Notes on the Teichmüller space of K3

Dietmar Salamon
ETH Zürich

1 May 2018

1 The cotangent bundle of the two-sphere

Identify the cotangent bundle of the 2-sphere with the manifold

$$T^*S^2 := \left\{\xi + \mathbf{i}\eta \in \mathbb{C}^3 \,|\, |\xi| = 1,\, \langle \xi, \eta \rangle = 0\right\}. \tag{1.1}$$

The canonical symplectic form on T^*S^2 is the restriction of the standard symplectic form $\omega_0 = \sum_{i=1}^3 d\xi_i \wedge d\eta_i$ on $\mathbb{C}^3$ to T^*S^2.

Lemma 1.1 (**Dehn Twist**). *Define the map $\tau : T^*S^2 \to T^*S^2$ by*

$$\tau(\xi, \eta) := (\xi', \eta')$$

for $(\xi, \eta) \in T^*S^2$, where $|\eta'| = |\eta|$, $\xi' := -\xi$ in the case $\eta = 0$, and

$$\xi' + \mathbf{i}\frac{\eta'}{|\eta'|} := -\exp\left(-\frac{2\pi\mathbf{i}|\eta|}{\sqrt{1+4|\eta|^2}}\right)\left(\xi + \mathbf{i}\frac{\eta}{|\eta|}\right) \tag{1.2}$$

in the case $\eta \neq 0$. Then τ is a symplectomorphism (called the **Dehn twist***).*

Proof. Equation (1.2) can be written in the form

$$\begin{aligned}
\xi' &= -\cos\left(\frac{2\pi|\eta|}{\sqrt{1+4|\eta|^2}}\right)\xi - \sin\left(\frac{2\pi|\eta|}{\sqrt{1+4|\eta|^2}}\right)\frac{\eta}{|\eta|}, \\
\eta' &= \sin\left(\frac{2\pi|\eta|}{\sqrt{1+4|\eta|^2}}\right)|\eta|\xi - \cos\left(\frac{2\pi|\eta|}{\sqrt{1+4|\eta|^2}}\right)\eta.
\end{aligned} \tag{1.3}$$

This shows that τ is a diffeomorphism. Now abbreviate

$$f := \frac{\pi}{2\sqrt{1+4|\eta|^2}}, \quad \theta := \frac{2\pi|\eta|}{\sqrt{1+4|\eta|^2}}, \quad df = -\frac{2\pi\sum_{i=1}^3 \eta_i d\eta_i}{(1+4|\eta|^2)^{3/2}} = -|\eta| d\theta.$$

Then $\eta_i' = |\eta|\sin(\theta)\xi_i - \cos(\theta)\eta_i$ and $\xi_i' = -\cos(\theta)\xi_i - |\eta|^{-1}\sin(\theta)\eta_i$. Thus

$$d\xi_i' = -\cos(\theta)d\xi_i + \sin(\theta)\frac{\sum_j(\eta_i\eta_j d\eta_j - \eta_j^2 d\eta_i)}{|\eta|^3} + \xi_i\sin(\theta)d\theta - \frac{\eta_i\cos(\theta)}{|\eta|}d\theta.$$

Using $\sum_i \xi_i^2 - 1 = \sum_i \xi_i\eta_i = 0$ and $\sum_i \xi_i d\xi_i = \sum_i\big(\xi_i d\eta_i + \eta_i d\xi_i\big) = 0$ we find

$$\begin{aligned}
\sum_i \eta_i' d\xi_i' &= \sin^2(\theta)\frac{\sum_{i,j}(\xi_i\eta_i\eta_j d\eta_j - \eta_j^2\xi_i d\eta_i)}{|\eta|^2} + |\eta|\sin^2(\theta)d\theta \\
&\quad + \cos^2(\theta)\sum_i \eta_i d\xi_i + |\eta|\cos^2(\theta)d\theta \\
&= \cos^2(\theta)\sum_i \eta_i d\xi_i - \sin^2(\theta)\sum_i \xi_i d\eta_i + |\eta|d\theta \\
&= \sum_i \eta_i d\xi_i - df
\end{aligned}$$

Thus the difference $\sum_i \eta_i' d\xi_i' - \sum_i \eta_i d\xi_i$ is exact and so $\tau : T^*S^2 \to T^*S^2$ is a symplectomorphism. This proves Lemma 1.1. $\square$

Lemma 1.2. *The set*

$$X := \left\{ z = x + \mathbf{i}y \in \mathbb{C}^3 \,\middle|\, |x|^2 - |y|^2 = 1,\ \langle x, y\rangle = 0 \right\}, \tag{1.4}$$

is a complex submanifold of $\mathbb{C}^3$*, the map* $\iota : X \to T^*S^2$ *defined by*

$$\iota(x + \mathbf{i}y) := |x|^{-1}x + \mathbf{i}|x|y \tag{1.5}$$

for $x + \mathbf{i}y \in X$ *is a symplectomorphism with the inverse*

$$\iota^{-1}(\xi + \mathbf{i}\eta) = \lambda\xi + \mathbf{i}\lambda^{-1}\eta, \qquad \lambda := \sqrt{\tfrac{1}{2}\left(1 + \sqrt{1+4|\eta|^2}\right)}, \tag{1.6}$$

and the symplectomorphism $\phi := \iota^{-1}\circ\tau\circ\iota$ *of* X *is given by* $\phi(x,y) = (x', y')$ *for* $z = x + \mathbf{i}y \in X$*, where* $x' = -x$*,* $y' = 0$ *in the case* $y = 0$ *and*

$$\frac{x'}{|x|} + \mathbf{i}\frac{y'}{|y|} = -\exp\left(-\frac{2\pi\mathbf{i}|x||y|}{|x|^2+|y|^2}\right)\left(\frac{x}{|x|} + \mathbf{i}\frac{y}{|y|}\right) \tag{1.7}$$

in the case $y \neq 0$.

Proof. We prove that the map $\iota : X \to T^*S^2$ in (1.5) is a diffeomorphism with the inverse given by (1.6). Let $x + \mathbf{i}y \in X$ with $y \neq 0$ and let

$$\xi + \mathbf{i}\eta := \iota(x + \mathbf{i}y) \in \mathbb{C}^3$$

be given by (1.5) so that $\xi = |x|^{-1}x$ and $\eta = |x|y$. Then

$$|\xi| = 1, \qquad \langle \xi, \eta \rangle = \langle x, y \rangle = 0$$

and so $\xi + \mathbf{i}\eta \in T^*S^2$. Moreover, $|\eta| = |x||y|$ and hence

$$\begin{aligned} 1 + 4|\eta|^2 &= 1 + 4|x|^2|y|^2 \\ &= 1 + 4|y|^2 + 4|y|^4 \\ &= (1 + 2|y|^2)^2 \\ &= (|x^2| + |y|^2)^2. \end{aligned} \tag{1.8}$$

This shows that

$$\lambda = \sqrt{\tfrac{1}{2}\left(1 + \sqrt{1 + 4|\eta|^2}\right)} = \sqrt{\tfrac{1}{2}\left(1 + |x|^2 + |y|^2\right)} = |x|$$

in (1.5). Thus the map $\iota : X \to T^*S^2$ is bijective and its inverse is given by (1.6). Moreover, both ι and ι^{-1} are smooth and so ι is a diffeomorphism. That ι is a symplectomorphism follows from the identity

$$\sum_i \eta_i d\xi_i = \sum_i |x| y_i d\frac{x_i}{|x|} = \sum_i y_i dx_i.$$

Here the last equation holds because $\sum_i y_i x_i = 0$ on X.

Now let $\phi : X \to \mathbb{C}^3$ be the map defined by (1.7). Let $x + \mathbf{i}y \in X$ and let $x' + \mathbf{i}y' := \phi(x + \mathbf{i}y) \in \mathbb{C}^3$. We prove first that

$$|x'| = |x|, \qquad |y'| = |y|, \qquad x' + \mathbf{i}y' \in X.$$

In the case $y = 0$ this follows directly from the definition. Thus assume $y \neq 0$. Since the vectors $|x|^{-1}x$ and $|y|^{-1}y$ in $\mathbb{R}^3$ are orthonormal it follows from (1.7) that the vectors $|x|^{-1}x'$ and $|y|^{-1}y'$ are also orthonormal. This implies

$$\langle x', y' \rangle = 0, \qquad |x'| = |x|, \qquad |y'| = |y|,$$

hence $|x'|^2 - |y'|^2 = 1$, and so $x' + \mathbf{i}y' \in X$.

We prove that $\iota \circ \phi = \tau \circ \iota$. Let $x + \mathbf{i}y \in X$ with $y \neq 0$ and define

$$x' + \mathbf{i}y' := \phi(x + \mathbf{i}y), \qquad \xi + \mathbf{i}\eta := \iota(x + \mathbf{i}y), \qquad \xi' + \mathbf{i}\eta' := \iota(x' + \mathbf{i}y').$$

Then $\xi = |x|^{-1}x$ and $\eta = |x|y$ and so it follows from (1.8) that

$$\alpha := \exp\left(-\frac{2\pi\mathbf{i}|\eta|}{\sqrt{1+4|\eta|^2}}\right) = \exp\left(-\frac{2\pi\mathbf{i}|x||y|}{|x|^2+|y|^2}\right).$$

Since $\xi' + \mathbf{i}\eta' = \iota(x' + \mathbf{i}y')$ and $(x' + \mathbf{i}y') = \phi(x + \mathbf{i}y)$, this implies

$$|\eta'| = |x'||y'| = |x||y| = |\eta| \tag{1.9}$$

and

$$\xi' + \mathbf{i}\frac{\eta'}{|\eta'|} = \frac{x'}{|x'|} + \mathbf{i}\frac{y'}{|y'|} = -\alpha\left(\frac{x}{|x|} + \mathbf{i}\frac{y}{|y|}\right) = -\alpha\left(\xi + \mathbf{i}\frac{\eta}{|\eta|}\right). \tag{1.10}$$

Here the second equality follows from (1.7). It follows from equations (1.9) and (1.10) and the definition of τ in (1.2) that $\xi' + \mathbf{i}\eta' = \tau(\xi + \mathbf{i}\eta)$. Thus

$$\iota \circ \phi(x + \mathbf{i}y) = \xi' + \mathbf{i}\eta' = \tau(\xi + \mathbf{i}\eta) = \tau \circ \iota(x + \mathbf{i}y)$$

for all $x + \mathbf{i}y \in X$ with $y \neq 0$. So $\iota \circ \phi = \tau \circ \iota$ and this proves Lemma 1.2. $\square$

Remark 1.3. The manifold X in Lemma 1.2 is the regular fiber $X = \pi^{-1}(1)$ of the Lefschetz fibration $\pi : \mathbb{C}^3 \to \mathbb{C}$ given by

$$\pi(z) := z_1^2 + z_2^2 + z_3^2$$

for $z = (z_1, z_2, z_3) \in \mathbb{C}^3$. The Dehn twist $\phi : X \to X$ in (1.7) is the monodromy around the unit circle in this Lefschetz fibration. More precisely, the parallel transport diffeomorphisms $\phi_t : \pi^{-1}(1) \to \pi^{-1}(e^{2\pi\mathbf{i}t})$ are given by

$$\begin{aligned} \phi_t(x + \mathbf{i}y) &= e^{\pi\mathbf{i}t}(u(t) + \mathbf{i}v(t)) \\ \frac{u(t)}{|x|} + \mathbf{i}\frac{v(t)}{|y|} &= \exp\left(-\frac{2\pi\mathbf{i}t|x||y|}{|x|^2+|y|^2}\right)\left(\frac{x}{|x|} + \mathbf{i}\frac{y}{|y|}\right) \end{aligned} \tag{1.11}$$

for $t \in \mathbb{R}$ and $x + \mathbf{i}y \in X$ with $y \neq 0$. This can be seen by noting that the function $w(t) := e^{-\pi\mathbf{i}t}z(t)$ satisfies the equation $\dot{w} = \pi\mathbf{i}(-w + \lambda\overline{w})$, where the coefficient $\lambda := |z(t)|^{-2}$ is independent of t. For $t = 1$ one obtains the diffeomorphism $\phi = \phi_1 : X \to X$ in Lemma 1.2. We emphasize that ϕ is a symplectomorphism but is not holomorphic.

A third model for the cotangent bundle of the 2-sphere is the total space of the second tensor power of the tautological line bundle over $\mathbb{C}\mathrm{P}^1$ or, equivalently, the resolution of the singularity $x^2+y^2+z^2=0$ in $\mathbb{C}^3$. Define

$$
\begin{aligned}
Z &:= \left\{ (x,y,z,[a:b]) \in \mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1 \,\middle|\, \begin{array}{l} x^2+y^2+z^2=0, \\ b(x+\mathbf{i}y)-az=0, \\ a(x-\mathbf{i}y)+bz=0 \end{array} \right\} \\
&= \left\{ (x,y,z,[a:b]) \in \mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1 \,\middle|\, \begin{array}{l} \exists\, \lambda,\mu \in \mathbb{C} \text{ such that} \\ x+\mathbf{i}y=\lambda a,\; z=\lambda b, \\ x-\mathbf{i}y=\mu b,\; -z=\mu a \end{array} \right\}.
\end{aligned}
\tag{1.12}
$$

This is a complex submanifold of $\mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1$ and hence it inherits a natural Kähler structure from the ambient manifold (with the standard symplectic form on $\mathbb{C}^3$ and the Fubini–Study form on $\mathbb{C}\mathrm{P}^1$). However, in contrast to the manifolds T^*S^2 in (1.1) and X in (1.4) where the zero section ($\eta=0$ in (1.1) and $y=0$ in (1.4)) is a Lagrangian submanifold, the zero section $x=y=z=0$ in the manifold Z in (1.12) is a holomorphic sphere with self-intersection number -2. Stereographic projection gives rise to an explicit diffeomorphism from Z to T^*S^2.

Lemma 1.4. *For $(x,y,z,[a:b]) \in Z$ define the pair*

$$
\jmath(x,y,z,[a:b]) := (\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3
$$

by

$$
\xi := \frac{1}{|a|^2+|b|^2} \begin{pmatrix} 2\mathrm{Re}(a\bar{b}) \\ 2\mathrm{Im}(a\bar{b}) \\ |b|^2-|a|^2 \end{pmatrix}, \qquad \eta := \begin{pmatrix} \mathrm{Im}(x) \\ \mathrm{Im}(y) \\ \mathrm{Im}(z) \end{pmatrix}. \tag{1.13}
$$

Then $\jmath : Z \to T^*S^2$ is an orientation preserving diffeomorphism. Its inverse is given by $\jmath^{-1}(\xi,\eta) = (x,y,z;[\xi_1+\mathbf{i}\xi_2 : 1+\xi_3])$ with $(x,y,z) = -\xi\times\eta + \mathbf{i}\eta$.

Proof. The square of the tautological line bundle over $\mathbb{C}\mathrm{P}^1$ is the quotient

$$
E := \left\{ (a,b,w) \in \mathbb{C}^3 \,|\, (a,b) \neq (0,0) \right\}/\!\sim, \qquad (a,b,w) \sim (\lambda a, \lambda b, \lambda^{-2} w).
$$

Denote the equivalence class of a triple (a,b,w) under the action of $\mathbb{C}^*$ by $[a:b;w] := \{(\lambda a,\lambda b,\lambda^{-2}w) \,|\, \lambda \in \mathbb{C}^*\}$. The line bundle E is diffeomorphic to Z via the diffeomorphism $[a:b;w] \mapsto (x,y,z,[a:b])$ given by

$$
x = \frac{b^2-a^2}{2}\mathbf{i}w, \qquad y = -\frac{a^2+b^2}{2}w, \qquad z = -\mathbf{i}abw. \tag{1.14}
$$

Note that $b(x+\mathbf{i}y) = -a^2b\mathbf{i}w = az$ and $a(x-\mathbf{i}y) = ab^2\mathbf{i}w = -bz$.

Now think of the complex plane as a subspace of the space of quaternions and denote by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the standard generators of the imaginary quaternions so that $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. For $(a, b) \in \mathbb{C}^2 \setminus \{0\}$ define the quaternion $q \in \mathbb{H}$ by

$$q := q(a,b) := \tfrac{1}{2}\Big((\bar{a} + \bar{b})(1 - \mathbf{i}) + (a - b)(\mathbf{j} + \mathbf{k})\Big). \tag{1.15}$$

Then a calculation shows that $|q|^2 = |a|^2 + |b|^2$ and

$$\begin{aligned} q\mathbf{i}\bar{q} &= 2\mathrm{Re}\big(a\bar{b}\big)\mathbf{i} + 2\mathrm{Im}\big(a\bar{b}\big)\mathbf{j} + \big(|b|^2 - |a|^2\big)\mathbf{k}, \\ q\mathbf{j}\bar{q} &= \mathrm{Re}\big(b^2 - a^2\big)\mathbf{i} - \mathrm{Im}\big(a^2 + b^2\big)\mathbf{j} - 2\mathrm{Re}\big(ab\big)\mathbf{k}, \\ q\mathbf{k}\bar{q} &= \mathrm{Im}\big(b^2 - a^2\big)\mathbf{i} + \mathrm{Re}\big(a^2 + b^2\big)\mathbf{j} - 2\mathrm{Im}\big(ab\big)\mathbf{k}. \end{aligned}$$

When $(a, b) \in \mathbb{C}^2$ is a unit vector, the coordinates of $q\mathbf{i}\bar{q}$ define the element of S^2 that correspond to the point a/b in the Riemann sphere under the stereographic projection from the south pole, and the vectors $q\mathbf{j}\bar{q}$ and $-q\mathbf{k}\bar{q}$ form a positive orthonormal basis of the cotangent space of S^2 at the point $\xi = q\mathbf{i}\bar{q}$. Thus a complex number $w = s + \mathbf{i}t$ determines a tangent vector

$$\begin{aligned} \eta &:= \tfrac{1}{2}q(a,b)\big(w\mathbf{j}\big)\overline{q(a,b)} \\ &= \tfrac{1}{2}q(a,b)\big(s\mathbf{j} - t\mathbf{k}\big)\overline{q(a,b)} \\ &= \mathrm{Re}\big(\tfrac{1}{2}(b^2 - a^2)w\big)\mathbf{i} - \mathrm{Im}\big(\tfrac{1}{2}(a^2 + b^2)w\big)\mathbf{j} - \mathrm{Re}\big(abw\big)\mathbf{k}. \end{aligned}$$

This gives rise to a vector bundle isomorphism from E to T^*S^2 which covers the inverse of the stereographic projection $S^2 \to \mathbb{C}\mathrm{P}^1$ and sends an element $[a : b; w]$ to the pair $(\xi, \eta) = (\xi(a, b), \eta(a, b; w)) \in T^*S^2$, defined by

$$\xi := \frac{1}{|a|^2 + |b|^2}\begin{pmatrix} 2\mathrm{Re}(a\bar{b}) \\ 2\mathrm{Im}(a\bar{b}) \\ |b|^2 - |a|^2 \end{pmatrix}, \qquad \eta := \begin{pmatrix} \mathrm{Re}\big(\frac{1}{2}(b^2 - a^2)w\big) \\ -\mathrm{Im}\big(\frac{1}{2}(a^2 + b^2)w\big) \\ -\mathrm{Re}\big(abw\big) \end{pmatrix}.$$

If $(x, y, z, [a : b])$ is the element of Z corresponding to the point $[a : b; w] \in E$ under the diffeomorphism in (1.14), then $\eta(a, b; w) = (\mathrm{Im}(x), \mathrm{Im}(y), \mathrm{Im}(z))$. The inverse map sends a point $(\xi, \eta) \in T^*S^2$ to $(x, y, z, [a : b]) \in Z$ with

$$\begin{aligned} [a : b] &= [\xi_1 + \mathbf{i}\xi_2 : 1 + \xi_3], \\ x &= \xi_3\eta_2 - \xi_2\eta_3 + \mathbf{i}\eta_1, \\ y &= \xi_1\eta_3 - \xi_3\eta_1 + \mathbf{i}\eta_2, \\ z &= \xi_2\eta_1 - \xi_1\eta_2 + \mathbf{i}\eta_3. \end{aligned} \tag{1.16}$$

This proves Lemma 1.4. $\square$

2 The Atiyah flop

The set

$$\mathcal{X} := \left\{(x,y,z,t) \in \mathbb{C}^4 \,|\, x^2+y^2+z^2+t=0\right\} \tag{2.1}$$

is a complex submanifold of $\mathbb{C}^4$, holomorphically diffeomorphic to $\mathbb{C}^3$, the projection $\pi : \mathcal{X} \to \mathbb{C}$ given by $\pi(x,y,z,t) := t$ is a Lefschetz fibration, the fiber over $t=1$ is the manifold X in (1.4), and the monodromy around the unit circle is the Dehn twist in Lemma 1.1. Now consider the singular variety

$$\mathcal{S} := \left\{(x,y,z,t) \in \mathbb{C}^4 \,\middle|\, x^2+y^2+z^2+t^2=0\right\}. \tag{2.2}$$

Blow up the origin to obtain a smooth manifold

$$\mathcal{Z} := \left\{(x,y,z,t,[a:b]) \in \mathbb{C}^4 \times \mathbb{C}\mathrm{P}^1 \,\middle|\, \begin{array}{c} b(x+\mathbf{i}y)-a(z+\mathbf{i}t)=0, \\ a(x-\mathbf{i}y)+b(z-\mathbf{i}t)=0 \end{array}\right\}. \tag{2.3}$$

Lemma 2.1. *The projection*

$$\mathcal{Z} \to \mathbb{C} : (x,y,z,t,[a:b]) \mapsto \pi(x,y,z,t,[a:b]) := t \tag{2.4}$$

is a holomorphic submersion.

Proof. If (x,y,z,t) is a nonzero vector in S, then one of the complex numbers x,y,z is nonzero. If $x \neq 0$, then the vector $(\widehat{x},\widehat{y},\widehat{z},\widehat{t})$ with

$$\widehat{x} = -\frac{t\widehat{t}}{x}, \qquad \widehat{y} = \widehat{z} = 0$$

is tangent to $\mathcal{S}$ at (x,y,z,t) and projects onto $\widehat{t}$ under the derivative of π. If $b \neq 0$, then the curve $(x(t),y(t),z(t),t,[a:b]) \in U$ with

$$x(t) = \frac{\mathbf{i}at}{b}, \qquad y(t) = \frac{at}{b}, \qquad z(t) = \mathbf{i}t$$

passes through $(0,0,0,0,[a:b])$ and satisfies $x(t)^2+y(t)^2+z(t)^2+t^2=0$ as well as $b(x+\mathbf{i}y)-a(z+\mathbf{i}t)=0$ and $a(x-\mathbf{i}y)+b(z-\mathbf{i}t)=0$. If $a=1$ and $b=0$, then the curve $(x(t),y(t),z(t),t,[1:0]) \in U$ with

$$x(t) = \mathbf{i}t, \qquad y(t) = t, \qquad z(t) = -\mathbf{i}t$$

satisfies the same conditions. This proves Lemma 2.1. $\square$

The central fiber of the fibration $\pi : \mathcal{Z} \to \mathbb{C}$ in (2.3) and (2.4) is the manifold $Z_0 := Z$ in (1.12) which is diffeomorphic to T^*S^2 by Lemma 1.4. By Lemma 2.1 the fibration $\mathcal{Z}$ admits a trivialization $\mathbb{C} \times Z \to \mathcal{Z}$. For $t \in \mathbb{C}\backslash\{0\}$ denote the fiber of $\mathcal{Z}$ over t by

$$Z_t := \left\{(x, y, z) \in \mathbb{C}^3 \,\middle|\, x^2 + y^2 + z^2 + t^2 = 0\right\}. \tag{2.5}$$

Lemma 2.2. *Fix a constant $\varepsilon > 0$. Then there exists a collection of diffeomorphisms $\psi_t : Z \to Z_t$ satisfying the following conditions.*
(I) *The map*

$$\mathbb{C} \times Z \to \mathcal{Z} : (t, (x, y, z, [a; b])) \mapsto (x', y', z', t, [a' : b']) \tag{2.6}$$

defined by $(x', y', z', [a' : b']) := (x, y, z, [a : b])$ for $t = 0$ and by

$$\begin{aligned} (x', y', z') &:= \psi_t(x, y, z, [a : b]), \\ [a' : b'] &:= \begin{cases} [x' + \mathbf{i}y' : z' + \mathbf{i}t], & \text{if } |x' + \mathbf{i}y'|^2 + |z' + \mathbf{i}t|^2 \neq 0, \\ [-z' + \mathbf{i}t : x' - \mathbf{i}y'], & \text{if } |-z' + \mathbf{i}t|^2 + |x' - \mathbf{i}y'|^2 \neq 0, \end{cases} \end{aligned} \tag{2.7}$$

for $t \neq 0$ is a diffeomorphism.
(II) *Let $(x, y, z, [a; b]) \in Z$ such that $|x|^2 + |y|^2 + |z|^2 \geq \varepsilon$ and let $t \in \mathbb{C} \setminus \{0\}$. Define $r := |t|$, choose $\theta \in \mathbb{R}$ such that $t = re^{\mathbf{i}\theta}$, and define*

$$\lambda := \sqrt{\frac{\sqrt{(|x|^2 + |y|^2 + |z|^2)^2 + r^4} + r^2}{|x|^2 + |y|^2 + |z|^2}} > 1. \tag{2.8}$$

Then $(x', y', z') = \psi_t(x, y, z, [a : b])$ is given by

$$(x', y', z') = \frac{\lambda + \lambda^{-1}}{2}(x, y, z) + e^{2\mathbf{i}\theta}\frac{\lambda - \lambda^{-1}}{2}(\bar{x}, \bar{y}, \bar{z}). \tag{2.9}$$

Proof. The map (2.9) is obtained by parallel transport in the fibration $\mathcal{Z}$ in Lemma 2.1 along the paths $r \mapsto re^{\mathbf{i}\theta}$ on the complement of the set

$$\mathcal{C} := \left\{(x, y, z, t, [a : b]) \in \mathcal{Z} \,\middle|\, |x|^2 + |y|^2 + |z|^2 = |t|^2\right\}.$$

Here the fiber Z_t is identified with the fiber X_{t^2} of the fibration $\mathcal{X}$ in (2.1) in the canonical way, the set $\mathcal{C}$ corresponds to the zero sections of the fibers under the identifications with T^*S^2, and parallel transport is understood with respect to the symplectic connection on $\mathcal{X}$ determined by the standard symplectic form on $\mathbb{C}^3$. The proof of Lemma 2.2 then follows by choosing a suitable symplectic connection form on $\mathcal{Z}$ which agrees with the standard symplectic form on $\mathbb{C}^3$ (with the coordinates (x, y, z)) outside of a sufficiently small neighborhood of the sphere $C := \{(0, 0, 0, 0)\} \times \mathbb{CP}^1 \subset \mathcal{Z}$. $\square$

Lemma 2.3. *For $t \in \mathbb{C}\backslash\{0\}$ let $\psi_t : Z \to Z_t$ be the trivialization of Lemma 2.2, let I_t be the standard complex structure on Z_t, and define*

$$J_t := \psi_t^* I_t \in \mathscr{J}_{\mathrm{int}}(Z).$$

Let J be the standard complex structure on Z and let $\tau : Z \to Z$ be a Dehn twist, localized near the (-2)-sphere $C = \{(0,0,0)\} \times \mathbb{C}\mathrm{P}^1 \subset Z$, under the identification of Z with T^*S^2 in Lemma 1.4. Then there exists a smooth family of diffeomorphisms $\mathbb{C} \setminus \{0\} \to \mathrm{Diff}_0(Z) : t \mapsto \phi_t$ with uniform compact support such that, for every $t \in \mathbb{C} \setminus \{0\}$, the diffeomorphism $\phi_t : Z \to Z$ is smoothly isotopic to the identity with uniform compact support and

$$\phi_t^* J_t = \tau^* J_{-t}.$$

Proof. For $t \in \mathbb{C} \setminus \{0\}$ we have $Z_t = Z_{-t}$ and denote by $\iota_t : Z_t \to Z_{-t}$ the identity map, so $\iota_t(x,y,z) = (x,y,z) \in Z_{-t}$ for $(x,y,z) \in Z_t$. We emphasize that the map $(x,y,z,t) \mapsto (x,y,z,-t)$ is a holomorphic diffeomorphism of $\mathcal{S} \setminus \{0\}$ and so induces a holomorphic diffeomorphism of $\mathcal{Z} \setminus C$, however, it does not extend to $\mathcal{Z}$. It follows from equation (2.9) in Lemma 2.2 that

$$\iota_{-t} \circ \psi_{-t} = \psi_t \qquad \text{on } \left\{ (x,y,z,[a:b]) \in Z \,\middle|\, |x|^2 + |y|^2 + |z|^2 \geq \varepsilon \right\}.$$

Thus the diffeomorphism

$$\tau_t := \psi_t^{-1} \circ \iota_{-t} \circ \psi_{-t} : Z \to Z$$

is equal to the identity on the subset $|x|^2 + |y|^2 + |z|^2 \geq \varepsilon$. By Lemma 1.2 and Remark 1.3 it is a Dehn twist, localized near the (-2)-sphere $C \subset Z$. Moreover, for all $t \in \mathbb{C} \setminus \{0\}$, we have $\tau_t \circ \tau_{-t} = \mathrm{id}$ and

$$\tau_t^* J_t = \psi_{-t}^* \iota_{-t}^* (\psi_t^{-1})^* J_t = \psi_{-t}^* \iota_{-t}^* I_t = \psi_{-t}^* I_{-t} = J_{-t}.$$

Now fix an element $t_0 \in \mathbb{C} \setminus \{0\}$ and take

$$\tau := \tau_{t_0}, \qquad \phi_t := \tau_t \circ \tau$$

for $t \in \mathbb{C} \setminus \{0\}$. Then, for every $t \in \mathbb{C} \setminus \{0\}$, we have $\phi_t^* J_t = \tau^* \tau_t^* J_t = \tau^* J_{-t}$ and ϕ_t is smoothly isotopic to the identity. An explicit isotopy with uniform compact support is given by $\phi_{s,t} := \tau_{\gamma_t(s)} \circ \tau_{t_0}$, where $\gamma_t : [0,1] \to \mathbb{C} \setminus \{0\}$ is a smooth curve satisfying $\gamma_t(0) = -t_0$ and $\gamma_t(1) = t$. If τ is any other Dehn twist about C, choose a smooth isotopy $[0,1] \to \mathrm{Diff}(Z) : s \mapsto \psi_s$ with uniform compact support joining $\psi_0 = \tau_{t_0}$ to $\psi_1 = \tau$ and take $\phi_{s,t} := \tau_{\gamma_t(s)} \circ \psi_s$. This proves Lemma 2.3. □

3 Teichmüller space of K3

For an oriented smooth manifold M of even dimension denote by $\mathscr{J}(M)$ the space of almost complex structures that are compatible with the orientation, by $\mathscr{J}_{\mathrm{int},0}(M) \subset \mathscr{J}(M)$ the subspace of integrable almost complex structures with vanishing real first Chern class, and by $\mathrm{Diff}_0(M)$ the group of diffeomorphisms of M that are isotopic to the identity.

Lemma 3.1. *Let M be a K3 surface, i.e. a closed oriented simply connected smooth four-manifold with $\mathscr{J}_{\mathrm{int},0}(M) \neq \emptyset$. Then the Teichmüller space*

$$\mathscr{T}_0(M) := \mathscr{J}_{\mathrm{int},0}(M)/\mathrm{Diff}_0(M)$$

is not Hausdorff.

Proof. Let $J \in \mathcal{J}_{\mathrm{int},0}(M)$ be a complex structure that admits an embedded holomorphic sphere $C \subset M$ with self-intersection number -2. An explicit example (taken from [1]) is the manifold

$$M := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; \begin{array}{l} \sum_{i=1}^3 z_i^2(z_i^2 - z_0^2) = 0, \\ |z_1|^2 + |z_2|^2 + |z_3|^2 \neq 0 \end{array} \right\} \cup U/\sim,$$

$$U := \left\{ (w_1, w_2, w_3, [a:b]) \in \mathbb{C}^3 \times \mathbb{CP}^1 \;\middle|\; \begin{array}{l} (w_1, w_2, w_3) \in W, \\ b(w_1 + \mathbf{i}w_2) - aw_3 = 0, \\ a(w_1 - \mathbf{i}w_2) + bw_3 = 0 \end{array} \right\},$$

where W is the set of all vectors $w = (\zeta_1\sqrt{1-\zeta_1^2}, \zeta_1\sqrt{1-\zeta_1^2}, \zeta_1\sqrt{1-\zeta_1^2})$ in $\mathbb{C}^3$ with $\zeta_i \in \mathbb{C}$ and $\sum_{i=1}^3 |\zeta_i|^2 < 1/2$, and the equivalence relation is given by $[z_0 : z_1 : z_2 : z_3] \cong (w_1, w_2, w_3)$ iff $0 < |z_1|^2 + |z_2|^2 + |z_3|^2 < |z_0|^2/2$ and

$$w_i = \frac{z_i}{z_0}\sqrt{1 - \left(\frac{z_i}{z_0}\right)^2} \qquad \text{for } i = 1, 2, 3.$$

In any such example a neighborhood U of C is holomorphically diffeomorphic to a neighborhood of the curve $C \subset Z$ in Lemma 2.3 by a theorem of Grauert. Let $\tau : M \to M$ denote the Dehn twist about C induced by such a diffeomorphism. Then, by Lemma 2.3, there exists a smooth family of complex structures $\mathbb{C} \to \mathcal{J}_{\mathrm{int},0}(M) : t \mapsto J_t$ and a smooth family of diffeomorphisms $\mathbb{C} \setminus \{0\} \to \mathrm{Diff}_0(M) : t \mapsto \phi_t$ such that $\phi_t^* J_t = \tau^* J_{-t}$ for all $t \in \mathbb{C} \setminus \{0\}$. Thus $\lim_{t\to 0} J_t = J$ and $\lim_{t\to 0} \phi_t^* J_t = \lim_{t\to 0} \tau^* J_{-t} = \tau^* J$. Since the homology class $A := [C] \in H_2(M;\mathbb{Z})$ is effective for J and the class $-A$ is effective for $\tau^* J$, the complex structures J and $\tau^* J$ do not represent the same equivalence class in $\mathscr{T}_0(M)$. This proves Lemma 3.1. $\square$

References

[1] V. Gritsenko & K. Hulek & G.K. Sankaran, Moduli of K3 surfaces and irreducible symplectic manifolds. *Handbook of Moduli, Vol I*, edited by Gavril Farkas and Ian Morrison, Advanced Lectures in Mathematics **24**, International Press, 2013, pp 459–526. `https://arxiv.org/abs/1012.4155v2`

[2] Daniel Huybrechts, Compact hyperKähler manifolds: Basic results. *Invent. Math.* **135** (1999), 63–113. Erratum: *Invent. Math.* **152** (2003), 209–212. `https://arxiv.org/abs/alg-geom/9705025`

[3] Daniel Huybrechts, *Lectures on K3 Surfaces.* Cambridge Studies in Adv. Math. **158**, CUP, 2016.

[4] Misha Verbitsky, Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Mathematical Journal* **162** (2013), 2929–2986. `https://arxiv.org/abs/0908.4121`

[5] Misha Verbitsky, Teichmüller spaces, ergodic theory and global Torelli theorem. *Proceedings of ICM* (2014), No. 2, 793–811. `https://arxiv.org/abs/1404.3847`