The Donaldson geometric flow for symplectic four-manifolds

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Abstract

This is an exposition of the Donaldson geometric flow on the space of symplectic forms on a closed smooth four-manifold, representing a fixed cohomology class. The original work appeared in [1].

1 Introduction

For any closed symplectic four-manifold (M, ω) it is an open question whether the space of symplectic forms on M representing the same cohomology class as ω is connected. By Moser isotopy a positive answer to this question is equivalent to the assertion that every symplectic form in the cohomology class of ω is diffeomorphic to ω via a diffeomorphism that is isotopic to the identity. In the case of the projective plane it follows from theorems of Gromov and Taubes that a positive answer is equivalent to the assertion that a diffeomorphism is isotopic to the identity if and only if it induces the identity on homology. In the case of the four-torus a positive answer is a longstanding conjecture in symplectic topology. This is part of the circle of questions around the uniqueness problem in symplectic topology as discussed in [5]. A remarkable geometric flow approach to the uniqueness problem in dimension four was explained by Donaldson in a lecture in Oxford in the spring of 1997 (attended by the second author) and written up in [1]. The purpose of this expository paper is to explain some of the details.

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The starting point of Donaldson's approach is the obervation that the space of diffeomorphism of a hyperKähler surface M can be viewed as an infinite-dimensional hyperKähler manifold, that the group of symplectomorphisms associated to a preferred symplectic structure ω acts on the right by hyperKähler isometries, and that this group action is generated by a hyperKähler moment map. In analogy to the finite-dimensional setting one can then study the gradient flow of the square of the hyperKähler moment map. Pushing ω forward under the diffeomorphisms of M one obtains a geometric flow on the space $\mathscr{S}_a$ of symplectic forms in the cohomology class $a := [\omega]$. It turns out that this geometric flow is well defined for each symplectic four-manifold (M, ω) equipped with a Riemannian metric g that is compatible with ω. It is the gradient flow of the energy functional

$$\mathscr{E}(\rho) := \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol}, \qquad \rho \in \mathscr{S}_a, \tag{1}$$

with respect to a suitable metric on $\mathscr{S}_a$. To describe this metric, we recall the well known observation (also used in [2]) that for every positive rank-3 subbundle $\Lambda^+ \subset \Lambda^2 T^*M$ and every positive volume form $\mathrm{dvol} \in \Omega^4(M)$ there is a unique Riemannian metric on M with volume form dvol such that Λ^+ is the bundle of self-dual 2-forms (Theorem A.1). Second, every $\rho \in \mathscr{S}_a$ determines an involution $R^\rho : \Omega^2(M) \to \Omega^2(M)$ which sends ρ to $-\rho$ and acts as the identity on the orthogonal complement of ρ with respect to the exterior product; it is given by $R^\rho\tau := \tau - 2\frac{\tau\wedge\rho}{\rho\wedge\rho}\rho$ and preserves the pairing. Thus every $\rho \in \mathscr{S}_a$ determines a unique Riemannian metric g^ρ on M with the same volume form as g such that τ is self-dual with respect to g if and only if $R^\rho\tau$ is self-dual with respect to g^ρ (Theorem A.2). For $\rho \in \mathscr{S}_a$ denote by $*^\rho : \Omega^k(M) \to \Omega^{4-k}(M)$ the Hodge $*$-operator of g^ρ. Then the **Donaldson metric** on the infinite-dimensional manifold $\mathscr{S}_a$ is given by

$$\|\widehat{\rho}\|_\rho^2 := \int_M \lambda \wedge *^\rho\lambda, \qquad d\lambda = \widehat{\rho}, \qquad *^\rho\lambda \text{ is exact}, \tag{2}$$

for $\widehat{\rho} \in T_\rho\mathscr{S}_a = \mathrm{im}(d : \Omega^1(M) \to \Omega^2(M))$ (see Definition 3.2). Now the differential of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ at a point $\rho \in \mathscr{S}_a$ is the linear map $\widehat{\rho} \mapsto \int_M \Theta^\rho \wedge \widehat{\rho}$ where the 2-form $\Theta^\rho \in \Omega^2(M)$ is given by

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad u := \frac{\rho\wedge\rho}{2\mathrm{dvol}}. \tag{3}$$

This is the pointwise orthogonal projection of the 2-form $u^{-1} * \rho$ onto the orthogonal complement of ρ with respect to the exterior product.

The negative gradient flow of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (1) with respect to the Donaldson metric (2) has the form

$$\partial_t \rho = d *^\rho d\Theta^\rho, \tag{4}$$

where $\Theta^\rho \in \Omega^2(M)$ is given by (3) (Proposition 3.4). This is the **Donaldson geometric flow**. The purpose of the present paper is to explain some of the geometric properties of this flow, and to give an exposition of the necessary background material. This includes a discussion of the Riemannian metrics g^ρ which is relegated to Appendix A. The Donaldson geometric flow in the original hyperKähler moment map setting is explained in Section 2, for general symplectic four-manifolds it is discussed in Section 3, and the Hessian of the energy functional is examined in Section 4.

The motivation for this study is the dream that the solutions of (4) can be used to settle the uniqueness problem for symplectic structures in dimension four in some favourable cases such as hyperKähler surfaces or the complex projective plane. This is backed up by the observations that the symplectic form ω is the unique absolute minimum of $\mathscr{E}$ (Corollary 3.5) and the Hessian of $\mathscr{E}$ at ω is positive definite (Corollary 4.4). For $M = \mathbb{C}\mathrm{P}^2$ we prove that the Fubini–Study form is the only critical point (Proposition 3.8). The present exposition also includes a proof of Donaldson's observation that higher critical points cannot be strictly stable in the hyperKähler setting (Theorem 4.5). Local existence and uniqueness and regularity for the solutions of (4) are estabished in the followup paper [3] for which the present paper provides the necessary background. Key problems for future research include long-time existence and to show that the solutions cannot *escape to infinity*.

Sign Conventions. Let (M, ω) be a symplectic manifold and let G be a Lie group with Lie algebra $\mathfrak{g} := \mathrm{Lie(G)}$ that acts covariantly on M by symplectomorphisms. Denote the infinitesimal action by $\mathfrak{g} \to \mathrm{Vect}(M) : \xi \mapsto X_\xi$. We use the sign convention $[X, Y] := \nabla_Y X - \nabla_X Y$ for the Lie bracket of vector fields so the infinitesimal action is a Lie algebra homomorphism. We use the sign convention $\iota(X_H)\omega = dH$ for Hamiltonian vector fields so the map $C^\infty(M) \to \mathrm{Vect}(M) : H \mapsto X_H$ is a Lie algebra homomorphism with respect to the Poisson bracket $\{F, G\} := \omega(X_F, X_G)$. The group action is called *Hamiltonian* if there is a G-equivariant *moment map* $\mu : M \to \mathfrak{g}^*$ such that X_ξ is the Hamiltonian vector field of $H_\xi := \langle \mu, \xi \rangle$ for $\xi \in \mathfrak{g}$. If $\mathfrak{g}$ is equipped with an invariant inner product it is convenient to write $\mu : M \to \mathfrak{g}$.

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2 The Moment Map Picture

Throughout this section M denotes a closed hyperKähler surface with symplectic forms $\omega_1, \omega_2, \omega_3$ and complex structures J_1, J_2, J_3. Thus each J_i is compatible with ω_i, the resulting Riemannian metric

$$\langle \cdot, \cdot \rangle := \omega_i(\cdot, J_i \cdot)$$

is independent of i, and the complex structures satisfy the quaternion relations $J_i J_j = -J_j J_i = J_k$ for every cyclic permutation i, j, k of $1, 2, 3$. Let (S, σ) be a symplectic four-manifold that is diffeomorphic to M and define

$$\mathscr{F} := \left\{ f : S \to M \,\middle|\, \begin{array}{l} f \text{ is a diffeomorphism and} \\ \text{the 2-form } f^*\omega_1 - \sigma \text{ is exact} \end{array} \right\}. \tag{5}$$

This space need not be connected. Assume it is nonempty. (Whether this implies that (S, σ) is symplectomorphic to (M, ω_1) is an open question.) Then the space $\mathscr{F}$ is a C^1 open set in the space of all smooth maps from S to M and can be viewed formally as an infinite-dimensional hyperKähler manifold. Its tangent space at $f \in \mathscr{F}$ is the space of vector fields along f and will be denoted by $T_f\mathscr{F} = \Omega^0(S, f^*TM)$. The three complex structures are given by $T_f\mathscr{F} \to T_f\mathscr{F} : \widehat{f} \mapsto J_i\widehat{f}$ and the three symplectic forms are given by

$$\Omega_i(\widehat{f_1}, \widehat{f_2}) := \int_S \omega_i(\widehat{f_1}, \widehat{f_2}) \, \mathrm{dvol}_\sigma, \qquad \mathrm{dvol}_\sigma := \frac{\sigma \wedge \sigma}{2} \tag{6}$$

for $\widehat{f_1}, \widehat{f_2} \in T_f\mathscr{F}$. The group

$$\mathscr{G} := \mathrm{Symp}(S, \sigma) := \left\{ \varphi \in \mathrm{Diff}(S) \,\middle|\, \varphi^*\sigma = \sigma \right\} \tag{7}$$

of symplectomorphism of (S, σ) acts contravariantly on $\mathscr{F}$ by composition on the right. This group action preserves the hyperKähler structure of $\mathscr{F}$. The quotient space $\mathscr{F}/\mathscr{G}$ is homeomorphic to the space $\mathscr{S}$ of all symplectic forms on M that are cohomologous to ω_1 and diffeomorphic to σ via the homeomorphism $\mathscr{F}/\mathscr{G} \to \mathscr{S} : [f] \mapsto (f^{-1})^*\sigma$. The action of the subgroup

$$\mathscr{G}_0 := \mathrm{Ham}(S, \sigma) \tag{8}$$

of Hamiltonian symplectomorphisms is Hamiltonian for all three symplectic forms on $\mathscr{F}$. This is the content of the next proposition. We identify the Lie algebra of $\mathscr{G}_0$ with the space of smooth real valued functions on S with mean value zero and its dual space with the quotient $\Omega^0(S)/\mathbb{R}$ via the L^2 inner product associated to the volume form dvol_σ.

Proposition 2.1 (**Moment Map**). *The map*

$$\mu_i : \mathscr{F} \to \Omega^0(S), \qquad \mu_i(f) := \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \tag{9}$$

is a moment map for the covariant action

$$\mathscr{G}_0 \times \mathscr{F} \to \mathscr{F} : (\varphi, f) \mapsto f \circ \varphi^{-1}$$

with respect to the symplectic form Ω_i.

Proof. The infinitesimal covariant action of a smooth function $H : S \to \mathbb{R}$ with mean value zero on $\mathscr{F}$ is given by the vector field on $\mathscr{F}$ which assigns to each $f \in \mathscr{F}$ the vector field $-df \circ X_H \in T_f\mathscr{F}$ along f. Here $X_H \in \mathrm{Vect}(S)$ is the Hamiltonian vector field on S associated to H and is determined by the equation $\iota(X_H)\sigma = dH$. The minus sign appears because composition on the right defines a contravariant action of $\mathscr{G}_0$ and the covariant action is given by composition with φ^{-1} on the right. By Cartan's formula, the differential of the map $\mu_i : \mathscr{F} \to \Omega^0(S)$ in (9) at f in the direction $\widehat{f} \in T_f\mathscr{F}$ is given by

$$d\mu_i(f)\widehat{f} = \frac{d\alpha_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \qquad \alpha_i := \omega_i(\widehat{f}, df\cdot) \in \Omega^1(S). \tag{10}$$

Now contract the vector field $f \mapsto -df \circ X_H$ on $\mathscr{F}$ with the symplectic form Ω_i to obtain

$$\begin{aligned}
\Omega_i(-df \circ X_H, \widehat{f}) &= \int_S \omega_i(\widehat{f}, df \circ X_H)\,\mathrm{dvol}_\sigma \\
&= \int_S (\iota(X_H)\alpha_i)\mathrm{dvol}_\sigma \\
&= \int_S \alpha_i \wedge \iota(X_H)\mathrm{dvol}_\sigma \\
&= \int_S \alpha_i \wedge dH \wedge \sigma \\
&= \int_S H d\alpha_i \wedge \sigma \\
&= \int_S H d\mu_i(f)\widehat{f}\,\mathrm{dvol}_\sigma.
\end{aligned}$$

The last term is the differential of the function $\mathscr{F} \to \mathbb{R} : f \mapsto \langle \mu_i(f), H\rangle_{L^2}$ at f in the direction $\widehat{f}$. This proves Proposition 2.1. □

The norm squared of the moment map in the hyperKähler setting is the function $\mathscr{E} := \frac{1}{2}(\|\mu_1\|^2 + \|\mu_2\|^2 + \|\mu_3\|^2)$, where the norm on the (dual of the) Lie algebra is associated to an invariant inner product. In the case at hand the invariant inner product is the L^2 inner product on $\Omega^0(S)$ and the norm squared of the moment map is the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ given by

$$\mathscr{E}(f) := \frac{1}{2}\int_S \sum_{i=1}^{3} |H_i|^2 \,\mathrm{dvol}_\sigma, \qquad H_i := \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}. \tag{11}$$

We next examine the negative gradient flow lines of $\mathscr{E}$ with respect to the hyperKähler metric on $\mathscr{F}$, given by

$$\left\langle \widehat{f}_1, \widehat{f}_2 \right\rangle_{L^2} := \int_S \langle \widehat{f}_1, \widehat{f}_2 \rangle \,\mathrm{dvol}_\sigma \qquad \text{for } \widehat{f}_1, \widehat{f}_2 \in T_f\mathscr{F}.$$

Proposition 2.2 (Gradient Flow). *An isotopy $\mathbb{R} \to \mathscr{F} : t \mapsto f_t$ is a negative L^2 gradient flow line of the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ in* (11) *if and only if it satisfies the partial differential equation*

$$\partial_t f_t = \sum_{i=1}^{3} J_i df_t \circ X_{H_{it}}, \qquad H_{it} := \frac{f_t^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \qquad \iota(X_{H_{it}})\sigma = dH_{it}. \tag{12}$$

Proof. The differential of the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ at $f \in \mathscr{F}$ in the direction $\widehat{f} \in T_f\mathscr{F}$ is given by

$$\begin{aligned} \delta\mathscr{E}(f)\widehat{f} &= \sum_{i=1}^{3} \left\langle d\mu_i(f)\widehat{f}, \mu_i(f) \right\rangle_{L^2} \\ &= \sum_{i=1}^{3} \Omega_i(-df \circ X_{H_i}, \widehat{f}) \\ &= -\sum_{i=1}^{3} \left\langle J_i df \circ X_{H_i}, \widehat{f} \right\rangle_{L^2}. \end{aligned}$$

Here $H_i := \mu_i(f) \in \Omega^0(S)$ is as in (11), the second equation follows from Proposition 2.1, and the third equation follows from the fact that $\omega_i = \langle J_i \cdot, \cdot \rangle$. Hence the L^2 gradient of $\mathscr{E}$ is given by

$$\mathrm{grad}\mathscr{E}(f) = -\sum_{i=1}^{3} J_i df \circ X_{H_i} \tag{13}$$

and this proves Proposition 2.2. □

The energy functional (11) and the L^2 metric on $\mathscr{F}$ are invariant under the action of the full group $\mathscr{G}$ of all symplectomorphisms of (S,σ) and so is the negative gradient flow (12). To eliminate the action of the infinite-dimensional symplectomorphism group it is convenient to replace the solutions $t \mapsto f_t$ of equation (12) by paths of symplectic forms $t \mapsto \rho_t$ on M obtained by pushing forward the symplectic form σ on S by the diffeomorphisms $f_t : S \to M$.

Proposition 2.3 (**Pushforward Gradient Flow**). *Let $\mathbb{R} \to \mathscr{F} : t \mapsto f_t$ be a solution of (12) and define the symplectic form $\rho_t \in \Omega^2(M)$ by*

$$\rho_t := (f_t^{-1})^* \sigma$$

for $t \in \mathbb{R}$. Then ρ_t is cohomologous to ω_1 for all t and the path $t \mapsto \rho_t$ satisfies the partial differential equation

$$\partial_t \rho_t = -\sum_{i=1}^{3} d(dK_i^{\rho_t} \circ J_i^{\rho_t}), \quad K_i^{\rho_t} := \frac{\omega_i \wedge \rho_t}{\mathrm{dvol}_{\rho_t}}, \quad \rho_t(J_i^{\rho_t}\cdot,\cdot) := \rho_t(\cdot, J_i\cdot). \tag{14}$$

Proof. Differentiate the equation $f_t^* \rho_t = \sigma$ using Cartan's formula to obtain

$$0 = f_t^* \partial_t \rho_t + d\beta_t, \qquad \beta_t := \rho_t(\partial_t f_t, df_t\cdot) \in \Omega^1(S). \tag{15}$$

Since f_t satisfies (12) it follows that

$$\begin{aligned}
\beta_t &= \sum_{i=1}^{3} \rho_t(J_i df_t \circ X_{H_{it}}, df_t\cdot) \\
&= \sum_{i=1}^{3} \rho_t(df_t \circ X_{H_{it}}, J_i^{\rho_t} df_t\cdot) \\
&= \sum_{i=1}^{3} \sigma(X_{H_{it}}, f_t^* J_i^{\rho_t}\cdot) \\
&= \sum_{i=1}^{3} dH_{it} \circ f_t^* J_i^{\rho_t} \\
&= \sum_{i=1}^{3} f_t^* \big(dK_i^{\rho_t} \circ J_i^{\rho_t}\big).
\end{aligned}$$

Here the last equation follows from the fact that $H_{it} = K_i^{\rho_t} \circ f_t = f_t^* K_i^{\rho_t}$. Now insert the formula $\beta_t = \sum_i f_t^*(dK_i^{\rho_t} \circ J_i^{\rho_t})$ into equation (15) to obtain (14). This proves Proposition 2.3. □

Equation (14) is the **Donaldson Geometric Flow** in the hyperKähler setting. It can be interpreted as the gradient flow of the pushforward energy functional on the space $\mathscr{S}_a$ of all symplectic forms on M representing the cohomology class $a := [\omega_1]$ with respect to a suitable Riemannian metric. (See Definition 3.2 below.) The energy functional and the Riemannian metric on $\mathscr{S}_a$ are independent of the choice of the symplectic four-manifold (S, σ).

Proposition 2.4 (**Pushforward Energy**). *Let $f \in \mathscr{F}$ and define*

$$\rho := (f^{-1})^*\sigma \in \Omega^2(M).$$

Then

$$\mathscr{E}(\rho) := \mathscr{E}(f) = \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol}, \tag{16}$$

where $\rho^\pm := \frac{1}{2}(\rho \pm *\rho)$ are the self-dual and anti-self-dual parts of ρ and $\mathrm{dvol} := \frac{1}{2}\omega_i \wedge \omega_i$ is the volume form of the hyperKähler metric.

Proof. Define

$$u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}, \qquad \mathrm{dvol}_\rho := \frac{\rho \wedge \rho}{2}, \tag{17}$$

and

$$K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad H_i := f^* K_i^\rho = \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}$$

as in (14) and (11). Then

$$\rho^+ = \frac{u}{2}\sum_i K_i^\rho \omega_i, \qquad 2|\rho^+|^2 = u^2 \sum_i |K_i^\rho|^2, \qquad |\rho^+|^2 - |\rho^-|^2 = 2u. \tag{18}$$

Divide and integrate to obtain

$$\begin{aligned}\int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\mathrm{dvol} &= \int_M \frac{u}{2}\sum_i |K_i^\rho|^2\,\mathrm{dvol} = \int_M \frac{1}{2}\sum_i |K_i^\rho|^2\,\mathrm{dvol}_\rho \\ &= \frac{1}{2}\sum_{i=1}^3 \int_S |H_i|^2\,\mathrm{dvol}_\sigma = \mathscr{E}(f).\end{aligned}$$

This proves Proposition 2.4. □

The energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (16) is well defined for symplectic forms on any closed oriented Riemannian four-manifold M. Moreover, the space $\mathscr{S}_a$ carries a natural Riemannian metric which in the hyperKähler case agrees with the pushforward of the L^2 metric on $\mathscr{F}$. Thus the Donaldson geometric flow extends to the general setting as explained in the next section.

3 General Symplectic Four-Manifolds

Let M be a closed oriented Riemannian four-manifold. Denote by g the Riemannian metric on M, denote by $\mathrm{dvol} \in \Omega^4(M)$ the volume form of g, and let $* : \Omega^k(M) \to \Omega^{4-k}(M)$ be the Hodge $*$-operator associated to the metric and orientation. Fix a cohomology class $a \in H^2(M;\mathbb{R})$ such that $a^2 > 0$ and consider the space

$$\mathscr{S}_a := \big\{ \rho \in \Omega^2(M) \,\big|\, d\rho = 0,\ \rho \wedge \rho > 0,\ [\rho] = a \big\}$$

of symplectic forms on M representing the class a. This is an infinite-dimensional manifold and the tangent space of $\mathscr{S}_a$ at any element $\rho \in \mathscr{S}_a$ is the space of exact 2-forms on M. The next proposition is of preparatory nature. It summarizes the properties of a family of Riemannian metrics g^ρ on M, one for each nondegenerate 2-form ρ (and for each fixed background metric g). These Riemannian metrics will play a central role in our study of the Donaldson geometric flow.

Proposition 3.1 (**Symplectic Forms and Riemannian Metrics**). *Fix a nondegenerate 2-form $\rho \in \Omega^2(M)$ such that $\rho \wedge \rho > 0$ and define the function $u : M \to (0,\infty)$ by (17). Then there exists a unique Riemannian metric g^ρ on M that satisfies the following conditions.*

(i) *The volume form of g^ρ agrees with the volume form of g.*

(ii) *The Hodge $*$-operator $*^\rho : \Omega^1(M) \to \Omega^3(M)$ associated to g^ρ is given by*

$$*^\rho \lambda = \frac{\rho \wedge *(\rho \wedge \lambda)}{u} \tag{19}$$

for $\lambda \in \Omega^1(M)$ and by $*^\rho \iota(X)\rho = -\rho \wedge g(X,\cdot)$ for $X \in \mathrm{Vect}(M)$.

(iii) *The Hodge $*$-operator $*^\rho : \Omega^2(M) \to \Omega^2(M)$ associated to g^ρ is given by*

$$*^\rho \omega = R^\rho * R^\rho \omega, \qquad R^\rho \omega := \omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho, \tag{20}$$

for $\omega \in \Omega^2(M)$. The linear map $R^\rho : \Omega^2(M) \to \Omega^2(M)$ is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of ρ with respect to the exterior product, and sends ρ to $-\rho$.

(iv) *Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form and let $J : TM \to TM$ be an almost complex structure such that $g = \omega(\cdot, J\cdot)$. Define the almost complex structure J^ρ by $\rho(J^\rho\cdot,\cdot) := \rho(\cdot, J\cdot)$ and define the 2-form $\omega^\rho \in \Omega^2(M)$ by $\omega^\rho := R^\rho\omega$. Then $g^\rho = \omega^\rho(\cdot, J^\rho\cdot)$ and so ω^ρ is self-dual with respect to g^ρ.*

Proof. See Theorem A.2. □

Definition 3.2. *Each nondegenerate 2-form $\rho \in \Omega^2(M)$ with $\rho^2 > 0$ determines an inner product $\langle \cdot, \cdot \rangle_\rho$ on the space of exact 2-forms defined by*

$$\langle \widehat{\rho}_1, \widehat{\rho}_2 \rangle_\rho := \int_M \lambda_1 \wedge *^\rho \lambda_2, \qquad d\lambda_i = \widehat{\rho}_i, \qquad *^\rho \lambda_i \in \operatorname{im} d. \tag{21}$$

These inner products determine a Riemannian metric on the infinite-dimensional manifold $\mathscr{S}_a$ called the **Donaldson metric**.

The Donaldson geometric flow on a general symplectic four-manifold is the negative gradient flow of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (16) with respect to the Donaldson metric in Definition 3.2. A central geometric ingredient in this flow is the following map $\Theta : \Omega^2_{\mathrm{ndg}}(M) \to \Omega^2(M)$. Its domain is the space $\Omega^2_{\mathrm{ndg}}(M) := \{\rho \in \Omega^2(M) \,|\, \rho \wedge \rho > 0\}$ of nondegenerate 2-forms compatible with the orientation and the map is given by

$$\Theta(\rho) := \Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}. \tag{22}$$

Proposition 3.3 (The Map Θ). *Let $\rho \in \Omega^2_{\mathrm{ndg}}(M)$ and define $u \in \Omega^0(M)$ and $\Theta^\rho \in \Omega^2(M)$ by (22). Then the following holds.*

(i) *Θ^ρ is the pointwise orthogonal projection of the 2-form $u^{-1} * \rho$ onto the orthogonal complement of ρ with respect to the wedge product. In particular*

$$\Theta^\rho \wedge \rho = 0. \tag{23}$$

(ii) *The 2-form Θ^ρ can be written as*

$$\Theta^\rho = \frac{2\rho^+}{u} - \left|\frac{\rho^+}{u}\right|^2 \rho = -\frac{|\rho^-|^2\rho^+ + |\rho^+|^2\rho^-}{u^2}. \tag{24}$$

(iii) *The square of Θ^ρ is given by*

$$\Theta^\rho \wedge \Theta^\rho = -\frac{2|\rho^+|^2|\rho^-|^2}{u^3}\mathrm{dvol}. \tag{25}$$

Thus $\Theta^\rho \wedge \Theta^\rho \leq 0$ with equality if and only if ρ is self-dual.

(iv) *Let $\rho_t : \mathbb{R} \to \Omega^2_{\mathrm{ndg}}(M)$ be a smooth path with $\rho_0 = \rho$ and $\partial_t \rho_t|_{t=0} = \widehat{\rho}$. Then*

$$\widehat{\Theta} := \frac{d}{dt}\bigg|_{t=0} \Theta^{\rho_t} = \frac{\widehat{\rho} + *^\rho \widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}. \tag{26}$$

(v) *Assume the hyperKähler case. Then*

$$\Theta^\rho = \sum_{i=1}^{3} \left(K_i^\rho \omega_i - \frac{1}{2}(K_i^\rho)^2 \rho \right), \qquad K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \tag{27}$$

and

$$d\Theta^\rho = *^\rho \sum_{i=1}^{3} dK_i^\rho \circ J_i^\rho, \qquad \rho(J_i^\rho \cdot, \cdot) := \rho(\cdot, J_i \cdot). \tag{28}$$

Proof. It follows directly from the definition of Θ^ρ in (22) that $\Theta^\rho \wedge \rho = 0$ and this proves part (i).

To prove part (ii), denote by $\Pi : \Omega^2(M) \to \Omega^2(M)$ the pointwise orthogonal projection onto the orthogonal complement of ρ with respect to the wedge product. Thus

$$\Pi(\tau) = \tau - \frac{\tau \wedge \rho}{\rho \wedge \rho}\rho \qquad \text{for } \tau \in \Omega^2(M).$$

Since $*\rho = \rho^+ - \rho^- = 2\rho^+ - \rho$ it follows from part (i) that

$$\Theta^\rho = \Pi\left(*\frac{\rho}{u}\right) = \Pi\left(\frac{2\rho^+}{u}\right).$$

This proves the first equation in (24). The second equation in (24) follows by direct calculation, using the identity $2u = |\rho^+|^2 - |\rho^-|^2$. This proves part (ii).

To prove part (iii), use the last term in equation (24) to obtain

$$\Theta^\rho \wedge \Theta^\rho = \frac{|\rho^-|^4 \rho^+ \wedge \rho^+ + |\rho^+|^4 \rho^- \wedge \rho^-}{u^4} = -\frac{2|\rho^+|^2|\rho^-|^2}{u^3}\mathrm{dvol}.$$

This proves equation (25) and part (iii).

To prove part (iv) choose a smooth path $\mathbb{R} \to \mathscr{S}_a : t \mapsto \rho_t$ such that $\rho_0 = \rho$ and $\partial_t \rho_t|_{t=0} = \widehat{\rho}$. Define

$$u_t := \frac{\rho_t \wedge \rho_t}{2\mathrm{dvol}}, \qquad \widehat{u} := \left.\frac{\partial}{\partial t}\right|_{t=0} u_t = \frac{\rho \wedge \widehat{\rho}}{\mathrm{dvol}}, \qquad \widehat{\Theta} := \left.\frac{\partial}{\partial t}\right|_{t=0} \Theta^{\rho_t}.$$

Then, by part (iii) of Proposition 3.1, we have

$$R^\rho \widehat{\rho} = \widehat{\rho} - \frac{\widehat{\rho} \wedge \rho}{\mathrm{dvol}_\rho}\rho = \widehat{\rho} - \frac{\widehat{u}}{u}\rho.$$

Hence

$$\begin{aligned} *^\rho\widehat{\rho} &= R^\rho * R^\rho\widehat{\rho} \\ &= *R^\rho\widehat{\rho} - \frac{\rho\wedge *R^\rho\widehat{\rho}}{\mathrm{dvol}_\rho}\rho \\ &= *\widehat{\rho} - \frac{\widehat{u}}{u}*\rho - \frac{\rho\wedge *\widehat{\rho}}{\mathrm{dvol}_\rho}\rho + \frac{|\rho|^2\widehat{u}}{u^2}\rho. \end{aligned} \tag{29}$$

This implies

$$\begin{aligned} \widehat{\Theta} &= \left.\frac{\partial}{\partial t}\right|_{t=0}\left(\frac{*\rho_t}{u_t} - \frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2\rho_t\right) \\ &= \frac{*\widehat{\rho}}{u} - \frac{\widehat{u}}{u}\frac{*\rho}{u} - \frac{\rho\wedge *\widehat{\rho}}{u^2\,\mathrm{dvol}}\rho + \frac{|\rho|^2\,\widehat{u}}{u^3}\rho - \frac{1}{2}\left|\frac{\rho}{u}\right|^2\widehat{\rho} \\ &= \frac{*^\rho\widehat{\rho}}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2\widehat{\rho} \\ &= \frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2\widehat{\rho}. \end{aligned}$$

Here the third step follows from (29) and the last step uses the identities $|\rho|^2 = |\rho^+|^2 + |\rho^-|^2$ and $2u = |\rho^+|^2 - |\rho^-|^2$ in (18). This proves part (iv).

Equation (27) follows from (24) and the identities

$$\frac{\rho^+}{u} = \frac{1}{2}\sum_i K_i^\rho\omega_i, \qquad \left|\frac{\rho^+}{u}\right|^2 = \frac{1}{2}\sum_i (K_i^\rho)^2.$$

To prove (28), define

$$\omega_i^\rho := \omega_i - K_i^\rho\rho, \qquad i = 1,2,3.$$

Then $\omega_i^\rho(\cdot, J_i^\rho\cdot) = g^\rho$ by part (iv) of Proposition 3.1 and hence

$$*^\rho(\lambda\circ J_i^\rho) = \lambda\wedge\omega_i^\rho$$

for all $\lambda\in\Omega^1(M)$ and all i. Take $\lambda = dK_i^\rho$ to obtain

$$*^\rho(dK_i^\rho\circ J_i^\rho) = dK_i^\rho\wedge\omega_i^\rho = d\left(K_i^\rho\omega_i - \frac{1}{2}(K_i^\rho)^2\rho\right).$$

Take the sum over all i and use equation (27) to obtain (28). This proves part (v) and Proposition 3.3. □

Proposition 3.4 (The Gradient of the Energy).
(i) *The differential of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ at $\rho \in \mathscr{S}_a$ and its gradient with respect to the Donaldson metric are given by*

$$\begin{aligned} \delta\mathscr{E}(\rho)\widehat{\rho} &= \int_M \Theta^\rho \wedge \widehat{\rho}, \\ \mathrm{grad}\mathscr{E}(\rho) &= -d *^\rho d\Theta^\rho. \end{aligned} \tag{30}$$

(ii) *Assume the hyperKähler case. Then*

$$\begin{aligned} \mathscr{E}(\rho) &= \frac{1}{2}\int_M \sum_{i=1}^3 |K_i^\rho|^2 \mathrm{dvol}_\rho, \qquad K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \\ \delta\mathscr{E}(\rho)\widehat{\rho} &= \int_M \sum_{i=1}^3 \left(K_i^\rho \omega_i - \frac{1}{2}(K_i^\rho)^2 \rho \right) \wedge \widehat{\rho}, \\ \mathrm{grad}\mathscr{E}(\rho) &= \sum_{i=1}^3 d\left(dK_i^\rho \circ J_i^\rho\right), \qquad \rho(J_i^\rho \cdot, \cdot) := \rho(\cdot, J_i \cdot). \end{aligned} \tag{31}$$

Proof. Let $\rho \in \mathscr{S}_a$ and define $u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}$. Then, by equation (18),

$$\mathscr{E}(\rho) - \mathrm{Vol}(M) = \int_M \frac{|\rho^+|^2 + |\rho^-|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} = \int_M \frac{\rho \wedge *\rho}{2u}.$$

Choose a path $\mathbb{R} \to \mathscr{S}_a : t \mapsto \rho_t$ and define

$$u_t := \frac{\rho_t \wedge \rho_t}{2\mathrm{dvol}}, \qquad \widehat{u}_t := \partial_t u_t = \frac{\rho_t \wedge \widehat{\rho}_t}{\mathrm{dvol}}, \qquad \widehat{\rho}_t := \partial_t \rho_t.$$

Then

$$\begin{aligned} \frac{d}{dt}\mathscr{E}(\rho_t) &= \frac{d}{dt}\int_M \frac{\rho_t \wedge *\rho_t}{2u_t} = \int_M \frac{\rho_t \wedge *\widehat{\rho}_t}{u_t} - \int_M \frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2 \widehat{u}_t \mathrm{dvol} \\ &= \int_M \frac{*\rho_t \wedge \widehat{\rho}_t}{u_t} - \int_M \frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2 \rho_t \wedge \widehat{\rho}_t = \int_M \Theta^{\rho_t} \wedge \widehat{\rho}_t. \end{aligned}$$

This proves the formula for $\delta\mathscr{E}(\rho)$. Now let $\widehat{\rho} \in T_\rho \mathscr{S}_a$ and choose $\lambda \in \Omega^1(M)$ such that $d\lambda = \widehat{\rho}$ and $*^\rho\lambda \in \mathrm{im}\, d$. Then

$$\delta\mathscr{E}(\rho)\widehat{\rho} = \int_M \Theta^\rho \wedge d\lambda = -\int_M d\Theta^\rho \wedge \lambda = \left\langle -d *^\rho d\Theta^\rho, \widehat{\rho} \right\rangle_\rho.$$

This proves part (i). In part (ii) the first equation in (31) follows from Proposition 2.4, the second equation follows from (30) and (27) and the third equation follows from (30) and (28). This proves Proposition 3.4. □

By part (i) of Proposition 3.4 a smooth path $\mathbb{R} \to \mathscr{S}_a : t \mapsto \rho_t$ is a negative gradient flow line of $\mathscr{E}$ with respect to the Donaldson metric if and only if it satisfies the partial differential equation

$$\partial_t \rho_t = d *^{\rho_t} d\Theta_t, \qquad \Theta_t := \frac{2\rho_t^+}{u_t} - \left|\frac{\rho_t^+}{u_t}\right|^2 \rho_t, \qquad u_t := \frac{\mathrm{dvol}_{\rho_t}}{\mathrm{dvol}}. \tag{32}$$

Equation (32) is the **Donaldson Geometric Flow**. By part (ii) of Proposition 3.4 it agrees with the geometric flow (14) in the hyperKähler case.

Corollary 3.5. (i) *A symplectic form $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in* (16) *if and only if the 2-form Θ^ρ is closed.*

(ii) *Suppose $\omega \in \mathscr{S}_a$ is compatible with the background metric g. Then ω is the unique absolute minimum of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$.*

(iii) *Assume the hyperKähler case. Then $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ if and only if $\sum_{i=1}^3 dK_i^\rho \circ J_i^\rho = 0$.*

Proof. Part (i) follows from equation (30) in Proposition 3.4. To prove (ii) observe that a symplectic form $\omega \in \mathscr{S}_a$ is compatible with the metric g if and only if it is self-dual. Moreover, every self-dual symplectic form is harmonic and the class a has a unique harmonic representative. Since

$$\frac{1}{2}\mathscr{E}(\rho) = \int_M \frac{|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \geq \int_M \mathrm{dvol} =: \mathrm{Vol}(M)$$

for all $\rho \in \mathscr{S}_a$, with equality if and only if $\rho^- = 0$, this proves (ii). Part (iii) follows from (i) and equation (28) in Proposition 3.3. □

The next proposition is an observation of Donaldson [1] which asserts that the energy controls the L^1 norm of ρ.

Proposition 3.6 (**Donaldson's L^1 Estimate**). *Every $\rho \in \mathscr{S}_a$ satisfies*

$$\|\rho\|_{L^1} \leq \sqrt{c(\mathscr{E}(\rho) - \mathrm{Vol}(M))}, \qquad c := \int_M \rho \wedge \rho = \langle a^2, [M] \rangle. \tag{33}$$

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\left(\int_M |\rho|\,\mathrm{dvol}\right)^2 &\leq \left(\int_M \left(|\rho^+|^2 - |\rho^-|^2\right)\,\mathrm{dvol}\right) \int_M \frac{|\rho|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \\
&= \left(\int_M \rho \wedge \rho\right) \int_M \frac{|\rho^+|^2 + |\rho^-|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \\
&= c(\mathscr{E}(\rho) - \mathrm{Vol}(M)).
\end{aligned}$$

This proves Proposition 3.6. □

Remark 3.7. **(i)** Donaldson's conjectural program involves a proof of long-time existence for all initial conditions, a proof that solutions cannot escape to infinity, and a proof that higher critical points can be *bypassed*, i.e. that they cannot be local minima. In those cases where this program can be carried out it would then follow that the space $\mathscr{S}_a$ is connected, which is an open question for all closed four-manifolds M and all cohomology classes a that can be represented by symplectic forms (see [5]). Short time existence and regularity, as well as long time existence for initial conditions sufficiently close to the absolute minimum, are established in [3].

(ii) In many situations (including certain Kähler classes on the K3-surface) a theorem of Seidel [6, 7, 8] asserts the existence of symplectomorphisms of (M,ω) that are smoothly, but not symplectically, isotopic to the identity. This implies the existence of noncontractible loops in $\mathscr{S}_a$. Hence, if the analytic difficulties in Donaldson's geometric flow approach can be settled, it would follow that in these cases the energy functional $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ must have critical points of index one, assuming that they are nondegenerate. Many other examples of nontrivial cohomology classes in $\mathscr{S}_a$ of all degrees were found by Kronheimer [4] using Seiberg–Witten theory.

(iii) By an observation of Donaldson [1] higher critical points of $\mathscr{E}$ (not equal to the absolute minimum) cannot be strictly stable in the hyperKähler case. We include a proof of this result in Section 4 (Theorem 4.5).

Proposition 3.8. *Let $M=\mathbb{C}\mathrm{P}^2$ be the complex projective plane with its standard Kähler metric, let ω_{FS} be the Fubini–Study form, and define*

$$a:=[\omega_{\mathrm{FS}}]\in H^2(M;\mathbb{R}).$$

Then ω_{FS} is the only critical point, and the absolute minimum, of the energy functional $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ in (16).

Proof. That ω_{FS} is the unique absolute minimum of $\mathscr{E}$ follows from part (ii) of Corollary 3.5. Now let $\rho\in\mathscr{S}_a$ be any critical point of $\mathscr{E}$. Then Θ^ρ is closed by part (i) of Proposition 3.4 and $\Theta^\rho\wedge\rho=0$ by part (i) of Proposition 3.3. Since $H^2(M;\mathbb{R})$ is one-dimensional, this implies that Θ^ρ is exact. Hence it follows from part (iii) of Proposition 3.3 that

$$0=\int_{\mathbb{C}\mathrm{P}^2}\Theta^\rho\wedge\Theta^\rho=-\int_{\mathbb{C}\mathrm{P}^2}\frac{2|\rho^+|^2|\rho^-|^2}{u^3}\,\mathrm{dvol}_{\mathrm{FS}},\qquad u:=\frac{\mathrm{dvol}_\rho}{\mathrm{dvol}_{\mathrm{FS}}}.$$

This shows that $\rho^-=0$. Thus ρ is self-dual and hence is harmonic. Since $[\rho]=a=[\omega_{\mathrm{FS}}]$ and ω_{FS} is also a harmonic 2-form it follows that $\rho=\omega_{\mathrm{FS}}$. □

4 The Hessian

The infinite-dimensional *manifold* $\mathscr{S}_a$ is an open set in an affine space. Hence the Hessian of $\mathscr{E}$ is well defined for every $\rho \in \mathscr{S}_a$ as the second derivative $\mathscr{H}_\rho(\widehat{\rho}) := \frac{d^2}{dt^2}|_{t=0}\mathscr{E}(\rho_t)$ along a curve $\mathbb{R} \to \mathscr{S}_a : t \to \rho_t$ satisfying $\rho_0 = \rho$, $\frac{d}{dt}|_{t=0}\rho_t = \widehat{\rho}$, and $\frac{d^2}{dt^2}|_{t=0}\rho_t = 0$.

Theorem 4.1. *Let $\rho \in \mathscr{S}_a$. Then the following holds.*

(i) *The Hessian of $\mathscr{E}$ at ρ is the quadratic form $\mathscr{H}_\rho : T_\rho\mathscr{S}_a \to \mathbb{R}$ given by*

$$\mathscr{H}_\rho(\widehat{\rho}) = \int_M \widehat{\Theta} \wedge \widehat{\rho}, \qquad \widehat{\Theta} := \frac{\widehat{\rho} + *^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}. \tag{34}$$

As a linear operator the Hessian is the map $T_\rho\mathscr{S}_a \to T_\rho\mathscr{S}_a : \widehat{\rho} \mapsto -d *^\rho d\widehat{\Theta}$.

(ii) *Assume the hyperKähler case and define*

$$K_i := K_i^\rho = \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \omega_i^\rho := \omega_i - K_i\rho. \tag{35}$$

Let $\widehat{\rho} \in T_\rho\mathscr{S}_a$, choose $X \in \mathrm{Vect}(M)$ such that $-d\iota(X)\rho = \widehat{\rho}$, define

$$\widehat{K}_i := \frac{(\omega_i - K_i\rho) \wedge \widehat{\rho}}{\mathrm{dvol}_\rho}, \qquad \widehat{H}_i := \frac{(d\iota(X)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}, \tag{36}$$

and let $\widehat{\Theta}$ be as in (34)*. Then*

$$\widehat{\Theta} = \sum_{i=1}^3 \widehat{K}_i\omega_i^\rho - \frac{1}{2}\sum_{i=1}^3 K_i^2\widehat{\rho}, \tag{37}$$

$$\mathscr{H}_\rho(\widehat{\rho}) = \int_M \sum_i \left(\widehat{K}_i^2\mathrm{dvol}_\rho - \frac{1}{2}K_i^2\widehat{\rho} \wedge \widehat{\rho}\right). \tag{38}$$

Moreover, if ρ is a critical point of $\mathscr{E}$, then

$$\begin{aligned}\mathscr{H}_\rho(\widehat{\rho}) &= \int_M \sum_{i=1}^3 \omega_i(X, X_{\widehat{H}_i} + \nabla_{X_{K_i}}X)\,\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 \left(\widehat{H}_i^2\mathrm{dvol}_\rho + \omega_i(X, \nabla_{X_{K_i}}X)\right)\,\mathrm{dvol}_\rho.\end{aligned} \tag{39}$$

Here ∇ denotes the Levi-Civita connection of the hyperKähler metric and X_F denotes the Hamiltonian vector field of a function $F : M \to \mathbb{R}$ with respect to ρ, i.e. $\iota(X_F)\rho = dF$.

Proof. See page 21. □

In [3] it is shown that, for every $\rho \in \mathscr{S}_a$, the quadratic form in (39) is the covariant Hessian of $\mathscr{E}$ with respect to the Donaldson metric in Definition 3.2. The proof of Theorem 4.1 relies on the following two lemmas.

Lemma 4.2. *Let ρ and ω be symplectic forms on M and define*

$$K := \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \omega^\rho := \omega - K\rho.$$

Then, for every vector field $X \in \mathrm{Vect}(M)$,

$$(\iota(X)\omega) \wedge \rho + \omega^\rho \wedge \iota(X)\rho = 0, \tag{40}$$

$$\frac{(d\iota(X)\omega) \wedge \rho}{\mathrm{dvol}_\rho} + \frac{\omega^\rho \wedge d\iota(X)\rho}{\mathrm{dvol}_\rho} = \mathcal{L}_X K. \tag{41}$$

If ω is self-dual and J is the almost complex structure such that $\omega(\cdot, J\cdot)$ is the background Riemannian metric on M, then

$$(\iota(X)\omega) \wedge \rho = - *^\rho \iota(JX)\rho. \tag{42}$$

Proof. Equation (40) follows by direct computation, i.e.

$$\begin{aligned} \big(\iota(X)\omega\big) \wedge \rho &= \iota(X)\big(\omega \wedge \rho\big) - \omega \wedge \iota(X)\rho \\ &= K\iota(X)\mathrm{dvol}_\rho - \omega \wedge \iota(X)\rho \\ &= -\big(\omega - K\rho\big) \wedge \iota(X)\rho. \end{aligned}$$

Now differentiate equation (40) and use the identity $d\omega^\rho = -dK \wedge \rho$ to obtain

$$\begin{aligned} 0 &= d\big(\big(\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge \iota(X)\rho\big) \\ &= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - dK \wedge \rho \wedge \iota(X)\rho \\ &= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - dK \wedge \iota(X)\mathrm{dvol}_\rho \\ &= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - (\iota(X)dK)\mathrm{dvol}_\rho. \end{aligned}$$

This proves (41). Now suppose ω is compatible with the almost complex structure J and $\omega(\cdot, J\cdot)$ is the background Riemannian metric. Define the almost complex structure J^ρ by $\rho(J^\rho\cdot, \cdot) := \rho(\cdot, J\cdot)$. Then $g^\rho = \omega^\rho(\cdot, J^\rho\cdot)$ by part (iv) of Theorem A.2. Hence it follows from (40) and Lemma A.4 that

$$(\iota(X)\omega) \wedge \rho = -\omega^\rho \wedge \iota(X)\rho = - *^\rho (\iota(X)\rho \circ J^\rho) = - *^\rho \iota(JX)\rho.$$

This proves equation (42) and Lemma 4.2 □

The identities in Lemma 4.2 are needed to establish the next result which relates equations (38) and (39) and is a key step in the proof of Theorem 4.1. It is shown in [3] that the last two integrals in equation (43) below arise from the Levi-Civita connection of the Donaldson metric in Definition 3.2 on the infinite-dimensional manifold $\mathscr{S}_a$. They vanish for critical points of $\mathscr{E}$. Both sides of equation (43) agree with the covariant Hessian of $\mathscr{E}$ at an arbitrary element $\rho \in \mathscr{S}_a$ (see [3]).

Lemma 4.3. *Let $\rho \in \mathscr{S}_a$, let $\widehat{\rho} \in T_\rho \mathscr{S}_a$ be an exact 2-form, let $K_i, \widehat{K}_i, \widehat{H}_i$ be as in equations* (35) *and* (36) *in Theorem 4.1, and let $X \in \mathrm{Vect}(M)$ be any vector field such that $-d\iota(X)\rho = \widehat{\rho}$. Then*

$$\begin{aligned}\int_M \sum_{i=1}^3 \Big(\widehat{H}_i^2 + \omega_i\big(X, \nabla_{X_{K_i}} X\big)\Big)\,\mathrm{dvol}_\rho &= \int_M \sum_{i=1}^3 \left(\widehat{K}_i^2 \mathrm{dvol}_\rho - \frac{1}{2} K_i^2 \widehat{\rho}^2\right) \\ + \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i) \wedge (\iota(X)\rho) \wedge \widehat{\rho} &+ \int_M \sum_{i=1}^3 \omega_i\big(X, \nabla_X X_{K_i}\big)\mathrm{dvol}_\rho.\end{aligned} \tag{43}$$

Here ∇ is the Levi-Civita connection of the Kähler metric and X_{K_i} is the Hamiltonian vector field of K_i associated to ρ so that $\iota(X_{K_i})\rho = dK_i$.

Proof. Equation (43) can be written in the form

$$\begin{aligned}\int_M \sum_{i=1}^3 \widehat{H}_i^2 \mathrm{dvol}_\rho = \int_M \sum_{i=1}^3 \left(\widehat{K}_i^2 \mathrm{dvol}_\rho - \frac{1}{2} K_i^2 \widehat{\rho} \wedge \widehat{\rho}\right) \\ + \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i) \wedge (\iota(X)\rho) \wedge \widehat{\rho} \\ + \int_M \sum_{i=1}^3 \omega_i\big(X, [X_{K_i}, X]\big)\mathrm{dvol}_\rho.\end{aligned} \tag{44}$$

To prove this formula we first observe that

$$\mathcal{L}_X K_i = \widehat{H}_i - \widehat{K}_i \tag{45}$$

for $i = 1, 2, 3$ by equation (41) in Lemma 4.2. This implies

$$\begin{aligned}\int_M \sum_{i=1}^3 \widehat{H}_i^2 \mathrm{dvol}_\rho = \int_M \sum_{i=1}^3 \widehat{K}_i^2 \mathrm{dvol}_\rho - \int_M \sum_{i=1}^3 (\mathcal{L}_X K_i)^2 \mathrm{dvol}_\rho \\ + 2\int_M \sum_{i=1}^3 \widehat{H}_i (\mathcal{L}_X K_i) \mathrm{dvol}_\rho.\end{aligned} \tag{46}$$

Define

$$\begin{aligned} A &:= -\frac{1}{2}\int_M \sum_{i=1}^3 K_i^2 \widehat{\rho}\wedge\widehat{\rho}, \\ B &:= \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i)\wedge(\iota(X)\rho)\wedge\widehat{\rho}, \\ C &:= \int_M \sum_{i=1}^3 \omega_i\big(X,[X_{K_i},X]\big)\mathrm{dvol}_\rho, \\ D &:= \int_M \sum_{i=1}^3 (\mathcal{L}_X K_i)^2\,\mathrm{dvol}_\rho, \\ E &:= \int_M \sum_{i=1}^3 \widehat{H}_i(\mathcal{L}_X K_i)\,\mathrm{dvol}_\rho. \end{aligned} \tag{47}$$

Then equation (46) shows that (44) is equivalent to the identity

$$A+B+C+D=2E.$$

To prove this, we first observe that

$$\begin{aligned} A &= \int_M \sum_{i=1}^3 \frac{1}{2}K_i^2(d\iota(X)\rho)\wedge\widehat{\rho} \\ &= -\int_M \sum_{i=1}^3 K_i dK_i\wedge(\iota(X)\rho)\wedge\widehat{\rho} \\ &= -\int_M \sum_{i=1}^3 K_i(\iota(X_{K_i})\rho)\wedge(\iota(X)\rho)\wedge\widehat{\rho} \\ &= -B+\int_M \sum_{i=1}^3 (\iota(X_{K_i})(\omega_i-K_i\rho))\wedge(\iota(X)\rho)\wedge\widehat{\rho}. \end{aligned}$$

Hence $A+B=F+G$, where

$$\begin{aligned} F &:= -\int_M \sum_{i=1}^3 (\omega_i-K_i\rho)\wedge(\iota(X_{K_i})\iota(X)\rho)\wedge\widehat{\rho}, \\ G &:= \int_M \sum_{i=1}^3 (\omega_i-K_i\rho)\wedge(\iota(X)\rho)\wedge(\iota(X_{K_i})\widehat{\rho}). \end{aligned} \tag{48}$$

Since $\iota(X_{K_i})\iota(X)\rho = -\mathcal{L}_X K_i$ and $(\omega_i - K_i\rho)\wedge\widehat{\rho} = \widehat{K}_i \mathrm{dvol}_\rho$, we have

$$\begin{aligned} F &= \int_M \sum_{i=1}^3 \widehat{K}_i(\mathcal{L}_X K_i)\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 \Big(\widehat{H}_i(\mathcal{L}_X K_i) - (\mathcal{L}_X K_i)^2\Big)\ \mathrm{dvol}_\rho \\ &= E - D. \end{aligned}$$

Here we have used equation (45). To sum up, we have proved that

$$A + B + D = D + F + G = E + G.$$

Thus it remains to prove that $C = E - G$. To see this, observe that

$$\iota(\mathcal{L}_{X_{K_i}} X)\mathrm{dvol}_\rho = \rho \wedge \iota(\mathcal{L}_{X_{K_i}} X)\rho = \rho \wedge \mathcal{L}_{X_{K_i}}(\iota(X)\rho)$$

and, by Cartan's formula,

$$\mathcal{L}_{X_{K_i}}\big(\iota(X)\rho\big) = d\iota(X_{K_i})\iota(X)\rho + \iota(X_{K_i})d\iota(X)\rho = -d(\mathcal{L}_X K_i) - \iota(X_{K_i})\widehat{\rho}.$$

Since $\omega_i\big(X, [X_{K_i}, X]\big)\mathrm{dvol}_\rho = -(\iota(X)\omega_i)\wedge\iota(\mathcal{L}_{X_{K_i}} X)\mathrm{dvol}_\rho$, this implies

$$\omega_i\big(X, [X_{K_i}, X]\big)\mathrm{dvol}_\rho = (\iota(X)\omega_i)\wedge\rho\wedge\Big(d(\mathcal{L}_X K_i) + \iota(X_{K_i})\widehat{\rho}\Big)$$

for $i = 1, 2, 3$. Integrate over M and take the sum over i to obtain

$$\begin{aligned} C &= \int_M \sum_{i=1}^3 \omega_i\big(X, [X_{K_i}, X]\big)\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge d(\mathcal{L}_X K_i) + \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge\iota(X_{K_i})\widehat{\rho} \\ &= \int_M \sum_{i=1}^3 \big(d(\iota(X)\omega_i)\wedge\rho\big)\wedge\mathcal{L}_X K_i + \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge\iota(X_{K_i})\widehat{\rho} \\ &= \int_M \sum_{i=1}^3 \widehat{H}_i(\mathcal{L}_X K_i)\mathrm{dvol}_\rho - \int_M \sum_{i=1}^3 (\omega_i - K_i\rho)\wedge(\iota(X)\rho)\wedge\iota(X_{K_i})\widehat{\rho} \\ &= E - G. \end{aligned}$$

Here the penultimate equation follows from the definition of $\widehat{H}_i$ and from equation (40) in Lemma 4.2. Thus $A + B + D = E + G$ and $C = E - G$, as claimed, and this completes the proof of Lemma 4.3. □

Proof of Theorem 4.1. Consider the map $\mathscr{S}_a \to \Omega^2(M) : \rho \mapsto \Theta^\rho$ in (22). By part (iv) of Proposition 3.3 its derivative at ρ in the direction $\widehat{\rho}$ is given by the 2-form $\widehat{\Theta}$ in (26). Hence equation (34) for the Hessian follows from the formula (30) for the differential of the energy functional $\mathscr{E}$ in (16).

Next we prove equation (37) and (38) in the hyperKähler case. The 2-forms $\omega_i^\rho = \omega_i - K_i\rho = R^\rho\omega_i$ span the space of self-dual 2-forms with respect to g^ρ by part (iii) of Proposition 3.1. Hence it follows from (36) that

$$\frac{\widehat{\rho} + *^\rho\widehat{\rho}}{u} = \sum_{i=1}^3 \widehat{K}_i\omega_i^\rho, \qquad \left|\frac{\rho^+}{u}\right|^2 = \frac{1}{2}\sum_{i=1}^3 K_i^2.$$

This proves (37) and (38). Alternatively, these two equations can be derived from the fact that Θ^ρ is given by equation (27) in the hyperkähler case. Namely, choose a smooth path $\rho_t \in \mathcal{S}_a$ such that $\rho_0 = \rho$ and $\frac{d}{dt}|_{t=0}\rho_t = \widehat{\rho}$. Then $\frac{d}{dt}|_{t=0}K_i^{\rho_t} = \widehat{K}_i$. Differentiate (27) to obtain that $\widehat{\Theta} = \frac{d}{dt}|_{t=0}\Theta^{\rho_t}$ is given by (37) and inserte this formula into (34) to obtain (38).

Next observe that the two integrals in (39) agree because

$$\begin{aligned}\int_M \widehat{H}_i^2\,\mathrm{dvol}_\rho &= \int_M \widehat{H}_i d(\iota(X)\omega_i)\wedge\rho = \int_M \iota(X)\omega_i\wedge d\widehat{H}_i\wedge\rho \\ &= \int_M \iota(X)\omega_i\wedge\iota(X_{\widehat{H}_i})\mathrm{dvol}_\rho = \int_M \omega_i(X, X_{\widehat{H}_i})\mathrm{dvol}_\rho.\end{aligned}$$

Here the first equation follows from the definition of the function $\widehat{H}_i$ in (36) and the third equation follows from the fact that $X_{\widehat{H}_i}$ is its Hamiltonian vector field with respect to ρ.

Now suppose that $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E}$ in (16). Then

$$\sum_{i=1}^3 \rho(J_iX_{K_i}, \cdot) = \sum_{i=1}^3 \rho(X_{K_i}, J_i^\rho\cdot) = \sum_{i=1}^3 dK_i\circ J_i^\rho = 0$$

by part (iii) of Corollary 3.5. Hence $\sum_{i=1}^3 J_iX_{K_i} = 0$ and this implies

$$\sum_{i=1}^3 J_i\nabla_X X_{K_i} = 0, \qquad \sum_{i=1}^3 \iota(X_{K_i})\omega_i = 0. \tag{49}$$

Thus the last two integrals in equation (43) vanish and so the right hand side of equation (38) agrees with the right hand side of equation (39) by Lemma 4.3. This proves Theorem 4.1. □

Corollary 4.4. *If $\rho = \omega \in \mathscr{S}_a$ is self-dual, then*

$$\mathscr{H}_\omega(\widehat{\rho}) = \int_M |\widehat{\rho}|^2 \,\mathrm{dvol}$$

for all $\widehat{\rho} \in T_\omega \mathscr{S}$.

Proof. This follows from (34) with $u = 1$, $|\rho^+|^2 = |\omega|^2 = 2$, and $g^\rho = g$. □

Theorem 4.5 (Donaldson). *Assume the hyperKähler case and $a := [\omega_1]$. If $\rho \in \mathscr{S}_a$ is a critical point of $\mathscr{E}$ and $\rho \neq \omega_1$, then the Hessian $\mathscr{H}_\rho$ is not positive definite.*

Proof. The proof has four steps.

Step 1. *Let $\rho \in \Omega^2_{\mathrm{ndg}}(M)$ and define J_i^ρ by $\rho(J_i^\rho \cdot, \cdot) := \rho(\cdot, J_i \cdot)$ for $i = 1, 2, 3$. Then the first order differential operator $D : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$ defined by*

$$D\lambda := \left(\frac{d *^\rho \lambda}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_1^\rho)}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_2^\rho)}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_3^\rho)}{\mathrm{dvol}} \right) \tag{50}$$

for $\lambda \in \Omega^1(M)$ is a Fredholm operator of Fredholm index $b_1 - 4$.

By part (iv) of Proposition 3.1 the 2-forms

$$\omega_i^\rho := \omega_i - \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho} \rho, \qquad i = 1, 2, 3,$$

form a basis of the space of self-dual 2-forms with respect to g^ρ and they satisfy $\omega_i^\rho(\cdot, J_i^\rho \cdot) = g^\rho$ for $i = 1, 2, 3$. Hence, for $\lambda \in \Omega^1(M)$, twice the self-dual part of $d\lambda$ with respect to g^ρ is the 2-form

$$d\lambda + *^\rho d\lambda = \sum_{i=1}^3 \frac{\omega_i^\rho \wedge d\lambda}{\mathrm{dvol}} \omega_i^\rho.$$

Hence the self-duality operator

$$\Omega^1(M) \to \Omega^0(M) \oplus \Omega^{2,+}_{g^\rho}(M) : \lambda \mapsto (d^{*^\rho} \lambda, d\lambda + *^\rho d\lambda)$$

of g^ρ is isomorphic to the operator $D' : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$ given by

$$D'\lambda := \left(\frac{d *^\rho \lambda}{\mathrm{dvol}}, \frac{\omega_1^\rho \wedge d\lambda}{\mathrm{dvol}}, \frac{\omega_2^\rho \wedge d\lambda}{\mathrm{dvol}}, \frac{\omega_3^\rho \wedge d\lambda}{\mathrm{dvol}} \right).$$

Hence D' is a Fredholm operator of index $b_1 - 4$. Since $\omega_i^\rho \wedge \lambda = *^\rho(\lambda \circ J_i^\rho)$ by Lemma A.4, we have

$$d *^\rho (\lambda \circ J_i^\rho) - \omega_i^\rho \wedge d\lambda = (d\omega_i^\rho) \wedge \lambda.$$

Hence $D - D'$ is a zeroth order operator and therefore is a compact operator between the appropriate Sobolev completions. Hence D is a Fredholm operator of index $b_1 - 4$. This proves Step 1.

Step 2. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$. Then at least one of the functions*

$$K_i^\rho = \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad i = 1, 2, 3,$$

is nonconstant.

Suppose by contradiction that K_i^ρ is constant for $i = 1, 2, 3$. Since $\rho - \omega_1$ is exact, we have

$$\int_M K_i^\rho \mathrm{dvol}_\rho = \int_M \omega_i \wedge \rho = \int_M \omega_i \wedge \omega_1 = \begin{cases} 2\mathrm{Vol}(M), & \text{if } i = 1, \\ 0, & \text{if } i = 2, 3, \end{cases}$$

and hence $K_1^\rho = 2$ and $K_2^\rho = K_3^\rho = 0$. This implies

$$u_1 = 2u, \qquad u_2 = u_3 = 0, \qquad u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}, \qquad u_i := \frac{\omega_i \wedge \rho}{\mathrm{dvol}}.$$

Hence it follows from (18) that

$$\begin{aligned} 0 &\le |\rho^-|^2 \\ &= |\rho^+|^2 - 2u \\ &= \frac{1}{2}\sum_{i=1}^3 u_i^2 - 2u \\ &= 2u(u-1). \end{aligned} \tag{51}$$

Hence $u \ge 1$ and

$$\int_M u \mathrm{dvol} = \int_M \mathrm{dvol}_\rho = \int_M \mathrm{dvol} = \mathrm{Vol}(M).$$

This shows that $u \equiv 1$, hence $\rho^- = 0$ by (51), and therefore $\rho = \omega_1$. This proves Step 2.

Step 3. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$ be a critical point of $\mathscr{E}$. Then there exists a 1-form $\lambda \in \Omega^1(M)$ such that*

$$d *^\rho \lambda = 0, \qquad d *^\rho (\lambda \circ J_j^\rho) = 0, \qquad j = 1, 2, 3 \tag{52}$$

and the exact 2-forms

$$\widehat{\rho}_0 := d\lambda, \qquad \widehat{\rho}_j := d(\lambda \circ J_j^\rho), \qquad j = 1, 2, 3. \tag{53}$$

are linearly independent.

By part (iii) of Corollary 3.5, we have

$$\sum_{i=1}^{3} dK_i^\rho \circ J_i^\rho = 0.$$

Hence the function

$$h := (0, K_1^\rho, K_2^\rho, K_3^\rho) : M \to \mathbb{R}^4$$

is L^2 orthogonal to the image of the operator

$$D : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$$

in Step 1, i.e.

$$\langle h, D\lambda \rangle_{L^2} = -\int_M \sum_{i=1}^{3} dK_i^\rho \wedge *^\rho (\lambda \circ J_i^\rho) = \int_M \sum_{i=1}^{3} (dK_i^\rho \circ J_i^\rho) \wedge *^\rho \lambda = 0$$

for all $\lambda \in \Omega^1(M)$. Since h is nonconstant by Step 2, this shows that the cokernel of D has dimension greater than four. Since D is a Fredholm operator of index $b_1 - 4$ by Step 1, its kernel has dimension greater than b_1. The kernel of D is a quaternionic vector space and each nonzero element $\lambda \in \ker D$ determines a four-dimensional quaternionic subspace

$$V_\lambda := \operatorname{span}\{\lambda, \lambda \circ J_1^\rho, \lambda \circ J_2^\rho, \lambda \circ J_3^\rho\} \subset \ker D.$$

Denote by

$$H^1_{g^\rho}(M) := \left\{\lambda \in \Omega^1(M) \,|\, d\lambda = 0,\, d *^\rho \lambda = 0\right\}$$

the space of harmonic 1-forms with respect to g^ρ. This space has dimension zero when M is a $K3$-surface and dimension four when M is a four-torus. Since $\ker D$ is a quaternionic vector space of dimension $4k$ with $k \geq 2$, it has a four-dimensional quaternionic subspace that is transverse to $H^1_{g^\rho}(M)$. Thus there exists a nonzero element $\lambda \in \ker D$ such that $V_\lambda \cap H^1_{g^\rho}(M) = 0$. (See Lemma B.1.) This proves Step 3.

Step 4. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$ be a critical point of $\mathscr{E}$ and let $\lambda \in \Omega^1(M)$ and $\widehat{\rho}_j$ for $j = 0, 1, 2, 3$ be as in Step 3. Then*

$$\sum_{j=0}^{3} \mathscr{H}_\rho(\widehat{\rho}_j) = 0.$$

Choose $X \in \mathrm{Vect}(M)$ such that $\iota(X)\rho = -\lambda$. Then

$$\widehat{\rho}_0 = -d\iota(X)\rho, \qquad \widehat{\rho}_j = -d(\iota(X)\rho \circ J_j^\rho) = -d\iota(J_j X)\rho, \qquad j = 1, 2, 3.$$

For $i, j = 1, 2, 3$ define

$$\widehat{H}_{0i} := \frac{(d\iota(X)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \widehat{H}_{ji} := \frac{(d\iota(J_j X)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}.$$

By equation (40) in Lemma 4.2, we have

$$(d\iota(Y)\omega_i) \wedge \rho = -d(\omega_i^\rho \wedge \iota(Y)\rho) = -d *^\rho ((\iota(Y)\rho) \circ J_i^\rho)$$

for every vector field $Y \in \mathrm{Vect}(M)$. Apply this formula to the vector fields $Y = X$ and $Y = J_j X$ and use Step 3 to obtain $\widehat{H}_{ji} = 0$ for $j = 0, 1, 2, 3$ and $i = 1, 2, 3$. Hence, by equation (39) in Theorem 4.1, we have

$$\begin{aligned}
\mathcal{H}_\rho(\widehat{\rho}_0) &= \sum_{i=1}^{3} \int_M \omega_i(X, \nabla_{X_{K_i^\rho}} X)\mathrm{dvol}_\rho, \\
\mathcal{H}_\rho(\widehat{\rho}_j) &= \sum_{i=1}^{3} \int_M \omega_i(J_j X, \nabla_{X_{K_i^\rho}}(J_j X))\mathrm{dvol}_\rho
\end{aligned}$$

for $j = 1, 2, 3$. Hence

$$\begin{aligned}
\sum_{j=1}^{3} \mathcal{H}_\rho(\widehat{\rho}_j) &= \sum_{i,j=1}^{3} \int_M \langle X, J_j J_i J_j \nabla_{X_{K_i^\rho}} X\rangle \mathrm{dvol}_\rho \\
&= \sum_{i=1}^{3} \int_M \langle X, J_i \nabla_{X_{K_i^\rho}} X\rangle \mathrm{dvol}_\rho \\
&= -\mathscr{H}_\rho(\widehat{\rho}_0)
\end{aligned}$$

This proves Step 4 and Theorem 4.5. □

Theorem 4.5 is an infinite-dimensional analogue of a general observation about finite-dimensional hyperKähler moment maps. Let (M, ω_i, J_i) be a hyperKähler manifold, let G be a compact Lie group that acts on M by hyperKähler isometries, and for $x \in M$ let $L_x : \mathfrak{g} \to T_xM$ denote the infinitesimal action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. Suppose $\mathfrak{g}$ is equipped with an invariant inner product and the group action is Hamiltonian for each ω_i. For $i = 1, 2, 3$ let $\mu_i : M \to \mathfrak{g}$ be an equivariant moment map so that $\langle d\mu_i(x)\widehat{x}, \xi \rangle = \omega_i(L_x\xi, \widehat{x})$ for $\xi \in \mathfrak{g}$ and $\widehat{x} \in T_xM$. Then the gradient of the function $\mathcal{E} := \frac{1}{2}\sum_i \|\mu_i\|^2$ is given by $\mathrm{grad}\mathcal{E}(x) = \sum_i J_i L_x \mu_i(x)$. Assume $\dim M \geq 4 \dim \mathrm{G}$ and let $x \in M$ be a critical point of $\mathcal{E}$ such that $\mathcal{E}(x) \neq 0$. Then the linear map $\mathfrak{g}^4 \to T_xM : (\xi_0, \xi_1, \xi_2, \xi_3) \mapsto L_x\xi_0 + \sum_i J_i L_x \xi_i$ is not injective and hence not surjective. Thus there exists a vector $\widehat{x} \in T_xM$ such that $L_x^*\widehat{x} = 0$ and $L_x^* J_i \widehat{x} = 0$ for all i. Denote by $\mathcal{H}_x : T_xM \to \mathbb{R}$ the Hessian of $\mathcal{E}$ at x. Then a calculation shows that $\mathcal{H}_x(\widehat{x}) + \sum_i \mathcal{H}_x(J_i\widehat{x}) = 0$. (See Donaldson [1, Proposition 6].) In the case at hand it would be interesting to find an exact 2-form $\widehat{\rho}$ such that $\mathcal{H}_\rho(\widehat{\rho}) < 0$.

A Four-Dimensional Linear Algebra

Let V be a 4-dimensional oriented real vector space and let $V^* := \mathrm{Hom}(V, \mathbb{R})$ be the dual space. Associated to an inner product $g : V \times V \to \mathbb{R}$ is the Hodge $*$-operator $*_g : \Lambda^k V^* \to \Lambda^{4-k} V^*$, the volume form $\mathrm{dvol}_g = *_g 1 \in \Lambda^4 V^*$, and the space

$$\Lambda_g^+ := \left\{ \omega \in \Lambda^2 V^* \,\middle|\, \omega = *_g \omega \right\}$$

of self-dual 2-forms. By a well known observation (which we learned from [2]) the inner product g is uniquely determined by dvol_g and Λ_g^+. This is the content of Theorem A.1 below. Call a linear subspace $\Lambda \subset \Lambda^2 V^*$ **positive** if the quadratic form $\Lambda \times \Lambda \to \mathbb{R} : (\omega, \tau) \mapsto \frac{\omega \wedge \tau}{\mathrm{dvol}}$ is positive definite for some (and hence every) positive volume form $\mathrm{dvol} \in \Lambda^4 V^*$. Denote by $\mathcal{G}(V)$ the space of all inner products $g : V \times V \to \mathbb{R}$, by $\mathcal{S}(V)$ the space of 2-forms $\rho \in \Lambda^2 V^*$ such that $\rho \wedge \rho > 0$, and by $\mathcal{J}(V)$ the set of linear complex structures $J : V \to V$ that are compatible with the orientation.

Theorem A.1. *For every positive volume form* $\mathrm{dvol} \in \Lambda^4 V^*$ *and every three-dimensional positive linear subspace* $\Lambda^+ \subset \Lambda^2 V^*$ *there exists a unique inner product* g *on* V *such that* $\mathrm{dvol}_g = \mathrm{dvol}$ *and* $\Lambda_g^+ = \Lambda^+$.

Proof. See page 31. □

Theorem A.2. *Let $g \in \mathcal{G}(V)$, $\rho \in \mathcal{S}(V)$, define $u > 0$ and $A \in \mathrm{GL}(V)$ by*

$$u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}_g}, \qquad \mathrm{dvol}_\rho := \frac{\rho \wedge \rho}{2}, \qquad g(A\cdot, \cdot) := \rho, \tag{54}$$

and define the linear map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ by

$$R\omega := \omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho \qquad \text{for all } \omega \in \Lambda^2 V^*. \tag{55}$$

Then R is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of ρ with respect to the exterior product, and $R\rho = -\rho$. Moreover, for every $\widetilde{g} \in \mathcal{G}(V)$, the following are equivalent.
(i) $\widetilde{g}(v, w) = u^{-1} g(Av, Aw)$ *for all* $v, w \in V$.
(ii) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\lambda = u^{-1}\rho \wedge *_g(\rho \wedge \lambda)$ *for all* $\lambda \in V^*$.
(iii) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\iota(v)\rho = -\rho \wedge g(v, \cdot)$ *for all* $v \in V$.
(iv) *Suppose $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$ satisfy $g = \omega(\cdot, J\cdot)$. Define $\widetilde{\omega} \in \Lambda^2 V^*$ and $\widetilde{J} \in \mathcal{J}(V)$ by $\widetilde{\omega} := R\omega$ and $\rho(\widetilde{J}\cdot, \cdot) := \rho(\cdot, J\cdot)$. Then $\widetilde{g} = \widetilde{\omega}(\cdot, \widetilde{J}\cdot)$.*
(v) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $\Lambda^+_{\widetilde{g}} = R\Lambda^+_g$.
(vi) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\omega = R *_g R\omega$ *for all* $\omega \in \Lambda^2 V^*$.

Proof. See page 32. □

The proofs of both theorems are based on the following six lemmas.

Lemma A.3. *For every $g \in \mathcal{G}(V)$ and every $v \in V$*

$$*_g\iota(v)\mathrm{dvol}_g = -g(v, \cdot), \qquad *_g g(v, \cdot) = \iota(v)\mathrm{dvol}_g.$$

Proof. Direct verification for the standard structures on $V = \mathbb{R}^4$. □

Lemma A.4. *Let $\omega \in \mathcal{S}(V)$, $g \in \mathcal{G}(V)$, $J \in \mathcal{J}(V)$. The following are equivalent.*
(i) $\omega(v, Jw) = g(v, w)$ *for all* $v, w \in V$.
(ii) $\mathrm{dvol}_\omega = \mathrm{dvol}_g$ *and* $*_g(\omega \wedge \lambda) = -\lambda \circ J$ *for all* $\lambda \in V^*$.

Proof. That (i) implies (ii) follows by direct verification for the standard structures on $V = \mathbb{C}^2$. We prove that (ii) implies (i). Assume ω, g, J satisfy (ii) and let $v \in V$. Then, by Lemma A.3 and (ii),

$$g(v, \cdot) = - *_g \iota(v)\mathrm{dvol}_\omega = - *_g \big(\omega \wedge \iota(v)\omega\big) = \iota(v)\omega \circ J = \omega(v, J\cdot).$$

Hence ω, g, J satisfy (i). This proves Lemma A.4. □

A symplectic form $\omega \in \mathcal{S}(V)$ is called **compatible with the inner product** $g \in \mathcal{G}(V)$ (respectively **compatible with the complex structure** $J \in \mathcal{J}(V)$) if there exists a $J \in \mathcal{J}(V)$ (respectively a $g \in \mathcal{G}(V)$) such that the equivalent conditions (i) and (ii) in Lemma A.4 are satisfied.

Lemma A.5. *Let $\omega \in \mathcal{S}(V)$ and $g \in \mathcal{G}(V)$. The following are equivalent.*

(i) *ω is compatible with g.*

(ii) $\mathrm{dvol}_\omega = \mathrm{dvol}_g$ *and* $\omega \in \Lambda^+_g$.

Proof. That (i) implies (ii) follows by direct verification for the standard structures on $V = \mathbb{C}^2$. To prove the converse, consider the standard inner product and orientation on the quaternions $V = \mathbb{H}$ with coordinates $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$. Define

$$\omega_i := dx_0 \wedge dx_i + dx_j \wedge dx_k$$

for $i = 1, 2, 3$ and i, j, k a cyclic permutation of $1, 2, 3$. If ω satisfies (ii), then

$$\omega = \sum_i t_i \omega_i, \qquad t_i \in \mathbb{R}, \qquad \sum_i t_i^2 = 1.$$

Hence ω is compatible with the inner product and the complex structure

$$J := t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$$

(acting on $\mathbb{H}$ on the left). This proves Lemma A.5. □

Lemma A.6. *Let $\rho \in \mathcal{S}(V)$ and $g \in \mathcal{G}(V)$. If u and A are defined by* (54)*, then*

$$\det(A) = u^2.$$

Proof. Assume $V = \mathbb{R}^4$ with the standard inner product and standard orientation. Denote the coordinates on $\mathbb{R}^4$ by $x = (x_0, x_1, x_2, x_3)$ and write

$$\rho = \sum_{i<j} \rho_{ij} dx_i \wedge dx_j, \qquad \rho_{ij} + \rho_{ji} = 0.$$

The nondegeneracy and orientation condition on ρ asserts that

$$u = \rho_{01}\rho_{23} + \rho_{02}\rho_{31} + \rho_{03}\rho_{12} > 0. \tag{56}$$

In the standard basis of $\mathbb{R}^4$ the linear operator A is represented by the matrix

$$A = \begin{pmatrix} 0 & \rho_{01} & \rho_{02} & \rho_{03} \\ -\rho_{01} & 0 & \rho_{12} & -\rho_{31} \\ -\rho_{02} & -\rho_{12} & 0 & \rho_{23} \\ -\rho_{03} & \rho_{31} & -\rho_{23} & 0 \end{pmatrix}. \tag{57}$$

It follows from equation (57) that

$$\begin{aligned} \det(A) &= \rho_{01} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ -\rho_{12} & 0 & \rho_{23} \\ \rho_{31} & -\rho_{23} & 0 \end{pmatrix} \\ &\quad - \rho_{02} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ 0 & \rho_{12} & -\rho_{31} \\ \rho_{31} & -\rho_{23} & 0 \end{pmatrix} \\ &\quad + \rho_{03} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ 0 & \rho_{12} & -\rho_{31} \\ -\rho_{12} & 0 & \rho_{23} \end{pmatrix} \\ &= \rho_{01} \left(\rho_{02}\rho_{23}\rho_{31} + \rho_{03}\rho_{12}\rho_{23} + \rho_{01}\rho_{23}^2 \right) \\ &\quad + \rho_{02} \left(\rho_{03}\rho_{12}\rho_{31} + \rho_{01}\rho_{23}\rho_{31} + \rho_{02}\rho_{31}^2 \right) \\ &\quad + \rho_{03} \left(\rho_{01}\rho_{23}\rho_{12} + \rho_{02}\rho_{31}\rho_{12} + \rho_{03}\rho_{12}^2 \right). \end{aligned}$$

Thus $\det(A) = u^2$ by (56) and this proves Lemma A.6. □

Lemma A.7. *Let $\omega_1, \omega_2, \omega_3 \in \mathcal{S}(V)$ and $J_1, J_2, J_3 \in \mathrm{GL}(V)$ such that*

$$\omega_2(\cdot, J_3\cdot) := \omega_1, \qquad \omega_3(\cdot, J_1\cdot) := \omega_2, \qquad \omega_1(\cdot, J_2\cdot) := \omega_3. \tag{58}$$

Then

$$\omega_i(J_j v, w) = \omega_i(v, J_j w) = \omega_k(v, w) \tag{59}$$

for every cyclic permutation i, j, k of $1, 2, 3$ and all $v, w \in V$. Moreover, the following are equivalent.

(i) *$\omega_i \wedge \omega_j = 0$ and $\omega_i \wedge \omega_i = \omega_j \wedge \omega_j$ for $1 \le i < j \le 3$.*

(ii) *$J_i^2 = -\mathbb{1}$ and $J_j J_k = -J_k J_j = J_i$ for cyclic permutations i, j, k of $1, 2, 3$.*

If these equivalent conditions are satisfied, then the following holds.

(a) *The vectors $v, J_1 v, J_2 v, J_3 v$ form a basis of V for every $v \in V \setminus \{0\}$.*

(b) *$\omega_1(v, J_1 w) = \omega_2(v, J_2 w) = \omega_3(v, J_3 w)$ for $v, w \in V$.*

(c) *$\omega_i(w, J_i v) = \omega_i(v, J_i w)$ for $i = 1, 2, 3$ and $v, w \in V$.*

(d) *$\omega_i(v, J_i v) \neq 0$ for $i = 1, 2, 3$ and $v \in V \setminus \{0\}$.*

Proof. That (58) implies (59) follows from the skew-symmetry of the ω_i.

(i) implies (ii). Since $\iota(J_j v)\omega_i = \iota(v)\omega_k$, it follows from (i) that $\omega_i \wedge \iota(v)\omega_i = \omega_k \wedge \iota(v)\omega_k = \omega_k \wedge \iota(J_j v)\omega_i = -\omega_i \wedge \iota(J_j v)\omega_k$ for $v \in V$ and every cyclic permutation i, j, k of $1, 2, 3$. Hence

$$\omega_k(J_j v, w) = \omega_k(v, J_j w) = -\omega_i(v, w). \tag{60}$$

Second, $\omega_2(\cdot, J_3 J_2 J_1 \cdot) = \omega_1(\cdot, J_2 J_1 \cdot) = \omega_3(\cdot, J_1 \cdot) = \omega_2$, by equation (59), and $\omega_2(\cdot, J_1 J_2 J_3 \cdot) = -\omega_3(\cdot, J_2 J_3 \cdot) = \omega_1(\cdot, J_3 \cdot) = -\omega_2$, by equation (60). Hence

$$J_3 J_2 J_1 = \mathbb{1} = -J_1 J_2 J_3. \tag{61}$$

Third, by (59) and (60), $\omega_j(\cdot, J_i^2 \cdot) = -\omega_k(\cdot, J_i \cdot) = -\omega_j$ and hence

$$J_1^2 = J_2^3 = J_3^2 = -\mathbb{1}. \tag{62}$$

Fourth, $J_2 J_1 = J_3^{-1} = -J_1 J_2$, by (61), and hence $J_2 J_1 = -J_3 = -J_1 J_2$, by (62). Multiply this equation by J_1 and J_2 on the left and right to obtain the quaternion relations $J_i J_j = -J_j J_i = J_k$ for i, j, k cyclic. This shows that (i) implies (ii).

(ii) implies (a). Let $v \in V \setminus \{0\}$ and $x_i \in \mathbb{R}$ such that $x_0 v + \sum_i x_i J_i v = 0$. Then

$$0 = \left(x_0 \mathbb{1} - \sum_{i=1}^{3} x_i J_i \right) \left(x_0 v + \sum_{i=1}^{3} x_i J_i v \right) = \left(\sum_{i=0}^{3} x_i^2 \right) v$$

and hence $x_0 = x_1 = x_2 = x_3 = 0$.

(ii) implies (b). It follows from equation (59) that, for i, j, k cyclic, $\omega_i(v, J_i w) = \omega_j(J_k v, J_i w) = \omega_j(v, J_k J_i w) = \omega_j(v, J_j w)$.

(ii) implies (60)**.** It follows from equation (59) that, for i, j, k cyclic, $\omega_k(J_j v, w) = \omega_i(J_j J_j v, w) = -\omega_i(v, w)$.

(ii) implies (c). It follows from equation (60) that, for i, j, k cyclic, $\omega_i(w, J_i v) = \omega_i(w, J_j J_k v) = \omega_i(J_k J_j w, v) = \omega_i(-J_i w, v) = \omega_i(v, J_i w)$.

(ii) implies (d). Fix a nonzero vector $v \in V$. Then $\omega_1(v, v) = 0$ and, by (59) and (60), $\omega_1(v, J_2 v) = \omega_3(v, v) = 0$ and $\omega_1(v, J_3 v) = -\omega_2(v, v) = 0$. Since ω_1 is nondegenerate, it follows from (a) that $\omega_1(v, J_1 v) \neq 0$.

(ii) implies (i). Fix a nonzero vector $v \in V$ and define $\Phi : \mathbb{H} \to V$ by $\Phi(x) := x_0 v + \sum_i x_i J_i v$. By (a) this is an isomorphism. By (b) and (d),

$$\lambda := \omega_1(v, J_1 v) = \omega_2(v, J_2 v) = \omega_3(v, J_3 v) \neq 0.$$

By (59) and (60), we have $\Phi^* \omega_i = \lambda(dx_0 \wedge dx_i + dx_j \wedge dx_k)$ for $i = 1, 2, 3$ and i, j, k a cyclic permutation of $1, 2, 3$. This shows that (ii) implies (i). $\square$

Lemma A.8. *Assume that $J_1, J_2, J_3 \in \mathcal{J}(V)$ are compatible with $g \in \mathcal{G}(V)$ and satisfy $J_iJ_j + J_jJ_i = 0$ for $i \neq j$. Then $J_3 = \pm J_1J_2$.*

Proof. Fix a unit vector v. Then $g(J_iv, J_jv) = g(v, J_jJ_iv) = -g(J_jv, J_iv)$. Hence v, J_1v, J_2v, J_3v form an orthonormal basis of V and J_1J_2v is orthogonal to v, J_1v, J_2v. Hence $J_1J_2v = \pm J_3v$. It follows that $J_1J_2 = \pm J_3$. □

Proof of Theorem A.1. **Existence.** Fix a basis $\omega_1, \omega_2, \omega_3$ of Λ^+ such that

$$\omega_i \wedge \omega_j = 2\delta_{ij}\mathrm{dvol}.$$

Choose $J_i \in \mathrm{GL}(V)$ such that (58) holds. By Lemma A.7, the bilinear map

$$V \times V \to \mathbb{R} : (v, w) \mapsto \omega_i(v, J_iw)$$

is independent of i, symmetric, and definite. Assume without loss of generality that $\omega_i(v, J_iv) > 0$ for all $v \in V \setminus \{0\}$. (Otherwise, replace the triple J_1, J_2, ω_3 by $-J_1, -J_2, -\omega_3$.) Then the inner product $g(v, w) := \omega_i(v, J_iw)$ is compatible with ω_i. Hence it follows from Lemma A.5 that $\mathrm{dvol}_g = \mathrm{dvol}_{\omega_i}$ and $\omega_i \in \Lambda^+_g$ for $i = 1, 2, 3$. Thus $\mathrm{dvol}_g = \mathrm{dvol}$ and $\Lambda^+_g = \Lambda^+$.

Uniqueness. Let $\widetilde{g} \in \mathcal{G}(V)$ such that $\Lambda^+_{\widetilde{g}} = \Lambda^+$ and $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}$. By Lemma A.5, the symplectic forms $\omega_1, \omega_2, \omega_3$ are compatible with $\widetilde{g}$. Hence there exist complex structures $\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3 \in \mathcal{J}(V)$ such that

$$\omega_i(\cdot, \widetilde{J}_i\cdot) = \widetilde{g}(\cdot, \cdot).$$

Thus

$$\omega_j(\cdot, \widetilde{J}_j\widetilde{J}_k\cdot) = \widetilde{g}(\cdot, \widetilde{J}_k\cdot) = -\omega_k(\cdot, \cdot) = \omega_j(\cdot, J_i\cdot)$$

by (60) and so $\widetilde{J}_j\widetilde{J}_k = J_i$ for i, j, k cyclic. Hence

$$\widetilde{J}_j\widetilde{J}_k + \widetilde{J}_k\widetilde{J}_j = J_i - \widetilde{J}_k\widetilde{J}_i\widetilde{J}_i\widetilde{J}_j = J_i - J_jJ_k = 0$$

for i, j, k cyclic. By Lemma A.8,

$$\widetilde{J}_3 = \pm\widetilde{J}_1\widetilde{J}_2 = \pm J_3.$$

Since $\omega_3(v, J_3v) > 0$ and $\omega_3(v, \widetilde{J}_3v) > 0$ for $v \neq 0$, we have $\widetilde{J}_3 = J_3$. Hence

$$\widetilde{g} = \omega_3(\cdot, \widetilde{J}_3\cdot) = \omega_3(\cdot, J_3\cdot) = g.$$

This proves Theorem A.1. □

Proof of Theorem A.2. That the linear map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ in (55) has the required properties follows by direct calculation.

We prove that (i) implies (ii). By Lemma A.6, $\det(-A^2) = \det(A)^2 = u^4$ and hence the inner product $\widetilde{g}(v,w) := u^{-1} g(Av, Aw)$ has the volume form

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g = u^{-1}\mathrm{dvol}_\rho.$$

Now let $\lambda \in V^*$ and choose $v \in V$ such that $\widetilde{g}(v,\cdot) = \lambda$. Then, by Lemma A.3,

$$*_{\widetilde{g}}\,\lambda = *_{\widetilde{g}}\widetilde{g}(v,\cdot) = \iota(v)\mathrm{dvol}_{\widetilde{g}} = u^{-1}\iota(v)\mathrm{dvol}_\rho = u^{-1}\rho \wedge \iota(v)\rho. \tag{63}$$

Since $\iota(v)\rho = g(Av,\cdot)$ it follows also from Lemma A.3 that

$$\begin{aligned} *_g\iota(v)\rho &= \iota(Av)\mathrm{dvol}_g = u^{-1}\iota(Av)\mathrm{dvol}_\rho = u^{-1}\rho \wedge \iota(Av)\rho \\ &= u^{-1}\rho \wedge g(A^2 v,\cdot) = -\rho \wedge \widetilde{g}(v,\cdot) = -\rho \wedge \lambda. \end{aligned}$$

Thus $\iota(v)\rho = *_g(\rho \wedge \lambda)$ and so $*_{\widetilde{g}}\lambda = u^{-1}\rho \wedge *_g(\rho \wedge \lambda)$ by (63). This shows that $\widetilde{g}$ satisfies (ii).

We prove that (ii) implies (iii). Assume $\widetilde{g}$ satisfies (ii) and let $v \in V$. Use the equation $u^{-1}(\rho \wedge \iota(v)\rho) = u^{-1}\iota(v)\mathrm{dvol}_\rho = \iota(v)\mathrm{dvol}_g$ to obtain

$$*_{\widetilde{g}}\iota(v)\rho = u^{-1}\rho \wedge *_g(\rho \wedge \iota(v)\rho) = \rho \wedge *_g\iota(v)\mathrm{dvol}_g = -\rho \wedge g(v,\cdot).$$

Here the last step follows from Lemma A.3. This shows that $\widetilde{g}$ satisfies (iii).

We prove that (iii) implies (iv). Assume $\widetilde{g}$ satisfies (iii). Let $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$ such that $\omega(\cdot, J\cdot) = g$. Then, by Lemma A.4,

$$\mathrm{dvol}_\omega = \mathrm{dvol}_g, \qquad *_g(\omega \wedge \lambda) = -\lambda \circ J \tag{64}$$

for every $\lambda \in V^*$. Define $\widetilde{\omega}$ and $\widetilde{J}$ by $\widetilde{\omega} := R\omega$ and $\rho(\widetilde{J}\cdot,\cdot) := \rho(\cdot, J\cdot)$. Then $\widetilde{\omega} \wedge \widetilde{\omega} = \omega \wedge \omega$ and so $\mathrm{dvol}_{\widetilde{\omega}} = \mathrm{dvol}_\omega = \mathrm{dvol}_g = \mathrm{dvol}_{\widetilde{g}}$ by (iii). Now let $\lambda \in V^*$ and choose $v \in V$ such that $\iota(v)\rho = \lambda$. Then

$$\lambda \circ \widetilde{J} = \iota(v)\rho \circ \widetilde{J} = \rho(v, \widetilde{J}\cdot) = \rho(Jv,\cdot) = \iota(Jv)\rho.$$

Abbreviate $K := \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}$. Then $\widetilde{\omega} = \omega - K\rho$ and so

$$\widetilde{\omega} \wedge \lambda = \omega \wedge \iota(v)\rho - K\iota(v)\mathrm{dvol}_\rho = \omega \wedge \iota(v)\rho - \iota(v)(\omega \wedge \rho) = -(\iota(v)\omega) \wedge \rho.$$

By (iii) this implies

$$*_{\widetilde{g}}(\widetilde{\omega} \wedge \lambda) = -*_{\widetilde{g}}(\rho \wedge \iota(v)\omega) = -*_{\widetilde{g}}(\rho \wedge g(Jv,\cdot)) = -\iota(Jv)\rho = -\lambda \circ \widetilde{J}.$$

Hence $\widetilde{\omega}(\cdot, \widetilde{J}\cdot) = \widetilde{g}$ by Lemma A.4. This shows that $\widetilde{g}$ satisfies (iv).

We prove that (iv) implies (v). Assume $\widetilde{g}$ satisfies (iv) and choose a symplectic form $\omega \in \mathcal{S}(V)$ that is compatible with g. Then $\widetilde{\omega} := R\omega$ is compatible with $\widetilde{g}$ by (iv), and hence

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_{\widetilde{\omega}} = \mathrm{dvol}_{\omega} = \mathrm{dvol}_{g}$$

by Lemma A.5. If $\omega \in \Lambda^+_g \setminus \{0\}$, then by Lemma A.5 there is a $c > 0$ such that $c\omega$ is compatible with g, hence $cR\omega$ is compatible with $\widetilde{g}$ by (iv), and hence $c\widetilde{\omega} \in \Lambda^+_{\widetilde{g}}$ by Lemma A.5. This shows that $R\Lambda^+_g \subset \Lambda^+_{\widetilde{g}}$. Since R is an involution of $\Lambda^2 V^*$, the subspace $R\Lambda^+_g$ has dimension three and hence agrees with $\Lambda^+_{\widetilde{g}}$. This shows that $\widetilde{g}$ satsfies (v).

We prove that (v) implies (vi). The map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ in (55) is an involution and preserves the exterior product, i.e.

$$R \circ R = \mathrm{id}, \qquad R\omega \wedge R\tau = \omega \wedge \tau$$

for all $\omega, \tau \in \Lambda^2 V^*$. By (v) it also satisfies

$$R\Lambda^+_g = \Lambda^+_{\widetilde{g}}.$$

If $\tau \in \Lambda^-_g$, then $R\tau \wedge R\omega = \tau \wedge \omega = 0$ for all $\omega \in \Lambda^+_g$, hence $R\tau \wedge \widetilde{\omega} = 0$ for every $\widetilde{\omega} \in \Lambda^+_{\widetilde{g}}$, and hence $R\tau \in \Lambda^-_{\widetilde{g}}$. Thus $R\Lambda^-_g = \Lambda^-_{\widetilde{g}}$. It follows that

$$R *_g \omega = R\omega = *_{\widetilde{g}} R\omega, \qquad R *_g \tau = -R\tau = *_{\widetilde{g}} R\tau$$

for all $\omega \in \Lambda^+_g$ and all $\tau \in \Lambda^-_g$. This shows that $R*_g = *_{\widetilde{g}} R$ on $\Lambda^2 V^*$ and hence $\widetilde{g}$ satisfies (vi).

We prove that (vi) implies (i). Let $\widetilde{g} \in \mathcal{G}(V)$ be any inner product that satisfies (vi) and let $h \in \mathcal{G}(V)$ be the inner product defined by the formula

$$h(v, w) := u^{-1} g(Av, Aw)$$

in (i). Since we have already proved that (i) implies (vi), the inner products $\widetilde{g}$ and h both satisfy (vi). Thus they have the same volume form and the same Hodge $*$-operator on 2-forms. Hence

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_h, \qquad \Lambda^+_{\widetilde{g}} = \Lambda^+_h$$

and so $\widetilde{g} = h$ by Theorem A.1. In other words, every inner product $\widetilde{g} \in \mathcal{G}(V)$ that satisfies (vi) is given by $\widetilde{g}(v, w) = u^{-1} g(Av, Aw)$. This completes the proof of Theorem A.2. $\square$

B Quaternionic Subspaces

Denote by $\mathbb{H} \cong \mathbb{R}^4$ the quaternions and by $\mathrm{Sp}(1) \cong S^3$ the unit quaternions. For $\lambda \in \mathbb{H}$ define $V_\lambda := \{(x, x\lambda) \,|\, x \in \mathbb{H}\}$. Thus V_λ is the unique quaternionic subspace of $\mathbb{H}^2$ of real dimension four that contains the pair $(1, \lambda)$.

Lemma B.1. *Let $W \subset \mathbb{H}^2$ be a real linear subspace of real dimension $\dim^{\mathbb{R}} W \leq 4$. Then there exists an element $\lambda \in \mathbb{H}$ such that $V_\lambda \cap W = 0$.*

Proof. The proof is a standard transversality argument and has two steps.

Step 1. *Define $f : \mathrm{Sp}(1) \times \mathbb{H} \to \mathbb{H}^2$ by $f(x, \lambda) := (x, x\lambda)$ for $x \in \mathrm{Sp}(1)$ and $\lambda \in \mathbb{H}$. Then f is transverse to every real linear subspace of $\mathbb{H}^2$.*

Let $W \subset \mathbb{H}^2$ be a real linear subspace and let $(x, \lambda) \in \mathrm{Sp}(1) \times \mathbb{H}$ such that $f(x, \lambda) \in W$. We must prove that $\mathrm{im} df(x, \lambda) + W = \mathbb{H}^2$. To see this, fix any pair $(\xi, \eta) \in \mathbb{H}^2$ and define $\widehat{x} := \xi - \langle \xi, x \rangle x$ and $\widehat{\lambda} := x^{-1}(\eta - \xi\lambda)$. Then

$$df(x, \lambda)(\widehat{x}, \widehat{\lambda}) - (\xi, \eta) = (\widehat{x} - \xi, \widehat{x}\lambda + x\widehat{\lambda} - \eta) = -\langle \xi, x \rangle (x, x\lambda) \in W.$$

This proves Step 1.

Step 2. *We prove the lemma.*

Let $W \subset \mathbb{H}^2$ be a real linear subspace of real dimension at most four. Then the set $\mathcal{M} := f^{-1}(W) = \{(x, \lambda) \in \mathrm{Sp}(1) \times \mathbb{H} \,|\, (x, x\lambda) \in W\}$ is a smooth submanifold of $\mathrm{Sp}(1) \times \mathbb{H}$ of (real) dimension at most three by Step 1. Hence the projection $\mathcal{M} \to \mathbb{H} : (x, \lambda) \mapsto \lambda$ is not surjective by Sard's theorem. Hence there exists an element $\lambda \in \mathbb{H}$ such that $\mathcal{M} \cap (\mathrm{Sp}(1) \times \{\lambda\}) = \emptyset$ and so $V_\lambda \cap W = 0$. This proves Lemma B.1. □

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