(\tau\psi^*X, \psi^*J)$ for any sufficiently small real number $\tau \neq 0$ and any symplectomorphism ψ.*

Proof: Recall from Theorem 6.1 that in the case $n \geq 2$ the set $\mathcal{S}_{\text{reg}} \subset \mathcal{S}$ is an open and dense set of symmetric matrices, characterized by the fact that the equations (19) and (20) have no solution (a,b,α,β,ζ) with $\zeta \neq 0$.

Now every symplectic vector field is locally Hamiltonian and so can be written in the form $X_H = -J_0\nabla H$ in local coordinates near a critical point. In this terminology condition (ii) in Definition 7.1 asserts that the Hessian of H at p_j can be represented by a regular symmetric matrix $S_j \in \mathcal{S}_{\text{reg}}$ in some (and hence every) unitary frame. This can be achieved by an arbitrarily small perturbation of the local Hamiltonian function H and hence of the symplectic vector field X. This proves (i). (ii) and (iii) follow from the fact that $\mathcal{S}_{\text{reg}}$ is open in $\mathcal{S}$. (iv) follows from the last statement in Theorem 6.1 and the fact that the eigenvalues of $\tau dX(p)$ for sufficiently small τ have modulus less that 2π. □

Remark 7.3 **(i)** The proof of the lemma shows in fact that admissability of X *with a given almost complex structure* J can be achieved by an arbitrarily small Hamiltonian perturbation of X. Thus the set of all pairs (X,J) which satisfy the conditions of Definition 7.1 is dense in $\mathcal{X}(M,\omega) \times \mathcal{J}(M,\omega)$.

(ii) The results of the previous section do not show whether admissability can be achieved by only perturbing J where we have to assume, of course, that the symplectic vector field X satisfies condition (i) of Definition 7.1. This question seems to be slightly more difficult than the one addressed in the previous section and be related to the linear Birkhoff normal form. Thus we do not know whether the set $\mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ is dense in $\mathcal{J}(M,\omega)$ for $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$.

Given an admissable symplectic vector field $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$ we denote by

$$\mathcal{J}_{\mathrm{reg}} = \mathcal{J}_{\mathrm{reg}}(M,\omega,X)$$

the set of all admissable almost complex structures $J \in \mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ such that the operator D_u is onto for all simple solutions $u \in \mathcal{M}^*(x^-,x^+,X,J)$ and all $x^\pm \in \mathrm{Fix}(\psi_X)$. These almost complex structures are called **regular** for X. Our goal in this section is to prove the following theorem.

Theorem 7.4 *Let (M,ω) be a compact symplectic manifold of dimension $2n \geq 4$ and let $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$. Then the set $\mathcal{J}_{\mathrm{reg}}(M,\omega,X)$ is of the second category in $\mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ with respect to the C^∞-topology (it contains a countable intersection of open and dense sets).*

The proof of this theorem relies on the following four lemmata. The first is concerned with a nonlinear version of the equations (19) and (20) in the previous section. Given $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$ and a sufficiently small neighbourhood V of the zero set of X there exists a unique Hamiltonian function $H : V \to \mathbb{R}$ such that $\iota(X)\omega = dH$ in V and $H(p_j) = 0$ for every zero p_j of X. (The condition $H(p_j) = 0$ is only used to simplify the notation in the proof.)

Lemma 7.5 *Let X, V, and H be as above and $J \in \mathcal{J}_{\mathrm{ad}}(M,\omega,X)$. Then there exists a neighbourhood $U \subset V$ of the zero set of X such that for any four real numbers α, β, $\hat{\alpha}$, $\hat{\beta}$ the equations*

$$[\nabla H, X] = \alpha \nabla H - \beta X \tag{25}$$

and

$$\nabla_{\nabla H}[\nabla H, X] = \alpha \nabla_{\nabla H}\nabla H + \beta \nabla_{\nabla H} X + \hat{\alpha}\nabla H + \hat{\beta} X \tag{26}$$

have no solution in U other than the zeros of X.

Proof: Choose local Darboux coordinates such that $z = 0$ is the critical point of H and such that the almost complex structure $J(z) \in \mathbb{R}^{2n\times 2n}$ satisfies $J(0) = J_0$. Then $J_0^T J(z)$ is a positive definite matrix for every z and

$$X(z) = -J_0 \partial H(z), \qquad \nabla H(z) = -J(z) J_0 \partial H(z)$$

where $\partial H(z)$ denotes the ordinary gradient of H and $\nabla H(z)$ denotes the gradient induced by the J-metric (11). Now suppose, by contradiction, that there exists a sequence $z_\nu \to 0$ and sequences α_ν, β_ν, $\hat{\alpha}_\nu$, $\hat{\beta}_\nu$ of real numbers which satisfy (25) and (26). Then there exists a constant $c > 0$ such that

$$\left(|\alpha_\nu| + |\beta_\nu| + |\hat{\alpha}_\nu| + \left|\hat{\beta}_\nu\right|\right)|z_\nu| \leq c.$$

Now define $\varepsilon_\nu = |z_\nu|$ and

$$H_\nu(\zeta) = \varepsilon_\nu^{-2} H(\varepsilon_\nu \zeta), \qquad X_\nu(\zeta) = \varepsilon_\nu^{-1} X(\varepsilon_\nu \zeta), \qquad J_\nu(\zeta) = J(\varepsilon_\nu \zeta).$$

Then the equations (25) and (26) are satisfied at the point $\zeta_\nu = \varepsilon_\nu^{-1} z_\nu$ with $\alpha = \varepsilon_\nu \alpha_\nu$, $\beta = \varepsilon_\nu \beta_\nu$ $\hat{\alpha} = \varepsilon_\nu \hat{\alpha}_\nu$ and $\hat{\beta} = \varepsilon_\nu \hat{\beta}_\nu$. Now take th limit $\nu \to \infty$. Then, since H_ν converges to the quadratic part of H and J_ν converges to J_0, we get a nontrivial solution of the equations (19) and (20) where $S = \partial^2 H(0) \in \mathbb{R}^{2n\times 2n}$ is the Hessian of H at 0. But by assumption this Hessian is in $\mathcal{S}_{\mathrm{reg}}$ and so no such solution exists. This contradiction proves the lemma. □

The next lemma asserts that every simple solution u of (21) is almost everywhere immersed.

Lemma 7.6 *Let $X \in \mathcal{X}(M,\omega)$ be a symplectic vector field with only non-degenerate critical points and $J \in \mathcal{J}(M,\omega)$ be an ω-compatible almost complex structure. If $u : \mathbb{R}^2 \to M$ is a simple solution of (21) and (22) with $u(s,t+1) = u(s,t)$ then the set of all points $(s,t) \in \mathbb{R}^2$ at which $\partial_s u$ and $\partial_t u$ are linearly independent is open and dense in $\mathbb{R}^2$.*

Proof: There are two kinds of special solutions of (21), namely those which are independent of t and hence satisfy $\partial_s u = J(u)X(u)$ and those which are independent of s and hence satisfy $\partial_t u = X(u)$. Our solution u cannot be of the first kind because it is simple and it cannot be of the second kind because then the limit condition (22) would imply that u is constant. Hence it follows from Corollary 2.3 that the set of all points (s,t) where either $\partial_s u = 0$ or $\partial_t u = 0$ is discrete. We must prove that $\partial_s u$ and $\partial_t u$ are linearly independent on a dense set. Suppose otherwise that there exists an open set $\Omega \subset \mathbb{R}^2$ on which $\partial_s u$ and $\partial_t u$ are linearly dependent. We assume without loss of generality that Ω is a neighbourhood of 0 and $\partial_s u$ and $\partial_t u$ do not vanish on Ω. Hence there exists a nonzero smooth function $\lambda : \Omega \to \mathbb{R}$ such that

$$\partial_t u(s,t) = \lambda(s,t)\partial_s u(s,t),$$

for $(s,t) \in \Omega$. We shall prove that λ is constant in Ω. If we choose Ω to be sufficiently small then the restriction of the symplectic vector field X to a neighbourhood U of $u(\Omega)$ is Hamiltonian and hence there exists a smooth function $H : U \to \mathbb{R}$ such that

$$\iota(X)\omega\Big|_U = dH$$

Now (21) implies that $\partial_s u + J(u)\partial_t u = \nabla H(u)$ and hence

$$\partial_s(H\circ u) = \langle \nabla H(u), \partial_s u\rangle = |\partial_s u|^2.$$

Similarly,

$$\partial_t(H\circ u) = \langle \nabla H(u), \partial_t u\rangle = \langle \partial_s u, \partial_t u\rangle = \lambda^{-1}|\partial_t u|^2.$$

Differentiate the last equation with respect to s and the first with respect to t to obtain

$$\begin{aligned}
2\langle \nabla_t \partial_s u, \partial_s u\rangle &= \partial_t\partial_s(H\circ u) \\
&= \partial_s\partial_t(H\circ u) \\
&= 2\lambda^{-1}\langle \nabla_s \partial_t u, \partial_t u\rangle + (\partial_s \lambda^{-1})|\partial_t u|^2 \\
&= 2\langle \nabla_t \partial_s u, \partial_s u\rangle - |\partial_s u|^2 \partial_s \lambda.
\end{aligned}$$

This shows that $\lambda(s,t) = \lambda(t)$ is independent of s. Now the identity $\partial_t u = \lambda \partial_s u$ shows that $u(s,t)$ is constant along the characteristic curves $t \mapsto (s(t), t)$ with $ds/dt = -\lambda(t)$ and hence

$$u(s,t) = \gamma\left(s + \int_0^t \lambda(\tau)\, d\tau\right)$$

for small s and t where, obviously, $\gamma(s) = u(s,0)$. Now use the formula $\partial_s u + J(u)\partial_t u = \nabla H(u)$ at the point $s = -\int_0^t \lambda(\tau)\, d\tau$ to obtain

$$\dot\gamma(0) + \lambda(t) J(\gamma(0))\dot\gamma(0) = \nabla H(\gamma(0)).$$

Since $\dot\gamma(0) \neq 0$ this implies that $\lambda(t) \equiv \lambda$ must be independent of t. The above formula now becomes $u(s,t) = \gamma(s + \lambda t)$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ is a solution of the ordinary differential equation

$$\dot\gamma(s) = (\mathbb{1} - \lambda J(\gamma))^{-1} X(\gamma(s))$$

for small s. Extend this solution to all of $\mathbb{R}$ and define $v(s,t) = \gamma(s + \lambda t)$. This function agrees with u in a neighbourhood of 0 and hence everywhere. Thus we have proved

$$u(s,t) = \gamma(s + \lambda t)$$

for all $t \in \mathbb{R}$. We have already ruled out the case $\lambda = 0$ because otherwise u would not be simple. But in the case $\lambda \neq 0$ we obtain $u(\lambda k, 0) = \gamma(\lambda k) = u(0,k) = u(0,0) = \gamma(0)$ for every integer k and in the limit as $k \to \infty$ we obtain $\gamma(0) = x^+$. Since $X(x^+) = 0$ this implies $\gamma(s) = x^+$ for all s. Hence u is constant and this again contradicts our assumption that u be simple. This proves the lemma. □

A point $(s,t) \in \mathbb{R}^2$ is called a **regular point** for u if the four vectors

$$\partial_s u,\ \partial_t u,\ X(u),\ \nabla H(u)$$

are linearly independent at (s,t). We denote by $R(u) \subset \mathbb{R}^2$ the set of regular points.

Lemma 7.7 *Assume $n \geq 2$ and let $(X, J) \subset \mathcal{X}(M,\omega) \times \mathcal{J}(M,\omega)$ be an admissable pair in the sense of Definition 7.1. Let U be the neighbourhood of the zero set of X in Lemma 7.5. Then for every simple solution $u : \mathbb{R}^2 \to M$ of (21) and (22) with $u(s,t+1) = u(s,t)$ the set*

$$\{(s,t) \in R(u) \,|\, u(s,t) \in U\}$$

is open and dense in $u^{-1}(U)$.

Proof: The set $R(u) \cap u^{-1}(U)$ is obviously open and we must prove that it is dense in $u^{-1}(U)$. Suppose otherwise that there exists an open set $\Omega \subset \mathbb{R}^2$ such that

$$\Omega \cap R(u) = \emptyset, \qquad u(\Omega) \subset U.$$

By Lemma 7.6 we may assume that $\partial_s u$ and $\partial_t u$ are linearly independent in Ω. We may also assume that $\nabla H(u) \neq 0$ in Ω. Since $\Omega \cap R(u) = \emptyset$ the vectors $\partial_s u$, $\partial_t u$, $\nabla H(u)$, and $X(u)$ are linearly dependent in Ω. This implies that there exist smooth functions $a, b : \Omega \to \mathbb{R}$ such that

$$\partial_s u = a\nabla H(u) + bX(u)$$

in Ω. Otherwise, by a general fact in complex linear algebra, the four vectors $\partial_s u$, $J(u)\partial_s u$, $X(u)$, and $\nabla H(u) = J(u)X(u)$ would be linearly independent and since $\partial_t u = J(u)\partial_s u + X(u)$ this would contradict our assumption. Now multiply the above formula by $J(u)$ to obtain $J(u)\partial_s u = b\nabla H(u) - aX(u)$. Since $\partial_t u = J(u)\partial_s u + X(u)$ this implies

$$\partial_t u = b\nabla H(u) + (1-a)X(u).$$

Denote by ∇ the Levi-Civita connection of the metric (11) (Note, that this time we have no t-dependence of the metric). Then $\nabla_s \partial_t u = \nabla_t \partial_s u$ and inserting the above two expression we obtain by a simple calculation that

$$[\nabla H, X] = \alpha \nabla H + \beta X$$

on $u(\Omega)$ where

$$\alpha = \frac{\partial_t a - \partial_s b}{a^2 + b^2 - a}, \qquad \beta = \frac{\partial_s a + \partial_t b}{a^2 + b^2 - a}.$$

Here we have $a^2 + b^2 - a \neq 0$ since $\partial_s u$ and $\partial_t u$ are linearly independent in Ω. Now differentiate the previous identity (covariantly) in the direction $\nabla H = \lambda \partial_s u + \mu \partial_t u$ with $\lambda = (a^2+b^2-a)^{-1}(a-1)$ and $\mu = (a^2+b^2-a)^{-1}b$ to obtain

$$\nabla_{\nabla H}[\nabla H, X] = \alpha \nabla_{\nabla H}\nabla H + \beta \nabla_{\nabla H} X + \hat{\alpha}\nabla H + \hat{\beta} X$$

where $\hat{\alpha} = \lambda \partial_s \alpha + \mu \partial_t \alpha$ and $\hat{\beta} = \lambda \partial_s \beta + mu \partial_t \beta$. By Lemma 7.5 there are no such numbers α, β, $\hat{\alpha}$, $\hat{\beta}$ which satisfy the last two equations as long as $u(s,t) \in U$. This contradiction shows that our assumption that $\partial_s u$, $\partial_t u$, $\nabla H(u)$, $X(u)$ were linearly independent on some open set in $u^{-1}(U)$ must have been wrong. □

Lemma 7.8 *Assume $n \geq 2$ and let $(X, J) \subset \mathcal{X}(M,\omega) \times \mathcal{J}(M,\omega)$ be an admissable pair in the sense of Definition 7.1. Let U be the neighbourhood of of the zero set of X in Lemma 7.5. Then for every simple solution $u : \mathbb{R}^2 \to M$ of (21) and (22) with $u(s, t+1) = u(s,t)$ the set of all points $(s_0, t_0) \in R(u) \cap u^{-1}(U)$ which satisfy*

$$u(s,t) = u(s_0, t_0) \qquad \Longrightarrow \qquad s = s_0,\ t - t_0 \in \mathbb{Z}$$

is open and dense in $u^{-1}(U)$.

Proof: Openness follows from a simple compactness argument as in the proof of Theorem 4.3. Suppose that the set in question was not dense. Then there would exist an open set

$$\Omega \subset R(u) \cap u^{-1}(U)$$

such that for every $(s,t) \in \Omega$ there exists a point $(s',t') \notin \Omega$ with $u(s',t') = u(s,t)$. Choose Ω so small that the restriction of u to Ω is an embedding. Denote

$$\Omega' = \left\{(s',t') \in \mathbb{R}^2 - \Omega \,|\, u(s',t') \in u(\Omega)\right\}.$$

We shall first use Sard's theorem and Baire's category theorem to conclude that Ω' must contain an open set. To see this denote

$$v = u\big|_{\Omega}, \qquad v' = u\big|_{\Omega'}.$$

Then $v'(\Omega') = v(\Omega)$ and the composite

$$\phi = v^{-1} \circ v' : \Omega' \to \Omega$$

extends to a smooth map on some neighbourhood of Ω'. (Just project $u(z)$ onto the submanifold $u(\Omega)$ for z near Ω' and then apply v^{-1}.) If $z = (s,t) \in \Omega$ is a regular value of ϕ then $\partial_s u(z')$ and $\partial_t u(z')$ are linearly independent for every $z' = (s',t') \in \Omega'$ with $u(z') = u(z)$. Hence for any such regular value the set of points $z' \in \Omega'$ with $u(z') = u(z)$ consists of isolated points and is therefore finite. By Sard's theorem, fix $z = z_0 \in \Omega$ to be such a regular value and let $z_1, \ldots, z_N \in \Omega'$ be the corresponding points with $u(z_j) = u(z_0)$. Now proceed as in the proof of Theorem 4.3 to conclude that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$u(B_\delta(z_0)) \subset \bigcup_{j=1}^{N} u(B_\varepsilon(z_\nu)).$$

Again, as in the proof of Theorem 4.3, the set $B_\delta(z_0)$ is covered by the finitely many sets

$$\Sigma_j = \left\{z' \in B_\varepsilon(z_\nu) \,|\, u(z') \in u(B_\delta(z_0))\right\} \subset \Omega'$$

and so, by Baire's category theorem, one of these sets must have a non-empty interior.

Having proved that Ω' contains an open set we consider again the map $\phi = v^{-1} \circ v' : \Omega' \to \Omega$. Write $\phi(s,t) = (\sigma(s,t), \tau(s,t))$ and use the fact that both u and $u \circ \phi$ satisfy (21) to obtain

$$\begin{aligned}
0 &= \partial_s(u \circ \phi) + J(u \circ \phi)\partial_t(u \circ \phi) - J(u \circ \phi)X(u \circ \phi) \\
&= \partial_s u(\phi)\partial_s\sigma + \partial_t u(\phi)\partial_s\tau + J(u(\phi))\partial_s u(\phi)\partial_t\sigma \\
&\quad + J(u(\phi))\partial_t u(\phi)\partial_t\tau - J(u(\phi))X(u(\phi)) \\
&= (\partial_s\sigma - \partial_t\tau)\partial_s u(\phi) + (\partial_s\tau + \partial_t\sigma)\partial_t u(\phi) \\
&\quad - X(u(\phi))\partial_s\tau - J(u(\phi))X(u(\phi))(1 - \partial_t\tau).
\end{aligned}$$

The last equation follows from $J(u)\partial_s u = \partial_t u - X(u)$ and $J(u)\partial_t u = J(u)X(u) + \partial_s u$. Since the image of ϕ is contained in $R(u)$ it follows that the vectors $\partial_s u(\phi)$, $\partial_t u(\phi)$, $X(u(\phi))$ and $J(u(\phi))X(u(\phi))$ are linearly independent and hence

$$\partial_s \sigma = \partial_t \tau = 1, \qquad \partial_s \tau = \partial_t \sigma = 0.$$

Hence ϕ is a translation. Since the domain and range of ϕ are disjoint in $\mathbb{R} \times S^1$ it follows that there exists a point $(s_0, t_0) \in \mathbb{R}^2 - \{0\} \times \mathbb{Z}$ such that

$$u(s,t) = u(s + s_0, t + t_0).$$

This holds on some open set and, by unique continuation, on all of $\mathbb{R}^2$. But this is impossible: if $s_0 \neq 0$ then $u(s,t) = u(s + ks_0, t + kt_0) \to x^+$ as $k \to +\infty$ and so u is constant, in contradiction to our assumption that u be simple. Hence $s = s_0$ and so $u(s, s + t_0) = u(s,t)$ where $t_0 \notin \mathbb{Z}$. But if t_0 is irrational then this condition together with $u(s, t+1) = u(s,t)$ implies that u must be independent of t, again contradicting simplicity. Finally, if t_0 is rational then we have $u(s, t + 1/m) = u(s,t)$ for some integer m and this contradicts again the definition of *simple*. This proves the lemma. □

We point out that all three lemmata remain valid for almost complex structures and symplectic vector fields of class C^ℓ. In this case all other functions, in particular J-holomorphic curves, will in general also be only of class C^ℓ.

Proof of Theorem 7.4: The basic strategy of the proof is the same as in Theorem 5.1. We fix a symplectic vector field $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$ denote by $\mathcal{B}_s \subset \mathcal{B} = \mathcal{B}^{1,p}(x^-, x^+, \mathrm{id})$ the open subset of all those maps $u \in \mathcal{B}$ for which there exists a point $(s,t) \in R(u)$ such that

$$u(s', t') = u(s,t) \qquad \Longrightarrow \qquad s' = s,\ t' - t \in \mathbb{Z}.$$

We shall call such a point a **regular injective point** for u. Note in particular that at any such point $X(u(s,t)) \neq 0$. It is a simple matter to prove that the set $\mathcal{B}_s$ is open in $\mathcal{B}$. Moreover, by Lemma 7.8, every simple solution of (21) and (22) with any almost complex structure $J \in \mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ admits a regular injective point and is therefore contained in $\mathcal{B}_s$.

Now denote by $\mathcal{J}_{\mathrm{ad}} = \mathcal{J}^\ell_{\mathrm{ad}}(M,\omega,X)$ the space of all almost complex structures $J \in \mathcal{J}^\ell(M,\omega)$ such that the pair (X,J) is admissable. This is an open set in the space $\mathcal{J}^\ell(M,\omega)$ (of all almost complex structures of class C^ℓ which are compatible with ω) and is therefore a Banach manifold. We consider the Banach vector bundle

$$\mathcal{E} \to \mathcal{B}_s \times \mathcal{J}_{\mathrm{ad}}$$

with fibers $\mathcal{E}_u = L^p(u^*TM)$. We shall prove that the section

$$\mathcal{F} : \mathcal{B}_s \times \mathcal{J}_{\mathrm{ad}} \to \mathcal{E}$$

defined by

$$\mathcal{F}(u,J) = \overline{\partial}_{X,J}(u) = \partial_s u + J(u)(\partial_t u - X(u))$$

is transverse to the zero section or, equivalently, the differential

$$D\mathcal{F}(u,J) : T_u\mathcal{B}_s + T_J\mathcal{J}_{\rm ad} \to \mathcal{E}_u,$$

given by

$$D\mathcal{F}(u,J)(\xi,Y) = D_u\xi + Y(u)(\partial_t u - X(u))$$

for $\xi \in W^{1,p}(u^*TM)$ and $Y \in T_J\mathcal{J}_{\rm ad}$, is surjective whenever $\overline{\partial}_{X,J}(u) = 0$. Since D_u is a Fredholm operator, it suffices to prove that $D\mathcal{F}(u,J)$ has a dense range. Now if $\eta \in L^q(u^*TM)$ with $1/p + 1/q = 1$ annihilates the range of $D\mathcal{F}(u,J)$ then η is of class C^ℓ with $D_u^*\eta = 0$ and, moreover,

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta, Y(u)J(u)\partial_s u\rangle \, dtds = 0.$$

for all $Y \in T_J\mathcal{J}_{\rm ad}$. But this last equation implies, by the same argument as in the proof of Theorem 5.1, that η must vanish at every regular injective point of u. Since the set of such points is open and nonempty it follows that η vanishes on some open set and, by unique continuation, $\eta = 0$. This proves that $\mathcal{F}$ is transverse to the zero section and so the universal moduli space of all pairs $(u,J) \in \mathcal{B}_s \times \mathcal{J}_{\rm ad}$ with $\overline{\partial}_{X,J}(u) = 0$ is a smooth Banach manifold. Now the regular values of the projection $(u,J) \mapsto J$, defined on this universal moduli space, are now the required regular almost complex structures. This proves the theorem in the C^ℓ-case. The details of this argument as well as the extension to the C^∞-case are precisely the same as in the proof of Theorem 5.1 and are left to the reader. □

8 Equivariant action functional

Denote by $\mathcal{L}$ the space of contractible loops on M and think of these loops as smooth maps $x : \mathbb{R} \to M$ which satisfy $x(t+1) = x(t)$. Given a symplectic vector field $X \in \mathcal{X}(M,\omega)$ there is a natural closed 1-form Ψ_X on $\mathcal{L}$ defined by

$$\Psi_X(x;\xi) = \int_0^1 \omega(\dot{x}(t) - X(x(t)), \xi(t))\, dt.$$

for $\xi \in T_x\mathcal{L} = C^\infty(x^*TM)$. The (negative) gradient flow lines of this 1-form with respect to the metric induced by an almost complex structure $J \in \mathcal{J}(M,\omega)$ are precisely the solutions of

$$\partial_s u + J(u)(\partial_t u - X(u)) = 0 \tag{27}$$

with boundary condition $u(s,t+1) = u(s,t)$. Connecting orbits also satisfy the limit condition

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm \tag{28}$$

where $x^\pm$ are zeros of the vector field X. These are the equations (21) and (22) studied in Section 7.

From a more abstract point of view the infinite dimensional manifold $\mathcal{L}$ carries a natural symplectic structure and the closed 1-form Ψ_X therefore generates a symplectic vector field $x \mapsto \dot{x} - X(x)$ on $\mathcal{L}$. The 1-form Ψ_X and the corresponding vector field on $\mathcal{L}$ are invariant under the natural S^1-action. In the case $X = 0$ this vector field in fact generates the S^1-action on $\mathcal{L}$ and the X-term can be considered as an equivariant perturbation. From this point of view the simple solutions of (27) and (28) are precisely those gradient trajectories u of the closed 1-form Ψ_X such that

(a) the limit points $x^\pm = \lim_{s\to\pm\infty} u(s,t)$ are fixed points of the S^1-action,

(b) S^1 acts freely on u.

Theorem 7.4 can be viewed as an equivariant transversality result for such gradient trajectories. We shall now use this result to prove that if the relative Morse index is less than or equal to 1 then all the connecting orbits between zeros of X are independent of the t-variable. Equivalently, if the limit points belong to the fixed point set of the S^1-action and have relative Morse index at most 1 then the connecting orbits also belong to the fixed point set of the S^1-action.

To make this precise we fix any symplectic vector field $X \in \mathcal{X}(M,\omega)$. By Lemma 7.2 (i) there exists an arbitrarily small Hamiltonian function $H \in C^\infty(M)$ such that the $X + X_H \in \mathcal{X}_{\mathrm{ad}}(M,\omega)$. It follows from Theorem 7.4 that the set $\mathcal{J}_{\mathrm{reg}}(M,\omega,X+X_H)$ is of the second category in the set $\mathcal{J}_{\mathrm{ad}}(M,\omega,X+X_H)$. Thus we have proved that for any cohomology class $a \in H^1(M)$ there exists a symplectic vector field $X \in \mathcal{X}(M,\omega,a)$, an open set $\mathcal{J}_{\mathrm{ad}}(M,\omega,X) \subset \mathcal{J}(M,\omega)$ and a generic set $\mathcal{J}_0(M,\omega,X) \subset \mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ (i.e. a set containing a countable intersection of open and dense sets in $\mathcal{J}_{\mathrm{ad}}(M,\omega,X)$) such that the following holds.

(1) The zeros of X are all nondegenerate.

(2) There exists a number $m_0 > 0$ such that for every integer $m \geq m_0$ the moduli spaces $\mathcal{M}^*(x^-,x^+,X/m,J)$ are finite dimensional manifolds of local dimension

$$\begin{aligned}\dim_u \mathcal{M}^*(x^-,x^+,X/m,J) &= \mu(u) \\ &= \mathrm{ind}_X(x^+) - \mathrm{ind}_X(x^-) + 2\int u^*c_1\end{aligned}$$

near u for any two zeros $x^\pm$ of X.

These assertions hold in fact for an open and dense set of symplectic vector fields in $\mathcal{X}(M,\omega,a)$. Note that (1) is slightly weaker than condition (i) in Definition 7.1. For the proof of (2) we note that if $X \in \mathcal{X}_{\mathrm{ad}}(M,\omega,a)$ then $X/m \in \mathcal{X}_{\mathrm{ad}}(M,\omega,a/m)$ for every sufficiently large integer $m \geq m_0$ and $\mathcal{J}_{\mathrm{ad}}(M,\omega,X/m) = \mathcal{J}_{\mathrm{ad}}(M,\omega,X)$. Thus $\mathcal{J}_0(M,\omega,X)$ can be defined as the intersection of the sets $\mathcal{J}_{\mathrm{reg}}(M,\omega,X/m)$ over all integers $m \geq m_0$.

These sets are all of the second category in $\mathcal{J}_{\mathrm{ad}}(M, \omega, X)$ in the sense of Baire and so is their intersection. The condition $m \geq m_0$ in (2) is required in order the Maslov index of $x^{\pm}$ to agree with the Morse index $\mathrm{ind}_X(x^{\pm})$. This number refers to the Morse index of $x^{\pm}$ as a critical point of H where H is a local Morse function near $x^{\pm}$ such that $\iota(X)\omega = dH$.

The following result was proved in [10] in the case where X is a Hamiltonian vector field.

Theorem 8.1 *Let (M, ω) be a compact symplectic manifold of dimension $2n \geq 4$. Assume either that M is monotone or $c_1(\pi_2(M)) = 0$ or the minimal Chern number is $N \geq n$. Assume also that $X \in \mathcal{X}(M, \omega)$ and $J \in \mathcal{J}(M, \omega)$ satisfy the conditions (1) and (2) above. Then there exists a constant $m_0 = m_0(X, J)$ such that every solution u of (27) and (28), with $\mu(u) \leq 1$ and X replaced by $(1/m)X$ with $m \geq m_0$, is independent of t.*

Proof: To prove this, one first uses a compactness argument to show that every solution with nonpositive area $\int u^*\omega \leq 0$ must be independent of t provided that m is sufficiently large, say $m \geq m_0$ (see [10], Lemma 7.1). Now let $u(s,t) = u(s, t+1)$ be a solution of (27) and (28) with X replaced by $(1/m)X$ where $m \geq m_0$ and hence

$$\int u^*\omega > 0.$$

Assume $\mu(u) \leq 1$ and, by contradiction, that $u(s,t)$ is not independent of t. If u is simple then u must be independent of t since otherwise the functions $(s,t) \mapsto u(s_0 + s, t_0 + t)$ form a 2-dimensional family of simple solutions in contradiction with the dimension formula of statement (2) above. If u is not simple then there exists an integer $k > 1$ such that

$$u(s, t + 1/k) \equiv u(s,t).$$

Let k be the largest such integer. (If there is no largest integer with this property then $u(s,t)$ is independent of t.) Then the function

$$v(s,t) = u(s/k, t/k) = v(s, t+1)$$

is a simple solution of (27) with X replaced by $(1/mk)X$ and index

$$\mu(v) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) + 2\int v^*c_1.$$

If $\int u^*c_1 \geq 0$ then

$$\int v^*c_1 = \frac{1}{k}\int u^*c_1 \leq \int u^*c_1.$$

and hence $\mu(v) \leq 1$. By (2) this implies that v, and hence u is independent of t. If, on the other hand, $\int u^*c_1 < 0$, then M is not monotone and hence must have minimal Chern number $N \geq n$ or $N = 0$. In the former case

$$\int v^*c_1 \leq -N \leq -n$$

and hence $\mu(v) \leq 0$. In the latter case $\mu(v) = \mu(u) \leq 1$. In both cases v is a simple solution of (27) and (22) with X replaced by $(1/mk)X$ and $\mu(v) \leq 1$. Since $((1/mk)X, J) \in \mathcal{X}\mathcal{J}_{\rm reg}$ it follows again that v is independent of t. This contradiction proves the theorem. □

In [10] the previous theorem was used in the Hamiltonian case to prove that the Floer homology groups $HF_*(M, \omega, H, J)$ are isomorphic to the ordinary homology of M, tensored by the Novikov ring associated to ω. In [11] LeHong Van and Kaoru Ono used a result similar to Theorem 8.1 to prove that, if the manifold M is monotone and $X_t = X_{t+1} : M \to TM$ is a time dependent family of symplectic vector fields, then the Floer homology groups $HF^*(M, \omega, X, J)$ are naturally isomorphic to the Novikov homology of the Calabi-invariant

$$\alpha = \int_0^1 \iota(X_t)\omega \, dt.$$

Their result is based on an ingenious continuation argument which allows them to rescale the vector field X (and hence the form α) by an arbitrarily small constant without changing the Floer homology groups.

Theorems 7.4 and 8.1 will also play an important role in studying equivariant Floer homology. For this it will be important to choose the almost complex structure $J \in \mathcal{J}_0(M, \omega, X)$ such that, in addition to (1) and (2) above the following conditions are satisfied

(3) The gradient flow JX of the 1-form $\alpha = \iota(X)\omega$ with respect to the metric induced by J is of Morse-Smale type.

(4) The zeros of X and the gradient flow lines of JX with index difference 1 do not intersect the holomorphic spheres of J with Chern number less than or equal to 1.

These conditions can be achieved by a generic perturbation of the almost complex structure. For (3) this follows from Theorem 8.1 in [20]. For (4) this follows from the fact that for a generic almost complex structure J the set of points which lie on J-holomorphic spheres of Chern number less than or equal to 1 form a set of codimension 2 in M (cf. [10] and [13]). In particular, conditions (3) and (4) can be used to prove that the set of simple solutions of (27) and (28) with relative Morse index $\mu(u) = 2$ is compact and hence there are only finitely many of these (modulo the action of S^1 and $\mathbb{R}$). The number of such orbits will play an important role in equivariant Floer homology.

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