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LAGRANGIAN INTERSECTIONS IN CONTACT GEOMETRY

Y. Eliashberg, H. Hofer and D. Salamon

1. Introduction

It is well-known that all problems of Contact geometry can be reformulated as problems of Symplectic geometry. This can be done via *symplectization* (see 2.1 below). In particular, the problem of Lagrangian intersections naturally arises in connection with several contact geometric questions (see 2.5 example, and below). However, there is one major difficulty when one tries to realize this approach: *the symplectizations of contact manifolds are non-compact* and, what is even worse, *non-convex* (see [EGr1]). This leads to the loss of compactness for the spaces of holomorphic curves and thus creates serious difficulties for the traditional Floer homology approach. The goal of this paper is to show that this problem can be successfully overcome by using an idea from [H].

We begin with an exposition of the main notions of contact geometry and their symplectic analogues. We develop then an analogue of Floer homology theory for the Lagrangian intersection problem in symplectizations of contact manifolds and give applications of this theory to contact geometry.

There exist other methods for handling similar problems in contact geometry. Let us mention here Givental's approach through the, so-called, non-linear Maslov index (see [G]), as well as the approach based on the theory of generating functions and hypersurfaces described in [EGr2]. Kaoru Ono ([On]) independently proved a result similar to our Theorem 2.5.4. All these methods, and the method considered in this paper, have common as well as complementary areas of application.

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2. Contact Geometry

2.1 Contact manifolds and their symplectizations. We recall in this section some basic definitions of contact geometry and their symplectic counterparts (see also [AG]). Let ζ be a *contact structure* on a $(2n+1)$-dimensional manifold M, i.e. ζ is a completely non-integrable tangent plane distribution of codimension 1. Thus, at least locally, ζ can be defined by the equation $\{\gamma = 0\}$ where the 1-form γ satisfies the condition $\gamma \wedge (d\gamma)^n \neq 0$. The global existence of such a form γ is equivalent to the coorientability of ζ.

Only cooriented contact structures are considered in this paper. The general case requires a $\mathbf{Z}/2$-equivariant analogue of the theory described here.

Let $S_\zeta(M)$ be the (trivial) subbundle of the cotangent bundle $T^*(M)$ whose fiber over a given point $q \in M$ consists of all non-zero linear forms from $T_q^*(M)$ which annihilate the hyperplane $\zeta_q \subset T_q(M)$ and define its given coorientation. The bundle $S_\zeta(M)$ is a principal $\mathbf{R}$-bundle where the action of $\mathbf{R}$ is defined by

$$\lambda * v = e^\lambda \cdot v \ , \qquad \lambda \in \mathbf{R} \ , \ v \in S_\zeta(M) \ .$$

Let us denote by α_ζ the restriction $pdq|_{S_\zeta(M)}$ of the canonical form pdq on T^*M to $S_\zeta(M) \subset T^*M$. Then the 2-form $\omega_\zeta = d\alpha_\zeta$ is a symplectic structure on $S_\zeta(M)$. The symplectic manifold $(S_\zeta(M), \omega_\zeta)$ is called the *symplectization* of the contact manifold (M, ζ).

Let us denote by X_ζ the vector field on $S_\zeta(M)$ which is ω_ζ-dual to α_ζ, i.e. $X_\zeta \rfloor \omega_\zeta = \alpha_\zeta$. The field X_ζ generates the $\mathbf{R}$-action described above:

$$(X_\zeta)^\lambda(v) = e^\lambda v \ , \qquad \lambda \in \mathbf{R} \ , \ v \in S_\zeta(M) \ .$$

The sections of the bundle $S_\zeta(M) \to M$ are called *contact forms*. The space of all contact forms will be denoted by $\mathrm{Cont}(\zeta)$.

A choice of a contact form $\gamma \in \mathrm{Cont}(\zeta)$ defines a splitting $H_\gamma : S_\zeta(M) \to M \times \mathbf{R}$. In terms of this splitting we have

$$\alpha_\zeta = e^\theta \gamma \ , \quad \omega_\zeta = d(e^\theta \gamma) \ , \quad X_\zeta = \frac{\partial}{\partial \theta} \ ,$$

where $\theta \in \mathbf{R}$ and we identify γ defined on M with its pullback on $M \times \mathbf{R}$.

It is useful to observe the following

PROPOSITION 2.1.1. *A fiberwise splitting $H : S_\zeta(M) \to M \times \mathbf{R}$ has the form H_γ for a contact form $\gamma \in \mathrm{Cont}(\zeta)$ if and only if H commutes with the $\mathbf{R}$-actions on $S_\zeta(M)$ and $M \times \mathbf{R}$.*

A diffeomorphism $f : M \to M$ lifts canonically to a symplectomorphism $F : T^*M \to T^*M$. Moreover, F preserves the 1-form pdq as well. If f is a contactomorphism of the contact manifold (M, ζ) then F leaves the subbundle $S_\zeta(M)$ invariant and thus induces an $\mathbf{R}$-equivariant symplectomorphism $S_\zeta(M) \to S_\zeta(M)$. We will denote this symplectomorphism by $\hat{f}$ and call it the *symplectization* of the contactomorphism f.

The converse is also true: *any* **R**-*equivariant symplectomorphism of* $S_\zeta M$ *has the form* $\hat{f}$ *for a uniquely defined suitable contactomorphism* $f : (M,\zeta) \to (M,\zeta)$.

The vector field X on (M,ζ) is called *contact* if the flow generated by X consists of contactomorphism $(M,\zeta) \to (M,\zeta)$. Equivalently, pick a 1-form $\gamma \in \mathrm{Cont}(\zeta)$. Then the vector field X is contact iff the Lie derivative $L_X\gamma$ is proportional to γ.

Each contact vector field X on (M,ζ) admits a lift to an **R**-invariant Hamiltonian vector field $\hat{X}$ on $(S_\zeta(M),\omega_\zeta)$. Conversely, each **R**-invariant Hamiltonian vector field Y on $(S_\zeta(M),\omega_\zeta)$ projects to a contact vector field on (M,ζ). An important example of a contact vector field is provided by the *Reeb vector field.* Notice that the choice of a contact form $\gamma \in \mathrm{Cont}(\zeta)$ defines on M a Hamiltonian flow which is transversal to the contact structure ζ. Indeed, there exists a unique vector field Y tangent to M such that $Y \rfloor d\gamma = 0$ and $\gamma(Y) = 1$. The vector field Y is called the Reeb vector field generated by the contact form γ. The field Y is contact. Indeed, we have $L_Y\gamma = d(\gamma(Y)) - Y\rfloor d\gamma = 0$.

2.2 Legendrian, Lagrangian, pre-Lagrangian. An n-dimensional submanifold $\Lambda \subset (M,\zeta)$ is called *Legendrian* if it is tangent to the distribution ζ. If γ is a contact form from $\mathrm{Cont}(\zeta)$ then Λ is Legendrian iff $\alpha|_\Lambda = 0$. The preimage $\hat{\Lambda} = \pi^{-1}(\Lambda) \subset S_\zeta M$ under the canonical projection $S_\zeta M \to M$ is an **R**-invariant Lagrangian cone. We call $\hat{\Lambda}$ the *symplectization* of Λ. Conversely, any Lagrangian cone in the symplectization projects onto a Legendrian submanifold in (M,ζ).

The following notion was suggested to us by D. Bennequin.

An $(n+1)$-dimensional submanifold L of the $(2n+1)$-dimensional contact manifold (M,ζ) is called pre-*Lagrangian* if it satisfies the following two conditions:

— L is transverse to ζ;

— The distribution $\zeta \cap T(L)$ is integrable and can be defined by a closed 1-form.

Remark 2.2.1: It is useful for applications to extend the definition of a pre-Lagrangian submanifold allowing certain types of tangency of L and ζ instead of their transversality. It will be done in one of our subsequent papers.

The motivation for the term pre-Lagrangian is provided by the following

PROPOSITION 2.2.2. *For any pre-Langrangian submanifold* $L \subset M$ *there exists a Lagrangian submanifold* $\hat{L} \subset S_\zeta M$ *such that* $\pi(\hat{L}) = L$. *The cohomology class* $\lambda \in H^1(L;\mathbf{R})$, *such that* $\pi^*\lambda = [\alpha_\zeta|_{\hat{L}}]$, *is defined uniquely up* *to multiplication by a non-zero constant. Conversely, if $L \subset M$ is the (embedded) image of a Lagrangian submanifold $\hat{L} \subset S_\zeta M$ under the projection $S_\zeta M \to M$ then L is pre-Lagrangian.*

Proof: By the definition of a pre-Lagrangian submanifold there exists a contact form $\beta \in \mathrm{Cont}_\zeta(M)$ whose restriction to L is closed. The required lift $\hat{L}$ of L is the graph of the form $\beta|_L$. Suppose that $\beta' \in \mathrm{Cont}(\zeta)$ is another form whose restriction to L is closed. Then $\beta'|_L = f\beta|_L$ for a non-vanishing function f, and we have $df \wedge \beta|L = 0$. Thus the function f must be constant on leaves of the foliation $\beta = 0$ on L. If the cohomology class $\lambda = [\beta|_L]$ is proportional to the integral class from $H^1(L;\mathbf{Z})$ then we can think that λ itself is integral and, therefore, $\beta|_L = h^*(d\theta)$, where h is a map $L \to S^1$ and the cohomology class of the closed form $d\theta$ generates $H^1(S^1)$. Thus the function f is constant on the fibers $h^{-1}(\theta)$, $\theta \in S^1$, i.e. f can be written as $\varphi \circ h$ for a function $\varphi : S^1 \to \mathbf{R}$. Set $C = \int_{S^1} \varphi d\theta$. Then there exists a diffeomorphism $g : S^1 \to S^1$ such that $g^*(d\theta) = (\varphi/C)d\theta$. Thus

$$\frac{1}{C}\beta'|_L = \frac{f}{C}\beta|_L = h^*\left(g^*(d\theta)\right) ,$$

and, therefore, the cohomology class $[\beta'|_L]$ coincides with $C\lambda$. If λ is not proportional to an integral class then the foliation defined by the form β on L has everywhere dense leaves. This implies that the function f has to be constant on all L. □

Thus with any pre-Lagrangian submanifold $L \subset M$ one can canonically associate a projective class of the form λ. A curve $\Gamma \subset L$ is called a *vanishing cycle* of L if its homology class annihilates λ. Examples of vanishing cycles are provided by curves which are contained in a Legendrian submanifold of L.

Let us recall that if $\delta : S^1 \to L$ is a loop in a Lagrangian, possibly immersed submanifold Lag of a symplectic manifold V then given a symplectic trivialization of the bundle $f^*T(V)$ one can define the Maslov index $\mu(\delta)$ (see, for instance, [RS1]). Of course, the index $\mu(\delta)$ depends on the trivialization. However, if $\Delta : S^1 \times [0,1] \to V$ is a homotopy between the loops $\delta_0 = \Delta|_{S^1\times 0} : S^1 \to Lag$, and $\delta_1 = \Delta|_{S^1\times 1} : S^1 \to Lag$ then the difference $\mu(\delta_0,\delta_1) = \mu(\delta_0) - \mu(\delta_1)$ can be invariantly defined. To do this one just needs to trivialize the bundle $\Delta^*T(V)$ over $S^1 \times [0,1]$.

The procedure of symplectization allows us to define the relative Maslov index $\mu(\delta_0,\delta_1)$ for a pair of homotopic loops in a contact manifold provided they are contained in its Legendrian or pre-Lagrangian submanifolds.

2.3 Contactization of symplectic manifolds. If a symplectic manifold (N,ω) is exact, i.e. $\omega = d\alpha$, then it can be contactized. The *contactization*

$C(N,\omega)$ is the manifold $M = N \times S^1$ (or $N \times \mathbf{R}$) endowed with the contact form $dz - \alpha$. Here we denote by z the projection to the second factor and still denote by α its pull-back under the projection $M \to N$.

However, the contactization can be defined sometimes, even when ω is not exact. Suppose that there exists an $\hbar > 0$ such that the form $\omega/\hbar$ represents an integral cohomology class $[\omega/\hbar] \in H^2(N)$. The *contactization* $C(N,\omega)$, or as it is also called, *pre-quantization* of the symplectic manifold (N,ω) can be constructed in this case as follows (see [W]). Let $M \to N$ be a principal circle bundle with the Euler class equal to $[\omega/\hbar]$. This bundle admits a connection whose curvature form equals $\omega/\hbar$. This connection can be viewed as a S^1-invariant 1-form α on M. The non-degeneracy of ω implies that α is a contact form and, therefore $\zeta = \{\alpha = 0\}$ is a contact structure on M. The contact manifold (M,ζ) is, by the definition, the contactization $C(N,\omega)$ of the symplectic manifold (N,ω). A change of the connection α leads to a contactomorphic manifold. However, a change of $\hbar$ (for instance, $\hbar \to \hbar/2$) affects not only the contact structure ζ but also the topology of the manifold M itself.

2.4 Examples. We give here examples of pre-Lagrangian and Legendrian submanifolds.

2.4.1 SYMPLECTIZATION OF THE SPACE OF CONTACT ELEMENTS. Let $M = P^+T^*N$ be the projectivized cotangent bundle of a n-manifold N, or the *space of cooriented contact elements* of N. Thus a point of M is a cooriented tangent hyperplane $T \subset T(V)$. The manifold M carries a canonical contact structure ζ (see [AG]) which is uniquely defined by the following property:

The symplectization $S_\zeta(M)$ coincides with $T^*N \setminus N$, the symplectic form ω_ζ is the restriction of the canonical symplectic form $d(pdq)$, and the $\mathbf{R}$-action is given by the multiplication by e^θ.

If we fix a Riemannian metric on N then the space P^+T^*N can be identified with the unit cotangent bundle. The restriction of the canonical 1-form pdq is a contact form for ζ. Thus the flow generated by the Reeb vector field for this contact form coincides with the geodesic flow.

Suppose now that α is a non-vanishing closed 1-form on N. Then it corresponds to a Lagrangian section $\hat{L}_\alpha \subset T^*N \setminus N = S_\zeta M$. The image $L_\alpha \subset M$ of $\hat{L}_\alpha$ under the canonical projection $S_\zeta M \to M$ is a pre-Lagrangian submanifold. The form α defines on L_α a foliation with Legendrian leaves. If a multiple $C\alpha$ for a constant $C > 0$ represents an integer cohomology class in $H^1(N;\mathbf{R})$ then all leaves of the foliation are closed Legendrian submanifolds of M.

Equivalently, the above example can be rephrased as follows. Suppose that a closed manifold N can be fibered over the circle S^1. Let $\pi : N \to S^1$ be the projection. Then $d\pi$ is a non-vanishing closed 1-form on N and its graph $\hat{L}$ is a Lagrangian submanifold in $T^*N \setminus N$. Then the image $L = L_{d\pi}$ of $\hat{L}_{d\pi}$ under the projection $T^*N \setminus N \to PT^*N$ is a pre-Lagrangian submanifold in the space of co-oriented contact elements of N. Notice that L is foliated by Legendrian lifts of hypersurfaces $\pi^{-1}(T) \subset N$, $T \in S^1$.

For instance, if N is the torus T^n, then the contact manifold $M = P^*T^*N$ admits a splitting $M = T^n \times S^{n-1}$ such that each torus $T^n \times p$, $p = (p_1, \ldots, p_n) \in S^{n-1}$, is a pre-Lagrangian torus of the form L_α for the non-vanishing closed 1-form $\alpha = \sum_1^n p_i dq_i$, $q_i \in S^1$. An everywhere dense set of these tori can be further split as products $T^{n-1} \times S^1$ where all tori $T^{n-1} \times q$, $q \in S^1$, are Legendrian.

2.4.2 PRE-LAGRANGIAN SURFACES IN 3-MANIFOLDS. Let (M, ζ) be a three-dimensional contact manifold and $T \subset M$ be an embedded 2-torus, transversal to ζ. The line bundle $T(T) \cap \zeta$ integrates to a 1-dimensional, so-called *characteristic* foliation $\mathcal{F}_\zeta$. The torus T is pre-Lagrangian if and only if the foliation $\mathcal{F}_\zeta$ is diffeomorphic to a linear foliation of the torus $T \cong \mathbf{R}^2/\mathbf{Z}^2$.

Remark 2.4.1: The above example indicates that the class of smoothness of the Lagrangian lift can be of crucial importance even in the case of a C^∞-smooth pre-Lagrangian manifold.

2.4.3 SYMPLECTIZATION OF CONTACTIZATION. Let (N, ω) be a symplectic manifold with the symplectic form $\omega/\hbar$ representing an integral cohomology class $[\omega/\hbar]$. Let (M, ζ) be the contactization $C(N, \omega)$ of the manifold (V, ω) and α be the connection on V as described in §2.3 above.

If $L \subset N$ is a Lagrangian submanifold then the connection α over it is flat. The pull-back $\pi^{-1}(L) \subset M$ under the projection $\pi : M \to N$ is a pre-Lagrangian submanifold L_0 foliated by Legendrian leaves obtained by integrating the flat connection over L. If this foliation is a fibration, i.e. when the holonomy defined by the connection α is integral over L then the pre-Lagrangian submanifold L is foliated by closed Legendrian manifolds. In particular, this is the case when the connection form is exact over L, i.e. the connection over L is trivial. If this condition is satisfied then L is called a *Bohr-Sommerfeld orbit*. In this case the pre-Lagrangian submanifold L_0 is foliated by closed Legendrian lifts of L. These lifts are called sometimes, *Planckian* submanifolds (see [W] and [So]). The integrality of the holonomy is independent of the choice of the connection α but the Bohr-Sommerfeld condition depends on this choice, unless the image

$$\mathrm{Im}\big(H_1(L; \mathbf{R}) \to H_1(N; \mathbf{R})\big)$$

is trivial.

2.5 Lagrangian intersections in contact manifolds. In this section, we formulate theorems which give lower bounds for the number of transversal intersection points of Legendrian and pre-Lagrangian submanifolds of a contact manifold. These estimates will be proven in §3.8 below as an application of Floer homology theory which we are going to build in the next sections.

2.5.1 INTERSECTIONS IN THE SPACE OF CONTACT ELEMENTS. Suppose that a closed manifold N admits a Riemannian metric without contractible closed geodesics (e.g. a metric of non-positive sectional curvature). Let $M = P^+T^*N$ be the space of co-oriented contact elements. Suppose that there exists a non-vanishing closed 1-form α which represents an integral class $[\alpha] \in H^1(N)$. Let L_α be the pre-Lagrangian submanifold constructed in §2.4.1. In other words, L_α is the image of the graph $\hat{L}_\alpha \subset T^*N$ of the form α under the projection $T^*N \setminus N \to M = P^+T^*N$. As explained in 2.4.1, L_α carries a foliation by closed Legendrian leaves. Let Λ be one of the leaves.

THEOREM 2.5.1. *Let $\varphi_t : M \to M$, $t \in [0,1]$, $\varphi_0 = \mathrm{Id}$, be a contact isotopy of M such that $\varphi_1(\Lambda)$ is transversal to L_α. Then*

$$\#\varphi_1(\Lambda) \cap L_\alpha \geq \operatorname{rank}(\mathrm{H}_*(\Lambda; \mathbf{Z}/2)) \ .$$

In particular, suppose $M = T^n$ is the n-torus. Then we have the splitting $P^+T^*T^n = T^n \times S^{n-1}$ and all tori $T^n \times a$, $a \in S^{n-1}$, are pre-Lagrangian. For an everywhere dense subset $A \subset S^{n-1}$, the tori $T^n \times a$, $a \in A$, are foliated by Legendrian $(n-1)$-dimensional tori. Let L be one of these pre-Lagrangian tori $T^n \times a$ and Λ, $\Lambda \subset L$, be one of its Legendrian subtori. Let $\varphi_t : P^+T^*T^n \to P^+T^*T^n$, $t \in [0,1]$, be a contact isotopy with $\varphi_0 = \mathrm{Id}$ such that $\varphi_1(\Lambda)$ is transversal to L.

Then we have

COROLLARY 2.5.2. $\#\varphi_1(\Lambda) \cap L \geq 2^{n-1}$.

Remark 2.5.3: A Legendrian submanifold $\Lambda \subset M$ has a neighborhood U contactomorphic to the 1-jet space $J^1(\Lambda)$. The pre-Lagrangian submanifold $L \cap U$ can be identified under the contactomorphism with the "0-wall" $W = \Lambda \times \mathbf{R} \subset J^1(\Lambda)$, i.e. the set of 1-jets of functions with zero differential. Thus, Theorem 2.5.1 can be considered as a global version of the well-known fact that Λ cannot be disjoined with W via a contact isotopy (Chekanov's theorem).

2.5.1 INTERSECTIONS IN THE SPACE OF PRE-QUANTIZATION. Let us now turn to the situation described in section 2.4.3. Let (N, ω) be a symplectic manifold such that the symplectic form $\omega/\hbar$ represents an integral cohomology class $[\omega/\hbar] \in H^2(N)$. Let $(M,\zeta) = C(N,\omega)$ be the contactization of (N,ω) (see 2.4.3 above) and $L \subset N$ be a Lagrangian submanifold which satisfies the Bohr-Sommerfeld condition. Let Λ_1, $\Lambda_1 \subset M$ be a Legendrian lift of L and $\Lambda_0 = \pi^{-1}(L)$ be the pre-Lagrangian pull-back of L under the projection $\pi : M \to N$.

Let $\varphi_t : M \to M$, $t \in [0,1]$, be a contact isotopy with $\varphi_0 = \mathrm{Id}$ such that $\varphi_1(\Lambda_1)$ is transversal to Λ_0.

THEOREM 2.5.4. *Suppose that $\pi_2(M,\Lambda_0) = 0$. Then*

$$\#\varphi_1(\Lambda_1) \cap \Lambda_0 \geq \text{ rank } H_*(\Lambda_1; \mathbf{Z}/2) .$$

For instance, let N be a surface of positive genus, ω an area form with $\int_N \omega = 1$ and $(M,\zeta) = C(N,\omega)$ be the contactization with $\hbar = 1/n$. Let $L \subset N$ be a non-contractible Bohr-Sommerfeld orbit, $\Lambda_1 \subset M$ its Legendrian lift and $\Lambda_0 = \pi^{-1}(L) \subset M$ its pre-Lagrangian pull-back. Then, we have

COROLLARY 2.5.5. *For the contact isotopy $\varphi_t : M \to M, t \in [0,1], \varphi_0 = \mathrm{Id}$, such that $\varphi_1(\Lambda_1)$ is transversal to Λ_0 we have*

$$\#\varphi_1(\Lambda_1) \cap \Lambda_0 \geq 2 .$$

3. Floer Homology

3.1 Admissible Legendrian and pre-Lagrangian submanifolds. Let Λ_0 and Λ_1 be a pre-Lagrangian and a Legendrian submanifold, respectively, of a contact manifold (M,ζ). We will always assume in what follows that the submanifold Λ_1 is connected.

Let us denote by $\mathcal{P}(\Lambda_0,\Lambda_1)$ the space of paths $\delta : [0,1] \to M$ with $\delta(0) \in \Lambda_0$ and $\delta(1) \in \Lambda_1$. A component $\mathcal{P}_0$ of the space $\mathcal{P}(\Lambda_0,\Lambda_1)$ is called *admissible* if it satisfies the following two conditions $\mathcal{P}_1$ and $\mathcal{P}_2$.

$\mathcal{P}_1$ For any map $\Delta : S^1 \times [0,1] \to M$ such that $\Delta(u,0) \in \Lambda_0$, $\Delta(u,1) \in \Lambda_1$, and $\Delta|_{u\times[0,1]} \in \mathcal{P}_0$ for $u \in S^1$, the curve $\Delta|_{S^1\times 0}$ is a vanishing cycle on Λ_0 (see 2.2).

$\mathcal{P}_2$ For any map $\Delta : S^1 \times [0,1] \to M$, as in $\mathcal{P}_1$, the relative Maslov class $\mu(\Delta|_{S^1\times 0}, \Delta|_{S^1\times 1})$ vanishes (see 2.2).

LEMMA 3.1.1. *The condition $\mathcal{P}_1$ implies that for any map $F : (D^2, \partial D^2) \to (M,\Lambda_0)$ the curve $F|_{\partial D^2} : \partial D^2 \to \Lambda_0$ is a vanishing cycle in Λ_0.*

Proof: Any such map can be deformed, keeping the boundary fixed, to a map $\widetilde{F}$ such that there exists a map Δ as in $\mathcal{P}_1$, which can factored as

$\Delta = \widetilde{F} \circ p$ where $p : S^1 \times [0,1] \to D^2$ is the projection which collapses the circle $S^1 \times 1$ to the center of the disc D^2. □

Of course, existence of an admissible component of the space $\mathcal{P}(\Lambda_0, \Lambda_1)$ is a very restrictive condition on manifolds M, Λ_0 and Λ_1. However, there is an important case when it does exist.

LEMMA 3.1.2. *Suppose that $\Lambda_1 \subset \Lambda_0$ and the boundary homomorphism*

$$\pi_2(M, \Lambda_0) \to \pi_1(\Lambda_0)$$

is trivial. Then the component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ which contains constant paths from Λ_1 is admissible.

Proof: The proof follows immediately from the observation that every loop in $\mathcal{P}_0$ is homotopic to a loop of constant paths. □

In order to develop a Floer homology theory for the intersection problem of Λ_0 and Λ_1 we fix a path component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ and impose two severe restrictions, including the admissibility of $\mathcal{P}_0$.

O_1 The path component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ is admissible.

O_2 There exists a contact form $\delta \in \mathrm{Cont}(\zeta)$ such that the flow defined by its Reeb vector field Y has no contractible periodic orbits and each orbit with two ends on Λ_1 represents a non-trivial class from $\pi_1(M, \Lambda_1)$.

The set of contact forms $\delta \in \mathrm{Cont}(\zeta)$ which satisfy the condition O_2 will be denoted by $\mathrm{Adm}(\zeta, \Lambda_0, \Lambda_1)$.

Our goal is to define Floer homology groups of the triple $\Lambda_0, \Lambda_1, \mathcal{P}_0$. To understand the relevance of the component $\mathcal{P}_0$ note that every intersection point $x \in \Lambda_0 \cap \Lambda_1$ determines a constant path $\delta(t) \equiv x$ and these constant paths may lie in different path components for different intersection points. The Floer homology groups $HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)$ will arise from a chain complex which is generated by all those intersection points which correspond to constant paths in $\mathcal{P}_0$. In most of our applications there is only one relevant path component which corresponds to all the fixed points and the Floer homology groups of all other path components are zero. Hence we shall sometimes neglect the dependence on $\mathcal{P}_0$ in our notation when the choice of the path component is clear from the context.

3.2 Examples of admissible submanifolds. We will verify in this section that the conditions O_1 and O_2 hold in all theorems from section 2.5.

LEGENDRIAN AND PRE-LAGRANGIAN SUBMANIFOLDS IN P^+T^*N. Let $M = P^+T^*N$, $\Lambda_1 = \Lambda$ and $\Lambda_0 = L_\alpha$ be as in Theorem 2.5.1. Fix a point $\tilde{q} \in \Lambda_0$. Let us denote by $p : M \to N$ the canonical projection and set $q = p(\tilde{q})$,

$S = p^{-1}(q)$. Let us verify that the boundary homomorphism $\pi_2(M, \Lambda_0) \to \pi_1(\Lambda_0, \tilde{q})$ is trivial. Indeed, let f be a map $(D^2, \partial D^2) \to (M, \Lambda_0)$ and $g_t : D^2 \to N$, $t \in [0,1]$, be a homotopy of the projection $g_0 = p \circ f$ to a constant map g_1 to the point $q \in Q$. This homotopy can be lifted, using the covering homotopy property, to a homotopy $f_t : (D^2, \partial D^2) \to (M, \Lambda_0)$. In particular, f_1 maps D^2 to the sphere $S = p^{-1}(q)$ and $f_1(\partial D^2)$ is the point $\tilde{q} = \Lambda_0 \cap p^{-1}(q)$. Thus the conditions of Lemma 3.1.2 are satisfied, and therefore the component $\mathcal{P}_0$ is admissible.

To verify the condition O_2 take a metric on N without contractible closed geodesics. Identifying $M = P^+T^*N$ with the unit cotangent bundle with respect to this metric we get a contact form β on M whose Reeb flow is the geodesic flow for the chosen metric. Thus the Reeb flow for the form β has no contractible periodic orbits. Let $q_\alpha : N \to S^1$ be the map corresponding to the form α, i.e. $\alpha = q_\alpha^*(d\theta)$. Notice that the projection $p : P^+T^*N \to N$ maps Λ onto one of the fibers $N_1 = q_\alpha^{-1}(\text{point})$. Let Γ be a piece of trajectory of the Reeb flow with two ends on Λ. Then the projection

$$(P^+T^*N, \Lambda) \xrightarrow{p} (N, N_1) \xrightarrow{q_\alpha} (S^1, \{\text{point}\})$$

projects Γ onto a non-trivial element of $\pi_1(S^1)$. Thus Γ represents a non-trivial element of $\pi_1(M, \Lambda)$ which verifies the condition O_2.

LEGENDRIAN AND PRE-LAGRANGIAN SUBMANIFOLDS IN THE SPACE OF CONTACTIZATION. Under the assumptions of Theorem 2.5.4 we have $\pi_2(M, \Lambda_0) = 0$ and thus, according to Lemma 3.1.2, the component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$, which contains constant paths from Λ_1, is admissible.

Let us check the condition O_2. Let us recall that the contact structure ζ on the space M of pre-quantization can be defined by an S^1-invariant contact form α on the principal S^1-bundle $M \to N$. The trajectories of the Reeb flow for the form α are fibers of the fibration. Thus all trajectories are closed and all simple closed trajectories are homotopic. Let Γ be one of the trajectories which is contained in Λ_0. Then $\int_\Gamma \alpha \neq 0$. Suppose that Γ bounds a disc $D \subset M$. Then $\int_D d\alpha \neq 0$ and, therefore, D represents a non-trivial element from $\pi_2(M, \Lambda_0)$. This contradicts the assumption of Theorem 2.5.4, and, therefore, Γ, and all its multiples, are non-contractible.

Notice that a trajectory of the Reeb flow with both ends on Λ_0 has to coincide with the periodic orbit Γ considered above. If Γ represents a trivial element of $\pi_1(M, \Lambda_1)$ then it bounds, together with a curve $\Gamma' \subset \Lambda_1$, a disc $D \subset M$, i.e. $\partial D = \Gamma \cup \Gamma'$. Then

$$\int_D d\alpha = \int_\Gamma \alpha + \int_{\Gamma'} \alpha \,.$$

But $\gamma|_{\Lambda_1} = 0$ and therefore the second integral equals 0. Thus, as in the case of the closed orbit, we have $\int_D d\alpha \neq 0$ and hence D represents a non-trivial element of $\pi_2(M, \Lambda_0)$ which again contradicts to assumption of Theorem 2.5.4.

3.3 Almost complex structures on the symplectization. Suppose that the contact manifold (M, ζ), its pre-Lagrangian submanifold Λ_0 and Legendrian submanifold Λ_1 satisfy the conditions O_1 and O_2. Let $(V = S_\zeta(M), \omega = \omega_\zeta)$ be the symplectization of (M, ζ).

Let us recall that an almost complex structure J is called *compatible* with ω, if the bilinear form $\langle v, w\rangle = \omega(v, Jw)$ is a metric, invariant under J.

A fiberwise splitting $H : V \to M \times \mathbf{R}$ is called *admissible* if it coincides at infinity with H_γ for an admissible form $\gamma \in \mathrm{Adm}(\zeta, \Lambda_0, \Lambda_1)$.

Notice that the push-forward $(H^{-1})^*\alpha_\zeta$ of the canonical form α_ζ on $S_\zeta(M)$ can be written as $\exp\theta\gamma_\theta$ where $\gamma_\theta \in \mathrm{Cont}(\zeta)$, $\theta \in \mathbf{R}$, and γ_θ coincides with γ when $|\theta|$ is sufficiently large. In other words, the pre-image $H^{-1}(M \times \theta) \subset S_\zeta(M)$, $\theta \in \mathbf{R}$, is the graph of the 1-form $\exp\theta\gamma_\theta$.

We also have $\omega = H^*(d(\exp\theta\gamma_\theta))$ and $dH(X_\zeta) = h\frac{\partial}{\partial\theta}$ for a positive function $h : M \times \mathbf{R} \to \mathbf{R}$ which is equal to 1 at infinity.

Having fixed an admissible splitting $H : V \to M \times \mathbf{R}$ we will not distinguish between an almost complex structure J on V and its push-forward $H_*(J)$ on $M \times \mathbf{R}$.

An almost complex structure J compatible with ω is called *admissible* for $(M, \zeta), \Lambda_0$ and Λ_1 if there exists an admissible splitting $H : S_\zeta(M) \to M \times \mathbf{R}$ of the space of symplectization such that

- for each $a \in \mathbf{R}$ the contact structure $\zeta = \{\gamma_a = 0\}$ on $M_a = M \times a$ is invariant under J;
- the vector field $J \cdot \frac{\partial}{\partial\theta}|_{M_a}$ belongs to the kernel of the form $(H^{-1})^*\omega|_{M_a} = d(\exp\theta\gamma_\theta)|_{M_a}$, $a \in \mathbf{R}$;
- J is invariant under the $\mathbf{R}$-action at infinity.

Notice that the above conditions imply that all the levels M_a, $a \in \mathbf{R}$, are J-convex being cooriented by the vector field $\frac{\partial}{\partial\theta}$.

Suppose we are given two admissible structures J and J'. Viewing them as defined on $M \times \mathbf{R}$ we say that a sequence of admissible almost complex structures J_n, $n = 1, \ldots$, *interpolates* between J' and J if there exists a constant $N > 0$ and an increasing sequence $d_n \to \infty$ such that $J_n = J$ on $M \times [-d_n, d_n]$, $J_n = J'$ outside of $M \times [-(d_n + N), d_n + N]$, and the restrictions $J_n|_{M \times [-(d_n+N), -d_n]}$ coincide up to translations for all $n = 1, \ldots$.

3.4 Action functional. Suppose that $(M, \zeta), \Lambda_0, \Lambda_1$ and the path component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ satisfy the condition O_1. Let (V, ω) be the symplectization of (M, ζ), L_1 the symplectization of Λ_1, and L_0 a Lagrangian lift of Λ_0. Denote by $\mathcal{P}(L_0, L_1)$ the space of paths $\delta : [0,1] \to V$ with $\delta(0) \in L_0$ and $\delta(1) \in L_1$. Note that every path $\delta \in \mathcal{P}(\Lambda_0, \Lambda_1)$ lifts to a path $\hat{\delta} : [0,1] \to V$ with $\hat{\delta}(0) \in L_0$ and $\hat{\delta}(1) \in L_1$ and that the homotopy class of the lift is uniquely determined by δ. Hence the path component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ determines a unique path component in $\mathcal{P}(L_0, L_1)$ which we shall also denote by $\mathcal{P}_0$. This slight abuse of notation should not create any confusion.

Fix a path $\delta_0 \in \mathcal{P}_0 \subset \mathcal{P}(L_0, L_1)$ and for any other path $\delta \in \mathcal{P}_0$ choose a homotopy $\delta_u \in \mathcal{P}_0$, $u \in [0,1]$, which connects δ_0 with $\delta_1 = \delta$. Set $\Delta(u,t) = \delta_u(t)$ for $(u,t) \in [0,1] \times [0,1]$. Define now the action

$$\mathcal{A}_{\delta_0}(\delta) = \int_{[0,1]\times[0,1]} \Delta^*\omega \ .$$

We will omit δ_0 in the notation for the action when the choice of the base path is clear or irrelevant.

The property O_1 ensures that $\mathcal{A}_{\delta_0}(\delta)$ does not depend on the choice of the homotopy Δ (but it does depend on the choice of the path δ_0).

Critical points of the functional $\mathcal{A}_{\delta_0}$ are constant paths corresponding to the intersection points of L_0 and L_1. In order to count their number we need to define (and compute) Floer homology groups for the action functional $\mathcal{A}_{\delta_0}$.

3.5 Gradient flow. Choose an admissible almost complex structure J on V. This choice allows us to define a quasi-Kählerian metric on V:

$$g(v,w) = \omega(v, Jw) \ , \qquad v, w \in T_x(V) \ , \ x \in V \ .$$

Given a family J^t, $t \in [0,1]$, of admissible almost complex structures, we can define a metric on the path space $\mathcal{P}(L_0, L_1)$ by the formula

$$||v||^2 = \int_0^1 \omega(v, J^t v)dt \ , \qquad v \in T_\delta \mathcal{P}(L_0, L_1) \ , \ \delta \in \mathcal{P}(L_0, L_1) \ .$$

The gradient of the symplectic action $\mathcal{A}_{\delta_0}$ with respect to this metric on $\mathcal{P}(L_0, L_1)$ is given by

$$\operatorname{grad} \mathcal{A}_{\delta_0}(\delta) = -J^t \dot{\delta} \ .$$

Thus a gradient flow line is a smooth map $u : \mathbf{R} \times [0,1] \to V$ which satisfies the partial differential equation

$$\frac{\partial u}{\partial s} + J^t(u) \cdot \frac{\partial u}{\partial t} = 0 \tag{1}$$

with boundary conditions

$$u(s,0) \in L_0 \ , \quad u(s,1) \in L_1 \text{ for } s \in \mathbf{R} \ . \tag{2}$$

When $J^t \equiv J$ then this is just the usual Cauchy-Riemann equations and, therefore, the gradient line u is a J-holomorphic curve in V with boundary in $L_0 \cup L_1$.

In the general case the gradient trajectories of the action functional can also be interpreted as holomorphic curves but in an auxiliary manifold, and not in the manifold V itself (comp., for instance, [Gr], [F3] and [SZ]).

3.6 Energy. Given a solution $u : \mathbf{R} \times [0,1] \to V$ of (1) and (2), the symplectic area $\int_B u^*\omega$ will be denoted by $E(u)$ and called the *energy* of the solution u. When $J^t \equiv J$ then the energy $E(u)$ coincides with the area of the J-holomorphic curve u computed in terms of the almost Kählerian metric

$$g(u,v) = \omega(u, Jv) .$$

The following proposition is standard in Floer homology theory (cf. [F1]) and me omit its proof here.

THEOREM 3.6.1. *Suppose that L_0 and L_1 intersect transversally and J^t is a family of admissible almost complex structures. Let u be a solution of (1) and (2) with $E(u) < \infty$. Then there exist the limits*

$$\lim_{t \to \pm\infty} u(s,t) = x^{\pm} , \qquad x^{\pm} \in L_0 \cap L_1 .$$

We will call such a u a *connecting orbit* between the two critical points x^+ and x^- of the action functional $\mathcal{A}_{\delta_0}$. The definition of the action functional implies that

$$E(u) = \mathcal{A}_{\delta_0}(x^+) - \mathcal{A}_{\delta_0}(x^-) .$$

If $J^t \equiv J$ then a solution u of (1) and (2) of finite energy can be viewed as a J-holomorphic disc with boundary in $L_0 \cup L_1$ passing through two points $x^{\pm} \in L_0 \cap L_1$.

A Floer complex can be defined now as usual by counting the connecting orbits when the relative Morse index is 1.

The crucial point for the construction of the theory is the following compactness theorem for the solutions of (1) and (2) with bounded energy. The proof will be given in §3.9.

THEOREM 3.6.2. *Assume that the contact manifold (M, ζ), its pre-Lagrangian submanifold Λ_0 and a Legendrian submanifold Λ_1 satisfy the hypotheses O_1 and O_2. Let J^t, $t \in [0,1]$, be a family of admissible almost complex structures on the symplectization V. Then for every $c > 0$ the space*

$$\mathcal{M}^c = \mathcal{M}^c(L_0, L_1; J^t)$$

of all smooth solutions u of the boundary value problem (1) and (2) which satisfy the energy bound

$$E(u) \le c$$

is compact (with respect to the topology of uniform convergence with all derivatives on compact sets).

We will need also a slightly stronger Theorem 3.6.3.

THEOREM 3.6.3. *Suppose that a sequence J_n^t, $t \in [0,1]$, $n = 1, \dots,$ of families of admissible almost complex structures on V interpolates between two families of admissible structures $(J')^t$ and J^t. Then given a sequence $u_n \in \mathcal{M}^c(L_0, L_1, J_n^t)$, $n = 1, \dots,$ one can find a subsequence which converges, uniformly on compact sets, to a solution $u \in \mathcal{M}^c(L_0, L_1, J^t)$.*

Remark 3.6.4: Theorem 3.6.3 holds even in a stronger form: it is sufficient to require that the sequence J_n^t converges to J^t uniformly on compact sets. However we will not need this stronger version in this paper.

Notice that the condition O_1 prohibits bubbling-off of the solutions at boundary points while the bubbling-off at interior points is impossible because the symplectic manifold (V, ω) is exact.

Thus, if we knew à priori that all the solutions of (1) and (2) would take values in a compact subset of V then the above theorem would follow directly from the usual compactness theory for Gromov's pseudoholomorphic curves (cf. [Gr] or [MS]). Hence our goal is to prove this bound for solutions from $\mathcal{M}^c$. The main ingredient to the proof is a rescaling trick which was first applied by Hofer in [H].

3.7 Floer homology. Suppose that (M, ζ), Λ_0 and Λ_1 satisfy the conditions $O_1 - O_2$. Let $(V, \omega), L_0, L_1$ be their symplectic counterparts and J^t a family of admissible almost complex structures. Pick an admissible component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ and a path $\delta_0 \in \mathcal{P}_0$. Let $\hat{\delta}_0$ be a lift of δ_0 to the space $\mathcal{P}(L_0, L_1)$. The component of $\hat{\delta}_0$ in $\mathcal{P}(L_0, L_1)$ will be still denoted by $\mathcal{P}_0$.

The Floer homology groups

$$HF_*(\Lambda_0, \Lambda_1; J^t) = HF_*(L_0, L_1; J^t) = HF_*(L_0, L_1, \mathcal{P}_0, J^t)$$

can roughly be described as the *middle dimensional homology groups* of the path space $\mathcal{P}_0 \subset \mathcal{P}(L_0, L_1)$ (compare [Wi]). They are obtained from the gradient flow of the symplectic action

$$\mathcal{A} : \mathcal{P}_0 \to \mathbf{R}$$

as in Floer's original work on Lagrangian intersections in compact symplectic manifolds [F1-3]. See also [O]. We summarize the main points of Floer's construction.

Assume that Λ_0 and Λ_1 (and hence L_0 and L_1) intersect transversally. Then all the critical points of $\mathcal{A}$ are nondegenerate. Given two intersection points $x^\pm \in L_0 \cap L_1$ denote by

$$\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+, J^t)$$

the space of all solutions $u : B \to V$ of (1) and (2) with limits (3.6.1). Linearizing the differential equation (1) gives rise to an operator

$$D_u : W_L^{1,2}(u^*(TV)) \to L^2(u^*(TV)) \ .$$

Here $W_L^{1,2}(u^*(TV))$ denotes the Sobolev space of all vector fields $Y(s,t) \in T_{u(s,t)}V$ along u which satisfy the boundary condition

$$Y(0,t) \in T_{u(0,t)}L_0 \ , \quad Y(1,t) \in T_{u(1,t)}L_1 \ .$$

The space $L^2(u^*(TV))$ is defined similarly and D_u is a Cauchy-Riemann operator. This operator is Fredholm whenever L_0 and L_1 intersect transversally. Its index is a relative Maslov class and can be defined as follows. Given $u \in \mathcal{M}(x^-, x^+)$ choose a symplectic trivialization

$$\Phi(s,t) : \mathbf{R}^{2n+2} \to T_{u(s,t)}V$$

of $u^*(TV)$ such that

$$\Phi(s,t)^*\omega = \sum_{j=0}^{n} dx_j \wedge dy_j$$

and there exist limits

$$\lim_{t\to\pm\infty} \Phi(s,t) = \Phi^\pm : \mathbf{R}^{2n+2} \to T_{x^\pm}V \ .$$

This gives rise to two Lagrangian paths in $\mathbf{R}^{2n+2}$:

$$\lambda_0(t) = \Phi(0,t)^{-1}T_{u(0,t)}(L_0)$$

and

$$\lambda_1(t) = \Phi(1,t)^{-1}\left(T_{u(1,t)}(L_1)\right) \ .$$

These paths are transverse at $t = \pm\infty$ and therefore have a relative Maslov index $\mu(\lambda_0, \lambda_1)$ (cf. [F1] and [RS1]). This index is independent of the choice of the trivialization. The Fredholm index of D_u agrees with this Maslov index

$$\text{INDEX } D_u = \mu(u) = \mu(\lambda_0, \lambda_1)$$

whenever u satisfies the boundary condition (2) and the limit condition (3.6.1) (cf. [F1] and [RS2]).

Now recall that not all the intersection points from $L_0 \cap L_1$, viewed as constant paths, belong to the component $\mathcal{P}_0$. We denote by $(L_0 \cap L_1)_0$ the subset of $L_0 \cap L_1$ which consists of those intersection points which belong to $\mathcal{P}_0$. The condition $\mathcal{P}_2$ implies:

LEMMA 3.7.1. *If* $x^- = x^+ \in (L_0 \cap L_1)_0$ *then* $\mu(u) = 0$.

The previous lemma shows that there exists a map $\mu : (L_0 \cap L_1)_0 \to \mathbb{Z}$ such that

$$\text{INDEX } D_u = \mu(x^-) - \mu(x^+)$$

whenever u and θ satisfy (2) and (3.6.1). Now everything is as usual. A family of admissible almost complex structures J^t, $t \in [0, 1]$, is called *regular* if the operator D_u is onto for every $u \in \mathcal{M}(x^-, x^+)$ and every pair of intersection points $x^\pm \in L_0 \cap L_1$. By the Sard-Smale theorem the set

$$\mathcal{REG} = \mathcal{REG}(L_0, L_1)$$

of regular J^t is dense in the set of all admissible families.

The argument is as in [F2] or [SZ]. See also [FSH] for a detailed discussion of transversality properties.

Remark 3.7.2: We need to consider families J^t rather than individual J just to ensure this genericity condition.

Now for every $J^t \in \mathcal{REG}$ the space $\mathcal{M}(x^-, x^+)$ is a finite dimensional manifold with

$$\dim \mathcal{M}(x^-, x^+) = \mu(x^-) - \mu(x^+) .$$

If $\mu(x^-) - \mu(x^+) = 1$ then, by Theorem 3.6.2, the quotient space $\mathcal{M}(x^-, x^+)/\mathbf{R}$ consists of finitely many orbits and the numbers

$$n_2(x^-, x^+) = \#\mathcal{M}(x^-, x^+)/\mathbf{R} \,(\mathrm{mod}\, 2)$$

determine the Floer chain complex as follows. The chain groups are defined by

$$CF_k = CF_k(L_0, L_1, \mathcal{P}_0) = \sum_{\substack{x \in (L_0 \cap L_1)_0 \\ \mu(x) = k}} \mathbb{Z}_2 \langle x \rangle .$$

and the boundary operator $\partial : CF_k \to CF_{k-1}$ is given by

$$\partial \langle x \rangle = \sum_{\mu(y) = k-1} n_2(x, y) \langle y \rangle$$

for $x \in (L_0 \cap L_1)_0$ with $\mu(x) = k$. As in Floer's original proof one uses gluing techniques to prove that $\partial \circ \partial = 0$ (cf. [F3] and [SZ]).

The *Floer homology groups* are now defined as the homology of this chain complex

$$HF_*(L_0, L_1; J^t) = HF_*(L_0, L_1, \mathcal{P}_0; J^t) := H_*(CF, \partial) .$$

The Floer homology groups depend on the path component $\mathcal{P}_0$ but when the choice of the path component is clear from the context we shall drop $\mathcal{P}_0$ from the notation.

THEOREM 3.7.3. (i) *For any two admissible families of almost complex structures J^t, $(J')^t \in \mathcal{REG}$ there is a natural isomorphism*

$$HF_*(L_0, L_1, \mathcal{P}_0; J^t) \to HF_*\big(L_0, L_1, \mathcal{P}_0; (J')^t\big) .$$

(ii) *For any $J^t \in \mathcal{REG}$ and any compactly supported Hamiltonian isotopy ψ_t, $t \in [0,1]$, there exists a natural isomorphism*

$$HF_*(L_0, L_1, \mathcal{P}_0; J^t) \to HF_*\big({\psi_0}^{-1}(L_0), {\psi_1}^{-1}(L_1), \psi^*\mathcal{P}_0; J^t\big)$$

where $\psi^*\mathcal{P}_0 \subset \mathcal{P}({\psi_0}^{-1}(L_0), {\psi_1}^{-1}(L_1))$ denotes the component of the path $t \mapsto {\psi_t}^{-1}(\delta(t))$ for $\delta \in \mathcal{P}_0 \subset \mathcal{P}(L_0, L_1)$.

(iii) *For any symplectomorphism $f : V \to V$, which commutes at infinity with the* **R***-action, there exists a natural isomorphism*

$$HF_*(L_0, L_1, \mathcal{P}_0; J^t) \to HF_*\big(f(L_0), f(L_1), f_*\mathcal{P}_0; f_*J^t\big) .$$

Proof: Statement (iii) is obvious. The invariance under compactly supported change of the regular family J^t is standard in Floer's theory. To prove the invariance under Hamiltonian isotopies of the Lagrangian submanifolds L_0 and L_1 it is convenient to introduce a Hamiltonian term in the action functional $\mathcal{A}$. Hence let $H^t = H^{t+1} : V \to \mathbf{R}$ be a smooth family of compactly supported Hamiltonian functions with corresponding Hamiltonian vector fields X^t. Then the critical points of the perturbed action functional are solutions $x : [0,1] \to V$ of $\dot{x}(t) = X^t(x(t))$ with $x(0) \in L_0$ and $x(t) \in L_1$ and the gradient flow lines are solutions $u : \mathbf{R} \times [0,1] \to V$ of

$$\partial_s u + J^t(u)\big(\partial_t u - X^t(u)\big) = 0 \tag{3}$$

with the same boundary conditions $u(s,0) \in L_0$ and $u(s,1) \in L_1$ (compare with equations (1) and (2)). This gives rise to Floer homology groups $HF_*(L_0, L_1, \mathcal{P}_0; J^t, H^t)$ and as in the usual Floer theory one proves that these groups are independent of J and H up to natural isomorphisms. Now let $\psi_t : V \to V$ be a Hamiltonian isotopy generated by X^t via $\frac{d}{dt}\psi_t = X^t \circ \psi_t$ and define $v(s,t) = \psi_t^{-1}(u(s,t))$ where u is a solution of (3). Then, by a simple calculation, we find that

$$\partial_s v + \psi_t^* J^t(v) \partial_t v = 0$$

and $v(s,0) \in \psi_0^{-1}(L_0)$, $v(s,1) \in \psi_1^{-1}(L_1)$. This shows that there is a natural isomorphism

$$HF_*(L_0, L_1, \mathcal{P}_0; J^t, H^t) \to HF_*\big(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1), \psi^*\mathcal{P}_0; \psi_t^* J^t, 0\big)$$

Thus we have proved (ii) as well as (i) for compactly supported variations of the almost complex structure. The only additional thing we need to check is that the groups $HF_*(L_0, L_1; J^t)$ and $HF_*(L_0, L_1; (J')^t)$ are isomorphic even when J^t and $(J')^t$ differ at infinity.

There exists a sequence J_n^t, $n = 1, \ldots,$ of admissible almost complex structures which interpolates between $(J')^t$ and J^t. In view of Theorem 3.6.3 one can find a compact set K such that all connecting orbits for all J_n^t, as well as for J^t, are contained in K. If n is sufficiently large then J_n^t coincides with J^t on K. Thus J^t and J_n^t have the same set of connecting orbits, and therefore the Floer homology groups $HF_*(L_0, L_1; J^t)$ and $HF_*(L_0, L_1; J_n^t)$ coincide. On the other hand, J_n^t coincides with $(J')^t$ at infinity. Thus we have a canonical isomorphism between the groups $HF_*(L_0, L_1; J^t)$ and $HF_*(L_0, L_1; J_n^t)$ in view of the argument above while the groups $HF_*(L_0, L_1; J_n^t)$ and $HF_*(L_0, L_1; (J')^t)$ are isomorphic according to the conventional Floer theory. □

Theorem 3.7.3 shows, in particular, that we can drop J^t from the notation of Floer homology groups and that the groups $HF_*(L_0, L_1, \mathcal{P}_0)$, also denoted by $HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)$, are well defined even when Λ_0 and Λ_1 are not transversal. It should be noted, however, that these groups do depend on the choice of the admissible path component $\mathcal{P}_0$.

3.8 Contact manifolds. Let us return now to the contact environment. Theorem 3.7.3 implies

THEOREM 3.8.1. *Suppose that the contact manifold (M, ξ), the pre-Lagrangian submanifold Λ_0, the Legendrian submanifold Λ_1, and the path component $\mathcal{P}_0 \subset \mathcal{P}(\Lambda_0, \Lambda_1)$ satisfy the conditions O_1 and O_2. Then the groups*

$$HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)$$

are well defined and invariant under Legendrian isotopy of the submanifold Λ_1 as well as under a contactomorphism $f : M \to M$, i.e.

$$HF_*\big(f(\Lambda_0), f(\Lambda_1), f_*\mathcal{P}_0\big) = HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)\ .$$

Theorem 3.8.1 has the following standard application for counting the number of intersection points $\#\Lambda_0 \cap \Lambda_1 = \#L_0 \cap L_1$.

THEOREM 3.8.2. *Let Λ_0, Λ_1, and $\mathcal{P}_0$ be as in Theorem 3.8.1. Suppose that Λ_0 and Λ_1 intersect transversally. Then*

$$\#\Lambda_0 \cap \Lambda_1 \geq \#(\Lambda_0 \cap \Lambda_1)_0 \geq \mathrm{rank} HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)\ .$$

In particular, if all path components are admissible, then

$$\#\Lambda_0 \cap \Lambda_1 \geq \sum_{\mathcal{P}_0} \mathrm{rank} HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0)\ .$$

We have to impose an additional restriction on Λ_1 and Λ_0 in order to be able to compute Floer homology groups $HF_*(\Lambda_0, \Lambda_1)$.

THEOREM 3.8.3. *Suppose that in addition to the assumptions of Theorem 3.8.2 we have $\Lambda_1 \subset \Lambda_0$. Then there is a natural isomorphism*

$$HF_*(\Lambda_0, \Lambda_1, \mathcal{P}_0) \to H_*(\Lambda_1; \mathbf{Z}/2)$$

where $\mathcal{P}_0$ denotes the component of the space of constant paths. In particular,

$$\#\Lambda_0 \cap \Lambda_1' \geq \operatorname{rank}(H_*(\Lambda_1; \mathbf{Z}/2))$$

for any Legendrian submanifold Λ_1' which is Legendrian isotopic to Λ_1 and transverse to Λ_0.

Proof: As has already been mentioned (see 2.5.3), a neighborhood U of the Legendrian submanifold Λ_1 in M is contactomorphic to a neighborhood of the 0-section in the 1-jet space $J^1(\Lambda_1)$. This contactomorphism moves $\Lambda_0 \cap U$ onto the 0-wall W, i.e. the space of 1-jets of functions with 0 differential. Thus a Legendrian submanifold Λ_1', which is C^1-close to Λ_1 and transverse to W, corresponds to a Morse function $\varphi : \Lambda_1 \to \mathbf{R}$ so that the intersection points of Λ_0 and Λ_1' are in one-to-one correspondence with the critical points of the function φ. One can explicitly choose a metric on Λ_1 and an admissible almost complex structure J on the symplectization of M in such a way that the connecting orbits of the action functional would be in one-to-one correspondence with the gradient trajectories of the function φ connecting the corresponding critical points of this function. This identifies the Floer chain complex $CF_*(\Lambda_0, \Lambda_1')$ with the Morse chain complex for the function φ (cf. [Sc]) and thus defines a canonical isomorphism between the groups $HF_*(\Lambda_0, \Lambda_1)$ and $H_*(\Lambda_1; \mathbf{Z}/2)$. See [P] for a detailed proof (in the general case of clean Lagrangian intersections). □

Proof of Theorems 2.5.1 and 2.5.4: We already verified in 3.2 the conditions O_1 and O_2 in the situation of 2.5.1 and 2.5.4. Thus both statement follow from Theorem 3.8.3. □

3.9 Compactness. To clarify the main ideas of the proof we will assume in this section that all considered families of almost complex structures are constant. Thus the solutions of (1) and (2) can be treated as holomorphic curves for the corresponding almost complex structures. The general case, when the almost complex structures may depend on the parameter t, is similar, but less geometrically transparent.

As it was mentioned in Section 3.6 a solution $u : B = \mathbf{R} \times [0,1] \to V$ from $\mathcal{M}^c(L_0, L_1, J)$ can be equivalently viewed as a J-holomorphic disc in V with boundary in $L_0 \cup L_1$. We will employ both points of view.

The Theorem 3.6.3 is an immediate corollary of the following

THEOREM 3.9.1. *Suppose that a contact manifold (M, ζ), a pre-Lagrangian* *submanifold $\Lambda_0 \subset M$ and a Legendrian submanifold $\Lambda_1 \subset M$ satisfy the conditions* O_1 *and* O_2. *Let $(V,\omega), L_1$ and L_0 be the symplectization of $(M,\zeta), \Lambda_1$ and a Lagrangian lift of Λ_0, respectively. Let J_n, $n = 1,\dots,$ be a sequence of admissible almost complex structures on V which interpolates between two admissible almost complex structures J' and J. Let $u_n : B = \mathbf{R} \times [0,1] \to V, n = 1,\dots,$ be a sequence of J_n-holomorphic curves from $\mathcal{M}^c(L_0, L_1, J_n)$. Then all discs $\Delta_n = u_n(B)$ are contained in a common compact set $K \subset V$.*

Proof: Set $J_0 = J'$, $J_\infty = J$. As in 3.3 we will consider the almost complex structures J, J' and J_n, $n = 0,\dots,\infty$, as defined on the product $M \times \mathbf{R}$ so that the following conditions are satisfied:

— there exists an integer $d > 0$ such that J, J' are invariant under the $\mathbf{R}$-action (by translations) outside of $M \times [-d,d]$;

— there exists a constant $N > 0$ and an increasing sequence $d_n \to \infty$ such that $d_1 = d$ and for all $n < \infty$ we have $J_n = J$ on $M \times [-d_n, d_n]$, $J_n = J'$ outside of $M \times [-(d_n + N), d_n + N]$, and the restrictions $J_n|_{[-(d_n+N),-d_n]}$ coincide up to translations for all $n = 1,\dots$;

— for each $n = 0,\dots,\infty$ the almost complex structure J_n is compatible with the symplectic form $\omega_n = d(\exp\theta\gamma_{n,\theta})$, $\gamma_{n,\theta} \in \mathrm{Cont}(\zeta)$; $\gamma_{n,\theta} = \gamma_\infty$ for $|\theta| \le d_n$, $\gamma_{n,\theta} = \gamma_0$ for $|\theta| \ge d_n + N$, and $\gamma_{n,\pm\theta\pm d_n} = \gamma_{m,\pm\theta\pm d_m}$ for all $m, n \ge 1$ and $\theta \ge 0$;

— for each $n = 0,\dots,\infty$ and $a \in \mathbf{R}$ the contact structure $\zeta = \{\gamma_{n,a} = 0\}$ on the level $M_a = M \times a$ is invariant under J_n, and $J_n \cdot \frac{\partial}{\partial\theta}|_{M_a}$ belongs to the kernel of the form $\omega_n|_{M_a}$.

The last condition implies, in particular, that all levels M_a, being cooriented by the vector field $\frac{\partial}{\partial\theta}$, are (pseudo)convex for each of the almost complex structures J_n.

Without loss of generality we can also assume that $L_0 \subset M \times (-d,d)$, $L_1 = \Lambda_1 \times \mathbf{R}$. According to Sard's theorem there exists a constant a, arbitrarily close to 1 such that u_n are transversal to M_{ka} for all integers k and all $n \ge 1$. To simplify the notation we will assume that $a = 1$.

Set $\Omega_{a,b} = M \times [a,b]$.

First we observe

LEMMA 3.9.2. *All discs Δ_n are contained in $\Omega_{-\infty,d}$.*

Proof: Suppose that a disc Δ_n intersects $\Omega_{d,\infty}$. Then we have $\sup\theta\circ u_n \ge d$. The maximum of the function $\theta|_{\Delta_n}$ is achieved in a point $p \in \Delta_n$ because u_n converges to $x^\pm$ at infinity, and, on the other hand, $\theta(x^\pm) < d$. Thus $a = \theta(p) \ge d$. The point p cannot be an interior point of Δ_n because this would contradict the pseudoconvexity of M_a (maximum principle). Suppose that $p \in \partial\Delta_n$. Let τ be a vector tangent to $\partial\Delta_n$ at the point p. Then τ is tangent to L_1 and, therefore, $\tau \in \zeta_p \subset T_p(M_a)$. By the assumption, ζ is J_n-invariant and hence we have $J_n \cdot \tau \in \zeta_p \subset T_p(M_a)$. Therefore, the disc Δ_n is tangent to M_a at the boundary point p. But this is again impossible in view of pseudoconvexity of M_a (strong maximum principle). □

Set $\bar\omega_n = d_M(\gamma_{n,\theta})$, $n = 0, \dots, \infty$. Here d_M denotes the differential with respect to the variable $x \in M$. Thus for a point $p = (x, a) \in M \times \mathbf{R}$ we have

$$\bar\omega_n|_{T_p(M\times\mathbf{R})} = \exp(-a)(d\pi)^*(\omega|_{T_p(M_a)})$$

where π is the projection $M \times \mathbf{R} \to M$.

Denote by κ_n the plane field formed by kernels of the form $\bar\omega_n$. It is generated by the vector field $X = \frac{\partial}{\partial\theta}$ and the vector field $Y_n = J_n \cdot X$. Notice that Y_n is tangent to the level-sets M_a and $Y_n|_{M_a}$ is proportional to the Reeb vector field of the form $\gamma_{n,a}$.

LEMMA 3.9.3. *For any J_n-holomorphic curve $v : C \to M \times \mathbf{R}$ we have*

$$v^*\bar\omega_n = h\exp(-\theta \circ v)v^*\omega_n|_C \quad \text{for a function} \quad h : C \to [0,1] \ .$$

The function h vanishes only at singular points of v and the points of tangency of the curve $h(C)$ and the vector field $X = \frac{\partial}{\partial\theta}$.

Proof: Outside the branching points of v, the function h is the determinant of the projection of $v(C)$ to the contact distribution ζ along the plane field κ_n. According to the choice of the J_n this is an orthogonal projection which is a pointwise complex linear map. Hence, $0 \le h \le 1$ and h vanishes only at the points where the vector field X is tangent to $v(C)$. □

Observe also

LEMMA 3.9.4. *For each $n = 1, \dots$ and $i \ge d$ the domain $C_n^i = u_n^{-1}(\Omega_{-\infty,-i})$ is a union of discs and the following inequality*

$$0 \le \int_{C_n^i} u_n^*\bar\omega_n \le \exp(i)\int_B u_n^*\omega_n < c\exp(i)$$

holds.

Proof: The first statement of the lemma follows from J_n-convexity of the levels M_a, similarly to the proof of Lemma 3.9.2. Set $P_n^i = u_n^{-1}(M_{-i})$ and $R_n^i = u_n^{-1}(L_1 \cap \Omega_{-\infty,-i})$. Thus $\partial C_n^i = P_n^i \cup R_n^i$. Taking into the account that $\gamma_{n,\theta}|_{L_1} = 0$ we get

$$\begin{aligned}\int_{C_n^i} u_n^*\bar\omega_n &= \int_{\partial C_n^i} u_n^*\gamma_{n,\theta} = \int_{P_n^i} u_n^*\gamma_{n,-i} \\ &= \exp(i)\int_{C_n^i} u_n^*\omega_n \le \exp(i)\int_B u_n^*\omega_n < c\exp(i) \ .\end{aligned}$$ □

Let u be a map $B \to V$. A subdomain $U \subset B$ is called a *special domain of level k for u* if
— U is either a disc or annulus;
— $u|_U$ is transversal to $M_{-k} \cup M_{-k-1}$;
— $u(\partial U) \subset M_{-k} \cup M_{-k-1} \cup L_1$, $f(\partial U \cap \partial B) \subset L_1$;
— $u(\partial U) \cap M_{-j} \neq \emptyset$ for $j = k, k+1$;
— $u(U) \subset \Omega_{-\infty,-d}$.

LEMMA 3.9.5. *Let U be a special domain of level k for a J_n-holomorphic map $u : B \to V$. Then*

$$\int_U u^*\omega_n \leq 2\exp(d-k)\int_B u^*\omega_n < 2c\exp(d-k) = C_1\exp(-k)\ .$$

Proof: Similarly to the proof of 3.9.4 set

$$P_+ = u^{-1}(M_{-k})\ ,\quad P_- = u^{-1}(M_{-k-1})\ ,\quad R = \partial U \setminus (P_+ \cup P_-)\ .$$

Notice that $f(R) \subset L_1$ and thus $(u^*\gamma_{n,\theta})|_R = 0$. Thus, properly orienting $P_\pm$ we get

$$\begin{aligned}
0 < \int_U u^*\omega_n &= \int_{\partial U} \exp(-\theta \circ f)u^*\gamma = \\
&= \exp(-k)\int_{P_+} u^*\gamma_{n,-k} + \exp(-k-1)\int_{P_-} u * \gamma_{n,-k-1} \leq \\
&\leq 2\exp(-k)\int_{C_n^k} u^*\bar{\omega}_n \leq \\
&\leq 2\exp(-k)\int_{C_n^d} u^*\bar{\omega}_n \leq \\
&\leq 2\exp(d-k)\int_B u^*\omega_n \leq 2c\exp(d-k)\ .
\end{aligned}$$

□

The following combinatorial lemma plays the crucial role in the proof of Theorem 3.9.1.

LEMMA 3.9.6. *Suppose that the sequence of J_n-holomorphic discs $u_n : B \to V$ is not contained in any compact set. Then there exists a subsequence u_{n_k}, $k = 1, \ldots$, and a sequence G_k, $G_k \subset B$, such that*
- G_k *is special for* u_{n_k};
- $\int_{G_k} u_{n_k}^*\bar{\omega}_{n_k} \underset{k\to\infty}{\longrightarrow} 0$.
- G_k *is either*
  - a) *on the level j, $d \leq j < d_{n_k}$ or $j \geq d_{n_k} + N$, and is contained in* $\Omega_{-(j+2),-j}$ *or*
  - b) *on the level d_{n_k}, and is contained in* $\Omega_{-(d_{n_k}+N+1),-d_{n_k}+1}$.

Proof: According to the assumption, the holomorphic discs Δ_n are not contained in any compact set. In view of Lemma 3.9.2 one can choose a subsequence u_{n_k}, $k = 1, \ldots$, such that $d_{n_k} \geq d + k + 1$ and $\Delta_{n_k} \cap M_{-k-1-d} \neq \emptyset$.

Let $d \leq i \leq d + k$. Set $\varphi_k = -\theta \circ u_{n_k}$ and $B_k^i = C_{n_k}^i \setminus \mathrm{Int} C_{n_k}^{i+1} = \{i \leq \varphi_k \leq i+1\}$. Let B be a component of B_k^i which has non-empty intersections with $\varphi_k^{-1}(i)$ and $\varphi_k^{-1}(i+1)$. Then B is a disc, possibly with several holes. One gets a *saturation* $\widehat{B}$ of the domain B by filling either all of these holes, or all but one in such a way that both intersections $\partial\widehat{B} \cap \varphi_k^{-1}(i)$ and $\partial\widehat{B} \cap \varphi_k^{-1}(i+1)$ are still non-empty. Notice that $\widetilde{B}$ is a *special domain of level* i for the map u_{n_k}.

For each $k \geq 1$ we can find a sequence of these special domains $\widetilde{B}_k^j$, $j = d, \ldots, d+k$, such that $\widetilde{B}_k^j$ is on the level j and $\mathrm{Int}\widetilde{B}_k^j \cap \mathrm{Int}\widetilde{B}_k^i = \emptyset$ for $i \neq j$. Notice that $\bigcup_{j=d}^{d+k} \widetilde{B}_k^j \subset C_{n_k}^d$. Thus according to 3.9.3 and 3.9.4 we have

$$\sum_{j=d}^{k+d} \int_{\widetilde{B}_k^j} u_{n_k}^* \bar{\omega}_{n_k} \leq \int_{C_{n_k}^d} u_{n_k}^* \bar{\omega}_{n_k} \leq c\exp(d) = C_1 \ ,$$

where all terms of the sum are positive. Hence, at least for some of the domains $\widetilde{B}_k^j$ we have $\int_{\widetilde{B}_k^j} u_{n_k}^* \bar{\omega}_{n_k} \leq C_1/k$.

Now choose a special domain G_k for u_{n_k} which has the smallest value of $\int_{G_k} u_{n_k}^* \bar{\omega}_{n_k}$ among all special domains on levels $j \in [d, d_{n_k}-1] \cup [d_{n_k}+N, \infty)$. Then we have $\int_{G_k} u_{n_k}^* \bar{\omega}_{n_k} \leq C_1/k$. Let $j = j(k)$ be the level of G_k. In all cases we have $G_k \cap M_{-j+1} = \emptyset$ in view of 3.9.4. If $j(k) < d_{n_k}$ or $j(k) \geq d_{n_k} + N$ then $u_{n_k}(G_k)$ does not intersect M_{-j-2} because otherwise we could choose a smaller special domain. By the same reason if $j(k) = d_{n_k}$ then $u_{n_k}(G_k)$ does not intersect $M_{-d_{n_k}-N-1}$ and thus $u_{n_k}(G_k) \subset \Omega_{-(d_{n_k}+N+1), -d_{n_k}}$. □

Now we apply the trick from [H]. Passing, if necessary, to a subsequence, we can think that all domains G_k were chosen either on the level

$(*)$ $j < d_k$, or

$(**)$ $j \geq d_k + N$ or

$(***)$ $j(k) = d_k$.

Let us denote by J'', ω'' and $\bar{\omega}''$ the almost complex structure $J_n|_{\Omega_{-d_n-N-1,-d_n}}$ and the forms $\omega_n|_{\Omega_{-d_n-N-1,-d_n}}$, $\bar{\omega}_n|_{\Omega_{-d_n-N-1,-d_n}}$, respectively, translated by the $\mathbf{R}$-action to the domain $\Omega = \Omega_{-d-N-1,-d}$. Set $\mu = \omega_\infty$, $\bar{\mu} = \bar{\omega}_\infty$ in the case $(*)$, $\mu = \omega_0$, $\bar{\mu} = \bar{\omega}_0$ in the case $(**)$ and $\mu = \bar{\omega}''$, $\bar{\mu} = \bar{\omega}''$ in the case $(***)$. Set also $\widetilde{J} = J$ in the case $(*)$, $\widetilde{J} = J'$ in the case $(**)$ and $\widetilde{J} = J''$ in the case $(***)$. Notice that J'', ω'' and $\bar{\omega}''$ coincide on $\Omega_{-d-1,-d}$ with $J = J_\infty, \omega_\infty$ and $\bar{\omega}_\infty$, respectively. Let us

translate now holomorphic maps $u_{n_k} : G_k \to V$ to the same common level d. Thus we get a sequence of maps $\tilde{u}_{n_k} : G_k \to \Omega$ such that
— each $\tilde{u}_{n_k}$ is holomorphic with respect to the almost complex structure $\widetilde{J}$;
— $\int_{G_k} \tilde{u}^*_{n_k}\bar{\mu} \underset{k\to\infty}{\to} = 0$.
We also have

$$\int_{G_k} \tilde{u}^*_{n_k}\mu = \exp\left(j(k)\right) \int_{G_k} u^*_{n_k}\omega_{n_k}$$

and in combination with Lemma 3.9.5 we get

$$\int_{G_k} \tilde{u}^*_{n_k}\mu < 2C_1 \ .$$

Let us consider all maps $\tilde{u}_{n_k}$ as being parametrized by the same unit disc Δ or a fixed annulus A (with a variable conformal structure). The sequence viewed this way will still be denoted by $\tilde{u}_{n_k}$.

We are now in a position to apply Gromov's compactness theorem (see [Gr]).

LEMMA 3.9.7. *There exists a subsequence of the sequence $\tilde{u}_{n_k}$ which converges uniformly on compact sets to a non-constant $\widetilde{J}$-holomorphic curve $\tilde{u}_\infty$. The set of boundary values of the map $\tilde{u}_\infty$ is contained in $L_1 \cup M_{-d} \cup M_{-d-1}$ and it is smooth at the boundary points which are maped into L_1.*

This lemma is a standard application of Gromov's theory (see [L] for the statement about the set of boundary values) for the case when the sequence $\tilde{u}_{n_k}$ is defined on the disc Δ, and would be for the case when it is defined on the annuli if we knew à priori that conformal moduli of the annuli were bounded. This is actually the case in our situation (see [La] for the proof). However, even without this knowledge Gromov's theory assures the convergence to a holomorphic *cusp*-curve. In our case the cusp degeneration would imply the existence of non-constant J-holomorphic discs with boundary values in $M_{-d-1} \cup M_{-d}$. The next lemma shows, in particular, that this is impossible.

LEMMA 3.9.8. *Let B be either a disc or an annulus and $u_\infty : \mathrm{Int}B \to \Omega$ be a non-constant $\widetilde{J}$-holomorphic curve with (possibly empty) boundary such that its boundary values are contained $L_1 \cap M_{-d} \cup M_{-d-1}$. Suppose that $\int_B u^*_\infty\bar{\mu} = 0$. Then $u_\infty(B)$ is a cylinder over an integral curve $P \subset M$ of the Reeb vector field of the contact form γ_0 in the case (**) and of the contact form γ_∞ in the cases (*) and (* * *). In other words,*

$$u_\infty(B) = P \times (-d-1, -d) \subset \mathrm{Int}\,\Omega_{-d-1,-d} \ .$$

The curve P is either a closed orbit or an arc connecting two points from Λ_1.

Proof: According to Lemma 3.9.3 we have $u_\infty^* \bar{\mu} = h u_\infty^* \mu$, where the function h takes values in $[0,1]$ and vanishes at the points where the vector field X is tangent to $u_\infty(B)$. Therefore the condition $\int_B u_\infty^* \bar{\mu} = 0$ implies that $h \equiv 0$ which means that $u_\infty(B)$ is a cylinder $P \times (-d-1, -d) \subset \mathrm{Int}\Omega_{-d-1,-d}$. The form $\bar{\mu}$ on $\Omega_{-d-1,-d}$ equals $d\gamma_0$ in the case (**) and $d\gamma_\infty$ in the cases (*) and (***). Thus the vector field $\widetilde{J} \cdot \frac{\partial}{\partial \theta}$ is proportional to the Reeb vector field for the contact forms γ_0 or γ_∞, respectively. P is a closed orbit if B is an annulus and P is an arc connecting two points of Λ_1 if B is a disc. □

Although Lemma 3.9.7 by itself does not provide any information about the boundary smoothness, or even continuity of the map u_∞ away from L_1, we can conclude from 3.8.9 that the curve B_∞ is smooth up to the boundary and transversal to M_{-d} and M_{-d-1}. This implies that the (subsequence of the) sequence $\tilde{u}_{n_k}$ converges to u_∞ on the closed domain B. In particular, the curve $P \times (-d)$ is a C^∞-limit of contractible loops in M_{-d} or arcs representing the trivial element of $\pi_1(M_{-d}, \Lambda_1 \times (-d))$. Summarizing we get that $P \subset M$ is a trajectory of the Reeb vector field of one of the forms γ_0 or γ_∞. P is either a closed contractible trajectory or an arc with ends on Λ_1 which represents the trivial class from $\pi_1(M, \Lambda_1)$. In both cases we get a contradiction with the admissibility of the almost complex structures $J_\infty = J$ or $J_0 = J'$.

This concludes the proof of Theorem 3.9.1.

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Y. Eliashberg
Dept. of Math.
Stanford University
Stanford, CA 94305
USA

H. Hofer
Mathematik
ETH-Zentrum
CH-8092 Zürich
Switzerland

D. Salamon
Mathematics Institute
University of Warwick
Coventry CV4 7AL
Great Britain

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