A construction of the Deligne–Mumford orbifold

Joel W. Robbin
University of Wisconsin

Dietmar A. Salamon
ETH-Zürich

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Abstract

The Deligne–Mumford moduli space is the space $\bar{\mathcal{M}}_{g,n}$ of isomorphism classes of stable nodal Riemann surfaces of arithmetic genus g with n marked points. A marked nodal Riemann surface is stable if and only if its isomorphism group is finite. We introduce the notion of a universal unfolding of a marked nodal Riemann surface and show that it exists if and only if the surface is stable. A natural construction based on the existence of universal unfoldings endows the Deligne–Mumford moduli space with an orbifold structure. We include a proof of compactness. Our proofs use the methods of differential geometry rather than algebraic geometry.

Contents

1 Introduction

According to Grothendieck [7], a moduli space is a space whose elements may be viewed as orbits of a groupoid.[1] In this paper, the main focus is on the *Riemann moduli space* $\mathcal{M}_g$ of (closed) Riemann surfaces of genus g and various related moduli spaces. We characterize the Deligne–Mumford compactification $\bar{\mathcal{M}}_g$ by a universal mapping property thus showing that it is (canonically) an orbifold. We also treat the related moduli spaces $\mathcal{M}_{g,n}$ and $\bar{\mathcal{M}}_{g,n}$.

The points in the moduli space $\mathcal{M}_g$ are in bijective correspondence with equivalence classes of Riemann surfaces where two Riemann surfaces are equivalent iff there is an isomorphism (holomorphic diffeomorphism)[2] between them; i.e. the Riemann moduli space is the orbit space of the groupoid whose objects are Riemann surfaces and whose morphisms are these isomorphisms. For applications it is important to refine these groupoids by considering Riemann surfaces with *marked points*. An object is now a *marked Riemann surface* of *type* (g, n), i.e. a Riemann surface of genus g equipped with a sequence of n distinct points in that surface. An isomorphism is an isomorphism of Riemann surfaces which carries the sequence of marked points in the source to the sequence in the target preserving the indexing. The corresponding moduli space is denoted $\mathcal{M}_{g,n}$ and of course $\mathcal{M}_{g,0} = \mathcal{M}_g$.

A Riemann surface is a smooth surface Σ equipped with a complex structure j. Since any two smooth surfaces of the same genus are diffeomorphic we may define the Riemann moduli space as the orbit space under the action of the diffeomorphism group $\mathrm{Diff}(\Sigma)$ of the space $\mathcal{J}(\Sigma)$ of complex structures j on Σ:

$$\mathcal{M}_g := \mathcal{J}(\Sigma)/\mathrm{Diff}(\Sigma).$$

The result is independent of the choice of the substrate Σ in the sense that any diffeomorphism $f : \Sigma \to \Sigma'$ induces a bijection $\mathcal{J}(\Sigma) \to \mathcal{J}(\Sigma')$ and a group isomorphism $\mathrm{Diff}(\Sigma) \to \mathrm{Diff}(\Sigma')$ intertwining the group actions. Similarly a marked[3] Riemann surface is a triple (Σ, s_*, j) where s_* is a finite sequence of

[1] By the term *groupoid*, we understand a category all of whose morphisms are isomorphisms.

[2] In the sequel, when no confusion can result, we will use the term *isomorphism* to signify any bijection between sets which preserve the appropriate structures.

[3] The reader is cautioned that the term *marked Riemann surface* is often used with another meaning in the literature.

n distinct points of Σ (i.e. $s_* \in \Sigma^n \setminus \Delta$ where Δ is the “fat” diagonal) so the corresponding moduli space is

$$\mathcal{M}_{g,n} := \big(\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)\big)/\mathrm{Diff}(\Sigma).$$

This can also be written as

$$\mathcal{M}_{g,n} = \mathcal{J}(\Sigma)/\mathrm{Diff}(\Sigma, s_*)$$

where $\mathrm{Diff}(\Sigma, s_*)$ is the subgroup of diffeomorphisms which fix the points of some particular sequence s_*. Thus in these cases we can replace the groupoid by a group action; the objects are the points of $\mathcal{J}(\Sigma)$.

An object in a groupoid is called *stable* iff its automorphism group is finite. A marked Riemann surface of type (g, n) is stable if and only if $n > \chi(\Sigma)$ where $\chi(\Sigma) = 2-2g$ is the Euler characteristic. In this case each automorphism group is finite, but (in the case $g \geq 1$) may be nontrivial. However, the only automorphism isotopic to the identity is the identity itself so the identity component $\mathrm{Diff}_0(\Sigma)$ of $\mathrm{Diff}(\Sigma)$ acts freely on $\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)$. The corresponding orbit space

$$\mathcal{T}_{g,n} := \big(\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)\big)/\mathrm{Diff}_0(\Sigma)$$

is called *Teichmüller space*. In [4] Earle and Eells showed that the projection $\mathcal{J}(\Sigma) \to \mathcal{T}_g$ is a principal fiber bundle with structure group $\mathrm{Diff}_0(\Sigma)$ and that the base $\mathcal{T}_g$ is a finite dimensional smooth manifold of real dimension $6g-6$. In other words, through each $j \in \mathcal{J}(\Sigma)$ there is a smooth slice for the action of $\mathrm{Diff}_0(\Sigma)$. (Similar statements hold for $\mathcal{T}_{g,n}$.) The total space $\mathcal{J}(\Sigma)$ is a complex manifold; the tangent space at a point $j \in \mathcal{J}(\Sigma)$ is the space

$$T_j\mathcal{J}(\Sigma) = \Omega^{0,1}_j(\Sigma, TM) := \{\hat{j} \in \Omega^0(\Sigma, \mathrm{End}(T\Sigma)) : j\hat{j} + \hat{j}j = 0\}$$

of $(0, 1)$ forms on (Σ, j) with values in the tangent bundle. This tangent space is clearly a complex vector space (the complex structure is $\hat{j} \mapsto j\hat{j}$) and it is not hard to show (see e.g. [20] or Section 7) that this almost complex structure on $\mathcal{J}(\Sigma)$ is integrable and that the action admits a holomorphic[4] slice through every point. Since the action of $\mathrm{Diff}_0(\Sigma)$ is (tautologically) by holomorphic diffeomorphisms of $\mathcal{J}(\Sigma)$, this defines a complex structure on the base $\mathcal{T}_g$ which is independent of the choice of the local slice used to define it. Thus $\mathcal{T}_g$ is a complex manifold of dimension $3g-3$. Again, similar results hold for $\mathcal{T}_{g,n}$. Earle and Eells also showed that all three spaces in the fibration

$$\mathrm{Diff}_0(\Sigma) \to \mathcal{J}(\Sigma) \to \mathcal{T}_g \qquad (EE)$$

are contractible so that the fibration is smoothly trivial and has a (globally defined) smooth section. In [3] Earle showed that there is no global holomorphic

[4] At this point in the discussion this means that the slice is a complex submanifold of $\mathcal{J}(\Sigma)$. After we define the complex structure on the base a holomorphic slice will be the same thing as the image of a holomorphic section.

section of $\mathcal{J}(\Sigma) \to \mathcal{T}_g$. The monograph of Tromba [20] contains a nice exposition of this point of view (and more) and the anthology [5] is very helpful for understanding the history of the subject and other points of view.

Now we take a different point of view. An *unfolding* is the germ of a pair (π_A, a_0) where $\pi_A : P \to A$ is a Riemann family and a_0 is a point of A. (The term *Riemann family* means that π_A is a proper holomorphic map and $\dim_{\mathbb{C}}(P) = \dim_{\mathbb{C}}(A)+1$. The term *germ* means that we do not distinguish between (π_A, a_0) and the unfolding which results by replacing A by a neighborhood of a_0 in A.) The fibers $P_a := \pi^{-1}(a)$ are then complex curves. The fiber P_{a_0} is called the *central fiber*. A *morphism* of unfoldings is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & Q \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

where Φ and ϕ are holomorphic, $\phi(a_0) = b_0$ and, for each $a \in A$, the restriction of Φ to the fiber P_a is an isomorphism. Again, this is to be understood in the sense of germs: ϕ need only be defined on a neighborhood of a_0 and two morphisms are the same iff they agree on a smaller neighborhood of a_0. An unfolding $(\pi_B : Q \to B, b_0)$ is called *universal* iff for every other unfolding (π_A, a_0) every isomorphism $f : P_{a_0} \to Q_{b_0}$ extends uniquely to a morphism (ϕ, Φ) from (π_A, a_0) to (π_B, b_0). From the uniqueness of the extension it follows that any two universal unfoldings with the same central fiber are isomorphic in the obvious sense.

Now assume that π_A is a submersion so that the fibers are Riemann surfaces. Using the holomorphic slices for the principal fiber bundle $\mathcal{J}(\Sigma) \to \mathcal{T}_g$ it is not hard to construct a universal unfolding of any Riemann surface of genus ≥ 2; similar results hold for $\mathcal{T}_{g,n}$ (see Section 8).

The spaces $\mathcal{M}_{g,n}$ are not compact. The *Deligne–Mumford* moduli space $\bar{\mathcal{M}}_{g,n}$ defined in Section 6 is a compactification of $\mathcal{M}_{g,n}$. The objects in the corresponding groupoid are commonly called *stable curves of type* (g, n). Two such curves need not be homeomorphic. This moduli space is still the orbit space of a groupoid but not (in any obvious way) the orbit space of a group action. We will characterize $\bar{\mathcal{M}}_{g,n}$ by the universal mapping property, but we will word the definitions so as to avoid the complexities of algebraic geometry and singularity theory.

It is a well known theorem of algebraic geometry that a complex curve C admits a desingularization $u : \Sigma \to C$. This means that Σ is a Riemann surface and that the restriction of u to the set of regular points of u is a holomorphic diffeomorphism onto the set of smooth points of the curve C. The desingularization is unique in the sense that if $u' : \Sigma' \to C$ is another desingularization, the holomorphic diffeomorphism $u^{-1} \circ u'$ extends to a holomorphic diffeomorphism $\Sigma' \to \Sigma$. A marked complex curve is one which is equipped with a finite sequence of distinct smooth points. A desingularization pulls pack the marking to a marking of Σ. That a marked complex curve C is of *type* (g, n) means that the arithmetic genus (see Definition 3.6) of C is g and the number of marked points is n. A *nodal curve* is a complex curve with at worst nodal singularities. For a nodal curve the desingularization u is an immersion and the critical points occur in pairs. This equips Σ with what we call a *nodal structure*. In Section 3 we use the term *marked nodal Riemann surface* to designate a surface Σ with these additional structures. A *stable curve* is a marked nodal curve whose corresponding marked nodal Riemann surface has a finite automorphism group. The main result of this paper extends the universal unfolding construction from the groupoid of stable Riemann surfaces to the groupoid of stable marked nodal Riemann surfaces.

Theorem A. *A marked nodal Riemann surface admits a universal unfolding if and only if it is stable.*

This theorem is an immediate consequence of Theorems 5.4 and 5.6 below. To avoid the intricacies of singularity theory our precise definitions (see Sections 4 and 5) involve only what we call *nodal families*. However, it is well known that (near its central fiber) an unfolding is a submersion if and only if its central fiber is a smooth complex curve and is a nodal family if and only if its central fiber is a nodal curve.

Now we describe the proof. First we consider the case of a Riemann surface without marked points or nodal points. In this case the sequence (EE) is a principal bundle if and only if $g \geq 2$, i.e. if and only if any Riemann surface of genus g is stable. Abbreviate

$$\mathcal{D}_0 := \mathrm{Diff}_0(\Sigma), \qquad \mathcal{J} := \mathcal{J}(\Sigma), \qquad \mathcal{T} := \mathcal{T}(\Sigma) := \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

Thus $\mathcal{T}_g := \mathcal{T}$ is Teichmüller space and the principal fiber bundle (EE) takes the form

$$\mathcal{D}_0 \to \mathcal{J} \to \mathcal{T}.$$

The associated fiber bundle

$$\pi_{\mathcal{T}} : \mathcal{Q} := \mathcal{J} \times_{\mathcal{D}_0} \Sigma \to \mathcal{T}$$

has fibers isomorphic to Σ. It is commonly called the *universal curve of genus g over Teichmüller space*. Choose a Riemann surface (Σ, j_0) and a holomorphic slice $B \subset \mathcal{J}$ through j_0. Let

$$\pi_B : Q \to B$$

be the restriction to B of the pull back of the bundle $\pi_{\mathcal{T}}$ to its total space. As B is a slice, the projection π_B is a trivial bundle (in the smooth sense). The map π_B is a holomorphic submersion. In Section 8 we show that it is a universal unfolding of j_0. Here's why (π_B, j_0) is universal. Let $\pi_A : P \to A$ be a holomorphic submersion whose fiber has genus g and whose central fiber over $a_0 \in A$ is isomorphic to (Σ, j_0). As a smooth map π_A is trivial so after shrinking A we have a smooth local trivialization $\tau : A \times \Sigma \to P$. Write $\tau_a(z) := \tau(a, z)$

for $a \in A$ so τ_a is a diffeomorphism from Σ to P_a. Denote the pull back by τ_a of the complex structure on P_a by j_a, i.e. $\tau_a : (\Sigma, j_a) \to P_a$ is an isomorphism. As B is a slice we can modify the trivialization τ so $j_a \in B$. The equation $\phi(a) = j_a$ defines a map $\phi : A \to B$. Using the various trivializations we then get a morphism (ϕ, Φ) from π_A to π_B. In Section 8 we show that these maps are holomorphic. We also carry out the analogous construction for $\mathcal{T}_{g,n}$.

It is now clear that (π_A, a_0) is universal if and only if $\phi : (A, a_0) \to (B, b_0)$ is the germ of a diffeomorphism. By the inverse function theorem this is so if and only if the linear operator $d\phi(a_0) : T_{a_0}A \to T_{b_0}B$ is invertible. This condition can be formulated as the unique solvability of a partial differential equation on P_{a_0}; we call an unfolding *infinitesimally universal* when it satisfies this unique solvability condition. The crucial point is that infinitesimal universality is meaningful even for nodal families, i.e. when there is no analog of the Earle–Eells principal fiber bundle. But we still have the following

Theorem B. *A nodal unfolding is universal if and only if it is infinitesimally universal.*

This is restated as Theorem 5.4 below. Here is the idea of the proof. Let $(\pi_A : P \to A, a_0)$ and $(\pi_B : Q \to B, b_0)$ be nodal unfoldings and $f_0 : P_{a_0} \to Q_{b_0}$ be an isomorphism of the central fibers. For simplicity assume there is at most one critical point in each fiber and no marked points. Essentially by the definition of nodal unfolding there is a neighborhood N of the set of critical points such that for $a \in A$ the intersection $N_a := N \cap P_a$ admits an isomorphism

$$N_a \cong \{(x, y) \in \mathbb{D}^2 : xy = z\}$$

where $\mathbb{D}$ is the closed unit disk in $\mathbb{C}$ and $z = z(a) \in \mathbb{D}$. Thus if $z(a) \neq 0$ the fiber N_a is an annulus whereas if $z(a) = 0$ it is a pair of transverse disks. In either case the boundary is a disjoint union $(\partial\mathbb{D} \sqcup \partial\mathbb{D})$ of two copies of the circle $S^1 := \partial\mathbb{D}$. The map $N \to A$ is therefore not trivializable as the topology of the fiber changes. However, the bundle $\partial N \to A$ is trivializable; choose a trivialization $A \times (\partial\mathbb{D} \sqcup \partial\mathbb{D}) \to \partial N$. Using this trivialization we will define (see Section 11) manifolds of maps

$$\mathcal{W} := \bigsqcup_{a \in A} \mathcal{W}_a, \qquad \mathcal{W}_a := \bigsqcup_{b \in B} \mathcal{W}(a, b), \qquad \mathcal{W}(a, b) := \mathrm{Map}(\partial N_a, Q_b \setminus C_B)$$

where C_B is the set of critical points of π_B and $\bigsqcup$ denotes disjoint union. Let $\mathcal{U}_a \subset \mathcal{W}$ be the set of all maps in $\mathcal{W}_a$ which extend to a holomorphic map $N_a \to Q$ and $\mathcal{V}_a \subset \mathcal{W}$ be the set of all maps in $\mathcal{W}_a$ which extend to a holomorphic map $P_a \setminus N_a \to Q$. We will replace A and $\mathcal{W}$ by smaller neighborhoods of a_0 and $f_0|\partial N_{a_0}$ as necessary. We show that $\mathcal{U}_a$ and $\mathcal{V}_a$ are submanifolds of $\mathcal{W}_a$. It is not too hard to show that the unfolding (π_B, b_0) is universal if and only if the manifolds $\mathcal{U}_a$ and $\mathcal{V}_a$ intersect in a unique point: the morphism

$(\phi, \Phi) : (\pi_A, a_0) \to (\pi_B, b_0)$ is then defined so that this intersection point γ lies in the fiber $\mathcal{W}_{\phi(a)}$ and Φ_a is the unique holomorphic map extending γ. We will see that the unfolding (π_B, b_0) is infinitesimally universal if and only if (for all (π_A, a_0) and f_0) the corresponding infinitesimal condition

$$T_{\gamma_0}\mathcal{W}_{a_0} = T_{\gamma_0}\mathcal{U}_{a_0} \oplus T_{\gamma_0}\mathcal{V}_{a_0}$$

holds where $\gamma_0 = f_0|\partial N_{a_0}$. This Hardy space decomposition is reminiscent of the construction of the moduli space of holomorphic vector bundles explained by Pressley & Segal in [18].

We have already explained why smooth marked Riemann surfaces have universal unfoldings. It is now easy to construct a universal unfolding of a stable marked nodal Riemann surface: it is constructed from a universal unfolding for the marked Riemann surface that results by replacing each nodal point by a marked point. Such an unfolding is a triple (π, S_*, b_0) where $\pi : Q \to B$ is a nodal family, S_* is a sequence of holomorphic sections of π corresponding to the marked points, and $b_0 \in B$. We call a pair (π, S_*) a *universal family of type* (g, n) iff (1) (π, S_*, b_0) is a universal unfolding for each $b_0 \in B$ and (2) every marked nodal Riemann surface of type (g, n) occurs as the domain of a desingularization of some fiber Q_b, $b \in B$. Theorem 5.3 (openness of transversality) says that if (π, S_*, b_0) is an infintesimally universal unfolding so is (π, S_*, b) for b near b_0. Together with Theorems A and B this implies

> **Theorem C.** *If $n > 2 - 2g$ there exists a universal family of type (g, n).*

This is restated as Proposition 6.3 below. It is not asserted that B is connected. Rather, the universal family should be viewed as a generalization of the notion of an atlas for a manifold. This generalization is called an *etale groupoid.* The Deligne Mumford orbifold $\bar{\mathcal{M}}_{g,n}$ is then the orbit space of this groupoid and the definitions are arranged so that the orbifold structure is independent of the choice of the universal family used to define it. See Section 6.

A consequence of our theorems is that other constructions of the Deligne–Mumford moduli space (and in particular of the Riemann moduli space) which have the universal unfolding property give the same space. However, in the case of a construction where the moduli space is given only a topology (or a notion of convergence of sequences as in [10]) we show that the topology determined by our construction agrees with the topology of the other construction (see Section 13). In Section 14 we prove that our $\bar{\mathcal{M}}_{g,n}$ is compact and Hausdorff by adapting the arguments of the monograph of Hummel [10].

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Notation. Throughout the closed unit disk in the complex plane is denoted by

$$\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| \le 1\}$$

and its interior is denoted by $\mathrm{int}(\mathbb{D}) := \{z \in \mathbb{C} \,:\, |z| < 1\}$. Thus $S^1 := \partial\mathbb{D}$ is the unit circle. Also

$$\mathbb{A}(r, R) := \{z \in \mathbb{C} : r \le |z| \le R\}$$

denotes the closed annulus with inner radius r and outer radius R.

2 Orbifold structures

In this section we review orbifolds. Our definitions are arranged so as to suit our ultimate objective of defining an orbifold structure on the Deligne–Mumford moduli space.

2.1. A **groupoid** is a category in which every morphism is an isomorphism. Let B be the set of objects of a groupoid and Γ denote the set of (iso)morphisms. For $a, b \in B$ let $\Gamma_{a,b} \subset \Gamma$ denote the isomorphisms from a to b; the group

$$\Gamma_a := \Gamma_{a,a}$$

is called the **automorphism group**[5] of a. The groupoid is called **stable** iff every automorphism group is finite. Define the **source** and **target** maps $s, t : \Gamma \to B$ by

$$s(g) = a \text{ and } t(g) = b \iff g \in \Gamma_{a,b}.$$

The map $e : B \to \Gamma$ which assigns to each object a the identity morphism of a is called the **identity section** of the groupoid and the map $i : \Gamma \to \Gamma$ which assigns to each morphism g its inverse $i(g) = g^{-1}$ is called the **inversion map**. Define the set $\Gamma\,{}_s\!\times_t \Gamma$ of **composable pairs** by

$$\Gamma\,{}_s\!\times_t \Gamma = \{(g, h) \in \Gamma \times \Gamma : s(g) = t(h)\}.$$

The map $m : \Gamma\,{}_s\!\times_t \Gamma \to \Gamma$ which assigns to each composable pair the composition $m(g, h) = gh$ is called the **multiplication map**. The five maps s, t, e, i, m are called the **structure maps** of the groupoid. Note that

$$\Gamma_{a,b} = (s \times t)^{-1}(a, b).$$

We denote the **orbit space** of the groupoid (B, Γ) by B/Γ:

$$B/\Gamma := \{[b] : b \in B\}, \qquad [b] := \{t(g) \in B : g \in \Gamma,\ s(g) = b\}.$$

2.2. A **Lie groupoid** is a groupoid (B, Γ) such that B and Γ are smooth manifolds[6], the structure maps are smooth, and the map $s : \Gamma \to B$ (and hence

[5] Also commonly called the *isotropy group* or *stabilizer group*.
[6] For us a manifold is always second countable and Hausdorff, unless otherwise specified.

also the map $t = s \circ i$) is a submersion. (The latter condition implies that $\Gamma_s\times_t\Gamma$ is a submanifold of $\Gamma \times \Gamma$ so that the condition that m be smooth is meaningful.) A **homomorphism** from a Lie groupoid (B, Γ) to a Lie groupoid (B', Γ') is a smooth functor, i.e. a pair of smooth maps $B \to B'$ and $\Gamma \to \Gamma'$, both denoted by ι, which intertwine the structure maps:

$$s' \circ \iota = \iota \circ s, \qquad t' \circ \iota = \iota \circ t, \qquad e' \circ \iota = \iota \circ e,$$

$$i' \circ \iota = \iota \circ i, \qquad m' \circ (\iota \times \iota) = \iota \circ m.$$

(The first two of these five conditions imply that $(\iota \times \iota)(\Gamma\ {}_s\times_t \Gamma) \subset \Gamma'\ {}_{s'}\times_{t'} \Gamma'$ so that the fifth condition is meaningful.) Similar definitions are used in the complex category reading *complex* for *smooth* (for manifolds) or *holomorphic* for smooth (for maps). A Lie groupoid (B, Γ) is called **proper** if the map $s \times t : \Gamma \to B \times B$ is proper.

2.3. An **etale groupoid** is a Lie groupoid (B, Γ) such that the map $s : \Gamma \to B$ (and hence also the map $t = s \circ i$) is a local diffeomorphism. A proper etale groupoid is automatically stable. A homomorphism $\iota : (B, \Gamma) \to (B', \Gamma')$ of etale groupoids is called a **refinement** iff the following holds.

(i) The induced map $\iota_* : B/\Gamma \to B'/\Gamma'$ on orbit spaces is a bijection.

(ii) For all $a, b \in B$, ι restricts to a bijection $\Gamma_{a,b} \to \Gamma'_{\iota(a),\iota(b)}$.

(iii) The map on objects (and hence also the map on morphisms) is a local diffeomorphism.

Two proper etale groupoids are called **equivalent** iff they have a common proper refinement.

Definition 2.4. Fix an abstract groupoid $(\mathcal{B}, \mathcal{G})$. This groupoid is to be viewed as the "substrate" for an additional structure to be imposed; initially it does not even have a topology. Indeed, the definitions are worded so as to allow for the possibility that $\mathcal{B}$ is not even a set but a proper class in the sense of Gödel Bernays set theory (see [12]).

An **orbifold structure** on the groupoid $(\mathcal{B}, \mathcal{G})$ is a functor σ from a proper etale groupoid (B, Γ) to $(\mathcal{B}, \mathcal{G})$ such that

(i) σ induces a bijection $B/\Gamma \to \mathcal{B}/\mathcal{G}$ of orbit spaces, and

(ii) for all $a, b \in B$, σ restricts to a bijection $\Gamma_{a,b} \to \mathcal{G}_{\sigma(a),\sigma(b)}$.

A **refinement** of orbifold structures is a refinement $\iota : (B, \Gamma) \to (B', \Gamma')$ of proper etale groupoids such that $\sigma = \sigma' \circ \iota$; as before we say that $\sigma : (B, \Gamma) \to (\mathcal{B}, \mathcal{G})$ is a refinement of $\sigma' : (B', \Gamma') \to (\mathcal{B}, \mathcal{G})$. Two orbifold structures are called **equivalent** iff they have a common refinement. An **orbifold** is an abstract groupoid $(\mathcal{B}, \mathcal{G})$ equipped with an orbifold structure $\sigma : (B, \Gamma) \to (\mathcal{B}, \mathcal{G})$.

Example 2.5. A smooth manifold M is a special case of an orbifold as follows: View $M =: \mathcal{B}$ as a trivial groupoid, i.e. the only morphisms are identity morphisms. Any countable open cover $\{U_\alpha\}_{\alpha \in I}$ on M determines an etale groupoid (B, Γ) with

$$B := \bigsqcup_{\alpha \in I} U_\alpha, \qquad \Gamma := \bigsqcup_{(\alpha,\beta) \in I \times I} U_\alpha \cap U_\beta,$$

$$s(\alpha, p, \beta) := (\alpha, p), \qquad t(\alpha, p, \beta) := (\beta, p), \qquad e(\alpha, p) := (\alpha, p, \alpha),$$

$$i(\alpha, p, \beta) := (\beta, p, \alpha), \qquad m((\beta, p, \gamma), (\alpha, p, \beta)) := (\alpha, p, \gamma).$$

Here $\bigsqcup$ denotes disjoint union. (The **disjoint union** $\bigsqcup_{\alpha \in I} X_\alpha$ of an indexed collection $\{X_\alpha\}_{\alpha \in I}$ of sets is the set of pairs (α, x) where $\alpha \in I$ and $x \in X_\alpha$.) A refinement of open covers in the usual sense determines a refinement of etale groupoids as in 2.3.

If $\{\phi_\alpha, U_\alpha\}_{\alpha \in I}$ is a countable atlas then an obvious modification of the above construction gives rise to an orbifold structure on M where B is a disjoint union of open subsets of Euclidean space, i.e. a manifold structure is a special case of an orbifold structure.

Example 2.6. A Lie group action $G \to \mathrm{Diff}(M)$ determines a Lie groupoid $(\mathcal{B}, \mathcal{G})$ where $\mathcal{B} = M$, $\mathcal{G} = \{(g, a, b) \in G \times M \times M : b = g(a)\}$, and the structure maps are defined by $s(g, a, b) := a$, $t(g, a, b) := b$, $e(a) := (\mathrm{id}, a, a)$, $i(g, a, b) := (g^{-1}, b, a)$, and $m((h, b, c), (g, a, b)) := (hg, a, c)$. The orbit space $\mathcal{B}/\mathcal{G}$ of this groupoid is the same as the orbit space M/G of the group action. The condition that this groupoid be proper is the usual definition of proper group action, i.e. the map $G \times M \to M \times M : (g, x) \mapsto (x, g(x))$ is proper.

Assume that the action is almost free [meaning that the isotropy group G_p of each point of M is finite] and sliceable [meaning that there is a slice through every point of M; a slice is a submanifold $S \subset M$ such that there is a neighborhood U of the identity in G with the property that the map $U \times S \to M : (g, x) \mapsto g(x)$ is a diffeomorphism onto a neighborhood of S in M]. Now let

$$B := \bigsqcup_{\alpha \in I} S_\alpha$$

be a disjoint union of slices such that every orbit passes through at least one slice. Let

$$\Gamma := \bigsqcup_{\alpha, \beta \in I} \Gamma_{\alpha\beta}, \qquad \Gamma_{\alpha\beta} := \{(g, a, b) \in \mathcal{G} : a \in S_\alpha, b \in S_\beta\}.$$

Then $\Gamma_{\alpha\beta}$ is a submanifold of $\mathcal{G}$. Moreover, if the group action is proper, then the obvious morphism $\sigma : (B, \Gamma) \to (\mathcal{B}, \mathcal{G})$ is an orbifold structure, and any two such orbifold structures are equivalent. Note that, if G is a discrete group acting properly on M, then $S := \mathcal{B} = M$ is a slice and $\sigma := \mathrm{id}$ is an orbifold structure.

Example 2.7. Consider the group action where $G := \mathbb{Z}$ acts on $M := S^1$ by $(k, z) \mapsto e^{2\pi i k \omega} z$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is irrational. Then the groupoid $(\mathcal{B}, \mathcal{G})$

constructed in Example 2.6 is etale but not proper. Note that the quotient $\mathcal{B}/\mathcal{G}$ is an uncountable set with the trivial topology (two open sets). The inclusion of any open set into S^1 is a refinement.

Example 2.8. Consider the group action where the multiplicative group $G := \mathbb{R}^*$ of nonzero real numbers acts on $M := \mathbb{R}^2 \setminus 0$ by $t \cdot (x, y) := (tx, t^{-1}y)$. The action is free and sliceable but not proper, and the quotient topology is non Hausdorff (every neighborhood of $\mathbb{R}^* \cdot (1, 0)$ intersects every neighborhood of $\mathbb{R}^* \cdot (0, 1)$). The groupoid constructed from the disjoint union $B := S_1 \bigsqcup S_2$ of the two slices $S_1 := \{1\} \times \mathbb{R}$, $S_2 := \mathbb{R} \times \{1\}$ is not proper. If we extend the group action, by adjoining the map $(x, y) \mapsto (y, x)$, the orbit space is $\mathbb{R}$ which is Hausdorff, but the new group action is still not proper.

2.9. Let (B, Γ) be a stable etale groupoid, $a, b \in B$, and $g \in \Gamma_{a,b}$. Then there exist neighborhoods U of a, V of b in B, and N of g in Γ such that s maps N diffeomorphically onto U and t maps N diffeomorphically onto V. Define $s_g := s|N$, $t_g := t|N$, and $\phi_g := t_g \circ s_g^{-1}$. Thus ϕ_g "extends" $g \in \Gamma_{a,b}$ to diffeomorphism $\phi_g : U \to V$. The following lemma says that when $a = b$ we may choose $U = V$ independent of g and obtain an action

$$\Gamma_a \to \mathrm{Diff}(U) : g \mapsto \phi_g$$

of the finite group Γ_a on the open set U.

Lemma 2.10. *Let (B, Γ) be a stable etale groupoid and $a \in B$. Then there exists a neighborhood U of a and pairwise disjoint neighborhoods N_g (for $g \in \Gamma_a$) of g in Γ such that both s and t map each N_g diffeomorphically onto U.*

Proof. Choose disjoint open neighborhoods P_g of $g \in \Gamma_a$ such that $s_g := s|P_g$ and $t_g := t|P_g$ are diffeomorphisms onto (possibly different) neighborhoods of a. By stability the group Γ_a is finite so there is a neighborhood V of a in B such that $V \subset s(P_g) \cap t(P_g)$ for $g \in \Gamma_a$. Define $\phi_g : V \to B$ by $\phi_g := t_g \circ s_g^{-1}$. Now choose $f, g \in \Gamma_a$ and let $h := m(f, g)$. We show that

$$\phi_h(x) = \phi_f \circ \phi_g(x) \tag{1}$$

for x in a sufficiently small neighborhood of a in V. For such x define $y := \phi_g(x) \in V$, $z := \phi_f(y) \in V$, $g' := s_g^{-1}(x) \in P_g$, and $f' := s_f^{-1}(y) \in P_f$. As $t(g') = s(f') = y$ we have $(f', g') \in \Gamma_s\!\times_t \Gamma$, i.e. $h' := m(f', g')$ is well defined. By continuity, $h' \in P_h$ and $s(h') = s(g') = x$ and $t(h') = t(f') = z$, and hence $z = \phi_h(x)$ as claimed. Using the finiteness of Γ_a again we may choose a neighborhood W of a so that (1) holds for all $f, g \in \Gamma_a$ and all $x \in W$. Now the intersection

$$U := \bigcap_{g \in \Gamma_a} \phi_g(W) \subset V$$

satisfies $\phi_f(U) = U$ for $f \in \Gamma_a$ so U and $N_g := s_g^{-1}(U)$ satisfy the conclusions of the lemma. □

Corollary 2.11. *Let (B, Γ) be a stable etale groupoid and $a, b \in B$. Then there exist neighborhoods U and V of a and b in B and pairwise disjoint neighborhoods N_f (for $f \in \Gamma_{a,b}$) of f in Γ such that s maps each N_f diffeomorphically onto U and t maps each N_f diffeomorphically onto V. The etale groupoid is proper if and only if these neighborhoods may be chosen so that in addition*

$$(s \times t)^{-1}(U \times V) = \bigcup_{f \in \Gamma_{a,b}} N_f. \qquad (*)$$

Proof. Choose disjoint neighborhoods P_f of $f \in \Gamma_{a,b}$ such that $s_f := s|P_f$ and $t_f := t|P_f$ are diffeomorphisms onto (possibly different) neighborhoods of a. Choose U as in Lemma 2.10 so small that $U \subset s(P_f)$ for all $f \in \Gamma_{a,b}$ and define $\phi_f : U \to B$ by

$$\phi_f := t_f \circ s_f^{-1}|U.$$

Define $N_f := s_f^{-1}(U)$. As in Lemma 2.10 we have $\phi_h = \phi_f \circ \phi_g$ for $g \in \Gamma_a$, $f \in \Gamma_{a,b}$, $h := m(f, g)$, so $t_h(N_h) = \phi_h(U) = \phi_f(U) = t_f(N_f)$. Any two elements $h, f \in \Gamma_{a,b}$ satisfy $h = m(f, g)$ for some $g \in \Gamma_a$ so $V := t_f(N_f)$ is independent of the choice of $f \in \Gamma_{a,b}$ used to define it. The condition that $s \times t$ is proper, is that for any sequence $\{f_\nu \in \Gamma_{a_\nu, b_\nu}\}_\nu$ such that the sequences $\{a_\nu\}_\nu$ and $\{b_\nu\}_\nu$ converge to a and b respectively, the sequence $\{f_\nu\}_\nu$ has a convergent subsequence. Condition $(*)$ implies this as f_ν must lie in some N_f for infinitely many values of ν. The converse follows easily by an indirect argument. □

2.12. Let (B, Γ) be an etale groupoid and equip the orbit space B/Γ with the quotient topology, i.e. a subset of B/Γ is open iff its preimage under the quotient map $\pi : B \to B/\Gamma$ is open. If $U \subset B$ is open then so is $\pi^{-1}(\pi(U)) = \{t(g) : g \in s^{-1}(U)\}$ so π is an open map. If $\iota : (B, \Gamma) \to (B', \Gamma')$ is a refinement of etale groupoids, then the induced bijection $\iota_* : B/\Gamma \to B'/\Gamma'$ is a homeomorphism. [The continuity of ι_* follows from the continuity of ι; the openness of ι_* follows from the openness of ι and the fact that if $U' \subset B'$ is open then so is ${\pi'}^{-1}(\pi'(U'))$.] Hence equivalent etale groupoids have homeomorphic orbit spaces. It follows that the topology induced on $\mathcal{B}/\mathcal{G}$ by an orbifold structure $\sigma : (B, \Gamma) \to (\mathcal{B}, \mathcal{G})$ depends only the equivalence class. This topology is called the **orbifold topology**.

Corollary 2.13. *For a proper etale groupoid the quotient topology on B/Γ is Hausdorff.*

Proof. In other words if $\Gamma_{a_0,b_0} = \emptyset$ then there are neighborhoods U of a and V of b such that $\Gamma(a, b) = \emptyset$ for $a \in U$ and $v \in V$. This is a special case of Corollary 2.11. □

3 Structures on surfaces

The phrase **surface** means *oriented smooth (i.e. C^∞) manifold of (real) dimension two, not necessarily connected.* Unless otherwise specified all surfaces are assume to be closed, i.e. compact and without boundary. The structures we impose on surfaces are complex structures, nodal structures, and point markings. Surfaces equipped with these structures form the objects of a groupoid. The objective of this paper is to equip the orbit space of this groupoid with an orbifold structure.

Definition 3.1. A **Riemann surface** is a pair (Σ, j) where Σ is a surface and $j : T\Sigma \to T\Sigma$ is a smooth complex structure on Σ which determines the given orientation of Σ. Since a complex structure on a surface is necessarily integrable, a Riemann surface may be viewed as a smooth complex curve, i.e. a compact complex manifold of (complex) dimension one. When there is no danger of confusion we denote a Riemann surface and its underlying surface by the same letter.

Definition 3.2. A **nodal surface** is a pair (Σ, ν) consisting of a surface Σ and a set

$$\nu = \big\{\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{y_{2k-1}, y_{2k}\}\big\}$$

where $y_1, y_2, \ldots, y_{2k}$ are distinct points of Σ; we also say ν is a **nodal structure** on Σ. The points $y_1, y_2, \ldots, y_{2k}$ are called the **nodal points** of the structure and the points y_{2j-1} and y_{2j} are called **equivalent nodal points**. The nodal structure should be viewed as an equivalence relation on Σ such that every equivalence class consists of either one or two points and only finitely many equivalence classes have two points. Hence we often abbreviate $\Sigma \setminus \cup\nu$ by

$$\Sigma \setminus \nu := \Sigma \setminus \{y_1, y_2, y_3, y_4, \ldots, y_{2k-1}, y_{2k}\}.$$

Definition 3.3. A **point marking** of a surface Σ is a sequence

$$r_* = (r_1, r_2, \ldots, r_n)$$

of distinct points of Σ; the points r_i are called **marked points**. A **marked nodal surface** is a triple (Σ, r_*, ν) where (Σ, ν) is a nodal surface and r_* is a point marking of Σ such that no marked point r_i is a nodal point of (Σ, ν); a **special point** of the marked nodal surface is a point which is either a nodal point or a marked point.

Definition 3.4. A marked nodal surface (Σ, r_*, ν) determines a labelled graph called the **signature** of (Σ, r_*, ν) as follows. The set of vertices of the graph label the connected components of Σ and there is one edge connecting vertices α and β for every pair of equivalent nodal points with one of the points in Σ_α and the other in Σ_β. More precisely, the number of edges from Σ_α to Σ_β is the number of pairs $\{x, y\}$ of equivalent nodal points with either $x \in \Sigma_\alpha$ and $y \in \Sigma_\beta$ or $y \in \Sigma_\alpha$ and $x \in \Sigma_\beta$. Each vertex α has two labels, the genus of the component Σ_α denoted by g_α and the set of indices of marked points which lie in the component Σ_α.

Remark 3.5. Two marked nodal surfaces are isomorphic if and only if they have the same signature.

Proof. In other words, (Σ, r_*, ν) and (Σ', r'_*, ν') have the same signature if and only if there is a diffeomorphism $\phi : \Sigma \to \Sigma'$ such that $\nu' = \phi_*\nu$ where

$$\phi_*\nu := \big\{\{\phi(y_1), \phi(y_2)\}, \{\phi(y_3), \phi(y_4)\}, \ldots, \{\phi(y_{2k-1}), \phi(y_{2k})\}\big\}$$

and $r'_i = \phi(r_i)$ for $i = 1, 2, \ldots, n = n'$. This is because two connected surfaces are diffeomorphic if and only if they have the same genus and any bijection between two finite subsets of a connected surface extends to a diffeomorphism of the ambient manifold. □

Definition 3.6. Define the **Betti numbers** of a graph by the formula

$$b_i := \operatorname{rank} H_i(K), \qquad i = 0, 1,$$

where $H_i(K)$ is the ith homology group of the cell complex K. Thus K is connected if and only if $b_0 = 1$ and

$$b_0 - b_1 = \text{ \# vertices - \# edges.}$$

Define the **genus** of the labelled graph by

$$g := b_1 + \sum_\alpha g_\alpha.$$

The **arithmetic genus** of a nodal surface (Σ, ν) is the genus of the signature of (Σ, ν). Note that the arithmetic genus can be different from the **total genus** $g' := \sum_\alpha g_\alpha$.

Definition 3.7. A marked nodal surface (Σ, r_*, ν) said to be of **type** (g, n) iff the length of the sequence r_* is n, the underlying graph K in the signature is connected, and the arithmetic genus of (Σ, ν) is g. A marked nodal Riemann surface (Σ, r_*, ν, j) is called **stable** iff its **automorphism group**

$$\operatorname{Aut}(\Sigma, r_*, \nu, j) := \{\phi \in \operatorname{Diff}(\Sigma) : \phi_* j = j,\ \phi_*\nu = \nu,\ \phi(r_*) = r_*\}$$

is finite. A stable marked nodal Riemann surface is commonly called a **stable curve**.

3.8. A marked nodal Riemann surface of type (g, n) is stable if and only if the number of special points in each component of genus zero is at least three and the number of special points in each component of genus one is at least one. This is an immediate consequence of the following:

(i) An automorphism of a surface of genus zero is a Möbius transformation; if it fixes three points it is the identity.

(ii) A surface of genus one is isomorphic to $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ and τ lies in the upper half plane.

(iii) The automorphisms of the Abelian group Λ of form $z \mapsto az$ where $a \in \mathbb{C} \setminus 0$ form a group of order at most six.

(iv) The automorphism group of a compact Riemann surface of genus greater than one is finite.

The proofs of these well known assertions can be found in any book on Riemann surfaces. It follows that for each pair (g, n) of nonnegative integers there are only finitely many labelled graphs which arise as the signature of a stable marked nodal Riemann surface of type (g, n).

Remark 3.9. A marked nodal surface has arithmetic genus zero if and only if each component has genus zero and the graph is a tree. The automorphism group of a stable marked nodal Riemann surface of arithmetic genus zero is trivial.

4 Nodal families

In this section we introduce the basic setup which will allow us to define the charts of the Deligne–Mumford orbifold.

4.1. Let P and A be complex manifolds with $\dim_{\mathbb{C}}(P) = \dim_{\mathbb{C}}(A) + 1$ and

$$\pi : P \to A$$

be a holomorphic map. By the holomorphic implicit function theorem a point $p \in P$ is a regular point of $\pi : P \to A$ if and only if there is a holomorphic coordinate system $(t_1, \dots, t_n)$ defined in a neighborhood of $\pi(p) \in A$, and a function z defined in a neighborhood of p in P such that $(z, t_1 \circ \pi, \dots, t_n \circ \pi)$ is holomorphic coordinate system. In other words, the point p is a regular point if and only if the germ of π at p is isomorphic to the germ at 0 of the projection

$$\mathbb{C}^{n+1} \to \mathbb{C}^n : (z, t_1, \dots, t_n) \mapsto (t_1, \dots, t_n).$$

Similarly, a point $p \in P$ is a called a **nodal point** of π if and only if the germ of π at p is isomorphic to the germ at 0 of the map

$$\mathbb{C}^{n+1} \to \mathbb{C}^n : (x, y, t_2, \dots, t_n) \mapsto (xy, t_2, \dots, t_n),$$

i.e. if and only if there are holomorphic coordinates $z, t_2, \dots, t_n$ on A at $\pi(p)$ and holomorphic functions x and y defined in a neighborhood of p such that $(x,\, y,\, t_2 \circ \pi,\, \dots,\, t_n \circ \pi)$ is a holomorphic coordinate system, $x(p) = y(p) = 0$, and $xy = z \circ \pi$. At a regular point p we have that $\dim_{\mathbb{C}} \ker(d\pi(p)) = 1$ and $\dim_{\mathbb{C}} \operatorname{coker}(d\pi(p)) = 0$ while at a nodal point we have that $\dim_{\mathbb{C}} \ker(d\pi(p)) = 2$ and $\dim_{\mathbb{C}} \operatorname{coker}(d\pi(p)) = 1$

Definition 4.2. A **nodal family** is a surjective proper holomorphic map

$$\pi : P \to A$$

between connected complex manifolds such that $\dim_{\mathbb{C}}(P) = \dim_{\mathbb{C}}(A) + 1$ and every critical point of π is nodal. We denote the set of critical points of π by

$$C_\pi := \{p \in P : d\pi(p) \text{ not surjective}\}.$$

It intersects each fiber

$$P_a := \pi^{-1}(a)$$

in a finite set. For each regular value $a \in A$ of π the fiber P_a is a compact Riemann surface. When $a \in A$ is a critical value of π we view the fiber P_a as a nodal Riemann surface as follows.

By the maximum principle the composition $\pi \circ u$ of π with a holomorphic map $u : \Sigma \to P$ defined on a compact Riemann surface Σ must be constant, i.e. $u(\Sigma) \subset P_a$ for some a. A **desingularization** of a fiber P_a is a holomorphic map $u : \Sigma \to P$ defined on a compact Riemann surface Σ such that

(1) $u^{-1}(C_\pi)$ is finite,

(2) the restriction of u to $\Sigma \setminus u^{-1}(C_\pi)$ maps this set bijectively to $P_a \setminus C_\pi$.

The restriction of u to $\Sigma \setminus u^{-1}(C_\pi)$ is an isomorphism between this open Riemann surface and $P_a \setminus C_\pi$ (because it is holomorphic, bijective, and proper).

Lemma 4.3. **(i)** *Every fiber of a nodal family admits a desingularization.*

(ii) *If $u_1 : \Sigma_1 \to P$ and $u_2 : \Sigma_2 \to P$ are two desingularizations of the same fiber, then the map*

$$u_2^{-1} \circ u_1 : \Sigma_1 \setminus u_1^{-1}(C_\pi) \to \Sigma_2 \setminus u_2^{-1}(C_\pi)$$

extends to an isomorphism $\Sigma_1 \to \Sigma_2$.

(iii) *A desingularization u of a fiber of a nodal family is an immersion and the preimage $u^{-1}(p)$ of a critical point $p \in C_\pi$ consists of exactly two points.*

Proof. Let $\pi : P \to A$ be a nodal family and $a \in A$. Each $p \in C_\pi \cap P_a$ has a small neighborhood intersecting P_a in two transverse embedded holomorphic disks intersecting at p. Define Σ set theoretically as the disjoint union of $P_a \setminus C_\pi$ with two copies of $P_a \cap C_\pi$ and use these disks as coordinates; the map $u : \Sigma \to P_a$ is the identity on $P_a \setminus C_\pi$ and sends each pair of nodal points to the point of C_p which gave rise to it. Assertion (ii) follows from the removable singularity theorem for holomorphic functions and (iii) follows from (ii) and the fact that the maps $x \mapsto (x, 0)$ and $y \mapsto (0, y)$ are immersions. □

Remark 4.4. We can construct a **canonical desingularization** of the fiber by replacing each point $p \in P_a \cap C_\pi$ by a point for each connected component of $U \setminus \{p\}$ where U is a suitable neighborhood of p in P_a and extending the smooth and complex structures in the only way possible.

Definition 4.5. Let $\pi_A : P \to A$ and $\pi_B : Q \to B$ be nodal families. For $a \in A$ and $b \in B$ a bijection $f : P_a \to Q_b$ is called a **fiber isomorphism** if for some (and hence every) desingularization $u : \Sigma \to P_a$ the map $f \circ u : \Sigma \to Q_b$

is a desingularization. A **pseudomorphism** from π_A to π_B is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & Q \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

where Φ and ϕ are smooth and, for each $a \in A$, the restriction of Φ to the fiber P_a is a fiber isomorphism. A **morphism** is a pseudo morphism such that both ϕ and Φ are holomorphic. For $a \in A$ and $b \in B$ the notation

$$(\Phi, \phi) : (\pi_A, a) \to (\pi_B, b)$$

indicates that the pseudo morphism (Φ, ϕ) satisfies $\phi(a) = b$.

Lemma 4.6. *Let $\pi : P \to A$ be a nodal family. Then the arithmetic genus (see Definition 3.6) of the fiber P_a is a locally constant function of $a \in A$.*

Proof. The arithmetic genus is the genus of the surface obtained by removing a small disk about each nodal point and identifying corresponding components. Hence it is equal to the ordinary genus of a regular fiber. □

Definition 4.7. A **marked nodal family** is a pair (π, R_*) where $\pi : P \to A$ is a nodal family and

$$R_* = (R_1, \dots, R_n)$$

is a sequence of complex submanifolds of P which are pairwise disjoint and such that $\pi|R_i$ maps R_i diffeomorphically onto A. It follows that R_i does not intersect the set C_π of critical points. A desingularization $u : \Sigma \to P$ of a fiber P_a of a marked nodal family (π, R_*) determines a point marking r_*: the point marking r_* is given by the formula

$$\{u(r_i)\} = R_i \cap P_a$$

for $i = 1, 2, \dots, n$. By Lemma 4.3 any two desingularizations of the same fiber give rise to isomorphic marked nodal Riemann surfaces. Thus the signature (see Definition 3.4) of the fiber $(P_a, P_a \cap R_*)$ is independent of the choice of the desingularization used to define it. In the context of marked nodal families, the term *fiber isomorphism* is understood to entail that the bijection f preserves the induced point markings; similarly pseudo morphisms and morphisms of marked nodal families preserve the corresponding point markings. We say that the marked nodal family (π, R_*) is of **type** (g, n) when each fiber is of type (g, n) (see Definition 3.7).

Definition 4.8. A fiber of a marked nodal family $\pi : P \to A$ is called **stable** iff its desingularization is stable. A marked nodal family is called **stable** iff each of its fibers is stable.

Remark 4.9. It is easy to see that stability is an open condition, i.e. every stable fiber has a neighborhood consisting of stable fibers. However, the open set of stable fibers can have unstable fibers in its closure. For example, consider the nodal family $(\pi, (R_1, R_2, R_3))$ with

$$P = \{([x,y,z],a) \in \mathbb{C}P^2 \times \mathbb{C} : xy = az^2\},$$

$A = \mathbb{C}$, $\pi([x,y,z],a) = a$, $R_1 = \{[1,0,0]\} \times A$, $R_2 = \{[0,1,0]\} \times A$, and $R_3 = \{([1,a,1],a) : a \in A\}$. The desingularization of the fiber over 0 consists of two components of genus zero and the regular fibers consist of one component of genus zero. The regular fibers all have three marked points and are thus stable; one of the two components of the (desingularized) singular fiber has fewer than three special points and is thus unstable.

5 Universal unfoldings

In this section we formulate the most important definitions and theorems of this paper. The key definition is that of a universal unfolding. Once we have established the existence of universal unfoldings, the definition of the orbifold structure on the Deligne–Mumford moduli space (which we carry out in the next section) becomes almost tautological. The most important theorem asserts that an unfolding is universal if and only if it satisfies a suitable infinitesimal conditon (which is easier to verify).

Definition 5.1. A **nodal unfolding** is a triple (π_B, S_*, b) consisting of a marked nodal family $(\pi_B : Q \to B, S_*)$ and a point $b \in B$ of the base B. The fiber Q_b is called the **central fiber** of the unfolding and the unfolding is said to be an unfolding of the marked nodal Riemann surface induced by any desingularization of this central fiber. The unfolding is called **universal** iff for every other nodal unfolding $(\pi_A : P \to A, R_*, a)$ and any fiber isomorphism $f : P_a \to Q_b$ there is a unique germ of a morphism

$$(\Phi, \phi) : (\pi_A, a) \to (\pi_B, b)$$

such that $\Phi(R_i) \subset S_i$ for all i and $\Phi|P_a = f$. The term *germ* means that ϕ is defined in a neighborhood of a in A and Φ is defined on the preimage of this neighborhood under π_A. The term *unique* means that if (Φ', ϕ') is another morphism with the same properties then it agrees with (Φ, ϕ) over a sufficiently small neighborhood of a.

Definition 5.2. Let $(\pi : Q \to B, S_*, b)$ be an unfolding of a marked nodal Riemann surface (Σ, s_*, ν, j) and $u : \Sigma \to Q_b$ be a desingularization. Let $\mathcal{X}_{u,b}$ denote the space

$$\mathcal{X}_{u,b} := \left\{ (\hat{u}, \hat{b}) \in \Omega^0(\Sigma, u^*TQ) \times T_b B \,\middle|\, \begin{array}{l} d\pi(u)\hat{u} = \hat{b}, \quad \hat{u}(s_i) \in T_{u(s_i)} S_i, \text{ and} \\ u(z_1) = u(z_2) \implies \hat{u}(z_1) = \hat{u}(z_2). \end{array} \right\}$$

Let $\mathcal{Y}_u$ denote the space

$$\mathcal{Y}_u := \left\{ \eta \in \Omega^{0,1}(\Sigma, u^*TQ) : d\pi(u)\eta = 0 \right\}.$$

For $(\hat{u}, \hat{b}) \in \mathcal{X}_{u,b}$ define

$$D_{u,b}(\hat{u}, \hat{b}) := D_u \hat{u}$$

where $D_u : \Omega^0(\Sigma, u^*TQ) \to \Omega^{0,1}(\Sigma, u^*TQ)$ is the linearized Cauchy–Riemann operator. We call the unfolding (π, S_*, b) **infinitesimally universal** if the operator $D_{u,b} : \mathcal{X}_{u,b} \to \mathcal{Y}_u$ is bijective for some (and hence every) desingularization of the central fiber. Theorems 5.3, 5.5, and 5.6 which follow are proved in Section 12 below.

Theorem 5.3 (Stability)**.** *Let (π, S_*, b_0) be an infinitesimally universal unfolding. Then (π, S_*, b) is infinitesimally universal for b sufficiently near b_0.*

Theorem 5.4 (Universal Unfolding)**.** *An unfolding (π, S_*, b) is universal if and only if it is infinitesimally universal.*

Proof. We prove 'if' in Section 12. For 'only if' we argue as follows. A composition of morphisms (of nodal unfoldings) is again a morphism. The only morphism which is the identity on the central fiber of a universal unfolding is the identity. It follows that any two universal unfoldings of the same marked nodal Riemann surface are isomorphic. By Theorem 5.6 below there is an infinitesimally universal unfolding and by 'if' it is universal and hence isomorphic to every other universal unfolding. Any unfolding isomorphic to an infinitesimally universal unfolding is itself infinitesimally universal. □

Theorem 5.5 (Uniqueness)**.** *Let (π_B, S_*, b_0) be an infinitesimally universal unfolding. Then every pseudomorphism from (π_A, R_*, a_0) to (π_B, S_*, b_0) is a morphism.*

Theorem 5.6 (Existence)**.** *A marked nodal Riemann surface admits an infinitesimally universal unfolding if and only if it is stable.*

Proof. We prove 'if' in Section 12. For 'only if' we argue as follows. Let (Σ, s_*, ν, j) be a marked nodal Riemann surface. Assume it is not stable. Then either Σ has genus one and has no special points or else Σ contains a component of genus zero with at most two special points. In either case there is an abelian complex Lie group A (namely $A = \Sigma$ in the former case and $A = \mathbb{C}^*$ in the latter) and an effective holomorphic action $A \times \Sigma \to \Sigma : (a, z) \mapsto a_\Sigma(z)$. Let $P := A \times \Sigma$ and π_A be the projection on the first factor. If $v : \Sigma \to Q$ is any desingularization of a fiber Q_b of an unfolding $\pi_B : Q \to B$, then $\Phi_1(a, z) := v(z)$ and $\Phi_2(a, z) := v(a_\Sigma(z))$ are distinct morphisms which extend the fiber isomorphism $(e, z) \mapsto v(z)$. Hence π_B is not universal. □

6 Universal families and the Deligne–Mumford moduli space

In this section we define the orbifold structure on the Deligne–Mumford moduli space. The proof of compactness will be relegated to Section 14. The results we prove in this section are easy consequences of Theorems 5.3 and 5.6.

6.1. Throughout this section g and n are nonnegative integers with $n > 2 - 2g$. Let $\bar{\mathcal{B}}_{g,n}$ denote the groupoid whose objects are stable marked nodal Riemann surfaces of type (g, n) and whose morphisms are isomorphisms of marked nodal Riemann surfaces. The **Deligne–Mumford moduli space** is the orbit space $\bar{\mathcal{M}}_{g,n}$ of this groupoid: a point of $\bar{\mathcal{M}}_{g,n}$ is an equivalence class[7] of objects of $\bar{\mathcal{B}}_{g,n}$ where two objects are equivalent if and only if they are isomorphic. We will introduce a canonical orbifold structure (see Definition 2.4) on this groupoid. The following definition is crucial.

Definition 6.2. A **universal marked nodal family** of type (g, n) is a marked nodal family $(\pi_B : Q \to B, S_*)$ satisfying the following conditions.

(1) (π_B, S_*, b) is a universal unfolding for every $b \in B$.

(2) Every stable marked nodal Riemann surface of type (g, n) is the domain of a desingularization of at least one fiber of π_B.

(3) B is second countable (but possibly disconnected).

Proposition 6.3. *For every pair (g, n) with $n > 2 - 2g$ there is a universal marked nodal family.*

Proof. By Theorems 5.6, 5.4, and 5.3, each stable marked nodal Riemann surface admits a universal unfolding satisfying (1) and (3). To construct a universal unfolding that also satisfies (2) we must cover $\bar{\mathcal{M}}_{g,n}$ by countably many such families. This is possible because $\bar{\mathcal{M}}_{g,n}$ is a union of finitely many strata, one for each stable signature, and each stratum is a separable topological space. □

Definition 6.4. Let $(\pi_B : Q \to B, S_*)$ be a universal marked nodal family. The **associated groupoid** is the tuple $(B, \Gamma, s, t, e, i, m)$, where Γ denotes the set of all triples (a, f, b) such that $a, b \in B$ and $f : Q_a \to Q_b$ is a fiber isomorphism, and the structure maps $s, t : \Gamma \to B$, $e : B \to \Gamma$, $i : \Gamma \to \Gamma$, and $m : \Gamma {}_s\times_t \Gamma \to \Gamma$ are defined by

$$s(a, f, b) := a, \qquad t(a, f, b) := b, \qquad e(a) := (a, \mathrm{id}, a),$$

$$i(a, f, b) := (b, f^{-1}, a), \qquad m((b, g, c), (a, f, b)) := (a, g \circ f, c).$$

The associated groupoid is equipped with a functor

$$B \to \bar{\mathcal{B}}_{g,n} : b \mapsto \Sigma_b$$

to the groupoid $\bar{\mathcal{B}}_{g,n}$ of 6.1. In other words, $\iota_b : \Sigma_b \to Q_b$ denotes the canonical desingularization defined in Remark 4.4. By definition the induced map

$$B/\Gamma \to \bar{\mathcal{M}}_{g,n} : [b]_B \mapsto [\Sigma_b]_{\bar{\mathcal{B}}_{g,n}}, \qquad [b]_B := \{t(f) : f \in \Gamma,\ s(f) = b\},$$

[7]Strictly speaking, the equivalence class is a proper class in the sense of set theory as explained in the appendix of [12] for example. One could avoid this problem by choosing for each stable signature (see Remark 3.5 and 3.8) a "standard marked nodal surface" with that signature and restricting the space of objects of the groupoid $\bar{\mathcal{B}}_{g,n}$ to those having a standard surface as substrate.

on orbit spaces is bijective. The next theorem asserts that the groupoid (B, Γ) equips the moduli space $\bar{\mathcal{M}}_{g,n}$ with an orbifold structure which is independent of the choice of the universal family. This is the **orbifold structure** on the Deligne–Mumford moduli space.

Theorem 6.5. **(i)** *Let $(\pi_B : Q \to B, S_*)$ be universal as in Definition 6.2 and (B, Γ) be the associated groupoid of Definition 6.4. Then there is a unique complex manifold structure on Γ such that (B, Γ) is a complex etale Lie groupoid with structure maps s, t, e, i, m.*

(ii) *A morphism between universal families $\pi_0 : Q_0 \to B_0$ and $\pi_1 : Q_1 \to B_1$ induces a refinement $\iota : (B_0, \Gamma_0) \to (B_1, \Gamma_1)$ of the associated etale groupoids.*

(iii) *The orbifold structure on $\bar{\mathcal{M}}_{g,n}$ introduced in Definition 6.4 is independent of the choice of the universal marked nodal family (π_B, S_*) used to define it.*

Proof. We prove (i). Uniqueness is immediate since part of the definition of complex etale Lie groupoid is that s is a local holomorphic diffeomorphism. We prove existence. It follows from the definition of universal unfolding that each triple $(a_0, f_0, b_0) \in \Gamma$ determines a morphism

$$\begin{array}{ccc} Q|U & \xrightarrow{\Phi} & Q|V \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\phi} & V \end{array}$$

for suitable neighborhoods $U \subset B$ of a_0 and $V \subset B$ of b_0 such that $\Phi|Q_{a_0} = f_0$. Every such morphism determines a chart $\iota_\Phi : U \to \Gamma$ given by

$$\iota_\Phi(a) := (a, \Phi_a, \phi(a)).$$

(In this context a chart is a bijection between an open set in a complex manifold and a subset of Γ.) By construction each transition map between two such charts is the identity. This defines the manifold structure on Γ. That the structure maps are holomorphic follows from the identities

$$s \circ \iota_\Phi = \mathrm{id}, \qquad t \circ \iota_\Phi = \phi, \qquad e = \iota_{\mathrm{id}},$$

$$i \circ \iota_\Phi = \iota_{\Phi^{-1}} \circ \phi, \qquad m \circ (\iota_\Psi \circ \phi \times \iota_\Phi) = \iota_{\Psi \circ \Phi}.$$

This proves (i).

We prove (ii). If (ϕ, Φ) is a morphism from π_0 to π_1 then the refinement $\iota : (B_0, \Gamma_0) \to (B_1, \Gamma_1)$ of (ii) is given by

$$(a_0, f_0, b_0) \mapsto (\phi(a_0), \Phi_{b_0} \circ f_0 \circ \Phi_{a_0}^{-1}, \phi(b_0)).$$

This proves (ii).

We prove (iii). Let $\pi_0 : Q_0 \to B_0$ and $\pi_1 : Q_1 \to B_1$ be universal families. For each $b \in B_0$ choose a neighborhood $U_b \subset B_0$ of b and a morphism $\Phi_b : Q_0|U_b \to Q_1$. Cover B_0 by countably many such neighborhoods U_{b_i}. Then the disjoint union B of the nodal families $Q_0|U_{b_i}$ defines another universal family $\pi : Q \to B$ equiped with morphisms to both π_0 and π_1 (to π_0 by inclusion and to π_1 by construction). Now each morphism of universal families induces a refinement of the corresponding orbifold structures. □

Theorem 6.6. *Let $(\pi_B : Q \to B, S_*)$ be a universal family. Then the etale groupoid (B, Γ) constructed in Definition 6.4 is proper and the quotient topology on B/Γ is compact.*

Proof. See Section 14 below. □

Example 6.7. Assume $g = 0$. Then the moduli space $\bar{\mathcal{M}}_{0,n}$ of marked nodal Riemann surfaces of genus zero (called the **Grothendieck–Knudsen compactification**) is a compact connected complex manifold (Knudsen's theorem). In our formulation this follows from the fact that the automorphism group of each marked nodal Riemann surface of genus zero consists only of the identity. In [16, Appendix D] the complex manifold structure on $\bar{\mathcal{M}}_{0,n}$ is obtained from an embedding into a product of 2-spheres via cross ratios. That the manifold structure in [16] agrees with ours follows from the fact that the projection $\pi : \bar{\mathcal{M}}_{0,n+1} \to \bar{\mathcal{M}}_{0,n}$ (with the complex manifold structures of [16]) is a universal family as in Definition 6.2.

7 Complex structures on surfaces

The sphere

In preparation for the construction of universal unfoldings (without nodes and marked points) we review the space of complex structures on a Riemann surface Σ in this and the following two sections. This section treats the case of genus zero. Denote by $\mathcal{J}(S^2)$ the space of complex structures on S^2 that induce the standard orientation and by $\mathrm{Diff}_0(S^2)$ the group of orientation preserving diffeomorphisms of S^2.

Theorem 7.1. *There is a fibration*

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{C}) & \to & \mathrm{Diff}_0(S^2) \\ & & \downarrow \\ & & \mathcal{J}(S^2) \end{array}$$

where the inclusion $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Diff}_0(S^2)$ is the action by Möbius transformations and the projection $\mathrm{Diff}_0(S^2) \to \mathcal{J}(S^2)$ sends ϕ to $\phi^* i$.

The theorem asserts that the map $\mathrm{Diff}_0(S^2) \to \mathcal{J}(S^2)$ has the path lifting property for smooth paths and that the lifting depends smoothly on the path. One consequence of this, as observed in [4], is the celebrated theorem of Smale [17] which asserts that $\mathrm{Diff}_0(S^2)$ retracts onto SO(3). Another consequence is that, up to diffeomorphism, there is a unique complex structure on the 2-sphere. Yet another consequence is that a proper holomorphic submersion whose fibers have genus zero is holomorphically locally trivial. (See Theorem 8.9.)

Proof of Theorem 7.1. Choose a smooth path $[0,1] \to \mathcal{J}(S^2) : t \mapsto j_t$. We will find an isotopy $t \mapsto \psi_t$ of S^2 such that

$$\psi_t{}^* j_t = j_0. \tag{2}$$

Suppose that the unknown isotopy ψ_t is generated by a smooth family of vector fields $\xi_t \in \mathrm{Vect}(S^2)$ via

$$\frac{d}{dt}\psi_t = \xi_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

Then (2) is equivalent to $\psi_t^*(\mathcal{L}_{\xi_t} j_t + \hat{j}_t) = 0$ and hence to

$$\mathcal{L}_{\xi_t} j_t + \hat{j}_t = 0, \tag{3}$$

where $\hat{j}_t := \frac{d}{dt} j_t \in C^\infty(\mathrm{End}(TS^2))$. As usual we can think of $\hat{j}_t$ as a $(0,1)$-form on S^2 with values in the complex line bundle

$$E_t := (TS^2, j_t).$$

The vector field ξ_t is a section of this line bundle. This line bundle is holomorphic and its Cauchy-Riemann operator

$$\bar{\partial}_{j_t} : C^\infty(E_t) \to \Omega^{0,1}(E_t)$$

has the form

$$\bar{\partial}_{j_t}\eta = \frac{1}{2}\left(\nabla\eta + j_t \circ \nabla\eta \circ j_t\right)$$

where ∇ is the Levi-Civita connection of the Riemannian metric $\omega(\cdot, j_t\cdot)$ on S^2 and $\omega \in \Omega^2(S^2)$ denotes the standard volume form. Now, for every vector field $\eta \in \mathrm{Vect}(S^2)$, we have

$$\begin{aligned}
(\mathcal{L}_{\xi_t} j_t)\eta &= \mathcal{L}_{\xi_t}(j_t\eta) - j_t\mathcal{L}_{\xi_t}\eta \\
&= [j_t\eta, \xi_t] - j_t[\eta, \xi_t] \\
&= \nabla_{\xi_t}(j_t\eta) - \nabla_{j_t\eta}\xi_t - j_t\nabla_{\xi_t}\eta + j_t\nabla_\eta\xi_t \\
&= j_t\nabla_\eta\xi_t - \nabla_{j_t\eta}\xi_t \\
&= 2j_t(\bar{\partial}_{j_t}\xi_t)(\eta).
\end{aligned}$$

The penultimate equality uses the fact that j_t is integrable and so $\nabla j_t = 0$. Hence equation (3) can be expressed in the form

$$\bar{\partial}_{j_t}\xi_t = \frac{1}{2} j_t \hat{j}_t. \tag{4}$$

Now the line bundle E_t has Chern number $c_1(E_t) = 2$ and hence, by the Riemann-Roch theorem, the Cauchy-Riemann operator $\bar{\partial}_{j_t}$ has real Fredholm index six and is surjective for every t. Denote by

$$\bar{\partial}_{j_t}^* : \Omega^{0,1}(E_t) \to C^\infty(E_t)$$

the formal L^2-adjoint operator of $\bar{\partial}_{j_t}$. By elliptic regularity, the formula

$$\xi_t := \frac{1}{2}\bar{\partial}^*_{\xi_t}\left(\bar{\partial}_{j_t}\bar{\partial}^*_{j_t}\right)^{-1}\left(j_t\hat{j}_t\right)$$

defines a smooth family of vector fields on S^2 and this family obviously satisfies (4). Hence the isotopy ψ_t generated by ξ_t satisfies (2). □

Lemma 7.2. *Let $\mathbb{C} \to \mathcal{J}(S^2) : s+it \mapsto j_{s,t}$ be holomorphic and $\mathbb{C} \to \mathrm{Diff}(S^2) : s+it \mapsto \phi_{s,t}$ be the unique family of diffeomorphisms satisfying*

$$\phi^*_{s,t}j_{s,t} = i, \qquad \phi_{s,t}(0) = 0, \qquad \phi_{s,t}(1) = 1, \qquad \phi_{s,t}(\infty) = \infty.$$

Then the map

$$\mathbb{C} \times S^2 \to \mathbb{C} \times S^2 : (s+it, z) \mapsto (s+it, \phi_{s,t}(z))$$

is holomorphic with respect to the standard complex structure at the source and the complex structure

$$J(s,t,z) := \begin{pmatrix} i & 0 \\ 0 & j_{s,t}(z) \end{pmatrix}$$

at the target.

Proof. Define $\xi_{s,t}, \eta_{s,t} \in \mathrm{Vect}(S^2)$ by

$$\partial_s\phi_{s,t} = \xi_{s,t} \circ \phi_{s,t}, \qquad \partial_y\phi_{s,t} = \eta_{s,t} \circ \phi_{s,t}.$$

Differentiating the identity $\phi^*_{s,t}j_{s,t} = i$ gives $\partial_s j + \mathcal{L}_\xi j = \partial_t j + \mathcal{L}_\eta j = 0$. Since $s+it \mapsto j_{s,t}$ is holomorphic we have

$$0 = \partial_s j + j\partial_t j = -\mathcal{L}_\xi j - j\mathcal{L}_\eta j = -\mathcal{L}_{\xi+j\eta} j$$

where the last equality uses the integrability of j. Thus $\xi_{s,t} + j_{s,t}\eta_{s,t}$ is a holomorphic vector field vanishing at three points so $\xi_{s,t} + j_{s,t}\eta_{s,t} = 0$ for all s,t. Hence by definition of ξ and η we have

$$\partial_s\phi + j\partial_t\phi = 0$$

as required. □

The torus

Continuing the preparatory discussion of the previous section we treat the case of genus one. Denote by $\mathcal{J}(\mathbb{T}^2)$ the space of complex structures on the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ that induce the standard orientation and by $\mathrm{Diff}_0(\mathbb{T}^2)$ the group of diffeomorphisms of $\mathbb{T}^2$ that induce the identity on homology. Denote the elements of the upper half plane $\mathbb{H}$ by $\lambda = \lambda_1 + i\lambda_2$ and consider the map $j : \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$, given by

$$j(\lambda) := \frac{1}{\lambda_2} \begin{pmatrix} -\lambda_1 & -\lambda_1^2 - \lambda_2^2 \\ 1 & \lambda_1 \end{pmatrix}. \tag{5}$$

Thus $j(\lambda)$ is the pullback of the standard complex structure under the diffeomorphism

$$f_\lambda : \mathbb{T}^2 \to \frac{\mathbb{C}}{\mathbb{Z} + \lambda\mathbb{Z}}, \qquad f_\lambda(x, y) := x + \lambda y.$$

A straight forward calculation shows that the map $j : \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$ is holomorphic as is the map

$$(\lambda, z + \mathbb{Z} + \lambda\mathbb{Z}) \mapsto (j(\lambda), f_\lambda^{-1}(z) + \mathbb{Z}^2)$$

from $\{(\lambda, z + \mathbb{Z} + \lambda\mathbb{Z}) : \lambda \in \mathbb{H},\, z \in \mathbb{C}\}$ to $\mathcal{J}(\mathbb{T}^2) \times \mathbb{T}^2$. The next theorem shows that the map $j : \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$ is a global slice for the action of $\mathrm{Diff}_0(\mathbb{T}^2)$.

Theorem 7.3. *There is a proper fibration*

$$\begin{array}{ccc} \mathbb{T}^2 & \to & \mathrm{Diff}_0(\mathbb{T}^2) \times \mathbb{H} \\ & & \downarrow \\ & & \mathcal{J}(\mathbb{T}^2) \end{array}$$

where the inclusion $\mathbb{T}^2 \to \mathrm{Diff}_0(\mathbb{T}^2)$ *is the action by translations and the projection* $\mathrm{Diff}_0(\mathbb{T}^2) \times \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$ *sends* (ϕ, λ) *to* $\phi^* j_\lambda$.

The theorem asserts that the map $\mathrm{Diff}_0(\mathbb{T}^2) \times \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$ has the path lifting property for smooth paths and that the lifting depends smoothly on the path. One consequence of this is that $\mathrm{Diff}_0(\mathbb{T}^2)$ retracts onto $\mathbb{T}^2$. Another consequence is that every complex structure on $\mathbb{T}^2$ is diffeomorphic to j_λ for some $\lambda \in \mathbb{H}$.

Proof of Theorem 7.3. The uniformization theorem asserts that for every $j \in \mathcal{J}(\mathbb{T}^2)$ there is a unique volume form $\omega_j \in \Omega^2(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} \omega_j = 1$ such that the metric $g_j = \omega_j(\cdot, j\cdot)$ has constant curvature zero. (A proof can be based on the Kazdan–Warner equation.) Hence it follows from the Cartan–Ambrose–Hicks theorem that, for every positive real number μ, there is an orientation preserving diffeomorphism $\psi_j : \mathbb{C} \to \mathbb{R}^2$, unique up to composition with a rotation, such that

$$\psi_j^* g_j = \mu g_0, \qquad \psi(0) = 0.$$

Here g_0 denotes the standard metric on $\mathbb{C}$. We can choose μ and the rotation such that $\psi_j(1) = (1, 0)$. This determines ψ_j (and μ) uniquely. The orientation preserving condition shows that $\lambda_j := \psi_j(i) \in \mathbb{H}$. Moreover, it follows from the invariance of g_j under the action of $\mathbb{Z}^2$ that

$$\psi_j(\mathbb{Z} + \lambda\mathbb{Z}) = \mathbb{Z}^2.$$

Hence ψ_j induces an isometry of flat tori $(\mathbb{C}/\mathbb{Z}+\lambda_j\mathbb{Z}, g_0) \to (\mathbb{T}^2, g_j)$ which will still be denoted by ψ_j. Let ϕ_j be the precomposition of this isometry with the map $\mathbb{T}^2 \to \mathbb{C}^2/\mathbb{Z}+\lambda_j\mathbb{Z} : (x,y) \mapsto x+\lambda_j y$. Then $\phi_j \in \mathrm{Diff}_0(\mathbb{T}^2)$ and $\phi_j^* j = j(\lambda_j)$. Thus we have proved that the map

$$\mathrm{Diff}_{00}(\mathbb{T}^2) \times \mathbb{H} \to \mathcal{J}(\mathbb{T}^2) : (\phi, \lambda) \mapsto \phi^* j(\lambda)$$

is a bijection, where $\mathrm{Diff}_{00}(\mathbb{T}^2)$ denotes the subgroup of all diffeomorphisms $\phi \in \mathrm{Diff}_0(\mathbb{T}^2)$ that satisfy $\phi(0) = 0$. That the map $\mathrm{Diff}_{00}(\mathbb{T}^2) \times \mathbb{H} \to \mathcal{J}(\mathbb{T}^2)$ is actually a diffeomorphism follows by examining the linearized operator at points (ϕ, λ) with $\phi = \mathrm{id}$ and noting that it is a bijection (between suitable Sobolev completions). This proves the theorem. □

Surfaces of higher genus

Continuing the preparatory discussion of the previous two sections we treat the case of genus bigger than one. Let Σ be a compact connected oriented 2-manifold of genus $g > 1$ and $\mathcal{J}(\Sigma)$ be the Frechét manifold of complex structures j on Σ, i.e. j is an automorphism of $T\Sigma$ such that $j^2 = -\mathbb{1}$. The identity component $\mathrm{Diff}_0(\Sigma)$ of the group of orientation preserving diffeomorphisms acts on $\mathcal{J}(\Sigma)$ by $j \mapsto \phi^* j$. The orbit space

$$\mathcal{T}(\Sigma) := \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma)$$

is called the **Teichmüller space** of Σ. For $j \in \mathcal{J}(\Sigma)$ the tangent space $T_j\mathcal{J}(\Sigma)$ is the space the space of endomorphisms $\hat{j} \in \Omega^0(\Sigma, \mathrm{End}(T\Sigma))$ that anti-commute with j, i.e. $j\hat{j} + \hat{j}j = 0$. Thus

$$T_j\mathcal{J}(\Sigma) = \Omega^{0,1}_j(\Sigma, T\Sigma).$$

Define an almost complex structure on $\mathcal{J}(\Sigma)$ by the formula $\hat{j} \mapsto j\hat{j}$. The next theorem shows that $\mathcal{T}(\Sigma)$ is a complex manifold of dimension $3g-3$.

Theorem 7.4. *For every $j_0 \in \mathcal{J}(\Sigma)$ there exists a holomorphic local slice through j_0. More precisely, there is an open neighborhood B of zero in $\mathbb{C}^{3g-3}$ and a holomorphic map $\iota : B \to \mathcal{J}(\Sigma)$ such that the map*

$$B \times \mathrm{Diff}_0(\Sigma) \to \mathcal{J}(\Sigma) : (b, \phi) \mapsto \phi^* \iota(b)$$

is a diffeomorphism onto a neighborhood of the orbit of j_0.

Proof. We first show that each orbit of the action of $\mathrm{Diff}_0(\Sigma)$ is an almost complex submanifold of $\mathcal{J}(\Sigma)$. (The complex structure on $\mathcal{J}(\Sigma)$ is integrable because $\mathcal{J}(\Sigma)$ is the space of sections of a bundle over Σ whose fibers are complex manifolds. However, we shall not use this fact.) The Lie algebra of $\mathrm{Diff}_0(\Sigma)$ is the space of vector fields

$$\mathrm{Vect}(\Sigma) = \Omega^0(\Sigma, T\Sigma).$$

Its infinitesimal action on $\mathcal{J}(\Sigma)$ is given by

$$\mathrm{Vect}(\Sigma) \to T_j\mathcal{J}(\Sigma) : \xi \mapsto \mathcal{L}_\xi j = 2j\bar{\partial}_j\xi.$$

Thus the tangent space of the orbit of j is the image of the Cauchy–Riemann operator $\bar{\partial}_j : \Omega^0(\Sigma, T\Sigma) \to \Omega^{0,1}_j(\Sigma, T\Sigma)$. Since j is integrable the operator $\bar{\partial}_j$ is complex linear and so its image is invariant under multiplication by j.

By the Riemann–Roch theorem the operator $\bar{\partial}_j$ has complex Fredholm index $3-3g$. It is injective because its kernel is the space of holomorphic sections of a holomorphic line bundle of negative degree. Hence its cokernel has dimension $3g-3$. Let $B \subset \Omega^{0,1}_{j_0}(\Sigma, T\Sigma)$ be an open neighborhood of zero in a complex subspace of dimension $3g-3$ which is a complement of the image of $\bar{\partial}_{j_0}$ and assume that $\mathbb{1}+\eta$ is invertible for every $\eta \in B$. Define $\iota : B \to \mathcal{J}(\Sigma)$ by

$$\iota(\eta) := (\mathbb{1}+\eta)^{-1} j_0 (\mathbb{1}+\eta).$$

Then

$$d\iota(\eta)\hat{\eta} = \left[\iota(\eta), (\mathbb{1}+\eta)^{-1}\hat{\eta}\right]$$

and an easy calculation shows that ι is holomorphic, i.e. $d\iota(\eta)j_0\hat{\eta} = \iota(\eta)d\iota(\eta)\hat{\eta}$ for all η and $\hat{\eta}$.

Let $p > 2$ and denote by $\mathrm{Diff}_0^{2,p}(\Sigma)$ and $\mathcal{J}^{1,p}(\Sigma)$ the appropriate Sobolev completions. Consider the map

$$\mathrm{Diff}_0^{2,p}(\Sigma) \times B \to \mathcal{J}^{1,p}(\Sigma) : (\phi, \eta) \mapsto \phi^*\iota(\eta).$$

This is a smooth map between Banach manifolds and, by construction, its differential at $(\mathrm{id}, 0)$ is bijective. Hence, by the inverse function theorem, it restricts to a diffeomorphism from an open neighborhood of $(\mathrm{id}, 0)$ in $\mathrm{Diff}_0^{2,p}(\Sigma) \times B$ to an open neighborhood of j_0 in $\mathcal{J}^{1,p}(\Sigma)$. The restriction of this diffeomorphism to the space of smooth pairs in $\mathrm{Diff}_0(\Sigma) \times B$ is a diffeomorphism onto an open neighborhood of j_0 in $\mathcal{J}(\Sigma)$. To see this, note that every element of $\mathrm{Diff}_0^{2,p}(\Sigma)$ is a C^1-diffeomorphism and that every C^1-diffeomorphism of Σ that intertwines two smooth complex structures is necessarily smooth. Shrink B so that $\{\mathrm{id}\} \times B$ is a subset of the neighborhood just constructed. The action of $\mathrm{Diff}_0(\Sigma)$ on $\mathcal{J}(\Sigma)$ is free and Lemma 7.5 below asserts that it is proper. Hence, by a standard argument, we may shrink B further so that the local diffeomorphism

$$\mathrm{Diff}_0 \times B \to \mathcal{J}(\Sigma) : (\phi, \eta) \mapsto \phi^*\iota(\eta)$$

is injective; it is the required diffeomorphism onto an open neighborhood of the orbit of j_0. □

Lemma 7.5. *Let Σ be a surface and $j_k, j_k' \in \mathcal{J}(\Sigma)$ and $\phi_k \in \mathrm{Diff}(\Sigma)$ be sequences such that j_k' converges to $j' \in \mathcal{J}(\Sigma)$ and $j_k = \phi_k^* j_k'$ converges to $j \in \mathcal{J}(\Sigma)$. Then ϕ_k has a subsequence which converges in* $\mathrm{Diff}(\Sigma)$.

Proof. Fix an embedded closed disk $D \subset \Sigma$ and two points $z_0 \in \mathrm{int}(D)$, $z_1 \in \partial D$. Let $\mathbb{D} \subset \mathbb{C}$ denote the closed unit disk. By the Riemann mapping theorem, there is a unique diffeomorphism $u_k : \mathbb{D} \to D$ such that

$$u_k^* j_k = i, \qquad u_k(0) = z_0, \qquad u_k(1) = z_1.$$

The standard bubbling and elliptic bootstrapping arguments for J-holomorphic curves (see [16, Appendix B]) show that u_k converges in the C^∞-topology. The same arguments show that the sequence $u_k' := \phi_k \circ u_k$ of j_k'-holomorphic disks has a subsequence which converges on every compact subset of the interior of $\mathbb{D}$. Thus we have proved that the restriction of ϕ_k to any embedded disk in Σ has a convergent subsequence. Hence ϕ_k has a convergent subsequence. The limit ϕ satisfies $\phi^* j' = j$ and has degree one. Hence ϕ is a diffeomorphism. □

8 Teichmüller space

The space $\mathcal{T}_g$

In this section we prove Theorems 5.3-5.6 for $g > 1$ in the case of surfaces without nodes or marked points.

8.1. Let A be a complex manifold and Σ be a surface. We denote the complex structure on A by $\sqrt{-1}$. An almost complex structure on $A \times \Sigma$ with respect to which the projection $A \times \Sigma \to A$ is holomorphic has the form

$$J = \begin{pmatrix} \sqrt{-1} & 0 \\ \alpha & j \end{pmatrix}, \tag{6}$$

where $j : A \to \mathcal{J}(\Sigma)$ is a smooth function with values in the space of (almost) complex structures on Σ and $\alpha \in \Omega^1(A, \mathrm{Vect}(\Sigma))$ is a smooth 1-form on A with values in the space of vector fields on Σ such that

$$\alpha(a, \sqrt{-1}\hat{a}) + j(a)\alpha(a, \hat{a}) = 0 \tag{7}$$

for all $a \in A$ and $\hat{a} \in T_a A$. This means that the 1-form α is complex anti-linear with respect to the complex structure on the vector bundle $A \times \mathrm{Vect}(\Sigma) \to A$ determined by j. From an abstract point of view it is useful to think of α as a connection on the (trivial) principal bundle $A \times \mathrm{Diff}(\Sigma)$ and of $j : A \to \mathcal{J}(\Sigma)$ as a section of the associated fiber bundle $A \times \mathcal{J}(\Sigma)$. This section is holomorphic with respect to the Cauchy–Riemann operator associated to the connection α if and only if

$$dj(a)\hat{a} + j(a)dj(a)\sqrt{-1}\hat{a} + j(a)\mathcal{L}_{\alpha(a,\hat{a})}j(a) = 0 \tag{8}$$

for all $a \in A$ and $\hat{a} \in T_a A$. (For a finite dimensional analogue see for example [2].)

Lemma 8.2. *J is integrable if and only if j and α satisfy (8).*

Proof. It suffices to consider the case $m = 1$, so $A \subset \mathbb{C}$ with coordinate $s + it$. Then the complex structure J on $A \times \Sigma$ has the form

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -j\xi & -\xi & j \end{pmatrix}, \tag{9}$$

where $A \to \mathcal{J}(\Sigma) : s + it \mapsto j_{s,t}$ and $A \to \mathrm{Vect}(\Sigma) : s + it \mapsto \xi_{s,t}$ are smooth maps. The equation (8) has the form

$$\partial_s j + j\partial_t j + \mathcal{L}_\xi j = 0. \tag{10}$$

To see that this is equivalent to integrability of J evaluate the Nijenhuis tensor $N_J(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$ on a pair of vectors of the form $X = (1, 0, 0)$, $Y = (0, 0, \hat{z})$. The condition $N_J(X, Y) = 0$ for all such vectors is equivalent to (10) and it is easy to see that $N_J = 0$ if and only if $N_J((1, 0, 0), (0, 0, \hat{z})) = 0$ for all $\hat{z} \in T\Sigma$. The latter assertion uses the facts that N_J is bilinear, $N_J(X, Y) = -N_J(Y, X) = JN_J(JX, Y)$, and every complex structure on a 2-manifold is integrable. This proves the lemma. □

Let A be a complex manifold and $\iota : A \to \mathcal{J}(\Sigma)$ be a holomorphic map. Consider the fibration

$$\pi_\iota : P_\iota := A \times \Sigma \to A$$

with almost complex structure

$$J_\iota(a, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \iota(a)(z) \end{pmatrix}. \tag{11}$$

By Lemma 8.2 the almost complex structure J_ι on P_ι is integrable.

Lemma 8.3. *Let $a \in A$. Then the pair (π_ι, a) is an infinitesimally universal unfolding if and only if the restriction of ι to a sufficiently small neighborhood of a is a local slice as in Theorem 7.4.*

Proof. Let $u : \Sigma \to P_a$ be the diffeomorphism $u(z) := (a, z)$ and denote $j := \iota(a)$. Then the linearized operator $D_{u,a}$ (at the pair (u, a) for the equation $\bar{\partial}_j u = 0$ with $j = \iota(a)$) has domain $\mathcal{X}_{u,a} = \Omega^0(\Sigma, T\Sigma) \times T_a A$, target space $\mathcal{Y}_u = \Omega^{0,1}_j(\Sigma, T\Sigma)$ and is given by

$$D_{u,a}(\hat{u}, \hat{a}) = \bar{\partial}_j \hat{u} - \frac{1}{2} j d\iota(a)\hat{a}.$$

(See the formula in [16, page 176] with $v = \mathrm{id}$.) This operator is bijective if and only if $d\iota(a)$ is injective and its image in $T_j\mathcal{J} = \Omega^{0,1}_j(\Sigma, T\Sigma)$ is a complement of $\mathrm{im}\, \bar{\partial}_j = T_j(\mathrm{Diff}_0(\Sigma)^* j)$ (see the proof of Theorem 7.4). This proves the lemma. □

Theorem 8.4. *Theorems 5.3-5.6 hold for Riemann surfaces of genus $g > 1$ without nodes and marked points.*

Proof. Let Σ be a surface of genus g. Abbreviate

$$\mathcal{D}_0 := \mathrm{Diff}_0(\Sigma), \qquad \mathcal{J} := \mathcal{J}(\Sigma), \qquad \mathcal{T} := \mathcal{T}(\Sigma) := \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

Thus $\mathcal{T}_g := \mathcal{T}$ is Teichmüller space. Consider the principal fiber bundle

$$\mathcal{D}_0 \to \mathcal{J} \to \mathcal{T}.$$

The associated fiber bundle

$$\pi_{\mathcal{T}} : \mathcal{Q} := \mathcal{J} \times_{\mathcal{D}_0} \Sigma \to \mathcal{T}$$

has fibers isomorphic to Σ.

Step 1. *$\mathcal{Q}$ and $\mathcal{T}$ are complex manifolds and $\pi_{\mathcal{T}}$ is a proper holomorphic submersion.*

By Lemma 8.2 with $A = \mathcal{J}$ and the map $A \to \mathcal{J}$ equal to the identity, the space $\mathcal{J} \times \Sigma$ is a complex manifold. Since $\mathcal{D}_0$ acts by holomorphic diffeomorphisms, so is the (finite dimensional) quotient $\mathcal{Q}$.

Step 2. *The projection $\pi_{\mathcal{T}}$ is an infinitesimally universal unfolding of each of its fibers.*

Choose $[j_0] \in \mathcal{T}$. Let B be an open neighborhood of 0 in $\mathbb{C}^{3g-3}$ and $\iota : B \to \mathcal{J}$ be a local holomorphic slice such that $\iota(0) = j_0$ (see [20] or Section 7). Then the projection $Q_\iota \to B$ is a local coordinate chart on $\mathcal{Q} \to \mathcal{T}$. Hence Step 2 follows from Lemma 8.3.

Step 3. *Every pseudomorphism from (π_A, a_0) to $(\pi_{\mathcal{T}}, [j_0])$ is a morphism.*

Let (ϕ, Φ) be a pseudomorphism from $(\pi_A : P \to A, a_0)$ to $(\pi_{\mathcal{T}}, [j_0])$ and $\iota : B \to \mathcal{J}$ be as the proof of in Step 2. Define (ψ, Ψ) to be the composition of (ϕ, Φ) with the obvious morphism from $(\pi_{\mathcal{T}}, [j_0])$ to $(Q_\iota, 0)$. Using the maps $\Psi_a : P_a \to \Sigma$ given by $\Psi(p) =: (\psi(a), \Psi_a(p))$ for $p \in P_a$ we construct a trivialization

$$\tau : A \times \Sigma \to P, \qquad \tau(a, z) := \tau_a(z) := \Psi_a^{-1}(z).$$

Then the pullback of the complex structure on P under τ has the form

$$J(a, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ \alpha & j(a)(z) \end{pmatrix}$$

where $j := \iota \circ \psi : A \to \mathcal{J}$ and $\alpha \in \Omega_j^{0,1}(A, \mathrm{Vect}(\Sigma))$ Since J is integrable it follows from Lemma 8.2 that j and α satisfy (8). Since the local slice is holomorphic the term $dj(a)\hat{a} + j(a)dj(a)\sqrt{-1}\hat{a}$ is tangent to the slice while the last summand $j(a)\mathcal{L}_{\alpha(a,\hat{a})}j(a) = -\mathcal{L}_{\alpha(a,\sqrt{-1}\hat{a})}j(a)$ is tangent to the orbit of $j(a)$ under $\mathcal{D}_0$. It follows that both terms vanish for all $a \in A$ and $\hat{a} \in T_aA$. Hence $\alpha = 0$ and the map $j : A \to \mathcal{J}$ is holomorphic. Hence $\psi : A \to B$ is holomorphic and hence so is Ψ.

Step 4. *$\pi_{\mathcal{T}}$ is a universal unfolding of each of its fibers.*

Choose an unfolding $(\pi_A : P \to A, a_0)$ and a holomorphic diffeomorphism $u_0 : (\Sigma, j_0) \to P_{a_0}$. Then u_0^{-1} is a fiber isomorphism from P_{a_0} to $\mathcal{Q}_{[j_0]}$. Trivialize P by a map $\tau : A \times \Sigma \to P$ such that $\tau_{a_0} = u_0$. Define $j : A \to \mathcal{J}$ so that $j(a)$ is the pullback of the complex structure on P_a under τ_a. Then $j(a_0) = j_0$. Define $\phi : A \to \mathcal{T}$ and $\Phi : P \to \mathcal{Q}$ by

$$\phi(a) := [j(a)], \qquad \Phi(p) := [j(a), z], \qquad p =: \tau(a, z)$$

for $a \in A$ and $p \in P_a$. This is a pseudomorphism and hence, by Step 3, it is a morphism.

To prove uniqueness, choose a local holomorphic slice $\iota : B \to \mathcal{J}$ such that $\iota(0) = j_0$. Choose two morphisms $(\psi, \Psi), (\phi, \Phi) : (\pi_A, a_0) \to (\pi_\iota, 0)$ such that $\Phi_{a_0} = \Psi_{a_0} = u_0^{-1} : P_{a_0} \to \Sigma$. If a is near a_0 then

$$\Psi_a \circ \Phi_a^{-1} : (\Sigma, \iota(\phi(a))) \to (\Sigma, \iota(\psi(a)))$$

is a diffeomorphism close to the identity and hence isotopic to the identity. Hence by the local slice property $\phi(a) = \psi(a)$ and $\Psi_a \circ \Phi_a^{-1} = \mathrm{id}$.

Step 5. *Let j_0 be a complex structure on Σ. Every infinitesimally universal unfolding $(\pi_B : Q \to B, b_0)$ of (Σ_0, j_0) is isomorphic to $(\pi_{\mathcal{T}}, [j_0])$.*

As in Step 3 we may assume that $Q = B \times \Sigma$ with complex structure

$$J(b, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \iota(b)(z) \end{pmatrix}$$

where $\iota : B \to \mathcal{J}$ is holomorphic and $\iota(b_0) = j_0$. By Lemma 8.2 this almost complex structure is integrable. Since $(\pi_B : Q \to B, b_0)$ is infinitesimally universal, it follows from Lemma 8.3 that the restriction of ι to a neighborhood of b_0 is a local slice. Hence (π_B, b_0) is isomorphic to $(\pi_{\mathcal{T}}, [j_0])$ by the local slice property. □

Remark 8.5. The universal unfolding $\pi_{\mathcal{T}} : \mathcal{Q} \to \mathcal{T}$ of Theorem 8.4 determines an etale groupoid (B, Γ) with $B := \mathcal{T} = \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ and

$$\Gamma := \{[j, \phi, j'] : j, j' \in \mathcal{J}(\Sigma),\ \phi \in \mathrm{Diff}(\Sigma),\ j = \phi^* j'\}.$$

Here $[j, \phi, j']$ denotes the equivalence class under the diagonal action of $\mathrm{Diff}_0(\Sigma)$ by $\psi^*(j, \phi, j') := (\psi^* j, \psi^{-1} \circ \phi \circ \psi, \psi^* j')$. By Lemma 7.5 this etale groupoid is proper.

The space $\mathcal{T}_{g,n}$

In this section we prove Theorems 5.3-5.6 for all stable marked Riemann surfaces without nodes. Let (Σ, s_*, j_0) be a stable marked Riemann surface of type (g, n) without nodes. We will construct an infinitesimally universal unfolding (π_B, S_*, b_0) of (Σ, j_0, s_*), prove that it is universal, and prove that every infinitesimally universal unfolding of (Σ, s_*, j_0) is isomorphic to the one we've constructed.

8.6. Let n and g be nonnegative integers such that $n > 2 - 2g$ and let Σ be a surface of genus g. Abbreviate

$$\mathcal{G} := \mathrm{Diff}_0(\Sigma), \qquad \mathcal{P} := \mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta), \qquad \mathcal{B} := \mathcal{P}/\mathcal{G},$$

where $\Delta \subset \Sigma^n$ denotes the fat diagonal, i.e. set of all n-tuples of points in Σ^n where at least two components are equal. Thus $\mathcal{B} = \mathcal{T}_{g,n}$ is the Teichmüller space of Riemann surfaces of genus g with n distinct marked points. Consider the principal fiber bundle

$$\mathcal{G} \to \mathcal{P} \to \mathcal{B}.$$

The associated fiber bundle

$$\pi_{\mathcal{B}} : \mathcal{Q} := \mathcal{P} \times_{\mathcal{G}} \Sigma \to \mathcal{B}$$

has fibers isomorphic to Σ and is equipped with n disjoint sections

$$\mathcal{S}_i := \{[j, s_1, \ldots, s_n, z] \in \mathcal{Q} \,:\, z = s_i\}, \qquad i = 1, \ldots, n.$$

It is commonly called the **universal curve** of genus g with n marked points.

8.7. Let $(j_0, r_*) \in \mathcal{P}$, A be a complex manifold, $a_0 \in A$, and

$$\iota = (\iota_0, \iota_1, \ldots, \iota_n) : A \to \mathcal{P}$$

be a holomorphic map such that

$$\iota_0(a_0) = j_0, \qquad \iota_i(a_0) = r_i, \qquad i = 1, \ldots, n. \tag{12}$$

Define the unfolding $(\pi_\iota : P_\iota \to A, R_{\iota,*}, a_0)$ by

$$P_\iota := A \times \Sigma, \qquad J_\iota(a, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \iota_0(a)(z) \end{pmatrix} \tag{13}$$

where $\sqrt{-1}$ denotes the complex structure on A and

$$R_{\iota,i} := \{(a, \iota_i(a)) \,:\, a \in A\}, \qquad i = 1, \ldots, n. \tag{14}$$

Lemma 8.8. *The unfolding $(\pi_\iota, R_{\iota,*}, a_0)$ is infinitesimally universal if and only if the restriction of ι to a sufficiently small neighborhood of a_0 is a (holomorphic) local slice for the action of $\mathcal{G}$ on $\mathcal{P}$.*

Proof. Let $u_0 : (\Sigma, j_0) \to A$ be the holomorphic embedding $u_0(z) := (a_0, z)$. Then the operator D_{u_0,a_0} has domain

$$\mathcal{X}_0 := \left\{(\hat{u}, \hat{a}) \in \Omega^0(\Sigma, T\Sigma) \times T_{a_0}A \,:\, \hat{u}(r_i) = d\iota_i(a_0)\hat{a}\right\},$$

target space $\mathcal{Y}_0 := \Omega^{0,1}_{j_0}(\Sigma, T\Sigma)$, and is given by

$$D_{u_0,a_0}(\hat{u}, \hat{a}) = \bar{\partial}_{j_0}\hat{u} - \frac{1}{2} j_0 d\iota_0(a_0)\hat{a}.$$

Now the tangent space of the group orbit $\mathcal{G}^*(j_0, r_*)$ at (j_0, r_*) is given by

$$T_{(j_0,r_*)}\mathcal{G}^*(j_0, r_*) = \left\{(2j_0\bar{\partial}_{j_0}\xi, -\xi(r_1), \dots, -\xi(r_n)) \,:\, \xi \in \Omega^0(\Sigma, T\Sigma)\right\}.$$

(See the proof of Theorem 7.1 for the formula $\mathcal{L}_\xi j_0 = 2j_0\bar{\partial}_{j_0}\xi$.) Hence the operator D_{u_0,a_0} is injective if and only if $\operatorname{im} d\iota(a_0) \cap T_{(j_0,r_*)}\mathcal{G}^*(j_0, r_*) = 0$ and $d\iota(a_0)$ is injective. It is surjective if and only if $\operatorname{im} d\iota(a_0) + T_{(j_0,r_*)}\mathcal{G}^*(j_0, r_*) = T_{(j_0,r_*)}\mathcal{P}$. This proves the lemma. □

Theorem 8.9. *Theorems 5.3-5.6 hold for marked Riemann surfaces without nodes.*

Proof. **Step 1.** *$\mathcal{Q}$ and $\mathcal{B}$ are complex manifolds, the projection $\pi_\mathcal{B}$ is a proper holomorphic submersion, and $\mathcal{S}_1, \dots, \mathcal{S}_n$ are complex submanifolds of $\mathcal{Q}$.*

Apply Lemma 8.2 to the complex manifold $A = \mathcal{J} = \mathcal{J}(\Sigma)$, replace the fiber Σ by $\Sigma^n \setminus \Delta$, and replace ι by the map $\mathcal{J} \to \mathcal{J}(\Sigma^n \setminus \Delta)$ which assigns to each complex structure $j \in \mathcal{J}(\Sigma)$ the corresponding product structure on $\Sigma^n \setminus \Delta$. Then (the proof of) Lemma 8.2 shows that $\mathcal{P} = \mathcal{J} \times (\Sigma^n \setminus \Delta)$ is a complex manifold. The group $\mathcal{G} = \mathrm{Diff}_0(\Sigma)$ acts on this space by the holomorphic diffeomorphisms

$$(j, s_1, \dots, s_n) \mapsto (f^*j, f^{-1}(s_1), \dots, f^{-1}(s_n))$$

for $f \in \mathcal{G}$. The action is free and admits holomorphic local slices for all g and n. It follows that the quotient $\mathcal{B} = \mathcal{P}/\mathcal{G}$ is a complex manifold. The same argument shows that the total space $\mathcal{Q}$ is a complex manifold and that the projection $\pi_\mathcal{B} : \mathcal{Q} \to \mathcal{B}$ is holomorphic. That it is a proper submersion is immediate from the definitions.

Here are more details on the holomorphic local slices for the action of $\mathcal{G}$ on $\mathcal{P}$. In the case $g > 1$ we will find a holomorphic local slice $\iota : B \to \mathcal{P}$, defined on $B := B_0 \times \mathrm{int}(\mathbb{D})^n$, which has the form

$$\iota(b_0, b_1, \dots, b_n) = (\iota_0(b_0), \iota_1(b_0, b_1), \dots, \iota_n(b_0, b_n)).$$

Here $\iota_0 : B_0 \to \mathcal{J}$ is a holomorphic local slice as in Theorem 7.4. For $i = 1, \dots, n$, the map $(b_0, b_i) \mapsto (b_0, \iota_i(b_0, b_i))$ is holomorphic with respect to the complex structure J_{ι_0} on $Q_0 := B_0 \times \Sigma$ defined by (13) and restricts to a holomorphic embedding from $b_0 \times \mathrm{int}(\mathbb{D})$ to (Σ, j) with $j = \iota_0(b_0)$. That such maps ι_i exist and can be chosen with disjoint images follows from Lemma 7.2.

In the case $g = 1$ and $n \geq 1$ with $\Sigma = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ an example of a holomorphic local slice is the map $\iota : B = B_0 \times B_1 \times \cdots \times B_{n-1} \to \mathcal{P}$ given by

$$\iota(\lambda_0, b_1, \dots, b_{n-1}) := (j(\lambda_0), f_{\lambda_0}^{-1}(b_1), \dots, f_{\lambda_0}^{-1}(b_{n-1}), f_{\lambda_0}^{-1}(s_n))$$

where $B_0 \subset \mathbb{H}$ and $B_i \subset \mathbb{C}$ are open sets such that closures of the $n-1$ sets $B_i + \mathbb{Z} + \lambda_0\mathbb{Z} \subset T_{\lambda_0} := \mathbb{C}/\mathbb{Z} + \lambda_0\mathbb{Z}$ are pairwise disjoint, none of these sets contains the point $s_n + \mathbb{Z} + \lambda_0\mathbb{Z}$, the complex structure $j(\lambda_0) \in \mathcal{J}(\mathbb{T}^2)$ is defined by (5), and the isomorphism $f_{\lambda_0} : (\mathbb{T}^2, j(\lambda_0)) \to T_{\lambda_0}$ is defined by $f_{\lambda_0}(x, y) := x + \lambda_0 y$.

That any such map is a holomorphic local slice for the action of $\mathcal{G} = \mathrm{Diff}_0(\mathbb{T}^2)$ follows from Theorem 7.3.

In the case $g = 0$ and $n \geq 3$ with $\Sigma = S^2$ an example of a holomorphic local slice is the map $\iota : B = \mathrm{int}(\mathbb{D})^{n-3} \to \mathcal{P}$ given by

$$\iota(b_1, \dots, b_{n-s}) := (j_0, \iota_1(b_1), \dots, \iota_{n-3}(b_{n-3}), s_{n-2}, s_{n-1}, s_n)$$

s_{n-2}, s_{n-1}, s_n are distinct points in S^2, $j_0 \in \mathcal{J}(S^2)$ denotes the standard complex structure, and the $\iota_i : \mathrm{int}(\mathbb{D}) \to S^2$ are holomorphic embeddings for $1 \leq i \leq n-3$ such that the closures of their images are pairwise disjoint and do not contain the points s_{n-2}, s_{n-1}, s_n. That any such map is a holomorphic local slice for the action of $\mathcal{G} = \mathrm{Diff}_0(S^2)$ follows from Theorem 7.1.

Thus we have constructed holomorphic local slices for the action of $\mathcal{G} = \mathrm{Diff}_0(\Sigma)$ on $\mathcal{P} = \mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)$ in all cases. Holomorphic slices for the action of $\mathcal{G}$ on $\mathcal{P} \times \Sigma$ can be constructed in a similar fashion. It then follows from the symmetry of the construction under permutations of the components in Σ that the sections $\mathcal{S}_i$ are complex submanifolds of $\mathcal{Q}$. This proves Step 1.

Step 2. *The pair $(\pi_\mathcal{B}, \mathcal{S}_*)$ is an infinitesimally universal unfolding of each of its fibers.*

Choose $[j_0, s_*] \in \mathcal{B}$. Let B be an open neighborhood of $b_0 = 0$ in $\mathbb{C}^{3g-3+n}$ and $\iota = (\iota_0, \iota_1, \dots, \iota_n) : B \to \mathcal{P}$ be a local holomorphic slice satisfying (12). Then the unfolding $(\pi_\iota : Q_\iota \to B, S_{\iota,*}, b_0)$ defined as in (13) and (14) is isomorphic to $(\pi_\mathcal{B}, \mathcal{S}_*, [j_0, s_*])$. Hence Step 2 follows from Lemma 8.8.

Step 3. *Every pseudomorphism from (π_A, R_*, a_0) to $(\pi_\mathcal{B}, \mathcal{S}_*, [j_0, s_*])$ is a morphism.*

Let (ϕ, Φ) be a pseudomorphism from $(\pi_A : P \to A, R_*, a_0)$ to $(\pi_\mathcal{B}, \mathcal{S}_*, [j_0, s_*])$ and $\iota = (\iota_0, \iota_1, \dots, \iota_n) : B \to \mathcal{P}$ be as the proof of in Step 2. Define (ψ, Ψ) be the composition of (ϕ, Φ) with the obvious morphism from $(\pi_\mathcal{B}, \mathcal{S}_*, [j_0, s_*])$ to the unfolding $(Q_\iota, S_{\iota,*}, b_0)$, defined as in (13) and (14). Using the maps $\Psi_a : P_a \to \Sigma$ given by $\Psi(p) =: (\psi(a), \Psi_a(p))$ for $p \in P_a$ we construct a trivialization

$$\tau : A \times \Sigma \to P, \qquad \tau(a, z) := \tau_a(z) := \Psi_a^{-1}(z).$$

Then the pullback of the section R_i is given by

$$\tau^{-1}(R_i) = \{(a, \sigma_i(a)) \,:\, a \in A\}, \qquad \sigma_i := \iota_i \circ \psi : A \to \Sigma$$

and the pullback of the complex structure on P under τ has the form

$$J(a, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ \alpha & j(a)(z) \end{pmatrix}$$

where $j := \iota_0 \circ \psi : A \to \mathcal{J}$ and $\alpha \in \Omega^{0,1}_j(A, \mathrm{Vect}(\Sigma))$. Since J is integrable it follows from Lemma 8.2 that j and α satisfy

$$dj(a)\hat{a} + j(a)dj(a)\sqrt{-1}\hat{a} - \mathcal{L}_{\alpha(a,\sqrt{-1}\hat{a})}j(a) = 0.$$

Since $\tau^{-1}(R_i)$ is a complex submanifold of $A \times \Sigma$, we have

$$d\sigma_i(a)\hat{a} + j(a)d\sigma_i(a)\sqrt{-1}\hat{a} + \alpha(a, \sqrt{-1}\hat{a})(\sigma_i(a)) = 0 \tag{15}$$

for $i = 1, \ldots, n$. Since ι is a local holomorphic slice these two equations together imply that

$$dj(a)\hat{a} + j(a)dj(a)\sqrt{-1}\hat{a} = 0, \qquad d\sigma_i(a)\hat{a} + j(a)d\sigma_i(a)\sqrt{-1}\hat{a} = 0,$$

and $\mathcal{L}_{\alpha(a,\sqrt{-1}\hat{a})}j(a) = 0$ and $\alpha(a, \sqrt{-1}\hat{a})(\sigma_i(a)) = 0$ for all $\hat{a} \in T_aA$. Since $n > 2 - 2g$ it follows that $\alpha \equiv 0$. Moreover, the map $(j, \sigma_1, \ldots, \sigma_n) = \iota \circ \psi : A \to \mathcal{P}$ is holomorphic. Since ι is a holomorphic local slice, this implies that ψ, and hence also Ψ, is holomorphic.

Step 4. *The pair $(\pi_{\mathcal{B}}, \mathcal{S}_*)$ is a universal unfolding of each of its fibers.*

Choose $[j_0, s_*] \in \mathcal{B}$ and let $(\pi_A : P \to A, a_0)$ which admits an isomorphism $u_0 : (\Sigma, j_0) \to P_{a_0}$ such that $u_0(s_i) := P_{a_0} \cap R_i$. Then u_0^{-1} is a fiber isomorphism from P_{a_0} to $\mathcal{Q}_{[j_0,s_*]}$. Trivialize P by a map $\tau : A \times \Sigma \to P$ such that $\tau_{a_0} = u_0$. Define $j : A \to \mathcal{J}$ and $\sigma_i : A \to \Sigma$ so that $j(a)$ is the pullback of the complex structure on P_a under τ_a and $\tau^{-1}(R_i) = \{(a, \sigma_i(a)) : a \in A\}$. Then $j(a_0) = j_0$ and $\sigma_i(a_0) = s_i$. Define $\phi : A \to \mathcal{B}$ and $\Phi : P \to \mathcal{Q}$ by

$$\phi(a) := [j(a), \sigma_*(a)], \qquad \Phi(p) := [j(a), \sigma_*(a), z], \qquad p =: \tau(a, z)$$

for $a \in A$ and $p \in P_a$. This is a pseudomorphism and hence, by Step 3, it is a morphism.

To prove uniqueness, choose a local holomorphic slice $\iota = (\iota_0, \iota_1, \ldots, \iota_n) : B \to \mathcal{P}$ such that $\iota(b_0) = (j_0, s_1, \ldots, s_n)$. Choose two morphisms

$$(\psi, \Psi), (\phi, \Phi) : (\pi_A, R_*, a_0) \to (\pi_\iota, S_{\iota,*}, b_0)$$

such that $\Phi_{a_0} = \Psi_{a_0} = u_0^{-1} : P_{a_0} \to \Sigma$. If a is near a_0 then

$$\Psi_a \circ \Phi_a^{-1} : (\Sigma, \iota_0(\phi(a))) \to (\Sigma, \iota_0(\psi(a)))$$

is a diffeomorphism isotopic to the identity that sends $\iota_i(\phi(a))$ to $\iota_i(\psi(a))$ for $i = 1, \ldots, n$. Hence by the local slice property $\phi(a) = \psi(a)$ and $\Phi_a = \Psi_a$.

Step 5. *Let j_0 be a complex structure on Σ and $s_1, \ldots, s_n$ be distinct marked points on Σ. Every infinitesimally universal unfolding $(\pi_B : Q \to B, S_*, b_0)$ of (Σ_0, s_*, j_0) is isomorphic to $(\pi_{\mathcal{B}}, \mathcal{S}_*, [j_0, s_*])$.*

As in Step 3 we may assume that $Q = B \times \Sigma$ with complex structure

$$J(b, z) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \iota_0(b)(z) \end{pmatrix}$$

and $S_i = \{(b, \iota_i(b)) : b \in B\}$ where $\iota = (\iota_0, \iota_1, \ldots, \iota_n) : B \to \mathcal{P}$ is holomorphic and $\iota(b_0) = (j_0, s_1, \ldots, s_n)$. By Lemma 8.2 the almost complex structure J is integrable. Since $(\pi_B : Q \to B, S_*, b_0)$ is infinitesimally universal, the restriction of ι to a neighborhood of b_0 is a local slice by Lemma 8.8. Hence (π_B, S_*, b_0) is isomorphic to $(\pi_{\mathcal{B}}, \mathcal{S}_*, [j_0, s_0])$ by the local slice property. □

Remark 8.10. The etale groupoid associated to the universal marked curve of 8.6 is proper as in Remark 8.5.

9 Nonlinear Hardy spaces

In this section we characterize infinitesimally universal unfoldings in terms of certain "nonlinear Hardy spaces" associated to a desingularization. The idea is to decompose a Riemann surface Σ as a union of submanifolds Ω and Δ, intersecting in their common boundary, and to identify holomorphic maps on Σ with pairs of holomorphic maps defined on Ω and Δ that agree on that common boundary.

9.1. Throughout this section we assume that

$$(\pi_B : Q \to B, S_*, b_0)$$

is a nodal unfolding of a marked nodal Riemann surface (Σ, s_*, ν, j) and that

$$w_0 : \Sigma \to Q_{b_0}$$

is a desingularization. Let $C_B \subset Q$ denote the set of critical points of π_B. Let U be a neighborhood of C_B equipped with nodal coordinates. This means

$$U = U_1 \cup \cdots \cup U_k$$

where the sets U_i have pairwise disjoint closures, each U_i is a connected neighborhood of one of the components of C_B, and for $i = 1, \ldots, k$ there is a holomorphic coordinate system

$$(\zeta_i, \tau_i) : B \to \mathbb{C} \times \mathbb{C}^{d-1}, \qquad d := \dim_{\mathbb{C}} B$$

and holomorphic functions $\xi_i, \eta_i : U_i \to \mathbb{C}$ such that

$$(\xi_i, \eta_i, \tau_i \circ \pi_B) : U_i \to \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{d-1}$$

is a holomorphic coordinate system and $\xi_i \eta_i = \zeta_i \circ \pi_B$. Assume that $\bar{U} \cap S_* = \emptyset$. Let $V \subset Q$ be an open set such that

$$Q = U \cup V, \qquad \bar{V} \cap C_B = \emptyset,$$

and $U_i \cap V$ intersects each fiber Q_b in two open annuli with $|\xi_i| > |\eta_i|$ on one component and $|\xi_i| < |\eta_i|$ on the other. Introduce the abbreviations

$$W := U \cap V, \quad W_i := U_i \cap V, \quad W_{i,1} := \{|\xi_i| > |\eta_i|\}, \quad W_{i,2} := \{|\xi_i| < |\eta_i|\},$$

$$U_b := U \cap Q_b, \qquad V_b := V \cap Q_b, \qquad W_b := W \cap Q_b.$$

9.2. We consider a decomposition

$$\Sigma = \Omega \cup \Delta, \qquad \partial\Omega = \partial\Delta = \Omega \cap \Delta =: \Gamma,$$

into submanifolds with boundary such that Δ is a disjoint union

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_k$$

where, for each i, the set Δ_i is either an embedded closed annulus or it is the union of two disjoint embedded closed disks centered at two equivalent nodal points and

$$w_0(\Omega) \subset V, \qquad w_0(\Delta_i) \subset U_i$$

for $i = 1, \ldots, k$. It follows that every pair of equivalent nodal points appears in some Δ_i. In case Δ_i is a disjoint union of two disks, say $\Delta_i = \Delta_{i,1} \cup \Delta_{i,2}$, choose holomorphic diffeomorphisms $x_i : \Delta_{i,1} \to \mathbb{D}$ and $y_i : \Delta_{i,2} \to \mathbb{D}$ which send the nodal point to 0. In case Δ_i is an annulus choose a holomorphic diffeomorphism $x_i : \Delta_i \to \mathbb{A}(\delta_i, 1)$ and define $y_i : \Delta_i \to \mathbb{A}(\delta_i, 1)$ by $y_i = \delta_i / x_i$. In both cases choose the names so that

$$w_0\big(x_i^{-1}(S^1)\big) \subset W_{i,1}, \qquad w_0\big(y_i^{-1}(S^1)\big) \subset W_{i,2}.$$

The curves $\xi_i \circ w_0 \circ x_i^{-1}$ and $\eta_i \circ w_0 \circ y_i^{-1}$ from S^1 to $\mathbb{C} \setminus 0$ both have winding number one about the origin.

9.3. Fix an integer $s+1/2 > 1$. Since $\pi_B|W$ is a submersion the space $H^s(\Gamma, W_b)$ is, for each $b \in B$, a submanifold of the Hilbert manifold of all H^s maps from Γ to W (see Appendix B). Define an open subset

$$\mathcal{W}(b) \subset H^s(\Gamma, W_b)$$

by the condition that for $\gamma \in H^s(\Gamma, W_b)$ we have $\gamma \in \mathcal{W}(b)$ iff

$$\gamma\big(x_i^{-1}(S^1)\big) \subset W_{i,1}, \qquad \gamma\big(y_i^{-1}(S^1)\big) \subset W_{i,2},$$

and the curves $\xi_i \circ \gamma \circ x_i^{-1}$ and $\eta_i \circ \gamma \circ y_i^{-1}$ from S^1 to $\mathbb{C} \setminus 0$ both have winding number one about the origin. Introduce sets

$$\mathcal{Z}(b) := \{v \in \mathrm{Hol}^{s+1/2}(\Omega, V_b) : v|\Gamma \in \mathcal{W}(b) \text{ and } v(s_* \cap \Omega) = S_* \cap Q_b\},$$
$$\mathcal{N}(b) := \{u \in \mathrm{Hol}^{s+1/2}(\Delta, U_b) : u|\Gamma \in \mathcal{W}(b) \text{ and } u \text{ preserves } \nu\}.$$

Here we use the notation

$$\mathrm{Hol}^{s+1/2}(X, Y) = \Big\{ f \in H^{s+1/2}(X, Y) : df \circ J_X = J_Y \circ df \Big\} \tag{16}$$

for a compact Riemann surface X with boundary and an almost complex manifold Y. By Theorem B.4 (i), restriction to the boundary defines a continuous map $\mathrm{Hol}^{s+1/2}(X, Y) \to H^s(\partial X, Y)$. The phrase "$u$ preserves ν" means that $\{x, y\} \in \nu \implies u(x) = u(y) \in C_B$. Define the **nonlinear Hardy spaces** by

$$\mathcal{U}(b) := \{u|\Gamma : u \in \mathcal{N}(b)\}, \qquad \mathcal{V}(b) := \{v|\Gamma : v \in \mathcal{Z}(b)\}.$$

Define

$$\mathcal{W}_0 := \bigsqcup_{b\in B} \mathcal{W}(b), \qquad \mathcal{V}_0 := \bigsqcup_{b\in B} \mathcal{V}(b), \qquad \mathcal{U}_0 := \bigsqcup_{b\in B} \mathcal{U}(b),$$

so that $(\gamma, b) \in \mathcal{W}_0 \iff \gamma \in \mathcal{W}(b)$, etc. The desingularization $w_0 : \Sigma \to Q_{b_0}$ determines a point

$$(\gamma_0, b_0) \in \mathcal{U}_0 \cap \mathcal{V}_0 \subset \mathcal{W}_0, \qquad \gamma_0 := w_0|\Gamma. \tag{17}$$

Lemma 9.4. *For every $(\gamma, b) \in \mathcal{U}_0 \cap \mathcal{V}_0$ there is a unique desingularization $w : \Sigma \to Q_b$ with $w|\Gamma = \gamma$.*

Proof. Uniqueness is an immediate consequence of unique continuation. To prove existence, let $(\gamma, b) \in \mathcal{U}_0 \cap \mathcal{V}_0$ be given. Then, by definition of $\mathcal{U}_0$ and $\mathcal{V}_0$, there is a continuous map $w : \Sigma \to Q_b$ which is holomorphic in $\mathrm{int}(\Omega)$ and in $\mathrm{int}(\Delta)$ with $w(s_*) = S_* \cap Q_b$ and $w(z_0) = w(z_1)$ for every nodal pair $\{z_0, z_1\} \in \nu$. The map $w : \Sigma \to Q$ is of class $H^{s+1/2}$ and is therefore holomorphic on all of Σ. We must prove that if $z_0 \neq z_1$ we have $w(z_0) = w(z_1)$ if and only if $\{z_0, z_1\} \in \nu$. Assume first that there are no nodes, i.e. $Q_b \cap C_B = \emptyset$. Then Δ is a union of disjoint annuli and, by the winding number assumption, the restriction of w to Δ is an embedding into Q_b and $w(\Delta) \cup V_b = Q_b$. Hence there is a point $q \in w(\Delta) \setminus V$. Hence the degree of w at q is one and hence $w : \Sigma \to Q_b$ is a holomorphic diffeomorphism. Now assume $Q_b \cap C_B \neq \emptyset$. Then $w^{-1}(C_B) = \cup\nu$, by the winding number assumption, and so the restriction $w : \Sigma \setminus \cup\nu \to Q_b \setminus C_B$ is proper. Hence the degree of this restriction is constant on each component of $\Sigma \setminus \cup\nu$. Now each component of Σ contains a component Δ' of Δ that is diffeomorphic a disc. By the winding number assumption, the restriction of w to Δ' is an embedding. Moreover, the images under w of the components of $\Delta \setminus C_B$ are disjoint and there is a point $q \in w(\Delta') \setminus (C_B \cup V)$. Since $w(\Omega) \subset V$, the degree of the restriction $w : \Sigma \setminus \cup\nu \to Q_b \setminus C_B$ at any such point q is one. Hence the degree of the restriction is one at every point and hence $w : \Sigma \setminus \cup\nu \to Q_b \setminus C_B$ is a holomorphic diffeomorphism. This proves the lemma. □

Theorem 9.5. *Fix an integer $s + 1/2 > 4$. Let $(\pi_B : Q \to B, S_*, b_0)$ be a nodal unfolding of a marked nodal Riemann surface (Σ, s_*, ν, j) and $w_0 : \Sigma \to Q_{b_0}$ be a desingularization as in 9.1. Let $\mathcal{U}_0, \mathcal{V}_0 \subset \mathcal{W}_0$ be the subspaces in 9.3 and $\gamma_0 := w_0|\Gamma$ as in (17). Let D_{w_0,b_0} be the Fredholm operator in Definition 5.2. Then the following holds.*

(i) *$\mathcal{U}_0$ and $\mathcal{V}_0$ are complex Hilbert submanifolds of $\mathcal{W}_0$.*

(ii) *The intersection $T_{(\gamma_0,b_0)}\mathcal{U}_0 \cap T_{(\gamma_0,b_0)}\mathcal{V}_0$ is isomorphic to the kernel of D_{w_0,b_0}.*

(iii) *The quotient $T_{(\gamma_0,b_0)}\mathcal{W}_0/(T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0)$ is isomorphic to the cokernel of D_{w_0,b_0}.*

(iv) *The unfolding (π_B, S_*, b_0) is infinitesimally universal if and only if*

$$T_{(\gamma_0,b_0)}\mathcal{W}_0 = T_{(\gamma_0,b_0)}\mathcal{U}_0 \oplus T_{(\gamma_0,b_0)}\mathcal{V}_0.$$

Proof. We prove that $\mathcal{U}_0$ is a complex Hilbert submanifold of $\mathcal{W}_0$.

Choose the indexing so that $z_i(b_0) = 0$ for $i \le \ell$ and $z_i(b_0) \ne 0$ for $i > \ell$. Abbreviate $H^s := H^s(S^1, \mathbb{C})$ for the Sobolev space in B.1 and consider the map

$$\mathcal{W}_0 \to (H^s)^{2k} \times B : \gamma \mapsto (\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k, b)$$

where $\gamma \in \mathcal{W}(b)$ and $\alpha_i = \xi_i \circ \gamma \circ x_i^{-1}$ and $\beta_i = \eta_i \circ \gamma \circ y_i^{-1}$. This maps $\mathcal{W}_0$ diffeomorphically onto an open set in a Hilbert space. The map sends $\mathcal{U}_0 \subset \mathcal{W}_0$ to the subset of all tuples $(\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k, b)$ such that all nonpositive coefficients of α_i and β_i vanish for $i \le \ell$ and such that

$$\beta_i(y) = \frac{\zeta_i(b)}{\alpha_i(\delta_i/y)}$$

for $i > \ell$. Thus the tuple $(\alpha_1, \beta_1, \ldots, \alpha_\ell, \beta_\ell)$ is restricted to a closed subspace of $(H^s)^{2\ell}$ and, for $i > \ell$, the component β_i can be expressed as a holomorphic function of α_i and b. This shows that $\mathcal{U}_0$ is a complex Hilbert submanifold of $\mathcal{W}_0$.

We will show that the restriction map

$$\mathcal{Z}_0 := \bigsqcup_{b \in B} \mathcal{Z}(b) \to \mathcal{W}_0 : v \mapsto v|\Gamma$$

is a holomorphic embedding. Since the image is precisely $\mathcal{V}_0$ by definition, this will show that $\mathcal{V}_0$ is a complex Hilbert submanifold of $\mathcal{W}_0$. Denote by $\mathcal{B}$ the space of all pairs (v, b), where $b \in B$ and $v : \Omega \to V_b$ is an $H^{s+1/2}$ map satisfying

$$v(s_*) = S_* \cap Q_b.$$

The space $\mathcal{B}$ is a complex Hilbert manifold whose tangent space at (v, b) is the Sobolev space

$$T_{v,b}\mathcal{B} = \left\{ (\hat{v}, \hat{b}) \in H^{s+1/2}(\Omega, v^*TQ) \times T_b B \; : \; d\pi_B(v)\hat{v} \equiv \hat{b}, \; \hat{v}(s_i) \in T_{v(s_i)} S_i \right\},$$

i.e. $\hat{v}$ is a section of class $H^{s+1/2}$ of the pullback tangent bundle v^*TQ that projects to a constant tangent vector of B and at the marked points is tangent to S_*. Consider the complex Hilbert space bundle $\mathcal{E} \to \mathcal{B}$ whose fiber

$$\mathcal{E}_{v,b} := H^{s-1/2}(\Omega, \Lambda^{0,1} T^*\Omega \otimes v^*TQ_b)$$

over $(v, b) \in \mathcal{B}$ is the Sobolev space of $(0,1)$-forms on Ω of class $H^{s-1/2}$ with values in the vertical pullback tangent bundle v^*TQ_b. The Cauchy–Riemann operator $\bar{\partial}$ is a section of this bundle and its zero set is the space $\mathcal{Z}_0 \subset \mathcal{B}$ defined above. The vertical derivative of $\bar{\partial}$ at a zero (v, b) is the restriction

$$D_{v,b} : T_{v,b}\mathcal{B} \to \mathcal{E}_{v,b}$$

of the Cauchy–Riemann operator of the holomorphic vector bundle $v^*TQ \to \Omega$ to the subspace $T_{v,b}\mathcal{B} \subset H^{s+1/2}(\Omega, v^*TQ)$. This operator is split surjective;

a right inverse can be constructed from an appropriate Lagrangian boundary condition (see [16, Appendix C]). Hence $\mathcal{Z}_0$ is a complex submanifold of $\mathcal{B}$.

We show that the restriction map is an injective holomorphic immersion.

By unique continuation at boundary points, the restriction map is injective, i.e. two elements of $\mathrm{Hol}^{s+1/2}(\Omega, V_b)$ that agree on the boundary agree everywhere. The derivative of the restriction map is also a restriction map; it is injective and has a left inverse, by Theorem B.4 (ii). Hence the restriction map $\mathcal{Z}_0 \to \mathcal{W}_0$ is a holomorphic immersion.

We show that the restriction map $\mathcal{Z}_0 \to \mathcal{W}_0$ is proper.

Suppose that $v_k \in \mathcal{Z}(b_k)$, that $\gamma_k := v_k|\Gamma$, that γ_k converges to $\gamma \in \mathcal{W}(b)$, and that $\gamma = v|\Gamma$ where $v \in \mathcal{Z}(b)$. We prove in four steps that v_k converges to v in $H^{s+1/2}(\Omega, Q)$.

Step 1. *We may assume without loss of generality that each v_k is an embedding for every k.*

After shrinking $\mathcal{V}_0$ we obtain that $\gamma : \Gamma \to Q_b$ is an embedding for every $(\gamma, b) \in \mathcal{V}_0$. (This makes sense because $s > 3/2$, so γ is continuously differentiable.) If $\gamma = v|\Gamma$ is an embedding and $v \in \mathrm{Hol}^{s+1/2}(\Omega, V_b)$ then v is an embedding. This is because $\#v^{-1}(q)$ (the number of preimages counted with multiplicity) for $q \in Q_b \setminus v(\Gamma)$ can only change as q passes through the image of γ. As γ is an embedding $\#v^{-1}(q)$ is either zero or one. Hence v is an embedding.

Step 2. *A subsequence of v_k converges in the C^∞ topology on every compact subset of* $\mathrm{int}(\Omega)$.

If the first derivatives of v_k are uniformly bounded then $v_k|\mathrm{int}(\Omega)$ has a C^∞ convergent subsequence (see [16, Appendix B]). Moreover, a nonconstant holomorphic sphere in Q bubbles off whenever the first derivatives of v_k are not bounded. But bubbling cannot occur in V. To see this argue as follows. Suppose z_k converges to $z_0 \in \mathrm{int}(\Omega)$ and the derivatives of v_k at z_k blow up. Then the standard bubbling argument (see [16, Chapter 4]) applies. It shows that, after passing to a subsequence and modifying z_k (without changing the limit), there are (i, j_k)-holomorphic embeddings ε_k from the disk $\mathbb{D}_k \subset \mathbb{C}$, centered at zero with radius k, to Q such that $\varepsilon_k(0) = z_k$, the family of disks $\varepsilon_k(\mathbb{D}_k)$ converges to z_0, and $v_k \circ \varepsilon_k$ converges to a nonconstant J-holomorphic sphere $v_0 : S^2 = \mathbb{C} \cup \infty \to Q_b$. (The convergence is uniform with all derivatives on every compact subset of $\mathbb{C}$.) The image of v_0 must intersect the nodal set $Q_b \cap C_B$. Hence there is a point $a \in \mathbb{C} = S^2 \setminus \{\infty\}$ such that $v_0(a) \in Q \setminus \bar{V}$. This implies $v_k(\varepsilon_k(a)) \notin V$ for k sufficiently large, contradicting the fact that $v_k(\Omega) \subset V$.

Step 3. *A subsequence of v_k converges to v in the C^0 topology.*

By Arzéla–Ascoli it suffices to show that the sequence v_k is bounded in C^1. We treat this as a Lagrangian boundary value problem. Choose $M \subset Q_b$ to be a submanifold with boundary that contains the image of v in its interior. Choose a smooth family of embeddings

$$\iota_a : M \to Q_a \setminus C_B, \qquad a \in B,$$

such that $\iota_b : M \to Q_b \setminus C_B$ is the inclusion. Then the image of ι_{b_k} contains the image of v_k for k sufficiently large. Think of M as a symplectic manifold and define the Lagrangian submanifolds $L \subset M$ and $L_k \subset M$ by

$$L := \gamma(\Gamma), \qquad L_k := \iota_{b_k}^{-1} \circ \gamma_k(\Gamma).$$

Since $s > 7/2$ the sequence $\iota_{b_k}^{-1} \circ \gamma_k : \Gamma \to M$ converges to γ in the C^3 topology. Hence there is a sequence of diffeomorphisms $\phi_k : M \to M$ such that ϕ_k converges to the identity in the C^3 topology and

$$\phi_k \circ \iota_{b_k}^{-1} \circ \gamma_k = \gamma, \qquad \phi_k(L_k) = L.$$

Define

$$\tilde{v}_k := \phi_k \circ \iota_{b_k}^{-1} \circ v_k, \qquad \tilde{J}_k := (\phi_k \circ \iota_{b_k}^{-1})_* J_{b_k},$$

where J_a denotes the complex structure on Q_a. Then $\tilde{J}_k$ converges to $\tilde{J} := J_b$ in the C^2 topology, $\tilde{v}_k : \Omega \to M$ is a $\tilde{J}_k$-holomorphic curve such that $\tilde{v}_k(\Gamma) \subset L$ and, moreover,

$$\tilde{v}_k|\Gamma = \gamma : \Gamma \to L \tag{18}$$

for all k. We must prove that the first derivatives of $\tilde{v}_k$ are uniformly bounded. Suppose by contradiction that that there is a sequence $z_k \in \Omega$ such that

$$c_k := |d\tilde{v}_k(z_k)| = \|d\tilde{v}_k\|_{L^\infty} \to \infty.$$

Now apply the standard rescaling argument: Assume w.l.o.g. that z_k converges to $z_0 \in \Omega$, choose a coordinate chart from a neighborhood of z_0 to upper half plane (sending z_k to ζ_k), and compose the resulting $\tilde{J}_k$-holomorphic curve with the rescaling map $\varepsilon_k(\zeta) := \zeta_k + \zeta/c_k$. Let d_k be the Euclidean distance of ζ_k from the boundary of the upper half plane. There are two cases. If $c_k \cdot d_k \to \infty$ then a nonconstant holomorphic sphere bubbles off and the same argument as in Step 2 leads to a contradiction. If the sequence $c_k \cdot d_k$ is bounded then, by [16, Theorem B.4.2], the rescaled sequence has a subsequence that converges in the C^1 topology to a holomorphic curve $\tilde{w} : \{\zeta \in \mathbb{C} : \mathrm{im}\,\zeta \geq 0\} \to M$ with $\tilde{w}(\mathbb{R}) \subset L$. The choice of the rescaling factor shows that the derivative of $\tilde{w}$ has norm one at some point and so $\tilde{w}$ is nonconstant. On the other hand, since ε_k converges to a constant, condition (18) implies that the restriction of this holomorphic curve to the boundary is constant; contradiction.

Step 4. *A subsequence of v_k converges to v in the $H^{s+1/2}$ topology.*

Let v be the limit in Step 3. Then $v|\Gamma = \gamma$ takes values in W_b. By Step 2 it is enough to show that v_k converges to v in some neighborhood of each boundary component in the $H^{s+1/2}$ topology. We can identify such a neighborhood holomorphically with $\mathbb{A}(r,1)$. Shrinking the neighborhood if necessary we may assume that v_k maps $\mathbb{A}(r,1)$ to W_b for k sufficiently large. By assumption and Step 2 the restriction of v_k to $\partial\mathbb{A}(r.1)$ converges in H^s. Hence Step 4 follows from the fact that the restriction map $H^{s+1/2}(\mathbb{A}(r,1)) \to H^s(\partial\mathbb{A}(r,1))$ is a linear embedding onto a closed subspace (see Theorem B.4 (ii)). In the notation of 10.5 below this subspace is the diagonal in $H^s_r \times H^s$.

Thus we have proved that every subsequence of v_k has a further subsequence converging to v in $H^{s+1/2}$. Hence the sequence v_k itself converges to v in the $H^{s+1/2}$ topology. This completes the proof of (i).

We prove (ii). It follows directly from the definitions that there is a map

$$\ker D_{w_0,b_0} \to T_{(\gamma_0,b_0)}\mathcal{U}_0 \cap T_{(\gamma_0,b_0)}\mathcal{V}_0 : (\xi,\hat{b}) \mapsto (\xi|\Gamma,\hat{b})$$

This map is surjective by elliptic regularity: if $\xi : \Sigma \to w_0^*TQ$ is a continuous section with $d\pi_B(w_0)\xi = \hat{b}$ whose restriction to both Δ and Ω is of class $H^{s+1/2}$ and belongs to the kernel of the differential operator D_{w_0}, then ξ is smooth. The map is injective by unique continuation: an element $(\xi,\hat{b}) \in \ker D_{w_0,b_0}$ vanishes identically if and only if the restriction of ξ to the disjoint union Γ of circles vanishes. (The fibers are connected and so Γ intersects each component of Σ in at least one circle.) This proves (ii).

To prove (iii), we define a map

$$\operatorname{coker} D_{w_0,b_0} \to \frac{T_{(\gamma_0,b_0)}\mathcal{W}_0}{T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0} : [\eta] \mapsto [\hat{\gamma},\hat{b}] \tag{19}$$

as follows. Given $\eta \in \mathcal{Y}_{w_0}$ (i.e. $\eta \in \Omega^{0,1}(\Sigma, w_0^*TQ)$ with $d\pi(w_0)\eta \equiv 0$) choose two vector fields ξ_u along $u_0 := w_0|\Delta$ and ξ_v along $v_0 := w_0|\Omega$ that project each to a constant vector $\hat{b}_u := d\pi_B(u_0)\xi_u$ and $\hat{b}_v := d\pi_B(v_0)\xi_v$ in $T_{b_0}B$ and satisfy

$$D_{u_0,b_0}(\xi_u,\hat{b}_u) = \eta|\Delta, \qquad D_{v_0,b_0}(\xi_v,\hat{b}_v) = \eta|\Omega. \tag{20}$$

The existence of ξ_u and ξ_v (with $\hat{b}_u = \hat{b}_v = 0$) can be proved by imposing a Lagrangian boundary condition with high Maslov index. Define

$$\hat{\gamma} := \xi_u|\Gamma - \xi_v|\Gamma : \Gamma \to \gamma_0^*TQ, \qquad \hat{b} := \hat{b}_u - \hat{b}_v. \tag{21}$$

Given η, the pair $(\xi_u|\Gamma,\hat{b}_u)$ is well defined up to an additive vector in $T_{(\gamma_0,b_0)}\mathcal{U}_0$ and $(\xi_v|\Gamma,\hat{b}_v)$ up to an additive vector in $T_{(\gamma_0,b_0)}\mathcal{V}_0$. Moreover, if $\eta \in \mathrm{im}D_{w_0,b_0}$ then there is a pair $(\xi,\hat{b}) \in \mathcal{X}_{w_0,b_0}$ such that $D_{w_0,b_0}(\xi,\hat{b}) = \eta$ (see Definition 5.2) and we may choose $\xi_u := \xi|\Delta$ and $\xi_v := \xi|\Omega$ in (20) so that $\hat{b}_u = \hat{b}_v = \hat{b}$ and $(\hat{\gamma},\hat{b}) = (0,0)$ in (21). This shows that the equivalence class of $(\hat{\gamma},\hat{b})$ is independent of the choice of ξ_u and ξ_v and depends only on the equivalence class of η in $\mathcal{Y}_{w_0}/\mathrm{im}D_{w_0,b_0}$. Hence the map (19) is well defined.

We prove that (19) is injective. Let $\eta \in \mathcal{Y}_{w_0}$ be given, choose $(\xi_u,\hat{b}_u)$ and $(\xi_v,\hat{b}_v)$ so as to satisfy (20), and define $(\hat{\gamma},\hat{b}) \in T_{(\gamma_0,b_0)}\mathcal{W}_0$ by (21). Assume

$(\hat{\gamma}, \hat{b}) \in T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0$. Then there are vector fields ξ'_u along u_0 and ξ'_v along v_0, as well as tangent vectors $\hat{b}'_u, \hat{b}'_v \in T_{b_0}B$ such that $d\pi(u_0)\xi'_u \equiv \hat{b}'_u$, $d\pi(u_0)\xi'_v \equiv \hat{b}'_v$, and

$$D_{u_0,b_0}(\xi'_u, \hat{b}'_u) = 0, \qquad D_{v_0,b_u}(\xi'_v, \hat{b}'_v) = 0,$$

$$\hat{\gamma} = \xi'_u|\Gamma - \xi'_v|\Gamma, \qquad \hat{b} := \hat{b}'_u - \hat{b}'_v.$$

Hence, replacing $(\xi_u, \hat{b}_u)$ by $(\xi_u - \xi'_u, \hat{b}_u - \hat{b}'_u)$ and $(\xi_v, \hat{b}_v)$ by $(\xi_v - \xi'_v, \hat{b}_v - \hat{b}'_v)$, we may assume without loss of generality that $\hat{\gamma} = 0$ and $\hat{b} = 0$ in (21). Thus $\xi_u|\Gamma = \xi_v|\Gamma$. In other words, there is a continuous vector field ξ along w_0 such that $\xi|\Delta = \xi_u$ and $\xi|\Omega = \xi_v$. Since $D_{w_0}\xi = \eta$ is smooth it follows from elliptic regularity that ξ is smooth and hence $\eta \in \mathrm{im} D_{w_0,b_0}$. This shows that the map (19) is injective as claimed.

That (19) is surjective follows from the next two assertions.

(a) Each element of the quotient space $T_{(\gamma_0,b_0)}\mathcal{W}_0/(T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0)$ can be represented by a smooth vertical vector field along γ_0.

(b) For every smooth vertical vector field $\hat{\gamma}$ along γ_0 there exist smooth vertical vector fields ξ_u along u_0 and ξ_v along v_0 such that $\hat{\gamma} = \xi_u|\Gamma - \xi_v|\Gamma$ and the $(0,1)$-form η along w_0 defined by $\eta|\Delta := D_{u_0,b_0}(\xi_u, \hat{b}_u)$ and $\eta|\Omega := D_{u_0,b_0}(\xi_v, \hat{b}_v)$ is smooth.

To prove (b) choose holomorphic coordinates $\tau + i\theta \in [-\delta, \delta] + i\mathbb{R}/2\pi\mathbb{Z}$ near each component of Γ so that that component is $\{\tau = 0\}$ and Ω is $\{\tau \geq 0\}$. Choose a complex trivialization of the vertical tangent bundle over this annulus. In this trivialization a $(0,1)$-form over the annulus has the form $\frac{1}{2}(\eta\, ds - i\eta\, dt)$ with $\eta : [-\delta, \delta] \times S^1 \to \mathbb{C}$ and the operator $D = D_{w_0,b_0}$ has the form

$$D\xi = \partial_\tau\xi + i\partial_\theta\xi + S\xi$$

where $S : [-\delta, \delta] \times S^1 \to \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ is a smooth map. We seek three smooth sections ξ_u over $\{\tau \leq 0\}$, ξ_v over $\{\tau \geq 0\}$, and η over the whole annulus (of the vertical tangent bundle) such that

$$\eta|\{\tau \leq 0\} = D\xi_u, \qquad \eta|\{\tau \geq 0\} = D\xi_v,$$

and

$$\xi_v(0,\theta) - \xi_u(0,\theta) = \xi_0(\theta) := -\hat{\gamma}(\theta). \tag{22}$$

We take $\xi_u := 0$ and ξ_v in the form

$$\xi_v(\tau,\theta) = \sum_{k=0}^{\infty} \frac{c_k(\tau,\theta)}{k!}\tau^k, \qquad c_k(\tau,\theta) = \begin{cases} c_k(0,\theta) & \text{for } 0 \leq \tau \leq \delta_k, \\ 0 & \text{for } \tau \geq 2\delta_k, \end{cases}$$

where $2\delta_k < \delta$ and each c_k is smooth on $[0, \delta] \times S^2$. To satisfy condition (22) we must choose $c_0(0,\theta) = \xi_0(\theta)$. To make η continuous we must choose ξ_v such that

$$\partial_\tau\xi_v(0,\theta) + i\partial_\theta\xi_v(0,\theta) + S(0,\theta)\xi_v(0,\theta) = 0$$

and hence

$$c_1(0,\theta) = \partial_\tau \xi_v(0,\theta) = -i\partial_\theta \xi_0(\theta) - S(0,\theta)\xi_0(\theta).$$

More generally, once $c_0, \ldots, c_k$ have been chosen to make η of class C^{k-1} the function $\theta \mapsto c_{k+1}(0,\theta)$ is uniquely determined by ξ_0 and the condition that η be of class C^k. Finally choose δ_k converging sufficiently rapidly to zero and define $c_k(\tau,\theta) := \beta(\tau/\delta_k)c_k(0,\theta)$ for a suitable cutoff function β so that the series for ξ_v converges in the C^ℓ-norm for every ℓ. Note that our argument follows the construction, due to Emile Borel, of a smooth function with a prescribed Taylor series at a point.

To prove (a) we first observe that every smooth pair $(\hat\gamma, \hat b) \in T_{(\gamma_0,b_0)}\mathcal{W}_0$ is equivalent in the quotient $T_{(\gamma_0,b_0)}\mathcal{W}_0/(T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0)$ to a smooth vertical vector field. Namely, choose any vector field ξ' along v_0 that projects to $\hat b$, and choose a vertical vector field ξ'' along v_0 such that $\xi := \xi' + \xi''$ satisfies $D_{v_0}\xi = 0$ (as we did in the proof of (b)). Then $(\xi|\Gamma, \hat b) \in T_{(\gamma_0,b_0)}\mathcal{V}_0$ and hence $(\hat\gamma - \xi|\Gamma, 0)$ is a vertical vector field equivalent to $(\hat\gamma, \hat b)$. Now consider the subspace of all elements of the quotient $T_{(\gamma_0,b_0)}\mathcal{W}_0/(T_{(\gamma_0,b_0)}\mathcal{U}_0 + T_{(\gamma_0,b_0)}\mathcal{V}_0)$ that can be represented by smooth vertical vector fields. By what we have just proved, this subspace is dense and, by (b), it is finite dimensional. Hence this subspace must be equal to the entire quotient and this proves (a). Thus we have proved (a) and (b) and hence the operator (19) is surjective. This proves (iii). Part (iv) is an immediate consequence of (ii) and (iii). This completes the proof of Theorem 9.5. □

Remark 9.6. The strategy for the proof of the universal unfolding theorem is to assign to each unfolding $(\pi_A : P \to A, R_*, a_0)$ of the marked nodal Riemann surface (Σ, s_*, ν, j) a family of Hilbert submanifolds $\mathcal{U}_a, \mathcal{V}_a \subset \mathcal{W}_a$ as in 9.3 parametrized by $a \in A$. Transversality will then imply that for each a near a_0 there is a unique intersection point $(\gamma_a, b_a) \in \mathcal{U}_a \cap \mathcal{V}_a$ near (γ_0, b_0). Then the fiber isomorphisms $f_a : P_a \to Q_{b_a}$ determined by the γ_a as in Lemma 9.4 will fit together to determine the required morphisms $P \to Q$ of nodal families. The key point is to show that the submanifolds $\mathcal{U}_a$ fit together to form a complex submanifold

$$\mathcal{U} := \bigsqcup_{a\in A} \mathcal{U}_a \quad\subset\quad \mathcal{W} := \bigsqcup_{a\in A} \mathcal{W}_a$$

(see Theorem 11.9 below). We begin by studying a local model near a given nodal point in the next section.

10 The local model

10.1. Consider the **standard node** defined as the map

$$N \to \mathrm{int}(\mathbb{D}) : (x,y) \mapsto xy, \quad N := \{(x,y) \in \mathbb{D}^2 : |xy| < 1\}.$$

For $a \in \mathrm{int}(\mathbb{D})$ and $b \in \mathbb{C}$ denote

$$N_a := \{(x,y) \in \mathbb{D}^2 : xy = a\}, \qquad Q_b := \{(x,y) \in \mathbb{C}^2 : xy = b\}.$$

We study the set of all quadruples (a, ξ, η, b) where $a, b \in \mathbb{C}$ are close to 0 and

$$(\xi, \eta) : N_a \to Q_b$$

is a holomorphic map. If $a \neq 0$ this means $\xi(z), \eta(z)$ are holomorphic functions on the annulus $|a| \leq |z| \leq 1$ that are close to the identity, and satisfy the condition

$$xy = a \implies \xi(x)\eta(y) = b \tag{23}$$

for $|a| \leq |x| \leq 1$ and $|a| \leq |y| \leq 1$. If $a \neq 0$, this condition implies that $b \neq 0$. When $a = 0$, the functions ξ and η are defined on the closed unit disk and vanish at the origin; hence $b = 0$.

10.2. Fix $s > 1/2$ and let $H^s = H^s(S^1, \mathbb{C}^n)$ be the Sobolev space in B.1. We think of the elements of H^s as power series

$$\zeta(z) = \sum_{n \in \mathbb{Z}} \zeta_n z^n. \tag{24}$$

For $r > 0$ the **rescaling map** $z \mapsto rz$ maps the unit circle to the circle of radius r. Denote by ζ_r the result of conjugating ζ by this map, i.e.

$$\zeta_r(z) := r^{-1}\zeta(rz).$$

The norm $\|\zeta_r\|_s$ is finite if and only if the series ζ converges to an H^s function on the circle of radius r.

10.3. For $\delta > 0$ define the open set $\mathcal{W}_\delta \subset \mathbb{C} \times H^s \times H^s \times \mathbb{C}$ by

$$\mathcal{W}_\delta := \left\{(a, \xi, \eta, b) : \|\xi - \mathrm{id}\|_s < \delta,\ \|\eta - \mathrm{id}\|_s < \delta,\ |a| < \delta\right\}.$$

Define $\mathcal{U}_\delta \subset \mathcal{W}_\delta$ to be the set of those quadruples $(a, \xi, \eta, b) \in \mathcal{W}_\delta$ which satisfy (23). More precisely if $(a, \xi, \eta, b) \in \mathcal{W}_\delta$, then for $a \neq 0$ we have

$$(a, \xi, \eta, b) \in \mathcal{U}_\delta \qquad \iff \qquad \begin{cases} \left\|\xi_{|a|}\right\|_s < \infty,\ \left\|\eta_{|a|}\right\|_s < \infty, \text{ and} \\ \\ \xi(x)\eta(ax^{-1}) = b \text{ for } |a| \leq |x| \leq 1 \end{cases}$$

while for $a = 0$ we have

$$(0, \xi, \eta, b) \in \mathcal{U}_\delta \qquad \iff \qquad b = 0 \text{ and } \xi(0) = \eta(0) = 0.$$

Thus $\mathcal{U}_\delta$ is the space of (boundary values of) local holomorphic fiber isomorphisms in the standard model. The main result of this section is that $\mathcal{U}_\delta$ is a manifold:

Theorem 10.4. *Let $s > 1/2$. Then, for $\delta > 0$ sufficiently small, the set $\mathcal{U}_\delta$ is a complex submanifold of the open set $\mathcal{W}_\delta \subset \mathbb{C} \times H^s \times H^s \times \mathbb{C}$.*

The proof occupies the rest of this section. Using the Hardy space decomposition defined in 10.5 we formulate three propositions which define a map $\mathcal{T}$ whose graph lies in $\mathcal{U}_\delta$. We then prove six lemmas, then we prove the three propositions, and finally we prove that the graph of $\mathcal{T}$ is exactly equal to $\mathcal{U}_\delta$.

10.5. A holomorphic function $\zeta(z)$ defined on an annulus centered at the origin has a Laurent expansion of the form (24). We write $\zeta = \zeta_+ + \zeta_-$ where

$$\zeta_+(z) := \sum_{n>0} \zeta_n z^n, \qquad \zeta_-(z) := \sum_{n\le 0} \zeta_n z^n.$$

For $r > 0$ and $s > 1/2$ introduce the norm

$$\|\zeta\|_{r,s} := \sqrt{\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} r^{2n-2} |\zeta_n|^2}$$

so that ζ_+ converges inside the circle of radius r if $\|\zeta_+\|_{r,s} < \infty$ and ζ_- converges outside the circle of radius r if $\|\zeta_-\|_{r,s} < \infty$. Let

$$H^s_r := \{\zeta : \|\zeta\|_{r,s} < \infty\}$$

and $H^s_{r,\pm}$ be the subspace of those ζ for which $\zeta = \zeta_\pm$ so we have the Hardy space decomposition

$$H^s_r = H^s_{r,+} \oplus H^s_{r,-}.$$

Then $H^s = H^s_1$ and $\|\cdot\|_s = \|\cdot\|_{1,s}$. We abbreviate

$$H^s_\pm := H^s_{1,\pm}.$$

We view the ball of radius δ about id in the Hilbert space H^s_r as a space of H^s-maps from the circle of radius r to a neighborhood of this circle; the norm on H^s_r is defined so that conjugation by the rescaling map $z \mapsto rz$ induces an isometry $H^s_r \to H^s : \zeta \to \zeta_r$, i.e.

$$\|\zeta\|_{r,s} = \|\zeta_r\|_s, \qquad \zeta_r(z) := r^{-1}\zeta(rz). \tag{25}$$

Proposition 10.6 (Existence)**.** *For every $s > 1/2$ there are positive constants δ and c such that the following holds. If $a \in \mathbb{C}$ with $0 < r := \sqrt{|a|} \le 1$ and $\xi_+, \eta_+ \in H^s_+$ satisfy*

$$\|\xi_+ - \mathrm{id}\|_s < \delta, \qquad \|\eta_+ - \mathrm{id}\|_s < \delta \tag{26}$$

then there exists a triple $(b, \xi_-, \eta_-) \in \mathbb{C} \times H^s_{r^2,-} \times H^s_{r^2,-}$ such that $\xi := \xi_+ + \xi_-$ and $\eta := \eta_+ + \eta_-$ satisfy the equation

$$\xi(x)\eta(ax^{-1}) = b \tag{27}$$

for $r^2 \le |x| \le 1$ and

$$\left|ba^{-1} - \xi_1\eta_1\right| + \|\xi_-\|_{r,s} + \|\eta_-\|_{r,s} \le 2cr\big(\|\xi_+ - \mathrm{id}\|_s + \|\eta_+ - \mathrm{id}\|_s\big). \tag{28}$$

Proposition 10.7 (Uniqueness)**.** *For every $s > 1/2$ there exist positive constants δ and ε such that the following holds. If $a, b, b' \in \mathbb{C}$, $\xi_+, \eta_+ \in H^s_+$ and $\xi_-, \eta_-, \xi'_-, \eta'_- \in H^s_{r,-}$ with $0 < r := \sqrt{|a|} < 1$ satisfy (26) and*

$$\|\xi_-\|_{r,s} < \varepsilon, \qquad \|\eta_-\|_{r,s} < \varepsilon, \qquad \sup_{|x|=r} \left|\xi'_-(x)\right| < r\varepsilon, \qquad \sup_{|y|=r} \left|\eta'_-(y)\right| < r\varepsilon,$$

and if $(a, \xi := \xi_+ + \xi_-, \eta := \eta_+ + \eta_-, b)$ and $(a, \xi' := \xi_+ + \xi'_-, \eta' := \eta_+ + \eta'_-, b')$ satisfy (27) for $|x| = r$ then $(\xi_-, \eta_-, b) = (\xi'_-, \eta'_-, b')$.

10.8. Fix a constant $s > 1/2$. Choose positive constants δ and ε such that Proposition 10.7 holds. Shrinking δ if necessary we may assume that Proposition 10.6 holds with the same constant δ and a suitable constant $c > 0$. Let

$$H^s_+(\mathrm{id}, \delta) := \{\zeta \in H^s_+ : \|\zeta - \mathrm{id}\|_s < \delta\}$$

and define

$$\mathcal{T} : \mathbb{D} \times H^s_+(\mathrm{id}, \delta) \times H^s_+(\mathrm{id}, \delta) \to \mathbb{C} \times H^s_- \times H^s_-$$

by the conditions that $\mathcal{T}(a, \xi_+, \eta_+) = (b, \xi_-, \eta_-)$ is the triple constructed in Proposition 10.6 for $a \neq 0$ and

$$\mathcal{T}(0, \xi_+, \eta_+) := (0, 0, 0).$$

(In defining $\mathcal{T}$ we used the fact that $H^s_{r,-} \subset H^s_-$ for $r \leq 1$.)

Proposition 10.9. *The map $\mathcal{T}$ is continuous. It is holomorphic for $|a| < 1$.*

Lemma 10.10 (A priori estimates)**.** *There is a constant $c > 0$ such that, for $\delta > 0$ sufficiently small, the following holds. If $(a, \xi, \eta, b) \in \mathcal{U}_\delta$ and $a \neq 0$ then*

$$\left|ba^{-1} - 1\right| < c\delta, \qquad \sup_{|a| \leq |x| \leq 1} \left|\xi(x)x^{-1} - 1\right| \leq c\delta, \qquad \sup_{|a| \leq |y| \leq 1} \left|\eta(y)y^{-1} - 1\right| \leq c\delta.$$

Proof. Rewrite $\xi(x)\eta(ax^{-1}) = b$ as

$$a\frac{\xi(x)}{x^2} = ab\frac{x^{-2}}{\eta(ax^{-1})}.$$

Using the substitution $y = ax^{-1}$, $dy = -ax^{-2}\,dx$ we get

$$a \oint_{|x|=1} \frac{\xi(x)\,dx}{x^2} = b \oint_{|x|=1} \frac{ax^{-2}\,dx}{\eta(ax^{-1})} = b \oint_{|y|=|a|} \frac{dy}{\eta(y)} = b \oint_{|y|=1} \frac{dy}{\eta(y)}$$

where all the contour integrals are counter clockwise. By the Sobolev embedding theorem, there is a constant c such that $|\zeta(z)| \leq c\|\zeta\|_s$ for $\zeta \in H^s$ and $|z| = 1$. This gives the estimate

$$\left|\frac{\xi(x)}{x^2} - \frac{1}{x}\right| = |\xi(x) - x| \leq c\|\xi - \mathrm{id}\|_s \leq c\delta$$

for $|x| = 1$. If $\|\eta - \mathrm{id}\|_s < \delta \le 1/2c$ then, by the Sobolev embedding theorem again, $|\eta(y) - y| < 1/2$ and so $|\eta(y)| > 1/2$ for $|y| = 1$. Hence

$$\left| \frac{1}{\eta(y)} - \frac{1}{y} \right| = \frac{|\eta(y) - y|}{|\eta(y)|} \le 2c\|\eta - \mathrm{id}\|_s \le 2c\delta$$

for $|y| = 1$. Hence the contour integrals are within $4\pi c\delta$ of $2\pi i$ and so, enlarging c, b/a is within $c\delta$ of 1 as required.

By symmetry the third inequality follows from the second; we prove the second. Using the Sobolev inequality we have

$$\sup_{|x|=1} \left| \frac{\xi(x)}{x} - 1 \right| \le c\delta, \qquad \sup_{|y|=1} \left| \frac{\eta(y)}{y} - 1 \right| \le c\delta.$$

Now let $y := ax^{-1}$ and $|x| = |a|$. Then $|y| = 1$ and

$$\frac{\xi(x)}{x} - 1 = \frac{b}{a} \frac{y - \eta(y)}{\eta(y)} + \frac{b}{a} - 1.$$

Hence

$$\sup_{|x|=|a|} \left| \frac{\xi(x)}{x} - 1 \right| \le \left| \frac{b}{a} \right| \sup_{|y|=1} \left| \frac{y - \eta(y)}{\eta(y)} \right| + \left| \frac{b}{a} - 1 \right| \le c\delta.$$

By the maximum principle this implies

$$\sup_{|a| \le |x| \le 1} \left| \frac{\xi(x)}{x} - 1 \right| \le c\delta.$$

This proves the lemma. □

10.11. The proofs of Propositions 10.6, 10.7, and 10.9 are based on a version of the implicit function theorem for the map

$$\mathcal{F}_r : \mathbb{C} \times H^s_r \times H^s_r \to H^s$$

defined by

$$\mathcal{F}_r(\lambda, \xi, \eta)(z) := r^{-2}\xi(rz)\eta(rz^{-1}) - \lambda \tag{29}$$

for $|z| = 1$ and $r > 0$. The zeros of $\mathcal{F}_r$ are solutions of (27) with $a = r^2$ and $b = \lambda a$. Note that $\mathcal{F}_r(1, \mathrm{id}, \mathrm{id}) = 0$ for every $r > 0$. The differential of $\mathcal{F}_r$ at the point $(1, \mathrm{id}, \mathrm{id})$ will be denoted by

$$\mathcal{D}_r := d\mathcal{F}_r(1, \mathrm{id}, \mathrm{id}) : \mathbb{C} \times H^s_r \times H^s_r \to H^s.$$

Thus

$$\mathcal{D}_r(\hat{\lambda}, \hat{\xi}, \hat{\eta})(z) = r^{-1}z^{-1}\hat{\xi}(rz) + r^{-1}z\hat{\eta}(rz^{-1}) - \hat{\lambda}.$$

We shall need six lemmata. They are routine consequences of well known facts and rescaling. To ease the exposition we relegate the proofs of the first five to the end of the section and omit the proof of the sixth entirely. (The proof of the sixth is just the proof of the implicit function theorem keeping track of the estimates.)

Lemma 10.12 (Sobolev Estimate)**.** *Denote by $\mathbb{A}(r,R)\subset\mathbb{C}$ the closed annulus $r\le|z|\le R$. For every $s>1/2$ there is a constant $c>0$ such that*

$$\|\zeta\|_{L^\infty(\mathbb{A}(r,R))}\le c\big(r\|\zeta_-\|_{r,s}+R\|\zeta_+\|_{R,s}\big)$$

for all r and R and every holomorphic function $\zeta(z)$ on the annulus $r\le|z|\le R$.

Proof. This follows from Lemma A.2 and the maximum principle. □

Lemma 10.13 (Product Estimate)**.** *For every $s>1/2$ there is a positive constant C such that, for any two functions $\xi,\eta\in H^s$, we have*

$$\|\xi\eta\|_s\le C\|\xi\|_s\|\eta\|_s,\qquad \|\xi\eta\|_s\le C\left(\|\xi\|_s\|\eta\|_{L^\infty(S^1)}+\|\xi\|_{L^\infty(S^1)}\|\eta\|_s\right).$$

Proof. Lemma A.3 and Corollary A.11. □

Lemma 10.14 (Linear Estimate)**.** *For $\hat\xi_-,\hat\eta_-\in H^s_{r,-}$ and $\hat\lambda\in\mathbb{C}$ we have*

$$\|\hat\xi_-\|^2_{r,s}+\|\hat\eta_-\|^2_{r,s}+|\hat\lambda|^2\le\|\mathcal{D}_r(\hat\lambda,\hat\xi_-,\hat\eta_-)\|^2_s.$$

Proof. The formula

$$\mathcal{D}_r(\hat\lambda,\hat\xi_-,\hat\eta_-)(z)=r^{-1}z^{-1}\hat\xi_-(rz)+r^{-1}z\hat\eta_-(rz^{-1})-\hat\lambda$$

shows that

$$\mathcal{D}_r(\hat\lambda,\hat\xi_-,\hat\eta_-)=\mathcal{D}_1(\hat\lambda,(\hat\xi_-)_r,(\hat\eta_-)_r).$$

Hence, by (25) it suffices to prove the lemma for $r=1$. Then

$$\mathcal{D}_1(\hat\lambda,\hat\xi_-,\hat\eta_-)(z)\quad=\quad\sum_{n<0}\hat\xi_{n+1}z^n-\hat\lambda+\sum_{n>0}\hat\eta_{1-n}z^n$$

so

$$\begin{aligned}\left\|\mathcal{D}_1(\hat\lambda,\hat\xi_-,\hat\eta_-)\right\|^2_s &= \sum_{n<0}(1+|n|)^{2s}|\hat\xi_{n+1}|^2+|\hat\lambda|^2+\sum_{n>0}(1+|n|)^{2s}|\hat\eta_{1-n}|^2\\ &= \sum_{n\le0}(2+|n|)^{2s}|\hat\xi_n|^2+|\hat\lambda|^2+\sum_{n\le0}(2+|n|)^{2s}|\hat\eta_n|^2\\ &\ge \|\hat\xi_-\|^2_s+|\hat\lambda|^2+\|\hat\eta_-\|^2_s.\end{aligned}$$

This proves the lemma. □

Lemma 10.15 (Approximate Solution)**.** *For every $s>1/2$ there is a constant $c>0$ such that*

$$\|\mathcal{F}_r(\xi_1\eta_1,\xi_+,\eta_+)\|_s\le cr\left(\|\xi_+-\mathrm{id}\|_s+\|\eta_+-\mathrm{id}\|_s\right)$$

for every pair $\xi_+,\eta_+\in H^s_+$ with $\|\xi_+\|_s\le1$, $\|\eta_+\|_s\le1$, and every $r\in(0,1]$.

Proof. The constant is $c = 4\sqrt{3}C$ where C is the constant of Lemma 10.13. We first prove the inequality

$$\left\|\mathcal{F}_1(\xi_1\eta_1, \xi_+, \eta_+)\right\|_s \leq 2\sqrt{3}C\left(\left\|\xi_+ - \xi_1\mathrm{id}\right\|_s + \left\|\eta_+ - \eta_1\mathrm{id}\right\|_s\right). \tag{30}$$

Since

$$\begin{aligned}\mathcal{F}_1(\xi_1\eta_1, \xi_+, \eta_+)(z) &= \xi_+(z)\eta_+(z^{-1}) - \xi_1\eta_1 \\ &= \sum_{k\neq 0}\left(\sum_{n>0}\xi_{n+k}\eta_n\right)z^k + \sum_{n>1}\xi_n\eta_n\end{aligned}$$

we have

$$\begin{aligned}\left\|\mathcal{F}_1(\xi_1\eta_1, \xi_+, \eta_+)\right\|_s^2 &= \sum_{k\neq 0}(1+|k|)^{2s}\left|\sum_{n>0}\xi_{n+k}\eta_n\right|^2 + \left|\sum_{n>1}\xi_n\eta_n\right|^2 \\ &\leq \sum_{k>0}(1+k)^{2s}\left(\sum_{n>0}|\xi_{n+k}\eta_n|\right)^2 + \left(\sum_{n>1}|\xi_n\eta_n|\right)^2 \\ &\quad + \sum_{k>0}(1+k)^{2s}\left(\sum_{n>k}|\xi_{n-k}\eta_n|\right)^2 \\ &\leq 3C^2\left(\left\|\xi_+ - \xi_1\mathrm{id}\right\|_s^2\left\|\eta_+\right\|_s^2 + \left\|\xi_+\right\|_s^2\left\|\eta_+ - \eta_1\mathrm{id}\right\|_s^2\right).\end{aligned}$$

The last inequality follows from Lemma 10.13; note that each sum omits either ξ_1 or η_1 or both. The inequality (30) follows by taking the square root of the last estimate and using the fact that $\|\xi_+\|_s \leq 2$ and $\|\eta_+\|_s \leq 2$.

The formula

$$\mathcal{F}_r(\xi_1\eta_1, \xi_+, \eta_+)(z) = r^{-2}\xi_+(rz)\eta_+(rz^{-1}) - \xi_1\eta_1$$

shows that

$$\mathcal{F}_r(\xi_1\eta_1, \xi_+, \eta_+) = \mathcal{F}_1(\xi_1\eta_1, (\xi_+)_r, (\eta_+)_r).$$

Note that the operation $\xi \mapsto \xi_r$ leaves the coefficient ξ_1 unchanged. Hence, by (30), we have

$$\begin{aligned}\left\|\mathcal{F}_r(\xi_1\eta_1, \xi_+, \eta_+)\right\|_s &= \left\|\mathcal{F}_1(\xi_1\eta_1, (\xi_+)_r, (\eta_+)_r)\right\|_s \\ &\leq 2\sqrt{3}C\big(\left\|(\xi_+)_r - \xi_1\mathrm{id}\right\|_s + \left\|(\eta_+)_r - \eta_1\mathrm{id}\right\|_s\big) \\ &= 2\sqrt{3}C\big(\left\|\xi_+ - \xi_1\mathrm{id}\right\|_{r,s} + \left\|\eta_+ - \eta_1\mathrm{id}\right\|_{r,s}\big) \\ &\leq 2\sqrt{3}Cr\big(\left\|\xi_+ - \xi_1\mathrm{id}\right\|_s + \left\|\eta_+ - \eta_1\mathrm{id}\right\|_s\big) \\ &\leq 4\sqrt{3}Cr\big(\left\|\xi_+ - \mathrm{id}\right\|_s + \left\|\eta_+ - \mathrm{id}\right\|_s\big).\end{aligned}$$

This proves the lemma. □

Lemma 10.16 (Quadratic Estimate)**.** *For every $s > 1/2$ there is a constant $c > 0$ such that*

$$\left\|(d\mathcal{F}_r(\lambda,\xi,\eta) - \mathcal{D}_r)\,(\hat\lambda,\hat\xi,\hat\eta)\right\|_s \le c\big(\|\eta - \mathrm{id}\|_{r,s}\|\hat\xi\|_{r,s} + \|\xi - \mathrm{id}\|_{r,s}\|\hat\eta\|_{r,s}\big)$$

for all $\xi,\eta,\hat\xi,\hat\eta \in H^s_r$ and $\lambda,\hat\lambda \in \mathbb{C}$.

Proof. We have

$$d\mathcal{F}_r(\lambda,\xi,\eta)(\hat\lambda,\hat\xi,\hat\eta)(z) = r^{-2}\hat\xi(rz)\eta(rz^{-1}) + r^{-2}\xi(rz)\hat\eta(rz^{-1}) - \hat\lambda$$

and hence

$$(d\mathcal{F}_r(\lambda,\xi,\eta) - \mathcal{D}_r)\,(\hat\lambda,\hat\xi,\hat\eta)(z) = r^{-2}\hat\xi(rz)(\eta-\mathrm{id})(rz^{-1}) + r^{-2}(\xi-\mathrm{id})(rz)\hat\eta(rz^{-1}).$$

So the result follows from Lemma 10.13 with $c = C$. □

Lemma 10.17 (Inverse Function Theorem)**.** *Let $f : U \to V$ be a smooth map between Banach spaces and $D : U \to V$ be a Banach space isomorphism. Let $u_0 \in U$ and suppose that there is a constant $\rho > 0$ such that*

$$\left\|D^{-1}\right\| \le 1, \qquad \|f(u_0)\|_V \le \frac{\rho}{2} \tag{31}$$

and, for every $u \in U$,

$$\|u - u_0\|_U \le \rho \qquad \Longrightarrow \qquad \|df(u) - D\| \le \frac{1}{2}. \tag{32}$$

Then there is a unique element $u \in U$ such that

$$\|u - u_0\|_U \le \rho, \qquad f(u) = 0.$$

Moreover, $\|u - u_0\|_U \le 2\,\|f(u_0)\|_V$.

Proof. Standard. □

Proof of Proposition 10.6. Throughout we fix a constant $s > 1/2$ and a constant $c \ge 1$ such that the assertions of Lemmata 10.12, 10.15 and 10.16 hold with these constants s and c. Choose positive constants ε, ρ, and δ such that

$$3c\varepsilon < \frac{1}{2}, \qquad c\sqrt{2\delta^2 + \rho^2} \le \frac{1}{2}, \qquad 2c\delta \le \frac{\varepsilon}{2}, \qquad \rho := \sqrt{3}\varepsilon. \tag{33}$$

We prove the assertion with these constants c and δ.

Assume first that a is a positive real number and denote $r := \sqrt{a}$. Fix a pair $(\xi_+,\eta_+) \in H^s_+ \times H^s_+$ satisfying (26). Let

$$U := \mathbb{C} \times H^s_{r,-} \times H^s_{r,-}, \qquad V := H^s$$

and consider the map $f : U \to V$ defined by

$$f(u) := \mathcal{F}_r(\lambda, \xi_+ + \xi_-, \eta_+ + \eta_-), \qquad u := (\lambda, \xi_-, \eta_-). \tag{34}$$

Let $D := \mathcal{D}_r : U \to V$ and $u_0 := (\xi_1\eta_1, 0, 0)$. Then, by Lemma 10.14,

$$\left\|D^{-1}\right\|_{\mathcal{L}(V,U)} \le 1 \tag{35}$$

and, by Lemma 10.15 and (33),

$$\|f(u_0)\|_V \le cr\left(\|\xi_+ - \mathrm{id}\|_s + \|\eta_+ - \mathrm{id}\|_s\right) \le 2c\delta \le \frac{\varepsilon}{2} \le \frac{\rho}{2}. \tag{36}$$

In this notation the operator $df(u) - D : U \to V$ is the restriction of the operator $d\mathcal{F}_r(\lambda, \xi, \eta) - \mathcal{D}_r : \mathbb{C} \times H^s_r \times H^s_r \to H^s$ to the subspace U, so by Lemma 10.16 we have

$$\|df(u) - D\|_{\mathcal{L}(U,V)} \le c\sqrt{\|\eta - \mathrm{id}\|^2_{r,s} + \|\xi - \mathrm{id}\|^2_{r,s}}$$

for $u = (\lambda, \xi_-, \eta_-) \in U$ and $\xi := \xi_+ + \xi_-$ and $\eta := \eta_+ + \eta_-$. Note that $\|\zeta\|_{r,s} \le \|\zeta\|_s$ for $\zeta \in H^s_+$ and $0 < r \le 1$. Hence

$$\|df(u) - D\|_{\mathcal{L}(U,V)} \le c\sqrt{\|\xi_+ - \mathrm{id}\|^2_s + \|\eta_+ - \mathrm{id}\|^2_s + \|\xi_-\|^2_{r,s} + \|\eta_-\|^2_{r,s}}$$

for $u = (\lambda, \xi_-, \eta_-) \in U$. Since $\|\xi_+ - \mathrm{id}\|_s \le \delta$ and $\|\eta_+ - \mathrm{id}\|_s \le \delta$ we have

$$\|u - u_0\|_U \le \rho \qquad \Longrightarrow \qquad \|df(u) - D\|_{\mathcal{L}(U,V)} \le c\sqrt{2\delta^2 + \rho^2} \le \frac{1}{2} \tag{37}$$

for every $u := (\lambda, \xi_-, \eta_-) \in U$. Here we have used (33).

It follows from (35), (36), and (37) that the assumptions of Lemma 10.17 are satisfied. Hence there is a unique point $u \in U$ such that

$$\|u - u_0\|_U \le \rho, \qquad f(u) = 0,$$

and this unique point satisfies

$$\|u - u_0\|_U \le 2\,\|f(u_0)\|_V \le \varepsilon. \tag{38}$$

Thus, for every $(\xi_+, \eta_+) \in H^s_+ \times H^s_+$ satisfying (26), we have found a unique triple $(\lambda, \xi_-, \eta_-) \in U$ such that $\xi := \xi_+ + \xi_-$ and $\eta := \eta_+ + \eta_-$ satisfy

$$\mathcal{F}_r(\lambda, \xi, \eta) = 0, \qquad \|\xi_-\|_{r,s} \le \varepsilon, \qquad \|\eta_-\|_{r,s} \le \varepsilon, \qquad |\lambda - \xi_1\eta_1| \le \varepsilon.$$

That the quadruple (a, ξ, η, b) also satisfies the estimate (28) follows from (36) and (38).

Next we prove that this quadruple (a, ξ, η, b) satisfies $\xi(z) \ne 0$ and $\eta(z) \ne 0$ for $r \le |z| \le 1$. To see this note that

$$\|(\zeta/\mathrm{id})_+\|_s = \sqrt{\sum_{n\ge 2} n^{2s}\,|\zeta_n|^2} \le \|\zeta_+\|_s$$

and

$$r\,\|(\zeta/\mathrm{id})_-\|_{r,s} = \sqrt{|\zeta_1|^2 + \sum_{n\le 0}(2-n)^{2s} r^{2n-2}\,|\zeta_n|^2} \le \|\zeta_+\|_s + 2\,\|\zeta_-\|_{r,s}\,.$$

Hence

$$\begin{aligned}
\sup_{r\le|x|\le 1}\left|\xi(x)x^{-1}-1\right| &\le c\left(\|(\xi/\mathrm{id}-1)_+\|_s + r\,\|(\xi/\mathrm{id}-1)_-\|_{r,s}\right)\\
&\le 2c\left(\|\xi_+-\mathrm{id}\|_s+\|\xi_-\|_{r,s}\right)\\
&\le 2c(\delta+\varepsilon)\\
&\le 1/2.
\end{aligned}$$

Here the first inequality follows from Lemma 10.12 and the last uses the fact that $2c\varepsilon\le 1/3$ and $2c\delta\le\varepsilon/2\le 1/6$. Thus we have proved that ξ and η do not vanish on the closed annulus $r\le|z|\le 1$. Now extend ξ and η to the annulus $r^2\le|z|\le 1$ by the formulas

$$\xi(x):=\frac{b}{\eta(ax^{-1})},\qquad \eta(y):=\frac{b}{\xi(ay^{-1})},\qquad r^2\le|x|,|y|\le r.$$

The resulting functions ξ and η are continuous across the circle of of radius r by (27). Hence they are holomorphic on the large annulus $r^2<|z|<1$. Since (27) holds on the middle circle $|x|=r$ it holds on the annulus $r^2\le|x|\le 1$. This proves the proposition for positive real numbers a.

To prove the proposition for general a we use the following "rotation trick". Fix a constant $\theta\in\mathbb{R}$. Given $\xi,\eta\in H^s_+\oplus H^s_{r,-}$ and $a,b\in\mathbb{C}$ define $\tilde\xi,\tilde\eta\in H^s_+\oplus H^s_{r,-}$ and $\tilde a,\tilde b\in\mathbb{C}$ by

$$\tilde\xi(z):=e^{-i\theta}\xi(e^{i\theta}z),\quad \tilde\eta(z):=e^{-i\theta}\eta(e^{i\theta}z),\quad \tilde a:=e^{-2i\theta}a,\quad \tilde b:=e^{-2i\theta}b.$$

Then a,b,ξ,η satisfy (27) if and only if $\tilde a,\tilde b,\tilde\xi,\tilde\eta$ satisfy (27). Hence the result for general a can be reduced to the special case case by choosing θ such that $\tilde a:=e^{-2i\theta}a$ is a positive real number. This proves the proposition. □

Proof of Proposition 10.7. The general case can be reduced to the case $a>0$ by the rotation trick in the proof of Proposition 10.6. Hence we assume $a=r^2$ and $r>0$. Choose positive constants c and C such that the assertions of Lemmata 10.12, 10.13, and 10.16 hold with these constants. Choose δ and ε such that

$$2c\delta\le 1,\qquad 8C(1+c)\varepsilon<1.$$

Let (a,ξ,η,b) and (a',ξ',η',b') satisfy the assumptions of Proposition 10.7 with these constants δ and ε and denote

$$\lambda:=b/a=b/r^2,\qquad \lambda':=b'/a'=b'/r^2.$$

Then

$$r^{-2}\xi(rz)\eta(rz^{-1})=\lambda,\qquad r^{-2}\xi'(rz)\eta'(rz^{-1})=\lambda',\qquad |z|=1. \tag{39}$$

Denote by $L_r:\mathbb{C}\times H^s_{r,-}\times H^s_{r,-}\to H^s$ the linear operator given by

$$L_r(\hat\lambda,\hat\xi_-,\hat\eta_-)(z):=r^{-2}\hat\xi_-(rz)\eta_+(rz^{-1})+r^{-2}\xi_+(rz)\hat\eta_-(rz^{-1})-\hat\lambda.$$

In the notation of 10.11 the operator L_r is the restriction of the differential of $\mathcal{F}_r$ at (λ, ξ_+, η_+) (for any λ) to the subspace $\mathbb{C} \times H^s_{r,-} \times H^s_{r,-}$. Since $c\delta \le 1/2$ it follows from Lemmata 10.14 and 10.16 that the operator L_r is invertible and the norm of the inverse is bounded by 2:

$$|\hat{\lambda}|^2 + \|\hat{\xi}_-\|^2_{r,s} + \|\hat{\eta}_-\|^2_{r,s} \le 4\|L_r(\hat{\lambda}, \hat{\xi}_-, \hat{\eta}_-)\|^2_s. \tag{40}$$

Let us denote by $Q_r : H^s_{r,-} \times H^s_{r,-} \to H^s$ the quadratic form

$$Q_r(\xi_-, \eta_-)(z) := r^{-2}\xi_-(rz)\eta_-(rz^{-1}).$$

Then, by Lemma 10.13, we have

$$\|Q_r(\xi_-, \eta_-)\|_s \le Cr^{-1}\left(\|\xi_-\|_{r,s} \sup_{|y|=r} |\eta_-(y)| + \|\eta_-\|_{r,s} \sup_{|x|=r} |\xi_-(x)|\right). \tag{41}$$

Now let

$$\hat{\lambda} := \lambda' - \lambda, \qquad \hat{\xi}_- := \xi' - \xi, \qquad \hat{\eta}_- := \eta' - \eta.$$

Then the difference of the two equations in (39) can be expressed in the form

$$\begin{aligned} L_r(\hat{\lambda}, \hat{\xi}_-, \hat{\eta}_-) &= Q_r(\xi_-, \eta_-) - Q_r(\xi'_-, \eta'_-) \\ &= -Q_r(\hat{\xi}_-, \eta_-) - Q_r(\xi_-, \hat{\eta}_-) - Q_r(\hat{\xi}_-, \hat{\eta}_-). \end{aligned}$$

Abbreviate

$$\hat{\zeta} := (\hat{\lambda}, \hat{\xi}_-, \hat{\eta}_-), \qquad \|\hat{\zeta}\|_{r,s} := \sqrt{|\hat{\lambda}|^2 + \|\hat{\xi}_-\|^2_{r,s} + \|\hat{\eta}_-\|^2_{r,s}}.$$

Then

$$\begin{aligned} \|\hat{\zeta}\|_{r,s} &\le 2\|L_r\hat{\zeta}\|_s \\ &\le 2\left(\|Q_r(\hat{\xi}_-, \eta_-)\|_s + \|Q_r(\xi_-, \hat{\eta}_-)\|_s + \|Q_r(\hat{\xi}_-, \hat{\eta}_-)\|_s\right) \\ &\le 2C\big(\|\xi_-\|_{r,s} + \|\eta_-\|_{r,s}\big)\|\hat{\zeta}\|_{r,s} \\ &\quad + 2Cr^{-1}\left(\sup_{|x|=r} |\hat{\xi}_-(x)| + \sup_{|y|=r} |\hat{\eta}_-(y)|\right)\|\hat{\zeta}\|_{r,s} \\ &\le 2C(1+c)\big(\|\xi_-\|_{r,s} + \|\eta_-\|_{r,s}\big)\|\hat{\zeta}\|_{r,s} \\ &\quad + 2Cr^{-1}\left(\sup_{|x|=r} |\xi'_-(x)| + \sup_{|y|=r} |\eta'_-(y)|\right)\|\hat{\zeta}\|_{r,s} \\ &\le 8C(1+c)\varepsilon\|\hat{\zeta}\|_{r,s}. \end{aligned}$$

Here the first inequality follows from (40), the second from the triangle inequality, the third from Lemma 10.13 and (41), the fourth from Lemma 10.12, and the last from the assumptions of the proposition. Since $8C(1+c)\varepsilon < 1$ it follows that $\hat{\zeta} = 0$. This proves the proposition. □

Proof of Proposition 10.9. **Step 1.** *The map $\mathcal{T}$ is continuous.*

Continuity for $a > 0$ is an easy consequence of the proof of Proposition 10.6. The map f defined in equation (34) depends continuously on the parameters ξ_+, η_+, r and $r = \sqrt{a}$ depends continuously on a. For complex nonzero a we can choose θ in the rotation trick to depend continuously on a. To prove continuity for $a = 0$ we deduce from (28) that there is a constant $c > 0$ such that

$$\|\mathcal{T}(a, \xi_+, \eta_+)\| := \sqrt{|b|^2 + \|\xi_-\|_s^2 + \|\eta_-\|_s^2} \leq c|a|.$$

Here we used the fact that $\|\zeta\|_s \leq r\|\zeta\|_{r,s}$ for $\zeta \in H^s_{r,-}$.

Step 2. *Let $\xi_+, \eta_+ \in H^s_+(\mathrm{id}, \delta)$ and $a \in \mathbb{D} \setminus 0$ and denote $\xi := \xi_+ + \xi_-$ and $\eta := \eta_+ + \eta_-$, where $(b, \xi_-, \eta_-) := \mathcal{T}(a, \xi_+, \eta_+)$. Then the linear operator*

$$L : \mathbb{C} \times H^s_{r,-} \times H^s_{r,-} \to H^s_r$$

defined by

$$L(\hat{b}, \hat{\xi}_-, \hat{\eta}_-)(x) := \hat{\xi}_-(x)\eta(ax^{-1}) + \xi(x)\hat{\eta}(ax^{-1}) - \hat{b}$$

is invertible.

In the notation of the proof of Proposition 10.6 we have that L is conjugate to the operator $df(u)$. Specifically,

$$L(\hat{\lambda}r^2, \hat{\xi}_-, \hat{\eta}_-)_r = r df(u)(\hat{\lambda}, \hat{\xi}_-, \hat{\eta}_-)$$

when $a = r^2$. The operator $df(u)$ is invertible by (35) and (37). For general a use the rotation trick from the end of the proof of Proposition 10.6.

Step 3. *The map $\mathcal{T}$ is continuously differentiable for $0 < |a| < 1$.*

We formulate a related problem. Define a partial rescaling operator

$$R_r : H^s \to H^s_+ \oplus H^s_{r,-}, \qquad (R_r\xi)(z) := \xi_+(z) + r\xi_-(r^{-1}z).$$

The operator R_r is a Hilbert space isometry for every $r \in (0, 1]$. Let

$$\mathcal{X} := \mathbb{C} \times \mathbb{C} \times H^s \times H^s, \qquad \mathcal{Y} := H^s.$$

There is a splitting $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ where

$$\mathcal{X}_+ := \mathbb{C} \times H^s_+ \times H^s_+, \qquad \mathcal{X}_- := \mathbb{C} \times H^s_- \times H^s_-.$$

Let $\mathcal{U} \subset \mathcal{X}$ denote the open set $\{0 < |a| < 1\}$ and define $\tilde{\mathcal{F}} : \mathcal{U} \to \mathcal{Y}$ by

$$\tilde{\mathcal{F}}(a, b, \xi, \eta)(z) := (R_r\xi)(rz) \cdot (R_r\eta)(ar^{-1}z^{-1}) - b, \qquad r := \sqrt{|a|}.$$

Define $\tilde{\mathcal{T}} : \mathcal{U} \cap \mathcal{X}_+ \to \mathcal{X}_-$ by

$$\tilde{\mathcal{T}}(a, \xi_+, \eta_+) := \big(b, (\xi_-)_r, (\eta_-)_r\big), \qquad (b, \xi_-, \eta_-) := \mathcal{T}(a, \xi_+.\eta_+).$$

(Recall that $\zeta_r(z) := r^{-1}\zeta(rz)$ for $\zeta \in H^s_r$ and $|z| = 1$.) By construction the graph of $\tilde{\mathcal{T}}$ is contained in the zero set of $\tilde{\mathcal{F}}$. By step 2 the derivative of $\tilde{\mathcal{F}}$ in the direction $\tilde{\mathcal{X}}_-$ is an invertible operator at every point in the graph of $\tilde{\mathcal{T}}$. Hence, by the implicit function theorem, $\tilde{\mathcal{T}}$ is continuously differentiable. Define the map $\mathcal{R} : \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{R}(a, b, \xi, \eta) := (a, b, R_r\xi, R_r\eta), \qquad r := \sqrt{|a|}.$$

This map is continuously differentiable and

$$\mathrm{graph}(\mathcal{T}) = \mathcal{R} \circ \mathrm{graph}(\tilde{\mathcal{T}}).$$

Here $\mathrm{graph}(\mathcal{T})$ denotes the map $(a, \xi_+, \eta_+) \mapsto (a, b, \xi, \eta)$ given by $(b, \xi_-, \eta_-) := \mathcal{T}(a, \xi_+, \eta_+)$. Similarly for $\mathrm{graph}(\tilde{\mathcal{T}})$. Hence $\mathrm{graph}(\mathcal{T})$ is continuously differentiable for $0 < |a| < 1$ and so is $\mathcal{T}$.

Step 4. *The map $\mathcal{T}$ is holomorphic for $0 < |a| < 1$.*

As $\mathcal{T}$ is differentiable we have

$$d\mathcal{T}(a, \xi_+, \eta_+)(\hat{a}, \hat{\xi}_+, \hat{\eta}_+) = (\hat{b}, \hat{\xi}_-, \hat{\eta}_-)$$

for $\hat{a} \in \mathbb{C}$ and $\hat{\xi}_+, \hat{\eta}_+ \in H^s_+$ where $\hat{\xi}_-, \hat{\eta}_- \in H^s_{r,-}$ and $\hat{b} \in \mathbb{C}$ are determined by the equation

$$L(\hat{b}, \hat{\xi}_-, \hat{\eta}_-)(x) = -\hat{\xi}_+(x)\eta(ax^{-1}) - \xi(x)\hat{\eta}_+(ax^{-1}) - \xi(x)\eta'(ax^{-1})\hat{a}x^{-1} \tag{42}$$

for $|x| = r := \sqrt{|a|}$. Here $L : \mathbb{C} \times H^s_{r,-} \times H^s_{r,-} \to H^s$ is the operator of Step 2. Since L is complex linear so is $d\mathcal{T}(a, \xi_+, \eta_+)$.

Step 5. *The map $\mathcal{T}$ is holomorphic for $|a| < 1$.*

That $\mathcal{T}$ is holomorphic near $a = 0$ follows from Step 4, continuity, and the Cauchy integral formula. More precisely, suppose X and Y are complex Hilbert spaces and $\mathcal{T} : \mathbb{C} \times X \to Y$ is a continuous map which is holomorphic on $(\mathbb{C} \setminus 0) \times X$. Then

$$\mathcal{T}(a, x) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{T}(a + e^{i\theta}\hat{a}, x + e^{i\theta}\hat{x})\, d\theta$$

and

$$d\mathcal{T}(a, x)(\hat{a}, \hat{x}) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}\mathcal{T}(a + e^{i\theta}\hat{a}, x + e^{i\theta}\hat{x})\, d\theta$$

for $a, \hat{a} \in \mathbb{C}$ and $x, \hat{x} \in X$ with $a \neq 0$. In the case at hand $\mathcal{T}(a, x)$ converges uniformly to zero as $|a|$ tends to zero (see the proof of Step 1). By the Cauchy integral formula, this implies that $d\mathcal{T}(a, x)$ converges uniformly to zero in the operator norm as $|a|$ tends to zero. This proves the proposition. □

Proof of Theorem 10.4. Fix a constant $s > 1/2$ and choose δ, c, and ε such that Propositions 10.6 and 10.7 hold. Shrinking δ we may assume $4c\delta < \varepsilon$. We prove that the graph of $\mathcal{T}$ intersects $\mathcal{W}_\delta$ in $\mathcal{U}_\delta$. By definition

$$\mathrm{graph}(\mathcal{T}) \cap \mathcal{W}_\delta \subset \mathcal{U}_\delta.$$

To prove the converse choose $(a, \xi, \eta, b) \in \mathcal{U}_\delta$. If $a = 0$ then $\xi_- = \eta_- = 0$ and $b = 0$ so (a, ξ, η, b) belongs to the graph of $\mathcal{T}$. Hence assume $a \neq 0$ and let $r := \sqrt{|a|}$. Then $\|\xi_+ - \mathrm{id}\|_s < \delta$ and $\|\eta_+ - \mathrm{id}\|_s < \delta$. So, by Proposition 10.6, there is an element $(a, \xi', \eta', b') \in \mathcal{W}_\delta \cap \mathrm{graph}(\mathcal{T})$ satisfying $\xi'_+ = \xi_+$, $\eta'_+ = \eta_+$, and

$$\left\|\xi'_-\right\|_{r,s} \leq 4cr\delta < \varepsilon, \qquad \left\|\eta'_-\right\|_{r,s} \leq 4cr\delta < \varepsilon.$$

We claim that $\xi = \xi'$, $\eta = \eta'$, and $b = b'$. By Proposition 10.7 it suffices to show that

$$\sup_{|x|=r} |\xi_-(x)| < r\varepsilon, \qquad \sup_{|y|=r} |\eta_-(y)| < r\varepsilon.$$

By symmetry we need only prove the first inequality. By the triangle inequality

$$\sup_{|x|=r} |\xi_-(x)| \leq \sup_{|x|=r} |\xi(x) - x| + \sup_{|x|=r} |\xi_+(x) - x|\,. \tag{43}$$

By Lemma 10.10 we estimate the first term on the right by

$$\sup_{|x|=r} |\xi(x) - x| \leq cr\delta. \tag{44}$$

For the second term we have by Lemma 10.12

$$\sup_{|x|=r} |\xi_+(x) - x| = r \sup_{|z|=1} |(\xi_+ - \mathrm{id})_r(z)| \leq rc \left\|(\xi_+ - \mathrm{id})_r\right\|_s = cr \left\|\xi_+ - \mathrm{id}\right\|_{r,s}$$

But the series for $\xi_+ - \mathrm{id}$ has only positive powers and $r \leq 1$ so

$$\|\xi_+ - \mathrm{id}\|_{r,s} \leq \|\xi_+ - \mathrm{id}\|_s \leq \|\xi - \mathrm{id}\|_s \leq \delta.$$

Combining the last two lines gives

$$\sup_{|x|=r} |\xi_+(x) - x| \leq cr\delta. \tag{45}$$

Now use (43), (44), and (45) and shrink δ so $2c\delta < \varepsilon$. □

We close this section with two lemmas that will be useful in the sequel.

Lemma 10.18. *Fix $s > 1/2$ and choose $\delta > 0$ as in Theorem 10.4. Let $A \subset \mathrm{int}(\mathbb{D}) \times \mathbb{C}^m$ be an open set and*

$$A \to \mathcal{U}_\delta : (a, t) \mapsto (a, \xi_{a,t}, \eta_{a,t}, b_{a,t})$$

be a holomorphic map. Then the map

$$\Phi : \{(x, y, t) \in \mathbb{C}^{2+m} \,:\, x, y \in \mathrm{int}(\mathbb{D}),\, (xy, t) \in A\} \to \mathbb{C} \times \mathbb{C}$$

given by

$$\Phi(x,y,t) := \Phi_t(x,y) := (\xi_{xy,t}(x), \eta_{xy,t}(y))$$

is holomorphic.

Proof. The evaluation map

$$H^s \cap H^s_{r^2} \times \left\{z \in \mathbb{C} : r^2 < |z| < 1\right\} \to \mathbb{C} : (\zeta, z) \mapsto \zeta(z)$$

is holomorphic. It follows that the map $(x,y,t) \mapsto \Phi_t(x,y)$ is holomorphic in the domain $xy \neq 0$. We prove that Φ is continuous. Suppose $x_i \to x \neq 0$, $y_i \to 0$, and $t_i \to t$. Then $\xi_{x_iy_i,t_i}$ converges to $\xi_{0,t}$, uniformly in a neighbourhood of x, and hence $\xi_{x_iy_i,t_i}(x_i)$ converges to $\xi_{0,t}(x)$. Moreover, if c and δ are the constants of Lemma 10.10, then

$$|\eta_{x_iy_i,t_i}(y_i)| \leq (c\delta + 1)|y_i|$$

and so $\eta_{x_iy_i,t_i}(y_i)$ converges to $\eta_{0,t}(0) = 0$. Hence $\Phi_{t_i}(x_i, y_i)$ converges to $\Phi_t(x,0) = (\xi_{0,t}(x), 0)$. Hence Φ is continuous at every point $(x, 0, t)$ with $x \neq 0$. By symmetry, Φ is continuous at every point $(0, y, t)$ with $y \neq 0$. That Φ is continuous at very point $(0,0,t)$ follows again from Lemma 10.10. Since Φ is continuous and is holomorphic in $xy \neq 0$, it follows from the Cauchy integral formula that Φ is holmorphic. □

Lemma 10.19. *Let* $\xi_0, \eta_0 : \mathrm{int}(\mathbb{D}) \to \mathbb{C}$ *be two holomorphic functions satisfying* $\xi_0(0) = \eta_0(0) = 0$ *and* $\xi_0'(0) \neq 0$, $\eta_0'(0) \neq 0$. *Then there are neighborhoods* U_1 *and* U_2 *of* $(0,0)$ *in* $\mathbb{C}^2$ *and* B_1 *and* B_2 *of* 0 *in* $\mathbb{C}$ *and holomorphic diffeomorphisms* $\Phi := (\xi, \eta) : U_1 \to U_2$ *and* $\zeta : B_1 \to B_2$ *such that*

$$\xi(x,0) = \xi_0(x), \qquad \eta(0,y) = \eta_0(y), \qquad \xi(x,y)\eta(x,y) = \zeta(xy)$$

for x, y *near* 0.

Proof. Replacing ξ_0 and η_0 by $\xi_0'(0)^{-1}\xi_0$ and $\eta_0'(0)^{-1}\eta_0$ we may assume w.l.o.g. that $\xi_0'(0) = \eta_0'(0) = 1$. Replacing $\xi_0(x)$ and $\eta_0(y)$ by $\varepsilon^{-1}\xi_0(\varepsilon x)$ and $\varepsilon^{-1}\eta_0(\varepsilon y)$ we may assume w.l.o.g. that the power series for ξ_0 and η_0 lie in $H^s_+(\mathrm{id}, \delta)$ with $\delta > 0$ as in 10.8. For $z \in \mathbb{D}$ define $\alpha_z, \beta_z \in H^s_-$ by $(\zeta(z), \alpha_z, \beta_z) := \mathcal{T}(z, \xi_0, \eta_0)$ and then define $\xi(x,y) := \xi_0(x) + \alpha_{xy}(x)$ and $\eta(x,y) := \eta_0(y) + \beta_{xy}(y)$. Then Φ is holomorphic by Lemma 10.18. A direct calculation shows that $d\Phi(0,0)$ is the identity so Φ is a local diffeomorphism. The desired identities follow from the definition of $\mathcal{T}$. To prove that $\zeta'(0) = 1$ differentiate the identity $\xi\eta = \zeta$ twice. □

11 Hardy decompositions

In this section we redo Section 9 in parametrized form. We will use the implicit function theorem on a manifold of maps. The main difficulty in defining a suitable manifold of maps is that the nodal family π_A is not locally trivial because the homotopy type of the fiber changes. To circumvent this difficulty we use the local model of Section 10 for a neighborhood of the nodal set and suitable trivializations for the complement (see Definitions 11.2 and 11.6).

11.1. Throughout this section $(\pi_A : P \to A, R_*, a_0)$ and $(\pi_B : Q \to B, S_*, b_0)$ are nodal unfoldings,

$$f_0 : P_{a_0} \to Q_{b_0}$$

is a fiber isomorphism, and $p_1, p_2, \dots, p_k$ are the nodal points of the central fiber P_{a_0}, so $q_i := f_0(p_i)$ (for $i = 1, \dots, k$) are the nodal points of the central fiber Q_{b_0}. Let $m := \dim_{\mathbb{C}}(A)$ and $d := \dim_{\mathbb{C}}(B)$. Let $C_A \subset P$ and $C_B \subset Q$ denote the critical points of π_A and π_B respectively.

Definition 11.2. A **Hardy decomposition** for (π_A, R_*, a_0) is a decomposition

$$P = M \cup N, \qquad \partial M = \partial N = M \cap N,$$

into manifolds with boundary such that

$$N = N_1 \cup \cdots \cup N_k,$$

each N_i is a neighborhood of p_i, the closures of the N_i are pairwise disjoint, N is disjoint from the elements of R_*, and N is the domain of a **nodal coordinate system**. This consists of three sequences of holomorphic maps

$$(x_i, y_i) : N_i \to \mathbb{D}^2, \qquad z_i : A \to \mathbb{C}, \qquad t_i : A \to \mathbb{C}^{m-1},$$

such that each map

$$A \to \mathbb{D} \times \mathbb{C}^{m-1} : a \mapsto (z_i(a), t_i(a))$$

is a holomorphic coordinate system, each map

$$N_i \to \mathbb{D}^2 \times \mathbb{C}^{m-1} : p \mapsto \big(x_i(p), y_i(p), t_i(\pi_A(p))\big)$$

is a holomorphic coordinate system and

$$x_i(p_i) = y_i(p_i) = 0, \qquad z_i \circ \pi_A = x_i y_i.$$

(Note that here N_i has a boundary whereas its analog U_i in 9.1 was open.) Restricting to a fiber gives a decomposition

$$P_a = M_a \cup N_a, \qquad M_a := M \cap P_a, \qquad N_a := N \cap P_a$$

where M_a is a Riemann surface with boundary and each component of N_a is either a closed annulus or a pair of transverse closed disks. The nodal coordinate system determines a trivialization

$$\iota : A \times \Gamma \to \partial N, \qquad \Gamma := \bigcup_{i=1}^{k} \{(i,1), (i,2)\} \times S^1, \tag{46}$$

where ι^{-1} is the disjoint union of the maps

$$\pi \times x_i : \partial_1 N_i \to A \times S^1, \qquad \partial_1 N_i := \{|x_i| = 1\},$$

$$\pi \times y_i : \partial_2 N_i \to A \times S^1, \qquad \partial_2 N_i := \{|y_i| = 1\}.$$

The indexing is so that $\iota(A \times (i,1) \times S^1) = \partial_1 N_i$ and $\iota(A \times (i,2) \times S^1) = \partial_2 N_i$. For $a \in A$ define $\iota_a : \Gamma \to \partial N$ by $\iota_a(\lambda) := \iota(a, \lambda)$.

Lemma 11.3. *After shrinking A and B if necessary, there is a Hardy decomposition $P = M \cup N$ as in 11.2 and there are open subsets*

$$U = U_1 \cup \cdots \cup U_k, \qquad V, \qquad W := U \cap V$$

of Q and functions $\xi_i, \eta_i, \zeta_i, \tau_i$ as described in 9.1 such that

$$f_0(M_{a_0}) \subset V_{b_0}, \qquad f_0(N_{a_0}) \subset U_{b_0},$$

and

$$\xi_i \circ f_0 \circ x_i^{-1}(x, 0, 0) = x, \qquad \eta_i \circ f_0 \circ y_i^{-1}(0, y, 0) = y$$

for $x, y \in \mathbb{D}$.

Proof. Choose any Hardy decomposition $P = M \cup N$ as in 11.2 as well as open subsets $U = U_1 \cup \cdots \cup U_k$, V, W of Q and functions $\xi_i, \eta_i, \zeta_i, \tau_i$ as described in 9.1. Read $\xi_i \circ f_0 \circ x_i^{-1}(x)$ for $\xi_0(x)$ and $\eta_i \circ f_0 \circ y_i^{-1}(y)$ for $\eta_0(y)$ in Lemma 10.19, let Φ and ζ be as in the conclusion of that Lemma, and replace (ξ_i, η_i) by $\Phi^{-1} \circ (\xi_i, \eta_i)$ and ζ_i by $\zeta^{-1} \circ \zeta_i$. This requires shrinking U_i (and B). Then shrink N so that $f_0(N_{a_0}) \subset U_{b_0}$ and enlarge V so that $f_0(M_{a_0}) \subset V_{b_0}$. □

11.4. We use a Hardy decomposition to mimic the construction of 9.3 with $a \in A$ as a parameter. Choose a Hardy decomposition $P = M \cup N$ for (π_A, R_*, a_0), open subsets $U = U_1 \cup \cdots \cup U_k$, V, W of Q, and functions $\xi_i, \eta_i, \zeta_i, \tau_i$ as described in 9.1, such that the conditions of Lemma 11.3 are satisfied. Fix an integer $s + 1/2 > 1$ and define an open subset

$$\mathcal{W}(a, b) \subset H^s(\partial N_a, W_b)$$

by the condition that for $\gamma \in H^s(\partial N_a, W_b)$ we have $\gamma \in \mathcal{W}(a, b)$ iff

$$\gamma\big(x_i^{-1}(S^1)\big) \subset W_{i,1}, \qquad \gamma\big(y_i^{-1}(S^1)\big) \subset W_{i,2},$$

(see 9.1 for the notation $W_{i,1}$ and $W_{i,2}$) and the curves $\xi_i \circ \gamma \circ x_i^{-1}$ and $\eta_i \circ \gamma \circ y_i^{-1}$ from S^1 to $\mathbb{C} \setminus 0$ both have winding number one about the origin. For $a \in A$ and $b \in B$ let

$$\mathcal{U}(a, b) := \left\{ \gamma = u|\partial N_a \in \mathcal{W}(a, b) : \begin{array}{l} u \in \mathrm{Hol}^{s+1/2}(N_a, U_b), \\ u(C_A \cap P_a) = C_B \cap Q_b \end{array} \right\},$$

$$\mathcal{V}(a, b) := \left\{ \gamma = v|\partial N_a \in \mathcal{W}(a, b) : \begin{array}{l} v \in \mathrm{Hol}^{s+1/2}(M_a, V_b), \\ v(R_* \cap P_a) = S_* \cap Q_b \end{array} \right\}.$$

Here $\mathrm{Hol}^{s+1/2}(X, Y)$ is defined by (16); holomorphicity at a nodal point is defined as in 10.1. Define

$$\mathcal{W}_a := \bigsqcup_{b \in B} \mathcal{W}(a, b), \qquad \mathcal{V}_a := \bigsqcup_{b \in B} \mathcal{V}(a, b), \qquad \mathcal{U}_a := \bigsqcup_{b \in B} \mathcal{U}(a, b),$$

$$\mathcal{W} := \bigsqcup_{a \in A} \mathcal{W}_a, \qquad \mathcal{V} := \bigsqcup_{a \in A} \mathcal{V}_a, \qquad \mathcal{U} := \bigsqcup_{a \in A} \mathcal{U}_a.$$

Our notation means that the three formulas $(a, \gamma, b) \in \mathcal{W}$, $(\gamma, b) \in \mathcal{W}_a$, and $\gamma \in \mathcal{W}(a, b)$ have the same meaning.

11.5. We use the trivialization $\iota : A \times \Gamma \to \partial N$ in (46) to construct an auxiliary Hilbert manifold structure on $\mathcal{W}$. Define an open set

$$\mathcal{W}_0 \subset \{(a, \gamma, b) \in A \times H^s(\Gamma, W) \times B : \pi_B \circ \gamma = b\}$$

by the condition that the map

$$\mathcal{W}_0 \to \mathcal{W} : (a, \gamma, b) \mapsto (a, \gamma \circ \iota_a^{-1}, b) \tag{47}$$

is a bijection. In particular $\gamma((i,1) \times S^1) \subset W_{i,1}$ and $\gamma((i,2) \times S^1) \subset W_{i,2}$ for $(a, \gamma, b) \in \mathcal{W}_0$. By a standard construction $H^s(\Gamma, W)$ is a complex Hilbert manifold and the subset $\{(a, \gamma, b) : \pi_B \circ \gamma = b\}$ is a complex Hilbert submanifold of $A \times H^s(\Gamma, W) \times B$. This is because the map $H^s(\Gamma, W) \to H^s(\Gamma, B)$ induced by π_B is a holomorphic submersion. Note that $\mathcal{W}_0$ is a connected component of $\{(a, \gamma, b) : \pi_B \circ \gamma = b\}$ and hence inherits its Hilbert manifold structure. We emphasize that the resulting Hilbert manifold structure on $\mathcal{W}$ depends on the choice of the trivialization. Two different trivializations give rise to a homeomorphism which is of class C^k on the dense subset $\mathcal{W} \cap H^{s+k}$.

Definition 11.6. A **Hardy trivialization** for $(\pi_A : P \to A, R_*, a_0)$ is a triple $(M \cup N, \iota, \rho)$ where $P = M \cup N$ is a Hardy decomposition with corresponding trivialization $\iota : A \times \Gamma \to \partial N$ as in 11.2 and

$$\rho : M \to \Omega := M_{a_0}$$

is a trivialization such that

$$\rho_a \circ \iota_a = \iota_{a_0}, \qquad \rho_a := \rho|M_a : M_a \to \Omega$$

for $a \in A$ and

$$\rho(R_*) = R_* \cap \Omega =: r_*.$$

We require further that ρ is holomorphic in a neighborhood of the boundary, more precisely that the coordinates x_i and y_i in Definition 11.2 extend holomorphically to a neighborhood of N_i and that $x_i \circ \rho = x_i$ near $\partial_1 N_i$ and $y_i \circ \rho = y_i$ near $\partial_2 N_i$.

11.7. The fiber isomorphism $f_0 : P_{a_0} \to Q_{b_0}$ determines a point

$$(a_0, \gamma_0 := f_0|\partial N_{a_0}, b_0) \in \mathcal{W};$$

this point lies in $\mathcal{U} \cap \mathcal{V}$ as

$$\gamma_0 = u_0|\partial N_{a_0} = v_0|\partial M_{a_0}, \qquad \text{where} \qquad u_0 := f_0|N_{a_0}, \quad v_0 := f_0|M_{a_0}.$$

In the sequel we will denote neighborhoods of a_0 in A and (a_0, γ_0, b_0) in $\mathcal{U}$, $\mathcal{V}$, or $\mathcal{W}$ by the same letters A, respectively $\mathcal{U}$, $\mathcal{V}$, or $\mathcal{W}$, and signal this with the text "shrinking A, $\mathcal{U}$, $\mathcal{V}$, or $\mathcal{W}$, if necessary".

Lemma 11.8. *For every $(a, \gamma, b) \in \mathcal{U} \cap \mathcal{V}$ there is a unique fiber isomorphism $f : P_a \to Q_b$ with $f|\partial N_a = \gamma$.*

Proof. This follows immediately from Lemma 9.4. □

Theorem 11.9. *Fix an integer $s + 1/2 > 4$. After shrinking A, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, if necessary, the following holds.*

(i) *For each $a \in A$, $\mathcal{U}_a$ and $\mathcal{V}_a$ are complex submanifolds of $\mathcal{W}_a$.*

(ii) *$\mathcal{U}$ and $\mathcal{V}$ are complex submanifolds of $\mathcal{W}$.*

(iii) *The projections $\mathcal{W} \to A$, $\mathcal{U} \to A$, $\mathcal{V} \to A$ are holomorphic submersions.*

(iv) *The unfolding (π_B, S_*, b_0) is infinitesimally universal if and only if*

$$T_{w_0}\mathcal{W}_{a_0} = T_{w_0}\mathcal{U}_{a_0} \oplus T_{w_0}\mathcal{V}_{a_0}, \qquad w_0 = (a_0, \gamma_0, b_0).$$

Proof. In 9.1 it was not assumed that k was precisely the number of nodal pairs so the arguments of Section 9 will apply when the central fiber Q_{b_0} is replaced by a nearby fiber Q_b with possibly fewer nodal points. Hence (i) and (iv) follow from Theorem 9.5. We prove (ii) and (iii) in four steps.

Step 1. *We prove that $\mathcal{U}$ is a complex Hilbert submanifold of $\mathcal{W}$.*

The image of the nodal coordinate system (x_i, y_i, t_i) on N_i in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{m-1}$ has the form

$$\{(x, y, t) \in \mathbb{D}^2 \times \mathbb{C}^{m-1} : (xy, t) \in A_i,\ |x| < 1,\ |y| < 1\}.$$

where $A_i \subset \mathbb{C} \times \mathbb{C}^{m-1}$ is contained in the open set $\{|z_i| < 1\} \times \mathbb{C}^{m-1}$. The image of the nodal coordinate systems (ξ_i, η_i, τ_i) on U_i in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{d-1}$ has the form

$$\left\{(\xi_i, \eta_i, \tau_i) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{d-1}\ :\ |\xi_i| < 2,\ |\eta_i| < 2, (\xi_i\eta_i, \tau_i) \in B_i\right\},$$

where $B_i \subset \mathbb{C} \times \mathbb{C}^{d-1}$ is contained in the open set $\{|\zeta_i| < 4\} \times \mathbb{C}^{d-1}$. By assumption (see 11.4), the fiber isomorphism $f_0 : P_0 \to Q_0$ between the fibers over the origin is the identity in these coordinates.

Consider the map

$$\mathcal{W} \to A \times (H^s)^{2k} \times B : (a, \gamma, b) \mapsto (a, \alpha_1, \beta_1, \dots, \alpha_k, \beta_k, b)$$

where $\gamma \in \mathcal{W}(a, b)$ and $\alpha_i = \xi_i \circ \gamma \circ x_i^{-1}$ and $\beta_i = \eta_i \circ \gamma \circ y_i^{-1}$. This map is a diffeomorphism from $\mathcal{W}$, with the manifold structure of 11.5, onto an open subset of the Hilbert manifold $A \times (H^s)^{2k} \times B$. The image of the subset $\mathcal{U} \subset \mathcal{W}$ under this diffomorphism consists of all tuples $(a, \alpha_1, \beta_1, \dots, \alpha_k, \beta_k, b)$ in the image of $\mathcal{W}$ such that

$$xy = z_i(a) \implies \alpha_i(x)\beta_i(y) = \zeta_i(b) \text{ for } i = 1, \dots, k.$$

That this subset is a complex submanifold of $A \times (H^s)^{2k} \times B$ follows from Theorem 10.4

Step 2. *Define*

$$\mathcal{B} := \left\{ (a,v,b) : \begin{array}{l} a \in A, \quad b \in B, \quad v \in H^{s+1/2}(M_a, V_b), \\ v(R_* \cap P_a) = S_* \cap Q_b, \ v|\partial N_a \in \mathcal{W}(a,b) \end{array} \right\}$$

and

$$\mathcal{Z} := \{(a,v,b) \in \mathcal{B} \,:\, v \in \mathrm{Hol}^{s+1/2}(M_a, V_b)\}. \tag{48}$$

We construct an auxiliary Hilbert manifold structure on $\mathcal{B}$ and show that $\mathcal{Z}$ is a smooth submanifold of $\mathcal{B}$.

In analogy with 11.4 denote

$$\mathcal{B}_0 := \left\{ (a,v,b) : \begin{array}{l} a \in A, \quad b \in B, \quad v \in H^{s+1/2}(\Omega, V_b), \\ v(r_*) = S_* \cap Q_b, \ v \circ \rho_a|\partial N_a \in \mathcal{W}(a,b) \end{array} \right\},$$

where $\Omega := M_{a_0}$ and $r_* := R_* \cap \Omega = \rho(R_*)$ as in Definition 11.6. This space is a Hilbert manifold and the Hardy trivialization $(P = N \cup M, \iota, \rho)$ induces a bijection

$$\mathcal{B}_0 \to \mathcal{B} : (a,v,b) \mapsto (a, v \circ \rho_a, b).$$

This defines the Hilbert manifold structure on $\mathcal{B}$. Note the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_0 & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{W}_0 & \longrightarrow & \mathcal{W} \end{array}$$

where the bijection $\mathcal{W}_0 \to \mathcal{W}$ is given by (47), the map $\mathcal{B} \to \mathcal{W}$ is given by restriction to the boundary, and the map $\mathcal{B}_0 \to \mathcal{W}_0$ is $(a,v,b) \mapsto (a, v \circ \iota_{a_0}, b)$. The bijection $\mathcal{B}_0 \to \mathcal{B}$ identifies the subset $\mathcal{Z} \subset \mathcal{B}$ with the subset $\mathcal{Z}_0 \subset \mathcal{B}_0$ given by

$$\mathcal{Z}_0 := \{(a,v,b) \in \mathcal{B}_0 : v \in \mathrm{Hol}^{s+1/2}((\Omega, j_a), Q_b)\},$$

where $j_a := (\rho_a)_* J|M_a$, $\rho_a : M_a \to \Omega$ is the Hardy trivialization, and J is the complex structure on P. (Note that the map $a \mapsto j_a$ need not be holomorphic.)

We prove that $\mathcal{Z}_0$ is a smooth Hilbert submanifold of $\mathcal{B}_0$. The tangent space of $\mathcal{B}_0$ at a triple (a,v,b) is

$$T_{a,v,b}\mathcal{B}_0 = T_aA \times \{(\hat{v}, \hat{b}) \in H^{s+1/2}(\Omega, v^*TQ) \times T_bB \,: \\ d\pi_B(v)\hat{v} \equiv \hat{b}, \ \hat{v}(r_i) \in T_{v(s_i)}S_i\}.$$

Let $\mathcal{E} \to \mathcal{B}_0$ be the complex Hilbert space bundle whose fiber

$$\mathcal{E}_{a,v,b} := H^{s-1/2}(\Omega, \Lambda^{0,1}_{j_a} T^*\Omega \otimes v^*TQ_b)$$

over $(a,v,b) \in \mathcal{B}_0$ is the Sobolev space of $(0,1)$-forms on (Ω, j_a) of class $H^{s-1/2}$ with values in the pullback tangent bundle v^*TQ_b. As before the Cauchy–Riemann operator defines a smooth section $\bar{\partial} : \mathcal{B}_0 \to \mathcal{E}$ given by

$$\bar{\partial}(a,v,b) := \bar{\partial}_{j_a,J}(v) = \frac{1}{2}\left(dv + J \circ dv \circ j_a\right). \tag{49}$$

Here J denotes the complex structure on Q. The zero set of this section is the set $\mathcal{Z}_0$ defined above. It follows as in the proof of Theorem 9.5 that the linearized operator $D_{a,v,b} : T_{a,v,b}\mathcal{B}_0 \to \mathcal{E}_{a,v,b}$ is surjective and has a right inverse. Hence the zero set $\mathcal{Z}_0$ is a smooth Hilbert submanifold of $\mathcal{B}_0$.

Step 3. *We prove that $\mathcal{V}$ is a complex Hilbert submanifold of $\mathcal{W}$.*

As in the proof of Theorem 9.5 restriction to the boundary gives rise to a smooth embedding

$$\mathcal{Z} \to \mathcal{W} : (a, v, b) \mapsto (a, \gamma, b), \qquad \gamma := v|\partial M_a,$$

whose image is $\mathcal{V}$. The only difference in the proof that the restriction map $\mathcal{Z} \to \mathcal{W}$ is proper is that now we have a C^∞ convergent sequence of complex structures on Ω. The proof is otherwise word for word the same. (Note that [16, Theorem B.4.2] allows for a sequence of complex structures on the domain.) Hence $\mathcal{V}$ is a smooth Hilbert submanifold of $\mathcal{W}$.

We prove that $T_{(a,\gamma,b)}\mathcal{V}$ is a complex subspace of $T_{(a,\gamma,b)}\mathcal{W}$ for each triple $(a, \gamma, b) \in \mathcal{V}$. For this we identify $\mathcal{W}$ with $\mathcal{W}_0$ and hence $\mathcal{V}$ with the image $\mathcal{V}_0$ of the embedding

$$\mathcal{Z}_0 \to \mathcal{W}_0 : (a, v, b) \mapsto (a, v \circ \iota_{a_0}, b). \tag{50}$$

The tangent space of $\mathcal{B}_0$ at a point (a, v, b) is the space of all triples $(\hat{a}, \hat{v}, \hat{b}) \in T_aA \times \Omega^0(\Omega, v^*TQ) \times T_bB$ that satisfy

$$d\pi(v)\hat{v} \equiv b, \qquad \hat{v}(r_i) \in T_{v(r_i)}S_i, \qquad i = 1, \dots, n.$$

The trivialization $\pi \times \rho : M \to A \times \Omega$ induces a complex structure of the form (6) on $A \times \Omega$ where $j : A \to \mathcal{J}(\Omega)$ is a smooth map and $\alpha : TA \to \mathrm{Vect}(\Sigma)$ a smooth 1-form satisfying (7) and (8). Since ρ is holomorphic near ∂M with respect to the complex structure $j(a_0)$ on $\Omega = M_{a_0}$ (see Definition 11.6) it follows that α vanishes near $A \times \partial\Omega$. Let $D_v : \Omega^0(\Omega, v^*TQ) \to \Omega^{0,1}_{j(a)}(\Omega, v^*TQ)$ denote the linearized Cauchy–Riemann operator associated to a $(j(a), J)$-holomorphic curve $v : \Omega \to Q$. Then the tangent space of $\mathcal{Z}_0$ at (a, v, b) is the kernel of the operator $\mathcal{D}_{a,v} : T_{(a,v,b)}\mathcal{B}_0 \to \Omega^{0,1}_{j(a)}(\Omega, v^*TQ)$ given by

$$\mathcal{D}_{a,v}(\hat{a}, \hat{v}, \hat{b}) := D_v\hat{v} + \frac{1}{2}J(v)dv \cdot dj(a)\hat{a}. \tag{51}$$

It follows from (15) with $\sigma_i(a) \equiv r_i$ that the vector field $\alpha(a, \hat{a})$ vanishes at the point r_i for every $a \in A$ and every $\hat{a} \in T_aA$. Hence the tangent space $T_{(a,v,b)}\mathcal{B}_0$ carries a complex structure

$$\mathcal{I}(a, v, b)(\hat{a}, \hat{v}, \hat{b}) := \left(\sqrt{-1}\hat{a}, J(v)\hat{v} - dv \cdot \alpha(a, \hat{a}), \sqrt{-1}\hat{b}\right)$$

and, by direct calculation using (8), we find

$$J(v) \circ \mathcal{D}_{a,v} = \mathcal{D}_{a,v} \circ \mathcal{I}(a, v, b).$$

Hence the (almost) complex structure $\mathcal{I}$ descends to $\mathcal{Z}_0$. Since α vanishes near the boundary the differential of the embedding (50) is complex linear and hence $\mathcal{V}_0$ is a complex submanifold of $\mathcal{W}_0$ as claimed.

Step 4. *We prove (iii).*

That the projections $\mathcal{W} \to A$, $\mathcal{U} \to A$, $\mathcal{V} \to A$ are holomorphic is obvious from the construction. We prove that these three maps are submersions. For the map $\mathcal{U} \to A$, and hence for $\mathcal{W} \to A$, this follows immediately from Proposition 10.9. For $\mathcal{V}$ observe that the linearized operator of the section (49) is the operator (51). Choose $\hat{a} \in T_aA$ and solve the equation $\mathcal{D}_{a,v}(\hat{a}, \hat{v}, 0) = 0$ for $\hat{v}$. This equation has a solution because D_v is surjective with domain the space of vertical vector fields that vanish at the points r_i and target the space of vertical $(0,1)$-forms. This proves (iii). □

12 Proofs of the main theorems

Definition 12.1. The set C of critical points of a nodal family $\pi : Q \to B$ is a submanifold of Q and the restriction of π to this set is an immersion. The family is said to be **regular nodal** at $b \in B$ if all self-intersections of $\pi(C)$ in $\pi^{-1}(b)$ are transverse, i.e.

$$\dim_{\mathbb{C}} (\operatorname{im} d\pi(q_1) \cap \cdots \cap \operatorname{im} d\pi(q_m)) = \dim_{\mathbb{C}}(B) - m$$

whenever $q_1, \ldots, q_m \in C$ are pairwise distinct and $\pi(q_1) = \cdots = \pi(q_m) = b$; the nodal family is called **regular nodal** if it is regular nodal at each $b \in B$.

Lemma 12.2. *Let u be a desingularization of the fiber Q_b, g be the arithmetic genus of the fiber, and n be the number of marked points. Then the following hold:*

(i) *We have $D_{u,b}(\hat{u}, \hat{b}) \in \mathcal{Y}_u$ for $(\hat{u}, \hat{b}) \in \mathcal{X}_{u,b}$.*

(ii) *The operator $D_{u,b} : \mathcal{X}_{u,b} \to \mathcal{Y}_u$ is Fredholm.*

(iii) *The Fredholm index satisfies*

$$\operatorname{index}_{\mathbb{C}}(D_{u,b}) \geq 3 - 3g - n + \dim_{\mathbb{C}}(B)$$

with equality if and only if π is regular nodal at b.

Proof. We prove (i). Choose $(\hat{u}, \hat{b}) \in \mathcal{X}_{u,b}$ and let

$$\bar{\partial} : \Omega^0(\Sigma, T_bB) \to \Omega^{0,1}(\Sigma, T_bB)$$

denote usual Cauchy–Riemann operator. Then $d\pi_B(u)D_u\hat{u} = \bar{\partial} d\pi_B(u)\hat{u} = 0$ since $d\pi_B(u)\hat{u}$ is a constant vector. Hence $D_u\hat{u} \in \mathcal{Y}_u$. Item (ii) is immediate as D_u is Fredholm as a map from vertical vector fields to vertical $(0,1)$-forms and $D_{u,b}$ is obtained from D_u by a finite dimensional modification of the domain. (A vertical vector field is an element $\hat{u} \in \Omega^0(\Sigma, u^*TQ)$ such that $d\pi(u)\hat{u} = 0$; a vertical $(0,1)$-form is an element $\eta \in \Omega^{0,1}(\Sigma, u^*TQ)$ such that $d\pi(u)\eta = 0$, i.e. an element of $\mathcal{Y}_u$.)

We prove (iii). The arithmetic genus g of the fiber is given by

$$g = \#\text{edges} - \#\text{vertices} + 1 + \sum_i g_i \tag{52}$$

where $\#\text{vertices} = \sum_i 1$ is the number of components of Σ, #edges is the number of pairs of nodal points, and g_i is the genus of the ith component. Now consider the subspace

$$\mathcal{X}_u := \{(\hat{u}, \hat{b}) \in \mathcal{X}_{u,b} : \hat{b} = 0\}$$

of all vertical vector fields along u satisfying the nodal and marked point conditions. If $(\hat{u}, \hat{b}) \in \mathcal{X}_{u,b}$, then the vector $\hat{b}$ belongs to the image of $d\pi(q)$ for every $q \in Q_b$. Hence the codimension of $\mathcal{X}_u$ in $\mathcal{X}_{u,b}$ is

$$\operatorname{codim}_{\mathcal{X}_{u,b}}(\mathcal{X}_u) = \dim \left(\bigcap_{q \in Q_b} \operatorname{im} d\pi(q) \right) \geq \dim_{\mathbb{C}}(B) - \#\text{edges}$$

with equality if and only if π_B is regular nodal at b. By Riemann-Roch the restricted operator has Fredholm index

$$\operatorname{index}_{\mathbb{C}}(D_u : \mathcal{X}_u \to \mathcal{Y}_u) = \sum_i (3 - 3g_i) - 2\,\#\text{edges} - n.$$

Here the summand $-2\,\#\text{edges}$ arises from the nodal point conditions in the definition of $\mathcal{X}_u$. To obtain the Fredholm index of $D_{u,b}$ we must add the codimension of $\mathcal{X}_u$ in $\mathcal{X}_{u,b}$ to the last identity. Hence

$$\begin{aligned} \operatorname{index}_{\mathbb{C}}(D_{u,b}) &\geq \sum_i (3 - 3g_i) - 3\,\#\text{edges} - n + \dim_{\mathbb{C}}(B) \\ &= 3 - 3g - n + \dim_{\mathbb{C}}(B). \end{aligned}$$

The last identity follows from equation (52). Again, equality holds if and only if π_B is regular nodal at b. □

Proof of Theorem 5.3. The proof is an easy application of the openness of transversality. Take $P = Q$, $A = B$, $\pi_A = \pi_B$, and $f_0 = \text{id}$, so γ_0 is the inclusion of ∂N_{b_0} in Q_{b_0}. Assume the unfolding $(\pi_B : Q \to B, S_*, b_0)$ is infinitesimally universal. Choose $b \in B$ near b_0, fix an integer $s + 1/2 > 4$, and let $\mathcal{U}_b$ and $\mathcal{V}_b$ be the manifolds in 11.4 with $P = Q$ and $a = b$. To show that (π_B, S_*, b) is infinitesimally universal we must show that $\mathcal{U}_b$ and $\mathcal{V}_b$ intersect transversally at γ where γ is the inclusion of ∂N_b in Q_b. Since b is near b_0, so also is γ near γ_0. Consider the three subspaces $T_\gamma\mathcal{W}_b$, $T_\gamma\mathcal{U}_b$, $T_\gamma\mathcal{V}_b$, of $T_\gamma\mathcal{W}$. We have that $T_\gamma\mathcal{U}_b = T_\gamma\mathcal{W}_b \cap T_\gamma\mathcal{U}$ and the intersection is transverse as the projection $\mathcal{U} \to B$ is a submersion. Similarly, $T_\gamma\mathcal{V}_b = T_\gamma\mathcal{W}_b \cap T_\gamma\mathcal{V}$. Hence the subspaces $T_\gamma\mathcal{U}_b$ and $T_\gamma\mathcal{V}_b$ depend contiuously on (b, γ). By part (iv) of Theorem 11.9 the submanifolds $\mathcal{U}_{b_0}$ and $\mathcal{V}_{b_0}$ intersect transversally at γ_0, i.e. $T_{\gamma_0}\mathcal{W}_{b_0} = T_{\gamma_0}\mathcal{U}_{b_0} + T_{\gamma_0}\mathcal{V}_{b_0}$. Hence $T_\gamma\mathcal{W}_b = T_\gamma\mathcal{U}_b + T_\gamma\mathcal{V}_b$ for (b, γ) near (b_0, γ_0). Hence the unfolding (π_B, S_*, b) is infinitesimally universal for b near b_0 by Theorem 11.9 again as required. □

Proof of Theorem 5.4. We proved 'only if' in Section 5. For 'if' assume that the unfolding (π_B, S_*, b_0) is infinitesimally universal. Let (π_A, R_*, a_0) be another unfolding and $f_0 : P_{a_0} \to Q_{b_0}$ be a fiber isomorphism. Assume the notation introduced in Section 11. In particular assume the hypotheses of Theorem 11.9.

Step 1. *We show that $\mathcal{U}$ and $\mathcal{V}$ intersect transversally at (a_0, γ_0, b_0).*

Abbreviate $w_0 := (a_0, \gamma_0, b_0)$. Choose $\hat{w} \in T_{w_0}\mathcal{W}$ and let $\hat{a} = d\pi(w_0)\hat{w}$. As the restriction of π to $\mathcal{U}$ is a submersion there is a vector $\hat{u} \in T_{w_0}\mathcal{U}$ with $d\pi(w_0)\hat{u} = \hat{a}$. Then $\hat{w} - \hat{u} \in T_{w_0}\mathcal{W}_{a_0}$ so by part (iv) of Theorem 11.9 there are vectors $\hat{u}_0 \in T_{w_0}\mathcal{U}_{a_0}$ and $\hat{v}_0 \in T_{w_0}\mathcal{V}_{a_0}$ with $\hat{w} - \hat{u} = \hat{u}_0 + \hat{v}_0$.

Step 2. *The projection $\mathcal{U} \cap \mathcal{V} \to A$ is a holomorphic diffeomorphism.*

By Step 1 the intersection $\mathcal{U} \cap \mathcal{V}$ is a smooth submanifold of $\mathcal{W}$ (after shrinking) and $T_w(\mathcal{U} \cap \mathcal{V}) = T_w\mathcal{U} \cap T_w\mathcal{V}$ for $w \in \mathcal{U} \cap \mathcal{V}$. By the inverse function theorem it is enough to show that $d\pi(w_0) : T_w(\mathcal{U} \cap \mathcal{V}) \to T_{a_0}A$ is bijective. Injectivity follows from part (iv) of Theorem 11.9 and the fact that $T_{w_0}\mathcal{U}_{a_0} = T_{w_0}\mathcal{U} \cap \ker d\pi(w_0)$ and $T_{w_0}\mathcal{V}_{a_0} = T_{w_0}\mathcal{V} \cap \ker d\pi(w_0)$. We prove surjectivity. Choose $\hat{a} \in T_{a_0}A$. Since the restrictions of π to $\mathcal{U}$ and $\mathcal{V}$ are submersions, there exist tangent vectors $\hat{u} \in T_{w_0}\mathcal{U}$ and $\hat{v} \in T_{w_0}\mathcal{V}$ with $d\pi(w_0)\hat{u} = d\pi(w_0)\hat{v} = \hat{a}$. The difference $\hat{u} - \hat{v}$ lies in $T_{w_0}\mathcal{W}_{a_0}$ so, by part (iv) of Theorem 11.9, there are vectors $\hat{u}_0 \in T_{w_0}\mathcal{U}_{a_0}$ and $\hat{v}_0 \in T_{w_0}\mathcal{V}_{a_0}$ with $\hat{u} - \hat{v} = \hat{u}_0 + \hat{v}_0$. Hence $\hat{u} - \hat{u}_0 = \hat{v} + \hat{v}_0$ lies in $T_{w_0}(\mathcal{U} \cap \mathcal{V})$ and projects to $\hat{a}$.

Now define $\Phi : P \to Q$ and $\phi : A \to B$ by $\phi(a) := b_a$ and $\Phi|P_a := f_a$, where $f_a : P_a \to Q_{b_a}$ is the unique fiber isomorphism that satisfies $f_a|\partial N_a = \gamma_a$. (See Lemma 11.8.)

Step 3. *The maps ϕ and Φ and holomorphic.*

The map ϕ is the composition of the inverse of the projection $\mathcal{U} \cap \mathcal{V} \to A$ with the projection $\mathcal{U} \cap \mathcal{V} \to B$ and is therefore holomorphic by Step 2. By Lemma 10.18 the restriction of Φ to $\mathrm{int}(N)$ is holomorphic and hence smooth. To prove that the restriction of Φ to $\mathrm{int}(M)$ is holomorphic we write it as the composition

$$\mathrm{int}(M) \to A \times \Omega \to (\mathcal{U} \cap \mathcal{V}) \times \Omega \to \mathcal{Z}_0 \times \Omega \to Q$$

where the first map is the product $\pi_A \times \rho$, the second map is the inverse of the projection $\mathcal{U} \cap \mathcal{V} \to A$ on the first factor, the third map is given by the obvious embedding of $\mathcal{U} \cap \mathcal{V}$ into $\mathcal{V}_0 \cong \mathcal{Z}_0$, and the fourth map is the evaluation map $(a, v, b, z) \mapsto v(z)$. All five spaces are complex manifolds. In particular, the complex structure on $\mathcal{Z}_0 \times \Omega$ is

$$\left(\hat{a}, \hat{v}, \hat{b}, \hat{z}\right) \mapsto \left(\sqrt{-1}\hat{a}, J(v)\hat{v} - dv \cdot \alpha(a, \hat{a}), \sqrt{-1}\hat{b}, j(a)(z)\hat{z} + \alpha(a, \hat{a})(z)\right).$$

All four maps are holomorphic and hence, so is the restriction of Φ to $\mathrm{int}(M)$. Thus we have proved that Φ is holomorphic on $P \setminus \partial N$. Since Φ is continuous, it follows that Φ is holomorphic everywhere. This proves the theorem. □

Proof of Theorem 5.5. Assume that the unfolding (π_B, S_*, b_0) is infinitesimally universal and let (ϕ, Φ) is a pseudomorphism from (π_A, R_*, a_0) to (π_B, S_*, b_0). Then, in the notation of the proof of Theorem 5.4, we have that $(\gamma_a, b_a) := (\Phi|\partial N_a, \phi(a))$ is the unique intersection point of $\mathcal{U}_a$ and $\mathcal{V}_a$. Hence (ϕ, Φ) agrees with the unique (holomorphic) morphism constructed in the proof of Theorem 5.4. □

Proof of Theorem 5.6. We proved 'only if' in Section 5. For 'if' assume that (Σ, s_*, ν, j) is stable. We first consider the case where Σ is disconnected and there are no nodal points. Let $\Sigma_1, \dots, \Sigma_k$ be the components of Σ, g_j be the genus of Σ_j, and n_j be the number of marked points on Σ_j. Let $I_j \subset \{1, \dots, n\}$ be the index set associated to the marked points in Σ_j. Then $\{1, \dots, n\}$ is the disjoint union of the sets $I_1, \dots, I_k$ and $n_j = |I_j| > 2 - 2g_j$. By Theorem 8.9 there exists, for each j, a universal unfolding $(\pi_j : Q_j \to B_j, \{S_{ji}\}_{i \in I_j}, b_{0j}, v_{0j})$ of Σ_j. Note that $\dim_{\mathbb{C}}(B_j) = 3g_j - 3 + n_j$. Define

$$B_0 := B_1 \times \cdots \times B_k,$$

$$Q_0 := \bigsqcup_{j=1}^{k} B_1 \times \cdots \times B_{j-1} \times Q_j \times B_{j+1} \times \cdots \times B_k,$$

$$\pi_0(b_1, \dots, b_{j-1}, q_j, b_{j+1}, \dots, b_k) := (b_1, \dots, b_{j-1}, \pi_j(q_j), b_{j+1}, \dots, b_k),$$

$$S_{0i} := \{(b_1, \dots, b_{j-1}, q_j, b_{j+1}, \dots, b_k) \, : \, q_j \in S_{ij}\}, \qquad i \in I_j,$$

$$b_0 := (b_{01}, \dots, b_{0k}),$$

$$v_0(z) := (b_{01}, \dots, b_{0,j-1}, v_{0j}(z), b_{0,j+1}, \dots, b_{0k}), \qquad z \in \Sigma_j.$$

Then the quadruple $(\pi_0, S_{0*}, b_0, v_0)$ is a universal unfolding of Σ.

Next consider the general case. Denote the nodal points on Σ by $\nu = \{\{r_1, s_1\}, \dots, \{r_m, s_m\}\}$ and the marked points by $t_1, \dots, t_n$. Assume, without loss of generality, that the signature of (Σ, t_*, ν, j) is a connected graph (see Definition 3.4). Replace all the nodal points by marked points. Then, by what we have just proved, there exists a universal unfolding $(\pi_0, R_{0*}, S_{0*}, T_{0*}, b_0, v_0)$ of $(\Sigma, r_*, s_*, t_*, j)$. Choose disjoint open sets $U_1, \dots, U_m, V_1, \dots, V_m \subset Q_0$ such that

$$R_{0i} \subset U_i, \qquad S_{0i} \subset V_i, \qquad U_i \cap T_{0j} = V_i \cap T_{0j} = \emptyset$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Choose holomorphic functions $x_i : U_i \to \mathbb{C}$ and $y_i : V_i \to \mathbb{C}$ such that

$$x_i(R_{0i}) = 0, \qquad y_i(S_{0i}) = 0$$

and (π_0, x_i) and (π_0, y_i) are coordinates on Q_0. Shrink B and the open sets U_i, V_i, if necessary. Assume without loss of generality that $x_i(U_i) = y_i(V_i) = \mathbb{D}$. Define

$$B := B_0 \times \mathbb{D}^m, \qquad Q := Q_0 \times \mathbb{D}^m / \sim .$$

Two points (q, z) and (q', z) with $q \in U_i$ and $q' \in V_i$ are identified if and only if $\pi_0(q) = \pi_0(q')$ and either

$$x_i(q)y_i(q') = z_i \neq 0 \qquad \text{or} \qquad x_i(q) = y_i(q') = z_i = 0.$$

The equivalence relation on $Q_0 \times \mathbb{D}^m$ is generated by these identifications. (Two points (q, z) and (q', z) with $\pi_0(q) = \pi_0(q')$, $q \in U_i$, $q' \in V_i$, $z_i = 0$ are *not* identified in the case $x_i(q) = 0$ and $y_i(q') \neq 0$ nor in the case $x_i(q) \neq 0$ and $y_i(q') = 0$.) The projection $\pi : Q \to B$ and the sections $T_j : B \to Q$ are defined by

$$\pi([q, z]) := (\pi_0(q), z), \qquad T_j := \{[q, z] \,:\, q \in T_{0j}\}$$

for $j = 1, \dots, n$.

We have thus defined Q as a set. The manifold structure is defined as follows. For $i \in \{1, \dots, m\}$ denote by $C_i \subset Q$ the set of all equivalence classes $[q, z] \in Q$ that satisfy $z_i = 0$ and $q \in R_{0i}$. Note that any such point is equivalent to the pair $[q', z]$ with $q' \in S_{0i}$ and $\pi_0(q') = \pi_0(q)$. Let

$$C := \bigcup_{i=1}^{m} C_i.$$

The manifold structure on $Q \setminus C$ is induced by the product manifold structure on $Q_0 \times \mathbb{D}^m$. We now explain the manifold structure near C_i. Fix a constant $0 < \varepsilon < 1$ and define an open neighborhood $N_i \subset Q$ of C_i by

$$\begin{aligned} N_i \quad := \quad & C_i \cup \left\{ [q, z] \in Q : q \in U_i,\ \frac{|z_i|}{\varepsilon} < |x_i(q)| < \varepsilon \right\} \\ & \cup \left\{ [q, z] \in Q : q \in V_i,\ \frac{|z_i|}{\varepsilon} < |y_i(q)| < \varepsilon \right\}. \end{aligned}$$

A coordinate chart on N_i is the map

$$[q, z] \mapsto (b_0, z_1, \dots, z_{i-1}, x_i, y_i, z_{i+1}, \dots, z_m),$$

where $b_0 := \pi_0(q) \in B_0$,

$$x_i := \begin{cases} x_i(q), & \text{if } q \in U_i, \\ z_i / y_i(q), & \text{if } q \in V_i,\ z_i \neq 0, \\ 0, & \text{if } q \in V_i,\ z_i = 0, \end{cases}$$

(if $[q, z] \in N_i$ and $q \in V_i$ then $z_i \neq 0$ implies $y_i(q) \neq 0$), and

$$y_i := \begin{cases} y_i(q), & \text{if } q \in V_i, \\ z_i / x_i(q), & \text{if } q \in U_i,\ z_i \neq 0, \\ 0, & \text{if } q \in U_i,\ z_i = 0. \end{cases}$$

With this construction the transition maps are holomorphic and so Q is a complex manifold. In the coordinate chart on N_i the projection π has the form

$$(b_0, z_1, \dots, z_{i-1}, x_i, y_i, z_{i+1}, \dots, z_m) \mapsto (b_0, z), \qquad z_i := x_i y_i.$$

It follows that π is holomorphic, the critical set of π is C, and each critical point is nodal. Moreover, π restricts to a diffeomorphism from C_i onto the submanifold $\{z_i = 0\} \subset B$. Hence π is a regular nodal family (see Definition 12.1).

Denote $b := (b_0, 0) \in B$, let $\iota : Q_0 \to Q$ be the holomorphic map defined by $\iota(q) := [q, 0]$, and define $v : \Sigma \to Q$ by $v := \iota \circ v_0$. Then v is a desingularization of the fiber $Q_b = \pi^{-1}(b)$.

We prove that the triple (π, T_*, b) is a universal unfolding. Since the signature of the marked nodal Riemann surface Σ is a connected graph, the first Betti number of this graph is $1 - k + m$ (since m is the number of edges, i.e. of equivalent pairs of nodal points, and k is the number of vertices, i.e. of components of Σ). Hence the arithmetic genus g (see Definition 3.6) of the central fiber Q_b is given by

$$g - 1 = m + \sum_{j=1}^{k} (g_j - 1).$$

Now recall that n_j is the number of special points on Σ_j and

$$\sum_{j=1}^{k} n_j = n + 2m.$$

Since $\dim_{\mathbb{C}}(B_j) = 3g_j - 3 + n_j$ this implies

$$\dim_{\mathbb{C}}(B) = \dim_{\mathbb{C}}(B_0) + m = \sum_{j=1}^{k} (3g_j - 3 + n_j) + m = 3g - 3 + n.$$

Since the Riemann family $\pi : Q \to B$ is regular nodal it follows from lemma 12.2 that the operator $D_{v,b}$ (see Definition 5.2) has Fredholm index zero. Hence $D_{v,b}$ is bijective if and only if it is injective.

We prove in three steps that $D_{v,b}$ is injective. First, every vector $(\hat{v}, \hat{b}) \in \mathcal{X}_{v,b}$ with $\hat{b} =: (\hat{b}_0, \hat{z})$ satisfies $\hat{z} = 0$. To see this note that $d\pi(v)\hat{v} \equiv \hat{b}$. For every i there is a unique pair of equivalent nodal points in Σ that are mapped to C_i under v. Since the image of $d\pi$ at each point in C_i is contained in the subspace $\{\hat{z}_i = 0\}$ it follows that $\hat{z} = 0$. Second, we define a linear operator

$$\mathcal{X}_{v_0,b_0} \to X_{v,b} : (\hat{v}_0, \hat{b}_0) \mapsto (\hat{v}, \hat{b})$$

by

$$\hat{b} := (\hat{b}_0, 0), \qquad \hat{v}(s) := (\hat{v}_0(s), 0) \in T_{v(s)}Q,$$

for $s \in \Sigma \setminus \{r_1, \ldots, r_m, s_1, \ldots, s_m\}$. Then $\hat{v}$ extends uniquely to a smooth vector field along v. In the above coordinates on N_i the tangent vector $\hat{v}(r_i) = \hat{v}(s_i) \in T_{v(r_i)}Q = T_{v(s_i)}Q$ has the form $(\hat{x}_i, \hat{y}_i, \hat{b}_0, 0)$, where $\hat{x}_i := dx_i(v_0(r_i))\hat{v}_0(r_i)$ and $\hat{y}_i := dy_i(v_0(s_i))\hat{v}_0(s_i)$. It is easy to see that this operator is bijective. Third, since the map $\iota : Q_0 \to Q$ is holomorphic, it follows that

$$D_{v,b}(\hat{v}, \hat{b}) = d\iota(v) D_{v_0,b_0}(\hat{v}_0, \hat{b}_0).$$

Hence the operator $\mathcal{X}_{v_0,b_0} \to X_{v,b}$ restricts to a vector space isomorphism from the kernel of D_{v_0,b_0} to the kernel of $D_{v,b}$. By construction, the operator D_{v_0,b_0} is injective and hence, so is $D_{v,b}$. Thus we have proved that $D_{v,b}$ is bijective. Hence, by Theorem 5.4, the quadruple (π, T_*, b, v) is a universal unfolding of (Σ, s_*, ν, j). □

13 Topology

The orbit space of a groupoid inherits a topology from an orbifold structure on the groupoid. This topology is independent of the choice of the structure in the sense that equivalent orbifold structures determine the same topology (see Section 2). In the case of the Deligne–Mumford orbifold $\bar{\mathcal{M}}_{g,n}$, the topology has as a basis for the open sets the collection of all sets $\{[\Sigma_b]_{\mathcal{B}} : b \in U\}$ where $(\pi_B : Q \to B, S_*)$ is a universal family as in Definition 6.2, the functor

$$B \to \bar{\mathcal{B}}_{g,n} : b \mapsto \Sigma_b$$

is the corresponding orbifold structure as in Definition 6.4, and U runs over all open subsets of B. In section 14 we show that $\bar{\mathcal{M}}_{g,n}$ is compact and Hausdorff. (See Example 2.8 for an example which shows why it is not obvious that the moduli space is Hausdorff.) For this purpose we introduce in this section a notion of convergence of sequences of marked nodal Riemann surfaces which we call DM-convergence.

13.1. Let Σ be a compact oriented surface and $\gamma \subset \Sigma$ be a disjoint union of embedded circles. We denote by Σ_γ the compact surface with boundary which results by **cutting open** Σ along γ. This implies that there is a local embedding

$$\sigma : \Sigma_\gamma \to \Sigma$$

which maps $\mathrm{int}(\Sigma_\gamma)$ one to one onto $\Sigma \setminus \gamma$ and maps $\partial\Sigma_\gamma$ two to one onto γ. One might call σ the *suture map* and γ the *incision.*

Definition 13.2. Let (Σ', ν') and (Σ, ν) be nodal surfaces. A smooth map $\phi : \Sigma' \setminus \gamma' \to \Sigma$ is called a (ν', ν)**-deformation** iff $\gamma' \subset \Sigma' \setminus \bigcup \nu'$ is a disjoint union of embedded circles such that (where $\sigma : \Sigma'_{\gamma'} \to \Sigma'$ is the suture map just defined) we have

- $\phi_*\nu' := \big\{\{\phi(y'_1), \phi(y'_2)\} : \{y'_1, y'_2\} \in \nu'\big\} \subset \nu.$
- ϕ is a diffeomorphism from $\Sigma' \setminus \gamma'$ onto $\Sigma \setminus \gamma$, where $\gamma := \bigcup(\nu \setminus \phi_*\nu')$.
- $\phi \circ \sigma|\mathrm{int}(\Sigma'_{\gamma'})$ extends to a continuous surjective map $\Sigma'_{\gamma'} \to \Sigma$ such that the preimage of each nodal point in γ is a component of $\partial\Sigma'_{\gamma'}$ and two boundary components which map under σ to the same component of γ' map to a nodal pair $\{x, y\} \in \gamma$.

A sequence $\phi_k : (\Sigma_k \setminus \gamma_k, \nu_k) \to (\Sigma, \nu)$ of (ν_k, ν)-deformations is called **monotypic** iff $(\phi_k)_*\nu_k$ is independent of k.

Definition 13.3. A sequence $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ of marked nodal Riemann surfaces of type (g, n) is said to **converge monotypically** to a marked nodal Riemann surface (Σ, s_*, ν, j) of type (g, n) iff there is a monotypic sequence $\phi_k : \Sigma_k \setminus \gamma_k \to \Sigma$ of (ν_k, ν)-deformations such that for $i = 1, \ldots, n$ the sequence $\phi_k(s_{k,i})$ converges to s_i in Σ, and the sequence $(\phi_k)_* j_k$ of complex structures on $\Sigma \setminus \gamma$ converges to $j|(\Sigma \setminus \gamma)$ in the C^∞ topology. The sequence $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ is said to **DM-converge** to (Σ, j, s, ν) iff, after discarding finitely many terms, it is the disjoint union of finitely many sequences which converge monotypically to (Σ, s, ν, j).

Remark 13.4. Assume that $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ DM-converges to (Σ, s_*, ν, j), that $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ is isomorphic to $(\Sigma'_k, s'_{k,*}, \nu'_k, j'_k)$, and that (Σ, s_*, ν, j) is isomorphic to $(\Sigma', s'_*, \nu', j')$. Then $(\Sigma'_k, s'_{k,*}, \nu'_k, j'_k)$ DM-converges to $(\Sigma', s'_*, \nu', j')$.

Remark 13.5. Our definition of deformation agrees with [10, page 79]. Our definition of monotypic convergence is Hummel's definition of weak convergence to cusp curves in [10, page 80] (with the target manifold M a point) except that he does not allow marked points. However, the conclusion of Proposition 5.1 in [10, page 71] allows marked points in the guise of what Hummel calls degenerate boundary components. We will apply Proposition 5.1 of [10] in the proof of Theorem 14.5 below after some preliminary constructions to fit its hypotheses.

Theorem 13.6. *Let $(\pi : Q \to B, S_*, b_0)$ be a universal unfolding of a marked nodal Riemann surface $(\Sigma_0, s_{0,*}, \nu_0, j_0)$ of type (g, n) and $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ be a sequence of marked nodal Riemann surfaces of type (g, n). Then the following are equivalent.*

(i) *The sequence $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ DM-converges to $(\Sigma_0, s_{0,*}, \nu_0, j_0)$.*

(ii) *After discarding finitely many terms there is a sequence $b_k \in B$ such that b_k converges to b_0 and $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ arises from a desingularization $u_k : \Sigma_k \to Q_{b_k}$.*

We postpone the proof of Theorem 13.6 till after we treat the analogous theorem for fiber isomorphisms.

Definition 13.7. Let $(\pi_A : P \to A, R_*, a_0)$ and $(\pi_B : Q \to B, S_*, b_0)$ be two unfoldings of type (g, n) and $f_k : P_{a_k} \to Q_{b_k}$ be a sequence of fiber isomorphisms. The sequence f_k **DM-converge** to a fiber isomorphism $f_0 : P_{a_0} \to Q_{b_0}$ iff $a_k \to a_0$, $b_k \to b_0$, and for every Hardy decomposition $P = M \cup N$ as in Definition 11.2 the sequence $f_k \circ \iota_{a_k}$ converges to $f_0 \circ \iota_{a_0}$ in the C^∞ topology.

Theorem 13.8. *Let $(\pi_A : P \to A, R_*, a_0)$ and $(\pi_B : Q \to B, S_*, b_0)$ be two universal unfoldings of type (g, n) and $a_k \to a_0$ and $b_k \to b_0$ be convergent sequences. Let $(\Phi, \phi) : (P, A) \to (Q, B)$ be the germ of a morphism satisfying $\phi(a_0) = b_0$ and $\Phi_{a_0} = f_0$. Then the following are equivalent.*

(i) *The sequence (a_k, f_k, b_k) DM-converges to (a_0, f_0, b_0).*

(ii) *For k sufficiently large we have $\phi(a_k) = b_k$ and $\Phi_{a_k} = f_k$.*

Proof. That (ii) implies (i) is obvious. We prove that (i) implies (ii). Recall the Hardy decomposition as in the definition of the spaces $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ in 11.4. The proof of Theorem 5.4 in Section 12 shows that

$$\big(a, \Phi_a|\partial N \cap P_a, \phi(a)\big) = \mathcal{U}_a \cap \mathcal{V}_a.$$

But $(a_k, f_k|\partial N \cap P_{a_k}, b_k) \in \mathcal{U}_{a_k} \cap \mathcal{V}_{a_k} \subset \mathcal{W}$ for k sufficiently large by DM-convergence. Both sequences $(a_k, \Phi_{a_k}|\partial N \cap P_{a_k}, \phi(a_k))$ and $(a_k, f_k|\partial N \cap P_{a_k}, b_k)$ converge to the same point $(a_0, f_0|\partial N \cap P_{a_0}, b_0)$. Hence by transversality in Theorem 11.9 they are equal for large k. Now use Lemma 11.8. □

Proof of Theorem 13.6. We prove that (ii) implies (i). Let $u_0 : \Sigma_0 \to Q_{b_0}$ be a desingularization. Assume that b_k converges to b and that $u_k : \Sigma_k \to Q_{b_k}$ is a sequence of desingularizations. Choose a Hardy trivialization $(Q = M \cup N, \iota, \rho)$ for (π, S_*, b_0) as in Definition 11.6. For each $b \in B$ choose a smooth map

$$\psi_b : Q_b \to Q_{b_0}$$

as follows. The restriction of ψ_b to M_b agrees with ρ_b. Next, using the nodal coordinates of Definition 11.2, extend ψ_b to a neighborhood of the common boundary of M and N via $(x_i, 0, t_i) \mapsto (x_i, 0, 0)$ for $2\sqrt{|z_i(b)|} \leq |x_i| \leq 1$ and $(0, y_i, t_i) \mapsto (0, y_i, 0)$ for $2\sqrt{|z_i(b)|} \leq |y_i| \leq 1$. Finally, when $z_i(b) \neq 0$, extend to a smooth map $Q_b \cap N_i \to Q_{b_0} \cap N_i$ that maps the circle $|x_i| = |y_i| = \sqrt{|z_i(b)|}$ onto the nodal point q_i and is a diffeomorphism from the complement of this circle in $Q_b \cap N_i$ onto the complement of q_i in $Q_{b_0} \cap N_i$. Denote

$$\gamma_k := \bigcup_{z_i(b_k) \neq 0} u_k^{-1}\big(\{q \in Q_b \cap N_i \,:\, |x_i(q)| = |y_i(q)| = \sqrt{|z_i(b_k)|}\}\big) \subset \Sigma_k.$$

Then, for every k, there is a unique smooth map $\phi_k : \Sigma_k \setminus \gamma_k \to \Sigma_0$ such that

$$u_0 \circ \phi_k = \psi_{b_k} \circ u_k : \Sigma_k \setminus \gamma_k \to Q_{b_0}.$$

It follows that ϕ_k is a sequence of deformations as in Definition 13.2 and that this sequence satisfies the requirements of Definition 13.3. (The sequence ϕ_k is monotypic whenever there is an index set I such that, for every k, we have $z_i(b_k) = 0$ for $i \in I$ and $z_k(b_i) \neq 0$ for $i \notin I$; after discarding finitely many terms, we can write ϕ_k as a finite union of monotypic sequences.) Hence the sequence $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ DM-converges to (Σ, j, s_*, ν). Thus we have proved that (ii) implies (i).

We prove that (i) implies (ii). Let $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ be a sequence of marked nodal Riemann surfaces of type (g, n) that DM-converges to (Σ, j, s_*, ν). If Σ has no nodes then Σ_k has no nodes and the maps $\phi_k : \Sigma_k \to \Sigma$ in the definition of DM-convergence are diffeomorphisms. Since $(\phi_k)_* j_k$ converges to j, assertion (ii) follows from the fact that a slice in $\mathcal{J}(\Sigma)$ determines a universal unfolding. The same reasoning works when $(\Sigma_k, s_{k,*}, \nu_k)$ has the same signature as (Σ, s_*, ν). To avoid excessive notation we consider the case where (Σ, ν) has precisely one node and (Σ_k, ν_k) has no nodes, i.e.

$$\nu_k = \emptyset, \qquad \nu =: \{\{z_0, z_\infty\}\}.$$

Choose holomorphic diffeomorphisms

$$x : (\Delta_0, z_0) \to (\mathbb{D}, 0), \qquad y : (\Delta_\infty, z_\infty) \to (\mathbb{D}, 0),$$

where $\Delta_0, \Delta_\infty \subset \Sigma$ are disjoint closed disks centered at z_0, z_∞ respectively and

$$\Delta := \Delta_0 \cup \Delta_\infty$$

does not contain any marked points. For $\delta \in (0,1)$ let $\Delta(\delta) := \Delta_0(\delta) \cup \Delta_\infty(\delta)$ where

$$\Delta_0(\delta) := \{p \in \Delta_0 : |x(p)| \leq \delta\}, \qquad \Delta_\infty(\delta) := \{q \in \Delta_\infty : |y(q)| \leq \delta\}.$$

A decreasing sequence $\delta_k \in (0,1)$ converging to zero determines a sequence of decompositions

$$\Sigma = \Omega_k \cup \Delta(\delta_k), \qquad \partial\Omega_k = \partial\Delta(\delta_k) = \Omega_k \cap \Delta(\delta_k).$$

Thus Ω_k is obtained from Σ by removing a nested sequence of pairs of open disks centered at the nodal points so $\bigcup_k \Omega_k = \Sigma \setminus \{z_0, z_\infty\}$, $\Omega_k \subset \Omega_{k+1}$, and $\Omega_k \cap \Delta = (\Omega_k \cap \Delta_0) \cup (\Omega_k \cap \Delta_\infty)$ is a disjoint union of two closed annuli.

Claim. *There are sequences of real numbers δ_k, r_k, θ_{0k}, $\theta_{\infty k}$, smooth embeddings*

$$f_k : \Omega_k \to \Sigma_k, \qquad \xi_k : \mathbb{A}(\delta_k, 1) \to \mathbb{A}(r_k, 1), \qquad \eta_k : \mathbb{A}(\delta_k, 1) \to \mathbb{A}(r_k, 1),$$

and holomorphic diffeomorphisms

$$h_k : \mathbb{A}(r_k, 1) \to A_k := \Sigma_k \setminus f_k(\Sigma \setminus \Delta),$$

satisfying the following conditions.

1) *$f_k^* j_k$ converges to j in the C^∞ topology on $\Sigma \setminus \{z_0, z_\infty\}$.*

2) *$f_k^* j_k$ is equal to j on $\Omega_k \cap \Delta(\frac{1}{2})$.*

3) $\xi_k(S^1) = \eta_k(S^1) = S^1$.

4) *$h_k(\xi_k(x(p))) = f_k(p)$ for $p \in \Omega_k \cap \Delta_0$.*

5) *$h_k\left(\dfrac{r_k}{\eta_k(y(q))}\right) = f_k(q)$ for $q \in \Omega_k \cap \Delta_\infty$.*

6) *$\xi_k(x(p)) = e^{i\theta_{0k}} x(p)$ for $p \in \Omega_k \cap \Delta_0(\frac{1}{2})$.*

7) *$\eta_k(y(q)) = e^{i\theta_{\infty k}} y(q)$ for $q \in \Omega_k \cap \Delta_\infty(\frac{1}{2})$.*

Proof of the Claim. Let $\delta_k \in (0,1]$ be any sequence decreasing to zero, for example $\delta_k := 1/k$, and denote $\Omega_k := \Sigma \setminus \mathrm{int}(\Delta(\delta_k))$. Define $f_k : \Omega_k \to \Sigma_k$ by $f_k := \left(\phi_k|\phi_k^{-1}(\Omega_k)\right)^{-1}$ where $\phi_k : \Sigma_k \setminus \gamma_k \to \Sigma$ is as in Definition 13.3. Then f_k satisfies 1). We will modify δ_k and f_k to satisfy the other conditions.

By the path lifting arguments used in the proof of Theorem 7.1 (see also Appendix C.5 of [16]) there is a sequence of holomorphic embeddings

$$g_k : (\Omega_k \cap \Delta, j) \to (\Omega_k, f_k^* j_k)$$

that converges to the identity in the C^∞ topology and preserves the boundary of Ω_k. Extend g_k to a diffeomorphism, still denoted by $g_k : \Omega_k \to \Omega_k$, so that the extensions converge to the identity in the C^∞ topology and replace f_k by $f_k \circ g_k$. This new sequence satisfies 1) and 2); in fact, f_k is now holomorphic on $\Omega_k \cap \Delta$. (Below we modify f_k again.)

The set $A_k \subset \Sigma_k$ is an annulus with boundary $f_k(\partial\Delta_0) \cup f_k(\partial\Delta_\infty)$ so there is a unique number $r_k > 0$ and a holomorphic diffeomorphism $h_k : \mathbb{A}(r_k, 1) \to A_k$, unique up to composition with a rotation, such that

$$h_k(S^1) = f_k(\partial\Delta_0), \qquad h_k(r_k S^1) = f_k(\partial\Delta_\infty).$$

The embeddings $\xi_k : \mathbb{A}(\delta_k, 1) \to \mathbb{A}(r_k, 1)$ and $\eta_k : \mathbb{A}(\delta_k, 1) \to \mathbb{A}(r_k, 1)$ defined by 4) and 5) satisfy 3); they are holomorphic because f_k is holomorphic on $\Omega_k \cap \Delta$. Hence by Lemma 13.10 below $r_k < \delta_k$ and so r_k converges to zero.

By Lemma 13.11 below, there are sequences $\varepsilon_k > 0$, $\rho_k > \delta_k$, and $\theta_{0k}, \theta_{\infty k} \in [0, 2\pi]$ such that ε_k and ρ_k converge to zero and

$$\left|x^{-1}\xi_k(x) - e^{i\theta_{0k}}\right| \le \varepsilon_k, \qquad \left|y^{-1}\eta_k(y) - e^{i\theta_{\infty k}}\right| \le \varepsilon_k,$$

for $x, y \in \mathbb{A}(\rho_k, 1)$. To see this let $\delta(m) > 0$ be the constant of Lemma 13.11 with $\varepsilon = \rho = 1/m$, choose an increasing sequence of integers k_m such that $\delta_k \le \delta(m)$ for $k \ge k_m$, and define $\varepsilon_k := \rho_k := 1/m$ for $k_m \le k < k_{m+1}$. We call this kind of argument *proof by patience*. It follows that the maps $x \mapsto e^{-i\theta_{0k}}\xi_k(x)$ and $y \mapsto e^{-i\theta_{\infty k}}\eta_k(y)$ converge to the identity uniformly with all derivatives on every compact subset of $\mathrm{int}(\mathbb{D}) \setminus 0$.

Next we construct two sequences of diffeomorphisms $\alpha_k, \beta_k : \mathbb{D} \to \mathbb{D}$, converging to the identity in the C^∞ topology, and an exhausting sequence of closed annuli $B_k \subset \mathrm{int}(\mathbb{D}) \setminus 0$, such that α_k and β_k are equal to the identity in a neighborhood of $S^1 = \partial\mathbb{D}$ and

$$\xi_k(\alpha_k(x)) = e^{i\theta_{0k}}x, \qquad \eta_k(\beta_k(y)) = e^{i\theta_{\infty k}}y$$

for $x, y \in B_k$. The assertion is obvious by an interpolation argument when the sequence B_k is replaced by a single closed annulus $B \subset \mathrm{int}(\mathbb{D}) \setminus 0$. Now argue by patience as above.

Increasing δ_k if necessary we may assume that $A(\delta_k, \frac{1}{2}) \subset B_k$ for all k. Denote $\Omega_k := \Sigma \setminus \mathrm{int}(\Delta(\delta_k))$ as above. Now replace ξ_k by $\xi_k \circ \alpha_k$ and η_k by $\eta_k \circ \beta_k$. These functions satisfy 6) and 7). Redefine f_k so that 4) and 5) hold with the new definitions of ξ_k and η_k. Thus we have proved the claim.

In the following we assume w.l.o.g. that $(\pi : Q \to B, S_*, b_0)$ is the universal unfolding of (Σ, s_*, ν, j) constructed in the proof of Theorem 5.6. Now define the marked Riemann surface $(\Sigma'_k, s'_{k,*}, j'_k)$ by

$$\Sigma'_k := \frac{\Sigma \setminus \mathrm{int}(\Delta(\sqrt{r_k}))}{\sim},$$

where

$$j'_k := f_k^* j_k, \qquad s'_{k,i} := f_k^{-1}(s_{k,i}),$$

and where the equivalence relation is defined by

$$p \sim q \qquad \iff \qquad x(p)y(q) = z_k, \qquad z_k := r_k e^{-i(\theta_{0k}+\theta_{\infty k})}$$

for $p \in \Delta_0$ and $q \in \Delta_\infty$ with $|x(p)| = |y(q)| = \sqrt{r_k}$. Then, after removing finitely many terms, there is a sequence of regular values $b_k \in B$ of $\pi : Q \to B$ and a sequence of desingularizations $u'_k : \Sigma'_k \to Q_{b_k}$ such that

$$u'_k(s'_{k,i}) = S_i \cap Q_{b_k}$$

and j'_k is the pullback of the complex structure on Q_{b_k} under u'_k. This follows from Theorem 8.9 and the construction of a universal unfolding in the proof of Theorem 5.6. Moreover, $(\Sigma'_k, s'_{k,*}, j'_k)$ is isomorphic to $(\Sigma_k, s_{k,*}, j_k)$. An explicit isomorphism is the map $\psi_k : \Sigma'_k \to \Sigma_k$ defined by

$$\psi_k(p) := \begin{cases} f_k(p) & \text{for } p \in \Sigma'_k \setminus \Delta, \\ h_k(\xi_k(x(p))) & \text{for } p \in \Delta_0 \text{ with } \delta_k \le |x(p)| \le 1, \\ h_k(e^{i\theta_{0k}} x(p)) & \text{for } p \in \Delta_0 \text{ with } \sqrt{r_k} \le |x(p)| \le \delta_k, \\ h_k\left(\dfrac{r_k}{\eta_k(y(p))}\right) & \text{for } p \in \Delta_\infty \text{ with } \delta_k \le |y(p)| \le 1, \\ h_k\left(\dfrac{r_k}{e^{i\theta_{\infty k}} y(p)}\right) & \text{for } p \in \Delta_\infty \text{ with } \sqrt{r_k} \le |y(p)| \le \delta_k. \end{cases}$$

That (i) implies (ii) in the case of a single node follows immediately with

$$u_k := u'_k \circ \psi_k^{-1} : \Sigma_k \to Q_{b_k}.$$

The case of several nodes is analogous. This proves Theorem 13.6. □

Remark 13.9. The sequence u_k just constructed is such that $u_k(\gamma_k)$ converges to the nodal set in Q_{b_0} and $u_k \circ \phi_k^{-1} : \Sigma \setminus \cup\nu \to Q$ converges to $u_0|(\Sigma \setminus \cup\nu)$ in the C^∞ topology. To prove this, note that

$$\psi_k^{-1}(\gamma_k) \subset \{[p] = [q] \in \Sigma'_k \,:\, \sqrt{r_k} \le |x(p)|, |y(q)| \le \delta_k\}.$$

Hence, by Step 1, $u_k(\gamma_k)$ converges to the nodal point in Q_{b_0}. Moreover, the main part of Σ'_k can be identified with the subset $\Omega_k \subset \Sigma$ (exhausting $\Sigma \setminus \cup\nu$ in the limit $k \to \infty$), the sequence u'_k converges to u_0 on $\Sigma \setminus \cup\nu$ under this identification, and $\phi_k \circ \psi_k : \Omega_k \to \Sigma$ converges to the identity under this identification.

Lemma 13.10. *If there is a holomorphic map $f : \mathbb{A}(r_1, R_1) \to \mathbb{A}(r_2, R_2)$ inducing an isomorphism of fundamental groups, then $R_1/r_1 \le R_2/r_2$.*

Proof. The result is due to Huber [9]; an exposition appears in [14, Theorem 6.1, page 14]. The proof uses the Schwarz Pick Ahlfors Lemma (a holomorphic map from the unit disk to itself is a contraction in the Poincaré metric). The circle of radius $\sqrt{r_1 R_1}$ is a geodesic in the hyperbolic metric of length $2\pi^2/\log(R_1/r_1)$; its image under f is shorter and hence so is the central geodesic in $\mathbb{A}(r_2, R_2)$. □

Lemma 13.11. *For every $\varepsilon > 0$ and every $\rho > 0$ there is a constant $\delta \in (0, \rho)$ such that the following holds. If $u : \mathbb{A}(\delta, 1) \to \mathbb{D} \setminus 0$ is a holomorphic embedding such that $u(S^1) = S^1$ then there is a real number θ such that*

$$x \in \mathbb{A}(\rho, 1) \qquad \Longrightarrow \qquad \left| x^{-1} u(x) - e^{i\theta} \right| < \varepsilon.$$

Proof. It suffices to assume $u(1) = 1$ and then prove the claim with $\theta = 0$. Suppose, by contradiction that there exist constants $\varepsilon > 0$ and $\rho > 0$ such that the assertion is wrong. Then there exists a sequence $\delta_i > 0$ converging to zero and a sequence of holomorphic embeddings $u_i : \mathbb{A}(\delta_i, 1) \to \mathbb{D} \setminus 0$ such that

$$u_i(S^1) = S^1, \qquad u_i(1) = 1, \qquad \sup_{\rho \le |x| \le 1} |u_i(x) - x| \ge \varepsilon\rho.$$

We claim that u_i converges to the identity, uniformly on every compact subset of $\mathbb{D} \setminus 0$. To see this extend u_i to the annulus $\mathbb{A}(\delta_i, 1/\delta_i)$ by the formula

$$u_i(z) := \frac{1}{\overline{u_i(1/\bar{z})}}$$

for $1 \le |z| \le 1/\delta_i$. Think of the extended map as a holomorphic embedding $u_i : \mathbb{A}(\delta_i, 1/\delta_i) \to S^2 \setminus \{0, \infty\}$. Next we claim that

$$\sup_i \sup_{z \in K} |du_i(z)| < \infty \tag{53}$$

for every compact subset $K \subset \mathbb{C} \setminus 0$. Namely, the energy of the holomorphic curve u_i is bounded by the area of the target manifold S^2. So if $|du_i(z_i)| \to \infty$ for some sequence $z_i \to z_0 \in \mathbb{C} \setminus 0$, then a holomorphic sphere bubbles off near z_0 and it follows that a subsequence of u_i converges to a constant, uniformly on every compact subset of $\mathbb{C} \setminus \{0, z_0\}$. But this contradicts the fact that $u_i(S^1) = S^1$. Thus we have proved (53). Now it follows from the standard elliptic bootstrapping techniques (or alternatively from Cauchy's integral formula and the Arzela–Ascoli theorem) that there is a subsequence, still denoted by u_i, that converges in the C^∞ topology to a holomorphic curve $u_0 : \mathbb{C} \setminus 0 \to S^2 \setminus \{0, \infty\}$. By the removable singularity theorem, u_0 extends to a holomorphic curve $u_0 : S^2 \to S^2$. Since u_i is an embedding for every i, it follows that u_0 is an embedding and hence a Möbius transformation. Since $u_i(S^1) = S^1$, $u_i(1) = 1$ and $0 \notin u_i(\mathbb{A}(\delta_i, 1))$, it follows that

$$u_0(S^1) = S^1, \qquad u_0(1) = 1, \qquad u_0(0) = 0.$$

This implies that $u_0 = \mathrm{id}$. Thus we have proved that u_i converges to the identity, uniformly on every compact subset of $\mathbb{D} \setminus 0$. This contradicts the inequality $\sup_{\rho \le |x| \le 1} |u_i(x) - x| \ge \varepsilon\rho$ and this contradiction proves the lemma. □

14 Compactness

In this section we prove that every sequence of stable marked nodal Riemann surfaces of type (g, n) has a DM-convergent subsequence. Our strategy is to perform some preliminary constructions to reduce our compactness theorem to Proposition 5.1 of [10, page 71]. We begin by rephrasing Hummel's result in a weaker form that we will apply directly (see Proposition 14.4 below).

14.1. Let W be a smooth oriented surface, possibly with boundary and not necessarily compact or connected. A **finite extension** of W is a smooth orientation preserving embedding $\iota : W \to S$ into a compact oriented surface S such that $S \setminus \iota(W)$ is finite. If $\iota_1 : W \to S_1$ and $\iota_2 : W \to S_2$ are two such extensions, the map $\iota_2 \circ \iota_1^{-1}$ extends to a homeomorphism, but not necessarily to a diffeomorphism. Let $W_1, \ldots, W_\ell$ be the components of W, $S_1, \ldots, S_\ell$ be the corresponding components of a finite extension S, g_i be the genus of S_i, m_i be the number of boundary components of W_i, and n_i be the number of points in $S_i \setminus \iota(W_i)$. The (unordered) list (g_i, m_i, n_i) is called the **signature** of W. Two surfaces of finite type are diffeomorphic if and only if they have the same signature. (Compare 3.4 and 3.5.) We say that W is of **stable type** if $n_i > \chi(S_i)$ (at least one puncture point on an annulus or torus, at least two on a disk, and at least three on a sphere).

14.2. A **hyperbolic metric** on W is a complete Riemannian metric h of constant curvature -1 such that each boundary component is a closed geodesic. A **finite extension** of a complex structure j on W is a finite extension $\iota : W \to S$ such that $\iota_* j$ extends to a complex structure on S; we say j has **finite type** if it admits a finite extension.

Proposition 14.3. *Let W be a surface of stable type. Then the operation which assigns to each hyperbolic metric on W its corresponding complex structure (rotation by 90°) is bijective. It restricts to a bijection between hyperbolic metrics of finite area and complex structures of finite type.*

Proof. The operation $h \mapsto j$ is injective by the removable singularities theorem and surjective by applying the uniformization theorem to the holomorphic double. If j is of finite type, then the area is finite by [10, Proposition 3.9 page 68]. If h is of finite area, then j is of finite type by [10, Proposition 3.6 page 65]. □

Proposition 14.4 (Mumford–Hummel)**.** *Let S be a compact connected surface with boundary and $x_1, \ldots, x_n$ be a sequence of marked points in the interior of S such that $W := S \setminus \{x_1, \ldots, x_n\}$ is of stable type. Denote*

$$\partial S =: \partial_1 S \cup \cdots \cup \partial_m S,$$

where each $\partial_i S$ is a circle. Let j_k be a sequence of complex structures on S and h_k be the corresponding sequence of hyperbolic metrics on W. Assume:

(a) *The lengths of the closed geodesics in $W \setminus \partial W$ are bounded away from zero.*

(b) *The lengths of the boundary geodesics converge to zero.*

Then there exists a subsequence, still denoted by (j_k, h_k), a closed Riemann surface (Σ, j) with distinct marked points $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m$, a hyperbolic metric h of finite area on $\Sigma \setminus \{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m\}$, and a sequence of continuous maps $\phi_k : S \to \Sigma$ satisfying the following conditions.

(i) *$\phi_k(x_i) = \xi_i$ for $i = 1, \ldots, n$ and $\phi_k(\partial_i S) = \eta_i$ for $i = 1, \ldots, m$.*

(ii) *The restriction of ϕ_k to $S \setminus \partial S$ is a diffeomorphism onto $\Sigma \setminus \{\eta_1, \ldots, \eta_m\}$.*

(iii) *$(\phi_k)_* j_k$ converges to j on $\Sigma \setminus \{\eta_1, \ldots, \eta_m\}$.*

(iv) *$(\phi_k)_* h_k$ converges to h on $\Sigma \setminus \{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m\}$.*

Proof. This follows from Proposition 5.1 in [10, page 71]. The discussion preceding Proposition 5.1 in [10] explains how to extract the subsequence and how to construct the Riemann surface (Σ, j) and the hyperbolic metric h. □

Theorem 14.5. *Every sequence of stable marked nodal Riemann surfaces of type (g, n) has a DM-convergent subsequence.*

Proof. Let $(\Sigma_k, s_{k,*}, \nu_k, j_k)$ be a sequence of marked nodal Riemann surfaces of type (g, n). Passing to a subsequence, if necessary, we may assume that all marked nodal surfaces in our sequence have the same signature (see Definition 3.4) and hence are diffeomorphic. Thus we assume that

$$(\Sigma_k, s_{k,*}, \nu_k) = (\Sigma, s_*, \nu)$$

is independent of k. Denote by Σ^* the possibly disconnected and noncompact surface obtained from Σ by removing the special points. Let h_k be the hyperbolic metric on Σ^* determined by j_k (see Proposition 14.3).

Let ℓ_k^1 be the length of the shortest geodesic in Σ^* with respect to h_k. If a subsequence of the ℓ_k^1 is bounded away from zero we can apply Proposition 14.4 to each component of Σ and the assertion follows. Namely, the maps ϕ_k in Proposition 14.4 are deformations as in Definition 13.2.

Hence assume ℓ_k^1 converges to zero as k tends to infinity and, for each k, choose a geodesic γ_k^1 with length ℓ_k^1. Passing to a further subsequence and, if necessary, modifying h_k by a diffeomorphism that fixes the marked and nodal points we may assume that the geodesics γ_k^1 are all homotopic and indeed equal. Thus

$$\gamma_k^1 = \gamma^1$$

for every k. Now let ℓ_k^2 be the length of the shortest geodesic in $\Sigma \setminus \gamma^1$ with respect to h_k. If a subsequence of ℓ_k^2 is bounded away from zero we cut open

Σ along γ^1. Again the assertion follows by applying Proposition 14.4 to each component of the resulting surface with boundary.

Continue by induction. That the induction terminates follows from the fact that the geodesics in (Σ^*, h_k) of lengths at most 2arcsinh(1) are pairwise disjoint and their number is bounded above by $3g - 3 + N$, where N is the number of special points (see [10, Lemma 4.1 page 68]). This proves the theorem. $\square$

Lemma 14.6. *Let $(\pi : P \to A, R_*, a_0)$ be a nodal unfolding and $C \subset P$ be the set of critical points of π. Then, after shrinking A if necessary, there exists a closed subset $V \subset P$ and a smooth map*

$$\rho : P \setminus V \to P_{a_0} \setminus V$$

satisfying the following conditions.

(i) *For every $a \in A$ we have $C \cap P_a \subset V \cap P_a =: V_a$; moreover $C \cap P_{a_0} = V_{a_0}$.*

(ii) *Each component of V intersects P_a either in a simple closed curve or in a nodal point.*

(iii) *For every $a \in A$ the restriction $\rho_a := \rho|P_a \setminus V_a : P_a \setminus V_a \to P_{a_0} \setminus V_{a_0}$ is a diffeomorphism; moreover $\rho_{a_0} = \mathrm{id}$.*

Proof. Choose a Hardy trivialization $(P = M \cup N, \iota, \rho)$ as in 11.6 and write

$$N = N_1 \cup \cdots \cup N_k.$$

as in Definition 11.2. Let $(z_i, t_i) : A \to U_i \subset \mathbb{D} \times \mathbb{C}^{m-1}$ and $(x_i, y_i, t_i) : N_i \to \mathbb{D}^2 \times \mathbb{C}^{m-1}$ be the holomorphic coordinates of Definition 11.2 so that $z_i(\pi(p)) = x_i(p) y_i(p)$ and the critical set $C \subset P$ has components

$$C_i := \{p \in N_i \, : \, x_i(p) = y_i(p) = 0\}.$$

Define

$$V := V_1 \cup \cdots \cup V_k, \qquad V_i := \left\{ p \in N_i \, : \, |x_i(p)| = |y_i(p)| = \sqrt{|z_i(\pi(p))|} \right\}.$$

This set satisfies (i) and (ii). The restriction of the trivialization $\rho : M \to M_{a_0}$ to $\partial N_i \subset \partial N = \partial M$ is, in the above coordinates, given by $\rho(x_i, y_i, t_i) = (x_i, 0, t_i)$ for $|x_i| = 1$ and by $\rho(x_i, y_i, t_i) = (0, y_i, t_i)$ for $|y_i| = 1$. We extend this map by an explicit formula. Choose a smooth cutoff function $\beta : [1, \infty) \to [0, 1]$ such that $\beta'(r) \geq 0$ for every r and

$$\beta(r) := \begin{cases} r - 1, & \text{for } 1 \leq r \leq 3/2, \\ 1, & \text{for } r \geq 2. \end{cases}$$

Then define the extension $\rho : N_i \setminus V_i \to P_{a_0}$ in local coordinates by

$$\rho(x_i, y_i, t) := \begin{cases} \left(\beta\left(\sqrt{|x_i|/|y_i|}\right) x_i, 0, t_i\right), & \text{if } |x_i| > |y_i|, \\ \left(0, \beta\left(\sqrt{|y_i|/|x_i|}\right) y_i, t_i\right), & \text{if } |y_i| > |x_i|. \end{cases}$$

The resulting map $\rho : P \setminus V \to P_{a_0}$ is smooth and satisfies (iii). This proves the lemma. $\square$

Proof of Theorem 6.6. Let $(\pi : Q \to B, S_*)$ be a universal family and denote by (B, Γ) the associated etale groupoid of Definition 6.4 (see Theorem 6.5). We prove that this groupoid is proper. Thus let (a_k, f_k, b_k) be a sequence in Γ such that a_k converges to a_0 and b_k converges to b_0. We must show that there is a fiber isomorphism $f_0 : Q_{a_0} \to Q_{b_0}$ such that a suitable subsequence of f_k DM-converges to f_0 (see Definition 13.7). To see this choose desingularizations

$$\iota : \Sigma \to Q_{a_0}, \qquad \iota' : \Sigma' \to Q_{b_0}.$$

Denote by (Σ, s_*, ν, j) and $(\Sigma', s'_*, \nu', j')$ the induced marked nodal Riemann surfaces. Consider the following diagram

$$\begin{array}{ccccccc}
 & & Q_{a_k} \setminus V_{a_k} & \xrightarrow{f_k} & Q_{b_k} \setminus V_{b_k} & & \\
 & \overset{\iota_k}{\nearrow} & \downarrow{\scriptstyle \rho_{a_k}} & & \downarrow{\scriptstyle \rho_{b_k}} & \overset{\iota'_k}{\nwarrow} & \\
\Sigma \setminus \nu & \xrightarrow{\iota} & Q_{a_0} \setminus V_{a_0} & & Q_{b_0} \setminus V_{b_0} & \xleftarrow{\iota'} & \Sigma' \setminus \nu'
\end{array}$$

Here the sets $V_a := V \cap Q_a$ and the diffeomorphisms $\rho_a : Q_a \setminus V_a \to Q_{a_0} \setminus V_{a_0}$ are as in Lemma 14.6 for a near a_0; similarly for b near b_0. Moreover,

$$\iota_k := \rho_{a_k}^{-1} \circ \iota, \qquad \iota'_k := \rho_{b_k}^{-1} \circ \iota'.$$

By definition, the pullback complex structures

$$j_k := \iota_k^* J | Q_{a_k}, \qquad j'_k := {\iota'_k}^* J | Q_{b_k}$$

converge to j, respectively j', in the C^∞ topology on every compact subset of $\Sigma \setminus \nu$, respectively $\Sigma' \setminus \nu'$. By Lemma 14.6, there exist exhausting sequences of open sets

$$U_k \subset \Sigma \setminus \nu, \qquad U'_k \subset \Sigma' \setminus \nu', \qquad f_k(U_k) \subset U'_k,$$

such that j_k can be modified outside U_k so as to converge in the C^∞ topology on all of Σ to j, and similarly for j'_k. Then

$$u_k := (\iota'_k)^{-1} \circ f_k \circ \iota_k : U_k \to \Sigma'$$

is a sequence of (j_k, j'_k)-holomorphic embeddings such that $u_k(s_*) = s'_*$. The argument in Remark 8.5 shows that, if the first derivatives of u_k are uniformly bounded, then u_k has a C^∞ convergent subsequence. It also shows that a nonconstant holomorphic sphere in Q bubbles off whenever the first derivatives of u_k are not bounded. But bubbling cannot occur (in $\Sigma \setminus \nu$). To see this argue as follows. Suppose z_k converges to $z_0 \in \Sigma \setminus \nu$ and the derivatives of u_k at z_k blow up. Then the standard bubbling argument (see [16, Chapter 4]) applies. It shows that, after passing to a subsequence and modifying z_k (without changing the limit), there are (i, j_k)-holomorphic embeddings ε_k from the disk $\mathbb{D}_k \subset \mathbb{C}$, centered at zero with radius k, to Σ such that $\varepsilon_k(0) = z_k$, the family of disks $\varepsilon_k(\mathbb{D}_k)$ converges to z_0, and $u_k \circ \varepsilon_k$ converges to a nonconstant J-holomorphic sphere $v_0 : S^2 = \mathbb{C} \cup \infty \to Q_{b_0}$. (The convergence is uniform with all derivatives on every compact subset of $\mathbb{C}$.) Hence the image of v_0 contains at least three special points. It follows that the image of $u_k \circ \varepsilon_k$ contains at least two special points for k sufficiently large. But the image of u_k contains no nodal points, the image of ε_k contains at most one marked point, and u_k maps the marked points of Σ bijectively onto the marked points of Σ'. Hence the image of $u_k \circ \varepsilon_k$ contains at most one special point, a contradiction.

This shows that bubbling cannot occur, as claimed, and hence a suitable subsequence of u_k converges in the C^∞-topology to a (j, j')-holomorphic curve $u_0 : \Sigma \setminus \nu \to \Sigma' \setminus \nu'$. Now the removable singularity theorem shows that u_0 extends to a (j, j')-holomorphic curve on all of Σ and maps ν to ν'. That u_0 is bijective follows by applying the same argument to f_k^{-1}. Hence there exists a unique fiber isomorphism $f_0 : Q_{a_0} \to Q_{b_0}$ such that $\iota' \circ u_0 = f_0 \circ \iota$. By construction, the subsequence of f_k DM-converges to f_0

Thus we have proved that the map $s \times t : \Gamma \to B \times B$ is proper. Hence, by Corollary 2.13, the quotient space B/Γ is Hausdorff. Moreover, by Theorems 13.6 and 14.5, it is sequentially compact. Since B is second countable it follows that B/Γ is compact. This completes the proof of Theorem 6.6. □

Corollary 14.7. *Suppose that a sequence of marked nodal Riemann surfaces of type (g, n) DM-converges to both (Σ, s_*, ν, j) and $(\Sigma', s'_*, \nu', j')$. Then (Σ, s_*, ν, j) and $(\Sigma', s'_*, \nu', j')$ are isomorphic.*

Proof. Let $(\pi_B : Q \to B, S_*)$ be a universal family. By Theorem 13.6 there exist points $a_0, b_0 \in B$ and sequences $a_k \to a_0$, $b_k \to b_0$ such that (Σ, s_*, ν, j) arises from a desingularization of Q_{a_0}, $(\Sigma', s'_*, \nu', j')$ arises from a desingularization of Q_{b_0}, and the fibers Q_{a_k} and Q_{b_k} are isomorphic. Hence, by Theorem 6.6, there exists a fiber isomorphism from Q_{a_0} to Q_{b_0}, and so (Σ, s_*, ν, j) and $(\Sigma', s'_*, \nu', j')$ are isomorphic. □

A Fractional Sobolev spaces

In this appendix and the next we summarize, for the convenience of the reader, the basic properties of fractional Sobolev spaces used in this article.

A.1. For $s \geq 0$ denote by $H^s(S^1)$ the Hilbert space of all power series

$$u(e^{i\theta}) = \sum_{n \in \mathbb{Z}} u_n e^{in\theta}$$

with coefficients $u_n \in \mathbb{C}$ whose norm

$$\|u\|_s := \|u\|_{H^s(S^1)} := \sqrt{\sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |u_n|^2}$$

is finite. Thus $H^0(S^1) = L^2(S^1)$ and, for $s > 0$, we have $H^s(S^1) \subset L^2(S^1)$ with a compact dense inclusion.

Lemma A.2 (Sobolev Estimate)**.** *For every $s > 1/2$ there is a constant $c > 0$ such that every smooth function $u : S^1 \to \mathbb{C}$ satisfies*

$$\|u\|_{L^\infty(S^1)} \le c\,\|u\|_{H^s(S^1)}\,.$$

Proof. The constant is $c = \sqrt{\sum_{n\in\mathbb{Z}}(1+|n|)^{-2s}}$. For $z = e^{ix} \in S^1$ we have

$$|u(z)| \le \sum_{n\in\mathbb{Z}} |u_n| = \sum_{n\in\mathbb{Z}} (1+|n|)^s\,|u_n|\,(1+|n|)^{-s} \le c\,\|u\|_{H^s(S^1)}$$

where the last step is by the Cauchy–Schwarz inequality. □

Lemma A.3 (Weak product estimate)**.** *For all $t \ge s \ge 0$ with $t > 1/2$ there is a constant $c > 0$ such that, for all $u \in H^s(S^1)$ and $v \in H^t(S^1)$, we have*

$$\|uv\|_{H^s(S^1)} \le c\,\|u\|_{H^s(S^1)}\,\|v\|_{H^t(S^1)}\,.$$

Proof. The constant is

$$c = \sup_{k\in\mathbb{Z}} \sqrt{\sum_{n\in\mathbb{Z}} \frac{(1+|k|)^{2s}}{(1+|k-n|)^{2s}\,(1+|n|)^{2t}}} < \infty.$$

To see that this constant is finite assume $k > 0$ and consider the sum over the four regions $n \le 0$, $0 \le n \le k/2$, $k/2 \le n \le k$, and $n \ge k$. Now

$$\begin{aligned}
\|uv\|_s^2 &\le \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \left(\sum_{n\in\mathbb{Z}} |u_{k-n}|\,|v_n|\right)^2 \\
&\le \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \frac{(1+|k|)^{2s}}{(1+|k-n|)^{2s}\,(1+|n|)^{2t}}\right) \cdot \\
&\quad \cdot \left(\sum_{n\in\mathbb{Z}} (1+|k-n|)^{2s}\,|u_{k-n}|^2\,(1+|n|)^{2t}\,|v_n|^2\right) \\
&\le c^2 \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} (1+|k-n|)^{2s}\,|u_{k-n}|^2\,(1+|n|)^{2t}\,|v_n|^2 \\
&= c^2\,\|u\|_s^2\,\|v\|_t^2\,.
\end{aligned}$$

The third step follows from the Cauchy–Schwarz inequality. This proves the lemma. □

A.4. The **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by

$$\mathcal{F}(f)(\nu) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\nu x} f(x)\,dx.$$

For $s \ge 0$ the $H^s(\mathbb{R})$ norm of f is defined by

$$\|f\|_s := \|f\|_{H^s(\mathbb{R})} := \sqrt{\int_{\mathbb{R}} (1+|\nu|)^{2s}\,|\mathcal{F}(f)(\nu)|^2\,d\nu}.$$

The space $H^s(\mathbb{R})$ is defined to be the completion of the space of smooth functions of compact support in this norm. It is a Hilbert space.

Lemma A.5. *For every closed interval $I \subset \mathbb{R}$ of length less than 2π and every $s \geq 0$ there is a constant $c > 0$ such that*

$$c^{-1} \|u\|_{H^s(S^1)} \leq \|f\|_{H^s(\mathbb{R})} \leq c \|u\|_{H^s(S^1)}$$

whenever f is supported in I and $u(e^{ix}) = f(x)$ for $x \in I$ and $u(e^{ix}) = 0$ for $x \notin I + 2\pi\mathbb{Z}$.

Proof. Choose a smooth function $[0,1] \times S^1 \to S^1 : (t, e^{ix}) \mapsto \phi_t(e^{ix})$ such that

$$\phi_t(e^{ix}) = e^{-itx} \qquad \text{for} \quad t \in [0,1] \quad \text{and} \quad x \in I.$$

Then the Fourier coefficients of $\phi_t u$ are

$$(\phi_t u)_n = \frac{1}{2\pi} \int_I e^{-inx} \phi_t(e^{ix}) u(e^{ix})\, dx = \frac{1}{\sqrt{2\pi}} \mathcal{F}(f)(n+t)$$

and hence

$$\|\phi_t u\|^2_{H^s(S^1)} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |\mathcal{F}(f)(n+t)|^2 .$$

By Lemma A.3, there is a constant $c > 0$ such that

$$c^{-1} \|u\|_{H^s(S^1)} \leq \|\phi_t u\|_{H^s(S^1)} \leq c \|u\|_{H^s(S^1)}$$

for every smooth function $u : S^1 \to \mathbb{R}$ and every $t \in [0,1]$. Squaring this inequality and integrating over the interval $0 \leq t \leq 1$ gives

$$c^{-2} \|u\|^2_{H^s(S^1)} \leq \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_0^1 (1 + |n|)^{2s} |\mathcal{F}(f)(n+t)|^2 \, dt \leq c^2 \|u\|^2_{H^s(S^1)} .$$

The assertion follows by comparing the factor $(1 + |n|)^{2s}$ with $(1 + |n+t|)^{2s}$; their ratio is bounded below by 2^{-2s} and is bounded above by 2^{2s} for all $n \in \mathbb{Z}$ and $t \in [0,1]$. □

Lemma A.6. *Assume $0 < s < 1$. Then there is a constant $c > 0$ such that*

$$\int_{\mathbb{R}} |\nu|^{2s} |\mathcal{F}(f)(\nu)|^2 \, d\nu = c \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x+t) - f(x)|^2}{|t|^{1+2s}} \, dx\, dt \tag{54}$$

for every compactly supported smooth function $f : \mathbb{R} \to \mathbb{C}$.

Proof. The constant is

$$\frac{1}{c} := 4 \int_0^\infty \frac{1 - \cos t}{t^{1+2s}} \, dt.$$

To see this we observe that the Fourier transform of $f_t(x) := f(x+t) - f(x)$ is given by $\mathcal{F}(f_t)(\nu) = \left(e^{i\nu t} - 1\right)\mathcal{F}(f)(\nu)$. Hence, by Plancherel's theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}\,dx\,dy &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x+t)-f(x)|^2}{|t|^{1+2s}}\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left|e^{i\nu t}-1\right|^2}{|t|^{1+2s}} \left|\mathcal{F}(f)(\nu)\right|^2\,d\nu\,dt \\
&= 4\int_{\mathbb{R}} \left(\int_0^\infty \frac{1-\cos|\nu|\,t}{|t|^{1+2s}}\,dt\right) \left|\mathcal{F}(f)(\nu)\right|^2\,d\nu \\
&= 4\int_{\mathbb{R}} \left(\int_0^\infty \frac{1-\cos t}{|t|^{1+2s}}\,dt\right) |\nu|^{2s} \left|\mathcal{F}(f)(\nu)\right|^2\,d\nu.
\end{aligned}$$

This proves the lemma. □

Corollary A.7. *When s is a nonnegative integer, the space $H^s(\mathbb{R})$ is the completion of the space of smooth functions of compact support in the norm $\sum_{j=0}^s \left\|d^j f\right\|_{L^2}$. If $s > 0$ is not an integer $H^s(\mathbb{R})$ is the completion of the space of smooth functions of compact support in the norm $\sum_{j<s}(\left\|d^j f\right\|_{L^2} + |||d^j f|||_{s-k,2})$ where k is the unique integer with $k < s < k+1$ and*

$$|||g|||_{\sigma,2} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|g(x+t)-g(x)|^2}{|t|^{1+2\sigma}}\,dx\,dt\right)^{1/2}.$$

Lemma A.8. *Let $\alpha_k : \mathbb{R} \to [0,1]$ be a sequence of smooth cutoff functions, supported in $\{2^{k-1} \le |\nu| \le 2^{k+1}\}$ for $k \ge 1$ and in $[-2,2]$ for $k=0$, such that $\sum_k \alpha_k \equiv 1$. Denote by $a_k := \mathcal{F}^{-1}(\alpha_k)$ the inverse Fourier transform of α_k. Then*

$$\sum_{k=0}^\infty 2^{2sk}\left\|a_k * f\right\|_{L^2}^2 \le 4^s \left\|f\right\|_s^2.$$

Proof. Abbreviate $f_k := a_k * f$ and $\phi_k := \mathcal{F}(f_k) = \alpha_k\mathcal{F}(f)$. Then

$$\langle f_j, f_k\rangle_s = \int_{-\infty}^{\infty} (1+|\nu|)^{2s}\,\alpha_j(\nu)\alpha_k(\nu)\left|\mathcal{F}(f)(\nu)\right|^2\,d\nu \ge 0$$

for all j and k and hence

$$\|f\|_s^2 \ge \sum_{k=0}^\infty \|f_k\|_s^2 \ge \sum_{k=0}^\infty 2^{2(k-1)s}\|f_k\|_s^2.$$

This proves the lemma. □

Lemma A.9. *For every $s > 0$ there is a constant $c > 0$ such that the following holds. If $f = \sum_{k\geq 0} f_k$ and $\mathcal{F}(f_k)$ is supported in the interval $[-2^{k+2}, 2^{k+2}]$ then*

$$\|f\|_s^2 \leq c \sum_{k=0}^{\infty} 2^{2sk} \|f_k\|_{L^2}^2 .$$

Proof. The constant is $c := 5^{2s}/(1-2^{-2s})$. Denote $\phi := \mathcal{F}(f)$ and $\phi_k := \mathcal{F}(f_k)$ and assume w.l.o.g. that ϕ_k is supported in the interval $[0, 2^{k+2}]$. Then

$$\begin{aligned}
\|f\|_s^2 &= \int_0^\infty (1+|\nu|)^{2s} \left|\sum_{k=0}^\infty \phi_k(\nu)\right|^2 d\nu \\
&= \int_0^4 (1+|\nu|)^{2s} \left|\sum_{k=0}^\infty \phi_k(\nu)\right|^2 d\nu \\
&\quad + \sum_{j=0}^\infty \int_{2^{j+2}}^{2^{j+3}} (1+|\nu|)^{2s} \left|\sum_{k=j+1}^\infty \phi_k(\nu)\right|^2 d\nu \\
&\leq 5^{2s} \left(\int_0^4 \left|\sum_{k=0}^\infty \phi_k(\nu)\right|^2 d\nu + \sum_{j=0}^\infty 2^{2(j+1)s} \int_{2^{j+2}}^{2^{j+3}} \left|\sum_{k=j+1}^\infty \phi_k(\nu)\right|^2 d\nu \right).
\end{aligned}$$

Now it follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
\int_0^4 \left|\sum_{k=0}^\infty \phi_k(\nu)\right|^2 d\nu &\leq \int_0^4 \left(\sum_{k=0}^\infty 2^{-2ks}\right)\left(\sum_{k=0}^\infty 2^{2ks} |\phi_k(\nu)|^2\right) d\nu \\
&= \frac{1}{1-2^{-2s}} \sum_{k=0}^\infty 2^{2ks} \int_0^4 |\phi_k(\nu)|^2 \, d\nu
\end{aligned}$$

and, similarly,

$$2^{2(j+1)s} \int_{2^{j+2}}^{2^{j+3}} \left|\sum_{k=j+1}^\infty \phi_k(\nu)\right|^2 d\nu \leq \frac{1}{1-2^{-2s}} \sum_{k=j+1}^\infty 2^{2ks} \int_{2^{j+2}}^{2^{j+3}} |\phi_k(\nu)|^2 \, d\nu.$$

The result follows by combining these three estimates. □

Theorem A.10 (Strong product estimate)**.** *For every $s \geq 0$ there is a constant $c > 0$ such that*

$$\|fg\|_{H^s(\mathbb{R})} \leq c \left(\|f\|_{H^s(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|g\|_{H^s(\mathbb{R})} \right)$$

for any two compactly supported smooth functions $f, g : \mathbb{R} \to \mathbb{C}$.

Proof. For $s > 1$ we follow the beautiful argument by Bourgain, Brezis, and Mironescu in [1, Lemma D.2] which is based on the Littlewood-Paley decomposition; it simplifies slightly in our special case. Choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that

$$\beta(\nu) = \begin{cases} 1, & \text{for } |\nu| \le 1, \\ 0, & \text{for } |\nu| \ge 2, \end{cases}$$

and denote by $b := \mathcal{F}^{-1}(\beta)$ its inverse Fourier transform. For every integer $k \ge 0$ define

$$b_k(t) := 2^k b(2^k t), \qquad \beta_k(\nu) := \beta(2^{-k}\nu)$$

so that $\beta_k = \mathcal{F}(b_k)$. Then there is a constant $c_0 > 0$ such that

$$\|b_k\|_{L^1} \le c_0, \qquad \|b_k'\|_{L^1} \le 2^k c_0 \tag{55}$$

for every k. Next define

$$\alpha_k := \beta_k - \beta_{k-1}, \qquad a_k := \mathcal{F}^{-1}(\alpha_k) = b_k - b_{k-1},$$

for $k \ge 1$ and $\alpha_0 := \beta$, $a_0 = b$ so that

$$\beta_k = \sum_{j=0}^{k} \alpha_j, \qquad b_k = \sum_{j=0}^{k} a_j.$$

Then, for every smooth function $f : \mathbb{R} \to \mathbb{C}$ with compact support and $s \ge 0$, the series

$$f = \sum_{k=0}^{\infty} a_k * f$$

is absolutely summable in $H^s(\mathbb{R})$. Namely, the Fourier transform $\phi := \mathcal{F}(f)$ lies in the Schwarz space and so decays faster than any rational function; hence the series $\phi = \sum_{k=0}^{\infty} \alpha_k \phi$ is absolutely summable in the weighted L^2 space of all functions $\psi : \mathbb{R} \to \mathbb{C}$ for which $\nu \mapsto (1 + |\nu|)^s \psi(\nu)$ is square integrable.

Given two smooth functions $f, g : \mathbb{R} \to \mathbb{C}$ with compact support write

$$\begin{aligned} fg' &= \sum_{j,k=0}^{\infty} (a_j * f)(a_k' * g) \\ &= \sum_{j \le k} (a_j * f)(a_k' * g) + \sum_{j > k} (a_j * f)(a_k' * g) \\ &= \sum_{k=0}^{\infty} (b_k * f)(a_k' * g) + \sum_{j=1}^{\infty} (a_j * f)(b_{j-1}' * g). \end{aligned} \tag{56}$$

The Fourier transform of $(b_k * f)(a_k' * g)$ is the convolution of the functions $\beta_k \mathcal{F}(f)$ and $\alpha_k \mathcal{F}(g')$, both supported in the interval $[-2^{k+1}, 2^{k+1}]$, and so it is supported in the interval $[-2^{k+2}, 2^{k+2}]$. Hence

$$\begin{aligned}
\|(b_k * f)(a_k' * g)\|_{L^2} &\leq \|b_k * f\|_{L^\infty} \|a_k * g'\|_{L^2} \\
&\leq c_0 \|f\|_{L^\infty} \|a_k' * g\|_{L^2} \\
&\leq c_0 2^{k+1} \|f\|_{L^\infty} \|a_k * g\|_{L^2} .
\end{aligned}$$

The last step follows from the fact that $\mathcal{F}(a_k')(\nu) = i\nu\alpha_k(\nu)$ and α_k is supported in the domain $\{2^{k-1} \leq |\nu| \leq 2^{k+1}\}$. Similarly,

$$\begin{aligned}
\left\|(a_k * f)(b_{k-1}' * g)\right\|_{L^2} &\leq \|a_k * f\|_{L^2} \left\|b_{k-1}' * g\right\|_{L^\infty} \\
&\leq c_0 2^k \|a_k * f\|_{L^2} \|g\|_{L^\infty} .
\end{aligned}$$

Now let c_1 be the constant of Lemma A.9 with s replaced by $s-1$. Then

$$\begin{aligned}
&\|fg'\|_{s-1}^2 \\
&\leq c_1 \sum_{k=0}^{\infty} 2^{2(s-1)k} \left\|(a_k * f)(b_{k-1}' * g) + (a_k * f)(b_{k-1}' * g)\right\|_{L^2}^2 \\
&\leq 2c_1 \sum_{k=0}^{\infty} 2^{2(s-1)k} \left(\left\|(a_k * f)(b_{k-1}' * g)\right\|_{L^2}^2 + \left\|(a_k * f)(b_{k-1}' * g)\right\|_{L^2}^2\right) \\
&\leq 2c_0c_1 \sum_{k=0}^{\infty} 2^{2sk} \left(4 \|f\|_{L^\infty}^2 \|a_k * g\|_{L^2}^2 + \|a_k * f\|_{L^2}^2 \|g\|_{L^\infty}^2\right) \\
&= 2c_0c_1c_2 \left(4 \|f\|_{L^\infty}^2 \|g\|_s^2 + \|g\|_{L^\infty}^2 \|f\|_s^2\right) .
\end{aligned}$$

The last inequality follows from Lemma A.8 with $c_2 := 4^s$. Interchanging f and g and using the Leibnitz rule we obtain

$$\|(fg)'\|_{s-1} \leq c' \left(\|f\|_{L^\infty} \|g\|_s + \|g\|_{L^\infty} \|f\|_s\right)$$

with $c' := \sqrt{10c_0c_1c_2}$. Since

$$\|fg\|_{L^2} \leq \|f\|_{L^\infty} \|g\|_{L^2} \leq \|f\|_{L^\infty} \|g\|_s ,$$

this proves the theorem for $s > 1$. For $0 < s < 1$ the result follows easily from Lemma A.6. For $s = 0$ and $s = 1$ the result is obvious. This proves the theorem. □

Corollary A.11. *For every $s \geq 0$ there is a constant $C > 0$ such that*

$$\|uv\|_{H^s(S^1)} \leq C \left(\|u\|_{H^s(S^1)} \|v\|_{L^\infty(S^1)} + \|u\|_{L^\infty(S^1)} \|v\|_{H^s(S^1)}\right)$$

for any two smooth functions $u, v : S^1 \to \mathbb{C}$.

Proof. Lemma A.5 and Theorem A.10. □

Lemma A.12. *Fix a constant $s > 1/2$. Let $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ be open sets and $f : X \to Y$ be a smooth map. Then*

$$H^s(S^1, X) := \{v \in H^s(S^1, \mathbb{C}^m) : v(S^1) \subset X\}$$

is an open subset of $H^s(S^1, \mathbb{C}^m)$ and $H^s(S^1, Y)$ is an open subset of $H^s(S^1, \mathbb{C}^n)$. Morever, composition with f defines a smooth map

$$H^s(S^1, X) \to H^s(S^1, Y) : u \mapsto f \circ u. \tag{57}$$

Proof. This is Lemma C.1 in [1]. We sketch the proof. That $H^s(S^1, X)$ and $H^s(S^1, Y)$ are open sets follows immediately from Lemma A.2. To prove that

$$u \in H^s(S^1, X) \implies f \circ u \in H^s(S^1, Y) \tag{58}$$

we argue by induction. For $1/2 < s \leq 1$ this follows from the estimates

$$|||f \circ u|||_{s,2} \leq c|||u|||_{s,2}, \qquad \|(f \circ u)'\|_{L^2(S^1)} \leq c \, \|u'\|_{L^2(S^1)}$$

where $|||u|||^2_{s,2}$ is as in Corollary A.7 and c is a Lipschitz constant for f on the image of u (which is compact by Lemma A.2); for $1 < s \leq 3/2$ it follows from the identity

$$(f \circ u)' = df(u)u'$$

with $df(u) \in H^1$, $u' \in H^{s-1}$ and so $(f \circ u)' \in H^{s-1}$, by Lemma A.3. Fix an integer $k \geq 2$ and suppose, by induction, that (58) holds for $s \leq k - 1/2$. Fix a real number s with $k - 1/2 < s \leq k + 1/2$. If $u \in H^s$ then, by the induction hypothesis $df(u) \in H^{s-1}$ and, since $u' \in H^{s-1}$ it follows from Lemma A.3 with s replaced by $s - 1 > 1/2$ that $(f \circ u)' = df(u)u' \in H^{s-1}$ and hence $f \circ u \in H^s$.

Thus we have proved (58). The same argument shows that the map (57) is bounded in the following sense: *For every constant $c_0 > 0$ and every compact subset $K \subset X$ there is a constant $c > 0$ such that*

$$u(S^1) \subset K, \ \|u\|_{H^s} \leq c_0 \qquad \implies \qquad \|f \circ u\|_{H^s} \leq c$$

for every $u \in H^s(S^1, X)$. It follows that the map

$$H^s(S^1, X) \to H^s(S^1, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m, \mathbb{C}^n)) : u \mapsto df(u) \tag{59}$$

is bounded as well. This in turn implies that the map (57), and hence also (59), is continuous. That (57) is differentiable follows from the continuity of (59) and the estimate

$$\begin{aligned} \|f(u + \xi) - f(u) - df(u)\xi\|_{H^s} &\leq \sup_{0 \leq t \leq 1} \|df(u + t\xi)\xi - df(u)\xi\|_{H^s} \\ &\leq c \sup_{\|\eta\|_{H^s} \leq \delta} \|df(u + \eta) - df(u)\|_{H^s} \|\xi\|_{H^s} \end{aligned}$$

for $u \in H^s(S^1, X)$ and $\xi \in H^s(S^1, \mathbb{C}^m)$ with $\|\xi\|_{H^s} \leq \delta$; here c is the constant in Lemma A.3 and we choose δ sufficiently small. Since the differential of (57) is a map of the same type it then follows by induction that (57) is smooth. This proves the lemma. □

B An elliptic boundary estimate

B.1. Let $s \in \mathbb{R}$ and V be a finite dimensional complex Hilbert space. Denote by $H^s(S^1, V)$ the Hilbert space of all power series

$$v(e^{i\theta}) = \sum_{n \in \mathbb{Z}} v_n e^{in\theta}$$

with coefficients $v_n \in V$ whose norm

$$\|v\|_s := \|v\|_{H^s(S^1)} := \sqrt{\sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |v_n|^2}$$

is finite. The crucial properties of these spaces are the following.

(i) If s is a nonnegative integer then the H^s norm is equivalent to the sum of the L^2 norms of the derivatives up to order s.

(ii) The elements of $H^s(S^1, V)$ are continuous for $s > 1/2$ and, in this case, the inclusion $H^s(S^1, V) \to C^0(S^1, V)$ is a compact operator (Lemma A.2).

(iii) Composition with a diffeomorphism of S^1 induces an automorphism of $H^s(S^1, V)$ for every $s \in \mathbb{R}$.

(iv) Multiplication by a smooth complex valued function on S^1 induces an automorphism of $H^s(S^1, V)$ for every $s \in \mathbb{R}$ (Lemma A.3).

(v) Let $X \subset V$ be open and assume $s > 1/2$. Then

$$H^s(S^1, X) := \{v \in H^s(S^1, V) : v(S^1) \subset X\}$$

is an open subset of $H^1(S^1, V)$ (Lemma A.2).

(vi) Let $X \subset V$ and $Y \subset W$ be open subsets of finite dimensional complex Hilbert spaces and $\psi : X \to Y$ be a smooth map. Assume $s > 1/2$. Then composition with ψ induces a smooth map from $H^s(S^1, X)$ to $H^s(S^1, Y)$ (Lemma A.12).

If $E = \bigsqcup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} E_\theta \subset S^1 \times V$ is a smooth (real) subbundle of the trivial bundle then, by (iv), the subspace

$$H^s(E) := \{v \in H^s(S^1, V) : v(e^{i\theta}) \in E_\theta \; \forall \, \theta \in \mathbb{R}\}$$

is a closed (real) subspace of $H^s(S^1, V)$.

B.2. For every $s > 1/2$ there is a unique operation which assigns to every compact 1-manifold Γ and every smooth manifold M (both without boundary) a real Hilbert manifold $H^s(\Gamma, M) \subset C^0(\Gamma, M)$ satisfying the following axioms.

(a) If $\Gamma = S^1$ and M is an open subset of a complex vector space then $H^s(\Gamma, M)$ is as above.

(b) If $\psi : M \to M'$ is a smooth map then $\gamma \in H^s(\Gamma, M) \implies \psi \circ \gamma \in H^s(\Gamma, M')$ and the resulting map $\psi_* : H^s(\Gamma, M) \to H^s(\Gamma, M')$ is smooth.

(c) If $\phi : \Gamma' \to \Gamma$ is a diffeomorphism then $\gamma \in H^s(\Gamma', M) \implies \gamma \circ \phi \in H^s(\Gamma, M)$ and the resulting map $\phi^* : H^s(\Gamma', M) \to H^s(\Gamma, M)$ is smooth.

(d) If Γ is the disjoint union of Γ_1 and Γ_2 then the map $\gamma \mapsto (\gamma|\Gamma_1, \gamma|\Gamma_2)$ is a diffeomorphism from $H^s(\Gamma, M)$ to $H^s(\Gamma_1, M) \times H^s(\Gamma_2, M)$.

The Hilbert manifold structure on $H^s(\Gamma, M)$ is given by the exponential maps $H^s(\gamma^* TM) \to H^s(\Gamma, M)$ along the smooth maps $\gamma : \Gamma \to M$. If M is a complex manifold so is $H^s(\Gamma, M)$.

B.3. Let X be a compact surface with smooth boundary, M be a smooth manifold, and for each integer $k \geq 2$ let $H^k(X, M)$ denote the space of maps from X to M with k derivatives in L^2. The elements of $H^k(X, M)$ are continuous and a well known construction, analogous to the one sketched in B.2, equips $H^k(X, M)$ with a smooth Hilbert manifold structure which is a complex Hilbert manifold structure when M is a complex manifold. In particular, the space of H^k sections of a vector bundle over X is a Hilbert space. There are various ways of defining a smooth Hilbert manifold structure on the space $H^s(X, M)$ when s is a real number greater than 1, but we shall avoid these spaces. This is why many of our earlier theorems begin with the hypothesis "Let $s + 1/2$ be an integer".

Theorem B.4. *Let X be a compact Riemann surface with boundary $\Gamma := \partial X$ and $E \to X$ be a complex vector bundle. Denote the complex structure on X by j and the complex structure on E by J. Fix an integer $k = s + 1/2 \geq 1$.*

(i) *There is a constant $c > 0$ (depending continuously on j an J) such that*

$$\|\xi\|_{H^s(\Gamma)} \leq c \, \|\xi\|_{H^{s+1/2}(X)}$$

for every $\xi \in \Omega^0(X, E)$.

(ii) *Assume that X is connected and $\Gamma \neq \emptyset$. Let $D : \Omega^0(X, E) \to \Omega^{0,1}(X, E)$ be a real linear Cauchy–Riemann operator. Then there is a constant $c > 0$ (depending continuously on j, J, and D) such that*

$$\|\xi\|_{H^{s+1/2}(X)} \leq c \left(\|D\xi\|_{H^{s-1/2}(X)} + \|\xi\|_{H^s(\Gamma)} \right)$$

for every $\xi \in \Omega^0(X, E)$.

Proof. These estimates are well known and it is not necessary to assume that $s + 1/2$ is an integer. We include a proof because we couldn't find an explicit reference for (ii) in the literature. Assertion (i) is proved by the same argument. Both assertions are easy for sections supported in the interior of X. To prove them in general we first consider the case where $X = [0, 1] \times S^1$ is an annulus with the standard complex structure. Fix a complex (not necessarily holomorphic) trivialization of E over the annulus.

As a warmup we prove (i) for any real number $s \geq 1/2$. Denote the inner product on $H^s(S^1,\mathbb{C}^n)$ by

$$\langle \xi, \eta\rangle_s := \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} \langle \xi_n, \eta_n\rangle ,$$

where $\xi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \xi_n e^{in\theta}$ and $\eta(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \eta_n e^{in\theta}$. Then, for all $\xi, \eta \in C^\infty(S^1,\mathbb{C}^n)$, the Schwartz inequality gives

$$|\langle \xi, \eta\rangle_s| \leq \|\xi\|_{s-1/2} \|\eta\|_{s+1/2} .$$

Hence every smooth function $[0,1] \to C^\infty(S^1,\mathbb{C}^n) : \tau \mapsto \xi(\tau)$ satisfies the inequality

$$\begin{aligned} \frac{d}{d\tau} \|\xi(\tau)\|_s^2 &= 2\langle \partial_\tau \xi(\tau), \xi(\tau)\rangle_s \\ &\leq 2\|\partial_\tau \xi(\tau)\|_{s-1/2} \|\xi(\tau)\|_{s+1/2} \\ &\leq \|\partial_\tau \xi(\tau)\|_{s-1/2}^2 + \|\xi(\tau)\|_{s+1/2}^2 . \end{aligned}$$

Integrating this inequality gives

$$\begin{aligned} \|\xi(1)\|_s^2 &\leq \int_0^1 \left(\|\partial_\tau \xi(\tau)\|_{s-1/2}^2 + \|\xi(\tau)\|_{s+1/2}^2 \right) d\tau \\ &\leq \|\xi\|_{H^{s+1/2}([0,1]\times S^1)} \end{aligned}$$

whenever $\xi(0) = 0$. The last inquality uses the asumption $s \geq 1/2$. This proves (i).

We prove (ii). The operator D has the form

$$D\xi = \frac{1}{2}(\partial_\tau \xi + i\partial_\theta \xi + S\xi)\, ds + \frac{1}{2}(\partial_\theta \xi - i\partial_\tau \xi - iS\xi)\, dt$$

where $S : [0,1]\times S^1 \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Assume first that $S \equiv 0$ and denote

$$f := \partial_\tau \xi + i\partial_\theta \xi.$$

We think of f and ξ as functions from $[0,1]$ to $H^s(S^1,\mathbb{C}^n)$. Consider the decomposition

$$H^s(S^1,\mathbb{C}^n) = E^- \oplus E^0 \oplus E^+$$

where $E^0 \cong \mathbb{C}^n$ denotes the space of constant functions and

$$E^\pm := \text{closed span}\left\{ e^{in\theta} : \pm n > 0 \right\} .$$

The components of an element $\xi \in H^s(S^1,\mathbb{C}^n)$ with respect to this decomposition will be denoted by ξ^-, ξ^0, ξ^+. Note that $e^{in\theta}$ is an eigenfunction of the operator $A := -i\partial_\theta$ with eigenvalue n and hence

$$\left\langle \xi^+, A\xi^+\right\rangle_s \geq \frac{1}{2}\left\|\xi^+\right\|_{s+1/2}^2 , \qquad \left\langle \xi^-, A\xi^-\right\rangle_s \leq -\frac{1}{2}\left\|\xi^-\right\|_{s+1/2}^2 .$$

Since $\partial_\tau \xi = A\xi + f$ we have

$$\begin{aligned}
\frac{d}{d\tau}\left\|\xi^+(\tau)\right\|_s^2 &= 2\left\langle \partial_\tau \xi^+(\tau), \xi^+(\tau)\right\rangle_s \\
&= 2\left\langle A\xi^+(\tau) + f^+(\tau), \xi^+(\tau)\right\rangle_s \\
&\geq \left\|\xi^+(\tau)\right\|_{s+1/2}^2 - 2\left\|f^+(\tau)\right\|_{s-1/2}\left\|\xi^+(\tau)\right\|_{s+1/2} \\
&\geq \frac{1}{2}\left\|\xi^+(\tau)\right\|_{s+1/2}^2 - 2\left\|f^+(\tau)\right\|_{s-1/2}^2 .
\end{aligned}$$

Integrating this inequality gives

$$\int_0^1 \left\|\xi^+(\tau)\right\|_{s+1/2}^2 \, d\tau \leq 4\int_0^1 \left\|f^+(\tau)\right\|_{s-1/2}^2 \, d\tau + 2\left\|\xi^+(1)\right\|_s^2 . \tag{60}$$

Similarly,

$$\int_0^1 \left\|\xi^-(\tau)\right\|_{s+1/2}^2 \, d\tau \leq 4\int_0^1 \left\|f^-(\tau)\right\|_{s-1/2}^2 \, d\tau + 2\left\|\xi^-(0)\right\|_s^2 \tag{61}$$

and, since $\xi^0(\tau) = \xi^0(0) + \int_0^\tau f^0(\sigma)\, d\sigma$, we have

$$\int_0^1 \left\|\xi^0(\tau)\right\|_{s+1/2}^2 \, d\tau \leq 2\int_0^1 \left\|f^0(\tau)\right\|_{s-1/2}^2 \, d\tau + 2\left\|\xi^0(0)\right\|_s^2 . \tag{62}$$

(The three norms agree on E^0.) Combining the inequalities (60-62) we obtain

$$\int_0^1 \|\xi(\tau)\|_{s+1/2}^2 \, d\tau \leq 4\left(\int_0^1 \|f(\tau)\|_{s-1/2}^2 \, d\tau + \|\xi(0)\|_s^2 + \|\xi(1)\|_s^2\right). \tag{63}$$

Now assume $k := s + 1/2$ is an integer. We prove by induction that

$$\int_0^1 \|\partial_\tau^\nu \xi(\tau)\|_{H^{k-\nu}(S^1)}^2 \, d\tau \leq c_\nu \left(\|f\|_{H^{k-1}([0,1]\times S^1)}^2 + \|\xi(0)\|_s^2 + \|\xi(1)\|_s^2\right) \tag{64}$$

for $\nu = 0, 1, \ldots, k$. For $\nu = 0$ this is (63) with $c_0 = 4$. Assuming that (64) has been established for some integer $\nu \in \{0, \ldots, k-1\}$, we use the inequality

$$\begin{aligned}
\left\|\partial_\tau^{\nu+1}\xi\right\|_{H^{k-\nu-1}(S^1)} &= \|\partial_\tau^\nu (A\xi + f)\|_{H^{k-\nu-1}(S^1)} \\
&\leq \|\partial_\tau^\nu \xi\|_{H^{k-\nu}(S^1)} + \|\partial_\tau^\nu f\|_{H^{k-\nu-1}(S^1)}
\end{aligned}$$

to obtain (64) with ν replaced by $\nu + 1$. This completes the induction. Now sum (64) for $\nu = 0, 1, \ldots, k$ to obtain the estimate in part (ii) for the case $S = 0$ and $X = [0,1] \times S^1$. In the case where $S \neq 0$ and X is a general compact Riemann surface we deduce, using partitions of unity and what we've already proved, that

$$\|\xi\|_{H^{s+1/2}(X)} \leq c\left(\|D\xi\|_{H^{s-1/2}(X)} + \|\xi\|_{H^{s-1/2}(X)} + \|\xi\|_{H^s(\Gamma)}\right) \tag{65}$$

for some constant $c > 0$ and every smooth section $\xi \in \Omega^0(X, E)$. This implies that the operator

$$H^{s+1/2}(E) \to H^{s-1/2}(\Lambda^{0,1}T^*X \otimes E) \times H^s(E|\Gamma) : \xi \mapsto (D\xi, \xi|\Gamma)$$

has a finite dimensional kernel and a closed image (see [16, Lemma A.1.1]). If X is connected and $\Gamma \neq \emptyset$ then, by unique continuation, this operator is injective and so the term $\|\xi\|_{H^{s-1/2}(X)}$ on the right hand side of (65) can be dropped, by the open mapping principle. This proves the theorem. □

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