Notes on complex Lie groups

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1 Complex Lie groups

Lemma 1.1. *Let* G *be a connected Lie group and* $A : \mathfrak{g} \to \mathfrak{g}$ *be a linear map on its Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$*. Then the following are equivalent.*

(i) *For all* $\xi \in \mathfrak{g}$ *and* $g \in \mathrm{G}$ *we have* $A(g\xi g^{-1}) = g(A\xi)g^{-1}$.

(ii) *For all* $\xi, \eta \in \mathfrak{g}$ *we have* $A[\xi, \eta] = [A\xi, \eta] = [\xi, A\eta]$.

Proof. To prove that (i) implies (ii) differentiate the itentity

$$A(\exp(t\xi)\eta\exp(-t\xi)) = \exp(t\xi)(A\eta)\exp(-t\xi)$$

with respect to t at $t = 0$. To prove the converse choose a path $g : [0, 1] \to \mathrm{G}$ such that $g(0) = \mathbb{1}$ and an element $\xi \in \mathrm{G}$. Define the maps $\eta, \zeta : [0, 1] \to \mathfrak{g}$ by $\eta(t) := g(t)^{-1}\xi g(t)$ and $\zeta(t) := g(t)^{-1}(A\xi)g(t)$. Then

$$\partial_t(A\eta) + [g^{-1}\dot{g}, A\eta] = 0, \qquad \partial_t\zeta + [g^{-1}\dot{g}, \zeta], \qquad A\eta(0) = A\xi = \zeta(0).$$

Here the first equation follows from (ii). It follows that $A\eta(t) = \zeta(t)$ for all t. This proves Lemma 1.1. □

Definition 1.2. A **complex Lie group** is a Lie group G equipped with the structure of a complex manifold such that the structure maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g, h) \mapsto gh, \qquad \mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$$

are holomorphic.

Proposition 1.3. *Let* G *be a connected Lie group. Assume that the Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is equipped with a complex structure* $\mathfrak{g} \to \mathfrak{g} : \xi \mapsto \mathbf{i}\xi$ *and define the almost complex structure* J *on* G *by*

$$J_g v := (\mathbf{i}vg^{-1})g$$

for $v \in T_g\mathrm{G}$*. Then the following are equivalent.*

(i) (G, J) *is a complex Lie group.*

(ii) *The Lie bracket* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ *is complex bilinear, i.e.*

$$[\mathbf{i}\xi, \eta] = [\xi, \mathbf{i}\eta] = \mathbf{i}[\xi, \eta]$$

for all $\xi, \eta \in \mathfrak{g}$.

Proof. For $\xi \in \mathfrak{g}$ define the vector fields $X_\xi, Y_\xi \in \mathrm{Vect}(\mathrm{G})$ by

$$X_\xi(g) := \xi g, \qquad Y_\xi(g) := g\xi.$$

We prove that

$$JX_\xi = X_{\mathbf{i}\xi}, \qquad (\mathcal{L}_{X_\eta}J)X_\xi = X_{[\mathbf{i}\xi,\eta]-\mathbf{i}[\xi,\eta]}, \qquad \mathcal{L}_{Y_\xi}J = 0. \tag{1}$$

Here the first equation is obvious from the definitions. The second equation follows from the first and the identities $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ and

$$(\mathcal{L}_{X_\eta}J)X_\xi = \mathcal{L}_{X_\eta}(JX_\xi) - J\mathcal{L}_{X_\eta}X_\xi = [JX_\xi, X_\eta] - J[X_\xi, X_\eta]$$

for $\xi, \eta \in \mathfrak{g}$. To prove the last equation in (1) note that

$$J_{gh}(vh) = (J_g v)h$$

for all $g, h \in \mathrm{G}$ and $v \in T_g\mathrm{G}$. Hence the diffeomorphism $\mathrm{G} \to \mathrm{G} : g \mapsto gh$ is holomorphic for every $h \in \mathrm{G}$. Differentiating with respect to h gives $\mathcal{L}_{Y_\xi}J = 0$ for every $\xi \in \mathfrak{g}$. Thus we have proved (1).

That (i) implies (ii) follows immediately from (1). Conversely assume (ii) and denote by N_J the Nijenhuis tensor of J. Then, for all $\xi, \eta \in \mathfrak{g}$,

$$\begin{aligned}
N_J(X_\xi, X_\eta) &= [X_\xi, X_\eta] + J[JX_\xi, X_\eta] + J[X_\xi, JX_\eta] - [JX_\xi, JX_\eta] \\
&= [X_\xi, X_\eta] + J[X_{\mathbf{i}\xi}, X_\eta] + J[X_\xi, X_{i\eta}] - [X_{\mathbf{i}\xi}, X_{\mathbf{i}\eta}] \\
&= X_{[\xi,\eta]} + JX_{[\mathbf{i}\xi,\eta]} + JX_{[\xi,\mathbf{i}\eta]} - X_{[\mathbf{i}\xi,\mathbf{i}\eta]} \\
&= X_{[\xi,\eta]+\mathbf{i}[\mathbf{i}\xi,\eta]+\mathbf{i}[\xi,\mathbf{i}\eta]-[\mathbf{i}\xi,\mathbf{i}\eta]} \\
&= 0.
\end{aligned}$$

Here the second and fourth equations follow from (1) and the last equation follows from (ii). Since the vector fields X_ξ span the tangent bundle this shows that $N_J = 0$ and so J is integrable. By Lemma 1.1 it follows also from (ii) that $g^{-1}(\mathbf{i}\xi)g = \mathbf{i}(g^{-1}\xi g)$ for all $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$ and hence

$$J_g v := (\mathbf{i}vg^{-1})g = g(\mathbf{i}g^{-1}v) \tag{2}$$

for $g \in \mathrm{G}$ and $v \in T_g\mathrm{G}$. This implies that the multiplication map is holomorphic. Since the the multiplication map is a submersion, the preimage of the neutral element $1 \in \mathrm{G}$ is a complex submanifold of $\mathrm{G} \times \mathrm{G}$ and it is the graph of the map $g \mapsto g^{-1}$. Hence this map is holomorphic as well. This proves Proposition 1.3. □

Theorem 1.4. *Let* G *be a compact Lie group and* G^c *be a complex Lie group with Lie algebras* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{g}^c = \mathrm{Lie}(\mathrm{G}^c)$. *Let* $\iota : \mathrm{G} \to \mathrm{G}^c$ *be a Lie group homomorphism. Then the following are equivalent.*

(i) *For every complex Lie group* H *and every Lie group homomorphism* $\rho : \mathrm{G} \to \mathrm{H}$ *there is a unique holomorphic homomorphism* $\rho^c : \mathrm{G}^c \to \mathrm{H}$ *such that* $\rho = \rho^c \circ \iota$.

(ii) ι *is injective, its image* $\iota(\mathrm{G})$ *is a maximal compact subgroup of* G^c, *the quotient* $\mathrm{G}^c/\iota(\mathrm{G})$ *is connected, and the differential* $d\iota(1) : \mathfrak{g} \to \mathfrak{g}^c$ *maps* $\mathfrak{g}$ *onto a totally real subspace of* $\mathfrak{g}^c$.

Proof. See pages 31 and 33. □

A Lie group homomorphism

$$\iota : \mathrm{G} \to \mathrm{G}^c$$

that satisfies the equivalent conditions of Theorem 1.4 is called a **complexification** of G. By the universality property in part (i) of Theorem 1.4, the complexification (G^c, ι) of a compact Lie group G is unique up to canonical isomorphism.

Theorem 1.5. *Every compact Lie group admits a complexification, unique up to canonical isomorphism.*

Proof. See page 33. □

Theorem 1.6 (Cartan). *Let* $\iota : \mathrm{G} \to \mathrm{G}^c$ *be a complexification of a compact Lie group. Then every compact subgroup of* G^c *is conjugate in* G^c *to a subgroup of* $\iota(\mathrm{G})$.

Proof. See page 33. □

Theorem 1.7 (Mumford). *Let* $\iota : \mathrm{G} \to \mathrm{G}^c$ *be a complexification of a compact Lie group. Identify* G *with the image of* ι *and denote*

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}), \qquad \mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c).$$

Let $\zeta \in \mathfrak{g}^c$ *such that* $\exp(\zeta) = 1$. *Then there exist elements* $p, p^+ \in \mathrm{G}^c$ *such that*

$$p^{-1}\zeta p \in \mathfrak{g}, \qquad \lim_{t\to\infty} \exp(\mathbf{i}t\zeta) p \exp(-\mathbf{i}t\zeta) = p^+.$$

Proof. See page 34. □

2 First existence proof

The archetypal example of a complexification is the inclusion of $\mathrm{U}(n)$ into $\mathrm{GL}(n,\mathbb{C})$. Polar decomposition gives rise to a diffeomorphism

$$\phi : \mathrm{U}(n) \times \mathfrak{u}(n) \to \mathrm{GL}(n,\mathbb{C}), \qquad \phi(g,\eta) := \exp(\mathbf{i}\eta)g. \tag{3}$$

This example extends to every Lie subgroup of $\mathrm{U}(n)$.

Theorem 2.1. *Let* $\mathrm{G} \subset \mathrm{U}(n)$ *be a Lie subgroup with Lie algebra* $\mathfrak{g} \subset \mathfrak{u}(n)$. *Then the set*

$$\mathrm{G}^c := \{\exp(\mathbf{i}\eta)g \,|\, g \in \mathrm{G},\ \eta \in \mathfrak{g}\} \subset \mathrm{GL}(n,\mathbb{C})$$

is a complex Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ *and the inclusion of* G *into* G^c *satisfies condition (ii) in Theorem 1.4.*

Proof. The proof has ten steps.

Step 1. G^c *is a closed submanifold of* $\mathrm{GL}(n,\mathbb{C})$.

This follows from the fact that (3) is a diffeomorphism.

Step 2. $\mathbb{1} \in \mathrm{G}^c$ *and* $T_{\mathbb{1}}\mathrm{G} = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g} =: \mathfrak{g}^c$.

For $\xi, \eta \in \mathfrak{g}$ consider the curve $\gamma(t) := \exp(\mathbf{i}t\eta)\exp(t\xi) \in \mathrm{G}^c$. It satisfies $\dot\gamma(0) = \xi + \mathbf{i}\eta$. Hence $\mathfrak{g}^c \subset T_{\mathbb{1}}\mathrm{G}^c$ and both spaces have the same dimension.

Step 3. $T_k\mathrm{G}^c = k\mathfrak{g}^c$ *for every* $k \in \mathrm{G}^c$.

Both spaces have the same dimension, so it suffices to prove that $T_k\mathrm{G}^c \subset k\mathfrak{g}^c$. Let ϕ be the diffeomorphism (3). Fix an element $(g,\eta) \in \mathrm{G} \times \mathfrak{g}$ and let

$$k := \phi(g,\eta) = \exp(\mathbf{i}\eta)g \in \mathrm{G}^c.$$

Then, for every $\widehat{\xi} \in \mathfrak{g}$, we obviously have $d\phi(g,\eta)(g\widehat{\xi},0) = \exp(\mathbf{i}\eta)g\widehat{\xi} \in k\mathfrak{g}^c$. Now let $\widehat{\eta} \in \mathfrak{g}$. We must prove that $d\phi(g,\eta)(0,\widehat{\eta}) \in k\mathfrak{g}^c$. To see this consider the map $\gamma : \mathbb{R}^2 \to \mathrm{G}^c$ defined by

$$\gamma(s,t) := \phi(g, t(\eta + s\widehat{\eta})) = \exp(\mathbf{i}t(\eta + s\widehat{\eta}))g$$

and denote $\xi := \gamma^{-1}\partial_s\gamma$ and $\eta := \gamma^{-1}\partial_t\gamma$. Then $\eta(s,t) = g^{-1}\mathbf{i}(\eta + s\widehat{\eta})g \in \mathfrak{g}^c$ for all s,t and $\partial_t\xi = \partial_s\eta + [\xi,\eta]$, $\xi(s,0) = 0$. Since $\eta(s,t) \in \mathfrak{g}^c$ this implies $\xi(s,t) \in \mathfrak{g}^c$ for all s,t and, in particular, $d\phi(g,\eta)(0,\widehat{\eta}) = \gamma(0,1)\xi(0,1) \in k\mathfrak{g}^c$. This proves Step 3.

Step 4. *Let $a \in \mathrm{GL}(n,\mathbb{C})$. Then $a \in \mathrm{G}^c$ if and only if there exists a smooth path $\alpha : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ satisfying $\alpha(0) \in \mathrm{G}$, $\alpha(1) = a$, and $\alpha(t)^{-1}\dot\alpha(t) \in \mathfrak{g}^c$ for every t.*

To prove that the condition is necessary let $a = \exp(i\eta)h \in \mathrm{G}^c$ be given. Then the path $\alpha(t) := \exp(\mathbf{i}t\eta)h$ satisfies the requirements of Step 4. To prove the converse suppose that $\alpha : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ is a smooth curve satisfying $\alpha(0) \in \mathrm{G}$, $\alpha(1) = a$, and $\alpha(t)^{-1}\dot\alpha(t) \in \mathfrak{g}^c$ for all t. Consider the set

$$I := \{t \in [0,1] \,|\, \alpha(t) \in \mathrm{G}^c\}.$$

This set is nonempty, because $0 \in I$. It is closed because G^c is a closed subset of $\mathrm{GL}(n,\mathbb{C})$, by Step 1. To prove it is open, denote $\eta(t) := \alpha(t)^{-1}\dot\alpha(t) \in \mathfrak{g}^c$ and consider the vector fields X_t on $\mathbb{C}^{n\times n}$ given by $X_t(A) := A\eta(t)$. By Step 3, these vector fields are all tangent to G^c. Hence every solution of the differential equation $\dot A(t) = A(t)\eta(t)$ that starts in G^c remains in G^c on a sufficiently small time interval. In particular this holds for the curve $t \mapsto \alpha(t)$ and so I is open. Thus $I = [0,1]$ and hence $a = \alpha(1) \in \mathrm{G}^c$.

Step 5. *If $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$ then $a^{-1}\xi a \in \mathfrak{g}^c$.*

Choose $\alpha : [0,1] \to \mathrm{G}^c$ as in Step 4 and denote

$$\zeta(t) := \alpha(t)^{-1}\xi\alpha(t), \qquad \eta(t) := \alpha(t)^{-1}\dot\alpha(t).$$

Then

$$\dot\zeta + [\eta,\zeta] = 0, \qquad \zeta(0) = \alpha(0)\xi\alpha(0)^{-1} \in \mathfrak{g}^c.$$

Here the second assertion holds because $\alpha(0) \in \mathrm{G}$. Since $\eta(t) \in \mathfrak{g}^c$ for all t this implies that $\zeta(t) \in \mathfrak{g}^c$ for all t and, in particular, $a^{-1}\xi a = \zeta(1) \in \mathfrak{g}^c$.

Step 6. *If $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$ then $a\xi a^{-1} \in \mathfrak{g}^c$.*

The linear map $\xi \mapsto a^{-1}\xi a$ maps $\mathfrak{g}^c$ to itself, by Step 5, and it is injective. Hence the map $\mathfrak{g}^c \to \mathfrak{g}^c : \xi \mapsto a^{-1}\xi a$ is bijective and this proves Step 6.

Step 7. *If $a, b \in \mathrm{G}^c$ then $ab \in \mathrm{G}^c$.*

Choose two curves $\alpha, \beta : [0,1] \to \mathrm{G}^c$ as in Step 4 with $\alpha(0), \beta(0) \in \mathrm{G}$ and $\alpha(1) = a$, $\beta(1) = b$. Then the curve $\gamma := \alpha\beta : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ satisfies

$$\gamma^{-1}\dot\gamma = \beta^{-1}\dot\beta + \beta^{-1}(\alpha^{-1}\dot\alpha)\beta, \qquad \gamma(0) \in \mathrm{G}.$$

By Step 5, $\gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c$ for all t and hence, by Step 4, $ab = \gamma(1) \in \mathrm{G}^c$.

Step 8. *If $a \in \mathrm{G}^c$ then $a^{-1} \in \mathrm{G}^c$.*

Let α as in Step 4 and denote $\gamma(t) := \alpha(t)^{-1}$. Then $\gamma(0) \in \mathrm{G}$ and

$$\gamma^{-1}\dot{\gamma} = \alpha \frac{d}{dt}\alpha^{-1} = -\dot{\alpha}\alpha^{-1} = \alpha(-\alpha^{-1}\dot{\alpha})\alpha^{-1}.$$

By Step 6, $\gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}^c$ for all t and hence, by Step 4, $a^{-1} = \gamma(1) \in \mathrm{G}^c$.

Step 9. G^c *is a complex Lie subgroup of* $\mathrm{GL}(n, \mathbb{C})$*.*

G^c is a complex submanifold of $\mathrm{GL}(n, \mathbb{C})$ by Step 3 and is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ by Step 7 and Step 8.

Step 10. G *is a maximal compact subgroup of* G^c*.*

Let $\mathrm{H} \subset \mathrm{G}^c$ be a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Choose an element $h \in \mathrm{H} \setminus \mathrm{G}$. Since $\mathrm{H} \subset \mathrm{G}^c$, there is a pair $(g, \eta) \in \mathrm{G} \times \mathfrak{g}$ such that $h = \exp(\mathbf{i}\eta)g$. since $\mathrm{G} \subset \mathrm{H}$ and H is a subgroup of G^c we have

$$P := \exp(\mathbf{i}\eta) \in \mathrm{H}.$$

The matrix P is Hermitian and positive definite. Since $h \notin \mathrm{G}$ we also have $P \notin \mathrm{G}$. But this implies $\eta \neq 0$ and so at least one eigenvalue of P is not equal to 1. Hence the sequence

$$P^k = \exp(\mathbf{i}k\eta) \in \mathrm{H}, \qquad k = 1, 2, 3, \ldots$$

has no subsequence that converges to an element of $\mathrm{GL}(n, \mathbb{C})$. Thus H is not compact and this proves Theorem 2.1. □

The tangent space of the submanifold $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ in Theorem 2.1 at the identity element is obviously equal to $T_{\mathbb{1}}\mathrm{G}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g} = \mathfrak{g}^c$. Since G^c is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$, the curve $t \mapsto \exp(-\mathbf{i}\eta)\exp(\mathbf{i}\eta + t\mathbf{i}\widehat{\eta})$ lies in G^c, for every pair $\eta, \widehat{\eta} \in \mathfrak{g}$, and hence

$$A(\eta)\widehat{\eta} := \left.\frac{d}{dt}\right|_{t=0} \exp(-\mathbf{i}\eta)\exp(\mathbf{i}\eta + t\mathbf{i}\widehat{\eta}) \in \mathfrak{g}^c.$$

It turns out that $A \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ is a flat connection 1-form that satisfies $A(\eta)\widehat{\eta} = \mathbf{i}\widehat{\eta}$ whenever η and $\widehat{\eta}$ commute. Conversely, Theorem 3.6 below shows that, for any Lie algebra $\mathfrak{g}$, the connection A is uniquely determined by these conditions and that the group multiplication on $\mathrm{G} \times \mathfrak{g}$ can be reconstructed from A. This gives rise to an intrinsic construction of a complexified Lie group for any compact Lie group G that does not rely on an embedding into the unitary group.

3 Second existence proof

The second existence proof applies to compact connected Lie groups in the intrinsic setting.

Definition 3.1. Let X be a connected smooth manifold and $\mathfrak{g}$ be a Lie algebra. A flat connection $A \in \Omega^1(X, \mathfrak{g})$ is called an **infinitesimal group law** if it satisfies the following conditions.

(Monodromy) The monodromy representation of A is trivial, i.e. for any two smooth paths $\gamma : [0,1] \to X$ and $\zeta : [0,1] \to \mathfrak{g}$ we have

$$\dot{\zeta} + [A(\gamma)\dot{\gamma}, \zeta] = 0, \quad \gamma(0) = \gamma(1) \qquad \Longrightarrow \qquad \zeta(0) = \zeta(1).$$

(Parallel) $A(x) : T_xX \to \mathfrak{g}$ is a vector space isomorphism for every $x \in X$.

(Complete) The vector fields $Y_\xi \in \mathrm{Vect}(X)$ defined by

$$A(x)Y_\xi(x) = \xi$$

are complete, i.e. for every smooth path $\mathbb{R} \to \mathfrak{g} : t \mapsto \xi(t)$ the solutions of the differential equation $\dot{\gamma}(t) = Y_{\xi(t)}(\gamma(t))$ exist for all time.

Example 3.2. Let G be a Lie group with Lie algebra $\mathfrak{g} := T_1\mathrm{G} = \mathrm{Lie}(\mathrm{G})$. Then the 1-form $A \in \Omega^1(\mathrm{G}, \mathfrak{g})$ defined by

$$A(g)v := g^{-1}v$$

is an infinitesimal group law. The vector fields Y_ξ are given by $Y_\xi(g) = g\xi$ and the curvature $F_A \in \Omega^2(X, \mathfrak{g})$ is

$$\begin{aligned}
F_A(Y_\xi, Y_\eta) &= dA(Y_\xi, Y_\eta) + [AY_\xi, AY_\eta] \\
&= \mathcal{L}_{Y_\xi}(AY_\eta) - \mathcal{L}_{Y_\eta}(AY_\xi) + A[Y_\xi, Y_\eta] + [\xi, \eta] \\
&= A[Y_\xi, Y_\eta] + [\xi, \eta] \\
&= 0
\end{aligned}$$

for $\xi, \eta \in \mathfrak{g}$. Thus the connection is flat. The *(Monodromy)* condition holds because, for any path $g : [0,1] \to \mathrm{G}$, the solutions of the equation

$$\dot{\xi} + [g^{-1}\dot{g}, \xi] = 0$$

have the form $\xi(t) = g(t)^{-1}\xi_0 g(t)$.

Example 3.3. Let $\mathfrak{g}$ be a Lie algebra. Then there is a unique flat connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ such that

$$[\xi, \widehat{\xi}] = 0 \qquad \Longrightarrow \qquad A(\xi)\widehat{\xi} = \widehat{\xi} \tag{4}$$

for all $\xi, \widehat{\xi} \in \mathfrak{g}$. In general, this connection is not an infinitesimal group law. The idea behind this example is as follows. If we have a Lie group G with Lie algebra $\mathfrak{g}$ we might attempt to reconstruct the group multiplication locally as an operation $m : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\exp(\xi)\exp(\eta) = \exp(m(\xi, \eta))$. While this is not possible globally in most cases, the associated connection $A(\xi)\widehat{\xi} := \exp(-\xi) d\exp(\xi)\widehat{\xi}$ does exist globally and satisfying (4).

To prove uniqueness note that a connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ is flat if and only if every smooth map $\gamma : \mathbb{R}^2 \to \mathfrak{g}$ satisfies the equation

$$\partial_s(A(\gamma)\partial_t\gamma) - \partial_t(A(\gamma)\partial_s\gamma) + [A(\gamma)\partial_s\gamma, A(\gamma)\partial_t\gamma] = 0. \tag{5}$$

If in addition the connection satisfies (4) then, with $\gamma(s,t) := t(\xi + s\widehat{\xi})$, we obtain

$$A(\gamma)\partial_t\gamma = \xi + s\widehat{\xi}, \qquad A(\gamma)\partial_s\gamma = A(t(\xi + s\widehat{\xi}))t\widehat{\xi}.$$

Setting $s = 0$ we find that the function $\zeta(t) := A(t\xi)t\widehat{\xi}$ satisfies the differential equation

$$\dot{\zeta} + [\xi, \zeta] = \widehat{\xi}, \qquad \zeta(0) = 0. \tag{6}$$

Thus

$$A(\xi)\widehat{\xi} = \zeta(1) = \int_0^1 \exp(-t\mathrm{ad}(\xi))\widehat{\xi}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\mathrm{ad}(\xi)^k\widehat{\xi},$$

where $\mathrm{ad}(\xi) := [\xi, \cdot]$. Conversely, let $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ be defined by this formula. If $[\xi, \widehat{\xi}] = 0$ then $\zeta(t) = t\widehat{\xi}$ is the unique solution of the differential equation (6) and so $A(\xi)\widehat{\xi} = \zeta(1) = \widehat{\xi}$. To prove that A is flat we fix three elements $\xi, \widehat{\xi}_1, \widehat{\xi}_2 \in \mathfrak{g}$, define $\zeta_j : [0,1] \to \mathfrak{g}$ as the solutions of (6) with $\widehat{\xi} = \widehat{\xi}_j$, and define $\zeta_{ij} : [0,1] \to \mathfrak{g}$ as the solution of the linearized equation

$$\dot{\zeta}_{ij} + [\xi, \zeta_{ij}] + [\widehat{\xi}_i, \zeta_j] = 0, \qquad \zeta_{ij}(0) = 0.$$

Then $A(\xi)\widehat{\xi}_j = \zeta_j(1)$ and $(dA(\xi)\widehat{\xi}_i)\widehat{\xi}_j = \zeta_{ij}(1)$. Moreover,

$$\dot{\zeta} + [\xi, \zeta] = 0, \qquad \zeta := \zeta_{12} - \zeta_{21} + [\zeta_1, \zeta_2],$$

so $\eta \equiv 0$ and thus A is flat.

Theorem 3.4. *Let X be a connected smooth manifold, $\mathfrak{g}$ be a Lie algebra, and $A \in \Omega^1(X, \mathfrak{g})$ be an infinitesimal group law. Fix an element $1 \in X$. Then there is a unique Lie group structure on X with unit 1 such that*

$$A(x)v = A(1)x^{-1}v$$

for $x \in X$ and $v \in T_xX$. Moreover, the map $A(1) : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra isomorphism.

Proof. The proof has seven steps. The first step constructs an analogue Φ of the map $\mathrm{G} \to \mathrm{Aut}(\mathfrak{g}) : g \mapsto \mathrm{Ad}(g^{-1})$ (see Example 3.2).

Step 1. *There is a unique function $\Phi : X \to \mathrm{Aut}(\mathfrak{g})$ satisfying*

$$\Phi(1) = \mathrm{id}, \qquad (d\Phi(x)v)\xi + [A(x)v, \Phi(x)\xi] = 0 \tag{7}$$

for all $x \in X$, $v \in T_xX$, and $\xi \in \mathfrak{g}$.

Given $x \in X$ and $\xi \in \mathfrak{g}$ choose a smooth path $\gamma : [0,1] \to X$ with endpoints $\gamma(0) = 1$ and $\gamma(1) = x$, let $\zeta : [0,1] \to \mathfrak{g}$ be the unique solution of the differential equation

$$\dot{\zeta} + [A(\gamma)\dot{\gamma}, \zeta] = 0, \qquad \zeta(0) = \xi, \tag{8}$$

and define $\Phi(x)\xi := \zeta(1)$. The *(Monodromy)* axiom guarantees that $\zeta(1)$ is independent of the choice of the path γ. The resulting function Φ is obviously smooth and satisfies (7).

Step 2. *For any two smooth paths $\beta, \gamma : [0,1] \to X$ we have*

$$\beta(0) = \beta(1), \quad A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta} \qquad \Longrightarrow \qquad \gamma(0) = \gamma(1).$$

Assume without loss of generality that $\beta(0) = \beta(1) = 1$. Choose a smooth path $[0,1] \to X : \lambda \mapsto x_\lambda$ such that $x_0 = 1$ and $x_1 = \gamma(0)$. For $\lambda \in [0,1]$ let $\gamma_\lambda : [0,1] \to X$ be the solution of the differential equation

$$A(\gamma_\lambda(t))\partial_t\gamma_\lambda(t) = A(\beta(t))\partial_t\beta(t), \qquad \gamma_\lambda(0) = x_\lambda.$$

Then $\lambda \mapsto \gamma_\lambda$ is a smooth homotopy from β to γ. We observe that

$$A(\gamma_\lambda(t))\partial_\lambda\gamma_\lambda(t) = \Phi(\gamma_0(t))A(x_\lambda)\partial_\lambda x_\lambda, \tag{9}$$

where Φ is as in Step 1. Namely, both the left and right hand side of (9), as functions of t, satisfy the differential equation $\dot{\zeta} + [A(\gamma_0)\dot{\gamma}_0, \zeta] = 0$ with initial condition $\zeta(0) = A(x_\lambda)\partial_\lambda x_\lambda$. It follows from (9) with $t = 1$ that $A(\gamma_\lambda(1))\partial_\lambda\gamma_\lambda(1) = A(x_\lambda)\partial_\lambda x_\lambda$ for all λ. Since $\gamma_0(1) = x_0 = 1$ we obtain $\gamma_\lambda(1) = x_\lambda = \gamma_\lambda(0)$ for all λ. This proves Step 2.

Step 3. *For any four smooth paths $\beta_0, \beta_1, \gamma_0, \gamma_1 : [0,1] \to X$ satisfying $\beta_0(0) = \beta_1(0)$ and $\beta_0(1) = \beta_1(1)$ and $A(\gamma_j)\dot{\gamma}_j = A(\beta_j)\dot{\beta}_j$ we have*

$$\gamma_0(0) = \gamma_1(0) \qquad \Longleftrightarrow \qquad \gamma_0(1) = \gamma_1(1).$$

Assume without loss of generality that γ_j and β_j are constant near the endpoints and that $\beta_0(0) = \beta_1(0) = 1$ and $\gamma_0(0) = \gamma_1(0)$. Define $\beta : [0,1] \to X$ by $\beta(t) := \beta_0(2t)$ for $0 \le t \le 1/2$ and $\beta(t) := \beta_1(2-2t)$ for $1/2 \le t \le 1$. Let $\gamma : [0,1] \to X$ be the unique solution of the differential equation $A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta}$ with initial condition $\gamma(0) = \gamma_0(0)$. Then $\gamma_0(t) = \gamma(t/2)$ for $0 \le t \le 1$. Moreover, since $\beta(0) = \beta(1) = 1$, it follows from Step 2 that $\gamma(1) = \gamma(0) = \gamma_0(0) = \gamma_1(0)$. Hence $\gamma_1(t) = \gamma((1-t)/2)$. with $t = 1$ we obtain that $\gamma_1(1)$ and $\gamma_0(1)$ both agree with $\gamma(1/2)$. This proves Step 3.

Step 4. *There is a unique smooth map*

$$X \times X \to X : (x,y) \mapsto \phi_x(y) = \psi_y(x)$$

such that $\phi_x(1) = x$ and $\phi_x^* A = A$ for every $x \in X$. Moreover, ϕ_x and ψ_y are diffeomorphisms for all x, y and $\psi_1 = \phi_1 = \mathrm{id}$.

Fix an element $x \in X$. It follows from Step 3 that, for every smooth path $\beta : [0,1] \to X$ with $\beta(0) = 1$, the endpoint of the path $\gamma : [0,1] \to X$, defined by

$$A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta}, \qquad \gamma(0) = x, \tag{10}$$

depends only on the endpoint of β. Hence there is a well defined map $\phi_x : X \to X$ satisfying

$$\phi_x(\beta(1)) = \gamma(1)$$

whenever $\beta(0) = 1$ and γ is given by (10). Since the solutions of a differential equation depend smoothly on the initial condition and the parameter it follows that the map $(x,y) \mapsto \phi_x(y)$ is smooth. (Namely, choose a local smooth family of paths $\beta_y : [0,1] \to X$ with $\beta_y(0) = 1$ and $\beta_y(1) = y$.) It follows directly from the construction that $\phi_x(1) = x$ and $\phi_x^* A = A$ for every x. That ϕ_x is a diffeomorphism follows by reversing the roles of the pairs $(1, \beta)$ and (x, γ) to construct an inverse. That ψ_y is a diffeomorphism follows by interchanging 1 and y and reversing β. That ϕ_1 is the identity is obvious from the definition (we get $\gamma = \beta$ when $x = 1$). That ψ_1 is the identity follows by choosing $\beta(t) \equiv 1$. Uniqueness is left as an exercise. This proves Step 4.

Step 5. *The map $(x,y) \mapsto \phi_x(y) = \psi_y(x) =: xy$ in Step 4 defines a Lie group structure on X with unit* 1.

It suffices to prove associativity, i.e.

$$\phi_x(\phi_y(z)) = \psi_z(\psi_y(x)) \tag{11}$$

for $x, y, z \in X$. That 1 is the unit follows then from the fact that $\phi_1 = \psi_1 = \mathrm{id}$ and that every element has an inverse follows from the fact that ϕ_x and ψ_y are diffeomorphisms. The inverse map $x \mapsto \phi_x^{-1}(1)$ is smooth by Step 4.

To prove (11) we fix $x, y, z \in X$ and choose paths $\beta, \gamma : [0,1] \to X$ with endpoints $\beta(0) = \gamma(0) = 1$ and $\beta(1) = y$, $\gamma(1) = z$. Define the paths $\beta', \gamma', \gamma'' : [0,1] \to X$ by

$$A(\beta')\dot{\beta}' = A(\beta)\dot{\beta}, \qquad A(\gamma'')\dot{\gamma}'' = A(\gamma')\dot{\gamma}' = A(\gamma)\dot{\gamma},$$

and

$$\beta'(0) = x, \qquad \gamma'(0) = y, \qquad \gamma''(0) = \beta'(1) = \phi_x(y) = \psi_y(x).$$

We claim that

$$\phi_x(\phi_y(z)) = \gamma''(1) = \psi_z(\psi_y(x)).$$

To prove the first identity note that $\gamma'(1) = \phi_y(z)$ and so the catenation $\beta \# \gamma'$ (first β then γ') runs from 1 to $\phi_y(z)$. The catenation $\beta' \# \gamma''$ is the lift of this path starting at x and hence ends at $\gamma''(1) = \phi_x(\phi_y(z))$, by definition of ϕ_x in the proof of Step 4. On the other hand γ'' is also the lift of γ starting at $\psi_y(x)$ and hence ends at $\gamma''(1) = \psi_z(\psi_y(x))$, by definition of ψ_z in the proof of Step 4. This proves Step 5.

Step 6. *The map $A(1) : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra homomorphism and satisfies $A(x)xv = A(1)v$ for $x \in X$ and $v \in T_1X$.*

The formula $A(x)xv = A(1)v$ with $xv := d\phi_x(1)v$ follows immediately from the fact that $\phi_x^*A = A$ and $\phi_x(1) = x$. This formula shows that the vector fields $Y_\xi \in \mathrm{Vect}(X)$ in Definition 3.1 satisfy $\xi = A(x)Y_\xi(x) = A(1)x^{-1}Y_\xi(x)$. Hence

$$Y_\xi(x) = xv, \qquad v := A(1)^{-1}\xi \in \mathrm{Lie}(X).$$

The map $\mathrm{Lie}(X) \to \mathrm{Vect}(X)$ that assigns to every tangent vector $v \in \mathrm{Lie}(X)$ the left invariant vector field $x \mapsto xv$ is a Lie algebra anti-homomorphism. Since A is flat we have

$$0 = F_A(Y_\xi, Y_\eta) = dA(Y_\xi, Y_\eta) + [AY_\xi, AY_\eta] = A[Y_\xi, Y_\eta] + [\xi, \eta].$$

Hence the map $\mathfrak{g} \to \mathrm{Vect}(X) : \xi \mapsto Y_\xi$ is also a Lie algebra anti-homomorphism and so $A(1)$ is a Lie algebra isomorphism. This completes the proof of the existence statement.

Step 7. *The Lie group structure on X is uniquely determined by A and 1.*

Let $X \times X \to X : (x, y) \mapsto xy$ be a Lie group structure with unit 1 such that $A(x)v = A(1)x^{-1}v$ for $x \in X$ and $v \in T_xX$. Fix two elements $x, y \in X$, choose a path $\beta : [0,1] \to X$ such that $\beta(0) = 1$ and $\beta(1) = y$, and define $\gamma(t) := x\beta(t)$. Then $A(\gamma)\dot{\gamma} = A(1)\gamma^{-1}\dot{\gamma} = A(1)\beta^{-1}\dot{\beta} = A(\beta)\dot{\beta}$. Hence the Lie group structure on X agrees with the one constructed in Step 5. This proves Theorem 3.4. □

Lemma 3.5. *Let $\mathfrak{g}$ be a Lie algebra with an inner product $\langle \cdot, \cdot \rangle$ such that*

$$\langle \xi, [\eta, \zeta] \rangle = \langle [\xi, \eta], \zeta \rangle$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$. Fix an element $\eta \in \mathfrak{g}$ and let $\xi : \mathbb{R} \to \mathfrak{g}$ be a solution of the second order differential equation

$$\ddot{\xi} + [\eta, [\eta, \xi]] = 0, \qquad \xi(0) = 0. \tag{12}$$

Then $|\xi(t)| \geq |t|\,|\dot{\xi}(0)|$ for every $t \in \mathbb{R}$.

Proof. We have

$$\frac{d}{dt}\left(|\dot{\xi}|^2 - |[\xi, \eta]|^2\right) = 2\langle \dot{\xi}, \ddot{\xi} \rangle + 2\langle [\dot{\xi}, \eta], [\eta, \xi] \rangle = 0.$$

Since $\xi(0) = 0$ this implies $|\dot{\xi}(t)|^2 = |\dot{\xi}(0)|^2 + |[\xi(t), \eta]|^2 \geq |\dot{\xi}(0)|^2$ for all $t \in \mathbb{R}$. Moreover, it follows from (12), by taking the inner product with ξ and integrating by parts, that

$$\begin{aligned} 0 &= \int_0^t \langle \xi(s), \ddot{\xi}(s) + [\eta, [\eta, \xi(s)]] \rangle \, ds \\ &= \langle \xi(t), \dot{\xi}(t) \rangle - \int_0^t \left(|\dot{\xi}(s)|^2 + |[\xi(s), \eta]|^2\right) ds \\ &\leq \langle \xi(t), \dot{\xi}(t) \rangle - t|\dot{\xi}(0)|^2. \end{aligned}$$

The last inequality holds for $t \geq 0$. Hence

$$|\xi(t)|^2 = 2\int_0^t \langle \xi(s), \dot{\xi}(s) \rangle \, ds \geq 2\int_0^t s|\dot{\xi}(0)|^2 \, ds = t^2|\dot{\xi}(0)|^2$$

for $t \geq 0$. Since equation (12) is time reversible, this proves Lemma 3.5. □

Theorem 3.6. *Let* G *be a compact connected Lie group with Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and denote* $\mathrm{G}^c := \mathrm{G} \times \mathfrak{g}$ *and* $\mathfrak{g}^c := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$*. Then the following holds.*

(i) *There is a unique flat connection* $A_1 \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ *such that, for all* $\eta, \widehat{\eta} \in \mathfrak{g}$*, we have*

$$[\eta, \widehat{\eta}] = 0 \qquad \Longrightarrow \qquad A_1(\eta)\widehat{\eta} = \mathbf{i}\widehat{\eta} \tag{13}$$

(ii) *If* A_1 *is as in (i) then the* 1*-form* $A \in \Omega^1(\mathrm{G}^c, \mathfrak{g}^c)$ *defined by*

$$A(g, \eta)(v, \widehat{\eta}) := g^{-1}v + g^{-1}(A_1(\eta)\widehat{\eta})g$$

is an infinitesimal group law.

(iii) *Suppose* G^c *is equipped with the Lie group structure associated to the infinitesimal group law in* A *in (ii) via Theorem 3.4. Then* G^c *is a complex Lie group and, for* $g, h \in \mathrm{G}$ *and* $\xi, \eta \in \mathfrak{g}$*,*

$$[\xi, g\eta g^{-1}] = 0 \qquad \Longrightarrow \qquad (g, \xi) \cdot (h, \eta) = (gh, \xi + g\eta g^{-1}). \tag{14}$$

(iv) *The inclusion* $\mathrm{G} \to \mathrm{G}^c : g \mapsto \iota(g) := (g, 0)$ *satisfies (ii) in Theorem 1.4.*

Proof. First assume that A_1 satisfies the requirements of (i). Let $\eta, \widehat{\eta} \in \mathfrak{g}$ and define $\zeta : \mathbb{R} \to \mathfrak{g}^c$ by $\zeta(t) := A_1(t\eta)t\widehat{\eta}$. Then

$$\dot{\zeta} + [\mathbf{i}\eta, \zeta] = \mathbf{i}\widehat{\eta}, \qquad \zeta(0) = 0. \tag{15}$$

(Apply equation (5) to the function $(s, t) \mapsto t(\eta + s\widehat{\eta})$ and set $s = 0$.) Thus we must define $A_1(\eta)\widehat{\eta} := \zeta(1)$ where $\zeta : \mathbb{R} \to \mathfrak{g}$ is the unique solution of (15). That this 1-form satisfies (13) follows from the fact that $\zeta(t) := \mathbf{i}t\widehat{\eta}$ satisfies (15) whenever η and $\widehat{\eta}$ commute. That it is flat follows from the same argument that was used in Example 3.3. This proves (i).

We prove (ii). First we observe that A is flat. Namely the $\mathfrak{g}$-connection A_0 on G defined by $A_0(g)v := g^{-1}v$ is flat by Example 3.2 and A_1 is flat by (i). Hence, for two tangent vectors $w_j = (v_j, \widehat{\eta}_j) \in T_{(g,\eta)}(\mathrm{G} \times \mathfrak{g})$, $j = 1, 2$, we obtain

$$\begin{aligned}
F_A(w_1, w_2) &= dA(w_1, w_2) + [A(g, \eta)w_1, A(g, \eta)w_2] \\
&= dA_0(v_1, v_2) + g^{-1}dA_1(\widehat{\eta}_1, \widehat{\eta}_2)g \\
&\quad + [g^{-1}A_1(\widehat{\eta}_2)g, g^{-1}v_1] - [g^{-1}A_1(\widehat{\eta}_1)g, g^{-1}v_2] \\
&\quad + [g^{-1}v_1 + g^{-1}(A_1(\eta)\widehat{\eta}_1)g, g^{-1}v_2 + g^{-1}(A_1(\eta)\widehat{\eta}_2)g] \\
&= F_{A_0}(v_1, v_2) + g^{-1}F_{A_1}(\widehat{\eta}_1, \widehat{\eta}_2)g \\
&= 0.
\end{aligned}$$

For the *(Monodromy)* axiom it suffices to consider curves based at 1. It is obviously satisfied for curves in G and hence follows from the fact that the connection A is flat and that every based curve in G^c is homotopic to one in G. The *(Parallel)* and *(Complete)* axioms follow from the inequality

$$|\widehat{\eta}| \le |\mathrm{Im}(A_1(\eta)\widehat{\eta})|\,. \tag{16}$$

This inequality shows that the linear map $A(g,\eta) : T_{(g,\eta)}\mathrm{G}^c \to \mathfrak{g}^c$ is invertible for every pair $(g,\eta) \in \mathrm{G} \times \mathfrak{g}$. It also shows that, for every curve $\zeta : \mathbb{R} \to \mathfrak{g}^c$, the solutions $[0,T] \to \mathrm{G}^c : t \mapsto (g(t),\eta(t))$ of the differential equation

$$g(t)^{-1}\dot{g}(t) + g(t)^{-1}(A_1(\eta(t))\dot{\eta}(t))g(t) = \zeta(t)$$

satisfy $\sup_{0\le t\le T}|\eta(t)-\eta(0)| \le cT$, where $c := \sup_{0\le t\le T}|\mathrm{Im}\zeta(t)|$. Hence the solutions must exist for all time. To prove (16) consider the imaginary part $\xi := \mathrm{Im}(\zeta)$ of a solution $\zeta : [0,1] \to \mathfrak{g}^c$ of equation (15). It satisfies the second order differential equation

$$\ddot{\xi} + [\eta,[\eta,\xi]] = 0, \qquad \xi(0) = 0, \qquad \dot{\xi}(0) = \widehat{\eta}.$$

By Lemma 3.5 every solution of this equation satisfies $|\xi(1)| \ge |\widehat{\eta}|$ and this is equivalent to (16). Thus we have proved (ii).

We prove (iii). That G^c is a complex Lie group follows from Proposition 1.3. Now let $g,h \in \mathrm{G}$ and $\xi,\eta \in \mathfrak{g}$ such that $[\xi, g\eta g^{-1}] = 0$. Choose a smooth path $\alpha : [0,1] \to \mathrm{G}$ such that $\alpha(0) = 1$ and $\alpha(1) = h$ and define $\beta,\gamma : [0,1] \to \mathrm{G} \times \mathfrak{g}$ by $\beta(t) := (\alpha(t), t\eta)$ and $\gamma(t) := (g\alpha(t), \xi + tg\eta g^{-1})$. Then $\beta(0) = (1,0)$, $\gamma(0) = (g,\xi)$, and $A(\beta)\dot{\beta} = \alpha^{-1}\dot{\alpha} + \mathbf{i}\alpha^{-1}\eta\alpha = A(\gamma)\dot{\gamma}$. Hence $\gamma(t) = (g,\xi)\cdot\beta(t)$ for all t. With $t = 1$ we obtain equation (14), namely $(gh, \xi + g\eta g^{-1}) = (g,\xi)\cdot(h,\eta)$. This completes the proof of (iii).

We prove (iv). First, it follows from (14) that the embedding $\iota : \mathrm{G} \to \mathrm{G}^c$, given by $\iota(g) := (g,0)$, is a Lie group homomorphism. Second, the image of the differential $d\iota(1) : \mathfrak{g} \to \mathfrak{g}^c$ is a totally real subspace of $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$. Third, $\iota(\mathrm{G}) = \mathrm{G} \times \{0\}$ is a maximal compact subgroup of G^c. To see this, let $\mathrm{H} \subset \mathrm{G}^c$ be a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Then H contains an element of the form (g,ξ) with $\xi \ne 0$. Hence, by (14), the pair $(1,\xi) = (g,\xi)\cdot(g^{-1},0)$ is also an element of H and hence, so is $(1,k\xi)$ for every integer $k \ge 1$. This sequence has no convergent subsequence and so H is not compact. This proves Theorem 3.6. $\square$

4 Hadamard's theorem

Theorem 4.1 (**Hopf-Rinow**). *Let M be a connected Riemannian manifold and denote by $d : M \times M \to [0, \infty)$ the distance function associated to the Riemannian metric. Fix a point $p_0 \in M$. Then the following are equivalent.*

(i) *The geodesics starting at p_0 exist for all time.*

(ii) *For every $p_1 \in M$ there exists a geodesic $\gamma : [0, 1] \to M$ such that*

$$\gamma(0) = p_0, \qquad \gamma(1) = p_1, \qquad L(\gamma) := \int_0^1 |\dot\gamma(t)|\ dt = d(p_0, p_1).$$

(iii) *Every closed and bounded subset of (M, d) is compact.*

(iv) *(M, d) is a complete metric space.*

Proof. See [11, Theorems 2.57 and 2.58]. □

A connected Rimannian manifold satisfying the conditions of Theorem 4.1 is called **complete**. In such a manifold any two points can be joined by a (minimal) geodesic. If, in addition, M is simply connected and has nonpositive sectional curvature, then Hadamard's theorem asserts that any two points can be joined by a unique geodesic.

Theorem 4.2 (**Hadamard**). *Let M be a complete, connected, simply connected Riemannian manifold with nonpositive sectional curvature. Then, for every $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism.*

Proof. (Explained to me by Urs Lang.) There are three steps. The first step asserts that there are no conjugate points. We denote by ∇ the Levi-Civita connection and by $R \in \Omega^2(M, \mathrm{End}(TM))$ the Riemann curvature tensor.

Step 1. *If $\gamma : [0, 1] \to M$ is a smooth curve and $X : [0, 1] \to TM$ is a vector field along γ (i.e. $X(t) \in T_{\gamma(t)}M$ for all t) satisfying the* **Jacobi equation**

$$\nabla_t\nabla_t X + R(X, \dot\gamma)\dot\gamma = 0 \tag{17}$$

and the boundary conditions $X(0) = 0$, $X(1) = 0$ then $X \equiv 0$.

We have

$$\frac{d}{dt}\langle\nabla_t X, X\rangle = |\nabla_t X|^2 + \langle\nabla_t\nabla_t X, X\rangle = |\nabla_t X|^2 - \langle R(X, \dot\gamma)\dot\gamma, X\rangle$$

and hence

$$\int_0^1 \left(|\nabla_t X|^2 - \langle R(X, \dot\gamma)\dot\gamma, X\rangle\right) dt = 0.$$

Since $\langle R(X, \dot\gamma)\dot\gamma, X\rangle \le 0$ everywhere, we obtain $\nabla_t X \equiv 0$ and hence $X \equiv 0$.

Step 2. *The differential $d\exp_p(v) : T_pM \to T_{\exp_p(v)}M$ of the exponential map is bijective for every $v \in T_pM$.*

Let $\widehat{v} \in T_pM$ be a tangent vector such that $d\exp_p(v)\widehat{v} = 0$. Define the map $\gamma : \mathbb{R}^2 \to M$ and the vector field $X : \mathbb{R}^2 \to TM$ along γ by

$$\gamma(s,t) := \exp_p(t(v + s\widehat{v})), \qquad X(s,t) := \partial_s\gamma(s,t) = d\exp_p(t(v + s\widehat{v}))t\widehat{v}.$$

Then

$$\begin{aligned} \nabla_t\nabla_t X &= \nabla_t\nabla_t\partial_s\gamma \\ &= \nabla_t\nabla_s\partial_t\gamma \\ &= \nabla_s\nabla_t\partial_t\gamma - R(\partial_s\gamma, \partial_t\gamma)\partial_t\gamma \\ &= -R(X, \partial_t\gamma)\partial_t\gamma. \end{aligned}$$

Here the second equation follows from the fact that the Levi-Civita connection is torsion free, the third equation follows from the definition of the Riemann curvature tensor, and the last equation from the fact that the curve $t \mapsto \gamma(s,t)$ is a geodesic for every s. Since $X(0,0) = 0$ and $X(0,1) = 0$, by assumption, it follows from Step 1 that $X(0,t) = 0$ for all t. By choosing t small we find that $\widehat{v} = 0$.

Step 3. *The exponential map $\exp_p : T_pM \to M$ is a covering, i.e. it is surjective and, for every continuous path $\gamma : [0,1] \to M$ and every $v_0 \in T_pM$ with $\gamma(0) = \exp_p(v_0)$ there is a unique continuous path $v : [0,1] \to T_pM$ such that $v(0) = v_0$ and $\gamma(t) = \exp_p(v(t))$ for every t.*

That the map $\exp_p : T_pM \to M$ is surjective follows immediately from the Hopf-Rinow theorem. By Step 2 we may consider the space T_pM with the pullback metric under the map $\exp_p$. Thus $\exp_p$ is a local isometry for this metric and so the rays $t \mapsto tv$ are geodesics in T_pM for this metric (because they are mapped to geodesics in M under $\exp_p$). Now we can apply the Hopf-Rinow theorem again to the pullback metric and obtain that it is complete (use the implication $(i) \implies (iv)$ in Theorem 4.1). This implies the covering property by a standard open and closed argument (given γ, let $I \subset [0,1]$ be the set of all t such that the lift exists on the interval $[0,1]$. Then I is obvious nonempty and open. That I is closed follows from completeness of T_pM with the pullback metric). This proves Step 3.

By Step 3, the map $\exp_p : T_pM \to M$ is a universal covering of M. Since M is simply connected, this implies that $\exp_p$ is a diffeomorphism. This proves Theorem 4.2. $\square$

5 Cartan's fixed point theorem

Theorem 5.1 (Cartan). *Let M be a complete connected simply connected Riemannian manifold with nonpositive sectional curvature. Let* G *be a compact Lie group that acts on M by isometries. Then there exists a point $p \in M$ such that $gp = p$ for every $g \in \mathrm{G}$.*

Proof. The proof has three steps and follows the argument given by Bill Casselmann in [1]. The second step is Serre's uniqueness result for the *circumcentre* of a bounded set in a *semi-hyperbolic space*.

Step 1. *Let $m \in M$ and $v \in T_mM$ and define*

$$p_0 := \exp_m(-v), \qquad p_1 := \exp_m(v).$$

Then

$$2d(m,q)^2 + \frac{d(p_0,p_1)^2}{2} \le d(p_0,q)^2 + d(p_1,q)^2$$

for every $q \in M$.

By Theorem 4.2 the exponential map $\exp_m : T_mM \to M$ is a diffeomorphism. Hence

$$d(p_0,p_1) = 2|v|.$$

Now let $q \in M$. Then there is a unique tangent vector $w \in T_mM$ such that

$$q = \exp_m(w), \qquad d(m,q) = |w|.$$

Moreover, the exponential map is expanding (e.g. [11, Theorem 4.112]). Thus

$$d(p_0,q) \ge |w+v|, \qquad d(p_1,q) \ge |w-v|.$$

Hence

$$\begin{aligned} d(m,q)^2 &= |w|^2 \\ &= \frac{|w+v|^2 + |w-v|^2}{2} - |v|^2 \\ &\le \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4}. \end{aligned}$$

This proves Step 1.

Step 2. *For $p \in M$ and $r \geq 0$ denote by $B(p,r) \subset M$ the closed ball of radius r centered at p. Let $\Omega \subset M$ be a nonempty bounded set and define*

$$r_\Omega := \inf \{ r > 0 \,|\, \text{there exists a } p \in M \text{ such that } \Omega \subset B(p,r) \}$$

Then there exists a unique point $p_\Omega \in M$ such that $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove existence. Choose a sequence $r_i > r_\Omega$ and a sequence $p_i \in M$ such that

$$\Omega \subset B(p_i, r_i), \qquad \lim_{i \to \infty} r_i = r_\Omega.$$

Choose $q \in \Omega$. Then $d(q, p_i) \leq r_i$ for every i. Since the sequence r_i is bounded and M is complete, it follows that p_i has a convergent subsequence, still denoted by p_i. Its limit $p_\Omega := \lim_{i \to \infty} p_i$ satisfies $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove uniqueness. Let $p_0, p_1 \in M$ such that $\Omega \subset B(p_0, r_\Omega) \cap B(p_1, r_\Omega)$. Since the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism, by Theorem 4.2, there exists a unique vector $v_0 \in T_{p_0}M$ such that $p_1 = \exp_{p_0}(v_0)$. Denote the midpoint between p_0 and p_1 by $m := \exp_{p_0}\left(\frac{1}{2}v_0\right)$. Then it follows from Step 1 that

$$\begin{aligned} d(m,q)^2 &\leq \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4} \\ &\leq r_\Omega^2 - \frac{d(p_0,p_1)^2}{4} \end{aligned}$$

for every $q \in \Omega$. Since $\sup_{q \in \Omega} d(m,q) \geq r_\Omega$, by definition of r_Ω, it follows that $d(p_0,p_1) = 0$ and hence $p_0 = p_1$. This proves Step 2.

Step 3. *We prove Theorem 5.1.*

Let $q \in M$ and consider the group orbit $\Omega := \{ gq \,|\, g \in \mathrm{G} \}$. Let $r_\Omega \geq 0$ and $p_\Omega \in M$ be as in Step 2. Then

$$\Omega \subset B(p_\Omega, r_\Omega).$$

Since G acts on M by isometries, this implies

$$\Omega = g\Omega \subset B(gp_\Omega, r_\Omega)$$

for every $g \in \mathrm{G}$. Hence it follows from the uniqueness statement in Step 2 that $gp_\Omega = p_\Omega$ for every $g \in \mathrm{G}$. This proves Step 3 and Theorem 5.1. $\square$

6 Cartan decomposition

Throughout this section we assume the following.

> **(C)** G^c *is a complex Lie group with Lie algebra* $\mathfrak{g}^c$ *and* $\mathrm{G} \subset \mathrm{G}^c$ *is a maximal compact Lie subgroup such that the quotient* G^c/G *is connected and* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is a totally real subspace of* $\mathfrak{g}^c$*, i.e.* $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$.

If G is any compact connected Lie group then the complex Lie group G^c constructed in Theorem 3.6 satisfies condition (C). If $\mathrm{G} \subset \mathrm{U}(n)$ is a (not necessarily connected) Lie subgroup then the complex Lie group $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ constructed in Theorem 2.1 satisfies condition (C).

Theorem 6.1 (Cartan). *Assume (C). Then the map*

$$\mathrm{G} \times \mathfrak{g} \to \mathrm{G}^c : (g, \eta) \mapsto \exp(\mathbf{i}\eta)g$$

is a diffeomorphism. In particular, G^c/G *is simply connected.*

Proof. See page 23. □

6.2. Assume (C). Define the quotient space G^c/G by

$$\mathrm{G}^c/\mathrm{G} := \{[k] \,|\, k \in \mathrm{G}^c\}, \qquad [k] := k\mathrm{G} = \{kg \,|\, g \in \mathrm{G}\}.$$

The tangent space of G^c/G at $[k]$ is the quotient of the tangent spaces

$$T_{[k]}\mathrm{G}^c/\mathrm{G} = \frac{T_k\mathrm{G}^c}{T_k k\mathrm{G}} = \frac{T_k\mathrm{G}^c}{\{k\xi \,|\, \xi \in \mathfrak{g}\}}.$$

Throughout we use the notation

$$\mathrm{Re}(\zeta) := \xi, \qquad \mathrm{Im}(\zeta) := \eta$$

for $\zeta = \xi + \mathbf{i}\eta \in \mathfrak{g}^c$ with $\xi, \eta \in \mathfrak{g}$. Thus the equivalence class of a tangent vector $[\zeta] \in T_{[k]}\mathrm{G}^c/\mathrm{G}$ is uniquely determined by $\mathrm{Im}(\zeta)$. Now choose an invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and define a Riemannian metric on G^c/G by

$$\langle [k\zeta], [k\zeta'] \rangle := \langle \eta, \eta' \rangle_{\mathfrak{g}}, \qquad \zeta, \zeta' \in \mathfrak{g}, \qquad \eta := \mathrm{Im}(\zeta), \qquad \eta' := \mathrm{Im}(\zeta').$$

It is sometimes convenient to leave out the square bracket when writing $[k\zeta]$ with $\zeta \in \mathbf{i}\mathfrak{g}$. Thus we write $k\mathbf{i}\eta \in T_{[k]}\mathrm{G}^c/\mathrm{G}$ instead of $[k\mathbf{i}\eta]$. In particular, we use this notation to avoid any possible confusion with the Lie bracket.

Lemma 6.3. *Assume (C). Then the following holds.*
(i) *The geodesics in* G^c/G *have the form*

$$\gamma(t) = [k_0 \exp(\mathbf{i}t\eta)]$$

for $k_0 \in \mathrm{G}^c$ *and* $\eta \in \mathfrak{g}$.
(ii) *The Riemann curvature tensor on* G^c/G *is given by*

$$R(k\mathbf{i}\xi, k\mathbf{i}\eta)k\mathbf{i}\zeta = k\mathbf{i}[[\xi,\eta],\zeta]$$

for $k \in \mathrm{G}^c$ *and* $\xi,\eta,\zeta \in \mathfrak{g}$. *Thus* G^c/G *has nonpositive sectional curvature.*

Proof. The proof has three steps. The first step gives a formula for the Levi–Civita connection on G^c/G.

Step 1. *Let* $k : \mathbb{R} \to \mathrm{G}^c$ *and* $\xi : \mathbb{R} \to \mathfrak{g}^c$ *be smooth curves and denote*

$$\gamma(t) := [k(t)] \in \mathrm{G}^c, \qquad X(t) := [k(t)\xi(t)] \in T_{\gamma(t)}\mathrm{G}^c/\mathrm{G}.$$

Then

$$\nabla_t X(t) = [k(t)\eta(t)], \qquad \eta(t) := \dot\xi(t) + [\mathrm{Re}(k(t)^{-1}\dot k(t), \xi(t)].$$

To prove that the formula is well defined we must choose a smooth map $g : \mathbb{R} \to \mathrm{G}$ and replace k, ξ, η by

$$\widetilde{k} := kg, \qquad \widetilde{\xi} := g^{-1}\xi g, \qquad \widetilde{\eta} := \partial_t\widetilde{\xi} + [\mathrm{Re}(\widetilde{k}^{-1}\partial_t\widetilde{k}), \widetilde{\xi}]$$

and show that

$$\widetilde{\eta} = g\eta g^{-1}.$$

We must then show that the connection is Riemannian, i.e.

$$\partial_t \langle X, Y\rangle = \langle \nabla_t X, Y\rangle + \langle X, \nabla_t Y\rangle$$

for ant two vector fields along a curve γ, and that it is torsion free, i.e.

$$\nabla_s\partial_t\gamma = \nabla_t\partial_s\gamma$$

for any smooth map $\gamma : \mathbb{R}^2 \to \mathrm{G}^c/\mathrm{G}$ of two variables. These assertions follow easily by direct calculations which are left to the reader.

Step 2. *We prove (i).*

A smooth curve $\gamma(t) = [k(t)]$ is a geodesic in G^c/G if and only if $\nabla_t\dot{\gamma} \equiv 0$. By Step 1 this is equivalent to the differential equation

$$\partial_t \mathrm{Im}(k^{-1}\partial_t k) + [\mathrm{Re}(k^{-1}\partial_t k), \mathrm{Im}(k^{-1}\partial_t k)] = 0. \tag{18}$$

A function $k : \mathbb{R} \to \mathrm{G}^c$ satisfies this equation if and only if it has the form $k(t) = k_0 \exp(\mathbf{i}t\eta)g(t)$ for some $k_0 \in \mathrm{G}^c$, $\eta \in \mathfrak{g}$, and $g : \mathbb{R} \to \mathrm{G}$.

Step 3. *We prove (ii).*

Choose maps $\gamma : \mathbb{R}^2 \to \mathrm{G}^c$ and $\zeta : \mathbb{R}^2 \to \mathfrak{g}^c$ and denote

$$\xi := k^{-1}\partial_s k, \qquad \eta := k^{-1}\partial_t,$$

and

$$\gamma := [k], \qquad X := [k\xi] = \partial_s\gamma, \qquad Y := [k\eta] = \partial_t\gamma, \qquad Z := [k\zeta].$$

Then $\partial_s\eta - \partial_t\xi + [\xi, \eta] = 0$ and

$$\begin{aligned} \nabla_s Z &= [k\zeta_s], \qquad \zeta_s := \partial_s\zeta + [\mathrm{Re}(\xi), \zeta], \\ \nabla_t Z &= [k\zeta_t], \qquad \zeta_t := \partial_t\zeta + [\mathrm{Re}(\eta), \zeta]. \end{aligned}$$

Hence we obtain

$$R(X, Y)Z = \nabla_s\nabla_t Z - \nabla_t\nabla_s Z = [k\rho],$$

where

$$\begin{aligned} \rho &= \partial_s\zeta_t + [\mathrm{Re}(\xi), \zeta_t] - \partial_t\zeta_s - [\mathrm{Re}(\eta), \zeta_s] \\ &= [\mathrm{Re}(\partial_s\eta), \zeta] + [\mathrm{Re}(\xi), [\mathrm{Re}(\eta), \zeta]] \\ &\quad - [\mathrm{Re}(\partial_t\xi), \zeta] - [\mathrm{Re}(\eta), [\mathrm{Re}(\xi), \zeta]] \\ &= -[\mathrm{Re}([\xi, \eta]), \zeta] - [\zeta, [\mathrm{Re}(\xi), \mathrm{Re}(\eta)]] \\ &= [[\mathrm{Im}(\xi), \mathrm{Im}(\eta)], \zeta]. \end{aligned}$$

Thus we have $R(X, Y)Z = k\mathbf{i}\mathrm{Im}(\rho) = k\mathbf{i}[[\mathrm{Im}(\xi), \mathrm{Im}(\eta)], \mathrm{Im}(\zeta)]$ and the sectional curvature is $\langle R(X, Y)Y, X\rangle = -\left|[\mathrm{Im}(\xi), \mathrm{Im}(\eta)]\right|^2 \leq 0$. This proves Lemma 6.3. □

Proof of Theorem 6.1. Assume (C). We prove in four steps that the map

$$\mathrm{G} \times \mathfrak{g} \to \mathrm{G}^c : (g, \eta) \mapsto \exp(\mathbf{i}\eta) g$$

is a diffeomorphism.

Step 1. *If* $\eta \in \mathfrak{g}$ *and* $\exp(\mathbf{i}\eta) \in \mathrm{G}$ *then* $[\xi, \eta] = 0$ *for every* $\xi \in \mathfrak{g}$.

Define $\gamma : \mathbb{R}^2 \to \mathrm{G}^c/\mathrm{G}$ by

$$\gamma(s,t) := [\exp(\mathbf{i}s\xi)\exp(\mathbf{i}t\eta)].$$

By Lemma 6.3 the curve $t \mapsto \gamma(s,t)$ is a geodesic for every s, and by assumption it is periodic with period 1. Denote

$$X(s,t) := \partial_s \gamma(s,t) \in T_{\gamma(s,t)}\mathrm{G}^c/\mathrm{G}.$$

Since $t \mapsto \gamma(s,t)$ is a geodesic for every s we have that X satisfies the Jacobi equation (17). Since $X(s,t+1) = X(s,t)$ we obtain, as in the proof of Theorem 4.2, that X satisfies

$$\begin{aligned} 0 &= \int_0^1 \partial_t \langle \nabla_t X, X \rangle \, dt \\ &= \int_0^1 \left(|\nabla_t X|^2 + \langle \nabla_t \nabla_t X, X \rangle \right) dt \\ &= \int_0^1 \left(|\nabla_t X|^2 - \langle R(X, \partial_t\gamma)\partial_t\gamma, X \rangle \right) dt \end{aligned}$$

for every s. Since G^c/G has nonpositive sectional curvature, by Lemma 6.3, we deduce that the function $\langle R(X, \partial_t\gamma)\partial_t\gamma, X\rangle$ vanishes identically. With $s = t = 0$ we have $X(0,0) = [\mathbf{i}\xi]$ and $\partial_t\gamma(0,0) = [\mathbf{i}\eta]$ and hence

$$0 = \langle R(i\xi, i\eta)i\eta, i\xi \rangle = -\left|[\xi,\eta]\right|^2.$$

Here the last equation follows from Lemma 6.3 (ii). This proves Step 1.

Step 2. *If* $\eta \in \mathfrak{g}$ *and* $\exp(i\eta) \in \mathrm{G}$ *then* $\eta = 0$.

This is the only place in the proof where we use the fact that G is a maximal compact subgroup of G^c. Suppose by contradiction that $\eta \neq 0$. Then $\exp(\mathbf{i}t\eta) \notin \mathrm{G}$ for small t and hence

$$0 < \lambda := \inf \{ t > 0 \,|\, \exp(\mathbf{i}t\eta) \in \mathrm{G} \} \leq 1.$$

Moreover, $[\xi,\eta]=0$ for all $\xi\in\mathfrak{g}$ by Step 1 and so $[\zeta,\mathbf{i}\eta]=0$ for all $\zeta\in\mathfrak{g}^c$. Denote by G_0^c the identity component of G^c. Then $g^{-1}\mathbf{i}\eta g=\mathbf{i}\eta$ for all $g\in\mathrm{G}_0^c$ and hence $\exp(\mathbf{i}t\eta)g=g\exp(\mathbf{i}t\eta)$ for all $t\in\mathbb{R}$ and all $g\in\mathrm{G}_0^c$. In particular, this holds for $t=\lambda/2$ and so the element

$$h:=\exp(\mathbf{i}\lambda\eta/2)\in\mathrm{G}_0^c\setminus\mathrm{G}$$

commutes with every element of G_0^c. Since $h^2\in\mathrm{G}\cap\mathrm{G}_0^c$ this shows that

$$\mathrm{H}:=(\mathrm{G}\cup h\mathrm{G})\cap\mathrm{G}_0^c=(\mathrm{G}\cap\mathrm{G}_0^c)\cup h(\mathrm{G}\cap\mathrm{G}_0^c)$$

is a compact subgroup of G^c. Next we will use the fact that the Riemannian manifold G^c/G is connected and geodesically complete by Lemma 6.3. Thus the Hopf–Rinow theorem asserts that any two elements in G^c/G can be joined by a geodesic and so the map $\mathrm{G}\times\mathfrak{g}\to\mathrm{G}^c:(u,\xi)\mapsto\exp(\mathbf{i}\xi)u$ is surjective. Hence G^c has finitely many connected components. Since G is a maximal compact subgroup of G^c, it then follows from the Cartan–Iwasawa–Malcev Theorem in [3, Thm 14.1.3] that there exists an element $g\in\mathrm{G}^c$ such that $g^{-1}Hg\subset\mathrm{G}$. Thus $g^{-1}(\mathrm{G}\cap\mathrm{G}_0^c)g\subset\mathrm{G}\cap\mathrm{G}_0^c$ and $g^{-1}hg\in\mathrm{G}\cap\mathrm{G}_0^c$, and hence $h\in g(\mathrm{G}\cap\mathrm{G}_0^c)g^{-1}=\mathrm{G}\cap\mathrm{G}_0^c$ in contradiction to the fact that $h\notin\mathrm{G}$. This proves Step 2.

Step 3. G^c/G *is simply connected.*

Suppose not. Then, by the usual variational argument, there exists a nonconstant geodesic $\gamma:[0,1]\to\mathrm{G}^c/\mathrm{G}$ based at $\gamma(0)=\gamma(1)=[1]$. By Lemma 6.3 the geodesic has the form

$$\gamma(t)=[\exp(\mathbf{i}t\eta)]$$

for some $\eta\in\mathfrak{g}$. Since $\gamma(1)=[1]$ we have $\exp(i\eta)\in\mathrm{G}$ and hence $\eta=0$ by Step 2. Thus the geodesic is constant, a contradiction. This proves Step 3.

Step 4. *The map* $\mathrm{G}\times\mathfrak{g}\to\mathrm{G}^c:(g,\eta)\mapsto\exp(\mathbf{i}\eta)g$ *is a diffeomorphism.*

By assumption the quotient manifold G^c/G is connected, by Step 3 it is simply connected, and by Lemma 6.3 it is complete and has nonpositive sectional curvature. Hence Step 4 follows from Hadamard's theorem. Namely, the exponential map

$$T_{[1]}\mathrm{G}^c/\mathrm{G}\to\mathrm{G}^c/\mathrm{G}:[\mathbf{i}\eta]\mapsto[\exp(\mathbf{i}\eta)]$$

(the Riemannian and Lie group meanings of the term coincide in this case) is a diffeomorphism by Theorem 4.2, and this is equivalent to Step 4. This proves Theorem 6.1. □

7 Matrix factorization

Theorem 7.1. *Assume (C) on page 20 and let* $\xi \in \mathfrak{g}$ *such that* $\exp(\xi) = 1$. *Then, for every* $g \in \mathrm{G}^c$, *there exists a pair* $p, p^+ \in \mathrm{G}^c$ *such that*

$$\lim_{t\to\infty} \exp(\mathbf{i}t\xi)p\exp(-\mathbf{i}t\xi) = p^+, \qquad pg^{-1} \in \mathrm{G}.$$

Proof. See page 30. □

The proof uses the fact that every compact Lie group embedds into $\mathrm{U}(n)$ for some integer $n \in \mathbb{N}$ and relies on the next four lemmas.

Lemma 7.2. *Assume (C) on page 20 and let* $\xi \in \mathfrak{g}$ *such that* $\exp(\xi) = 1$. *Then the set*

$$\mathrm{P} := \left\{ p \in \mathrm{G}^c \,|\, \textit{the limit } \lim_{t\to\infty} \exp(\mathbf{i}t\xi)p\exp(-\mathbf{i}t\xi) \textit{ exists in } \mathrm{G}^c \right\} \tag{19}$$

is a Lie subgroup of G^c *with Lie algebra*

$$\mathfrak{p} := \left\{ \zeta \in \mathfrak{g}^c \,|\, \textit{the limit } \lim_{t\to\infty} \exp(\mathbf{i}t\xi)\zeta\exp(-\mathbf{i}t\xi) \textit{ exists in } \mathfrak{g}^c \right\}. \tag{20}$$

Proof. Assume $\mathrm{G} \subset \mathrm{U}(n)$ and $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$. Then $\mathbf{i}\xi$ is a Hermitian matrix with eigenvalues in $2\pi\mathbb{Z}$. Consider a decomposition $\mathbb{C}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ into eigenspaces $E_j \subset \mathbb{C}^n$ of $\mathbf{i}\xi$ with eigenvalues λ_j and choose the ordering such that $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Write a matrix $\zeta \in \mathfrak{g}^c \subset \mathfrak{gl}(n, \mathbb{C})$ in the form

$$\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1k} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{k1} & \zeta_{k2} & \cdots & \zeta_{kk} \end{pmatrix}, \qquad \zeta_{ij} \in \mathrm{Hom}(E_j, E_i).$$

Then

$$\exp(\mathbf{i}t\xi)\zeta\exp(-\mathbf{i}t\xi) = \begin{pmatrix} \zeta_{11} & e^{(\lambda_1-\lambda_2)t}\zeta_{12} & \cdots & e^{(\lambda_1-\lambda_k)t}\zeta_{1k} \\ e^{(\lambda_2-\lambda_1)t}\zeta_{21} & \zeta_{22} & \cdots & e^{(\lambda_2-\lambda_k)t}\zeta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(\lambda_k-\lambda_1)t}\zeta_{k1} & e^{(\lambda_k-\lambda_2)t}\zeta_{k2} & \cdots & \zeta_{kk} \end{pmatrix}.$$

Thus $\zeta \in \mathfrak{p}$ if and only if $\zeta \in \mathfrak{g}$ and $\zeta_{ij} = 0$ for $i > j$. Likewise, $g \in \mathrm{P}$ if and only if $g \in \mathrm{G}^c$ and $g_{ij} = 0$ for $i > j$. Hence P is a closed subset of G^c. Since every closed subgroup of a Lie group is a Lie subgroup, this proves Lemma 7.2. □

Lemma 7.3. *Let $N \in \mathbb{N}$. There exist real numbers*

$$\beta_0(N), \beta_1(N), \dots, \beta_{2N-1}(N)$$

such that $\beta_\nu(N) = 0$ when ν is even and, for $k = 1, 3, 5, \dots, 4N-1$,

$$\sum_{\nu=0}^{2N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = \begin{cases} \mathbf{i}, & \text{if } 0 < k < 2N, \\ -\mathbf{i}, & \text{if } 2N < k < 4N. \end{cases} \tag{21}$$

Proof. Define $\lambda := \exp(\frac{\pi\mathbf{i}}{2N})$ and consider the Vandermonde matrix

$$\Lambda := \begin{pmatrix} \lambda & \lambda^3 & \lambda^5 & \dots & \lambda^{2N-1} \\ \lambda^3 & \lambda^9 & \lambda^{15} & \dots & \lambda^{6N-3} \\ \lambda^5 & \lambda^{15} & \lambda^{25} & \dots & \lambda^{10N-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{2N-1} & \lambda^{6N-3} & \lambda^{10N-5} & \dots & \lambda^{(2N-1)^2} \end{pmatrix} \in \mathbb{C}^{N\times N}.$$

Its complex determinant is

$$\det^c(\Lambda) = \lambda^{(2N-1)(N-1)} \prod_{0\le i<j\le N-1} \left(\lambda^{4j} - \lambda^{4i}\right).$$

Since λ is a primitive $4N$th root of unity, the numbers λ^{4i}, $i = 0, \dots, N-1$, are pairwise distinct. Hence Λ is nonsingular. Hence there exists a unique vector $z = (z_1, z_3, \dots, z_{2N-1}) \in \mathbb{C}^N$ such that

$$\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) z_\nu = \mathbf{i}, \qquad k = 1, 3, \dots, 2N-1. \tag{22}$$

The numbers z_ν also satisfy the equation

$$\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \exp\left(\frac{(2N-k)\nu\pi\mathbf{i}}{2N}\right) \overline{z}_\nu = - \overline{\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) z_\nu} = \mathbf{i}$$

for $k = 1, 3, \dots, 2N-1$ and hence they are real.

Define $\beta_\nu(N) := z_\nu$ for $\nu = 1, 3, \dots, 2N-1$ and $\beta_\nu(N) := 0$ for ν even. These numbers satisfy (21) for $k = 1, 3, \dots, 2N-1$ by (22). That equation (21) also holds for $k = 2N+1, 2N+3, \dots, 4N-1$ follows from the fact that $\exp(k\pi\mathbf{i}) = -1$ whenever k is odd. This proves Lemma 7.3. $\square$

Lemma 7.4. *Let $m \in \mathbb{N}$ and $N := 2^m$. There exist real numbers*

$$\alpha_0(N), \alpha_1(N), \ldots, \alpha_{2N-1}(N)$$

such that $\alpha_\nu(N) = 0$ when ν is even and, for every $k \in \{0, 1, \ldots, 2N-1\}$,

$$\sum_{\nu=0}^{2N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{N}\right) = \begin{cases} \mathbf{i}, & \text{if } 1 \le k \le N-1, \\ -\mathbf{i}, & \text{if } N+1 \le k \le 2N-1, \\ 0, & \text{if } k = 0 \text{ or } k = N. \end{cases} \tag{23}$$

Proof. The proof is by induction on m. For $m = 1$ and $N = 2^m = 2$ choose $\alpha_1(2) := 1/2$ and $\alpha_3(2) := -1/2$. Then

$$\sum_{\nu=0}^{3} \alpha_\nu(2) \exp\left(\frac{k\nu\pi\mathbf{i}}{2}\right) = \frac{\mathbf{i}^k - (-\mathbf{i})^k}{2} = \begin{cases} \mathbf{i}, & \text{for } k = 1, \\ -\mathbf{i}, & \text{for } k = 3, \\ 0, & \text{for } k = 0, 2. \end{cases}$$

Now let $m \in \mathbb{N}$ and define $N := 2^m$. Assume, by induction, that the numbers $\alpha_\nu(N)$, $\nu = 0, 1, \ldots, 2N-1$, have been found such that (23) holds for $k = 0, 1, \ldots, 2N-1$. Let $\beta_\nu(N)$, $\nu = 0, 1, \ldots, 2N-1$, be the constants of Lemma 7.3. Define

$$\alpha_{2N+\nu}(N) := \alpha_\nu(N), \qquad \beta_{2N+\nu}(N) := -\beta_\nu(N),$$

for $\nu = 0, 1, 2, \ldots, 2N-1$ and

$$\alpha_\nu(2N) := \frac{\alpha_\nu(N) + \beta_\nu(N)}{2}, \qquad \nu = 0, 1, 2, \ldots, 4N-1. \tag{24}$$

Then

$$\sum_{\nu=0}^{4N-1} \alpha_\nu(2N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = A_k + B_k,$$

where

$$\begin{aligned} A_k &:= \frac{1}{2} \sum_{\nu=0}^{4N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right), \\ B_k &:= \frac{1}{2} \sum_{\nu=0}^{4N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right). \end{aligned}$$

Since $\alpha_{2N+\nu}(N) = \alpha_\nu(N)$, we have

$$\begin{aligned} A_k &= \frac{1}{2}\sum_{\nu=0}^{4N-1} \alpha_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1+\exp(k\pi\mathbf{i})}{2}\sum_{\nu=0}^{2N-1} \alpha_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1+(-1)^k}{2}\sum_{\nu=0}^{2N-1} \alpha_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right). \end{aligned}$$

If k is odd then the right hand side vanishes. If k is even it follows from the induction hypothesis that

$$A_k = \sum_{\nu=0}^{2N-1} \alpha_\nu(N)\exp\left(\frac{(k/2)\nu\pi\mathbf{i}}{N}\right) = \begin{cases} \mathbf{i}, & \text{for } k = 2,4,\ldots,2N-2, \\ -\mathbf{i}, & \text{for } k = 2N+2,\ldots,4N-2, \\ 0, & \text{for } k = 0, 2N. \end{cases}$$

Since $\beta_{2N+\nu}(N) = -\beta_\nu(N)$, we have

$$\begin{aligned} B_k &= \frac{1}{2}\sum_{\nu=0}^{2N-1} \beta_\nu(N)\left(\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) - \exp\left(\frac{k(2N+\nu)\pi\mathbf{i}}{2N}\right)\right) \\ &= \frac{1-\exp(k\pi\mathbf{i})}{2}\sum_{\nu=0}^{2N-1} \beta_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1-(-1)^k}{2}\sum_{\nu=0}^{2N-1} \beta_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right). \end{aligned}$$

If k is even then the right hand side vanishes. If k is odd it follows from Lemma 7.3 that

$$B_k = \sum_{\nu=0}^{2N-1} \beta_\nu(N)\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = \begin{cases} \mathbf{i}, & \text{if } k = 1,3,\ldots,2N-1, \\ -\mathbf{i}, & \text{if } k = 2N+1,\ldots,4N-1. \end{cases}$$

Combining the formulas for A_k and B_k we find

$$A_k + B_k = \begin{cases} \mathbf{i}, & \text{for } k = 1,2,3,\ldots,2N-1, \\ -\mathbf{i}, & \text{for } k = 2N+1, 2N+2,\ldots,4N-1, \\ 0, & \text{for } k = 0, 2N, \end{cases}$$

and this proves Lemma 7.4. □

Lemma 7.5. *Assume (C) on page 20 and let $\xi \in \mathfrak{g}$ such that $\exp(\xi) = 1$. Then, for every $\eta \in \mathfrak{g}$, there exists a pair $\zeta, \zeta^+ \in \mathfrak{g}^c$ such that*

$$\lim_{t\to\infty} \exp(\mathbf{i}t\xi)\zeta\exp(-\mathbf{i}t\xi) = \zeta^+, \qquad \zeta - \mathbf{i}\eta \in \mathfrak{g}.$$

Proof. Assume without loss of generality that $\mathrm{G} \subset \mathrm{U}(n)$ and $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$. Let $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$ and $\lambda_1 < \cdots < \lambda_k$ be as in Lemma 7.2. Then

$$\lambda_i - \lambda_j = 2\pi m_{ij}, \qquad m_{ij} \in \mathbb{Z},$$

with $m_{ij} > 0$ for $i > j$ and $m_{ij} < 0$ for $i < j$. Choose $m \in \mathbb{N}$ such that

$$N := 2^m > m_{k1} = \frac{\lambda_k - \lambda_1}{2\pi}.$$

Choose $\alpha_0, \ldots, \alpha_{2N-1} \in \mathbb{R}$ as in Lemma 7.4. Let

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1k} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k1} & \eta_{k2} & \cdots & \eta_{kk} \end{pmatrix} \in \mathfrak{g},$$

where $\eta_{ij} \in \mathrm{Hom}(E_j, E_i)$. Define

$$\zeta := \mathbf{i}\eta - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(-\frac{\nu}{2N}\xi\right)\eta\exp\left(\frac{\nu}{2N}\xi\right) \in \mathfrak{g}^c.$$

Then, for $i > j$, we have

$$\begin{aligned} \zeta_{ij} &= \mathbf{i}\eta_{ij} - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(\frac{\nu}{2N}\mathbf{i}(\lambda_i - \lambda_j)\right)\eta_{ij} \\ &= \left(\mathbf{i} - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(\frac{m_{ij}\nu\pi\mathbf{i}}{N}\right)\right)\eta_{ij} \\ &= 0. \end{aligned}$$

The last equation follows from Lemma 7.4 and the fact that $1 \le m_{ij} \le N-1$ for $i > j$. Since $\zeta_{ij} = 0$ for $i > j$ it follows from the proof of Lemma 7.2 that $\zeta \in \mathfrak{p}$. Moreover, by construction $\mathbf{i}\eta - \zeta \in \mathfrak{g}$. This proves Lemma 7.5. $\square$

Proof of Theorem 7.1. Let $\mathrm{P} \subset \mathrm{G}^c$ and $\mathfrak{p} \subset \mathfrak{g}^c$ be defined by (19) and (20) (see Lemma 7.2). We prove in three steps that, for every $g \in \mathrm{G}^c$, there exists a $p \in \mathrm{P}$ such that $pg^{-1} \in \mathrm{G}$.

Step 1. *The set*

$$A := \left\{ g \in \mathrm{G}^c \,|\, \exists\, p \in \mathrm{P} \text{ such that } pg^{-1} \in \mathrm{G} \right\}$$

is (relatively) closed in G^c*.*

Let $g_i \in A$ be a sequence which converges to an element $g \in \mathrm{G}^c$. Then there exists a sequence $p_i \in \mathrm{P}$ such that $u_i := p_i g_i^{-1} \in \mathrm{G}$. Since G is compact there exists a subsequence (still denoted by u_i) which converges to an element $u \in \mathrm{G}$. Since P is a closed subset of G^c, by Step 1 in the proof of Lemma 7.4, we have $p := ug = \lim_{i\to\infty} u_i g_i = \lim_{i\to\infty} p_i \in \mathrm{P}$. Hence $pg^{-1} = u \in \mathrm{G}$ and hence $g \in A$. This proves Step 1.

Step 2. *The function* $f : P \times \mathfrak{g} \to \mathrm{G}^c$*, defined by*

$$f(p, u) := pu$$

for $p \in \mathrm{P}$ *and* $u \in \mathrm{G}$*, is a submersion.*

Let $p \in \mathrm{P}$ and $u \in \mathrm{G}$ and denote $g := f(p, u) = pu$. Let $\widehat{g} \in T_g\mathrm{G}^c$ and denote

$$\widetilde{\zeta} := p^{-1}\widehat{g}u^{-1} = u(g^{-1}\widehat{g})u^{-1} \in \mathfrak{g}^c. \tag{25}$$

Let $\eta \in \mathfrak{g}$ be the imginary part of $\widetilde{\zeta}$ so that $\widetilde{\zeta} - \mathbf{i}\eta \in \mathfrak{g}$. By Lemma 7.5, there exists an element $\zeta \in \mathfrak{p}$ such that $\zeta - \mathbf{i}\eta \in \mathfrak{g}$ and hence $\widetilde{\zeta} - \zeta \in \mathfrak{g}$. Define

$$\widehat{p} := p\zeta, \qquad \widehat{u} := \left(\widetilde{\zeta} - \zeta\right) u.$$

Then $\widehat{p} \in T_p\mathrm{P}$, $\widehat{u} \in T_u\mathrm{G}$, and

$$df(p, u)(\widehat{p}, \widehat{u}) = \widehat{p}u + p\widehat{u} = p\widetilde{\zeta}u = \widehat{g}.$$

Here the last equation follows from (25). Thus we have proved that the differential $df(p, u) : T_p\mathrm{P} \times T_u\mathrm{G} \to T_{pu}\mathrm{G}^c$ is surjective for every $p \in \mathrm{P}$ and every $u \in \mathrm{G}$. Hence f is a submersion and this proves Step 2.

Step 3. $A = \mathrm{G}^c$*.*

The set A contains G by definition. By Step 1 it is closed and by Step 2 it is the image of a submersion and hence is open. Since G^c is homeomorphic to $\mathrm{G} \times \mathfrak{g}$ (see Theorem 6.1) and A contains $\mathrm{G} \cong \mathrm{G} \times \{0\}$, it follows that A intersects each connected component of G^c in a nonempty open and closed set. Hence $A = \mathrm{G}^c$. This proves Step 3 and Theorem 7.1. $\square$

8 Proof of the main theorems

Proof of Theorem 1.4 "(ii) $\Longrightarrow$ (i)". Assume (C) on page 20. (In the notation of Theorem 1.4 the map ι is the inclusion of G into G^c.) Then the quotient G^c/G is simply connected by Theorem 6.1. Let H be a complex Lie group with Lie algebra $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$ and let

$$\rho : \mathrm{G} \to \mathrm{H}$$

be a Lie group homomorphism. We use the following two basic facts to construct the homomorphism $\rho^c : \mathrm{G}^c \to \mathrm{H}$ that extends ρ.

Fact 1. *Since G^c/G is connected there exists, for every $a \in \mathrm{G}^c$, a smooth path $\alpha : [0,1] \to \mathrm{G}^c$ such that $\alpha(0) \in \mathrm{G}$ and $\alpha(1) = a$.*

Fact 2. *Since G^c/G is simply connected, any two paths $\alpha_0, \alpha_1 : [0,1] \to \mathrm{G}^c$ as in Fact 1 can be joined by a smooth homotopy $\{\alpha_s\}_{0\le s\le 1}$ satisfying $\alpha_s(0) \in \mathrm{G}$ and $\alpha_s(1) = a$ for every $s \in [0,1]$.*

We define ρ^c as follows. Let $\Phi := d\rho(1) : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie algebra homomorphism and define $\Phi^c : \mathfrak{g}^c \to \mathfrak{h}$ as the complexification of Φ. Given an element $a \in \mathrm{G}^c$ choose α as in Fact 1, let $\beta : [0,1] \to \mathrm{H}$ be the unique solution of the differential equation

$$\beta^{-1}\dot{\beta} = \Phi^c(\alpha^{-1}\dot{\alpha}), \qquad \beta(0) = \rho(\alpha(0)), \tag{26}$$

and define

$$\rho^c(a) := \beta(1).$$

We prove first that ρ^c is well defined, i.e. that the endpoint $\beta(1)$ does not depend on the choice of the path α. By Fact 2 any two paths α_0 and α_1 with $\alpha_0(0), \alpha_1(0) \in \mathrm{G}$ and $\alpha_0(1) = \alpha_1(1) = a$ can be joined by a smooth homotopy $[0,1]^2 \to \mathrm{G}^c : (s,t) \mapsto \alpha_s(t) = \alpha(s,t)$ such that $\alpha_s(0) \in \mathrm{G}$ and $\alpha_s(1) = a$ for all s. Define $\beta : [0,1]^2 \to \mathrm{H}$ by

$$\beta^{-1}\partial_t\beta = \Phi^c(\alpha^{-1}\partial_t\alpha), \qquad \beta(s,0) = \rho(\alpha(s,0)).$$

We claim that

$$\beta^{-1}\partial_s\beta = \Phi^c(\alpha^{-1}\partial_s\alpha). \tag{27}$$

To see this, abbreviate

$$\xi := \alpha^{-1}\partial_s\alpha, \qquad \eta := \alpha^{-1}\partial_t\alpha, \qquad \xi' := \beta^{-1}\partial_s\beta, \qquad \eta' := \beta^{-1}\partial_t\beta.$$

Then

$$\partial_t \xi' = \partial_s \eta' + [\xi', \eta'], \qquad \partial_t \Phi^c(\xi) = \partial_s \Phi^c(\eta) + [\Phi^c(\xi), \Phi^c(\eta)].$$

Moreover, when $t = 0$ we have $d\rho(\alpha)\alpha\xi = \rho(\alpha)\Phi\xi$ and hence

$$\xi'(s,0) = \beta(s,0)^{-1}\partial_s\beta(s,0) = \Phi(\alpha(s,0)^{-1}\partial_s\alpha(s,0)) = \Phi(\xi(s,0)).$$

Hence both functions $t \mapsto \xi'(s,t)$ and $t \mapsto \Phi^c(\xi(s,t))$ satisfy the same initial value problem and hence agree. This proves (27). It follows that $\partial_s\beta(1,s) = 0$ and this shows that ρ^c is well defined.

We prove that, for $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$,

$$\Phi^c(a^{-1}\xi a) = \rho^c(a)^{-1}\Phi^c(\xi)\rho^c(a). \tag{28}$$

Choose α and β as in the definition of $\rho^c(a)$ and define

$$\eta(t) := \alpha(t)^{-1}\xi\alpha(t), \qquad \eta'(t) := \beta(t)^{-1}\Phi^c(\xi)\beta(t).$$

Then $\Phi^c(\eta)$ and η' satsify the same differential equation

$$\dot{\eta}' + [\beta^{-1}\dot{\beta}, \eta'] = 0$$

and the same initial condition and hence have the same endpoints. This proves equation (28).

We prove that ρ^c is a group homomorphism. Let $a_1, a_2 \in \mathrm{G}^c$ and choose α_j and β_j as in the definition of $\rho^c(a_j)$ for $j = 1,2$. Then $\rho_c(\alpha_j(t)) = \beta_j(t)$ for $0 \le t \le 1$ and $j = 1,2$. Define

$$\alpha := \alpha_1\alpha_2, \qquad \beta := \beta_1\beta_2.$$

Then

$$\begin{aligned}
\beta^{-1}\dot{\beta} &= \beta_2^{-1}\dot{\beta}_2 + \beta_2^{-1}\beta_1^{-1}\dot{\beta}_1\beta_2 \\
&= \Phi^c(\alpha_2^{-1}\dot{\alpha}_2) + \rho^c(\alpha_2)^{-1}\Phi^c(\alpha_1^{-1}\dot{\alpha}_1)\rho^c(\alpha_2) \\
&= \Phi^c(\alpha_2^{-1}\dot{\alpha}_2 + \alpha_2^{-1}\alpha_1^{-1}\dot{\alpha}_1\alpha_2) \\
&= \Phi^c(\alpha^{-1}\dot{\alpha}).
\end{aligned}$$

Here we have used (28). It follows that $\rho^c(a_1a_2) = \beta(1) = \rho^c(a_1)\rho^c(a_2)$ and so ρ^c is a group homomorphism.

We prove that ρ^c is smooth. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{G}^c \arrow[r, "\rho^c"] & \mathrm{H}\,, \\ \mathrm{G}\times\mathfrak{g} \arrow[u, "\approx"] \arrow[ur] & \end{tikzcd}$$

where the map $\mathrm{G}\times\mathfrak{g}\to\mathrm{G}^c$ is the diffeomorphism of Theorem 6.1 and the map $\mathrm{G}\times\mathfrak{g}\to\mathrm{H}$ is given by $(g,\eta)\mapsto\exp(\mathbf{i}\Phi^c(\eta))\rho(g)$ and hence is smooth. That the differential of ρ^c at 1 is equal to Φ^c follows also from this diagram. Thus we have proved that (ii) implies (i) in Theorem 1.4. $\square$

Proof of Theorem 1.5. By Theorem 3.6 (the intrinsic construction for compact connected Lie groups), respectively Theorem 2.1 (for possibly disconnected Lie subgroups of $\mathrm{U}(n)$), there is an embedding $\iota:\mathrm{G}\to\mathrm{G}^c$ into a complex Lie group (diffeomorphic to $\mathrm{G}\times\mathfrak{g}$) that satisfies condition (ii) in Theorem 1.4. Since (ii) implies (i) in Theorem 1.4, the embedding $\iota:\mathrm{G}\to\mathrm{G}^c$ satisfies both (i) and (ii) in Theorem 1.4 and hence is a complexification. Moreover, any two embeddings of G into a complex Lie group that satisfy (i) in Theorem 1.4 are naturally isomorphic. This proves Theorem 1.5. $\square$

Proof of Theorem 1.4 "(i) $\Longrightarrow$ (ii)". Let $\iota:\mathrm{G}\to\mathrm{G}^c$ be an embedding into a complex Lie group that satisfies (i). By Theorem 1.5 there exists an embedding $\widetilde{\iota}:\mathrm{G}\to\widetilde{\mathrm{G}}^c$ into a complex Lie group that satisfies both (i) and (ii). Since both embeddings satisfy (i), there exists a unique holomorphic Lie group isomorphism $\phi:\mathrm{G}^c\to\widetilde{\mathrm{G}}^c$ such that $\phi\circ\iota=\widetilde{\iota}$. Since the embedding $\widetilde{\iota}$ satisfies (ii), so does ι. This proves Theorem 1.4. $\square$

Proof of Theorem 1.6. Assume (C) on page 20. Let $\mathrm{K}\subset\mathrm{G}^c$ be a compact subgroup and consider the map

$$\mathrm{K}\times\mathrm{G}^c/\mathrm{G}\to\mathrm{G}^c/\mathrm{G}:(k,[g])\mapsto[kg].$$

By definition of the Riemannian metric on G^c/G in 6.2 this is a group action by isometries. By Theorem 6.1 and Lemma 6.3, the quotient space G^c/G is a complete connected simply connected Riemannian manifold with nonpositive sectional curvature. Hence it follows from Theorem 5.1 that the action of K on G^c/G has a fixed point. Let $[g]\in\mathrm{G}^c/\mathrm{G}$ be such a fixed point. Then $[kg]=[g]$ for every $k\in\mathrm{K}$. In other words, for every $k\in K$ there exists an $h\in\mathrm{G}$ such that $kg=gh$. This means that $g^{-1}kg\in\mathrm{G}$ for every $k\in K$ or, equivalently, $g^{-1}Kg\subset\mathrm{G}$. This proves Theorem 1.6. $\square$

Proof of Theorem 1.7. Assume (C) on page 20. Let $\zeta \in \mathfrak{g}^c$ such that

$$\exp(\zeta) = 1.$$

We prove in three steps that there exist two elements $p, p^+ \in \mathrm{G}^c$ such that $p^{-1}\zeta p \in \mathfrak{g}$ and $\lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta) = p^+$.

Step 1. *There exists an element $g \in \mathrm{G}^c$ such that $g^{-1}\zeta g \in \mathfrak{g}$.*

The set $S := \{\exp(s\zeta) \,|\, s \in \mathbb{R}\}$ is a compact subgroup of G^c. Hence it follows from Theorem 1.6 that there exists an element $g \in \mathrm{G}^c$ such that $g^{-1}Sg \subset \mathrm{G}$. Thus $g^{-1}\zeta g = \frac{d}{ds}\big|_{s=0} g^{-1}\exp(s\zeta)g \in \mathfrak{g}$ and this proves Step 1.

Step 2. *Let $g \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}$ such that $\exp(\xi) = 1$. Then there exist two element $q, q^+ \in \mathrm{G}^c$ such that $qg^{-1} \in \mathrm{G}$ and $\lim_{t\to\infty} \exp(\mathbf{i}t\xi)q\exp(-\mathbf{i}t\xi) = q^+$.*

This is the assertion of Theorem 7.1.

Step 3. *There exist two elements $p, p^+ \in \mathrm{G}^c$ such that $p^{-1}\zeta p \in \mathfrak{g}$ and $\lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta) = p^+$.*

Let $g \in \mathrm{G}^c$ be as in Step 1, denote

$$\xi := g^{-1}\zeta g \in \mathfrak{g},$$

choose $q, q^+ \in \mathrm{G}^c$ as in Step 2, and define

$$p := gqg^{-1}, \qquad p^+ := gq^+g^{-1}.$$

Then

$$\begin{aligned} p^+ &= gq^+g^{-1} \\ &= \lim_{t\to\infty} g\exp(\mathbf{i}t\xi)q\exp(-\mathbf{i}t\xi)g^{-1} \\ &= \lim_{t\to\infty} \exp(\mathbf{i}tg\xi g^{-1})(gqg^{-1})\exp(-\mathbf{i}tg\xi g^{-1}) \\ &= \lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta). \end{aligned}$$

Moreover $g^{-1}p = qg^{-1} \in \mathrm{G}$ and hence

$$p^{-1}\zeta p = (g^{-1}p)^{-1}\xi(g^{-1}p) \in \mathfrak{g}.$$

Thus p satisfies the requirements of Step 3 and this proves Theorem 1.7. $\square$

Comments on the literature

The group $\mathrm{P}(\xi)$ in Lemma 7.2 was introduced by Mumford. In [8, Proposition 2.6] he proved that it is a *parabolic subgroup* of G^c. If $\mathrm{T} \subset \mathrm{G}$ is a maximal torus whose Lie algebra contains ξ then there exists a *Borel subgroup* $\mathrm{B} \subset \mathrm{G}^c$ such that $\mathrm{T} \subset \mathrm{B} \subset \mathrm{P}(\xi)$. In this situation $\mathrm{B} \cap \mathrm{G} = \mathrm{T}$ and it then follows that the inclusion of G into G^c descends to a diffeomorphism $\mathrm{G}/\mathrm{T} \cong \mathrm{G}^c/\mathrm{B}$ (see Schmid [12, Lemma 2.4.6]). This implies Theorem 7.1. The proof of Theorem 7.1 given above uses direct arguments, and does not rely on the structure theory for Lie groups.

The discussion on page 34 shows that Theorem 1.7 is an easy consequence of Theorem 7.1 and Cartan's uniqueness theorem for maximal compact subgroups of G^c (see Theorem 1.6). Theorem 1.7 is mentioned in the work of Ness [10, page 1292] as a consequence of Mumford's result that $\mathrm{P}(\xi)$ is parabolic. It plays a central role in the study by Kempf and Ness of *Mumford's numerical function* and of the *Hilbert–Mumford stability criterion* for linear G^c-actions (see [4, 5, 9, 10]). Specifically, Theorem 1.7 is needed in the proof of the *moment-weight* inequality (see Ness [10, Lemma 3.1 (iv)] and Szekelyhidi [13, Theorem 1.3.6]). The moment-weight inequality implies the necessity of the Hilbert–Mumford criterion for semistability. It also implies the *Kirwan–Ness inequality* in [10, Theorem 1.2] (and implicit in [6]), which asserts that the restriction of the moment map squared to the complexified group orbit of a critical point attains its minimum at that critical point.

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