MORSE INEQUALITIES AND ZETA FUNCTIONS

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§0 INTRODUCTION

Morse inequalities for diffeomorphisms of a compact manifold were first proved by Smale [21] under the assumption that the nonwandering set is finite. We call these the integral Morse inequalities. They were generalized by Zeeman in an unpublished work cited in [25] to diffeomorphisms with a hyperbolic chain recurrent set that is axiom-A-diffeomorphisms which satisfy the no cycle condition[1]. For the same class of diffeomorphisms Franks derived in [7] and [8] the polynomial Morse inequalities. These contain the integral Morse inequalities as a special case. In this paper we generalize the polynomial Morse inequalities to Morse decompositions of isolated invariant sets.

To explain the relationship between the integral and the polynomial Morse inequalities we consider a diffeomorphism $f : M \to M$ of a compact manifold M with a hyperbolic chain recurrent set R_f. By the shadowing lemma R_f decomposes into finitely many chain transitive components $\{\Lambda_p\}_{p \in P}$ satisfying the no cycle condition. Under these assumptions all the periodic points of f are nondegenerate. Counting the periodic points in the basic set Λ_p algebraically gives the homology zeta function [22]

$$\zeta_j(t; \Lambda_p) = \exp\left(\sum_{k=1}^{\infty} \frac{\nu_k(\Lambda_p)t^k}{k}\right), \qquad \nu_k(\Lambda_p) = \sum_{f^k(x)=x\in\Lambda_p} \nu(x; f^k).$$

Here j is the dimension of the associated expanding bundle E^u and $\nu(x; f^k) = \pm 1$ according to whether the restriction $df^k(x)|_{E_x^u}$ is orientation preserving or orientation reversing. Thus $(-1)^j \nu(x; f^k) = \text{sign} \det(I - df^k(x))$. The homology zeta function is the inverse of a polynomial. It is convenient to define $\zeta_j(t; \Lambda_p) = 1$ for $j \neq \dim E^u$.

Following Franks [7] we define the relation $\zeta_0 \prec \zeta_1$ for rational functions ζ_0, ζ_1 if the quotient ζ_0/ζ_1 is a polynomial. This defines a partial order on the field $\mathbb{Z}(t)$

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[1]This class of diffeomorphisms is C^1-open and C^0-dense in the space of all diffeomorphisms [25].

of rational functions with integer coefficients. The polynomial Morse inequalities of [7] can be written as

$$(0.1) \qquad \prod_{i=0}^{j} \zeta_{j-i}(t; f)^{(-1)^i} \quad \prec \quad \prod_{i=0}^{j} \left(\prod_{p \in P} \zeta_{j-i}(t; \Lambda_p) \right)^{(-1)^i}$$

for $j = 0, 1, \ldots, n$ with $\zeta_j(t; f) = \det(I - tH_j(f))^{-1}$. Here H_* denotes singular homology with integer coefficients[2]. For $j = n$ one gets equality in (0.1) and this is equivalent to the Lefschetz fixed point formula for f^k.

To obtain the integral Morse inequalities consider the homomorphism

$$\deg : \mathbb{Z}(t) \to \mathbb{Z}$$

which associates with every rational function ζ the degree of the denominator minus the degree of the numerator. According to this convention the degree of a polynomial is negative. So the degree homomorphism is order preserving. Taking degrees in (0.1) we obtain Morse inequalities in the familiar form

$$(0.2) \qquad \sum_{i=0}^{j} (-1)^i \beta_{j-i} \leq \sum_{i=0}^{j} (-1)^i \sum_{p \in P} d_{j-i}(\Lambda_p)$$

where $d_j(\Lambda_p) = \deg \zeta_j(\cdot; \Lambda_p)$ and $\beta_j = \deg \zeta_j(\cdot; f) = \operatorname{rank} H_j(f)$. For diffeomorphisms with a finite nonwandering set these are Smale's Morse inequalities [21].

We generalize the polynomial Morse inequalities (0.1) (and thus the weaker integral Morse inequalities (0.2)) to the case where the Λ_p are arbitrary isolated invariant sets satisfying the no cycle condition. Also M itself will be replaced by an isolated invariant set S. The homology zeta functions $\zeta_j(t; \Lambda)$ of an isolated invariant set Λ are defined as invariants of the shape index. The shape index was introduced in [18] as a generalization of the Conley index to discrete-time dynamical systems. The ζ_j are inverse polynomials which may be nontrivial for many j. In general they cannot be defined by counting the periodic points.

Throughout the paper we fix a principal ideal domain R and denote by H_* the singular homology functor with coefficients in R. We prove the generalized polynomial Morse inequalities (0.1) for coefficients in R. For this $\zeta_j(t; \Lambda_p)$ will be defined as a rational function with coefficients in R. The reason for choosing R to be a principal ideal domain is that for any finitely generated module X over R the quotient $X/\operatorname{Tor} X$ is free. So it is meaningful to define $\operatorname{rank} X = \dim(X/\operatorname{Tor} X)$. Also for any module endomorphism $A : X \to X$ we denote by $\operatorname{trace} A$ and $\det A$ the trace or determinant of the induced endomorphism on $X/\operatorname{Tor} X$. With these definitions the familiar identities remain valid like for example $\operatorname{rank} X = \operatorname{rank} X' + \operatorname{rank}(X/X')$ for any submodule $X' \subset X$. Also $\operatorname{trace} A = 0$ whenever A is nilpotent.

If f is the time-1-map of a flow then the polynomial and the integral Morse inequalities for isolated invariant sets are equivalent and they agree with the Morse inequalities for the Conley index derived by Conley and Zehnder [4]. If in addition the flow is a gradient flow of a Morse function then these reduce to the classical Morse inequalities. In [15] Mrozek generalized the integral Morse inequalities

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[2]In [7] Franks worked with coefficients in a field and in [8] with coefficients in $\mathbb{Z}_2$.

(0.2) to isolated invariant sets for homeomorphisms, however, without proving the finitedimensionality.

In section 1 we introduce the homology zeta functions associated to an isolated invariant set. This requires the existence of a polyhedral index pair [18]. We prove that the zeta function is independent of the choice of the index pair and invariant under continuation. In section 2 we prove the Morse inequalities. This requires the existence of a polyhedral index filtration [19]. There are two ways to relate discrete-time dynamical systems (diffeomorphisms) and continuous-time dynamical systems (flows). On the one hand the time-1-map of a flow is a diffeomorphism and on the other hand the suspension [22] of a diffeomorphism is a flow. In both cases there is a one-to-one correspondence of isolated invariant sets. We will discuss this correspondence in section 3 and section 4. In particular, using the Meyer-Vietoris sequence we relate the homology zeta function of an isolated invariant set S for f to the Conley index of its suspension $\Sigma_f S$. In section 5 we discuss several examples.

§1 Zeta functions

Let $f : M \to M$ be a diffeomorphism of a boundaryless, possibly noncompact, n-dimensional manifold M. A compact invariant set $S \subset M$ for f is called **isolated** if it admits a neighborhood V with compact closure such that

$$S = \bigcap_{k \in \mathbb{Z}} f^k(cl(V));$$

any neighborhood V of S with this property is called **isolating** for S. An **index pair** for an isolated invariant set S is a pair of compact sets $L \subset N$ in M such that $N \backslash L$ is an isolating neighborhood for S and the induced map on the topological quotient

$$f_{N,L} : N/L \to N/L$$

defined by

$$f_{N,L}(x) = \begin{cases} f(x) & \text{if } x, f(x) \in N \setminus L, \\ * & \text{otherwise} \end{cases}$$

is continuous. An isolated invariant set for f is automatically an isolated invariant set for f^k where $k \neq 0$ and an index pair for f is automatically an index pair for f^k where $k > 0$.

In [18] it is shown that a polyhedral index pair always exists. In particular, this implies that the homology of the pair (N, L) is finite dimensional and agrees with the homology of the topological quotient N/L. Also $H_j(N, L) = \{0\}$ for $j > n$.

The map $f_{N,L}$ will in general not be homotopic to the identity and one can easily construct two index pairs (N_1, L_1) and (N_2, L_2) for S such that N_1/L_1 is not homotopy equivalent to N_2/L_2 (see Examples 5.1, 5.2 and 5.3). However, an algebraic invariant of the isolated invariant set S is given by the trace of the induced map on the homology of the index pair.

Proposition 1.1. *Let $S \subset M$ be an isolated invariant set for f. Then the traces*

$$\tau_j^k(S) = \text{trace}\left(H_j(f_{N,L}^k) : H_j(N, L) \to H_j(N, L)\right) \qquad \in \qquad R$$

are independent of the choice of the index pair (N, L) for S.

Proof. If (N_α, L_α) and (N_β, L_β) are two index pairs for S then the map

$$f_{\beta\alpha}^\ell : N_\alpha/L_\alpha \to N_\beta/L_\beta$$

defined by

$$f_{\beta\alpha}^\ell(x) = \begin{cases} f^\ell(x) & \text{if } f^j(x) \in N_\alpha \setminus L_\alpha \text{ for } 0 \le j \le \frac{2\ell}{3} \\ & \quad \text{and } f^j(x) \in N_\beta \setminus L_\beta \text{ for } \frac{\ell}{3} \le j \le \ell, \\ * & \text{otherwise} \end{cases}$$

is continuous for sufficiently large $\ell \in \mathbb{N}$ [18] and satisfies

(1.2) $$f_{\gamma\beta}^m \circ f_{\beta\alpha}^\ell = f_{\gamma\alpha}^{\ell+m}.$$

Note that

$$f_{\alpha\alpha} = f_{N_\alpha/L_\alpha}$$

and let $f_\alpha = f_{\alpha\alpha}$.

We claim that $A = H_j(f_\alpha)$ and $B = H_j(f_\beta)$ are shift equivalent in the sense of Williams [23], [24] (see also [16]). To see this apply the homology functor H_j to the following commutative diagram:

(1.3)
$$\begin{array}{ccccc}
N_\alpha/L_\alpha & \xrightarrow{f_{\beta\alpha}^\ell} & N_\beta/L_\beta & \xrightarrow{f_{\gamma\beta}^m} & N_\gamma/L_\gamma \\
f_\alpha \downarrow & & f_\beta \downarrow & & \downarrow f_\gamma \\
N_\alpha/L_\alpha & \xrightarrow[f_{\beta\alpha}^\ell]{} & N_\beta/L_\beta & \xrightarrow[f_{\gamma\beta}^m]{} & N_\gamma/L_\gamma
\end{array}$$

Take $\gamma = \alpha$, $S = H_j(f_{\beta\alpha}^\ell)$, $T = H_j(f_{\alpha\beta}^m)$ to obtain $SA = BS$ and $AT = TB$. Now apply the functor H_j to (1.2) to obtain $TS = A^{\ell+m}$. Reverse α and β to obtain $ST = B^{\ell+m}$.

Now shift equivalent module endomorphisms have the same trace. To see this note that $\ker S \subset X = H_j(N_\alpha, L_\alpha)$ is invariant under A and the restriction of A to $\ker S$ is nilpotent. So the induced endomorphism on $X/\ker S$ has the same trace as A. It follows that $\text{trace } A = \text{trace}(B|_{\text{range } S})$. Since $\text{range } S \supset \text{range } B^{\ell+m}$ we conclude that $\text{trace } A = \text{trace } B$. □

In [18] it is shown that the shape equivalence class of the inverse system

$$N/L \xleftarrow{f_{N,L}} N/L \xleftarrow{f_{N,L}} \cdots$$

is independent of the choice of the index pair and is moreover invariant under continuation. This follows from (1.2) and (1.3) together with the fact that an index pair (N, L) for (S, f) can be chosen such that $g_{N,L} : N/L \to N/L$ is continuous for every diffeomorphism $g : M \to M$ sufficiently (C^0) close to f. We also point out that the trace of $H_j(f_{N,L}^k)$ is a shape invariant and so is the direct limit of

(1.4) $$H_j(N, L) \xrightarrow{H_j(f_{N,L})} H_j(N, L) \xrightarrow{H_j(f_{N,L})} \cdots.$$

This direct limit will be denoted by $\mathcal{H}_j(N, L)$. It is defined as the set of equivalence classes $[a, k]$ where $a \in H_j(N, L)$ and $k \in \mathbb{N}$ under the equivalence relation

$$[a, k] = [a', k'] \qquad \Longleftrightarrow \qquad H_j(f_{N,L}^{k'})a = H_j(f_{N,L}^k)a'.$$

The induced map

$$(1.5) \qquad \mathcal{H}_j(f_{N,L}) : \mathcal{H}_j(N, L) \to \mathcal{H}_j(N, L), \qquad [a, k] \mapsto [H_j(f_{N,L})a, k],$$

is a module automorphism whose inverse is given by $[a, k] \mapsto [a, k + 1]$. So it follows from (1.2) and (1.3) that the isomorphism class of the R-module $\mathcal{H}_j(N, L)$ is independent of the choice of the index pair. For any two index pairs (N_α, L_α) and (N_β, L_β) there is a natural isomorphism

$$(1.6) \qquad \mathcal{H}_j(f_{\beta\alpha}) : \mathcal{H}_j(N_\alpha, L_\alpha) \to \mathcal{H}_j(N_\beta, L_\beta), \qquad [a, k] \mapsto [H_j(f_{\beta\alpha}^\ell)a, k + \ell],$$

whose inverse is given by $\mathcal{H}_j(f_{\alpha\beta})$. Abstractly speaking we have constructed a connected simple system $\mathcal{H}_*(S, f)$ whose objects are the shift automorphisms (1.5) associated to index pairs for (S, f) and whose morphisms are the isomorphisms (1.6) associated to ordered pairs of index pairs.

If the homomorphism $H_j(f_{N,L})$ is unimodular then $\mathcal{H}_j(N, L) = H_j(N, L)$. Also in the case of coefficients in a field $\mathcal{H}_j(N, L)$ can be identified with the generalized range[3] of $H_j(f_{N,L})$. In general the direct limit $\mathcal{H}_j(N, L)$ may not be finitely generated even if $H_j(N, L)$ is finitely generated (see Example 5.3).

Both $\tau_j^k(S)$ and $\mathcal{H}_*(S, f)$ are continuation invariants. Recall from [18] that (S_0, f_0) and (S_1, f_1) are **related by continuation** if there exists a collection of pairs

$$(S_\lambda, f_\lambda), \qquad 0 \le \lambda \le 1,$$

such that the dynamical system ϕ on $M \times [0, 1]$ defined by

$$\phi(x, \lambda) = (f_\lambda(x), \lambda)$$

is a diffeomorphism and $\Sigma = \{(x, \lambda) \,|\, x \in S_\lambda\}$ is an isolated invariant set for ϕ.

Proposition 1.7. *Let $(S_\lambda, f_\lambda), 0 \le \lambda \le 1$, be a continuation of isolated invariant sets. Then*

$$\tau_j^k(S_0) = \tau_j^k(S_1)$$

for all j and k. Also for any pair of index pairs (N_0, L_0) for (S_0, f_0) and (N_1, L_1) for (S_1, f_1) there exists a natural isomorphism $\Phi_j : \mathcal{H}_j(N_0, L_0) \to \mathcal{H}_j(N_1, L_1)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{H}_j(N_0, L_0) & \xrightarrow{\ \Phi_j\ } & \mathcal{H}_j(N_1, L_1) \\
{\scriptstyle \mathcal{H}_j(f_{0_{N_0, L_0}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{H}_j(f_{1_{N_1, L_1}})} \\
\mathcal{H}_j(N_0, L_0) & \xrightarrow[\ \Phi_j\ ]{} & \mathcal{H}_j(N_1, L_1)
\end{array}
$$

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[3]The generalized kernel (resp. range) of an endomorphism A is the kernel (resp. range) of a high power.

Proof. By Corollary 5.5 in [18] there exists a covering $J_1, J_2, \ldots, J_m$ of the unit interval by open intervals and there exist compact pairs (N_ν, L_ν), $\nu = 1, \ldots, m$, such that (N_ν, L_ν) is an index pair for (S_λ, f_λ) for $\lambda \in J_\nu$. Thus $\tau_j^k(S_\lambda)$ is independent of λ. Moreover, Φ_j can be constructed as a composition of finitely many maps of the form (1.6). $\square$

Denote by $R[[t]]$ the power series with coefficients in R and by $R(t)$ the rational functions with coefficients in R that is the quotient field of the polynomial ring $R[t]$. Denote by $R_1(t)$ those rational functions ζ for which denominator and numerator both have constant term 1 that is $\zeta, \zeta^{-1} \in R[[t]]$. Define the relation

$$\zeta_0 \prec \zeta_1 \qquad \Longleftrightarrow \qquad \frac{\zeta_0}{\zeta_1} \in R[t]$$

for $\zeta_0, \zeta_1 \in R_1(t)$. This defines a partial order on $R_1(t)$. The degree homomorphism

$$\deg : R_1(t) \to \mathbb{Z}$$

associates with every rational function the degree of the denominator minus the degree of the numerator. This homomorphism is order preserving. It also defines a discrete valuation map in the sense of [1].

The divided power algebra $\Gamma_R[[t]]$ is defined as an algebra over R with generators $t_0, t_1, t_2, \ldots$ which satisfy the relations

$$t_k t_\ell = \binom{k + \ell}{k} t_{k+\ell}.$$

So t_k can be thought of as the polynomial $t^k/k!$. An element of $\Gamma_R[[t]]$ can be thought of as a formal power series in t where the coefficient of t^k is an element of R divided by $k!$. If R is a field of characteristic zero then the divided power algebra $\Gamma_R[[t]]$ agrees with $R[[t]]$. In general $\Gamma_R[[t]]$ properly contains $R[[t]]$. In this extension there is an exponential map

$$\exp : \Gamma_R[[t]] \to \Gamma_R[[t]]$$

defined by

$$\exp(t_k) = \sum_{\nu=0}^{\infty} \binom{k\nu}{\nu} t_{k\nu}.$$

Let X be a finitely generated R module and let $A : X \to X$ be a module endomorphism. Then the familiar formula

$$(1.8) \qquad \exp\left(\sum_{k=1}^{\infty} \operatorname{trace}(A^k)(k-1)! t_k\right) = \frac{1}{\det(I - tA)}$$

remains valid. It can be proved by extending the coefficient ring to the algebraic closure of the quotient field F of R. Note that (1.8) defines a rational function in $R_1(t)$.

Define the **j-dimensional homology zeta function** of the pair (S, f) by

$$\zeta_j(t; S) = \exp\left(\sum_{k=1}^{\infty} \tau_j^k(S)(k-1)! t_k\right).$$

By Proposition 1.7 this is a continuation invariant. Also it follows from (1.8) that

$$\zeta_j(t; S) = \frac{1}{\det(I - tH_j(f_{N,L}))} \qquad \in \qquad R_1(t).$$

The reciprocal of ζ_j is, up to multiplication by a power of t, the characteristic polynomial of the shift automorphism $\mathcal{H}_j(f_{N,L})$.

Taking the alternating product we obtain the **homology zeta function** of the pair (S, f) defined by

$$\zeta(t; S) = \prod_{j=0}^{n} \zeta_j(t; S)^{(-1)^j};$$

In other words,

(1.9)
$$\zeta(t; S) = \exp\left(\sum_{k=1}^{\infty} L(S, f^k)(k-1)! t_k \right)$$

where

$$L(S, f) = \sum_{j=0}^{n} (-1)^j \operatorname{trace} H_j(f_{N,L})$$

denotes the **Lefschetz number** of the pair (S, f).

We also introduce the **Poincaré polynomial** of the pair (S, f) by taking degrees in t. More precisely,

$$p(s; S) = \sum_{j=0}^{n} d_j s^j$$

where $d_j = d_j(S) = \deg \zeta_j$. If R is a field then $d_j(S)$ is the dimension of the direct limit $\mathcal{H}_j(N, L)$. The **Euler characteristic** of the pair (S, f) is given by specializing to $s = -1$:

$$\chi(S) = p(-1; S) = \sum_{j=0}^{n} d_j(-1)^j.$$

In the case of rational coefficients both the Poincaré polynomial and the Lefschetz number of an isolated invariant set appear in the work of Mrozek [13], [14]. In particular in [13] Mrozek proves a localized version of the Lefschetz fixed point formula for isolated invariant sets. Mrozek assumes the existence of an index pair such that N and L are ANR's. For diffeomorphisms the existence of such an index pair follows from Theorem 5.3 in [18].

Theorem 1.10 (Mrozek). *Assume $R = \mathbb{Z}$. Let $S \subset M$ be an isolated invariant set for f such that every fixed point $x = f(x) \in S$ is nondegenerate. Then*

$$L(S, f) = \sum_{x=f(x) \in S} \operatorname{sign} \det(I - df(x)).$$

In particular, if $L(S, f) \neq 0$ then f has a fixed point in S.

This result gives rise to a proof of Fried's generalization [10] of Manning's theorem [11] on the rationality of the zeta function which does not rely on a suspension argument. For this we recall that an invariant set $S \subset M$ is called **expansive** if

$$\Delta_S = \{(x, x) \mid x \in S\} \subset M \times M$$

is an isolated invariant set for $f \times f$

Theorem 1.11 (Manning, Fried). *Let $S \subset M$ be an expansive, isolated invariant set for f such that every periodic point in S is nondegenerate. Then the zeta function*

$$Z(t; S) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} t^k\right), \qquad N_k = \sharp\{x \in S \mid f^k(x) = x\},$$

is rational.

Proof. For every $x \in S$ with $f^k(x) = x$ the fixed point index of (x, x) with respect to $f^k \times f^k$ is the square of the fixed point index of x with respect to f^k and hence equal to 1. Thus it follows from Theorem 1.10 that

$$L(\Delta_S, f^k \times f^k) = N_k$$

and therefore $Z(t; S) = \zeta(t; \Delta_S)$ is rational. $\square$

In [14] the previous theorem has been generalized to C^1 maps on compact manifolds using open index pairs.

§2 MORSE INEQUALITIES

Let S be an isolated invariant set for a diffeomorphism $f : M \to M$. A **Morse decomposition** of S is a finite collection $\{\Lambda_p\}_{p \in P}$ of disjoint compact invariant sets indexed by a partially ordered set $(P, \preccurlyeq)$ such that

$$S \setminus \bigcup_{p \in P} \Lambda_p = \bigcup_{p \preccurlyeq q, p \neq q} W^u(\Lambda_q, S) \cap W^s(\Lambda_p, S).$$

This is called the **no cycle** condition. It implies that the Λ_p are isolated invariant sets for f in M.

Theorem 2.1 (Polynomial Morse inequalities). *Let $\{\Lambda_p\}_{p \in P}$ be a partially ordered Morse decomposition of the isolated invariant set S. Then*

$$\prod_{p \in P} \zeta_0(t; \Lambda_p) \succ \zeta_0(t; S),$$

$$\prod_{p \in P} \zeta_1(t; \Lambda_p) \left(\prod_{p \in P} \zeta_0(t; \Lambda_p)\right)^{-1} \succ \zeta_1(t; S)\zeta_0(t; S)^{-1},$$

$$\vdots$$

(2.2)
$$\prod_{i=0}^{j} \left(\prod_{p \in P} \zeta_{j-i}(t; \Lambda_p)\right)^{(-1)^i} \succ \prod_{i=0}^{j} \zeta_{j-i}(t; S)^{(-1)^i},$$

$$\vdots$$

$$\prod_{p \in P} \zeta(t; \Lambda_p) = \zeta(t; S).$$

The last equation in (2.2) can be rewritten as the additivity of the Lefschetz numbers.

Corollary 2.3. *Let $\{\Lambda_p\}_{p\in P}$ be a partially ordered Morse decomposition of the isolated invariant set S. Then*

$$\sum_{p\in P} L(\Lambda_p, f^k) = L(S, f^k)$$

for $k = 1, 2, \ldots$.

Proof. Theorem 2.1 and formula (1.9). □

Taking degrees in (2.2) we obtain Morse inequalities in the classical form. Under the assumption of finitedimensionality and for rational coefficients these have been proved by Mrozek [15].

Corollary 2.4 (Integral Morse inequalities). *Let $\{\Lambda_p\}_{p\in P}$ be a partially ordered Morse decomposition of the isolated invariant set S. Then*

$$\sum_{p\in P} d_0(\Lambda_p) \geq d_0(S),$$

$$\sum_{p\in P} d_1(\Lambda_p) - \sum_{p\in P} d_0(\Lambda_p) \geq d_1(S) - d_0(S),$$

$$\vdots$$

(2.5)
$$\sum_{i=0}^{j} (-1)^i \sum_{p\in P} d_{j-i}(\Lambda_p) \geq \sum_{i=0}^{j} (-1)^i d_{j-i}(S),$$

$$\vdots$$

$$\sum_{p\in P} \chi(\Lambda_p) = \chi(S).$$

Equivalently

$$\sum_{p\in P} p(s; \Lambda_p) = p(s; S) + (1 + s)q(s)$$

where $q(s)$ is a polynomial with nonnegative integer coefficients.

Proof. Define $q_j = \deg \eta_j$ for $j = 0, 1, \ldots, n-1$. Since η_j^{-1} is a polynomial we have $q_j \geq 0$ and it follows from (2.2) that

$$\sum_{i=0}^{j} (-1)^i \sum_{p\in P} d_{j-i}(\Lambda_p) - \sum_{i=0}^{j} (-1)^i d_{j-i}(S) = q_j. \quad \square$$

As an example consider a diffeomorphism $f : M \to M$ of a compact Manifold M with a hyperbolic chain recurrent set R_f [2], [17], [22]. Then

$$R_f = \bigcup_{\nu=0}^{m} \Lambda_\nu$$

where the compact invariant sets Λ_ν each have a dense orbit [22]. These are the **basic sets** and suitably ordered they satisfy the no cycle condition. For each basic

set the zeta function $\zeta_j(t; \Lambda_\nu)$ is nontrivial in at most one dimension, that is the dimension j_ν of the unstable subbundle of $T_{\Lambda_\nu} M$. Under these conditions Theorem 2.1 was proved by Franks [7], [8] and Corollary 2.4 was proved by Zeeman in an unpublished work which was announced in [25]. Those basic sets with $j_\nu = 0$ are attracting periodic orbits [9] and hence

$$\zeta_0(t; \Lambda_\nu) = \frac{1}{1 - t^{p_\nu}} \qquad \text{if} \qquad j_\nu = 0,$$

where p_ν is the period (see Example 5.5). Similarly, the basic sets with $j_\nu = n$ are repelling periodic orbits and hence

$$\zeta_n(t; \Lambda_\nu) = \frac{1}{1 \pm t^{p_\nu}} \qquad \text{if} \qquad j_\nu = n.$$

If all the basic sets Λ_p are hyperbolic periodic orbits then Corollary 2.4 reduces to Smale's Morse inequalities in [21].

We examine the special case of a diffeomorphism $f : S \to S$ of a compact orientable surface $M = S$ with a hyperbolic chain recurrent set. Theorem 2.1 with $R = \mathbb{Z}$ shows that

$$\frac{\zeta_j(t; S)}{\prod_{\nu=0}^m \zeta_j(t; \Lambda_\nu)} \quad \in \quad \mathbb{Z}[t]$$

is a polynomial for $j = 0, 1, 2$ and

$$\frac{\zeta_1(t; S)}{\prod_{\nu=0}^m \zeta_1(t; \Lambda_\nu)} = \frac{\zeta_0(t; S)\zeta_2(t; S)}{\prod_{\nu=0}^m \zeta_0(t; \Lambda_\nu)\zeta_2(t; \Lambda_\nu)}$$

The zeroes of the right hand side are all roots of unity. So the figure G horseshoe of Example 5.3 cannot occur as a basic set for f (see also [9] and [22]). Otherwise its characteristic polynomial $\zeta_1(t; \Lambda)^{-1} = 1 - 2t$ would have to divide $\zeta_1(t; S)^{-1}$ that is the unimodular integer matrix $H_1(f)$ would have an integer eigenvalue not equal to 1 or -1. This is impossible.

In contrast, it is easy to construct a diffeomorphism of the 2-sphere with the Smale horseshoe of Example 5.2 as a basic set [22].

To prove Theorem 2.1 we first examine attractor-repeller pairs in S. These consist of two disjoint isolated invariant sets $A, A^* \subset S$ such that

$$S \setminus (A \cup A^*) = W^s(A, S) \cap W^u(A^*, S).$$

Such an attractor-repeller pair is uniquely determined by the attractor A.[4] An **index triple** for the attractor-repeller pair (A, A^*) in S is a triple $L \subset N \subset N^*$ of compact sets such that (N^*, L) is an index pair for S, (N, L) is an index pair for A and (N^*, N) is an index pair for A^*. In [19] it is shown that for every attractor-repeller pair (A, A^*) in S there exists a polyhedral index triple $L \subset N \subset N^*$. This means that N^* is a finite polyhedron and L, N are subpolyhedra. So, in particular, the homology groups of the index pairs (N^*, N), (N^*, L) and (N, L) are of finite rank and vanish in dimensions larger than n.

---

[4]See [19] for the relevant definitions.

Proposition 2.6. *Let (A, A^*) be an attractor-repeller pair for the isolated invariant set S. Then the traces*[5]

$$\tau_j^k(A, S) = \text{trace } H_j(f_{N,L}^k)|_{\text{range } \partial} \qquad \in \qquad R$$

where ∂ denotes the boundary operator $\partial : H_{j+1}(N^*, N) \to H_j(N, L)$ are independent of the choice of the index triple $L \subset N \subset N^*$ used to define them. Also they are invariant under continuation of the attractor repeller pair.

Proof. Proposition 1.1 and Proposition 1.7 □

Define the *j-dimensional homology eta function* of the attractor-repeller pair
(A, A^*) in S by

$$\eta_j(t; A, S) = \exp\left(\sum_{k=1}^{\infty} \tau_j^k(A, S)(k-1)! t_k\right).$$

By Proposition 2.6 this is a continuation invariant. Also it follows from (1.8) that

$$\eta_j(t; A, S) = \det\left(I - t H_j(f_{N,L})|_{\text{range } \partial}\right)^{-1} \qquad \in \qquad R_1(t).$$

We also introduce the **connection polynomial** of an attractor repeller pair (A, A^*) in S by taking degrees. More precisely,

$$q(s; A, S) = \sum_{j=0}^{n} d_j s^j$$

where $d_j = d_j(A, S) = \deg \eta_j(\cdot; A, S)$. If R is a field then $d_j(A, S)$ is the rank of the induced boundary homomorphism $\partial : \mathcal{H}_{j+1}(N^*, N) \to \mathcal{H}_j(N, L)$.

Proposition 2.7. *Let (A, A^*) be an attractor repeller pair in the isolated invariant set S. Then*

$$\zeta_j(t; A)\zeta_j(t; A^*) = \zeta_j(t; S)\eta_j(t; A, S)\eta_{j-1}(t; A, S).$$

Proof. For any morphism of exact sequences

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{\nu+1} & \xrightarrow{\phi_{\nu+1}} & X_\nu & \xrightarrow{\phi_\nu} & X_{\nu-1} & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle A_{\nu+1}} & & \downarrow{\scriptstyle A_\nu} & & \downarrow{\scriptstyle A_{\nu-1}} & & \\
\cdots & \longrightarrow & X_{\nu+1} & \xrightarrow[\phi_{\nu+1}]{} & X_\nu & \xrightarrow[\phi_\nu]{} & X_{\nu-1} & \longrightarrow & \cdots
\end{array}
$$

where the X_ν are modules over a principal ideal domain we have

$$\text{trace } A_{\nu-1}|_{\text{range } \phi_\nu} = \text{trace } A_\nu - \text{trace } A_\nu|_{\text{range } \phi_{\nu+1}}$$

and hence

$$\text{trace } A_{\mu-1}|_{\text{range } \phi_\mu} + \text{trace } A_{\mu+2}|_{\text{range } \phi_{\mu+3}} = \text{trace } A_\mu - \text{trace } A_{\mu+1} + \text{trace } A_{\mu+2}.$$

---

[5]These traces depend on S, see Example 5.10.

Apply this to the commuting diagram

$$\cdots H_j(N,L) \longrightarrow H_j(N^*,L) \longrightarrow H_j(N^*,N) \xrightarrow{\;\partial\;} H_{j-1}(N,L) \cdots$$

$$\downarrow H_j(f_{N,L}^k) \qquad\qquad \downarrow H_j(f_{N^*,L}^k) \qquad\qquad \downarrow H_j(f_{N^*,N}^k) \qquad\qquad \downarrow H_{j-1}(f_{N,L}^k)$$

$$\cdots H_j(N,L) \longrightarrow H_j(N^*,L) \longrightarrow H_j(N^*,N) \xrightarrow[\;\partial\;]{} H_{j-1}(N,L) \cdots$$

to obtain

$$\tau_{j-1}^k(A,S) + \tau_j^k(A,S) = \tau_j^k(A^*) - \tau_j^k(S) + \tau_j^k(A). \quad \square$$

Proof of Theorem 2.1. Let $\{\Lambda_p\}_{p\in P}$ be a Morse decomposition indexed by the partially ordered set $(P,\preccurlyeq)$. In [19] it is shown that

$$A_I = \bigcup_{p\in I} W^u(\Lambda_p, S).$$

is an attractor for every initial interval $I \subset P$ (that is: if $p \in I$ and $q \preccurlyeq p$ then $q \in I$). A total order $p_0, p_1, \ldots, p_m$ of P is called **admissable** if $p_\mu \preccurlyeq p_\nu$ implies $\mu \leq \nu$. Fix such an admissable order of P and define

$$A_\nu = A_{\{p_0,\ldots,p_\nu\}}$$

for $\nu = 0, 1, \ldots, m$. Then A_ν is an isolated invariant set for f and $(A_\nu, \Lambda_{p_{\nu+1}})$ is an attractor-repeller pair in $A_{\nu+1}$ (see Proposition 2.3 and Lemma 2.5 in [19]). Define

$$(2.8) \qquad\qquad \eta_j(t) = \prod_{\nu=0}^{m-1} \eta_j(t; A_\nu, A_{\nu+1})$$

and

$$(2.9) \qquad\qquad q(s) = \sum_{\nu=0}^{m-1} q(s; A_\nu, A_{\nu+1}).$$

Then Proposition 2.7 shows that

$$\zeta_j(t; A_\nu)\zeta_j(t; \Lambda_{p_{\nu+1}}) = \zeta_j(t; A_{\nu+1})\eta_j(t; A_\nu, A_{\nu+1})\eta_{j-1}(t; A_\nu, A_{\nu+1})$$

for $\nu = 0, 1, \ldots, m-1$. Thus by induction

$$\prod_{\nu=0}^{\ell} \zeta_j(t; \Lambda_{p_\nu}) = \zeta_j(t; A_\ell) \prod_{\nu=0}^{\ell-1} \eta_j(t; A_\nu, A_{\nu+1})\eta_{j-1}(t; A_\nu, A_{\nu+1})$$

For $\ell = m$ this yields

$$\prod_{p\in P} \zeta_j(t; \Lambda_p) = \zeta_j(t; S)\eta_j(t)\eta_{j-1}(t)$$

and hence (2.2). $\quad \square$

The above proof shows that the connection polynomials (2.8) and (2.9) of the Morse decomposition $\{\Lambda_p\}_{p\in P}$ are independent of the choice of the admissable ordering of P used to define them.

§3 Flows

Let $\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto f^t$ be a flow. As in the case of diffeomorphisms a compact invariant set S is called **isolated** if there exists a neighborhood V of S such that

$$S = \bigcap_{t \in \mathbb{R}} f^t(cl(V));$$

every neighborhood V of S with this property is called isolating for S. An **index pair** for an isolated invariant set S is a pair of compact sets $L \subset N$ in M such that $N \setminus L$ is an isolating neighborhood for S and the induced semiflow

$$N/L \times [0, \infty) \to N/L : (x, t) \mapsto f^t_{N,L}(x)$$

defined by

$$f^t_{N,L}(x) = \begin{cases} f^t(x) & \text{if } f^s(x) \in N \setminus L \text{ for } 0 \le s \le t, \\ * & \text{otherwise} \end{cases}$$

is continuous.

Proposition 3.1. *Let $f = f^1$ be the time-1-map of a flow $f^t : M \to M$. Then $S \subset M$ is an isolated invariant set for f^t in the sense of Conley [2] if and only if S is an isolated invariant set for f. Moreover, if (N, L) is an index pair for the flow then it is an index pair for the diffeomorphism and*

$$(3.2) \qquad \zeta_j(t; S) = (1 - t)^{\mathrm{rank}\, H_j(N,L)}, \qquad \zeta(t; S) = (1 - t)^{\chi(S)}.$$

Proof. Suppose S is an isolated invariant set for the diffeomorphism. Then the set of all $t \in \mathbb{R}$ such the $f^t(S) \subset S$ is open (isolation) and closed so that S is invariant under the flow. Let V be an isolating neighborhood for S with respect to the diffeomorphism. If the $\mathbb{R}$-orbit of a point x lies in V then the $\mathbb{Z}$-orbit lies in V so $x \in S$. Thus S is isolated for the flow.

Conversely, suppose that S is an isolated invariant set for the flow with isolating neighborhood V. Choose a smaller isolating neighborhood W so that

$$\bigcup_{0 \le s \le 1} f^s(W) \subset V.$$

If $f^k(x) \in W$ for all $k \in \mathbb{Z}$ then $f^t(x) \in V$ for all $t \in \mathbb{R}$ so $x \in S$. Hence S is an isolated invariant set for f with isolating neighborhood W.

Now let (N, L) be an index pair for S with respect to the flow. Then the induced semiflow $(x, t) \mapsto f^t_{N,L}(x)$ on $N/L \times [0, \infty)$ is continuous. Hence $f_{N,L} = f^1_{N,L}$ is continuous and so (N, L) is an index pair for S with respect to the diffeomorphism. Furthermore, $f_{N,L}$ is homotopic to the identity and this proves (3.2). $\square$

The previous result shows that in the case of a flow the Poincaré polynomial $p(s; S)$ carries the same information as the double homology zeta function $\zeta(s, t; S)$. So in this case Theorem 2.1 and Corollary 2.4 are equivalent and they agree with Theorem 3.3 in [4].

As a special case we obtain the classical Morse inequalities for the gradient flow of a Morse function $g : M \to \mathbb{R}$ on a compact Riemannian manifold $M = S$. In this case the Morse sets Λ_p are the critical points of g and

$$\sum_{p \in P} p(s; \Lambda_p) = \sum_{j=0}^{n} c_j s^j$$

where c_j is the number of critical points x_0 of g of Morse index $\mathrm{ind}(x_0) = j$. Here $\mathrm{ind}(x_0)$ is the dimension of the unstable manifold $W^u(x_0)$ with respect to the gradient flow $\dot{x} = -\nabla g(x)$. Since $N = M, L = \emptyset$ is an index pair for $S = M$ the index polynomial $p(s; S)$ agrees with the Poincaré polynomial of the compact manifold M. Thus Corollary 2.4 gives the familiar Morse inequalities for a gradient flow.

As another example consider the flow on $S = M = \mathbb{R}P^3$ discussed in Example 5.11. There is a 3-set Morse decomposition $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ where Λ_0 is an attracting fixed point and Λ_2 is a repelling fixed point. The set Λ_1 is a normally hyperbolic periodic orbit with a 1-dimensional stable and a 1-dimensional unstable bundle. For any index pair (N, L) the homotopy type of the quotient is that of $\mathbb{R}P^2$, considered as a pointed space. So the the Morse inequalities are satisfied with $q(s) = 0$. (Note that the Poincaré polynomials of both Λ_1 and S depend on the coefficient ring.)

§4 SUSPENSION

Let $f : M \to M$ be a homeomorphism of a topological space M. The suspension of f is the flow

$$\Sigma_f M \times \mathbb{R} \to \Sigma_f M : ([x, \tau], t) \mapsto \Phi_f^t([x, \tau])$$

where $\Sigma_f M = \{[x, \tau] \mid x \in M, \tau \in \mathbb{R}\}$ is the set of equivalence classes $[x, \tau]$ under the equivalence relation

$$(f^k(x), \tau) \equiv (x, \tau + k), \qquad k \in \mathbb{Z}.$$

and

$$\Phi_f^t([x, \tau]) = [x, \tau + t], \qquad t \in \mathbb{R}.$$

For any subset $A \subset M$ define

$$\Sigma_f A = \{[x, \tau] \mid x \in A, 0 \le \tau \le 1\}.$$

(Note that if A is f-invariant then $\Sigma_f A = \Sigma_{f|A} A$.)

Let $\phi : X \to X$ be a base point preserving map of a pointed topological space $(X, *)$. The suspension of ϕ is the semiflow

$$\Sigma_\phi X \times [0, \infty) \to \Sigma_\phi X : ([x, \tau], t) \mapsto \Phi_\phi^t([x, \tau]).$$

Here the pointed topological space

$$\Sigma_\phi X = \{[x, \tau] \mid x \in X, \tau \ge 0\}$$

is the set of equivalence classes $[x, \tau]$ under the smallest equivalence relation implied by the conditions

$$(\phi^k(x), \tau) \equiv (x, \tau + k), \qquad k \in \mathbb{N},$$
$$(*, \tau) \equiv (*, \sigma), \qquad \tau, \sigma \ge 0$$

and

$$\Phi_\phi^t([x, \tau]) = [x, \tau + t], \qquad t \ge 0.$$

Proposition 4.1. *Let $f : M \to M$ be a diffeomorphism and let $S \subset M$.*

(i) *The set S is invariant under f if and only if $\Sigma_f S$ is invariant under the flow Φ_f^t.*

(ii) *The set S is isolated invariant for f if and only if $\Sigma_f S$ is isolated invariant for the flow Φ_f^t.*

(iii) *If S is an isolated invariant set for f then there exists an index pair (N, L) satisfying the conditions[6]*

$$f(L) \cap N \subset L, \qquad f(N \setminus L) \subset N.$$

Such index pairs are called **suspendable***.*

(iv) *For any suspendable index pair (N, L) the suspended pair $(\Sigma_f N, \Sigma_f L)$ is an index pair for $\Sigma_f S$ and*

$$\Sigma_f N / \Sigma_f L = \Sigma_{f_{N,L}} N / L, \qquad (\Phi_f)_{\Sigma_f N, \Sigma_f L}^t = \Phi_{f_{N,L}}^t$$

Proof. Part (i) is obvious.

We prove part (ii). Assume S is an isolated invariant set for f with isolating neighborhood V. Then $\Sigma_f V$ is an isolating neighborhood for $\Sigma_f S$. Indeed, suppose that $[x, \tau + t] \in \Sigma_f V$ for all $t \in \mathbb{R}$. Taking $t = k - \tau + \frac{1}{2}$ with $k \in \mathbb{Z}$ we get $[f^k(x), \frac{1}{2}] \in \Sigma_f V$ and hence $f^k(x) \in V$ for all $k \in \mathbb{Z}$. Thus $x \in S$ so $[x, \tau] \in \Sigma_f S$. This proves that $\Sigma_f S$ is isolated.

Conversely, assume $\Sigma_f S$ is an isolated invariant set for the flow Φ_f^t with isolating neighborhood $\tilde{V}$. Then

$$V = \{x \in M \mid [x, \tau] \in \tilde{V} \text{ for } 0 \leq \tau \leq 1\}$$

is a neighborhood of S with $\Sigma_f V \subset \tilde{V}$. If the f-orbit of x lies in V then the Φ_f-orbit of $[x, 0]$ lies in $\tilde{V}$ and hence $x \in S$. Thus S is an isolated invariant set for f.

We prove part (iii). Recall from [18] that there exists a smooth Lyapunov function $\theta : V \cup f(V) \cup f^{-1}(V) \to \mathbb{R}$ for any isolating neighborhood V of S. This means that $S \subset \theta^{-1}(0)$ and $\theta(f(x)) < \theta(x) < \theta(f^{-1}(x))$ for every $x \in V \setminus S$. Define

$$N = \{x \in V \mid -\varepsilon \leq \theta(x) \leq \theta(f^{-1}(x)) \leq \varepsilon\}$$
$$L = \{x \in N \mid \theta(f(x)) \leq -\varepsilon\}.$$

For $\varepsilon > 0$ sufficiently small N is a compact subset of the interior of V. As in [18] (N, L) is a polyhedral pair for a generic small ε. We show that $f(L) \cap N \subset L$: Let $y \in f(L) \cap N$ so $y = f(x)$ for some $x \in L$. Then

$$\theta(f(y)) \leq \theta(y) = \theta(f(x)) \leq -\varepsilon$$

and hence $y \in L$. We show that $f(N \setminus L) \subset N$: Let $y \in f(N \setminus L)$ so $y = f(x)$ where $x \in N \setminus L$. Then

$$-\varepsilon < \theta(f(x)) \leq \theta(x) \leq \theta(f^{-1}(x)) \leq \varepsilon$$

---

[6]These conditions say that L is positively invariant relative to N and every orbit which exits N does so through L.

so $y = f(x) \in N$. By Corollary 4.4 of [18] (N, L) is an index pair.

We prove part (iv). As in part (ii) $\Sigma_f(N \setminus L)$ is an isolating neighborhood of $\Sigma_f S$. Hence the smaller set $\Sigma_f N \setminus \Sigma_f L$ is also an isolating neighborhood for $\Sigma_f S$. From $f(L) \cap N \subset L$ it follows that if $[x, \tau] \in \Sigma_f L$ and $\Phi_f^{[0,t]}([x, \tau]) \subset \Sigma_f N$ then $\Phi_f^t([x, \tau]) \subset \Sigma_f L$. Thus $\Sigma_f L$ is positively invariant in $\Sigma_f N$. From $f(N \setminus L) \subset N$ it follows that every Φ_f-orbit which exists $\Sigma_f N$ does so through $\Sigma_f L$. Hence by Theorem 4.2 of [18] $(\Sigma_f N, \Sigma_f L)$ is an index pair for $\Sigma_f S$. The associated topological quotient is given by $\Sigma_f N / \Sigma_f L = \Sigma_f N \setminus \Sigma_f L \cup \{*\}$ where

$$\Sigma_f N \setminus \Sigma_f L = \Sigma_f(N \setminus L) \setminus \{[x, 1] \mid x \in N \setminus L, f(x) \in L\} = \Sigma_{f_{N,L}} N/L \setminus \{*\}.$$

This shows that $\Sigma_f N / \Sigma_f L = \Sigma_{f_{N,L}} N/L$. The associated semiflow $(\Phi_f)_{\Sigma_f N, \Sigma_f L}^t$ is induced by the right shift $([x, \tau], t) \mapsto [x, \tau + t]$ and therefore agrees with $\Phi_{f_{N,L}}^t$. $\square$

Part (iv) of Theorem 4.1 allows us to relate the Conley index of an isolated invariant set $S \subset M$ of a diffeomorphism $f : M \to M$ to the Conley index of its suspension. We make use of a long exact sequence which also appears in the work of Floer [5] on normally hyperbolic invariant submanifolds. We include a proof of this result since it is short.

Theorem 4.2. *Let $\phi : X \to X$ be a base point preserving map of a pointed topological space $(X, *)$. Then there is a commutative diagram as follows in which the horizontal sequences are exact.*

$$(4.3) \quad
\begin{array}{ccccccc}
\cdots H_j(X) & \xrightarrow{I - \phi_*} & H_j(X) & \longrightarrow & H_j(\Sigma_\phi X) & \xrightarrow{\partial} & H_{j-1}(X) \cdots \\
\phi_* \downarrow & & \downarrow \phi_* & & \downarrow I & & \downarrow \phi_* \\
\cdots H_j(X) & \xrightarrow[I - \phi_*]{} & H_j(X) & \longrightarrow & H_j(\Sigma_\phi X) & \xrightarrow[\partial]{} & H_{j-1}(X) \cdots
\end{array}$$

Proof. For any real numbers $b \geq a \geq 0$ define

$$X_{a,b} = \{[x, \tau] \mid x \in X, \, a < \tau < b\}.$$

If $b - a < 1$ and $a < \tau < b$ then the map $X \to X_{a,b} : x \mapsto [x, \tau]$ is a homotopy equivalence.

Cover $\Sigma_\phi X$ by the open sets

$$U = X_{0, \frac{3}{4}}, \qquad V = X_{\frac{1}{2}, \frac{5}{4}}.$$

Then U and V are homotopy equivalent to X and

$$U \cap V = X_{\frac{1}{2}, \frac{3}{4}} \cup X_{1, \frac{5}{4}}$$

is homotopy equivalent to the sum $X \vee X$. This induces an isomorphism of long exact sequences

$$(4.4) \quad
\begin{array}{ccccc}
\cdots H_j(U \cap V) & \longrightarrow & H_j(U) \oplus H_j(V) & \longrightarrow & H_j(U \cup V) \cdots \\
\simeq \uparrow & & \uparrow \simeq & & \uparrow = \\
\cdots H_j(X) \oplus H_j(X) & \xrightarrow[T_\phi]{} & H_j(X) \oplus H_j(X) & \longrightarrow & H_j(\Sigma_\phi X) \cdots
\end{array}$$

where

$$T_\phi = \begin{pmatrix} I & \phi_* \\ I & I \end{pmatrix} : H_*(X) \oplus H_*(X) \to H_*(X) \oplus H_*(X).$$

The isomorphism (4.4) intertwines the endomorphisms

$$
\begin{array}{ccccc}
\cdots H_j(U \cap V) & \longrightarrow & H_j(U) \oplus H_j(V) & \longrightarrow & H_j(U \cup V) \cdots \\
(\Phi_\phi^1)_* \downarrow & & \downarrow (\Phi_\phi^1)_* & & \downarrow (\Phi_\phi^1)_* \\
\cdots H_j(U \cap V) & \longrightarrow & H_j(U) \oplus H_j(V) & \longrightarrow & H_j(U \cup V) \cdots
\end{array}
$$

and

(4.5)
$$
\begin{array}{ccccc}
\cdots H_j(X) \oplus H_j(X) & \xrightarrow{T_\phi} & H_j(X) \oplus H_j(X) & \longrightarrow & H_j(\Sigma_\phi X) \cdots \\
\phi_* \oplus \phi_* \downarrow & & \downarrow \phi_* \oplus \phi_* & & \downarrow I \\
\cdots H_j(X) \oplus H_j(X) & \xrightarrow[T_\phi]{} & H_j(X) \oplus H_j(X) & \longrightarrow & H_j(\Sigma_\phi X) \cdots
\end{array}
$$

Since

$$\text{range}\, \partial = \ker T_\phi \subset \{(\alpha, \beta) \in H_*(X) \oplus H_*(X) \,|\, \alpha = -\beta\}$$

and

$$\{T_\phi(\alpha, \beta) \,|\, \alpha = -\beta\} \subset H_*(X) \oplus \{0\}$$

the commuting diagram (4.5) reduces to (4.3). □

Applying Theorem 4.2 to the induced map $f_{N,L} : N/L \to N/L$ of a suspendable index pair we obtain the following result.

Theorem 4.6. *Let S be an isolated invariant set for a diffeomorphism $f : M \to M$ with an index pair (N, L). Then*

$$p(s; \Sigma_f S) = (1 + s) p_1(s; S, f)$$

where

$$p_1(s; S, f) = \sum_{j=0}^{n} \text{rank}\, \ker(I - H_j(f_{N,L})) s^j.$$

Proof. Choose a suspendable index pair (N, L). Recall from Theorem 4.1 that the pair $(\Sigma_f N, \Sigma_f L)$ is an index pair for $\Sigma_f S$ and $\Sigma_f N / \Sigma_f L = \Sigma_{f_{N,L}} N/L$. This shows that

$$p(s; \Sigma_f S) = \sum_{j=0}^{n} \kappa_j s^j$$

where

$$\kappa_j = \text{rank}\, K_j, \qquad K_j = H_j(\Sigma_{f_{N,L}} N/L).$$

Also

$$p_1(s; S, f) = \sum_{j=0}^{n} \rho_j s^j$$

where

$$\rho_j = \operatorname{rank} \ker(I - h_j), \qquad h_j = H_j(f_{N,L}) : H_j \to H_j, \qquad H_j = H_j(N/L).$$

Read N/L for X and $f_{N,L}$ for ϕ in Theorem 2.4 to obtain the exact sequence

$$(4.7) \qquad \cdots \xrightarrow{\partial} H_j \xrightarrow{I - h_j} H_j \to K_j \xrightarrow{\partial} H_{j-1} \xrightarrow{I - h_{j-1}} \cdots$$

For any exact sequence

$$\cdots \to X_\nu \xrightarrow{\phi_\nu} X_{\nu-1} \to \cdots$$

of modules over a principal ideal domain we have

$$\operatorname{rank} \ker \phi_{\nu-1} = \operatorname{rank} X_\nu - \operatorname{rank} \ker \phi_\nu$$

and hence

$$\operatorname{rank} \ker \phi_\mu + \operatorname{rank} \ker \phi_{\mu-3} = \operatorname{rank} X_\mu - \operatorname{rank} X_{\mu-1} + \operatorname{rank} X_{\mu-2}.$$

Apply this to the exact sequence (4.7) to obtain

$$\kappa_j = \rho_j + \rho_{j-1}. \quad \square$$

The previous result shows that the Conley index of the suspension $\Sigma_f S$ carries much less information than the homology zeta functions of the pair (S, f).

For example, if S is the figure G horseshoe then there exists an index pair (N, L) such that $H_*(N, L) = (0, \mathbb{Z}, 0, \dots)$ and $H_1(f_{N,L})$ acts by multiplication with 2 (see Example 5.3). So in this case $\zeta_1(t; S) = (1 - 2t)^{-1}$ but $p(s; \Sigma_{f^k} S) = 0$ for all $k \in \mathbb{Z}$.

As another example consider a stable periodic orbit $S = \{x, f(x)\}$ of period 2 and choose an index pair (N, L) with $H_*(N, L) = (\mathbb{Z} \oplus \mathbb{Z}, 0, \dots)$ (see Example 5.6). Then $\zeta_0(t; S) = (1 - t^2)^{-1}$ and $p(s; \Sigma_f S) = 1 + s$, $p(s; \Sigma_{f^2} S) = 2 + 2s$.

§5 Examples

In [18] it is shown that for a compact pair $L \subset N$ the induced semidynamical system $f_{N,L} : N/L \to N/L$ is continuous whenever

$$f(N) \cap \partial N \subset L \subset N \setminus f^{-1}(N).$$

Given N one can find such an L if and only if

$$\partial N \cap f(N) \cap f^{-1}(N) = \emptyset.$$

This criterion goes back to Conley and turns out to be most useful for constructing index pairs explicitly.

Example 5.1 : A hyperbolic fixed point. Let x be a hyperbolic fixed point of a diffeomorphism $f : M \to M$ and let E_x^s and E_x^u be the contracting and expanding subspaces for $df(x) : T_x M \to T_x M$. Then $S = \{x\}$ is an isolated invariant set and an index pair for S is given by

$$N = \{\exp_x(\xi^s + \xi^u) \,|\, \xi^s \in E_x^s, \, \xi^u \in E_x^u, \, |\xi^s| \leq \varepsilon, \, |\xi^u| \leq \varepsilon\},$$
$$L = \{\exp_x(\xi^s + \xi^u) \in N \,|\, |\xi^u| = \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small. (See Figure 1 for $n_u = dim E_x^u = 1$.) So $H_j(N, L) = 0$ for $j \neq n_u$ and $H_j(N, L) = \mathbb{Z}$ for $j = n_u$. The induced map $H_j(f_{N,L})$ for $j = n_u$ is the identity whenever the restriction of f to $W^u(x)$ is orientation preserving and is minus the identity if this restriction is orientation reversing. Thus

$$\zeta_j(t; S) = \frac{1}{1 - \nu(x; f)t}, \qquad j = n_u.$$

In connection with Theorem 1.10 his shows that the definition of $\zeta_j(t; S)$ given in section 1 agrees (in the case $R = \mathbb{Z}$) with the one given in the introduction whenever the isolated invariant set S is hyperbolic.
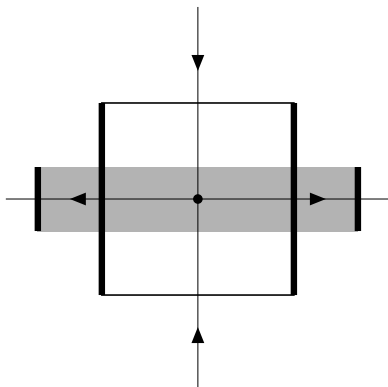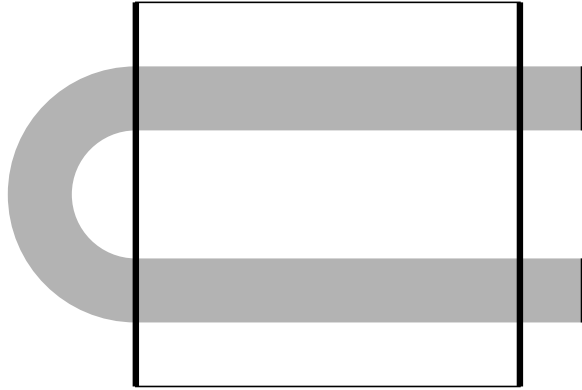

Figure 1

Example 5.2 : The Smale horseshoe. Consider a diffeomorphism which deforms the square N and maps it as indicated in Figure 2. The square is an isolating neighborhood and the isolated invariant set $S \subset N$ is a Cantor set [22]. The one point compactification of its unstable manifold is the Knaster continuum. An index pair is given by the square N with the exit set consisting of the thick lines $L \subset \partial N$. This pair satisfies the above conditions. Its homology $H_j(N, L)$ vanishes for $j \neq 1$ and $H_1(N, L) = \mathbb{Z}$. The induced map $f_{N,L}$ acts trivially on $H_1(N, L)$ so $\zeta_j(t; S) = 1$ for every j. This might seem a bit odd since S is a rather large invariant set. However, this invariant set can be perturbed away and the zeta function is a continuation invariant. So it must be trivial.

Example 5.3 : The figure G horseshoe. A similar example is the mapping of the square indicated in Figure 3. Here the invariant set is in fact the same as in Example 5.2 and so is the index pair (N, L). However, the two invariant sets differ
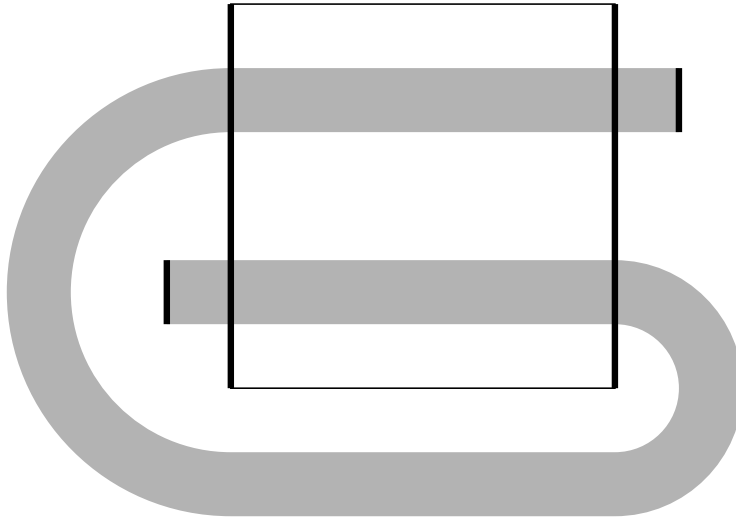
Figure 2

in their unstable manifolds. This is reflected by the induced map $H_1(f_{N,L})$ which in the second example acts by multiplication with 2. So

$$\zeta_1(t; S) = \frac{1}{1 - 2t}$$

and $\zeta_j(t; S) = 1$ for $j \neq 1$. The direct system (1.4) is given by

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots$$

and its direct limit $\mathcal{H}_1(N, L) = \mathbb{Q}_2$ is the set of rational numbers whose denominator is a power of 2.

Figure 3

An alternative exit set is the closure of the set of points which leave N that is $L = \mathrm{cl}\left(N \setminus f^{-1}(N)\right)$. So (N, L) is a suspendable index pair. (See Figure 4.) Then $H_1(N, L) = \mathbb{Z} \oplus \mathbb{Z}$ and

$$H_1(f_{N,L}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

This matrix is shift equivalent to the 1×1-matrix 2.

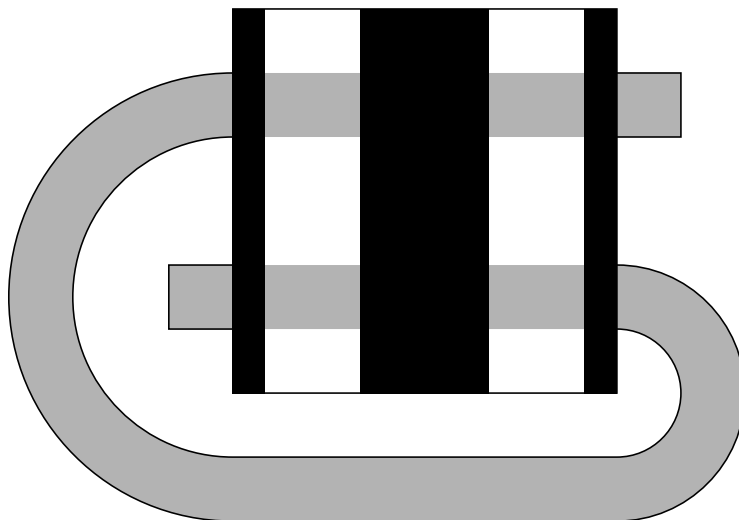In this example the Conley index of the suspension in trivial (see section 4).

Figure 4

Example 5.4 : The propellor. The mapping indicated in Figure 5 has an isolated invariant set S with a degenerate fixed point in the centre. So the invariant set S is not hyperbolic. An index pair with $L \subset \partial N$ can be constructed as in the previous examples. The index homology is trivial in dimensions $j \neq 1$ and $H_1(N, L) = \mathbb{Z} \oplus \mathbb{Z}$. The induced map is

$$H_1(f_{N,L}) = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

so

$$\zeta_1(t; S) = \det\left(I - tH_1(f_{N,L})\right)^{-1} = \frac{1}{1 - 5t + 7t^2}.$$
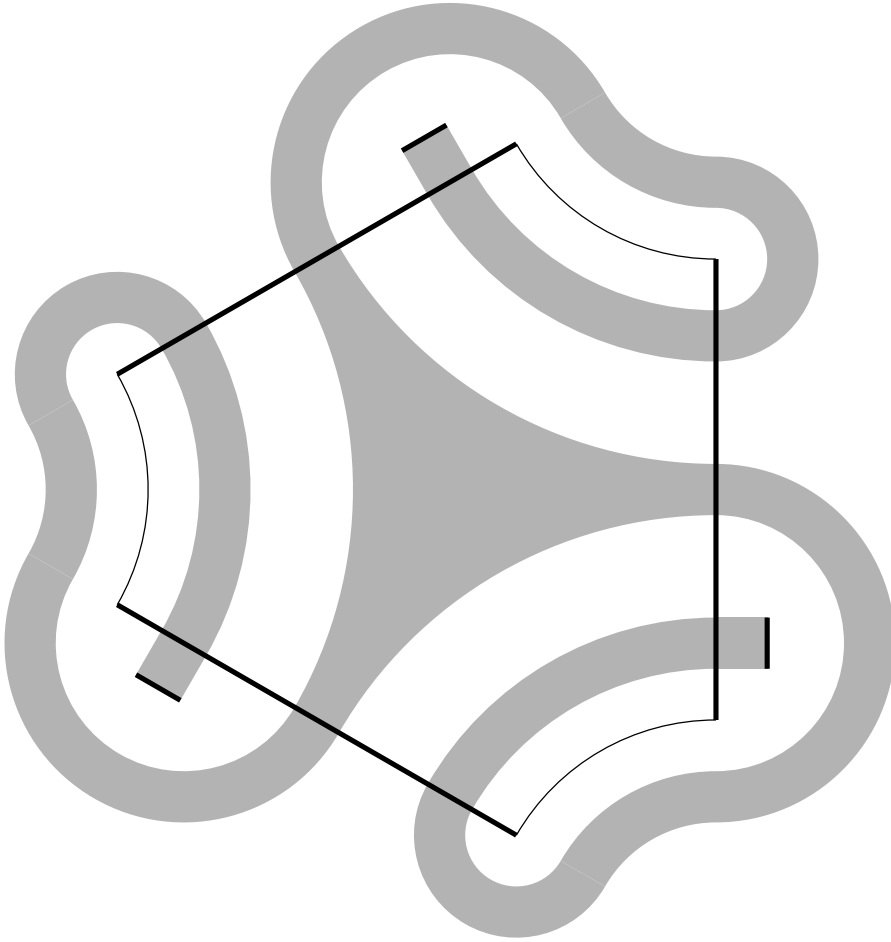
The Lefschetz number of this isolated invariant set is

$$L(S, f) = -\operatorname{trace} H_1(f_{N,L}) = -5.$$

Corrsepondingly the set S contains three fixed points of index -1 and one (in the centre) of index -2.

Example 5.5 : A heteroclinic orbit. Consider two hyperbolic fixed points in the plane where the stable manifold of one intersects the unstable manifold of the other transversally. Let S be the invariant set consisting of the two fixed points together with one connecting orbit. Then for any $m \geq 2$ there exists an index pair (N, L) for S such that $L \subset \partial N$ and N consists of m components (see Figure 6 for $m = 3$). So in this case the index homology is trivial in dimensions $j \neq 1$ and $H_1(N, L) = \mathbb{Z}^m$. The induced map is given by

$$H_1(f_{N,L}) = \begin{pmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & 1 & 0 \\ \cdot & & & & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix}.$$
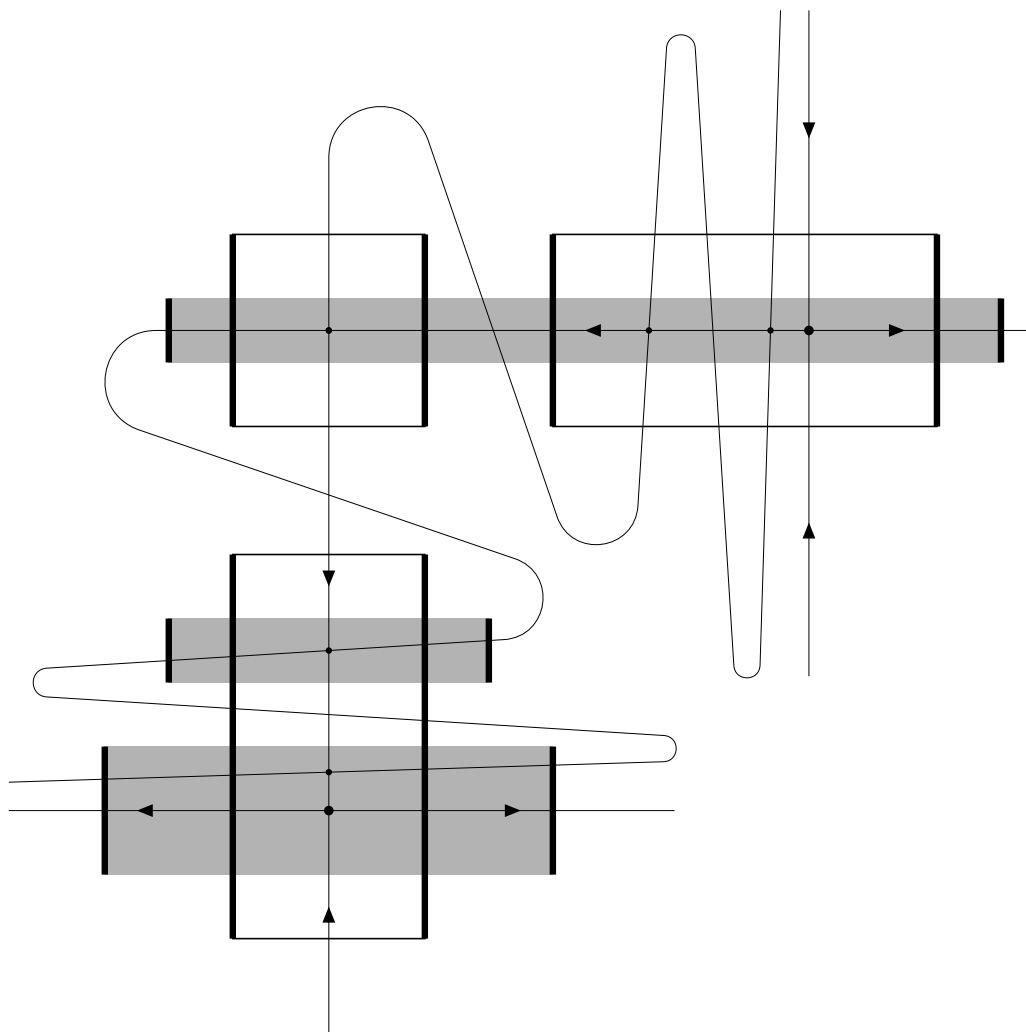
Figure 5

Thus the zeta function

$$\zeta_1(t; S) = \frac{1}{(1-t)^2}$$

is the same as for the invariant set consisting only of the two fixed points. However, the connecting orbit is represented by the off-diagonal entries in the induced map $H_1(f_{N,L})$. Also

$$\mathcal{H}_1(N, L) = \mathbb{Z} \oplus \mathbb{Z}, \qquad \mathcal{H}_1(f_{N,L}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this example there is another connecting orbit not belonging to S. The two connecting orbits can be cancelled by a suitable perturbation. So the index map for the invariant set S' consisting of the two fixed points together with both connecting orbits will be the identity.

Example 5.6 : A hyperbolic periodic orbit. Consider a hyperbolic periodic orbit S of period p in the plane. Then there is an index pair (N, L) for S with N

Figure 6

having p components (see Figure 7 for $p = 5$). So $H_1(N, L) = \mathbb{Z}^p$ and

$$H_1(f_{N,L}) = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

It follows that

$$\zeta_1(t; S) = \frac{1}{1 - t^p}.$$

If the restriction of the diffeomorphism f^p to the unstable manifold is orientation reversing then
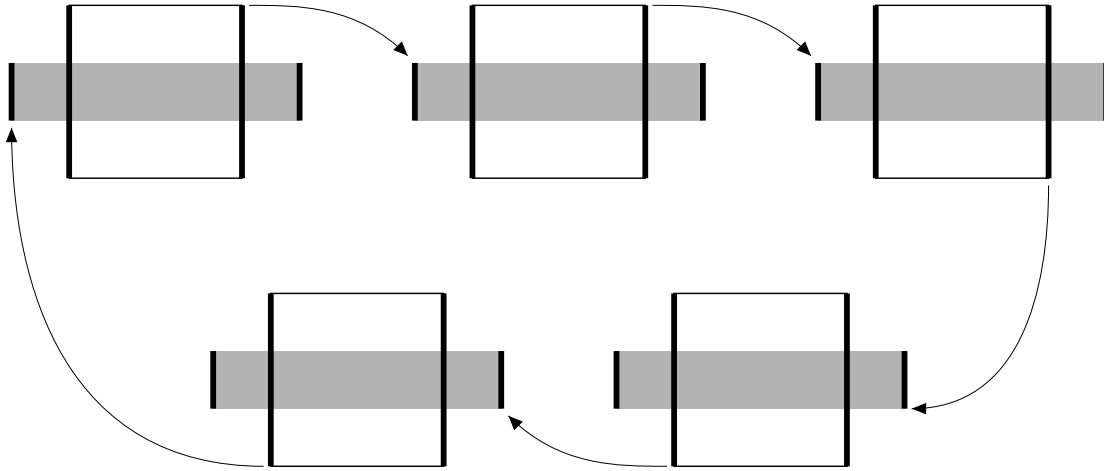
$$\zeta_1(t; S) = \frac{1}{1 + t^p}.$$

Figure 7

Example 5.7 : A homoclinic orbit. Consider a hyperbolic fixed point in the plane whose stable and unstable manifold intersect transversally. The invariant set S consisting of the fixed point together with a homoclinic orbit is not isolated. (It follows from the shadowing lemma that the homoclinic orbit can be approximated by periodic orbits.) However there is a sequence of isolated invariant sets S_m whose intersection is the invariant set S. Each invariant set S_m has an isolating neighborhood N_m consisting of m components (see Figure 8 for $m = 3$).

The associated index homology vanishes in dimensions $j \neq 1$ and $H_1(N, L) = \mathbb{Z}^m$. In a suitable basis the induced map acts by multiplication with the matrix

$$
H_1(f_{N,L}) = \begin{pmatrix}
1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 1 & \cdot & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & 1 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\
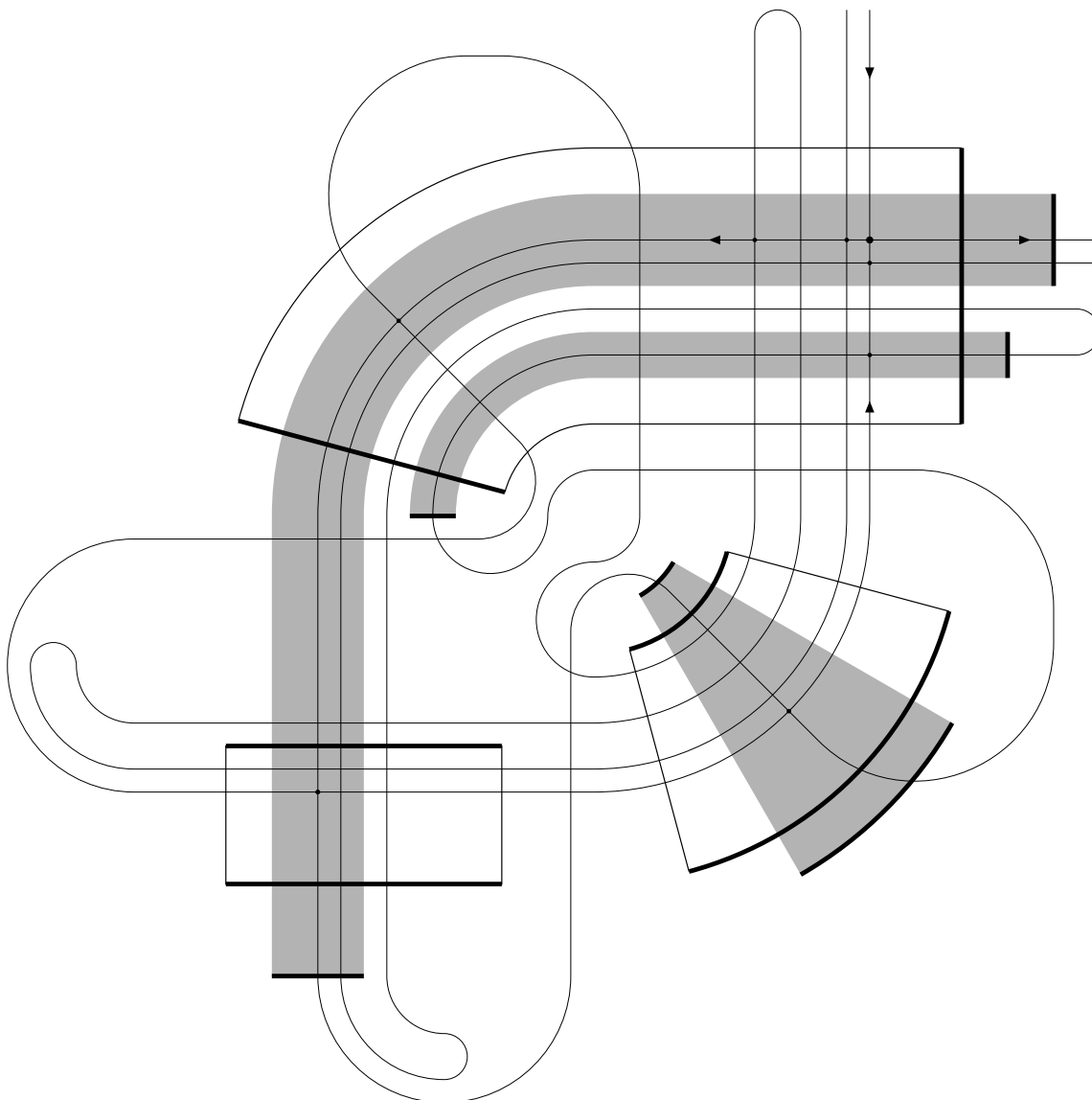-1 & 0 & \cdot & \cdot & \cdot & 0 & 0
\end{pmatrix}.
$$

It follows that

$$
\zeta_1(t; S_m) = \frac{1}{1 - t + t^m}.
$$

A similar phenomenon occurs in the case of two hyperbolic fixed points with heteroclinic orbits in both directions. These occur inside the horseshoes of Example 5.1 and 5.2. If both fixed points have index -1 then

$$
\zeta_1(t; S_m) = \frac{1}{(1 - t)^2 - t^{2m}}.
$$

The case $m = 1$ is precisely Example 5.3.

Figure 8

Example 5.8 : Normally hyperbolic invariant submanifolds. Let $S \subset M$ be a normally hyperbolic invariant submanifold for the diffeomorphism $f : M \to M$. This means that the the restricted tangent bundle splits into 3 invariant subbundles

$$T_S M = E^s \oplus TS \oplus E^u$$

such that df uniformly contracts E^s and expands E^u in a suitable metric. An index pair is then given by

$$N = \{\exp_x(\xi^s + \xi^u) \,|\, x \in S,\, \xi^s \in E^s_x,\, \xi^u \in E^u_x,\, |\xi^s| \leq \varepsilon,\, |\xi^u| \leq \varepsilon\}\,,$$
$$L = \{\exp_x(\xi^s + \xi^u) \in N \,|\, |\xi^u| = \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small. So the index cohomology is given by

$$H^*(N, L) \simeq H^*(E^u, E^u \setminus S).$$

If the submanifold S and the unstable bundle E^u are orientable then there is a Thom isomorphism[7]

$$\Phi : H^*(S) \to H^{*+n_u}(N, L), \qquad \Phi(a) = r^*(a) \cup U, \qquad a \in H^*(S).$$

Here n_u is the fibre dimension of E^u. The smooth map $r : N \to S$ is a retraction, so its restriction to S is the identity. This retraction r is induced by the projection of the restricted tangent bundle $T_S M$ onto its zero section. The homology class $U \in H^{n_u}(N, L)$ is the image of the Thom class in $H^{n_u}(E^u, E^u \setminus S)$ under the above isomorphism.

In [5] Floer proved a local and a global perturbation theorem for normally hyperbolic invariant manifolds. The proof of his local result can be considerably simplified using the discrete time Conley index rather than suspensions.

Theorem 5.9. *Let S be a normally hyperbolic compact invariant submanifold for a diffeomorphism f. Suppose that S and the unstable bundle $E^u \subset T_S M$ are orientable. Let N be an isolating neighborhood for S which admits a retraction $r : N \to S$. Then for every diffeomorphism $f' : M \to M$ sufficiently C^0 close to f the induced map on Alexander cohomology*

$$(r|_{S'})^* : H^*(S) \to H^*(S')$$

is injective. Here S' denotes the maximal invariant set for f' in N.

Proof. In [18] it is shown that an index pair (N, L) for S can be chosen such that it remains an index pair for every diffeomorphism f' sufficiently C^0 close to f. (In fact this is the case for the index pair constructed above.) Define $W' = W^u(S', N)$. The crucial fact is that the inclusion induced map $H^*(W') \to H^*(S')$ is an isomorphism due to the continuity property of Alexander cohomology. In view of the commuting diagram

$$
\begin{array}{ccc}
H^*(S) & \xrightarrow{(r|_{S'})^*} & H^*(S') \\
r^* \downarrow & & \uparrow \simeq \\
\end{array}
$$

$$
H^*(N, W') \xrightarrow{\quad j^* \quad} H^*(N) \xrightarrow{\quad i^* \quad} H^*(W')
$$

it remains to be shown that $i^* \circ r^*$ is injective. Let $a \in H^*(S)$ such that

$$i^* \circ r^*(a) = 0.$$

By exactness there exists a $b \in H^*(N, W')$ such that $r^*(a) = j^*(b)$. Thus

$$r^*(a) \cup U = j^*(b) \cup U = k^*(b \cup U)$$

where $k^* : H^*(N, W' \cup L) \to H^*(N)$. But it follows again from the continuity property of Alexander cohomology that $H^*(N, W' \cup L) = \{0\}$. Thus $\Phi(a) = r^*(a) \cup U = 0$ and hence $a = 0$. $\square$

The proof of Floer's global perturbation theorem in [5] can be simplified along similar lines. We will not carry out this argument.

Theorem 5.9 cannot be generalized to singular cohomology. An example is an attracting circle of fixed points in the plane which after a perturbation degenerates into a Warsaw circle.

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[7]Even without the orientability assumption this remains valid for $\mathbb{Z}_2$-coefficients.

Example 5.10 : Attractor-repeller pairs for gradient flows. Let $g : M \to \mathbb{R}$ be a Morse function on a compact Riemannian manifold. Choose a metric on M such that the gradient flow

$$\dot{x} = -\nabla g(x)$$

is a Morse Smale flow. This means that the stable and unstable manifolds $W^u(y)$ and $W^s(x)$ intersect transversally for any two critical points x and y of g. So in particular $W^u(y) \cap W^s(x)$ is empty if $\mathrm{ind}(y) \leq \mathrm{ind}(x)$ and consists of finitely many orbits if $\mathrm{ind}(y) - \mathrm{ind}(x) = 1$. Consider the latter case and let S consist of the points x, y together with some finite set of connecting orbits. Then $A = \{x\}$ and $A^* = \{y\}$ form an attractor-repeller pair in S. Let $L \subset N_x \subset N_y$ be an index triple. If $\mathrm{ind}(x) = j$ then $H_j(N_x, L) \simeq H_{j+1}(N_y, N_x) \simeq \mathbb{Z}$ and the boundary map $\partial : H_{j+1}(N_y, N_x) \to H_j(N_x, L)$ is given by the number of connecting orbits, counted with appropriate signs [6], [12], [20]. Thus the boundary map depends on the invariant set S and not only on the pair (A, A^*).

Example 5.11 : A flow on $\mathbb{R}P^3$. On 3-dimensional real projective space

$$M = \mathbb{R}P^3 = \{[w : x : y : z] \,|\, w, x, y, z \in \mathbb{R}\}$$

consider the flow induced by the linear differential equation

$$\dot{w} = w$$
$$\dot{x} = y$$
$$\dot{y} = -x$$
$$\dot{z} = -z$$

on R^4. In this flow the z-achsis is a repelling fixed point Λ_2 while the w-achsis is an attracting fixed point Λ_0. The lines in the (x, y)-plane form a hyperbolic periodic orbit Λ_1. The unstable subbundle of $T_{\Lambda_1} M$ is a Moebius strip. Both the stable and unstable manifold form a pointed projective plane, embedded in $\mathbb{R}P^3$. The Poincaré polynomial of Λ_1 depends on the coefficients:

$$p(s; \Lambda_1; \mathbb{Z}) = 0, \qquad p(s; \Lambda_1; \mathbb{Z}_2) = s + s^2.$$

So

$$p(s; M; R) = \sum_{\nu=0}^{2} p(s; \Lambda_\nu; R)$$

both for $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$.

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