

A moment map interpretation of the Ricci form, Kähler–Einstein structures, and Teichmüller spaces

Oscar García-Prada and Dietmar Salamon

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ABSTRACT. This paper surveys the role of moment maps in Kähler geometry. The first section discusses the Ricci form as a moment map and then moves on to moment map interpretations of the Kähler–Einstein condition and the scalar curvature (Quillen–Fujiki–Donaldson). The second section examines the ramifications of these results for various Teichmüller spaces and their Weil–Petersson symplectic forms and explains how these arise naturally from the construction of symplectic quotients. The third section discusses a symplectic form introduced by Donaldson on the space of Fano complex structures.

1. The Ricci form

This section explains how the Ricci form appears as a moment map for the action of the group of exact volume preserving diffeomorphisms on the space of almost complex structures. A direct consequence of this observation is the Quillen–Fujiki–Donaldson Theorem about the scalar curvature as a moment map for the action of the group of Hamiltonian symplectomorphisms on the space of compatible almost complex structures on a symplectic manifold. This section also discusses how the Kähler–Einstein condition can be interpreted as a moment map equation.

1.1. The Ricci form as a moment map. Let M be a closed oriented $2n$ -manifold equipped with a positive volume form $\rho \in \Omega^{2n}(M)$. Then the space $\mathcal{J}(M)$ of all almost complex structures on M that are compatible with the orientation can be thought of as an infinite-dimensional symplectic manifold. Its tangent space at $J \in \mathcal{J}(M)$ is the space of all complex anti-linear endomorphisms $\widehat{J} : TM \rightarrow TM$ of the tangent bundle (see [32, Section 2]) and thus can be identified with the space $\Omega_J^{0,1}(M, TM)$ of complex anti-linear 1-forms on M with values in the tangent bundle. The symplectic form Ω_ρ is given by

$$(1.1) \quad \Omega_{\rho, J}(\widehat{J}_1, \widehat{J}_2) := \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 J \widehat{J}_2) \rho$$

for $J \in \mathcal{J}(M)$ and $\widehat{J}_1, \widehat{J}_2 \in T_J \mathcal{J}(M) = \Omega_J^{0,1}(M, TM)$.

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The symplectic form is preserved by the action of the group $\text{Diff}(M, \rho)$ of volume preserving diffeomorphisms. Denote the identity component by $\text{Diff}_0(M, \rho)$ and the subgroup of exact volume preserving diffeomorphisms (that are isotopic to the identity via an isotopy that is generated by a smooth family of exact divergence-free vector fields) by $\text{Diff}^{\text{ex}}(M, \rho)$.

Consider the submanifold $\mathcal{J}_0(M) \subset \mathcal{J}(M)$ of all almost complex structures that are compatible with the orientation and have real first Chern class zero. It was shown in [32] that the action of $\text{Diff}^{\text{ex}}(M, \rho)$ on $\mathcal{J}_0(M)$ is Hamiltonian and that twice the Ricci form appears as a moment map. To make this precise, note that the Lie algebra of $\text{Diff}^{\text{ex}}(M, \rho)$ is the space of exact divergence-free vector fields and can be identified with the quotient space $\Omega^{2n-2}(M)/\ker d$ via the correspondence $\Omega^{2n-2}(M) \rightarrow \text{Vect}^{\text{ex}}(M, \rho) : \alpha \mapsto Y_\alpha$, defined by

$$\iota(Y_\alpha)\rho = d\alpha.$$

The dual space of the quotient $\Omega^{2n-2}(M)/\ker d$ can formally be thought of as the space of exact 2-forms, in that every exact 2-form $\tau \in d\Omega^1(M)$ gives rise to a continuous linear functional $\Omega^{2n-2}(M)/\ker d \rightarrow \mathbb{R} : [\alpha] \mapsto \int_M \tau \wedge \alpha$.

The **Ricci form** $\text{Ric}_{\rho, J} \in \Omega^2(M)$ associated to a volume form ρ and an almost complex structure J , both inducing the same orientation of M , is defined by

$$(1.2) \quad \text{Ric}_{\rho, J}(u, v) := \frac{1}{2}\text{trace}(JR^\nabla(u, v)) + \frac{1}{4}\text{trace}((\nabla_u J)J(\nabla_v J)) + \frac{1}{2}d\lambda_J^\nabla(u, v)$$

for $u, v \in \text{Vect}(M)$. Here ∇ is a torsion-free connection on TM that preserves the volume form ρ and the 1-form $\lambda_J^\nabla \in \Omega^1(M)$ is defined by

$$\lambda_J^\nabla(u) := \text{trace}((\nabla J)u)$$

for $u \in \text{Vect}(M)$. The Ricci form is independent of the choice of the torsion-free ρ -connection used to define it and it is closed and represents the cohomology class $2\pi c_1^{\mathbb{R}}(J)$. Its dependence on the volume form is governed by the identity

$$(1.3) \quad \text{Ric}_{e^f \rho, J} = \text{Ric}_{\rho, J} + \frac{1}{2}d(df \circ J).$$

for $J \in \mathcal{J}(M)$ and $f \in \Omega^0(M)$, and the map $(\rho, J) \mapsto \text{Ric}_{\rho, J}$ is equivariant under the action of the diffeomorphism group, i.e.

$$(1.4) \quad \text{Ric}_{\phi^* \rho, \phi^* J} = \phi^* \text{Ric}_{\rho, J}$$

for all $J \in \mathcal{J}(M)$ and all $\phi \in \text{Diff}(M)$.

The definition of the Ricci form in (1.2) arises as a special case of a general moment map identity in [23] for sections of certain $\text{SL}(2n, \mathbb{R})$ fiber bundles. If ρ is the volume form of a Kähler metric and ∇ is the Levi-Civita connection, then $\nabla J = 0$ and hence the last two terms in (1.2) vanish and $\text{Ric}_{\rho, J}$ is the standard Ricci form. In general, the second summand in (1.2) is a correction term which gives rise to a closed 2-form that represents 2π times the first Chern class, and the last summand is a further correction term that makes the Ricci form independent of the choice of the torsion-free ρ -connection ∇ . If J is compatible with a symplectic form ω and ∇ is the Levi-Civita connection of the Riemannian metric $\omega(\cdot, J\cdot)$, then $\lambda_J^\nabla = 0$. In the integrable case the 2-form $\mathbf{i}\text{Ric}_{\rho, J}$ is the curvature of the Chern connection on the canonical bundle associated to the Hermitian structure determined by ρ and hence is a (1, 1)-form.

THEOREM 1.1 ([32]). *The action of the group $\text{Diff}^{\text{ex}}(M, \rho)$ on the space $\mathcal{J}_0(M)$ with the symplectic form (1.1) is a Hamiltonian group action and is generated by the $\text{Diff}(M, \rho)$ -equivariant moment map $\mathcal{J}_0(M) \rightarrow d\Omega^1(M) : J \mapsto 2\text{Ric}_{\rho, J}$, i.e.*

$$(1.5) \quad \int_M 2\widehat{\text{Ric}}_{\rho}(J, \widehat{J}) \wedge \alpha = \Omega_{\rho, J}(\widehat{J}, \mathcal{L}_{Y_{\alpha}} J)$$

for all $J \in \mathcal{J}_0(M)$, all $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ and all $\alpha \in \Omega^{2n-2}(M)$, where

$$\widehat{\text{Ric}}_{\rho}(J, \widehat{J}) := \left. \frac{d}{dt} \right|_{t=0} \text{Ric}_{\rho, J_t}$$

for any smooth path $\mathbb{R} \rightarrow \mathcal{J}(M) : t \mapsto J_t$ satisfying $J_0 = J$ and $\left. \frac{d}{dt} \right|_{t=0} J_t = \widehat{J}$.

Theorem 1.1 is based on ideas in [23]. We emphasize that equation (1.5) does not require the vanishing of the first Chern class. Its proof in [32] relies on the construction of a 1-form Λ_{ρ} on $\mathcal{J}(M)$ with values in the space of 1-forms on M . For $J \in \mathcal{J}(M)$ and $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ the 1-form $\Lambda_{\rho}(J, \widehat{J}) \in \Omega^1(M)$ is defined by

$$(1.6) \quad (\Lambda_{\rho}(J, \widehat{J}))(u) := \text{trace}((\nabla \widehat{J})u + \frac{1}{2} \widehat{J} J \nabla_u J)$$

for $u \in \text{Vect}(M)$, where ∇ is a torsion-free ρ -connection. As before, $\Lambda_{\rho}(J, \widehat{J})$ is independent of the choice of ∇ . Moreover, Λ_{ρ} satisfies the following identities.

PROPOSITION 1.2 ([32]). *Let $J \in \mathcal{J}(M)$, $\widehat{J} \in \Omega_J^{0,1}(M, TM)$, and $v \in \text{Vect}(M)$. Denote the divergence of v by $f_v := d\iota(v)\rho/\rho$. Then*

$$(1.7) \quad d(\Lambda_{\rho}(J, \widehat{J})) = 2\widehat{\text{Ric}}_{\rho}(J, \widehat{J}),$$

$$(1.8) \quad \int_M \Lambda_{\rho}(J, \widehat{J}) \wedge \iota(v)\rho = \frac{1}{2} \int_M \text{trace}(\widehat{J} J \mathcal{L}_v J)\rho,$$

$$(1.9) \quad \Lambda_{\rho}(J, \mathcal{L}_v J) = 2\iota(v)\text{Ric}_{\rho, J} - df_v \circ J + df_{Jv}.$$

For a proof of Proposition 1.2 see [32, Theorems 2.6 & 2.7], and note that equation (1.5) in Theorem 1.1 follows directly from (1.7) and (1.8) with $v = Y_{\alpha}$.

REMARK 1.3. Two useful equations (see [32, Lemma 2.12]) are

$$(1.10) \quad \mathcal{L}_X J = 2J\bar{\partial}_J X, \quad \Lambda_{\rho}(J, \widehat{J}) = \iota(2J\bar{\partial}_J^* \widehat{J}^*)\omega$$

for $J \in \mathcal{J}(M)$, $\widehat{J} \in \Omega_J^{0,1}(M, TM)$, $X \in \text{Vect}(M)$. Here ω is a nondegenerate 2-form on M such that $\omega^n/n! = \rho$ and $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is a Riemannian metric.

For any Hamiltonian group action the zero set of the moment map is invariant under the group action and its orbit space is called the Marsden–Weinstein quotient. In the case at hand this quotient is the space of exact volume preserving isotopy classes of Ricci-flat almost complex structures given by

$$(1.11) \quad \begin{aligned} \mathcal{W}_0(M, \rho) &:= \mathcal{J}_0(M, \rho) / \text{Diff}^{\text{ex}}(M, \rho), \\ \mathcal{J}_0(M, \rho) &:= \{J \in \mathcal{J}_0(M) \mid \text{Ric}_{\rho, J} = 0\}. \end{aligned}$$

In the finite-dimensional setting it follows directly from the definitions that an element of the zero set of the moment map is a regular point for the moment map (i.e. its derivative is surjective) if and only if the isotropy subgroup is discrete. It was shown in [32, Theorem 2.11] that this carries over to the present situation.

PROPOSITION 1.4 ([32]). *Fix an element $J \in \mathcal{J}(M)$.*

(i) *Let $\widehat{\lambda} \in \Omega^1(M)$. Then there exists a $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ such that $\widehat{\text{Ric}}_\rho(J, \widehat{J}) = d\widehat{\lambda}$ if and only if $\int_M d\widehat{\lambda} \wedge \alpha = 0$ for all $\alpha \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_\alpha} J = 0$.*

(ii) *Let $\widehat{J} \in \Omega_J^{0,1}(M, TM)$. Then there exists a $\alpha \in \Omega^{2n-2}(M)$ such that $\mathcal{L}_{Y_\alpha} J = \widehat{J}$ if and only if $\Omega_{\rho, J}(\widehat{J}, \widehat{J}') = 0$ for all $\widehat{J}' \in \Omega_J^{0,1}(M, TM)$ with $\widehat{\text{Ric}}_\rho(J, \widehat{J}') = 0$.*

Call an almost complex structure $J \in \mathcal{J}(M)$ **regular** if there is no nonzero exact divergence-free J -holomorphic vector field. The set of regular almost complex structures is open and the next proposition gives a regularity criterion. It shows that every Kählerable complex structure with real first Chern class zero is regular.

PROPOSITION 1.5 ([32]). *Assume that $J \in \mathcal{J}_0(M)$ satisfies $\text{Ric}_{\rho, J} = 0$ and is compatible with a symplectic form $\omega \in \Omega^2(M)$ such that $\omega^n/n! = \rho$ and the homomorphism $H^1(M; \mathbb{R}) \rightarrow H^{2n-1}(M, \mathbb{R}) : [\lambda] \mapsto [\lambda \wedge \omega^{n-1}/(n-1)!]$ is bijective. Then $\mathcal{L}_{Y_\alpha} J = 0$ implies $Y_\alpha = 0$ for every $\alpha \in \Omega^{2n-2}(M)$.*

PROOF. Assume $\mathcal{L}_{Y_\alpha} J = 0$. Then $\iota(Y_\alpha)\omega$ is harmonic by [32, Lemma 3.9(ii)] and is exact because $\iota(Y_\alpha)\omega \wedge \omega^{n-1}/(n-1)! = d\alpha$. Thus $Y_\alpha = 0$. \square

By part (i) of Proposition 1.4 an almost complex structure $J \in \mathcal{J}_0(M)$ is regular if and only if the linear map $\Omega_J^{0,1}(M, TM) \rightarrow d\Omega^1(M) : \widehat{J} \mapsto \widehat{\text{Ric}}_\rho(J, \widehat{J})$ is surjective. By the implicit function theorem in appropriate Sobolev completions this implies that the regular part of $\mathcal{J}_0(M, \rho)$ is a co-isotropic submanifold of $\mathcal{J}_0(M)$ whose isotropic fibers are the $\text{Diff}^{\text{ex}}(M, \rho)$ -orbits. One would like to deduce that the regular part of $\mathcal{W}_0(M, \rho)$ is a symplectic orbifold with the tangent spaces

$$T_{[J]_\rho} \mathcal{W}_0(M, \rho) = \frac{\left\{ \widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \widehat{\text{Ric}}_\rho(J, \widehat{J}) = 0 \right\}}{\left\{ \mathcal{L}_{Y_\alpha} J \mid \alpha \in \Omega^{2n-2}(M) \right\}}$$

at regular elements $J \in \mathcal{J}_0(M, \rho)$. Indeed, the 2-form (1.1) is nondegenerate on this quotient by part (ii) of Proposition 1.4. However, the action of $\text{Diff}^{\text{ex}}(M, \rho)$ on $\mathcal{J}_0(M, \rho)$ is not always proper and the quotient $\mathcal{W}_0(M, \rho)$ need not be Hausdorff. The archetypal example is the K3-surface [34, 58].

EXAMPLE 1.6. Let (M, J) be a K3-surface that admits an embedded holomorphic sphere C with self-intersection number $C \cdot C = -2$, and let $\tau : M \rightarrow M$ be the Dehn twist about C . Then there exists a smooth family of complex structures $\{J_t\}_{t \in \mathbb{C}}$ and a smooth family of diffeomorphisms $\{\phi_t \in \text{Diff}_0(M)\}_{t \in \mathbb{C} \setminus \{0\}}$ such that $J_0 = J$ and $\phi_t^* J_t = \tau^* J_{-t}$ for $t \neq 0$. Thus the complex structures J_t and $\tau^* J_{-t}$ represent the same equivalence class in $\mathcal{W}_0(M)$, however, their limits $\lim_{t \rightarrow 0} J_t = J_0$ and $\lim_{t \rightarrow 0} \tau^* J_{-t} = \tau^* J_0$ do not represent the same class in $\mathcal{W}_0(M)$ because the homology class $[C]$ belongs to the effective cone of J_0 while the class $-[C]$ belongs to the effective cone of $\tau^* J_0$. This shows that the action of $\text{Diff}_0(M)$ on $\mathcal{J}_0(M)$ is not proper and neither is the action of $\text{Diff}_0(M, \rho) = \text{Diff}^{\text{ex}}(M, \rho)$ on $\mathcal{J}_0(M, \rho)$.

1.2. Symplectic Einstein structures. Let M be a closed oriented $2n$ -manifold and fix a nonzero real number \hbar . A **tame symplectic Einstein structure** on M is a pair (ω, J) consisting of a symplectic form $\omega \in \Omega^2(M)$ and an almost complex structure J tamed by ω (i.e. $\omega(v, Jv) > 0$ for $0 \neq v \in TM$) such that

$$(1.12) \quad \text{Ric}_{\rho, J} = \omega/\hbar, \quad \rho := \omega^n/n!.$$

Every such structure satisfies $2\pi\hbar c_1^{\mathbb{R}}(\omega) = [\omega]$. In the case $\dim(M) = 4$ the integral first Chern class of a symplectic form ω depends only on the cohomology class of ω (see [44, Proposition 13.3.11]), while in higher dimensions this is an open question.

The purpose of this subsection is to exhibit the space of equivalence classes of tame symplectic Einstein structures in a fixed cohomology class $[\omega] = a \in H^2(M; \mathbb{R})$ and with a fixed volume form $\omega^n/n! = \rho$, modulo the action of the group of exact volume preserving diffeomorphisms, as a symplectic quotient.

A pair (a, ρ) consisting of a cohomology class $a \in H^2(M; \mathbb{R})$ and a positive volume form ρ is called a **Lefschetz pair** if it satisfies the following conditions.

(V) $V := \langle a^n/n!, [M] \rangle > 0$ and $\int_M \rho = V$.

(L) The homomorphism $H^1(M; \mathbb{R}) \rightarrow H^{2n-1}(M; \mathbb{R}) : b \mapsto a^{n-1} \cup b$ is bijective.

Fix a Lefschetz pair (a, ρ) . Then the space

$$\mathcal{S}_{a,\rho} := \{ \omega \in \Omega^2(M) \mid d\omega = 0, [\omega] = a, \omega^n/n! = \rho \}$$

of symplectic forms on M in the cohomology class a with volume form ρ is an infinite-dimensional manifold whose tangent space at $\omega \in \mathcal{S}_{a,\rho}$ is given by

$$T_\omega \mathcal{S}_{a,\rho} = \{ \widehat{\omega} \in d\Omega^1(M) \mid \widehat{\omega} \wedge \omega^{n-1} = 0 \}.$$

The proof uses the fact that the map $\mathcal{S}_a \rightarrow \mathcal{V}_a : \omega \mapsto \omega^n/n!$ from the space \mathcal{S}_a of symplectic forms in the class a to the space \mathcal{V}_a of volume forms with total volume V is a submersion. (It is surjective by Moser isotopy whenever $\mathcal{S}_a \neq \emptyset$.) It was shown by Trautwein [57, Lemma 6.4.2] that $\mathcal{S}_{a,\rho}$ carries a symplectic structure

$$(1.13) \quad \Omega_\omega(\widehat{\omega}_1, \widehat{\omega}_2) := \int_M \widehat{\lambda}_1 \wedge \widehat{\lambda}_2 \wedge \frac{\omega^{n-1}}{(n-1)!}$$

for $\omega \in \mathcal{S}_{a,\rho}$ and $\widehat{\omega}_1, \widehat{\omega}_2 \in T_\omega \mathcal{S}_{a,\rho}$, where $\widehat{\lambda}_i \in \Omega^1(M)$ is chosen such that

$$(1.14) \quad d\widehat{\lambda}_i = \widehat{\omega}_i, \quad \widehat{\lambda}_i \wedge \omega^{n-1} \in d\Omega^{2n-2}(M).$$

Here the existence of $\widehat{\lambda}_i$ and the nondegeneracy of (1.13) both require the Lefschetz condition (L). In the rational case a result of Fine [27] shows that $\mathcal{S}_{a,\rho}$ is a symplectic quotient via the action of the group of gauge transformations on the space of connections with symplectic curvature on a suitable line bundle.

Now consider the space

$$\mathcal{P}(M, a, \rho) := \{ (\omega, J) \in \mathcal{S}_{a,\rho} \times \mathcal{J}(M) \mid \omega(v, Jv) > 0 \text{ for all } 0 \neq v \in TM \}$$

of all pairs (ω, J) consisting of a symplectic form ω in the class a with volume form ρ and an ω -tame almost complex structure J . This is an open subset of the product space $\mathcal{S}_{a,\rho} \times \mathcal{J}(M)$, and the symplectic forms (1.13) on $\mathcal{S}_{a,\rho}$ and (1.1) on $\mathcal{J}(M)$ together determine a natural product symplectic structure on $\mathcal{P}(M, a, \rho)$, given by

$$(1.15) \quad \Omega_{\omega,J}((\widehat{\omega}_1, \widehat{J}_1), (\widehat{\omega}_2, \widehat{J}_2)) := \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 J \widehat{J}_2) \frac{\omega^n}{n!} - \frac{2}{\hbar} \int_M \widehat{\lambda}_1 \wedge \widehat{\lambda}_2 \wedge \frac{\omega^{n-1}}{(n-1)!}$$

for $(\omega, J) \in \mathcal{P}(M, a, \rho)$ and $(\widehat{\omega}_i, \widehat{J}_i) \in T_{(\omega,J)} \mathcal{P}(M, a, \rho) = T_\omega \mathcal{S}_{a,\rho} \times \Omega_J^{0,1}(M, TM)$, where the $\widehat{\lambda}_i$ are as in (1.14).

Throughout we will use the notation Y_α for the exact divergence-free vector field associated to a $(2n-2)$ -form $\alpha \in \Omega^{2n-2}(M)$ via $\iota(Y_\alpha)\rho = d\alpha$. When the choice of the symplectic form ω is clear from the context, we will use the notation v_H for the Hamiltonian vector field associated to a function $H \in \Omega^0(M)$ via $\iota(v_H)\omega = dH$.

THEOREM 1.7 (Trautwein). *The symplectic form (1.15) on $\mathcal{P}(M, a, \rho)$ is preserved by the action of $\text{Diff}(M, \rho)$. The action of the subgroup $\text{Diff}^{\text{ex}}(M, \rho)$ is a Hamiltonian group action and is generated by the $\text{Diff}(M, \rho)$ -equivariant moment map $\mathcal{P}(M, a, \rho) \rightarrow d\Omega^1(M) : (\omega, J) \mapsto 2(\widehat{\text{Ric}}_{\rho, J} - \widehat{\omega}/\hbar)$, i.e.*

$$(1.16) \quad \int_M 2(\widehat{\text{Ric}}_{\rho}(J, \widehat{J}) - \widehat{\omega}/\hbar) \wedge \alpha = \Omega_{\omega, J}((\widehat{\omega}, \widehat{J}), (\mathcal{L}_{Y_{\alpha}}\omega, \mathcal{L}_{Y_{\alpha}}J))$$

for all $(\omega, J) \in \mathcal{P}(M, a, \rho)$, all $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$, and all $\alpha \in \Omega^{2n-2}(M)$.

PROOF. Equation (1.16) follows from Theorem 1.1 and the identity

$$\int_M \widehat{\omega} \wedge \alpha = \int_M \widehat{\lambda} \wedge \iota(Y_{\alpha})\omega \wedge \frac{\omega^{n-1}}{(n-1)!} = \Omega_{\omega}(\widehat{\omega}, \mathcal{L}_{Y_{\alpha}}\omega)$$

for all $\omega \in \mathcal{S}_{a, \rho}$, all $\widehat{\omega} = d\widehat{\lambda} \in T_{\omega}\mathcal{S}_{a, \rho}$, and all $\alpha \in \Omega^{2n-2}(M)$. \square

PROPOSITION 1.8. *The action of $\text{Diff}^{\text{ex}}(M, \rho)$ on $\mathcal{P}(M, a, \rho)$ is proper.*

PROOF. The properness proof by Fujiki–Schumacher [29] carries over to the present situation. Choose sequences $(\omega_i, J_i) \in \mathcal{P}(M, a, \rho)$ and $\phi_i \in \text{Diff}^{\text{ex}}(M, \rho)$ such that the limits $(\omega, J) = \lim_{t \rightarrow \infty} (\omega_i, J_i)$ and $(\omega', J') = \lim_{t \rightarrow \infty} (\phi_i^*\omega_i, \phi_i^*J_i)$ exist in the C^{∞} topology and both belong to $\mathcal{P}(M, a, \rho)$. Define the Riemannian metrics g_i by $g_i(u, v) := \frac{1}{2}(\omega_i(u, J_i v) + \omega_i(v, J_i u))$ and similarly for g, g' . Then g_i converges to g and $\phi_i^*g_i$ converges to g' in the C^{∞} topology. Thus by [29, Lemma 3.8] a subsequence of ϕ_i converges to a diffeomorphism $\phi \in \text{Diff}(M, \rho)$ in the C^{∞} topology. Now the flux homomorphism $\text{Flux}_{\rho} : \pi_1(\text{Diff}_0(M, \rho)) \rightarrow H^{2n-1}(M; \mathbb{R})$ has a discrete image by [44, Exercise 10.2.23(v)]. Thus it follows from standard arguments as in [44, Theorem 10.2.5 & Proposition 10.2.16] that $\text{Diff}^{\text{ex}}(M, \rho)$ is a closed subgroup of $\text{Diff}(M, \rho)$ with respect to the C^{∞} topology. Hence $\phi \in \text{Diff}^{\text{ex}}(M, \rho)$. \square

In the present setting the Marsden–Weinstein quotient is the space of exact volume preserving isotopy classes of tame symplectic Einstein structures given by

$$(1.17) \quad \begin{aligned} \mathcal{W}_{\text{SE}}(M, a, \rho) &:= \mathcal{P}_{\text{SE}}(M, a, \rho) / \text{Diff}^{\text{ex}}(M, \rho), \\ \mathcal{P}_{\text{SE}}(M, a, \rho) &:= \{(\omega, J) \in \mathcal{P}(M, a, \rho) \mid \widehat{\text{Ric}}_{\rho, J} = \widehat{\omega}/\hbar\}. \end{aligned}$$

The next result is the analogue of Proposition 1.4 in the symplectic Einstein setting. In particular, part (i) asserts that a pair $(\omega, J) \in \mathcal{P}(M, a, \rho)$ is a regular point for the moment map if and only if the isotropy subgroup is discrete. While the finite-dimensional analogue follows directly from the definitions, in the present situation the proof requires elliptic regularity (in the guise of Proposition 1.4).

PROPOSITION 1.9. *Fix a pair $(\omega, J) \in \mathcal{P}(M, a, \rho)$.*

(i) *Let $\widehat{\lambda} \in \Omega^1(M)$. Then there exists a pair $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$ that satisfies $\widehat{\text{Ric}}_{\rho}(J, \widehat{J}) - \widehat{\omega}/\hbar = d\widehat{\lambda}$ if and only if $\int_M \widehat{\lambda} \wedge \iota(v_H)\rho = 0$ for all $H \in \Omega^0(M)$ with $\mathcal{L}_{v_H}J = 0$.*

(ii) *Let $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$. Then there exists a $(2n-2)$ -form $\alpha \in \Omega^{2n-2}(M)$ that satisfies $\mathcal{L}_{Y_{\alpha}}\omega = \widehat{\omega}$ and $\mathcal{L}_{Y_{\alpha}}J = \widehat{J}$ if and only if $\Omega_{\omega, J}((\widehat{\omega}, \widehat{J}), (\widehat{\omega}', \widehat{J}')) = 0$ for all $(\widehat{\omega}', \widehat{J}') \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$ with $\widehat{\text{Ric}}_{\rho}(J, \widehat{J}') = \widehat{\omega}'/\hbar$.*

The necessity of the conditions in (i) and (ii) follows directly from (1.16). The proof of the converse implications relies on the following three lemmas, which allow us to reduce the result to Proposition 1.4.

LEMMA 1.10. *Let $(\omega, J) \in \mathcal{P}(M, a, \rho)$ and let $\alpha \in \Omega^{2n-2}(M)$. Then Y_α is Hamiltonian if and only if $\int_M \lambda \wedge d\alpha = 0$ for every 1-form λ with $d\lambda \wedge \omega^{n-1} = 0$.*

PROOF. That the condition is necessary follows directly from the definitions. Conversely, assume that $\int_M \lambda \wedge d\alpha = 0$ for all $\lambda \in \Omega^1(M)$ with $d\lambda \wedge \omega^{n-1} = 0$. Let β be a closed $(2n-1)$ -form and choose $\lambda \in \Omega^1(M)$ with $\beta = \lambda \wedge \omega^{n-1}/(n-1)!$. Then

$$\int_M \beta \wedge ((*d\alpha) \circ J) = \int_M \lambda \wedge ((*d\alpha) \circ J) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_M \lambda \wedge d\alpha = 0.$$

Hence $(*d\alpha) \circ J$ is an exact 1-form. Choose $H \in \Omega^0(M)$ such that $(*d\alpha) \circ J = dH$. Then $*d\alpha = -dH \circ J$, hence $d\alpha = *(dH \circ J) = dH \wedge \omega^{n-1}/(n-1)! = \iota(v_H)\rho$ and so $Y_\alpha = v_H$ is a Hamiltonian vector field. \square

LEMMA 1.11. *Let $(\omega, J) \in \mathcal{P}(M, a, \rho)$ and let $\mathcal{Y} \subset \text{Vect}^{\text{ex}}(M, \rho)$ be a finite-dimensional subspace that contains no nonzero Hamiltonian vector field. Then, for every linear functional $\Phi : \mathcal{Y} \rightarrow \mathbb{R}$, there exists a 1-form λ such that $d\lambda \wedge \omega^{n-1} = 0$ and $\int_M \lambda \wedge \iota(Y)\rho = \Phi(Y)$ for all $Y \in \mathcal{Y}$.*

PROOF. Define $\mathcal{L} := \{\lambda \in \Omega^1(M) \mid d\lambda \wedge \omega^{n-1} = 0\}$ and consider the linear map $\mathcal{L} \rightarrow \mathcal{Y}^* : \lambda \mapsto \Phi_\lambda$ defined by $\Phi_\lambda(Y) := \int_M \lambda \wedge \iota(Y)\rho$ for $\lambda \in \mathcal{L}$ and $Y \in \mathcal{Y}$. Then the dual map $\mathcal{Y} \rightarrow \mathcal{L}^*$ is injective by assumption and Lemma 1.10. Since \mathcal{Y} is finite-dimensional, this implies that the map $\mathcal{L} \rightarrow \mathcal{Y}^*$ is surjective. \square

LEMMA 1.12. *Let $(\omega, J) \in \mathcal{P}(M, a, \rho)$ and let $\alpha \in \Omega^{2n-2}(M)$. Then the following assertions are equivalent.*

- (a) *There exists a function $H \in \Omega^0(M)$ such that $\mathcal{L}_{v_H}J = \mathcal{L}_{Y_\alpha}J$.*
- (b) *If $\lambda \in \Omega^1(M)$ satisfies $d\lambda \wedge \omega^{n-1} = 0$ and $\int_M d\lambda \wedge \beta = 0$ for all $\beta \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_\beta}J = 0$, then $\int_M d\lambda \wedge \alpha = 0$.*

PROOF. Assume (a) and define $\beta := \alpha - H\omega^{n-1}/(n-1)!$. Then $Y_\beta = Y_\alpha - v_H$, hence $\mathcal{L}_{Y_\beta}J = 0$ by (a), and so each λ as in (b) satisfies

$$\int_M d\lambda \wedge \alpha = \int_M d\lambda \wedge \left(\beta + H \frac{\omega^{n-1}}{(n-1)!} \right) = 0.$$

Conversely, assume (a) does not hold and choose a subspace $\mathcal{Y}_0 \subset \text{Vect}^{\text{ex}}(M, \rho)$ such that $\{Y_\beta \mid \mathcal{L}_{Y_\beta}J = 0\} = \mathcal{Y}_0 \oplus \{v_H \mid \mathcal{L}_{v_H}J = 0\}$. Then $\mathcal{Y} := \mathcal{Y}_0 \oplus \mathbb{R}Y_\alpha$ does not contain nonzero Hamiltonian vector fields and so, by Lemma 1.11 there is a $\lambda \in \Omega^1(M)$ such that $d\lambda \wedge \omega^{n-1} = 0$, $\int_M \lambda \wedge \iota(Y_\alpha)\rho = 1$, and $\int_M \lambda \wedge \iota(Y)\rho = 0$ for all $Y \in \mathcal{Y}_0$. This implies $\int_M d\lambda \wedge \beta = 0$ for all $\beta \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_\beta}J = 0$ and $\int_M d\lambda \wedge \alpha = 1$. Hence (b) does not hold. \square

PROOF OF PROPOSITION 1.9. We prove part (i). Assume that $\widehat{\lambda} \in \Omega^1(M)$ satisfies $\int_M \widehat{\lambda} \wedge \iota(v_H)\rho = 0$ for all $H \in \Omega^0(M)$ with $\mathcal{L}_{v_H}J = 0$. Then by Lemma 1.11 there exists a $\lambda \in \Omega^1(M)$ such that $d\lambda \wedge \omega^{n-1} = 0$ and $\int_M (\lambda + \widehat{\lambda}) \wedge \iota(Y_\alpha)\rho = 0$ for all $\alpha \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_\alpha}J = 0$. By part (i) of Proposition 1.4 there exists a $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ such that $\widehat{\text{Ric}}_\rho(J, \widehat{J}) = d(\lambda + \widehat{\lambda})$. This proves (i) with $\widehat{\omega} := h d\lambda$.

To prove (ii), let $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$ such that $\Omega_{\omega, J}((\widehat{\omega}, \widehat{J}), (\widehat{\omega}', \widehat{J}')) = 0$ for all $(\widehat{\omega}', \widehat{J}') \in T_{(\omega, J)}\mathcal{P}(M, a, \rho)$ with $\widehat{\text{Ric}}_\rho(J, \widehat{J}') = \widehat{\omega}'/h$. Then by part (ii) of Proposition 1.4 there exist α, β with $\widehat{\omega} = d\iota(Y_\alpha)\omega$ and $\widehat{J} = \mathcal{L}_{Y_\beta}J$. Let $\lambda \in \Omega^1(M)$

such that $d\lambda \wedge \omega^{n-1} = 0$ and $\int_M d\lambda \wedge \alpha' = 0$ for all $\alpha' \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_{\alpha'}} J = 0$. By part (i) of Proposition 1.4 choose \widehat{J}' such that $\widehat{\text{Ric}}_{\rho}(J, \widehat{J}') = d\lambda$. Then

$$\begin{aligned} 2 \int_M d\lambda \wedge (\beta - \alpha) &= \frac{1}{2} \int_M \text{trace}(\widehat{J}' J \mathcal{L}_{Y_{\beta}} J) \frac{\omega^n}{n!} - \frac{2}{\hbar} \int_M \hbar \lambda \wedge (\iota(Y_{\alpha})\omega) \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &= \Omega_{\omega, J}((\widehat{\omega}', \widehat{J}'), (\widehat{\omega}, \widehat{J})) = 0, \end{aligned}$$

where $\widehat{\omega}' := \hbar d\lambda = \hbar \widehat{\text{Ric}}_{\rho}(J, \widehat{J}')$. Thus by Lemma 1.12 there exists a function H such that $\mathcal{L}_{Y_{\beta-\alpha}} J = \mathcal{L}_{v_H} J$, so $\mathcal{L}_{Y_{\alpha+v_H}} J = \widehat{J}$ and $d\iota(Y_{\alpha} + v_H)\omega = \widehat{\omega}$. This proves (ii). \square

Call a pair $(\omega, J) \in \mathcal{P}(M, a, \rho)$ **regular** if there are no nonzero J -holomorphic Hamiltonian vector fields. By part (i) of Proposition 1.9 a pair (ω, J) is regular if and only if the map $T_{(\omega, J)} \mathcal{P}(M, a, \rho) \rightarrow d\Omega^1(M) : (\widehat{\omega}, \widehat{J}) \mapsto \widehat{\text{Ric}}_{\rho}(J, \widehat{J}) - \widehat{\omega}/\hbar$ is surjective. Thus by Proposition 1.8 (and a suitable local slice theorem that requires a Nash–Moser type proof) the regular part of $\mathcal{W}_{\text{SE}}(M, a, \rho)$ is a symplectic orbifold whose tangent space at the equivalence class of a regular element $(\omega, J) \in \mathcal{P}_{\text{SE}}(M, a, \rho)$ is the quotient

$$T_{[\omega, J]_{\rho}} \mathcal{W}_{\text{SE}}(M, a, \rho) = \frac{\{(\widehat{\omega}, \widehat{J}) \mid \widehat{\omega} \wedge \omega^{n-1} = 0, \widehat{\text{Ric}}_{\omega}(J, \widehat{J}) = \widehat{\omega}/\hbar\}}{\{(\mathcal{L}_{Y_{\alpha}}\omega, \mathcal{L}_{Y_{\alpha}}J) \mid \alpha \in \Omega^{2n-2}(M)\}}.$$

The 2-form (1.15) is nondegenerate on this quotient by part (ii) of Proposition 1.9.

REMARK 1.13 (Compatible pairs). The space $\mathcal{C}(M, a, \rho) \subset \mathcal{P}(M, a, \rho)$ of compatible pairs is a submanifold of $\mathcal{P}(M, a, \rho)$ and $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)} \mathcal{P}(M, a, \rho)$ is tangent to $\mathcal{C}(M, a, \rho)$ at a compatible pair (ω, J) if and only if

$$(1.18) \quad \widehat{\omega}(u, v) - \widehat{\omega}(Ju, Jv) = \omega(\widehat{J}u, Jv) + \omega(Ju, \widehat{J}v).$$

It is an open question whether the restriction of the 2-form (1.15) to $\mathcal{C}(M, a, \rho)$ is nondegenerate. A pair $(\widehat{\omega}, \widehat{J}) \in T_{(\omega, J)} \mathcal{C}(M, a, \rho)$ belongs to its kernel if and only if

$$(1.19) \quad \widehat{J} + \widehat{J}^* = 0, \quad \widehat{\text{Ric}}_{\rho}(J, \widehat{J}) = \widehat{\omega}/\hbar.$$

If this holds, then there exists a vector field $X \in \text{Vect}(M)$ such that $\widehat{\omega} = d\iota(X)\omega$ and $\widehat{J} = \frac{1}{2}(\mathcal{L}_X J - (\mathcal{L}_X J)^*)$ (respectively $\widehat{J} = \mathcal{L}_X J$ in the Kähler–Einstein case). Thus the solutions of (1.19) form the kernel of a Fredholm operator, so nondegeneracy is an open condition. It follows also that the restriction of the 2-form (1.15) to $\mathcal{C}(M, a, \rho)$ is nondegenerate at a Kähler–Einstein pair (ω, J) if and only if

$$(1.20) \quad \mathcal{L}_X J + (\mathcal{L}_X J)^* = 0 \quad \implies \quad \mathcal{L}_X J = 0.$$

Now fix a Kähler manifold (M, ω, J) . Then Proposition 1.2 yields the equation

$$\begin{aligned} \frac{1}{2} \langle (\mathcal{L}_X J)^*, \mathcal{L}_X J \rangle &= -\frac{1}{2} \int_M \text{trace}((\mathcal{L}_X J)J(\mathcal{L}_{JX}J))\rho = - \int_M \Lambda_{\rho}(J, \mathcal{L}_X J) \wedge \iota(JX)\rho \\ &= \int_M \left(df_X \circ J - df_{JX} - 2\iota(X)\text{Ric}_{\rho, J} \right) \wedge \iota(JX)\rho \end{aligned}$$

for every vector field X , and this implies the Weitzenböck formula

$$(1.21) \quad \frac{1}{4} \|\mathcal{L}_X J + (\mathcal{L}_X J)^*\|^2 = \frac{1}{2} \|\mathcal{L}_X J\|^2 + \int_M (f_X^2 + f_{JX}^2 - 2\text{Ric}_{\rho, J}(X, JX))\rho.$$

In the Kähler–Einstein case $\text{Ric}_{\rho, J} = \omega/\hbar$ with $\hbar < 0$ this identity implies (1.20). In the Fano case $\hbar > 0$ it is an open question whether (1.20) holds for all Kähler–Einstein pairs.

1.3. Scalar curvature. Let (M, ω) be a closed symplectic $2n$ -manifold with the volume form $\rho := \omega^n/n!$. For $F, G \in \Omega^0(M)$ we denote by v_F the Hamiltonian vector field of F and by $\{F, G\} := \omega(v_F, v_G)$ the Poisson bracket. Let $\mathcal{J}(M, \omega)$ be the space of all almost complex structures that are compatible with ω , i.e. the bilinear form $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is a Riemannian metric. This is an infinite-dimensional manifold whose tangent space at $J \in \mathcal{J}(M, \omega)$ is given by

$$T_J \mathcal{J}(M, \omega) = \left\{ \widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \omega(J\cdot, \widehat{J}\cdot) + \omega(\widehat{J}\cdot, J\cdot) = 0 \right\}.$$

Here the condition $\omega(J\cdot, \widehat{J}\cdot) + \omega(\widehat{J}\cdot, J\cdot) = 0$ holds if and only if \widehat{J} is symmetric with respect to the Riemannian metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$. In particular, $\mathcal{L}_v J$ is symmetric for every symplectic vector field v . The symplectic form on $\mathcal{J}(M, \omega)$ is given by

$$(1.22) \quad \Omega_{\omega, J}(\widehat{J}_1, \widehat{J}_2) := \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 J \widehat{J}_2) \frac{\omega^n}{n!}$$

for $J \in \mathcal{J}(M, \omega)$ and $\widehat{J}_i \in T_J \mathcal{J}(M, \omega)$ and the complex structure is $\widehat{J} \mapsto -J\widehat{J}$. With these structures $\mathcal{J}(M, \omega)$ is an infinite-dimensional Kähler manifold.

The group $\text{Symp}(M, \omega)$ of symplectomorphisms acts on $\mathcal{J}(M, \omega)$ by Kähler isometries. By a theorem of Donaldson [19], the action of the subgroup $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms on $\mathcal{J}(M, \omega)$ is a Hamiltonian group action and the scalar curvature appears as a moment map. Earlier versions of this result were proved by Quillen (for Riemann surfaces) and Fujiki [28] (in the integrable case). Below we derive it as a direct consequence of Theorem 1.1.

To explain this, we first observe that the Lie algebra of the group $\text{Ham}(M, \omega)$ is the space of Hamiltonian vector fields and hence can be identified with the quotient space $\Omega^0(M)/\mathbb{R}$. Its dual space can formally be thought of as the space $\Omega_\rho^0(M)$ of all functions $f \in \Omega^0(M)$ with mean value zero, in that every such function determines a continuous linear functional $\Omega^0(M)/\mathbb{R} \rightarrow \mathbb{R} : [H] \mapsto \int_M f H \rho$. Now the scalar curvature of an almost complex structure $J \in \mathcal{J}(M, \omega)$ is defined by

$$(1.23) \quad S_{\omega, J} \rho := 2\text{Ric}_{\rho, J} \wedge \omega^{n-1} / (n-1)!.$$

Its mean value is the topological invariant $c_\omega := \frac{4\pi}{V} \langle c_1(\omega) \cup \frac{[\omega]^{n-1}}{(n-1)!}, [M] \rangle$, $V := \int_M \rho$. In the integrable case $S_{\omega, J}$ is the standard scalar curvature of the Kähler metric.

THEOREM 1.14 (Quillen–Fujiki–Donaldson). *The action of $\text{Ham}(M, \omega)$ on the space $\mathcal{J}(M, \omega)$ is Hamiltonian and is generated by the $\text{Symp}(M, \omega)$ -equivariant moment map $\mathcal{J}(M, \omega) \rightarrow \Omega_\rho^0(M) : J \mapsto S_{\omega, J} - c_\omega$, i.e.*

$$(1.24) \quad \int_M \widehat{S}_\omega(J, \widehat{J}) H \frac{\omega^n}{n!} = \Omega_{\omega, J}(\widehat{J}, \mathcal{L}_{v_H} J)$$

for all $J \in \mathcal{J}(M, \omega)$, all $\widehat{J} \in T_J \mathcal{J}(M, \omega)$, and all $H \in \Omega^0(M)$, where

$$\widehat{S}_\omega(J, \widehat{J}) := \left. \frac{d}{dt} \right|_{t=0} S_{\omega, J_t}$$

for any smooth path $\mathbb{R} \rightarrow \mathcal{J}(M, \omega) : t \mapsto J_t$ satisfying $J_0 = J$ and $\left. \frac{d}{dt} \right|_{t=0} J_t = \widehat{J}$.

PROOF. By (1.23) we have

$$\int_M \widehat{S}_\omega(J, \widehat{J}) H \frac{\omega^n}{n!} = \int_M 2\widehat{\text{Ric}}_\rho(J, \widehat{J}) \wedge H \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \int_M \text{trace}(\widehat{J} J \mathcal{L}_{v_H} J) \frac{\omega^n}{n!}.$$

The last equation follows from Theorem 1.1 with $Y_\alpha = v_H$ and $\alpha = H \frac{\omega^{n-1}}{(n-1)!}$. \square

In the present situation one proves exactly as in Proposition 1.8 that the action of the group $\text{Ham}(M, \omega)$ on $\mathcal{J}(M, \omega)$ is proper. Here the argument uses a theorem of Ono [47] which asserts that $\text{Ham}(M, \omega)$ is a closed subgroup of $\text{Symp}(M, \omega)$ with respect to the C^∞ topology. Now the Marsden–Weinstein quotient is the space of Hamiltonian isotopy classes of ω -compatible almost complex structures with constant scalar curvature given by

$$(1.25) \quad \begin{aligned} \mathcal{W}_{\text{csc}}(M, \omega) &:= \mathcal{J}_{\text{csc}}(M, \omega) / \text{Ham}(M, \omega), \\ \mathcal{J}_{\text{csc}}(M, \omega) &:= \{J \in \mathcal{J}(M, \omega) \mid S_{\omega, J} = c_\omega\}. \end{aligned}$$

The next result is the analogue of Proposition 1.4 in the present setting.

PROPOSITION 1.15. *Fix an element $J \in \mathcal{J}(M, \omega)$.*

(i) *Let $f \in \Omega^0(M)$. Then there exists a $\widehat{J} \in T_J \mathcal{J}(M, \omega)$ such that $\widehat{S}_\omega(J, \widehat{J}) = f$ if and only if $\int_M f H \rho = 0$ for all $H \in \Omega^0(M)$ with $\mathcal{L}_{v_H} J = 0$.*

(ii) *Let $\widehat{J} \in T_J \mathcal{J}(M, \omega)$. Then there exists an $H \in \Omega^0(M)$ such that $\mathcal{L}_{v_H} J = \widehat{J}$ if and only if $\Omega_{\omega, J}(\widehat{J}, \widehat{J}') = 0$ for all $\widehat{J}' \in T_J \mathcal{J}(M, \omega)$ with $\widehat{S}_\omega(J, \widehat{J}') = 0$.*

PROOF. That the conditions in (i) and (ii) are necessary follows from (1.24). To prove the converse, define the operator $\mathbf{L} : \Omega^0(M) \rightarrow \Omega^0(M)$ by

$$(1.26) \quad \begin{aligned} \mathbf{L}F &:= \widehat{S}_\omega(J, J\mathcal{L}_{v_F} J) = -d^*(\Lambda_\rho(J, J\mathcal{L}_{v_F} J) \circ J) \\ &= d^* dd^* dF - 2d^*(\iota(Jv_F)\text{Ric}_{\rho, J} \circ J) + d^*(\Lambda_\rho(J, N_J(v_F, \cdot)) \circ J) \end{aligned}$$

for $F \in \Omega^0(M)$. Here the last equality follows by a calculation which uses the identities $f_{Jv_F} = -d^* dF$ and $N_J(u, v) = J(\mathcal{L}_v J)u - (\mathcal{L}_{Jv} J)u$ for the Nijenhuis tensor. The operator \mathbf{L} is a fourth order self-adjoint Fredholm operator and, by (1.24),

$$(1.27) \quad \int_M (\mathbf{L}F)G \rho = \frac{1}{2} \int_M \text{trace}((\mathcal{L}_{v_F} J)(\mathcal{L}_{v_G} J)) \rho$$

for all $F, G \in \Omega^0(M)$. Thus $H \in \ker \mathbf{L}$ if and only if $\mathcal{L}_{v_H} J = 0$. Hence, if $f \in \Omega^0(M)$ satisfies $\int_M f H \rho = 0$ for all H with $\mathcal{L}_{v_H} J = 0$, then $f \in \text{im } \mathbf{L}$, and this proves (i).

To prove part (ii), assume that $\widehat{J} \in T_J \mathcal{J}(M, \omega)$ satisfies $\Omega_{\omega, J}(\widehat{J}, \widehat{J}') = 0$ for all $\widehat{J}' \in T_J \mathcal{J}(M, \omega)$ with $\widehat{S}_\omega(J, \widehat{J}') = 0$. Then, by part (ii) of Proposition 1.4, there exists an α such that $\mathcal{L}_{Y_\alpha} J = \widehat{J}$. Now let $\lambda \in \Omega^1(M)$ such that $d\lambda \wedge \omega^{n-1} = 0$ and $\int_M d\lambda \wedge \beta = 0$ for all $\beta \in \Omega^{2n-2}(M)$ with $\mathcal{L}_{Y_\beta} J = 0$. Choose $\widehat{J}' \in \Omega_J^{0,1}(M, TM)$ with $\text{Ric}_\rho(J, \widehat{J}') = d\lambda$ by part (i) of Proposition 1.4. Then $\widehat{S}_\omega(J, \widehat{J}') = 0$ and hence

$$2 \int_M d\lambda \wedge \alpha = \int_M 2\widehat{\text{Ric}}_\rho(J, \widehat{J}') \wedge \alpha = \Omega_{\rho, J}(\widehat{J}', \mathcal{L}_{Y_\alpha} J) = \Omega_{\omega, J}(\widehat{J}', \widehat{J}) = 0.$$

Thus by Lemma 1.12 there exists an $H \in \Omega^0(M)$ such that $\mathcal{L}_{v_H} J = \mathcal{L}_{Y_\alpha} J = \widehat{J}$. \square

PROPOSITION 1.16. *Let $J \in \mathcal{J}(M, \omega)$ and let $\widehat{J} \in T_J \mathcal{J}(M, \omega)$. Then there exists a function $H \in \Omega^0(M)$ such that $\widehat{S}_\omega(J, \widehat{J} - J\mathcal{L}_{v_H} J) = 0$. Moreover, $\mathcal{L}_{v_H} J$ is uniquely determined by this condition.*

PROOF. Define $f \in \Omega^0(M)$ by $f\rho := d\Lambda_\rho(J, \widehat{J}) \wedge \omega^{n-1}/(n-1)!$ and let \mathbf{L} be as in Proposition 1.15. Then $(\mathbf{L}H)\rho = d\Lambda_\rho(J, J\mathcal{L}_{v_H} J) \wedge \omega^{n-1}/(n-1)!$ and, by (1.24),

$$\int_M f H \rho = \int_M d\Lambda_\rho(J, \widehat{J}) \wedge H \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \int_M \text{trace}(\widehat{J} J \mathcal{L}_{v_H} J) \rho = 0$$

for all $H \in \ker \mathbf{L}$. Thus f belongs to the image of \mathbf{L} . \square

Call an almost complex structure $J \in \mathcal{J}(M, \omega)$ **regular** if there are no nonzero J -holomorphic Hamiltonian vector fields. By part (i) of Proposition 1.15 J is regular if and only if the map $T_J \mathcal{J}(M, \omega) \rightarrow \Omega_\rho^0(M) : \widehat{J} \mapsto \widehat{S}_\omega(J, \widehat{J})$ is surjective. Hence, since the action is proper, it follows again from a suitable local slice theorem that the regular part of the quotient $\mathcal{W}_{\text{csc}}(M, \omega)$ is a Kähler orbifold. It is infinite-dimensional when $\dim(M) > 2$, and its tangent space at the equivalence class of a regular element $J \in \mathcal{J}_{\text{csc}}(M, \omega)$ is the quotient

$$(1.28) \quad T_{[J]_\omega} \mathcal{W}_{\text{csc}}(M, \omega) = \frac{\left\{ \widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \widehat{J} = \widehat{J}^*, \widehat{S}_\omega(J, \widehat{J}) = 0 \right\}}{\{ \mathcal{L}_{v_H} J \mid H \in \Omega^0(M) \}}.$$

The 2-form (1.22) is nondegenerate on this quotient by part (ii) of Proposition 1.15 and the complex structure is given by $[\widehat{J}]_\omega \mapsto [-J(\widehat{J} - \mathcal{L}_{v_H} J)]_\omega$, where H is chosen as in Proposition 1.16 so that $\widehat{S}_\omega(J, J\widehat{J} - J\mathcal{L}_{v_H} J) = 0$.

COROLLARY 1.17. *Let $J \in \mathcal{J}(M, \omega)$ and $F, G \in \Omega^0(M)$. Then*

$$(1.29) \quad \Omega_{\rho, J}(\mathcal{L}_{v_F} J, \mathcal{L}_{v_G} J) = \int_M S_{\omega, J} \{F, G\} \frac{\omega^n}{n!}.$$

In particular, if $S_{\omega, J} = c_\omega$, then the L^2 -norm of the endomorphism $\mathcal{L}_{v_F} J + J\mathcal{L}_{v_G} J$ is given by $\|\mathcal{L}_{v_F} J + J\mathcal{L}_{v_G} J\|^2 = \|\mathcal{L}_{v_F} J\|^2 + \|\mathcal{L}_{v_G} J\|^2$.

PROOF. Let ϕ_t be the flow of v_F . Then $S_{\omega, \phi_t^* J} = S_{\omega, J} \circ \phi_t$ for all t and hence differentiation with respect to t yields the identity

$$(1.30) \quad \widehat{S}_\omega(J, \mathcal{L}_{v_F} J) = \{S_{\omega, J}, F\}.$$

Insert equation (1.30) into (1.24) with $\widehat{J} = \mathcal{L}_{v_F} J$ and $H = G$ to obtain (1.29). \square

Let (M, ω, J) be a closed Kähler manifold with constant scalar curvature such that $H^1(M; \mathbb{R}) = 0$. Then every holomorphic vector field is the sum of a Hamiltonian and a gradient vector field by [32, Lemma 3.7(ii)], and for all $F, G \in \Omega^0(M)$ we have $\|\mathcal{L}_{v_F + Jv_G} J\|^2 = \|\mathcal{L}_{v_F} J\|^2 + \|\mathcal{L}_{v_G} J\|^2$ by Corollary 1.17. Hence the Lie algebra of holomorphic vector fields is the complexification of the Lie algebra of Killing fields and is therefore reductive. This is the content of **Matsushima's Theorem**.

REMARK 1.18. Non-integrable almost complex structures in $\mathcal{J}(M, \omega)$ with vanishing scalar curvature need not be Ricci-flat. To see this, assume $\dim(M) \geq 4$ and $H^1(M; \mathbb{R}) = 0$, and that there exists a $J \in \mathcal{J}_{\text{int}}(M, \omega)$ such that $\text{Ric}_{\rho, J} = 0$ with $\rho := \omega^n/n!$. Then there exists a $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ such that

$$(1.31) \quad \widehat{J} = \widehat{J}^*, \quad \widehat{S}_\omega(J, \widehat{J}) = 0, \quad \widehat{\text{Ric}}_\rho(J, \widehat{J}) \neq 0.$$

Namely, choose a nonzero 1-form $\widehat{\lambda}$ such that $d^* \widehat{\lambda} = 0$ and $d^*(\widehat{\lambda} \circ J) = 0$. Then $\widehat{\lambda}$ is not closed and there is no nonzero J -holomorphic vector field by [32, Lemma 3.9]. Thus, by (1.21) with $\text{Ric}_{\rho, J} = 0$ the operator $X \mapsto \mathcal{L}_X J + (\mathcal{L}_X J)^*$ is injective and hence, by the closed image theorem the dual operator $\widehat{J} = \widehat{J}^* \mapsto \widehat{\partial}_J^* \widehat{J}$ is surjective. Thus, by Remark 1.3 there exists a $\widehat{J} = \widehat{J}^* \in \Omega_J^{0,1}(M, TM)$ such that $\Lambda_\rho(J, \widehat{J}) = \widehat{\lambda}$. This implies $\widehat{S}_\omega(J, \widehat{J}) = -d^*(\widehat{\lambda} \circ J)$ and $2\widehat{\text{Ric}}_\rho(J, \widehat{J}) = d\widehat{\lambda}$, so \widehat{J} satisfies (1.31).

By (1.31) there exists a smooth curve $\mathbb{R} \rightarrow \mathcal{J}(M, \omega) : t \mapsto J_t$ such that $J_0 = J$ and $\partial_t|_{t=0} J_t = \widehat{J}$ and $S_{\omega, J_t} = 0$ for all t . Since $\widehat{\text{Ric}}_\rho(J, \widehat{J}) \neq 0$, this curve also satisfies $\text{Ric}_{\rho, J_t} \neq 0$ for small nonzero t . Note that $\widehat{\text{Ric}}_\rho(J, \widehat{J})$ is not a $(1, 1)$ -form, hence $\widehat{\partial}_J \widehat{J} \neq 0$ by [32, Lemma 3.6], and so J_t is not integrable for small nonzero t .

2. Teichmüller spaces

In this section we consider integrable complex structures and examine the Teichmüller spaces of Calabi–Yau structures, of Kähler–Einstein structures, and of constant scalar curvature Kähler metrics. Such Teichmüller spaces have been studied by many authors, see e.g. [7, 12, 28, 29, 30, 34, 38, 39, 40, 45, 46, 49, 50, 53, 55, 56, 60] and the references therein. The regular part of each Teichmüller space is a finite-dimensional symplectic submanifold of the relevant symplectic quotient in Section 1. It thus acquires a natural symplectic structure that descends to the Weil–Petersson form on the corresponding moduli space (the quotient of Teichmüller space by the mapping class group).

In our formulation the ambient manifold M is fixed and the Weil–Petersson form arises via symplectic reduction of a Hamiltonian group action by an infinite-dimensional group on an infinite-dimensional space with a finite-dimensional quotient. In the original algebro-geometric approach the moduli space is directly characterized in finite-dimensional terms via a Torelli type theorem and the Weil–Petersson form arises from the natural homogeneous symplectic form on the relevant period domain.

It seems to be an open question whether there exist closed Kähler manifolds that admit holomorphic diffeomorphisms that are smoothly isotopic to the identity, but not through holomorphic diffeomorphisms. (For nonKähler examples see [45].) If they do exist, then the regular parts of the Teichmüller spaces examined here are orbifolds rather than manifolds. When discussing Teichmüller spaces as manifolds, we tacitly assume that such automorphisms do not exist, as is the case for Riemann surfaces.

2.1. The Teichmüller space of Calabi–Yau structures. Let M be a closed connected oriented $2n$ -manifold. Then the **Teichmüller space of Calabi–Yau structures** on M is the space of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone. It is denoted by

$$(2.1) \quad \begin{aligned} \mathcal{T}_0(M) &:= \mathcal{I}_{\text{int},0}(M)/\text{Diff}_0(M), \\ \mathcal{I}_{\text{int},0}(M) &:= \{J \in \mathcal{I}_{\text{int}}(M) \mid c_1^{\mathbb{R}}(J) = 0 \text{ and } J \text{ admits a Kähler form}\}. \end{aligned}$$

Associated to a complex structure $J \in \mathcal{I}_{\text{int},0}(M)$ there is the Dolbeault complex

$$(2.2) \quad \Omega^0(M, TM) \xrightarrow{\bar{\partial}_J} \Omega_J^{0,1}(M, TM) \xrightarrow{\bar{\partial}_J} \Omega_J^{0,2}(M, TM),$$

where the first operator corresponds to the infinitesimal action of the vector fields on $T_J \mathcal{I}(M) = \Omega_J^{0,1}(M, TM)$ by Remark 1.3, and the second operator corresponds to the derivative of the map which assigns to an almost complex structure J its Nijenhuis tensor N_J by [32, (3.2)]. Thus the tangent space of the Teichmüller space $\mathcal{T}_0(M)$ at the equivalence class of an element $J \in \mathcal{I}_{\text{int},0}(M)$ can formally be identified with the cohomology of the Dolbeault complex (2.2), i.e.

$$(2.3) \quad T_{[J]} \mathcal{T}_0(M) = \frac{\ker(\bar{\partial}_J : \Omega_J^{0,1}(M, TM) \rightarrow \Omega_J^{0,2}(M, TM))}{\text{im}(\bar{\partial}_J : \Omega^0(M, TM) \rightarrow \Omega_J^{0,1}(M, TM))}.$$

The proof requires a local slice theorem for the action of the diffeomorphism group on the space of integrable complex structures.

For every $J \in \mathcal{J}_{\text{int},0}(M)$ the space of holomorphic vector fields is isomorphic to the space of harmonic 1-forms by [32, Lemma 3.9]. Moreover, the Bogomolov–Tian–Todorov theorem asserts that the obstruction class vanishes [6, 55, 56], so the cohomology of the Dolbeault complex (2.2) has constant dimension, and that $\mathcal{T}_0(M)$ is indeed a smooth manifold whose tangent space at the equivalence class of $J \in \mathcal{J}_{\text{int},0}(M)$ is the cohomology group (2.3). The Teichmüller space is in general not Hausdorff, even for the K3 surface (see [34, 58] and also Example 1.6). For hyperKähler manifolds the Teichmüller space becomes Hausdorff after identifying inseparable complex structures (see Verbitsky [58, 59]) which are biholomorphic by a theorem of Huybrechts [37].

Now fix a positive volume form $\rho \in \Omega^{2n}(M)$. Then another description of the Teichmüller space of Calabi–Yau structures is as the quotient

$$(2.4) \quad \begin{aligned} \mathcal{T}_0(M, \rho) &:= \mathcal{J}_{\text{int},0}(M, \rho) / \text{Diff}_0(M, \rho) = \mathcal{J}_{\text{int},0}(M, \rho) / \text{Diff}^{\text{ex}}(M, \rho), \\ \mathcal{J}_{\text{int},0}(M, \rho) &:= \{J \in \mathcal{J}_{\text{int},0}(M) \mid \text{Ric}_{\rho,J} = 0\}. \end{aligned}$$

Here the two quotients agree because the quotient group $\text{Diff}_0(M, \rho) / \text{Diff}^{\text{ex}}(M, \rho)$ acts trivially on $\mathcal{J}_{\text{int},0}(M, \rho) / \text{Diff}^{\text{ex}}(M, \rho)$ (see [32, Lemma 3.9]). The tangent space of $\mathcal{T}_0(M, \rho)$ at the equivalence class of $J \in \mathcal{J}_{\text{int},0}(M, \rho)$ is the quotient

$$(2.5) \quad T_{[J]_\rho} \mathcal{T}_0(M, \rho) = \frac{\{\hat{J} \in \Omega_J^{0,1}(M, TM) \mid \bar{\partial}_J \hat{J} = 0, \widehat{\text{Ric}}_\rho(J, \hat{J}) = 0\}}{\{\mathcal{L}_{Y_\alpha} J \mid \alpha \in \Omega^{2n-2}(M)\}}.$$

The inclusion $\iota_\rho : \mathcal{T}_0(M, \rho) \rightarrow \mathcal{T}_0(M)$ is a diffeomorphism by [32, Lemma 4.2]. The proof uses Moser isotopy and the formula (1.3). The Teichmüller space $\mathcal{T}_0(M, \rho)$ is a submanifold of the infinite-dimensional symplectic quotient $\mathcal{W}_0(M, \rho)$ in (1.11) and it turns out that the 2-form (1.1) descends to a symplectic form on $\mathcal{T}_0(M, \rho)$ and thus induces a symplectic form on $\mathcal{T}_0(M)$. An explicit formula for this symplectic form relies on the following two observations. First, it follows from (1.3) that, for every $J \in \mathcal{J}_{\text{int},0}(M)$, there exists a unique positive volume form $\rho_J \in \Omega^{2n}(M)$ that satisfies

$$(2.6) \quad \text{Ric}_{\rho_J, J} = 0, \quad \int_M \rho_J = V := \int_M \rho.$$

Second, for every $J \in \mathcal{J}_{\text{int},0}(M)$ and every $\hat{J} \in \Omega_J^{0,1}(M, TM)$ with $\bar{\partial}_J \hat{J} = 0$, there exist unique functions $f, g \in \Omega^0(M)$ that satisfy

$$(2.7) \quad \Lambda_{\rho_J}(J, \hat{J}) = -df \circ J + dg, \quad \int_M f \rho_J = \int_M g \rho_J = 0.$$

(See [32, Lemma 3.8].) With this understood, define

$$(2.8) \quad \Omega_J^{\text{WP}}(\hat{J}_1, \hat{J}_2) := \int_M \left(\frac{1}{2} \text{trace}(\hat{J}_1 J \hat{J}_2) - f_1 g_2 + f_2 g_1 \right) \rho_J$$

for $J \in \mathcal{J}_{\text{int},0}(M)$ and $\hat{J}_i \in \Omega_J^{0,1}(M, TM)$ with $\bar{\partial}_J \hat{J}_i = 0$, where f_i, g_i are as in (2.7).

THEOREM 2.1 ([32]). *Equation (2.8) defines a closed 2-form Ω^{WP} on $\mathcal{J}_{\text{int},0}(M)$ that descends to a symplectic form, still denoted by Ω^{WP} , on the Teichmüller space $\mathcal{T}_0(M)$. Its pullback under the diffeomorphism $\iota_\rho : \mathcal{T}_0(M, \rho) \rightarrow \mathcal{T}_0(M)$ is the symplectic form induced by (1.1) and renders $\mathcal{T}_0(M, \rho)$ into a symplectic submanifold of the infinite-dimensional symplectic quotient $\mathcal{W}_0(M, \rho)$ in (1.11).*

Here is why this result is not quite as obvious as it may seem at first glance. The space $\mathcal{J}_0(M)$ admits a complex structure

$$\widehat{J} \mapsto -J\widehat{J}$$

and the symplectic form Ω_ρ in equation (1.1) is a $(1, 1)$ -form for this complex structure. However, when $\dim(M) > 2$, it is not a Kähler form because the symmetric bilinear form

$$\Omega_{\rho, J}(\widehat{J}_1, -J\widehat{J}_2) = \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 \widehat{J}_2) \rho$$

is indefinite on $\Omega_J^{0,1}(M, TM)$. Thus a complex submanifold of $\mathcal{J}_0(M)$ need not be symplectic. This is precisely the case for the submanifold $\mathcal{J}_{\text{int},0}(M)$ in (2.1), because the restriction of the 2-form $\Omega_{\rho, J}$ to $T_J \mathcal{J}_{\text{int},0}(M) = \ker \bar{\partial}_J$ has a nontrivial kernel in the case $\dim(M) > 2$, which in the case $\text{Ric}_{\rho, J} = 0$ consists of all infinitesimal deformations $\widehat{J} = \mathcal{L}_X J$ of complex structures such that both X and JX are divergence-free. A key ingredient in the proof of this assertion is the fact that the space of all $\bar{\partial}_J$ -harmonic $(0, 1)$ -forms $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ on a closed Ricci-flat Kähler manifold is invariant under the homomorphism $\widehat{J} \mapsto \widehat{J}^*$ (see [32, Lemma 3.10]). It then follows that the kernel of the 2-form Ω_J^{WP} in (2.8) on $T_J \mathcal{J}_{\text{int},0}(M) = \ker \bar{\partial}_J$ is the image of $\bar{\partial}_J$ and hence Ω_J^{WP} descends to a nondegenerate 2-form on the tangent space $T_{[J]} \mathcal{T}_0(M) = \ker \bar{\partial}_J / \text{im} \bar{\partial}_J$ for each $J \in \mathcal{J}_{\text{int},0}(M)$. This shows that Ω^{WP} descends to a symplectic form on $\mathcal{T}_0(M)$ (see [32, Theorem 4.4]).

Theorem 2.1 gives an alternative construction of the Weil–Petersson symplectic form on the Teichmüller space of Calabi–Yau structures (see [38, 40, 46, 50, 55, 56] for the polarized case and [26, Ch 16] for the K3 surface). The Teichmüller space $\mathcal{T}_0(M)$ carries a natural complex structure

$$[\widehat{J}] \mapsto [-J\widehat{J}]$$

and the Weil–Petersson symplectic form Ω^{WP} is of type $(1, 1)$, however, it is not a Kähler form in general. The complex dimension of the negative part is the Hodge number $h^{2,0}$ and the total dimension is the Hodge number $h^{n-1,1}(M, L)$, where $L = \Lambda_J^{n,0} TM$. These Hodge numbers are deformation invariant by [61, Proposition 9.30].

If $a \in H^2(M; \mathbb{R})$ is a Kähler class, then the tangent spaces of the polarized Teichmüller space $\mathcal{T}_{0,a}(M) \subset \mathcal{T}_0(M)$ in Remark 2.6 below are positive subspaces for the Weil–Petersson symplectic form and so Ω^{WP} restricts to a Kähler form on $\mathcal{T}_{0,a}(M)$. If $h^{2,0} = 0$, then $\mathcal{T}_0(M)$ is Hausdorff and Kähler and each polarized space $\mathcal{T}_{0,a}(M)$ is an open subset of $\mathcal{T}_0(M)$. Here is a list of the real dimensions for the $2n$ -torus, the K3 surface, the Enriques surface, the quintic in $\mathbb{C}P^4$, and the banana manifold B in [8]. The last column lists the dimensions of the Kähler cones.

M	$\mathcal{T}_0(M)$ $2h^{n-1,1}(M, L)$	$\mathcal{T}_{0,a}(M)$ $2h^{n-1,1}(M, L) - 2h^{2,0}$	$2h^{2,0}$	\mathcal{K}_J $h^{1,1}$
\mathbb{T}^{2n}	$2n^2$	$n^2 + n$	$n^2 - n$	n^2
K3	40	38	2	20
Enriques	20	20	0	10
Quintic	202	202	0	1
B	16	16	0	20

2.2. The Teichmüller space of Kähler–Einstein structures. Let M be a closed connected oriented $2n$ -manifold, let $c \in H^2(M; \mathbb{R})$ be a nonzero cohomology class that admits an integral lift, and let \hbar be a real number such that $(2\pi\hbar c)^n > 0$. Consider the space

$$\mathcal{J}_{\text{int},c}(M) := \{J \in \mathcal{J}_{\text{int}}(M) \mid c_1^{\mathbb{R}}(J) = c, 2\pi\hbar c \in \mathcal{K}_J\}$$

of all complex structures J on M whose real first Chern class is c and whose Kähler cone \mathcal{K}_J contains the cohomology class $2\pi\hbar c$. For $J \in \mathcal{J}_{\text{int},c}(M)$ denote by

$$(2.9) \quad \mathcal{S}_J := \{\omega \in \Omega^2(M) \mid d\omega = 0, \omega^n > 0, [\omega] = 2\pi\hbar c, J \in \mathcal{J}_{\text{int}}(M, \omega)\}$$

the space of all symplectic forms ω on M that are compatible with J and represent the cohomology class $2\pi\hbar c$. By the Calabi–Yau Theorem [10, 62] each volume form $\rho \in \Omega^{2n}(M)$ with $\int_M \rho = \langle (2\pi\hbar c)^n / n!, [M] \rangle$ and each $J \in \mathcal{J}_{\text{int},c}(M)$ determine a unique symplectic form $\omega_{\rho,J} \in \mathcal{S}_J$ whose volume form is ρ , i.e.

$$(2.10) \quad \omega_{\rho,J}^n / n! = \rho.$$

In the case of general type with $\hbar < 0$ a theorem of Yau [63, 64] asserts that every complex structure $J \in \mathcal{J}_{\text{int},c}(M)$ admits a unique symplectic form $\omega \in \mathcal{S}_J$ that satisfies the Kähler–Einstein condition

$$\text{Ric}_{\omega^n/n!,J} = \omega/\hbar.$$

In the Fano case with $\hbar > 0$ the Chen–Donaldson–Sun Theorem [15, 16] asserts that a complex structure $J \in \mathcal{J}_{\text{int},c}(M)$ admits a symplectic form $\omega \in \mathcal{S}_J$ that satisfies the Kähler–Einstein condition $\text{Ric}_{\omega^n/n!,J} = \omega/\hbar$ if and only if it satisfies the K-polystability condition of Yau–Tian–Donaldson [24, 54, 64]. The “only if” statement was proved earlier by Berman [2]. Moreover, it was shown by Berman–Berndtsson [3] that, if $\omega, \omega' \in \mathcal{S}_J$ both satisfy the Kähler–Einstein condition, then there exists a holomorphic diffeomorphism $\psi \in \text{Aut}_0(M, J)$ such that $\omega' = \psi^*\omega$.

Fix a volume form $\rho \in \Omega^{2n}(M)$ with $\int_M \rho = \langle (2\pi\hbar c)^n / n!, [M] \rangle$ and consider the following models for the **Teichmüller space of Kähler–Einstein structures**:

$$(2.11) \quad \begin{aligned} \mathcal{T}_c(M, \rho) &:= \mathcal{J}_{\text{KE},c}(M, \rho) / \text{Diff}_0(M, \rho), \\ \mathcal{J}_{\text{KE},c}(M, \rho) &:= \{J \in \mathcal{J}_{\text{int},c}(M) \mid \text{Ric}_{\rho,J} = \omega_{\rho,J}/\hbar\}, \\ \mathcal{T}_c(M) &:= \mathcal{J}_{\text{KE},c}(M) / \text{Diff}_0(M), \\ \mathcal{J}_{\text{KE},c}(M) &:= \{J \in \mathcal{J}_{\text{int},c}(M) \mid J \text{ is K-polystable}\}. \end{aligned}$$

In general these spaces may be singular. At the equivalence class of a regular element $J \in \mathcal{J}_{\text{KE},c}(M, \rho)$, respectively $J \in \mathcal{J}_{\text{KE},c}(M)$, the tangent spaces are

$$(2.12) \quad \begin{aligned} T_{[J]_\rho} \mathcal{T}_c(M, \rho) &= \frac{\{\widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \bar{\partial}_J \widehat{J} = 0, \widehat{\text{Ric}}_\rho(J, \widehat{J}) \wedge \omega_{\rho,J}^{n-1} = 0\}}{\{\mathcal{L}_X J \mid X \in \text{Vect}(M), d\iota(X)\rho = 0\}}, \\ T_{[J]} \mathcal{T}_c(M) &= \frac{\{\widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \bar{\partial}_J \widehat{J} = 0\}}{\{\mathcal{L}_X J \mid X \in \text{Vect}(M)\}}. \end{aligned}$$

Combining the theorems of Yau, Berman–Berndtsson, and Chen–Donaldson–Sun with Moser isotopy, we find that the natural map $\iota_\rho : \mathcal{T}_c(M, \rho) \rightarrow \mathcal{T}_c(M)$ is a bijection (and a diffeomorphism on the smooth part). Thus $\mathcal{T}_c(M, \rho)$ inherits the complex structure from $\mathcal{T}_c(M)$, and $\mathcal{T}_c(M)$ inherits the symplectic form from $\mathcal{T}_c(M, \rho)$.

LEMMA 2.2 (Hodge decomposition). *Let $J \in \mathcal{J}_{\text{KE},c}(M)$, choose $\omega \in \mathcal{S}_J$ with $\text{Ric}_{\omega^n/n!,J} = \omega/\hbar$, define $\rho := \omega^n/n!$, and let $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ with $\bar{\partial}_J \widehat{J} = 0$. Then there exist $X \in \text{Vect}(M)$, $F, G \in \Omega^0(M)$, and $A \in \Omega_J^{0,1}(M, TM)$ such that*

$$(2.13) \quad \widehat{J} = \mathcal{L}_X J + \mathcal{L}_{v_F} J + \mathcal{L}_{Jv_G} J + A,$$

$$(2.14) \quad d\iota(X)\rho = d\iota(JX)\rho = 0, \quad A = A^*, \quad \bar{\partial}_J A = 0, \quad \bar{\partial}_J^* A = 0.$$

Thus $\Lambda_\rho(J, A) = 0$ by Remark 1.3. Moreover, X and A are uniquely determined by \widehat{J} , the four summands in (2.13) are pairwise L^2 orthogonal, and

$$(2.15) \quad \Lambda_\rho(J, \widehat{J}) = \frac{2}{\hbar} \iota(X)\omega + d\left(\frac{2}{\hbar}F - d^*dF\right) - d\left(\frac{2}{\hbar}G - d^*dG\right) \circ J.$$

PROOF. The proof has five steps.

Step 1. *If $F, G \in \Omega^0(M)$ and $X \in \text{Vect}(M)$ satisfies $d\iota(X)\rho = d\iota(JX)\rho = 0$, then*

$$\Lambda_\rho(J, \mathcal{L}_X J + \mathcal{L}_{v_F} J + \mathcal{L}_{Jv_G} J) = \frac{2}{\hbar} \iota(X)\omega + d\left(\frac{2}{\hbar}F - d^*dF\right) - d\left(\frac{2}{\hbar}G - d^*dG\right) \circ J.$$

This follows from (1.9) with $f_{v_F} = 0$ and $f_{Jv_F} = -d^*dF$.

Step 2. *There exist functions $\Phi, \Psi \in \Omega^0(M)$ and vector fields $X, Y \in \text{Vect}(M)$ such that $d\iota(X)\rho = d\iota(JX)\rho = 0$ and*

$$(2.16) \quad \Lambda_\rho(J, \widehat{J}) = \frac{2}{\hbar} \iota(Y)\omega = \frac{2}{\hbar} (\iota(X)\omega + d\Phi - d\Psi \circ J), \quad Y = X + v_\Phi + Jv_\Psi.$$

Choose $Y \in \text{Vect}(M)$ such that $\iota(Y)\omega = \frac{\hbar}{2} \Lambda_\rho(J, \widehat{J})$ and then choose $\Phi, \Psi \in \Omega^0(M)$ and $X \in \text{Vect}(M)$ such that $d\iota(X)\rho = d\iota(JX)\rho = 0$ and $Y = X + v_\Phi + Jv_\Psi$.

Step 3. *Let $H \in \Omega^0(M)$. Then $d(d^*dH - \frac{2}{\hbar}H) = 0$ if and only if $\mathcal{L}_{v_H} J = 0$.*

We have $\iota(Jv_H)\rho = *dH$ and $\Lambda_\rho(J, \mathcal{L}_{v_H} J) = d\left(\frac{2}{\hbar}H - d^*dH\right)$ by (1.9). Hence

$$(2.17) \quad \frac{1}{2} \|\mathcal{L}_{v_H} J\|^2 = - \int_M \Lambda_\rho(J, \mathcal{L}_{v_H} J) \wedge \iota(Jv_H)\rho = \int_M (d^*dH - \frac{2}{\hbar}H)(d^*dH)\rho$$

by (1.8) and this proves Step 3.

Step 4. *Let $H \in \Omega^0(M)$ such that $d^*dH = \frac{2}{\hbar}H$. Then $\int_M \Phi H \rho = \int_M \Psi H \rho = 0$.*

By Step 3 we have $\mathcal{L}_{v_H} J = 0$ and $\mathcal{L}_{Jv_H} J = 0$. Hence, by (1.8) and (2.16),

$$\begin{aligned} 0 &= \frac{\hbar}{2} \int_M \Lambda_\rho(J, \widehat{J}) \wedge \iota(Jv_H)\rho = \int_M d\Phi \wedge \iota(Jv_H)\rho = \int_M \Phi(d^*dH)\rho = \frac{2}{\hbar} \int_M \Phi H \rho, \\ 0 &= -\frac{\hbar}{2} \int_M \Lambda_\rho(J, \widehat{J}) \wedge \iota(v_H)\rho = \int_M d\Psi \wedge \iota(Jv_H)\rho = \int_M \Psi(d^*dH)\rho = \frac{2}{\hbar} \int_M \Psi H \rho. \end{aligned}$$

This proves Step 4.

Step 5. *We prove the lemma.*

By Step 4 there exist $F, G \in \Omega^0(M)$ such that $\frac{2}{\hbar}F - d^*dF = \frac{2}{\hbar}\Phi$, $\frac{2}{\hbar}G - d^*dG = \frac{2}{\hbar}\Psi$. Thus $Z := X + v_F + Jv_G$ satisfies $\Lambda_\rho(J, \mathcal{L}_Z J) = \Lambda_\rho(J, \widehat{J})$ by Step 1 and Step 2, and so (2.13) and (2.15) hold with $A := \widehat{J} - \mathcal{L}_Z J$. Let $\widehat{\omega} := d\iota(Y)\omega = \hbar \widehat{\text{Ric}}_\rho(J, \widehat{J})$. Then, since $\bar{\partial}_J \widehat{J} = 0$, it follows from [32, Lemma 3.6] that

$$\omega(\widehat{J}u, Jv) + \omega(Ju, \widehat{J}v) = \widehat{\omega}(u, v) - \widehat{\omega}(Ju, Jv) = \omega((\mathcal{L}_Y J)u, Jv) + \omega(Ju, (\mathcal{L}_Y J)v)$$

for all $u, v \in \text{Vect}(M)$. Hence the endomorphism $A = (\widehat{J} - \mathcal{L}_Y J) + \mathcal{L}_{Y-Z} J$ of TM is symmetric. This proves (2.14) and Lemma 2.2. \square

THEOREM 2.3 (Weil–Petersson symplectic form). *Let $J \in \mathcal{J}_{\text{KE},c}(M)$ and let $\omega, \rho, \widehat{J}_i, X_i, F_i, G_i, A_i$ for $i = 1, 2$ be as in Lemma 2.2. Then*

$$\begin{aligned}
(2.18) \quad \Omega_J^{\text{WP}}(\widehat{J}_1, \widehat{J}_2) &:= \frac{1}{2} \int_M \text{trace}(A_1 J A_2) \rho \\
&= \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 J \widehat{J}_2) \rho - \frac{2}{\hbar} \int_M \omega_{\rho, J}(X_1, X_2) \rho \\
&\quad + \int_M \left((d^* d F_1 - \frac{2}{\hbar} F_1) d^* d G_2 - (d^* d G_1 - \frac{2}{\hbar} G_1) d^* d F_2 \right) \rho.
\end{aligned}$$

The formula (2.18) defines a Kähler form on the Teichmüller space $\mathcal{T}_c(M)$ and the mapping class group $\text{Diff}_c(M)/\text{Diff}_0(M)$ of isotopy classes of diffeomorphisms that preserve the cohomology class c acts on $\mathcal{T}_c(M)$ by Kähler isometries. The pullback of Ω^{WP} to $\mathcal{T}_c(M, \rho)$ is the 2-form $\Omega_{\rho}^{\text{WP}}$ given by

$$\begin{aligned}
(2.19) \quad \Omega_{\rho, J}^{\text{WP}}(\widehat{J}_1, \widehat{J}_2) &= \frac{1}{2} \int_M \text{trace}(\widehat{J}_1 J \widehat{J}_2) \rho \\
&\quad - \frac{\hbar}{2} \int_M \Lambda_{\rho}(J, \widehat{J}_1) \wedge \Lambda_{\rho}(J, \widehat{J}_2) \wedge \frac{\omega_{\rho, J}^{n-1}}{(n-1)!}
\end{aligned}$$

for $J \in \mathcal{J}_{\text{KE},c}(M, \rho)$ and $\widehat{J}_i \in \Omega_J^{0,1}(M, TM)$ with $\bar{\partial}_J \widehat{J}_i = 0$, $\widehat{\text{Ric}}_{\rho}(J, \widehat{J}_i) \wedge \omega_{\rho, J}^{n-1} = 0$.

PROOF. The second equality in (2.18) follows from (2.17) and the fact that the terms $\mathcal{L}_X J$, $\mathcal{L}_{v_F} J$, $\mathcal{L}_{J v_G} J$, A in Lemma 2.2 are pairwise L^2 orthogonal. Also, for each $J \in \mathcal{J}_{\text{KE},c}(M)$, it follows directly from the definition that the skew-symmetric bilinear form Ω_J^{WP} on the kernel of $\bar{\partial}_J$ in $\Omega_J^{0,1}(M, TM)$ descends to a nondegenerate form on the quotient space $T_{[J]} \mathcal{T}_c(M) = \ker \bar{\partial}_J / \text{im } \bar{\partial}_J$ that is compatible with the linear complex structure $[\widehat{J}] \mapsto [-J\widehat{J}]$. That the 2-form Ω^{WP} descends to $\mathcal{T}_c(M)$ and is preserved by the action of the mapping class group follows from the naturality of the decomposition in Lemma 2.2. Its pullback to $\mathcal{T}_c(M, \rho)$ is given by (2.19), because $\widehat{\text{Ric}}_{\rho}(J, \widehat{J}) \wedge \omega^{n-1} = 0$ implies $d(\frac{2}{\hbar} G - d^* d G) = 0$ in Lemma 2.2. That this pullback is closed follows from the equations $\partial_t \omega_{\rho, J} = \frac{\hbar}{2} d \Lambda_{\rho}(J, \partial_t J)$ and $\partial_s \Lambda_{\rho}(J, \partial_t J) - \partial_t \Lambda_{\rho}(J, \partial_s J) = -\frac{1}{2} d \text{trace}((\partial_s J) J (\partial_t J))$ (in [32, Theorem 2.7]) for every smooth map $\mathbb{R}^2 \rightarrow \mathcal{J}_{\text{KE},c}(M, \rho) : (s, t) \mapsto J_{s,t}$. \square

The space $\mathcal{K}(M, a, \rho)$ of all Kähler pairs (ω, J) that satisfy $[\omega] = a := 2\pi\hbar c$ and $\omega^n/n! = \rho$ is not a symplectic submanifold of $\mathcal{P}(M, a, \rho)$ whenever $\dim(M) > 2$. In fact, using Lemma 2.2 one can show that the kernel of the restriction of the 2-form (1.15) to $\mathcal{K}(M, a, \rho)$ at every Kähler–Einstein pair $(\omega, J) \in \mathcal{K}(M, a, \rho)$ with $\text{Ric}_{\rho, J} = \omega/\hbar$ is the set of all pairs $(\mathcal{L}_X \omega, \mathcal{L}_X J)$ such that both X and JX are divergence-free. Nevertheless, if $H^1(M; \mathbb{R}) = 0$, then Theorem 2.3 shows that the regular part of $\mathcal{T}_c(M, \rho)$ embeds via $[J] \mapsto [\omega_{\rho, J}, J]$ as a symplectic submanifold into the symplectic quotient $\mathcal{W}_{\text{SE}}(M, a, \rho)$ in (1.17). The condition $H^1(M; \mathbb{R}) = 0$ is necessarily satisfied in the Fano case $\hbar > 0$. In the case $\hbar < 0$ the group $\text{Diff}_0(M)$ acts on $\mathcal{J}_{\text{KE},c}(M) = \mathcal{J}_{\text{int},c}(M)$ with finite isotropy, the quotient

$$\mathcal{L}_c(M, \rho) := \mathcal{J}_{\text{KE},c}(M, \rho) / \text{Diff}^{\text{ex}}(M, \rho)$$

embeds symplectically into $\mathcal{W}_{\text{SE}}(M, a, \rho)$ via $[J] \mapsto [\omega_{\rho, J}, J]$, and the space $\mathcal{L}_c(M, \rho)$ fibers over $\mathcal{T}_c(M, \rho)$ with symplectic fibers. If the action of the group $\text{Diff}_0(M, \rho)$ on $\mathcal{J}_{\text{KE},c}(M, \rho)$ is free, then each fiber is isomorphic to $H^{2n-1}(M; \mathbb{R})/\Gamma_{\rho}$, where Γ_{ρ} is the image of the flux homomorphism $\text{Flux}_{\rho} : \pi_1(\text{Diff}_0(M, \rho)) \rightarrow H^{2n-1}(M; \mathbb{R})$.

2.3. Constant scalar curvature Kähler metrics. Let (M, ω) be a closed connected symplectic $2n$ -manifold with the volume form $\rho := \omega^n/n!$ and denote by $\mathcal{J}_{\text{int}}(M, \omega)$ the space of all complex structures on M that are compatible with ω . This space is connected for rational and ruled surfaces [1], however, in general this is an open question. The regular part of $\mathcal{J}_{\text{int}}(M, \omega)$ is an infinite-dimensional Kähler submanifold of $\mathcal{J}(M, \omega)$ whose tangent space at a regular element J is

$$T_J \mathcal{J}_{\text{int}}(M, \omega) = \left\{ J \in \Omega_J^{0,1}(M, TM) \mid \widehat{J} = \widehat{J}^*, \bar{\partial}_J \widehat{J} = 0 \right\}.$$

Consider the symplectic quotient

$$(2.20) \quad \begin{aligned} \mathcal{Z}(M, \omega) &:= \mathcal{J}_{\text{cscK}}(M, \omega)/\text{Ham}(M, \omega), \\ \mathcal{J}_{\text{cscK}}(M, \omega) &:= \{ J \in \mathcal{J}_{\text{int}}(M, \omega) \mid S_{\omega, J} = c_\omega \}. \end{aligned}$$

The regular part of this space is a complex submanifold of the infinite-dimensional quotient $\mathcal{W}_{\text{csc}}(M, \omega)$ in (1.25) with the tangent spaces

$$(2.21) \quad T_{[J]_\omega} \mathcal{Z}(M, \omega) = \frac{\left\{ \widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \bar{\partial}_J \widehat{J} = 0, \widehat{J} = \widehat{J}^*, \widehat{S}_\omega(J, \widehat{J}) = 0 \right\}}{\left\{ \mathcal{L}_{v_H} J \mid H \in \Omega^0(M) \right\}}$$

for $J \in \mathcal{J}_{\text{cscK}}(M, \omega)$. So $\mathcal{Z}(M, \omega)$ inherits the Kähler structure of $\mathcal{W}_{\text{csc}}(M, \omega)$ with the symplectic form (1.22) and the complex structure $[\widehat{J}]_\omega \mapsto [-J(\widehat{J} - \mathcal{L}_{v_H} J)]_\omega$, where $H \in \Omega^0(M)$ satisfies $\widehat{S}_\omega(J, J\widehat{J} - J\mathcal{L}_{v_H} J) = 0$ as in Proposition 1.16. To describe $\mathcal{Z}(M, \omega)$ as a complex quotient, we digress briefly into GIT.

REMARK 2.4 (Geometric invariant theory). Let (X, ω, J) be a closed Kähler manifold and let G be a compact Lie group which acts on X by Kähler isometries. Assume that the Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is equipped with an invariant inner product and that the action is Hamiltonian and is generated by an equivariant moment map $\mu : X \rightarrow \mathfrak{g}$. Then the complexified group G^c acts on X by holomorphic diffeomorphisms and the symplectic quotient $X//G := \mu^{-1}(0)/G$ is naturally isomorphic to the complex quotient X^{ps}/G^c of the set $X^{\text{ps}} := \{x \in X \mid G^c(x) \cap \mu^{-1}(0) \neq \emptyset\}$ of μ -**polystable** elements of X by the complexified group. The set of μ -polystable elements can be characterized in terms of **Mumford's numerical invariants**

$$(2.22) \quad w_\mu(x, \xi) := \lim_{t \rightarrow \infty} \langle \mu(\exp(it\xi)x), \xi \rangle$$

associated to $x \in X$ and $0 \neq \xi \in \mathfrak{g}$. The **moment-weight inequality** asserts that

$$(2.23) \quad \sup_{0 \neq \xi \in \mathfrak{g}} \frac{-w_\mu(x, \xi)}{|\xi|} \leq \inf_{g \in G^c} |\mu(gx)|$$

and that equality holds whenever the right hand side is positive. An immediate consequence is that an element $x \in X$ is μ -**semistable** (i.e. $\overline{G^c(x)} \cap \mu^{-1}(0) \neq \emptyset$) if and only if $w_\mu(x, \xi) \geq 0$ for all $\xi \in \mathfrak{g} \setminus \{0\}$. For μ -polystability the additional condition is required that $w_\mu(x, \xi) = 0$ implies $\lim_{t \rightarrow \infty} \exp(it\xi)x \in G^c(x)$. This is the content of the **Hilbert–Mumford criterion**. Its proof is based on the study of the gradient flow of the moment map squared $f := \frac{1}{2}|\mu|^2 : X \rightarrow \mathbb{R}$ and the related gradient flow of the **Kempf–Ness function** $\Phi_x : G^c/G \rightarrow \mathbb{R}$, defined by

$$(2.24) \quad d\Phi_x(g)\widehat{g} = -\langle \mu(g^{-1}x), \text{Im}(g^{-1}\widehat{g}) \rangle, \quad \Phi_x(u) = 0$$

for $\widehat{g} \in T_g G^c$ and $u \in G$. The symmetric space G^c/G is a Hadamard space and the Kempf–Ness function is convex along geodesics. By the **Kempf–Ness Theorem** it is bounded below if and only if x is μ -semistable. For an exposition see [33].

REMARK 2.5 (The space of Kähler potentials). It was noted by Donaldson in his landmark paper [20] that much of geometric invariant theory carries over (in part conjecturally) to the infinite-dimensional setting where X is replaced by the space $\mathcal{J}(M, \omega)$ and the compact Lie group G by $\mathcal{G} = \text{Ham}(M, \omega)$. While in this situation there is no complexified group there do exist complexified group orbits. In the integrable case the complexified group orbit of $J \in \mathcal{J}_{\text{int}}(M, \omega)$ is the space $\mathcal{G}^c(J)$ of all elements $J' \in \mathcal{J}_{\text{int}}(M, \omega)$ that are **exact isotopic to J** (i.e. there exists a smooth path $[0, 1] \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : s \mapsto J_s$ and a smooth family of vector fields $[0, 1] \rightarrow \text{Vect}(M, \omega) : s \mapsto v_s$ such that $J_0 = J$, $J_1 = J'$, $\partial_s J_s = \mathcal{L}_{v_s} J_s$, and $\iota(v_s)\omega = d\Phi_s - d\Psi_s \circ J_s$ for $\Phi_s, \Psi_s \in \Omega^0(M)$). In this situation the role of the symmetric space G^c/G is played by the **space of Kähler potentials**

$$(2.25) \quad \mathcal{H}_J := \left\{ h \in \Omega^0(M) \mid \begin{array}{l} \text{The 2-form } \omega_h := \omega + \frac{1}{2}d(dh \circ J) = \omega - i\partial\bar{\partial}h \\ \text{satisfies } \omega_h(\hat{x}, J\hat{x}) > 0 \forall 0 \neq \hat{x} \in TM \end{array} \right\}.$$

This space has been studied by Calabi, Chen [9, 11, 13, 14], Mabuchi [41, 42], Semmes [52], Donaldson [20] and others. It is an infinite-dimensional symmetric space of nonpositive sectional curvature with the Mabuchi metric

$$(2.26) \quad \langle \hat{h}_1, \hat{h}_2 \rangle_h := \int_M \hat{h}_1 \hat{h}_2 \frac{\omega_h^n}{n!}$$

and geodesics are the solutions $t \mapsto h_t$ of the **Monge–Ampère equation**

$$(2.27) \quad \partial_t \partial_t h_t + \frac{1}{2} |d\partial_t h_t|_{h_t}^2 = 0.$$

In [13] Chen proved that any two elements of \mathcal{H}_J can be joined by a weak $C^{1,1}$ geodesic. As noted by Donaldson [19, 20], the analogues of Mumford’s numerical invariants in this setting are the **Futaki invariants** [31], the analogue of the Kempf–Ness function is the **Mabuchi functional** [41] $\mathcal{M}_J : \mathcal{H}_J \rightarrow \mathbb{R}$ defined by

$$(2.28) \quad d\mathcal{M}_J(h)\hat{h} = \int_M (S_{\omega_{h,J}} - c_\omega)\hat{h} \frac{\omega_h^n}{n!}, \quad \mathcal{M}_J(0) = 0,$$

and the analogue of the gradient flow of the moment map squared is the Calabi flow. After earlier results by Donaldson [21, 22], Mabuchi [43], and Chen–Tian [17] it was shown by Berman–Berndtsson [3] that the Mabuchi functional is convex along weak geodesics, that every Kähler potential $h \in \mathcal{H}_J$ with constant scalar curvature $S_{\omega_{h,J}} = c_\omega$ minimizes the Mabuchi functional, and that constant scalar curvature Kähler metrics are unique up to holomorphic diffeomorphism, i.e. if two Kähler potentials $h, h' \in \mathcal{H}_J$ have constant scalar curvature $S_{\omega_{h,J}} = S_{\omega_{h',J}} = c_\omega$, then there exists a holomorphic diffeomorphism $\psi \in \text{Aut}_0(M, J)$ such that $\omega_{h'} = \psi^* \omega_h$.

In the present setting the analogue of the Hilbert–Mumford criterion is the Yau–Tian–Donaldson conjecture [24, 54, 64] which relates the existence of a constant scalar curvature Kähler potential to K-polystability. For Fano manifolds it was confirmed by Chen–Donaldson–Sun [15, 16], while in general it is an open question.

Remark 2.5 shows that, according to the YTD conjecture, the space $\mathcal{Z}(M, \omega)$ can be expressed as the quotient $\mathcal{Z}_K(M, \omega) := \mathcal{J}_K(M, \omega) / \sim$, where $\mathcal{J}_K(M, \omega)$ is the space of K-polystable complex structures that are compatible with ω , and the equivalence relation is exact isotopy as in Remark 2.5. The formal (Zariski type) tangent space of $\mathcal{Z}_K(M, \omega)$ at the equivalence class of an element $J \in \mathcal{J}_K(M, \omega)$ is the quotient $T_{[J]}\mathcal{Z}_K(M, \omega) = T_J \mathcal{J}_{\text{int}}(M, \omega) / \{\mathcal{L}_{v_F + Jv_G} J \mid F, G \in \Omega^0(M)\}$ and the complex structure is $[\hat{J}] \mapsto [-J\hat{J}]$.

It is also of interest to consider the **Teichmüller space of constant scalar curvature Kähler metrics**, defined by

$$(2.29) \quad \mathcal{T}(M, \omega) := \mathcal{I}_{\text{cscK}}(M, \omega) / \text{Symp}_0(M, \omega).$$

If $H^1(M; \mathbb{R}) = 0$, then $\text{Ham}(M, \omega) = \text{Symp}_0(M, \omega)$ and hence the regular part of the Teichmüller space $\mathcal{T}(M, \omega) = \mathcal{Z}(M, \omega)$ is Kähler. The two spaces also agree in the Calabi–Yau case, where $\text{Symp}_0(M, \omega) / \text{Ham}(M, \omega)$ acts trivially on $\mathcal{Z}(M, \omega)$. In the Kähler–Einstein case with $2\pi\hbar c_1^{\mathbb{R}}(\omega) = [\omega]$ and $\hbar < 0$ the group $\text{Symp}_0(M, \omega)$ acts on $\mathcal{I}_{\text{cscK}}(M, \omega)$ with finite isotropy, the Teichmüller space $\mathcal{T}(M, \omega)$ carries a Kähler form given by (2.19) with $\omega_{\rho, J} = \omega$, and $\mathcal{Z}(M, \omega)$ fibers over $\mathcal{T}(M, \omega)$ with symplectic fibers. If the action of $\text{Symp}_0(M, \omega)$ on $\mathcal{I}_{\text{cscK}}(M, \omega)$ is free, then each fiber is isomorphic to the space $H^1(M; \mathbb{R}) / \Gamma_\omega$, where Γ_ω is the image of the flux homomorphism $\text{Flux}_\omega : \pi_1(\text{Symp}_0(M, \omega)) \rightarrow H^1(M; \mathbb{R})$.

REMARK 2.6. Fix a symplectic form ω on M that admits a compatible complex structure J with $S_{\omega, J} = c_\omega$ and $\text{Aut}(M, J) \cap \text{Diff}_0(M) = \text{Aut}_0(M, J)$. Define

$$\rho := \omega^n / n!, \quad a = [\omega] \in H^2(M; \mathbb{R}), \quad c := c_1^{\mathbb{R}}(\omega) \in H^2(M; \mathbb{R}).$$

Assume the Calabi–Yau case $c = 0$ and consider the **polarized Teichmüller space**

$$(2.30) \quad \mathcal{T}_{0,a}(M, \rho) := \{J \in \mathcal{I}_{\text{int},0}(M) \mid \text{Ric}_{\rho, J} = 0, a \in \mathcal{K}_J\} / \text{Diff}_0(M, \rho)$$

of all isotopy classes of Ricci-flat complex structures that contain the cohomology class a in their Kähler cone. This space is Hausdorff and the Weil–Petersson metric on $\mathcal{T}_{0,a}(M, \rho)$ is Kähler. Moreover, there is a natural holomorphic map

$$\iota_\omega : \mathcal{T}(M, \omega) \rightarrow \mathcal{T}_{0,a}(M, \rho)$$

which pulls back the Weil–Petersson symplectic form on $\mathcal{T}_{0,a}(M, \rho)$ in Theorem 2.1 to the symplectic form (1.22) on $\mathcal{T}(M, \omega)$. The map ι_ω need not be injective or surjective. It is injective if and only if $\text{Symp}(M, \omega) \cap \text{Diff}_0(M) = \text{Symp}_0(M, \omega)$. In [51] Seidel found many examples of symplectic four-manifolds that admit symplectomorphisms that are smoothly, but not symplectically, isotopic to the identity, including K3-surfaces with embedded Lagrangian spheres. The map ι_ω is surjective if and only if the space $\mathcal{S}_{0,a}$ of symplectic forms in the class a with real first Chern class zero that admit compatible complex structures is connected. By a theorem of Hajduk–Tralle [35] the space $\mathcal{S}_{0,a}$ is disconnected for the $8k$ -torus with $k \geq 1$.

Now assume the Kähler–Einstein case $2\pi\hbar c = a$ for some nonzero real number \hbar . Then there is again a natural holomorphic map

$$\iota_\omega : \mathcal{T}(M, \omega) \rightarrow \mathcal{T}_c(M, \rho)$$

which pulls back the Weil–Petersson symplectic form on $\mathcal{T}_c(M, \rho)$ in Theorem 2.3 to the symplectic form (2.19) on $\mathcal{T}(M, \omega)$. As before, this map need not be injective or surjective. Seidel’s examples in dimension four include as Fano manifolds the k -fold blowup of the projective plane with $5 \leq k \leq 8$ and many Kähler surfaces of general type, so in these cases the map ι_ω is not injective. It is surjective if and only if the space $\mathcal{S}_{c,a}$ of symplectic forms in the class a with first Chern class c that admit compatible K-polystable complex structures is connected. By a theorem of Randal-Williams [48] every complete intersection M with $\dim(M) = 16k \geq 16$ and $\dim(H^{8k}(M; \mathbb{R})) \geq 6$ admits a diffeomorphism ϕ that acts as the identity on homology such that ω is not isotopic to $\phi^*\omega$ for every symplectic form ω . This includes Kähler–Einstein examples, and in these cases $\mathcal{S}_{c,a}$ is disconnected.

3. Fano manifolds

This section explains how the symplectic form introduced by Donaldson [25] on the space of Fano complex structures fits into the present setup. We begin by giving another proof of nondegeneracy, and then discuss Berndtsson convexity for the Ding functional and the Donaldson–Kähler–Ricci flow.

3.1. The Donaldson symplectic form. Let (M, ω) be a closed connected symplectic $2n$ -manifold that satisfies the Fano condition

$$(3.1) \quad 2\pi c_1^{\mathbb{R}}(\omega) = [\omega] \in H^2(M; \mathbb{R}).$$

As in Subsection 1.3 we denote by v_F the Hamiltonian vector field of F and by $\{F, G\}$ the Poisson bracket of $F, G \in \Omega^0(M)$. Let $\mathcal{J}_{\text{int}}(M, \omega)$ be the space of ω -compatible complex structures on M . If this space is nonempty, then M is simply connected. Throughout we will ignore all regularity issues and treat $\mathcal{J}_{\text{int}}(M, \omega)$ as a submanifold of $\mathcal{J}(M, \omega)$ whose tangent space at $J \in \mathcal{J}_{\text{int}}(M, \omega)$ is

$$T_J \mathcal{J}_{\text{int}}(M, \omega) = \left\{ \widehat{J} \in \Omega_J^{0,1}(M, TM) \mid \bar{\partial}_J \widehat{J} = 0, \widehat{J} = \widehat{J}^* \right\}.$$

This is a complex subspace of $T_J \mathcal{J}(M, \omega)$ and so inherits the symplectic form (1.22) from the ambient Kähler manifold $\mathcal{J}(M, \omega)$. In [25] Donaldson introduced another symplectic form on $\mathcal{J}_{\text{int}}(M, \omega)$ which we explain next.

It follows from (1.3) that, for every complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$, there exists a unique positive volume form $\rho_J \in \Omega^{2n}(M)$ that satisfies

$$(3.2) \quad \text{Ric}_{\rho_J, J} = \omega, \quad \int_M \rho_J = 1.$$

Moreover, $\widehat{\text{Ric}}_{\rho_J}(J, \widehat{J})$ is an exact $(1, 1)$ -form for every $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$ by (3.2) and [32, Lemma 3.6]. Thus, by the $\partial\bar{\partial}$ -lemma, for every $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and every $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, there exist unique functions $f, g \in \Omega^0(M)$ that satisfy

$$(3.3) \quad \Lambda_{\rho_J}(J, \widehat{J}) = -df \circ J + dg, \quad \int_M f \rho_J = \int_M g \rho_J = 0.$$

The **Donaldson symplectic form** on $\mathcal{J}_{\text{int}}(M, \omega)$ is defined by

$$(3.4) \quad \Omega_J^D(\widehat{J}_1, \widehat{J}_2) := \int_M \left(\frac{1}{2} \text{trace}(\widehat{J}_1 J \widehat{J}_2) - f_1 g_2 + g_1 f_2 \right) \rho_J$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\widehat{J}_i \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, where ρ_J, f_i, g_i are as in (3.2) and (3.3). The fact that the 2-form (3.4) is nondegenerate is far from trivial and is one of the main results in [25]. Indeed, as noted by Donaldson, nondegeneracy can be viewed as a reformulation of Berndtsson’s convexity theorem [4, 5] for the Ding functional [18] on the space of Kähler potentials (see Subsection 3.2 below).

The symplectic form Ω^D in (3.4) is given by essentially the same formula as the Weil–Petersson symplectic form Ω^{WP} on $\mathcal{T}_0(M)$ in (2.8). In contrast to the Calabi–Yau case, where the lifted 2-form on $\mathcal{J}_{\text{int}, 0}(M)$ has the kernel $\text{im} \bar{\partial}_J$ on each tangent space $T_J \mathcal{J}_{\text{int}, 0}(M)$, the 2-form Ω^D is nondegenerate in the Fano case.

The definition of the symplectic form in Donaldson’s paper [25] uses the existence of a holomorphic n -form with values in a suitable holomorphic line bundle to define the volume form denoted by ρ_J in (3.2). That the 2-form (3.4) agrees with the symplectic form in [25] (up to a factor $1/4$) then follows from the discussion in [32, Appendix D].

THEOREM 3.1 (Donaldson [25]). Ω^D is a $\text{Symp}(M, \omega)$ -invariant symplectic form on $\mathcal{J}_{\text{int}}(M, \omega)$ and is compatible with the complex structure $\widehat{J} \mapsto -J\widehat{J}$. The action of $\text{Ham}(M, \omega)$ on $\mathcal{J}_{\text{int}}(M, \omega)$ is Hamiltonian and is generated by the $\text{Symp}(M, \omega)$ -equivariant moment map $\mu : \mathcal{J}_{\text{int}}(M, \omega) \rightarrow (\Omega^0(M)/\mathbb{R})^*$, given by

$$(3.5) \quad \langle \mu(J), H \rangle := 2 \int_M H \left(\frac{1}{V} \frac{\omega^n}{n!} - \rho_J \right), \quad V := \int_M \frac{\omega^n}{n!},$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $H \in \Omega^0(M)$, where $\rho_J \in \Omega^{2n}(M)$ is as in (3.2). Thus

$$(3.6) \quad \Omega_J^D(\widehat{J}, \mathcal{L}_{v_H} J) = -2 \int_M H f \rho_J = \langle d\mu(J)\widehat{J}, H \rangle$$

for $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$ and $H \in \Omega^0(M)$, where f is as in (3.3).

REMARK 3.2 (Weil–Petersson symplectic form). The zero set of the moment map in Theorem 3.1 is the space

$$\mathcal{J}_{\text{KE}}(M, \omega) := \{J \in \mathcal{J}_{\text{int}}(M, \omega) \mid \text{Ric}_{\omega^n/n!, J} = \omega\} = \mathcal{J}_{\text{cscK}}(M, \omega)$$

of Kähler–Einstein complex structures compatible with ω . Since M is simply connected, the quotient $\mathcal{T}_{\text{KE}}(M, \omega) := \mathcal{J}_{\text{KE}}(M, \omega)/\text{Ham}(M, \omega) = \mathcal{T}(M, \omega)$ is the Teichmüller space in (2.29) and the symplectic form on this space induced by (3.4) is $1/V$ times the Weil–Petersson symplectic form induced by (1.22). For the relation to the Teichmüller space $\mathcal{T}_c(M, \rho) \cong \mathcal{T}_c(M)$ in Theorem 2.3 see Remark 2.6.

Below we give a proof of nondegeneracy of (3.4) which amounts to translating the argument in [25] into our notation. The heart of the proof is Lemma 3.7.

DEFINITION 3.3. Fix a complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and let $\rho := \omega^n/n!$. The **Kähler–Ricci potential** of J is the function $\Theta_{\omega, J} := \Theta_J : M \rightarrow (0, \infty)$ defined by $\Theta_J := \rho_J/\rho$. Hence $\text{Ric}_{\Theta_J \rho, J} = \omega$ by (3.2) and so by (1.3), we have

$$(3.7) \quad \frac{1}{2} d(d \log(\Theta_J) \circ J) = \omega - \text{Ric}_{\rho, J}, \quad \int_M \Theta_J \rho = 1.$$

Thus $\Theta_J = 1/V$ if and only if (M, ω, J) is a Kähler–Einstein manifold. Denote by $d^*d : \Omega^0(M) \rightarrow \Omega^0(M)$ the Laplace–Beltrami operator of the Riemannian metric $\langle \cdot, \cdot \rangle := \omega(\cdot, J \cdot)$ and define the linear operators $\mathbf{L}, \mathbf{B} : \Omega^0(M) \rightarrow \Omega^0(M)$ by

$$(3.8) \quad \mathbf{L}F := d^*dF - \frac{\langle v_{\Theta_J}, v_F \rangle}{\Theta_J}, \quad \mathbf{B}F := \frac{\{\Theta_J, F\}}{\Theta_J}$$

for $F \in \Omega^0(M)$. Thus \mathbf{L} is a self-adjoint Fredholm operator and \mathbf{B} is skew-adjoint with respect to the L^2 inner product $\langle F, G \rangle_J := \int_M FG \rho_J$ on $\Omega^0(M)$.

LEMMA 3.4 (Kähler–Ricci potential). Choose elements $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$ and let f and Θ_J be as in (3.3) and (3.7). Then

$$(3.9) \quad \widehat{\Theta}_\omega(J, \widehat{J}) := \frac{\partial}{\partial t} \Big|_{t=0} \Theta_{J_t} = f \Theta_J$$

for every smooth path $\mathbb{R} \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : t \mapsto J_t$ with $J_0 = J$ and $\frac{\partial}{\partial t} \Big|_{t=0} J_t = \widehat{J}$.

PROOF. By Proposition 1.2 and (1.3) the derivative of any path $t \mapsto \text{Ric}_{\rho_t, J_t}$ is given by $\partial_t \text{Ric}_{\rho_t, J_t} = \widehat{\text{Ric}}_{\rho_t}(J_t, \partial_t J_t) + \frac{1}{2} d(d(\partial_t \rho_t/\rho_t) \circ J_t)$. In the case at hand with $J_t \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\rho_t = \rho_{J_t} = \Theta_{J_t} \omega^n/n!$ this yields the equation

$$0 = \widehat{\text{Ric}}_{\rho_J}(J, \widehat{J}) + \frac{1}{2} d(d(\widehat{\Theta}_\omega(J, \widehat{J})/\Theta_J) \circ J) = \frac{1}{2} d(d(\widehat{\Theta}_\omega(J, \widehat{J})/\Theta_J - f) \circ J).$$

Since $\widehat{\Theta}_\omega(J, \widehat{J})/\Theta_J - f$ has mean value zero for ρ_J , this proves the lemma. \square

LEMMA 3.5 (**Holomorphic vector fields**). *Let $J \in \mathcal{J}_{\text{int}}(M, \omega)$, choose the quadruple $\rho_J, \Theta_J, \mathbf{L}, \mathbf{B}$ as in Definition 3.3, and choose functions $F, G \in \Omega^0(M)$ such that $\int_M F \rho_J = \int_M G \rho_J = 0$. Then the following holds.*

- (i) $F = G = 0$ if and only if $\mathbf{L}F + \mathbf{B}G = 0$ and $\mathbf{L}G - \mathbf{B}F = 0$.
- (ii) $\Lambda_{\rho_J}(J, \mathcal{L}_{v_F + Jv_G}J) = -d(2G - \mathbf{L}G + \mathbf{B}F) \circ J + d(2F - \mathbf{L}F - \mathbf{B}G)$.
- (iii) $\mathcal{L}_{v_F + Jv_G}J = 0$ if and only if $\mathbf{L}F + \mathbf{B}G = 2F$ and $\mathbf{L}G - \mathbf{B}F = 2G$.

PROOF. Throughout the proof we use the notation

$$\|F\| := \sqrt{\int_M F^2 \rho_J}, \quad \|v\| := \sqrt{\int_M \omega(v, Jv) \rho_J}, \quad \|\widehat{J}\| := \sqrt{\int_M \text{trace}(\widehat{J}^2) \rho_J}$$

for $F \in \Omega^0(M)$, $v \in \text{Vect}(M)$, and $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ with $\widehat{J} = \widehat{J}^*$.

To prove part (i), we observe that

$$(3.10) \quad \begin{aligned} \int_M (\mathbf{L}F)G \rho_J &= \int_M (dF \circ J) \wedge dG \wedge \Theta_J \frac{\omega^{n-1}}{(n-1)!} = \int_M \omega(v_F, Jv_G) \rho_J, \\ \int_M (\mathbf{B}F)G \rho_J &= \int_M dF \wedge dG \wedge \Theta_J \frac{\omega^{n-1}}{(n-1)!} = \int_M \omega(v_F, v_G) \rho_J \end{aligned}$$

for all $F, G \in \Omega^0(M)$, and hence

$$(3.11) \quad \|v_F + Jv_G\|^2 = \int_M \left(F(\mathbf{L}F + \mathbf{B}G) + G(\mathbf{L}G - \mathbf{B}F) \right) \rho_J.$$

Now let $F, G \in \Omega^0(M)$ such that $\mathbf{L}F + \mathbf{B}G = \mathbf{L}G - \mathbf{B}F = 0$. Then $v_F + Jv_G = 0$ by (3.11) and hence $\int_M \omega(v_F, Jv_F) \omega^n / n! = \int_M \omega(v_F, v_G) \omega^n / n! = 0$. Thus F and G are constant and this proves (i). Part (ii) follows from (1.9), (3.2), and

$$(3.12) \quad du(v_F) \rho_J = (\mathbf{B}F) \rho_J, \quad du(Jv_G) \rho_J = -(\mathbf{L}G) \rho_J.$$

Moreover, $\frac{1}{2} \|\mathcal{L}_{v_F + Jv_G}J\|^2 = -\int_M \Lambda_{\rho_J}(J, \mathcal{L}_{v_F + Jv_G}J) \wedge \iota(J(v_F + Jv_G)) \rho_J$ by (1.8), and hence (iii) follows from (ii). \square

LEMMA 3.6 (**Decomposition Lemma**). *Let $J \in \mathcal{J}_{\text{int}}(M, \omega)$, let ρ_J be as in (3.2), and let $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ such that $\bar{\partial}_J \widehat{J} = 0$ and $\widehat{J} = \widehat{J}^*$ with respect to the metric $\omega(\cdot, J\cdot)$. Then there exist $F, G \in \Omega^0(M)$ and $A \in \Omega_J^{0,1}(M, TM)$ such that*

$$(3.13) \quad \widehat{J} = \mathcal{L}_{v_F}J + \mathcal{L}_{Jv_G}J + A,$$

and

$$(3.14) \quad \int_M F \rho_J = \int_M G \rho_J = 0, \quad A = A^*, \quad \bar{\partial}_J A = 0, \quad \Lambda_{\rho_J}(J, A) = 0.$$

Moreover, A and $\mathcal{L}_{v_F + Jv_G}J$ are L^2 orthogonal. If \widehat{J} satisfies (3.13) and (3.14), then $\Lambda_{\rho_J}(J, \widehat{J})$ is given by (3.3) with

$$(3.15) \quad f = 2G - \mathbf{L}G + \mathbf{B}F, \quad g = 2F - \mathbf{L}F - \mathbf{B}G,$$

and \widehat{J} satisfies the equation

$$(3.16) \quad \begin{aligned} &\int_M \left(\frac{1}{2} \text{trace}(\widehat{J}^2) - f^2 - g^2 \right) \rho_J \\ &= \int_M \frac{1}{2} \text{trace}(A^2) \rho_J + 2 \int_M \left(|v_F + Jv_G|^2 - 2(F^2 + G^2) \right) \rho_J. \end{aligned}$$

PROOF. Let $f, g \in \Omega^0(M)$ be as in (3.3). We prove first that all $F, G \in \Omega^0(M)$ satisfies the identity

$$(3.17) \quad \frac{1}{2} \int_M \text{trace}(\widehat{J}\mathcal{L}_{v_F+Jv_G}J)\rho_J = - \int_M \left(g(\mathbf{L}F + \mathbf{B}G) + f(\mathbf{L}G - \mathbf{B}F) \right) \rho_J.$$

To see this, abbreviate $v := v_F + Jv_G$. Then we have $du(v)\rho_J = (-\mathbf{L}G + \mathbf{B}F)\rho_J$ and $du(Jv)\rho_J = (-\mathbf{L}F - \mathbf{B}G)\rho_J$ by (3.12). Hence, by (1.8) and (3.3),

$$\begin{aligned} \frac{1}{2} \int_M \text{trace}(\widehat{J}\mathcal{L}_vJ)\rho_J &= \int_M (df \circ J - dg) \wedge \iota(Jv)\rho_J \\ &= \int_M (fdu(v)\rho_J + gdu(Jv)\rho_J) \\ &= \int_M (f(-\mathbf{L}G + \mathbf{B}F) + g(-\mathbf{L}F - \mathbf{B}G))\rho_J. \end{aligned}$$

This proves (3.17).

Now choose functions $F, G \in \Omega^0(M)$ such that

$$\mathbf{L}F + \mathbf{B}G = 2F, \quad \mathbf{L}G - \mathbf{B}F = 2G.$$

Then $\mathcal{L}_{v_F+Jv_G}J = 0$ by part (iii) of Lemma 3.5 and hence $\int_M (gF + fG)\rho_J = 0$ by equation (3.17). Thus the pair (g, f) is L^2 orthogonal to the kernel of the self-adjoint Fredholm operator $(F, G) \mapsto (2F - \mathbf{L}F - \mathbf{B}G, 2G - \mathbf{L}G + \mathbf{B}F)$ and so belongs to its image. Hence there exist smooth functions $F, G \in \Omega^0(M)$ such that

$$2F - \mathbf{L}F - \mathbf{B}G = g, \quad 2G - \mathbf{L}G + \mathbf{B}F = f.$$

By part (ii) of Lemma 3.5 this implies $\Lambda_{\rho_J}(J, \widehat{J} - \mathcal{L}_{v_F+Jv_G}J) = 0$. Since $\mathcal{L}_{v_F}J$ and $\mathcal{L}_{Jv_G}J = J\mathcal{L}_{v_G}J$ are symmetric, by [32, Lemma 3.7], this proves (3.14).

Now assume that F, G, A have been found such that \widehat{J} satisfies (3.13) and (3.14). Then (3.15) follows directly from part (ii) of Lemma 3.5. Moreover,

$$\int_M \text{trace}(A\mathcal{L}_{v_F+Jv_G}J)\rho_J = 0$$

by (1.8) and (3.14). Hence, by (3.17) we have

$$\begin{aligned} &\frac{1}{2} \int_M \text{trace}(\widehat{J}^2)\rho_J - \frac{1}{2} \int_M \text{trace}(A^2)\rho_J \\ &= \frac{1}{2} \int_M \text{trace}(\widehat{J}(\mathcal{L}_{v_F+Jv_G}J))\rho_J \\ &= \int_M (f(-\mathbf{L}G + \mathbf{B}F) + g(-\mathbf{L}F - \mathbf{B}G))\rho_J \\ &= \int_M (f(f - 2G) + g(g - 2F))\rho_J \\ &= \int_M (f^2 + g^2)\rho_J + 2 \int_M (F(\mathbf{L}F + \mathbf{B}G - 2F) + G(\mathbf{L}G - \mathbf{B}F - 2G))\rho_J \\ &= \int_M (f^2 + g^2)\rho_J + 2 \int_M (|v_F + Jv_G|^2 - 2(F^2 + G^2))\rho_J. \end{aligned}$$

Here the last step uses (3.11). This proves (3.16). \square

LEMMA 3.7 (Berndtsson Inequality). *Let $\mathcal{J} \in \mathcal{J}_{\text{int}}(M, \omega)$, choose ρ_J as in (3.2), and let $F, G \in \Omega^0(M)$ such that $\int_M F \rho_J = \int_M G \rho_J = 0$. Then*

$$(3.18) \quad \int_M |v_F + Jv_G|^2 \rho_J \geq 2 \int_M (F^2 + G^2) \rho_J,$$

and equality holds in (3.18) if and only if $\mathcal{L}_{v_F + Jv_G} J = 0$. If $\mathcal{L}_{v_F + Jv_G} J = 0$, then every pair of functions $\widehat{F}, \widehat{G} \in \Omega^0(M)$ with $\int_M \widehat{F} \rho_J = \int_M \widehat{G} \rho_J = 0$ satisfies

$$(3.19) \quad \int_M \langle v_F + Jv_G, v_{\widehat{F}} + Jv_{\widehat{G}} \rangle = 2 \int_M (F\widehat{F} + G\widehat{G}) \rho_J.$$

PROOF. Since F and G have mean value zero, it follows from part (i) of Lemma 3.5 that there exists a unique pair of functions $\Phi, \Psi \in \Omega^0(M)$ such that

$$(3.20) \quad \mathbf{L}\Phi + \mathbf{B}\Psi = F, \quad \mathbf{L}\Psi - \mathbf{B}\Phi = G, \quad \int_M \Phi \rho_J = \int_M \Psi \rho_J = 0.$$

Continue the notation in the proof of Lemma 3.5 and define

$$(3.21) \quad u := v_\Phi + Jv_\Psi, \quad v := v_F + Jv_G.$$

Then Lemma 3.6 with $\widehat{J} = \mathcal{L}_u J$, $f = 2\Psi - G$, $g = 2\Phi - F$, $A = 0$ yields

$$(3.22) \quad \begin{aligned} \frac{1}{2} \|\mathcal{L}_u J\|^2 &= \int_M \left(2|u|^2 - 4(\Phi^2 + \Psi^2) + (2\Phi - F)^2 + (2\Psi - G)^2 \right) \rho_J \\ &= \int_M \left(2|u|^2 + F^2 + G^2 - 4\Phi F - 4\Psi G \right) \rho_J \\ &= \int_M (F^2 + G^2) \rho_J - 2\|u\|^2. \end{aligned}$$

The last step uses the formula $\|u\|^2 = \int_M (\Phi F + \Psi G) \rho_J$ in (3.11). Now, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|v\|^2 - \|v - \lambda u\|^2 &= 2\lambda \int_M \omega(u, Jv) \rho_J - \lambda^2 \|u\|^2 \\ &= 2\lambda \int_M \left(\omega(v_\Phi, Jv_F) + \omega(v_\Psi, v_F) + \omega(v_\Psi, Jv_G) - \omega(v_\Phi, v_G) \right) \rho_J - \lambda^2 \|u\|^2 \\ &= 2\lambda \int_M \left((\mathbf{L}\Phi + \mathbf{B}\Psi)F + (\mathbf{L}\Psi - \mathbf{B}\Phi)G \right) \rho_J - \lambda^2 \|u\|^2 \\ &= 2\lambda \int_M (F^2 + G^2) \rho_J - \lambda^2 \|u\|^2 \\ &= \left(2\lambda - \frac{\lambda^2}{2} \right) \int_M (F^2 + G^2) \rho_J + \frac{\lambda^2}{4} \|\mathcal{L}_u J\|^2. \end{aligned}$$

Here the second equality follows from (3.21), the third from (3.10), the fourth from (3.20), and the last from (3.22). With $\lambda = 2$ this yields

$$(3.23) \quad \|v\|^2 - 2 \int_M (F^2 + G^2) \rho_J = \|\mathcal{L}_u J\|^2 + \|v - 2u\|^2 \geq 0.$$

This proves (3.18). Moreover, equality in (3.18) implies $v = 2u$ and $\mathcal{L}_u J = 0$, and so $\mathcal{L}_v J = 0$. Conversely, if $\mathcal{L}_v J = 0$, then $\mathbf{L}F + \mathbf{B}G = 2F$ and $\mathbf{L}G - \mathbf{B}F = 2G$ by part (iii) of Lemma 3.5, hence the unique solution of (3.20) is given by $\Phi = \frac{1}{2}F$ and $\Psi = \frac{1}{2}G$, which implies $u = \frac{1}{2}v$ and $\mathcal{L}_u J = 0$, so equality in (3.18) follows from (3.23). To prove the last assertion, define $F_t := F + t\widehat{F}$ and $G_t := G + t\widehat{G}$ and differentiate the function $t \mapsto \int_M (\frac{1}{2}|v_{F_t} + Jv_{G_t}|^2 - F_t^2 - G_t^2) \rho_J$ at $t = 0$. \square

PROOF OF THEOREM 3.1. Fix an element $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and let ρ_J be as in (3.2). We show first that the 2-form (3.4) is nondegenerate and compatible with the complex structure $\widehat{J} \mapsto -J\widehat{J}$. To see this, let $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, let f, g be as in (3.3), and let F, G, A be as in Lemma 3.6. Then (3.4) and (3.16) yield

$$(3.24) \quad \begin{aligned} \Omega_J^D(\widehat{J}, -J\widehat{J}) &= \int_M \left(\frac{1}{2} \text{trace}(\widehat{J}^2) - f^2 - g^2 \right) \rho_J \\ &= \int_M \frac{1}{2} \text{trace}(A^2) \rho_J + 2 \int_M \left(|v_F + Jv_G|^2 - 2(F^2 + G^2) \right) \rho_J. \end{aligned}$$

By Lemma 3.7 the right hand side in (3.24) is nonnegative and vanishes if and only if $A = 0$ and $\mathcal{L}_{v_F + Jv_G} J = 0$ or, equivalently, $\widehat{J} = 0$. This proves nondegeneracy.

To prove (3.6), fix an element $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, let f, g be as in (3.3), and let $H \in \Omega^0(M)$ such that $\int_M H \rho_J = 0$. Then, by Lemma 3.5 and (3.10), we have

$$\begin{aligned} \Omega_J^D(\widehat{J}, \mathcal{L}_{v_H} J) &= \frac{1}{2} \int_M \text{trace}(\widehat{J} J \mathcal{L}_{v_H} J) \rho_J - \int_M f(2H - \mathbf{L}H) \rho_J + \int_M g(\mathbf{B}H) \rho_J \\ &= \int_M \Lambda_{\rho_J}(J, \widehat{J}) \wedge \iota(v_H) \rho_J + \int_M dH \wedge dg \wedge \Theta_J \frac{\omega^{n-1}}{(n-1)!} \\ &\quad + \int_M (df \circ J) \wedge dH \wedge \Theta_J \frac{\omega^{n-1}}{(n-1)!} - 2 \int_M H f \rho_J \\ &= -2 \int_M H f \rho_J. \end{aligned}$$

Here the last equality holds because $\Lambda_{\rho_J}(J, \widehat{J}) = -df \circ J + dg$. This proves the first equality in (3.6) and the second follows from Lemma 3.4.

It remains to prove that the 2-form (3.4) is closed. In Donaldson's formulation this follows directly from the definition, while in our formulation this requires proof. Here is the outline. First, let $\mathbb{R}^2 \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : (s, t) \mapsto J_{s,t}$ be a smooth map and, for $s, t \in \mathbb{R}$, define the functions $f_s, g_s, f_t, g_t \in \Omega^0(M)$ such that they have mean value zero with respect to ρ_J for $J = J_{s,t}$ and

$$\Lambda_{\rho_J}(J, \partial_s J) = -df_s \circ J + dg_s, \quad \Lambda_{\rho_J}(J, \partial_t J) = -df_t \circ J + dg_t.$$

Here we have dropped the subscripts s, t for J and observe that f_s, g_s, f_t, g_t depend also on s and t . Then by [32, Theorem 2.7] and Lemma 3.4 we have

$$\partial_s \Lambda_{\rho_J}(J, \partial_t J) - \partial_t \Lambda_{\rho_J}(J, \partial_s J) + \frac{1}{2} d \text{trace}((\partial_s J) J (\partial_t J)) = df_s \circ \partial_t J - df_t \circ \partial_s J.$$

Hence a calculation shows that

$$(3.25) \quad \partial_s f_t - \partial_t f_s = 0, \quad d \left(\partial_s g_t - \partial_t g_s + \frac{1}{2} \text{trace}((\partial_s J) J (\partial_t J)) \right) = 0.$$

Now let $\mathbb{R}^3 \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : (r, s, t) \mapsto J(r, s, t)$ be a smooth map and define the functions $f_r, f_s, f_t, g_r, g_s, g_t$ as before. Then $\partial_r \rho_J = f_r \rho_J$ by Lemma 3.4 and hence

$$\begin{aligned} \partial_r \Omega_J^D(\partial_s J, \partial_t J) &= \frac{1}{2} \int_M f_r \text{trace}((\partial_s J) J (\partial_t J)) \rho_J \\ &\quad + \frac{1}{2} \int_M \text{trace}((\partial_r \partial_s J) J (\partial_t J)) \rho_J + \frac{1}{2} \int_M \text{trace}((\partial_s J) J (\partial_r \partial_t J)) \rho_J \\ &\quad + \int_M \left((\partial_r g_s) f_t + g_s (\partial_r f_t) + f_r g_s f_t - (\partial_r f_s) g_t - f_s (\partial_r g_t) - f_r f_s g_t \right) \rho_J. \end{aligned}$$

Take a cyclic sum and use (3.25) to obtain $(d\Omega^D)_J(\partial_r J, \partial_s J, \partial_t J) = 0$. \square

3.2. The Ding functional and the Kähler–Ricci flow. Fix a complex structure $J \in \mathcal{I}_{\text{int}}(M, \omega)$ and denote by \mathcal{H}_J the space of Kähler potentials as in Remark 2.5. The analogue of the Mabuchi functional in the present setting is the **Ding functional** $\mathcal{F}_J : \mathcal{H}_J \rightarrow \mathbb{R}$, defined by

$$(3.26) \quad \mathcal{F}_J(h) := \mathcal{I}_J(h) - \log \left(\int_M e^h \rho_J \right)$$

for $h \in \mathcal{H}_J$, where $\mathcal{I}_J : \mathcal{H}_J \rightarrow \mathbb{R}$ is the unique functional that satisfies

$$\mathcal{I}_J(0) = 0, \quad d\mathcal{I}_J(h)\widehat{h} = \frac{1}{V} \int_M \widehat{h} \rho_h, \quad \rho_h := \frac{\omega_h^n}{n!},$$

for all $h \in \mathcal{H}_J$ and all $\widehat{h} \in \Omega^0(M)$. An explicit formula is $\mathcal{I}_J(h) := \int_0^1 \frac{1}{V} \int_M h \rho_{th} dt$. For $h \in \mathcal{H}_J$ define $\theta_h := \Theta_{\omega_h, J} : M \rightarrow (0, \infty)$ (see Definition 3.3). Then

$$(3.27) \quad \text{Ric}_{\theta_h \rho_h, J} = \omega_h, \quad \int_M \theta_h \rho_h = 1, \quad \rho_h := \frac{\omega_h^n}{n!}.$$

Since $\text{Ric}_{\rho_J, J} = \omega$, we have $\text{Ric}_{e^h \rho_J, J} = \omega_h = \text{Ric}_{\theta_h \rho_h, J}$ by (1.3), and hence

$$(3.28) \quad \theta_h \rho_h = \frac{e^h \rho_J}{\int_M e^h \rho_J}$$

for all $h \in \mathcal{H}_J$. This implies

$$(3.29) \quad d\mathcal{F}_J(h)\widehat{h} = \frac{1}{V} \int_M \widehat{h} \rho_h - \frac{\int_M \widehat{h} e^h \rho_J}{\int_M e^h \rho_J} = \int_M \widehat{h} \left(\frac{1}{V} - \theta_h \right) \rho_h$$

for $h \in \mathcal{H}_J$ and $\widehat{h} \in \Omega^0(M)$. Thus the gradient of the Ding functional \mathcal{F}_J with respect to the Riemannian metric (2.26) on \mathcal{H}_J is given by

$$(3.30) \quad \text{grad} \mathcal{F}_J(h) = \frac{1}{V} - \theta_h$$

for $h \in \mathcal{H}_J$. In [4] Berndtsson proved the following.

THEOREM 3.8 (Berndtsson). *The Ding functional is convex along geodesics.*

PROOF. Let $I \rightarrow \mathcal{H}_J : t \mapsto h_t$ be a geodesic so that $\partial_t \partial_t h + \frac{1}{2} |d\partial_t h|_h^2 = 0$. Then it follows from (3.29) that

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{F}_J(h) &= \frac{1}{V} \frac{d}{dt} \int_M (\partial_t h) \rho_h - \frac{d}{dt} \frac{\int_M (\partial_t h) e^h \rho_J}{\int_M e^h \rho_J} \\ &= - \frac{\int_M (\partial_t \partial_t h) e^h \rho_J}{\int_M e^h \rho_J} - \frac{\int_M (\partial_t h)^2 e^h \rho_J}{\int_M e^h \rho_J} + \left| \frac{\int_M (\partial_t h) e^h \rho_J}{\int_M e^h \rho_J} \right|^2 \\ &= \frac{1}{2} \frac{\int_M |d\partial_t h|_h^2 e^h \rho_J}{\int_M e^h \rho_J} - \frac{\int_M (\partial_t h)^2 e^h \rho_J}{\int_M e^h \rho_J} + \left| \frac{\int_M (\partial_t h) e^h \rho_J}{\int_M e^h \rho_J} \right|^2 \\ &= \frac{1}{2} \int_M |d\partial_t h|_h^2 \theta_h \rho_h - \int_M (\partial_t h)^2 \theta_h \rho_h + \left| \int_M (\partial_t h) \theta_h \rho_h \right|^2 \\ &\geq 0. \end{aligned}$$

Here the second equality holds because $\int_M (\partial_t h) (\partial_t \rho_h) = \frac{1}{2} \int_M |d\partial_t h|_h^2 \rho_h$. The last inequality holds by Lemma 3.7 with ω replaced by ω_h , with ρ_J replaced by $\theta_h \rho_h$, and with $F := \partial_t h - \int_M (\partial_t h) \theta_h \rho_h$ and $G := 0$. \square

In finite-dimensional GIT the gradient flow of the moment map squared translates into the gradient flow of the Kempf–Ness function. In the present setting the moment map is given by $\mathcal{J}_{\text{int}}(M, \omega) \rightarrow \Omega_\rho^0(M) : J \mapsto 2(1/V - \Theta_J)$, where Θ_J is defined by (3.7). It is convenient to take one eighth (instead of one half) of the square of the moment map to obtain the energy functional $\mathcal{E}_\omega : \mathcal{J}_{\text{int}}(M, \omega) \rightarrow \mathbb{R}$ defined by

$$(3.31) \quad \mathcal{E}_\omega(J) := \frac{1}{2} \int_M \left(\frac{1}{V} - \Theta_J \right)^2 \frac{\omega^n}{n!}$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$. Consider the Riemannian metric on $\mathcal{J}_{\text{int}}(M, \omega)$ determined by the symplectic form (3.4) and the complex structure $\hat{J} \mapsto -J\hat{J}$. It is given by

$$(3.32) \quad \langle \hat{J}_1, \hat{J}_2 \rangle_J := \Omega_J^D(\hat{J}_1, -J\hat{J}_2) = \int_M \left(\frac{1}{2} \text{trace}(\hat{J}_1 \hat{J}_2) - f_1 f_2 - g_1 g_2 \right) \rho_J$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\hat{J}_i \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, where ρ_J, f_i, g_i are as in (3.2) and (3.3). By Lemma 3.4 the differential of the functional \mathcal{E}_ω in (3.31) is given by

$$d\mathcal{E}_\omega(J)\hat{J} = \int_M f \Theta_J \rho_J = -\frac{1}{2} \Omega_J^D(\hat{J}, \mathcal{L}_v J) = \langle \hat{J}, -\frac{1}{2} J \mathcal{L}_v J \rangle_J,$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\hat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, where f is as in (3.3), v is the Hamiltonian vector field of Θ_J . and the second equality follows from (3.6). This shows that the gradient of \mathcal{E}_ω at J with respect to the metric (3.32) is given by

$$(3.33) \quad \text{grad} \mathcal{E}_\omega(J) = -\frac{1}{2} J \mathcal{L}_v J, \quad \iota(v)\omega = d\Theta_J.$$

Thus a complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$ is a critical point of \mathcal{E}_ω if and only if the Hamiltonian vector field of Θ_J is holomorphic. Such a complex structure is called a **Donaldson–Kähler–Ricci soliton**. By (3.33) a negative gradient flow line of \mathcal{E}_ω is a solution $I \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : t \mapsto J_t$ of the partial differential equation

$$(3.34) \quad \partial_t J_t = \frac{1}{2} J_t \mathcal{L}_{v_t} J_t, \quad \iota(v_t)\omega = d\Theta_{J_t}.$$

If $t \mapsto J_t$ is a solution of (3.34) on an interval $I \subset \mathbb{R}$ containing zero with $J_0 = J$, and $I \rightarrow \text{Diff}_0(M) : t \mapsto \phi_t$ is the isotopy defined by $\partial_t \phi_t + \frac{1}{2} J_t v_t \circ \phi_t = 0$, $\phi_0 = \text{id}$, then $\phi_t^* J_t = J$ for all t and the paths $\omega_t := \phi_t^* \omega$ and $\theta_t := \Theta_{J_t} \circ \phi_t$ satisfy

$$(3.35) \quad \partial_t \omega_t = \frac{1}{2} d(d\theta_t \circ J), \quad \frac{1}{2} d(d \log(\theta_t) \circ J) = \omega_t - \text{Ric}_{\omega_t^n/n!, J}, \quad \int_M \theta_t \frac{\omega_t^n}{n!} = 1,$$

This is the **Donaldson–Kähler–Ricci flow**. Here J is a Fano complex structure and (3.35) is understood as an equation for paths in the space \mathcal{S}_J in (2.9) of all symplectic forms that are compatible with J and represent the cohomology class $2\pi c_1^{\mathbb{R}}(J)$. When $\omega \in \mathcal{S}_J$ is fixed, a solution of (3.35) has the form $\omega_t = \omega_{h_t}$, where $I \rightarrow \mathcal{H}_J : t \mapsto h_t$ is a smooth path satisfying

$$(3.36) \quad \partial_t h_t = \theta_{h_t} - \frac{1}{V}.$$

By (3.30) the solutions of (3.36) are the negative gradient flow lines of the Ding functional $\mathcal{F}_J : \mathcal{H}_J \rightarrow \mathbb{R}$ in (3.26). The next remark shows that (3.36) is a second order parabolic partial differential equation.

REMARK 3.9. Let ∇ be the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ and let $h \in \mathcal{H}_J$. Then $\rho_h = \det(\mathbb{1} - \frac{1}{2} \nabla^2 h + \frac{1}{2} J(\nabla^2 h)J)^{1/2} \omega^n / n!$ and hence it follows from (3.28) that $\theta_h = (\int_M e^h \rho_J)^{-1} e^h \Theta_J \det(\mathbb{1} - \frac{1}{2} \nabla^2 h + \frac{1}{2} J(\nabla^2 h)J)^{-1/2}$.

Define the functional $\mathcal{H}_\omega : \mathcal{J}_{\text{int}}(M, \omega) \rightarrow \mathbb{R}$ by

$$(3.37) \quad \mathcal{H}_\omega(J) := \int_M \log(V\Theta_J) \Theta_J \frac{\omega^n}{n!}$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$. This functional was introduced by Weiyong He [36] (as a functional on the space of Kähler potentials for a fixed complex structure). It is nonnegative and vanishes on a complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$ if and only if it satisfies the Kähler–Einstein condition $\text{Ric}_{\omega^n/n!, J} = \omega$ (see part (i) of Theorem 3.10 below). By Lemma 3.4 the differential of the functional \mathcal{H}_ω is given by

$$d\mathcal{H}_\omega(J)\widehat{J} = \int_M f \log(\Theta_J) \rho_J = \langle \widehat{J}, -\frac{1}{2}J\mathcal{L}_v J \rangle_J$$

for $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $\widehat{J} \in T_J \mathcal{J}_{\text{int}}(M, \omega)$, where f is as in (3.3), v is the Hamiltonian vector field of $\log(\Theta_J)$, and the second equality follows from (3.6). This shows that the gradient of \mathcal{H}_ω at J with respect to the metric (3.32) is given by

$$(3.38) \quad \text{grad}\mathcal{H}_\omega(J) = -\frac{1}{2}J\mathcal{L}_v J, \quad \iota(v)\omega = d\log(\Theta_J).$$

Thus a complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$ is a critical point of \mathcal{H}_ω if and only if the Hamiltonian vector field of $\log(\Theta_J)$ is holomorphic. Such a complex structure is called a **Kähler–Ricci soliton**. By (3.38) a negative gradient flow line of \mathcal{H}_ω is a solution $I \rightarrow \mathcal{J}_{\text{int}}(M, \omega) : t \mapsto J_t$ of the partial differential equation

$$(3.39) \quad \partial_t J_t = \frac{1}{2}J_t \mathcal{L}_{v_t} J_t, \quad \iota(v_t)\omega = d\log(\Theta_{J_t}),$$

If $t \mapsto J_t$ is a solution of (3.39) on an interval $I \subset \mathbb{R}$ containing zero with $J_0 = J$, and $I \rightarrow \text{Diff}_0(M) : t \mapsto \phi_t$ is the isotopy defined by $\partial_t \phi_t + \frac{1}{2}J_t v_t \circ \phi_t = 0$, $\phi_0 = \text{id}$, then $\phi_t^* J_t = J$ for all t and the paths $\omega_t := \phi_t^* \omega$ and $\theta_t := \Theta_{J_t} \circ \phi_t$ satisfy the equation $\partial_t \omega_t = \frac{1}{2}d(d\log(\theta_t) \circ J)$. With $\rho_t := \omega_t^n/n!$ we also have $\theta_t \rho_t = \phi_t^* \rho_{J_t}$, hence $\text{Ric}_{\rho_t, J} + \frac{1}{2}d(d\log(\theta_t) \circ J) = \text{Ric}_{\theta_t \rho_t, J} = \phi_t^* \text{Ric}_{\rho_{J_t}, J_t} = \phi_t^* \omega = \omega_t$, and so

$$(3.40) \quad \partial_t \omega_t = \omega_t - \text{Ric}_{\rho_t, J}, \quad \rho_t := \omega_t^n/n!.$$

This is the standard **Kähler–Ricci flow** on the space \mathcal{S}_J of all J -compatible symplectic forms in the class $2\pi c_1^{\mathbb{R}}(J)$ associated to a Fano complex structure J . When a symplectic form $\omega \in \mathcal{S}_J$ is fixed, a solution of (3.40) has the form $\omega_t = \omega_{h_t}$, where $I \rightarrow \mathcal{H}_J : t \mapsto h_t$ is a smooth path satisfying

$$(3.41) \quad \partial_t h_t = \log(V\theta_{h_t}).$$

Now define the functional $\mathcal{H}_J : \mathcal{H}_J \rightarrow \mathbb{R}$ as in Weiyong He’s original paper [36] by

$$(3.42) \quad \mathcal{H}_J(h) := \mathcal{H}_{\omega_h}(J) = \int_M \log(V\theta_h) \theta_h \rho_h$$

for $h \in \mathcal{H}_J$, where θ_h and ρ_h are as in (3.27). The properties of this functional with regard to the Kähler–Ricci flow are summarized in the following theorem. The first two assertions are due to He [36] and the last inequality is due to Donaldson [25].

THEOREM 3.10 (He, Donaldson). *Fix a complex structure $J \in \mathcal{J}_{\text{int}}(M, \omega)$.*

- (i) *Let $h \in \mathcal{H}_J$. Then $\mathcal{H}_J(h) \geq 0$ with equality if and only if $\text{Ric}_{\rho_h, J} = \omega_h$.*
- (ii) *A Kähler potential $h \in \mathcal{H}_J$ is a critical point of \mathcal{H}_J if and only if it is a Kähler–Ricci soliton, i.e. the ω_h -Hamiltonian vector field of $\log(\theta_h)$ is holomorphic.*
- (iii) *Every solution $I \rightarrow \mathcal{H}_J : t \mapsto h_t$ of (3.41) satisfies the inequalities*

$$(3.43) \quad \frac{d}{dt} \mathcal{H}_J(h_t) \leq 0, \quad \frac{d}{dt} \mathcal{F}_J(h_t) \leq -\mathcal{H}_J(h_t).$$

PROOF. Following [25], we define the function $B : (0, \infty) \rightarrow \mathbb{R}$ by

$$(3.44) \quad B(x) := x \log(Vx) + \frac{1}{V} - x$$

for $x > 0$. Then $B'(x) = \log(Vx)$ and $B''(x) = 1/x$. Hence $B(1/V) = 0$ and B is strictly convex. This implies $B(x) > 0$ for every positive real number $x \neq 1/V$ and

$$(3.45) \quad \left(x - \frac{1}{V}\right) \log(Vx) = \left(x - \frac{1}{V}\right) B'(x) \geq B(x) = x \log(Vx) + \frac{1}{V} - x.$$

To prove part (i), fix an element $h \in \mathcal{H}_J$. Then, by (3.27), (3.42), and (3.44),

$$\mathcal{H}_J(h) = \int_M \left(\theta_h \log(V\theta_h) + \frac{1}{V} - \theta_h \right) \rho_h = \int_M B(\theta_h) \rho_h.$$

Hence $\mathcal{H}_J(h) \geq 0$ with equality if and only if $\theta_h = 1/V$. This proves (i).

We prove part (ii). A calculation shows that

$$(3.46) \quad \begin{aligned} d\mathcal{H}_J(h)\widehat{h} &= -\frac{1}{2} \int_M \langle d\widehat{h}, d\log(V\theta_h) \rangle_h \theta_h \rho_h + \int_M \widehat{h} \log(V\theta_h) \theta_h \rho_h \\ &\quad - \left(\int_M \log(V\theta_h) \theta_h \rho_h \right) \left(\int_M \widehat{h} \theta_h \rho_h \right) \end{aligned}$$

for $h \in \mathcal{H}_J$ and $\widehat{h} \in \Omega^0(M)$. This implies

$$(3.47) \quad \begin{aligned} d\mathcal{H}_J(h) \log(V\theta_h) &= -\frac{1}{2} \int_M |d\log(V\theta_h)|_h^2 \theta_h \rho_h + \int_M (\log(V\theta_h))^2 \theta_h \rho_h \\ &\quad - \left(\int_M \log(V\theta_h) \theta_h \rho_h \right)^2 \leq 0 \end{aligned}$$

for all $h \in \mathcal{H}_J$. Here the inequality follows from Lemma 3.7, with ω, ρ_J replaced by $\omega_h, \theta_h \rho_h$ and $F := \log(V\theta_h) - \int_M \log(V\theta_h) \theta_h \rho_h$ and $G := 0$. It follows also from Lemma 3.7 that $d\mathcal{H}_J(h) = 0$ if and only if $d\mathcal{H}_J(h) \log(V\theta_h) = 0$ if and only if the vector field v defined by $\iota(v)\omega_h = d\log(V\theta_h)$ satisfies $\mathcal{L}_v J = 0$. This proves (ii).

We prove part (iii). The first inequality in (3.43) follows directly from (3.47). To prove the second inequality, recall from equation (3.29) that

$$\begin{aligned} d\mathcal{F}_J(h) \log(V\theta_h) &= \int_M \log(V\theta_h) \left(\frac{1}{V} - \theta_h \right) \rho_h \\ &\leq - \int_M \left(\theta_h \log(V\theta_h) + \frac{1}{V} - \theta_h \right) \rho_h = -\mathcal{H}_J(h). \end{aligned}$$

Here the second step follows from (3.45). This proves (iii) and the theorem. \square

In [25] Donaldson noted the following. If $[0, \infty) \rightarrow \mathcal{H}_J : t \mapsto h_t$ is a solution of the Kähler–Ricci flow (3.41) and the limit $h := \lim_{t \rightarrow \infty} h_t$ exists in \mathcal{H}_J , but the pair (ω_h, J) is not a Kähler–Einstein structure, then it follows from Theorem 3.10 that $\mathcal{H}_J(h_t) \geq \mathcal{H}_J(h) > 0$ and hence the Ding functional $\mathcal{F}_J(h_t)$ diverges to minus infinity as t tends to infinity. This corresponds to the observation in GIT that the Kempf–Ness function of an unstable point is unbounded below. The analogue of the Kempf–Ness Theorem in the present setting would be the assertion

$$(3.48) \quad \inf_{h \in \mathcal{H}_J} \int_M \left(\frac{1}{V} - \theta_h \right)^2 \rho_h > 0 \quad \iff \quad \inf_{h \in \mathcal{H}_J} \mathcal{F}_J(h) = -\infty$$

for every $J \in \mathcal{I}_{\text{int}}(M, \omega)$. This seems to be an open question.

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ICMAT MADRID

E-mail address: oscar.garcia-prada@icmat.es

ETH ZÜRICH

E-mail address: salamond@math.ethz.ch