A survey of symmetric functions, Grassmannians, and representations of the unitary group

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Contents

1 Introduction

Associated to a partition $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ is a symmetric function

$$\theta_a = \det\left(({t_i}^{a_i+k-j})_{i,j=1}^k\right) / \det\left(({t_i}^{k-j})_{i,j=1}^k\right),$$

an irreducible representation $\rho_a : \mathrm{U}(k) \to \mathrm{Aut}(V_a)$ whose highest weight is determined by a, and a Schubert cycle Σ_a in the Grassmannian $\mathrm{G}(k,n)$ of complex k-planes in $\mathbb{C}^n$. The Weyl character formula asserts that θ_a is the character of ρ_a, Chern-Weil theory associates a characteristic class $[\theta(F_A/2\pi i)] \in H^*(\mathrm{G}(k,n))$ of the tautological bundle bundle $E \to \mathrm{G}(k,n)$ to every symmetric polynomial θ, and Agnihotri observed that the class $\omega_a = [\theta_a(F_A/2\pi i)]$ is Poincaré dual to the Schubert cycle Σ_a. The correspondence $\rho \mapsto [\theta_\rho(F_A/2\pi i)]$ was introduced by Witten. This gives rise to a triangle of ring isomorphisms

$$\begin{array}{ccccc} & & \mathcal{S}(k,n-k) & & \\ & {\scriptstyle Weyl} \nearrow & & \searrow {\scriptstyle Chern-Weil} & \\ \mathcal{R}(k,n-k) & & \xrightarrow{Witten} & & H^*(\mathrm{G}(k,n)) \end{array}$$

Here $\mathcal{S}(k,n-k)$ is a finite dimensional quotient of the ring of symmetric functions in k arguments and $\mathcal{R}(k,n-k)$ is a corresponding finite dimensional quotient of the representation ring of $\mathrm{U}(k)$. These isomorphisms are uniquely determined by the correspondence

$$\begin{array}{ccccc} & & \phi_i & & \\ & \nearrow & & \searrow & \\ \Lambda^i\mathbb{C}^k & & \longrightarrow & & c_i(E^*) \end{array}$$

where the ϕ_i are the elementary symmetric functions. Witten's motivation for considering the isomorphism $\mathcal{R}(k,n-k) \to H^*(\mathrm{G}(k,n))$ is his conjecture that it should identify the Verlinde algebra with the quantum cohomology of the Grassmannian. The purpose of the present survey is to describe this picture in the *classical* context.

2 Symmetric polynomials

2.1 Elementary symmetric functions

This section contains some foundational material about the ring of symmetric polynomials. An excellent reference is MacDonald [6]. Let $\mathcal{S} = \mathcal{S}(k)$ denote the ring of symmetric polynomials in the variables $t_1, \ldots, t_k$ with complex coefficients. Of fundamental importance are the **elementary symmetric functions** ϕ_i for $1 \leq i \leq k$ and the **complete symmetric functions** ψ_j for $j \geq 1$. They are defined by

$$\phi_i = \sum_{1 \leq \nu_1 < \cdots < \nu_i \leq k} t_{\nu_1} \cdots t_{\nu_i}, \qquad \psi_j = \sum_{1 \leq \nu_1 \leq \cdots \leq \nu_j \leq k} t_{\nu_1} \cdots t_{\nu_j}.$$

The power series

$$\phi(\lambda) = \sum_{i=0}^{k} \phi_i \lambda^i, \qquad \psi(\lambda) = \sum_{j=0}^{\infty} \psi_j \lambda^j$$

can be expressed in the form

$$\phi(\lambda) = \prod_{\nu=1}^{k} (1 + t_\nu \lambda), \qquad \psi(\lambda) = \prod_{\nu=1}^{k} \frac{1}{1 - t_\nu \lambda}. \tag{1}$$

The expression for ϕ is obvious and the one for ψ is equivalent to the following identity.

Lemma 2.1

$$\psi(\lambda)\phi(-\lambda) = 1. \tag{2}$$

Proof: Fix $j \geq 1$ and denote

$$\gamma_i = \sum_{\nu_1 > \cdots > \nu_i \leq \nu_{i+1} \leq \cdots \leq \nu_j} t_{\nu_1} \cdots t_{\nu_j}, \qquad i = 1, \ldots, j.$$

Then $\gamma_1 = \psi_j$, $\gamma_j = \phi_j$, and $\gamma_i + \gamma_{i+1} = \phi_i \psi_{j-i}$ for $i = 1, \ldots, j-1$. Hence

$$\sum_{i=0}^{j} (-1)^i \phi_i \psi_{j-i} = \psi_j + \sum_{i=1}^{j-1} (-1)^i (\gamma_i + \gamma_{i+1}) + (-1)^j \phi_j = 0. \qquad \square$$

In terms of the coefficients the previous lemma shows that each ϕ_i is a polynomial in the $\psi_1, \ldots, \psi_k$, namely

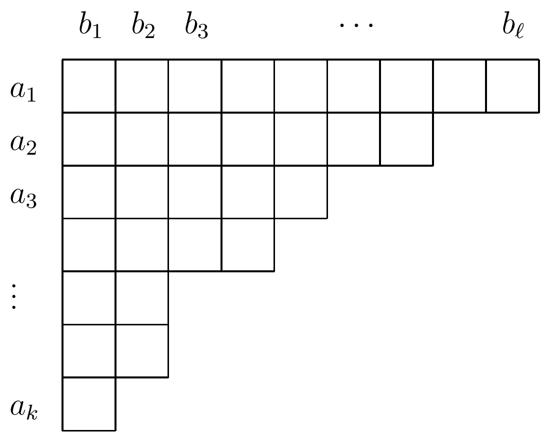
$$\phi_i = \begin{vmatrix} \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_i \\ 1 & \psi_1 & \psi_2 & \cdots & \psi_{i-1} \\ 0 & 1 & \psi_1 & \cdots & \psi_{i-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \psi_1 \end{vmatrix}, \qquad \psi_j = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_j \\ 1 & \phi_1 & \phi_2 & \cdots & \phi_{j-1} \\ 0 & 1 & \phi_1 & \cdots & \phi_{j-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \phi_1 \end{vmatrix} \tag{3}$$

for all $i, j \geq 0$. These equations are easily seen to be equivalent to (2). The first identity continues to hold for $i > k$ with $\phi_i = 0$ and hence the ψ_j satisfy the relations

$$\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_i \\ 1 & \psi_1 & \psi_2 & \cdots & \psi_{i-1} \\ 0 & 1 & \psi_1 & \cdots & \psi_{i-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \psi_1 \end{vmatrix} = 0, \qquad \text{for all} \quad i > k. \tag{4}$$

2.2 Schur polynomials and Young diagrams

A **partition** is a finite sequence of nonnegative integers $a_1 \geq \cdots \geq a_k \geq 0$. Associated to every partition is a **Young diagram** Y_a with a_i squares in the i-th row. The rows are understood to be aligned on the left. The dual partition $b_1 \geq \cdots \geq b_\ell \geq 0$ is obtained by transposing the Young diagram.



Thus b_j is the number of squares in the j-th column and $\ell \geq a_1$. (The numbers a_i and b_j are not required to be nonzero.) Explicitly, the numbers b_j are defined by

$$b_j = \#\{i \mid a_i \geq j\}, \qquad j \geq 1. \tag{5}$$

The next lemma shows how the Schur polynomials in the ψ_j are related to Schur polynomials in the ϕ_i via transposition of Young diagrams. Note that (3) appears as a special case.

Lemma 2.2 *Let $a_1 \geq \cdots \geq a_k \geq 0$ and $b_1 \geq \cdots \geq b_\ell \geq 0$ be related by (5). Then*

$$\begin{vmatrix} \psi_{a_1} & \psi_{a_1+1} & \cdots & \psi_{a_1+k-1} \\ \psi_{a_2-1} & \psi_{a_2} & \cdots & \psi_{a_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_k-k+1} & \psi_{a_k-k+2} & \cdots & \psi_{a_k} \end{vmatrix} = \begin{vmatrix} \phi_{b_1} & \phi_{b_1+1} & \cdots & \phi_{b_1+\ell-1} \\ \phi_{b_2-1} & \phi_{b_2} & \cdots & \phi_{b_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{b_\ell-\ell+1} & \phi_{b_\ell-\ell+2} & \cdots & \phi_{b_\ell} \end{vmatrix}.$$

Proof: Equation (2) is equivalent to

$$\sum_{i=1}^{j}(-1)^i\phi_i\psi_{j-i} = 0$$

for every $j \geq 0$ and this can be expressed in the form $\Psi = \Phi^{-1}$ where

$$\Psi = \begin{pmatrix} 1 & \psi_1 & \cdots & \psi_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \psi_1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 1 & -\phi_1 & \cdots & (-1)^{n-1}\phi_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\phi_1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

for every n. Thus every minor of Ψ agrees with the complementary minor of Φ^T (the transpose of Φ). Suppose $n = k + \ell$ and define

$$\alpha_i = i + a_{k-i}, \qquad \beta_j = j + k - b_{j+1}$$

for $i = 0, \ldots, k-1$ and $j = 0, \ldots, \ell-1$. Then $\beta_j = j + \#\{i \mid \alpha_i \leq i + j\}$ and hence

$$\{0, \ldots, n-1\} = \{\alpha_0, \ldots, \alpha_{k-1}\} \cup \{\beta_0, \ldots, \beta_{\ell-1}\}.$$

Now consider the minor of Ψ with rows $\alpha_0, \ldots, \alpha_{k-1}$ and columns $0, \ldots, k-1$. This agrees up to a sign with the minor of Φ with rows $k, \ldots, n-1$ and columns $\beta_0, \dots, \beta_{\ell-1}$. In fact, the signs in the definition of Φ cancel with the sign in this identification and we obtain

$$\det\left((\psi_{\alpha_i - j})_{i,j=0}^{k-1}\right) = \det\left((\phi_{k+i-\beta_j})_{i,j=0}^{\ell-1}\right).$$

In terms of the coefficients a_i and b_j this becomes

$$\det\left((\psi_{a_{k-i}+i-j})_{i,j=0}^{k-1}\right) = \det\left((\phi_{b_{j+1}+i-j})_{i,j=0}^{\ell-1}\right).$$

This is the required identity. □

Corollary 2.3 *If $a_1 \geq \cdots \geq a_m > 0$ with $m > k$ then*

$$\begin{vmatrix} \psi_{a_1} & \psi_{a_1+1} & \cdots & \psi_{a_1+m-1} \\ \psi_{a_2-1} & \psi_{a_2} & \cdots & \psi_{a_2+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_m-m+1} & \psi_{a_m-m+2} & \cdots & \psi_{a_m} \end{vmatrix} = 0. \tag{6}$$

Proof: Suppose $\ell \geq a_1$ and let $b_1 \geq \cdots \geq b_\ell \geq 0$ be defined by (5). Then the identity of Lemma 2.2 holds with k replaced by m. But $b_1 = m > k$ and hence the first row on the right hand side is zero. □

Remark 2.4 Let $\mathbf{R}$ be any commutative ring with unit and

$$\psi(\lambda) = \sum_{j=0}^{\infty} \psi_j \lambda^j \in \mathbf{R}(\lambda)$$

be a power series with coefficients in $\mathbf{R}$. Suppose that the ψ_j satisfy (4) for $i > k$ with $\psi_0 = 1$. Then there exists a polynomial

$$\phi(\lambda) = \sum_{i=1}^{k} \phi_i \lambda^i \in \mathbf{R}[\lambda]$$

which satisfies (2). In fact the coefficients $\phi_0 = 1, \phi_1, \dots, \phi_k$ are given by (1). Hence Lemma 2.2 and Corollary 2.3 continue to hold in this case. In particular, the ψ_j satisfy the relation (6). □

2.3 Jacobi-Trudi identity

Consider the symmetric polynomials

$$\theta_{a_1,\ldots,a_k} = \frac{\begin{vmatrix} {t_1}^{a_1+k-1} & {t_2}^{a_1+k-1} & \cdots & {t_k}^{a_1+k-1} \\ {t_1}^{a_2+k-2} & {t_2}^{a_2+k-2} & \cdots & {t_k}^{a_2+k-2} \\ \vdots & \vdots & & \vdots \\ {t_1}^{a_k} & {t_2}^{a_k} & \cdots & {t_k}^{a_k} \end{vmatrix}}{\begin{vmatrix} {t_1}^{k-1} & {t_2}^{k-1} & \cdots & {t_k}^{k-1} \\ {t_1}^{k-2} & {t_2}^{k-2} & \cdots & {t_k}^{k-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}} \tag{7}$$

for $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. The **Jacobi-Trudi identity** expresses these functions explicitly as Schur polynomials in the ψ_j. The proof is taken from MacDonald [6],page 25,I (3.4) (see also Fulton [3],Lemma A.9.3).

Lemma 2.5 (Jacobi-Trudi) *For any integers $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ we have*

$$\theta_{a_1,\ldots,a_k} = \begin{vmatrix} \psi_{a_1} & \psi_{a_1+1} & \cdots & \psi_{a_1+k-1} \\ \psi_{a_2-1} & \psi_{a_2} & \cdots & \psi_{a_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_k-k+1} & \psi_{a_k-k+2} & \cdots & \psi_{a_k} \end{vmatrix}. \tag{8}$$

Proof: We follow the argument in [6]. Write $\alpha_i = a_i + k - i$ and denote by $\phi_i^{(j)}$ the i-th symmetric function of the variables $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k$ (with t_j omitted). Then the matrices

$$T_\alpha = \begin{pmatrix} {t_1}^{\alpha_1} & {t_2}^{\alpha_1} & \cdots & {t_k}^{\alpha_1} \\ {t_1}^{\alpha_2} & {t_2}^{\alpha_2} & \cdots & {t_k}^{\alpha_2} \\ \vdots & \vdots & & \vdots \\ {t_1}^{\alpha_k} & {t_2}^{\alpha_k} & \cdots & {t_k}^{\alpha_k} \end{pmatrix}, \quad \Psi_\alpha = \begin{pmatrix} \psi_{\alpha_1-k+1} & \psi_{\alpha_1-k+2} & \cdots & \psi_{\alpha_1} \\ \psi_{\alpha_2-k+1} & \psi_{\alpha_2-k+2} & \cdots & \psi_{\alpha_2} \\ \vdots & \vdots & & \vdots \\ \psi_{\alpha_k-k+1} & \psi_{\alpha_k-k+2} & \cdots & \psi_{\alpha_k} \end{pmatrix},$$

$$\Phi = \begin{pmatrix} (-1)^{k-1}\phi_{k-1}^{(1)} & (-1)^{k-1}\phi_{k-1}^{(2)} & \cdots & \cdots & (-1)^{k-1}\phi_{k-1}^{(k)} \\ (-1)^{k-2}\phi_{k-2}^{(1)} & (-1)^{k-2}\phi_{k-2}^{(2)} & \cdots & \cdots & (-1)^{k-2}\phi_{k-2}^{(k)} \\ \vdots & \vdots & & & \vdots \\ -\phi_1^{(1)} & -\phi_1^{(2)} & \cdots & \cdots & -\phi_1^{(k)} \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}$$

satisfy

$$T_\alpha = \Psi_\alpha \Phi. \tag{9}$$

To see this consider the polynomial

$$\phi^{(j)}(\lambda) = \sum_{i=0}^{k-1} \phi_i^{(j)} \lambda^i = \prod_{\nu \neq j} (1 + t_\nu \lambda).$$

By (3), we have

$$\psi(\lambda)\phi^{(j)}(-\lambda) = \frac{1}{1 - t_j\lambda} = \sum_{\nu=0}^{\infty} {t_j}^\nu \lambda^\nu.$$

Comparing coefficients one finds

$${t_j}^\nu = \psi_\nu - \psi_{\nu-1}\phi_1^{(j)} + \psi_{\nu-2}\phi_2^{(j)} \mp \cdots + (-1)^{k-1}\psi_{\nu-k+1}\phi_{k-1}^{(j)}$$

and this equivalent to (9). Taking determinants we obtain

$$\det(T_\alpha) = \det(\Psi_\alpha)\det(\Phi).$$

The result now follows from the identity $\det(\Phi) = \det(T_\delta)$ which is obtained by specializing to $\alpha = \delta = (k-1, k-2, \ldots, 1, 0)$ with $\det(\Psi_\delta) = 1$. □

Exercise 2.6 Use Lemma 2.5 to prove (6) for the complete symmetric functions. □

Exercise 2.7 Let θ_a be given by (7). Then

$$\theta_{1,\ldots,1,0,\ldots,0} = \phi_i, \qquad \theta_{j,0,\ldots,0} = \psi_j.$$

Give a direct proof of the first identity. **Hint:** Consider the coefficient of ${t_0}^{k-i}$ in the polynomial $\det({t_i}^j)_{i,j=0}^k = \prod_{0 \le i < j \le k}(t_j - t_i)$. □

Exercise 2.8 Prove that the polynomials θ_a are linearly independent. **Hint:** Denote by

$$\Theta_a = \det\left(({t_j}^{a_i+k-i})_{i,j=1}^k\right)$$

the numerator in (7). Use the identity

$$\Theta_{a_1,\ldots,a_k} = \sum_{i=1}^{k} (-1)^{i-1} {t_1}^{a_i+k-i} \Theta_{a_1+1,\ldots,a_{i-1}+1,a_{i+1},\ldots,a_k}(t_2,\ldots,t_k)$$

and prove, by induction over k, that the Θ_a with $|a| = d$ are linearly independent for each d. $\square$

Proposition 2.9 *The polynomials θ_a for $a_1 \geq \cdots \geq a_k \geq 0$ form a basis of the vector space $\mathcal{S}(k)$ of symmetric polynomials with complex coefficients. Hence every symmetric polynomial can be expressed as a polynomial in the ϕ_i or, alternatively, (nonuniquely) as a polynomial in the ψ_j.*

Proof: A basis of the space $\mathcal{S}_d(k)$ of symmetric polynomials of degree d is given by the polynomials

$$p_a(t) = \sum_{\sigma \in S_k} {t_{\sigma(1)}}^{a_1} \cdots {t_{\sigma(k)}}^{a_k}$$

where $a_1 \geq \cdots \geq a_k \geq 0$ with $|a| = \sum_i a_i = d$. Thus the dimension of $\mathcal{S}_d(k)$ is equal to the number of θ_a's of degree d and the result follows from the linear independence of the θ_a (Exercise 2.8). $\square$

2.4 Littlewood-Richardson rule

Structure constants

The product in $\mathcal{S}(k)$ can be expressed in terms of the structure constants N_{ab}^c defined by

$$\theta_a \theta_b = \sum_c N_{ab}^c \theta_c. \tag{10}$$

It turns out that these constants are uniquely determined by the relations (4) and the formula (8).

Lemma 2.10 *Let* $\mathbf{R}$ *be a commutative ring with unit. Suppose that the sequence $\psi_0 = 1, \psi_1, \psi_2, \ldots$ in* $\mathbf{R}$ *satisfies the relations (4). Then the elements*

$$\theta_{a_1,\ldots,a_k} = \det(\psi_{a_i+j-i})_{i,j=1}^k$$

satisfy (10) for any two partitions a, b of length k where the constants N_{ab}^c are the same as those in the Ring $\mathcal{S}(k)$ of symmetric polynomials.

Proof: The structure constants can be obtained in three steps. Firstly, use (8) to write θ_a and θ_b as sums of products of the form $\psi_{j_1} \cdots \psi_{j_k}$ with $j_1 \geq j_2 \geq \cdots \geq j_k \geq 0$ (at most k factors). Then the product $\theta_a \theta_b$ is a sum of products of the ψ_j with at most $2k$ factors in each summand. The second and crucial step is to express a product $\psi_{j_1} \cdots \psi_{j_m}$ with $m > k$ factors as a sum of such products with at most k factors. This can be done by induction over m and j_m using the relations (6). (It follows from Corollary 2.3 and Remark 2.4 that the ψ_j satisfy (6).) The third step is to express any product $\psi_{j_1} \cdots \psi_{j_k}$ as a linear combination of the polynomials θ_c for certain partitions c of length k. This third step follows by induction from (8). (See Exercise 2.11 below.) Combining these three steps one obtains an expression for $\theta_a \theta_b$ as a linear combination of the θ_c. The resulting coefficients are the structure constants N_{ab}^c. In all three steps the constants depend only on the formula (8) for the θ_a and on the relations (6) between the ψ_j but not on the particular ring in question. This proves the lemma. □

Exercise 2.11 Use (8) to prove that

$$\psi_j \psi_i = \theta_{j,i,0,\ldots,0} + \theta_{j+1,i-1,0,\ldots,0} + \cdots + \theta_{j+i,0,\ldots,0}, \qquad i \leq j.$$

Find an expression for $\psi_{j_1} \psi_{j_2} \psi_{j_3}$ as a linear combination of the θ_a. □

Littlewood-Richardson

Although the structure constants N_{ab}^c can, in principle, be determined from the proof of Lemma 2.10 this is, in practice, a highly nontrivial task. A beautiful algorithm for determining these constants was found by Littlewood and Richardson. To state the result we need some notation. The set of partitions $a \in \mathbb{Z}^k$ carries a natural partial order

$$a \prec c \qquad \Longleftrightarrow \qquad a_i \leq c_i \ \forall \ i.$$

Thus, if Y_a denotes denotes the Young diagram determined by a, then $a \prec c$ iff $Y_a \subset Y_c$. The set theoretic difference $Y_c - Y_a$ is called a **skew diagram**. A **tableau** T **of shape** $c - a$ is a labelling of the squares in the skew diagram $Y_c - Y_a$ by positive integers such that the labels are nondecreasing from left to right and strictly increasing from top to bottom. The **weight** of T is the vector $b = (b_1, \ldots, b_\ell)$ where b_i is the number of occurences of i in the tableau. Obviously the weight satisfies

$$|b| = |c| - |a|.$$

Associated to every tableau T is a **word** $w(T) = \lambda_1\lambda_2\cdots\lambda_N$ where the integers $\lambda_i > 0$ are obtained by reading the labels in the squares from right to left and in successive rows from top to bottom. The word $w(T)$ is called **monotone** if the number of occurences of the symbol i in each substring $\lambda_1\lambda_2\cdots\lambda_\nu$ is greater than or equal to the number of occurences of $i+1$. Note that if the word $w(T)$ is monotone then the number ℓ of labels is bounded above by the number k of rows. Moreover, in this case the weight b is a partition and it satisfies $b \prec c$. The following formula for the structure constants N_{ab}^c is the **Littlewood-Richardson rule**. A proof can be found in MacDonald [6], Section I.9. The result shows, in particular, that N_{ab}^c is always nonnegative and can only be nonzero if $a \prec c$, $b \prec c$, and $|a|+|b| = |c|$.

Theorem 2.12 (Littlewood-Richardson) *The constant N_{ab}^c is the number of Young tableaus T of shape $c-a$ and weight b such that the word $w(T)$ is monotone.*

2.5 Quotient rings

Consider the subspace

$$\mathcal{I}_\ell = \operatorname{span}\left\{\theta_a \,|\, a_1 > \ell\right\}.$$

Since $N_{ab}^c = 0$ whenever $a \not\prec c$ this subspace is an ideal and the quotient will be denoted by

$$\mathcal{S}(k,\ell) = \mathcal{S}(k)/\mathcal{I}_\ell.$$

This quotient can be identified with the subspace spanned by the θ_a with $\ell \geq a_1 \geq \cdots \geq a_k \geq 0$. With this interpretation the product is given by multiplication in $\mathcal{S}(k)$ followed by projection onto $\mathcal{S}(k,\ell)$. The dimension of $\mathcal{S}(k,\ell)$ as a complex vector space is

$$\dim \mathcal{S}(k,\ell) = \binom{k+\ell}{k}.$$

As a ring $\mathcal{S}(k,\ell)$ is generated by the polynomials $\phi_1,\dots,\phi_k$. But while $\mathcal{S}(k)=\mathbb{C}[\phi_1,\dots,\phi_k]$ is freely generated by the ϕ_i there are now relations $\theta_{a_1,\dots,a_k}=0$ whenever $a_1\geq\cdots\geq a_k\geq 0$ with $a_1>\ell$. One checks easily that these relations are equivalent to

$$\psi_j=\begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_j \\ 1 & \phi_1 & \phi_2 & \cdots & \phi_{j-1} \\ 0 & 1 & \phi_1 & \cdots & \phi_{j-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \phi_1 \end{vmatrix}=0 \quad \text{for} \quad j=\ell+1,\dots,\ell+k. \tag{11}$$

Namely, if (11) holds then, by developing the determinant expression for ψ_j with respect to the first row, we see that $\psi_j=0$ for all $j>\ell$. If this holds then the formula (8) shows that $\theta_a=0$ whenever $a_1>\ell$. (Compare with Remark 2.4.) It follows that the ring $\mathcal{S}(k,\ell)$ can be naturally identitified with the quotient

$$\mathcal{S}(k,\ell)\cong\frac{\mathbb{C}[\phi_1,\dots,\phi_k]}{\langle\psi_{\ell+1}=0,\dots,\psi_{\ell+k}=0\rangle}.$$

From an algebraic point of view there is now a natural symmetry between the variables $\phi_1,\dots,\phi_k$ and $\psi_1,\dots,\psi_\ell$.

Remark 2.13 There is a natural isomorphism

$$\mathcal{S}(k,\ell)\xrightarrow{\cong}\mathcal{S}(\ell,k)$$

which interchanges the roles of ϕ_i and ψ_j. In other words the isomorphism maps the elementary symmetric functions in $t_1,\dots,t_k$ to the complete symmetric functions in the variables $u_1,\dots,u_\ell$ and vice versa. By Lemma 2.2, this isomorphism maps

$$\theta_{a_1,\dots,a_k}(t_1,\dots,t_k)\mapsto\theta_{b_1,\dots,b_\ell}(u_1,\dots,u_\ell)$$

where the $b_j=\#\{i\,|\,a_i\geq j\}$ are given by the transpose Young diagram. $\Box$

3 Grassmannian

3.1 Symplectic quotient

Let $\mathrm{G}(k,n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. This manifold can be described as a symplectic quotient as follows. Consider the space $M = \mathbb{C}^{n\times k}$ of complex $n\times k$-matrices as a symplectic manifold with its standard complex and symplectic structures. The unitary group $\mathrm{U}(k)$ acts on $\mathbb{C}^{n\times k}$ on the right by $\Phi \mapsto \Phi U^{-1}$ for $U \in \mathrm{U}(k)$. This action is Hamiltonian and a moment map $\mu : \mathbb{C}^{n\times k} \to \mathfrak{u}(k)$ is given by

$$\mu(\Phi) = \frac{i}{2}(\Phi^*\Phi - \mathbb{1}).$$

Here we identify the Lie algebra $\mathfrak{g} = \mathfrak{u}(k)$ with its dual via the inner product $\langle \xi, \eta \rangle = \mathrm{trace}(\xi^*\eta)$ for $\xi, \eta \in \mathfrak{u}(k)$. The moment map has been normalized (by adding a central element) so that the zero set $\mu^{-1}(0)$ consists of unitary k-frames, i.e.

$$\mu^{-1}(0) = \mathcal{F}(k,n) = \left\{ \Phi \in \mathbb{C}^{n\times k} \,|\, \Phi^*\Phi = \mathbb{1} \right\}.$$

Thus the quotient is diffeomorphic the Grassmannian

$$\mathrm{G}(k,n) \cong \mathcal{F}(k,n)/\mathrm{U}(k) = \mathbb{C}^{n\times k}/\!\!/\mathrm{U}(k).$$

The diffeomorphism $\mathcal{F}(k,n)/\mathrm{U}(k) \to \mathrm{G}(k,n)$ is given by $\Phi \mapsto \Lambda = \mathrm{im}\,\Phi$.

3.2 Schubert cycles

Fix a complete flag

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

with $\dim V_\nu = \nu$. For any k-dimensional subspace $\Lambda \subset \mathbb{C}^n$ consider the subspaces $\Lambda \cap V_\nu$. Their dimensions form a nondecreasing sequence of integers $\lambda_\nu = \dim(\Lambda \cap V_\nu)$ with $\lambda_\nu \le \lambda_{\nu+1} \le \lambda_\nu + 1$ and $\lambda_0 = 0$, $\lambda_n = k$. Thus the jumps in the λ_ν form a strictly increasing sequence $0 < \nu_1 < \cdots < \nu_k \le n$ such that

$$\dim(\Lambda \cap V_\nu) = i, \qquad \nu_i \le \nu < \nu_{i+1},$$

for $i = 0, \ldots, k$ where $\nu_0 = 0$ and $\nu_{k+1} = n$. Thus $\nu_i \geq i$ is the smallest integer with $\dim(\Lambda \cap V_{\nu_i}) = i$. It is convenient to characterize the jumps by the decreasing sequence $a_i = n - k + i - \nu_i$. These numbers form a partition

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$$

and they are characterized by the condition

$$\dim(\Lambda \cap V_{n-k+i-a_i}) = i, \qquad \dim(\Lambda \cap V_{n-k+i-a_i-1}) = i - 1. \tag{12}$$

The **Schubert cycle** associated to the integer vector $a = (a_1, \ldots, a_k)$ and the flag $V = (V_0, \ldots, V_n)$ is the set of all k-planes $\Lambda \in \mathrm{G}(k, n)$ which satisfy (12). This set is a smooth submanifold of $\mathrm{G}(k, n)$ denoted by

$$\Sigma_a = \Sigma_a(V) = \Sigma^{k,n}_{a_1,\ldots,a_k}(V_0, \ldots, V_n)$$

For generic flags V and W the Schubert cycles $\Sigma_a(V)$ and $\Sigma_b(W)$ are transverse. Moreover, the Schubert cycles represent homology classes and these generate the integral homology of $\mathrm{G}(k, n)$ additively. More precisely, they have the following fundamental properties. Proofs can be found in Griffiths-Harris [4] and Milnor-Stasheff [7].

Theorem 3.1 **(i)** *Each Σ_a is a smooth submanifold of $\mathrm{G}(k, n)$ with*

$$\operatorname{codim}^c \Sigma_a = a_1 + \cdots + a_k = |a|.$$

(ii) *The closure of Σ_a is given by*

$$\overline{\Sigma}_a = \bigcup_{\substack{a' \\ a \prec a'}} \Sigma_{a'}$$

where $a \prec a'$ iff $a_i \leq a'_i$ for all i. Thus the Σ_a represent homology classes $\sigma_a = [\Sigma_a] \in H_{2k(n-k)-2|a|}(\mathrm{G}(k, n); \mathbb{Z})$.

(iii) *The classes σ_a generate $H_*(\mathrm{G}(k, n); \mathbb{Z})$ additively and, in particular,*

$$\dim\, H_*(\mathrm{G}(k, n); \mathbb{Z}) = \binom{n}{k}.$$

(iv) *The intersection number of 2-Schubert cycles σ_a and σ_b is given by*

$$\sigma_a \cdot \sigma_b = \begin{cases} 1, & \text{if } a_{k+1-i} + b_i = n - k \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

3.3 Duality

It is useful to examine the duality between $\mathrm{G}(k,n)$ and $\mathrm{G}(n-k,n)$ via the obvious diffeomorphism

$$f:\mathrm{G}(k,n)\to\mathrm{G}(n-k,n),\qquad f(\Lambda)=\Lambda^{\perp}.$$

The action on Schubert cycles corresponds to transposition of Young diagrams as follows.

Proposition 3.2 *Let the partitions $a\in\mathbb{Z}^k$ and $b\in\mathbb{Z}^{n-k}$ be related by $b_j=\#\{i\,|\,a_i\geq j\}$ as in (5) and define the flag W by $W_\nu=V_{n-\nu}^{\perp}$. Then the diffeomorphism $f:\mathrm{G}(k,n)\to\mathrm{G}(n-k,n)$ maps the Schubert cycle $\Sigma_a(V)$ to $\Sigma_b(W)$:*

$$f(\Sigma_a(V_0,\ldots,V_n))=\Sigma_b(V_n^{\perp},\ldots,V_0^{\perp}).$$

Proof: Let $\alpha_i=i+n-k-a_i$ and $\beta_j=j+b_{n-k+1-j}$. Then $\beta_j=j+\#\{i\,|\,\alpha_i\leq i+j-1\}$ and hence

$$\{1,\ldots,n\}=\{\alpha_1,\ldots,\alpha_k\}\cup\{\beta_1,\ldots,\beta_{n-k}\}.$$

Now let $\Lambda\in\Sigma_a(V)$ and denote

$$\lambda_\nu=\dim\left(\Lambda\cap V_\nu\right),\qquad \lambda'_\nu=\dim\left(\Lambda^{\perp}\cap V_{n-\nu}^{\perp}\right)=\nu-k+\lambda_{n-\nu}.$$

for $\nu=1,\ldots,n$. Then

$$\lambda_\nu\neq\lambda_{\nu-1}\quad\Longleftrightarrow\quad\nu\in\{\alpha_i\}.$$

Hence

$$\begin{aligned}
\lambda'_\nu\neq\lambda'_{\nu-1} &\Longleftrightarrow \lambda'_\nu-1=\lambda'_{\nu-1}\\
&\Longleftrightarrow \lambda_{n-\nu+1}=\lambda_{n-\nu}\\
&\Longleftrightarrow n-\nu+1\notin\{\alpha_i\}\\
&\Longleftrightarrow n-\nu+1\in\{\beta_j\}\\
&\Longleftrightarrow \nu\in\{n+1-\beta_j\}\\
&\Longleftrightarrow \nu\in\{n+1-j-b_{n-k+1-j}\}\\
&\Longleftrightarrow \nu\in\{j+k-b_j\}
\end{aligned}$$

This means that $\Lambda^{\perp}$ is an element of the Schubert cycle in $\mathrm{G}(n-k,n)$ associated to b and W. $\square$

3.4 Tautological bundles

Consider the canonical vector bundles

$$E \to \mathrm{G}(k,n), \qquad F \to \mathrm{G}(k,n)$$

with fibres $E_\Lambda = \Lambda$ and $F_\Lambda = \Lambda^\perp$ over $\Lambda \in \mathrm{G}(k,n)$. Thus E can be identified with the quotient

$$E \cong \frac{\mathcal{F}(k,n) \times \mathbb{C}^k}{\mathrm{U}(k)}$$

where $\mathrm{U}(k)$ acts on $(\Phi, z) \in \mathcal{F}(k,n) \times \mathbb{C}^k$ via $[\Phi, z] \equiv [\Phi U, U^{-1}z]$. The correspondence is given by $[\Phi, z] \mapsto (\operatorname{im}\Phi, \Phi z)$. Let us denote the Chern classes of the dual bundles by

$$c_i = c_i(E^*), \qquad d_j = c_j(F^*)$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$. These classes are related to the homology classes σ_a as follows. For a proof see Milnor-Stasheff [7].

Proposition 3.3 *The Chern classes of E^* and F^* are given by*

$$c_i(E^*) = \mathrm{PD}(\xi_i), \qquad c_j(F^*) = (-1)^j \mathrm{PD}(\eta_j)$$

where ξ_i and η_j denote the homology classes of the special Schubert cycles

$$\xi_i = \sigma_{1,\ldots,1,0,\ldots,0}, \qquad \eta_j = \sigma_{j,0,\ldots,0}$$

(with 1 occurring i times in the first case).

3.5 Giambelli's formula

Each homology class σ_a can be expressed as a polynomial in the ξ_i or respectively in the η_j. An explicit formula for this polynomial was found by Giambelli and this is the contents of the following theorem. A proof of the first identity can be found in Griffiths-Harris [4], page 205, and the second identity follows from the first and Lemma 2.2.

Theorem 3.4 (Giambelli) *Fix integers $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ and let $k \geq b_1 \geq \cdots \geq b_{n-k} \geq 0$ be defined by (5). Then*

$$
\begin{aligned}
\sigma_{a_1,\dots,a_k} &= \begin{vmatrix} \eta_{a_1} & \eta_{a_1+1} & \eta_{a_1+2} & \cdots & \eta_{a_1+k-1} \\ \eta_{a_2-1} & \eta_{a_2} & \eta_{a_2+1} & \cdots & \eta_{a_2+k-2} \\ \eta_{a_3-2} & \eta_{a_3-1} & \eta_{a_3} & \cdots & \eta_{a_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_{a_k-k+1} & \eta_{a_k-k+2} & \eta_{a_k-k+3} & \cdots & \eta_{a_k} \end{vmatrix} \\
&= \begin{vmatrix} \xi_{b_1} & \xi_{b_1+1} & \xi_{b_1+2} & \cdots & \xi_{b_1+n-k-1} \\ \xi_{b_2-1} & \xi_{b_2} & \xi_{b_2+1} & \cdots & \xi_{b_2+n-k-2} \\ \xi_{b_3-2} & \xi_{b_3-1} & \xi_{b_3} & \cdots & \xi_{b_3+n-k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{b_{n-k}-n+k+1} & \xi_{b_{n-k}-n+k+2} & \xi_{b_{n-k}-n+k+3} & \cdots & \xi_{b_{n-k}} \end{vmatrix}.
\end{aligned}
$$

Here multiplication is to be understood as the intersection product (i.e. the Poincaré dual of the cup-product of the Poincaré duals).

Giambelli's theorem shows that the c_i generate the cohomology of $\mathrm{G}(k,n)$ multiplicatively. Relations arise from the identities

$$\sum_{i=1}^{j} c_i d_{j-i} = 0$$

for $j = 1, \dots, n$. For $j = 1, \dots, n-k$ these identities determine the d_j as functions of the c_i and for $j = n-k+1, \dots, n$ they become relations for the c_i which can be expressed in the form

$$d_j = \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_j \\ 1 & c_1 & c_2 & \cdots & c_{j-1} \\ 0 & 1 & \psi_1 & \cdots & c_{j-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & c_1 \end{vmatrix} = 0, \qquad \text{for all} \quad j > n-k. \tag{13}$$

(Compare with equation (4).) That these are the only relations follows from dimensional considerations. Thus the cohomology ring of the Grassmannian can be naturally identified with

$$H^*(\mathrm{G}(k,n);\mathbb{Z}) = \frac{\mathbb{Z}[c_1, \dots, c_k]}{\langle d_{n-k+1} = 0, \dots, d_n = 0 \rangle}.$$

Giambelli's formula can also be used to prove that the products

$$c^m = {c_1}^{m_1} \wedge \ldots \wedge {c_k}^{m_k}$$

with $m_1 + \cdots + m_k \leq n - k$ form an additive basis of the cohomology of $\mathrm{G}(k,n)$. Moreover, Mielke and Whitehouse conjectured that the quantum product of up to $n-k$ of the classes c_i agrees with the ordinary cup-product (c.f. [9]). Combining this with the computation by Witten [12] and Siebert-Tian [10] of the quantum cohomology of $\mathrm{G}(k,n)$ as an abstract ring would then determine the quantum product structure in the basis $\mathrm{PD}(\sigma_a)$.

Exercise 3.5 Check that Giambelli's formula for $a = (j, 0, \ldots, 0)$ is equivalent to $\sum_{i=0}^{j} c_i(E^*)c_{j-i}(F^*) = 0$.

Exercise 3.6 Prove that $\int_{\mathrm{G}(k,n)} c_k(E^*)^{n-k} = 1$.

4 Representations

4.1 Root systems

Let G be a compact Lie group and $T \subset \mathrm{G}$ be a maximal torus. Denote the corresponding Lie algebras by $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{t} = \mathrm{Lie}(T)$ and let $\Lambda \subset \mathfrak{t}$ denote the integral lattice (i.e. $\tau \in \Lambda$ iff $\exp(\tau) = 1$). Consider the root space decomposition

$$\mathfrak{g}^c = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

of the complexified Lie algebra where the α are linear functionals $\alpha : \mathfrak{t} \to i\mathbb{R}$ with $\alpha(\Lambda) \subset 2\pi i\mathbb{Z}$ and

$$[\tau, \xi] = \alpha(\tau)\xi \qquad \text{for} \qquad \tau \in \mathfrak{t},\ \xi \in \mathfrak{g}^\alpha.$$

The basic facts are that $\alpha \in \Delta$ iff $-\alpha \in \Delta$ with $\mathfrak{g}^{-\alpha} = \bar{\mathfrak{g}}^\alpha$ and if $[\xi, \eta] \neq 0$ for $\xi \in \mathfrak{g}^\alpha$ and $\eta \in \mathfrak{g}^\beta$ then $\alpha + \beta \in \Delta$ with $\mathfrak{g}^{\alpha+\beta} = [\mathfrak{g}^\alpha, \mathfrak{g}^\beta]$. Thus there is a decomposition $\Delta = \Delta^+ \cup \Delta^-$ such that

$$\alpha \in \Delta^+ \qquad \Longleftrightarrow \qquad -\alpha \in \Delta^-$$

and

$$\alpha, \beta \in \Delta^+, \alpha + \beta \in \Delta \qquad \Longrightarrow \qquad \alpha + \beta \in \Delta^+.$$

Call Δ^+ the **set of positive roots**. Choosing a set of positive roots is equivalent to choosing a **Borel subgroup** $\mathrm{B} \subset \mathrm{G}^c$ with Lie algebra

$$\mathfrak{b} = \mathrm{Lie}(\mathrm{B}) = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha.$$

Note that this subgroup is invariant under conjugation by elements of the torus. Denote the opposite Borel subgroup by $\bar{\mathrm{B}} \subset \mathrm{G}^c$. Its Lie algebra is

$$\bar{\mathfrak{b}} = \mathrm{Lie}(\bar{\mathrm{B}}) = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^\alpha.$$

Note that both G^c/B and $\mathrm{G}^c/\bar{\mathrm{B}}$ can be identified with G/T and this determines two opposite complex structures on this quotient. We shall denote by J_B the complex structure obtained by the identification

$$\mathrm{G}/T \cong \mathrm{G}^c/\bar{\mathrm{B}}.$$

This corresponds to the identification of the tangent space of G/T at 1 with the complex vector space $\oplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$.

Weyl group

Different choices of Borel subgroups containing T are related by the conjugate action of the Weyl group. Denote by

$$N(T) = \{g \in \mathrm{G} \mid g\mathfrak{t}g^{-1} = \mathfrak{t}\}$$

the normalizer of the torus. Any $g \in N(T)$ gives rise to an automorphism $s_g : T \to T$ defined by

$$s_g(t) = gtg^{-1}$$

for $t \in T$. The **Weyl group** W is the finite group of these automorphisms. Thus

$$W = N(T)/T = \{s_g \mid g \in N(T)\}$$

where T acts on the right. For $s \in W$ the induced automorphism of the Lie algebra will still be denoted by $s : \mathfrak{t} \to \mathfrak{t}$ and $\det(s) \in \{\pm 1\}$ denotes its determinant. The proof of the following proposition is an easy exercise.

Proposition 4.1 **(i)** *If $g \in N(T)$ and $\alpha \in \Delta$ then $\alpha \circ s_g \in \Delta$ with*

$$\mathfrak{g}^{\alpha \circ s_g} = g^{-1} \mathfrak{g}^{\alpha} g.$$

(ii) *If $g \in N(T)$ and $\mathrm{B} \subset \mathrm{G}^c$ is a Borel subgroup for T then so is $g^{-1}\mathrm{B}g$. The corresponding system of positive roots is given by*

$$\Delta^{+}_{g^{-1}\mathrm{B}g} = \left\{ \alpha \circ s_g \,|\, \alpha \in \Delta^{+}_{\mathrm{B}} \right\}.$$

Moreover, $g^{-1}\mathrm{B}g = \mathrm{B}$ if and only if $g \in T$.

(iii) *For any two Borel subgroups $\mathrm{B}, \mathrm{B}' \subset \mathrm{G}^c$ containing the same torus T there exists a $g \in N(T)$ such that $\mathrm{B}' = g^{-1}\mathrm{B}g$. In particular, there exists a $g \in N(T)$ such that*

$$\bar{\mathrm{B}} = g^{-1}\mathrm{B}g, \qquad g^2 = 1.$$

Simple roots

A positive root $\alpha \in \Delta^+$ is called **simple** if it cannot be written as a sum $\alpha = \alpha' + \alpha''$ with $\alpha', \alpha'' \in \Delta^+$. Let us fix a collection of simple roots $\alpha_1, \ldots, \alpha_\ell$. These roots together with the centre span the dual torus $\mathfrak{t}^*$. Let us now fix an invariant inner product on $\mathfrak{g}$ and for each root $\alpha \in \Delta$ choose a vector $\eta_\alpha \in \mathfrak{t}^c$ such that

$$\alpha(\tau) = \langle \eta_\alpha, \tau \rangle$$

for $\tau \in \mathfrak{t}^c$ and define

$$h_\alpha = \frac{2\eta_\alpha}{|\eta_\alpha|^2} \in \mathfrak{t}^c. \tag{14}$$

Then the vectors h_{α_i} together with the center span the complexified torus $\mathfrak{t}^c$.

4.2 Irreducible representations

Let us fix a maximal torus $T \subset \mathrm{G}$ and a Borel subgroup $\mathrm{B} \subset \mathrm{G}^c$ with corresponding system Δ^+ of positive roots. Let V be a Hermitian vector space and

$$\rho : \mathrm{G} \to \mathrm{Aut}(V)$$

be a unitary representation. Denote by $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ the corresponding Lie algebra homomorphism. There is a natural decomposition

$$V = \bigoplus_{\lambda \in \Sigma} V^{\lambda}$$

into the eigenspaces under the action of the torus T. The subspaces V^λ are labelled by the linear maps $\lambda : \mathfrak{t} \to i\mathbb{R}$ which satisfy

$$\dot\rho(\tau)v = \lambda(\tau)v \qquad \text{for} \qquad \tau \in \mathfrak{t},\ v \in V^\lambda.$$

The λ's are called the **weights** of the representation. It is easy to check that if $\lambda \in \Sigma$ is a weight and $\alpha \in \Delta$ is a root such that $\dot\rho(\mathfrak{g}^\alpha)V^\lambda \neq \{0\}$ then $\lambda+\alpha$ is again a weight and

$$\dot\rho(\mathfrak{g}^\alpha)V^\lambda \subset V^{\lambda+\alpha}.$$

The following proposition summarizes the fundamental properties of the weight systems of irreducible representations.

Theorem 4.2 *Let $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ be an irreducible representation with weight system $\Sigma \subset \mathrm{Hom}(\mathfrak{t}, i\mathbb{R})$ and fix a Borel subgroup $\mathrm{B} \subset \mathrm{G}^c$ for the maximal torus $T \subset \mathrm{G}$. Then the following holds.*

(i) *There exists a unique weight $\lambda \in \Sigma$ (called the* **highest weight** *with respect to B) such that $\lambda+\alpha \notin \Sigma$ for all $\alpha \in \Delta_{\mathrm{B}}^+$. The corresponding* **weight space** *V^λ is one-dimensional.*

(ii) *Two irreducible representations with the same highest weights are isomorphic.*

(iii) *If $h_\alpha \in \mathfrak{t}^c$ is defined by (14) for $\alpha \in \Delta$ then $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\lambda \in \Sigma$ and all $\alpha \in \Delta$. Moreover, if λ is the highest weight then*

$$\lambda(h_\alpha) > 0 \qquad \textit{for} \qquad \alpha \in \Delta_{\mathrm{B}}^+.$$

(iv) *If G has a discrete centre and $\alpha_1, \ldots, \alpha_\ell \in \Delta_{\mathrm{B}}^+$ are the simple roots, then for any nonnegative integers $m_1, \ldots, m_\ell \geq 0$ there exists a unique irreducible representation (up to isomorphism) with highest weight λ satisfying*

$$\lambda(h_{\alpha_i}) = m_i$$

for $i = 1, \ldots, \ell$.

The previous theorem shows that every highest weight is a nonnegative linear combination of the minimal highest weights $\mu_i : \mathfrak{t} \to \sqrt{-1}\mathbb{R}$ defined by

$$\mu_i(h_{\alpha_j}) = \delta_{ij}.$$

With m_i as above the highest weight λ is given by

$$\lambda = \sum_{i=1}^{\ell} m_i \mu_i.$$

Sometimes it is convenient to denote a representation $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ simply by the vector space V. In many cases the action is obvious (e.g. the unitary group $\mathrm{U}(k)$ acts in an obvious way on $\Lambda^i \mathbb{C}^k$ and $S^j \mathbb{C}^k$). The highest weight of a representation V with respect to B will sometimes be denoted by $\lambda_{V,\mathrm{B}}$. Also, if the Borel subgroup $\mathrm{B} \subset \mathrm{G}^c$ is clear from the context (such as the group of upper triangular matrices in the unitary case) then we shall denote by V_λ the representation with highest weight λ. The next proposition concerns the action of the Weyl group on the the weight spaces. The proof is an easy exercise.

Proposition 4.3 **(i)** *If λ is a weight of V and $g \in N(T)$ then $\lambda \circ s_g$ is a weight of V with weight space*

$$V^{\lambda \circ s_g} = \rho(g^{-1}) V^{\lambda}.$$

(ii) *For every $g \in N(T)$ and every Borel subgroup $B \subset \mathrm{G}^c$ with $T \subset \mathrm{B}$*

$$\lambda_{V, g^{-1}\mathrm{B}g} = \lambda_{V,\mathrm{B}} \circ s_g.$$

Duality

Associated to a representation $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ is the dual or *contragredient* representation $\bar{\rho} : \mathrm{G} \to \mathrm{Aut}(\bar{V})$ where one can think of $\bar{V}$ either as the real vector space V with the reversed complex structure or as the dual space and then

$$\bar{V} = \mathrm{Hom}(V, \mathbb{C}), \qquad \bar{\rho}(g) = \rho(g)^{*-1}.$$

Thus, if $\phi : V \to \mathbb{C}$ is a complex linear map then $\bar{\rho}(g)\phi = \phi \circ \rho(g)^{-1}$. It is interesting to determine the highest weight of $\bar{V}$ with respect to the original Borel subgroup B. Firstly, note that the weights of $\bar{V}$ are given by

$$\Sigma_{\bar{V}} = \{-\lambda \,|\, \lambda \in \Sigma_V\}.$$

This implies that $\lambda_{\bar{V},\bar{\mathrm{B}}} = -\lambda_{V,B}$. Hence, by Proposition 4.3 the highest weight of $\bar{V}$ with respect to B is given by

$$\lambda_{\bar{V},\mathrm{B}} = -\lambda_{V,\mathrm{B}} \circ \mathrm{ad}(g), \tag{15}$$

where $g \in N(T)$ is chosen such that $g^{-1}\mathrm{B}g = \bar{\mathrm{B}}$.

4.3 Borel-Weil theory

Fix a maximal torus $T \subset \mathrm{G}$. A linear functional $\lambda : \mathfrak{t} \to i\mathbb{R}$ is called a **weight** if $\exp(\tau) = 1$ implies $\lambda(\tau) \in 2\pi i\mathbb{Z}$. For each weight λ there exists a unique homomorphism

$$\chi_\lambda : T \to S^1$$

such that $\dot{\chi}_\lambda = \lambda$. Thus every character λ gives rise to a complex line bundle

$$L = L_\lambda = \mathrm{G} \times_{\chi_\lambda} \mathbb{C} \to \mathrm{G}/T$$

where T acts on $\mathbb{C}$ by ${\chi_\lambda}^{-1}$. A point in L is an equivalence class of pairs $[g, z]$ where $g \in \mathrm{G}^c$ and $z \in \mathbb{C}$ under the equivalence relation $[g, z] \equiv [gt, \chi_\lambda(t)^{-1}z]$ for $t \in T$. Now fix a system Δ^+ of positive roots and denote the corresponding Borel subgroup by B. Recall that B determines a complex structure J_B on $\mathrm{G}/T \cong \mathrm{G}^c/\bar{\mathrm{B}}$. The Borel-Weil theorem asserts that a representation V with highest weight λ is isomorphic to the space $H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L_\lambda)$ of holomorphic sections of L_λ with a suitable holomorphic structure.

Remark 4.4 **(i)** There is a conjugation $\mathrm{G}^c \mapsto \mathrm{G}^c : g \mapsto \bar{g}$ on the complexified Lie group which extends the obvious conjugation on the Lie algebra $\mathfrak{g}^c$. This conjugation maps the Borel subgroup B to $\bar{\mathrm{B}}$ and the extension $\rho : \mathrm{G}^c \to \mathrm{Aut}(V)$ of a unitary representation satisfies

$$\rho(\bar{g})^{-1} = \rho(g)^*$$

for $g \in \mathrm{G}^c$.
(ii) For any weight λ the homomorphism $\chi_\lambda : T \to S^1$ extends uniquely to B (but not in general to G^c). This extension is determined by the extension of λ to $\mathfrak{b}$ via $\lambda(\mathfrak{g}^\alpha) = 0$ for $\alpha \in \Delta^+$. Similarly, there is an extension to $\bar{\mathrm{B}}$ which will also be denoted by $\chi_\lambda : \bar{\mathrm{B}} \to \mathbb{C}^*$. Both extensions evidently agree on $\mathrm{B} \cap \bar{\mathrm{B}} = T^c$ and they are related under conjugation by

$$\chi_\lambda(\bar{b})^{-1} = \overline{\chi_\lambda(b)}$$

for $b \in B$. □

The above line bundle $L = L_\lambda \to \mathrm{G}/T$ can be identified with the quotient

$$L_\lambda = \frac{\mathrm{G} \times \mathbb{C}}{T} \cong \frac{\mathrm{G}^c \times \mathbb{C}}{\bar{\mathrm{B}}}$$

where $\bar{\mathrm{B}}$ and T act on $\mathbb{C}$ via ${\chi_\lambda}^{-1}$. Thus a point in L is an equivalence class of pairs $[g, z]$ where $g \in \mathrm{G}^c$ and $z \in \mathbb{C}$ under the equivalence relation $[g, z] \equiv [g\bar{b}, \chi_\lambda(\bar{b})^{-1} z]$ for $\bar{b} \in \bar{\mathrm{B}}$. A section of L can be expressed as a function $f : \mathrm{G}^c \to \mathbb{C}$ which satisfies

$$f(g\bar{b}) = \chi_\lambda(\bar{b})^{-1} f(g) \tag{16}$$

for $g \in \mathrm{G}^c$ and $\bar{b} \in \bar{\mathrm{B}}$. A holomorphic section is simply a holomorphic map which satisfies (16). The space of holomorphic sections will be denoted by $H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L_\lambda)$.

Theorem 4.5 (Borel-Weil) *Let $\rho_\lambda : \mathrm{G} \to \mathrm{Aut}(V_\lambda)$ be an irreducible representation with highest weight λ. Then there exists an isomorphism*

$$V_\lambda \cong H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L_\lambda).$$

It is easy to write down a linear map $V \to H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L)$ which is equivariant with respect to the two actions of G^c. Just fix a highest weight vector $v_0 \in V^\lambda$ with respect to B such that $\dot{\rho}(\tau)v_0 = \lambda(\tau)v_0$ for $\tau \in \mathfrak{t}$. The definition of *highest weight* then implies that $\dot{\rho}(\mathfrak{g}^\alpha)v_0 = 0$ for $\alpha \in \Delta^+$ and it follows that

$$\rho(b)v_0 = \chi_\lambda(b)v_0$$

for $b \in \mathrm{B}$. Now consider the map $V \to H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L) : v \mapsto f_v$ defined by

$$f_v(g) = \langle \rho(\bar{g})v_0, v \rangle \tag{17}$$

for $v \in V$ and $g \in \mathrm{G}^c$. Then it is easy to check that f_v satisfies (16) and that the map $v \mapsto f_v$ intertwines the two actions of G^c. That this map is bijective is a consequence of the fact that irreducible representations are uniquely determined by their highest weights and this will not be discussed here.

Remark 4.6 In terms of Borel-Weil theory the dual representation of $V \cong H^0(\mathrm{G}^c/\bar{\mathrm{B}}, L)$ (with highest weight λ with respect to B) is given by

$$\bar{V} \cong H^0(\mathrm{G}^c/\mathrm{B}, \bar{L}).$$

Here the bundle $\bar{L} \to \mathrm{G}^c/\mathrm{B}$ can be explicitly represented as the quotient $\bar{L} = \mathrm{G}^c \times \mathbb{C}/\mathrm{B}$ under the equivalence relation $[g, z] \equiv [gb, \chi_\lambda(b)z]$ for $b \in \mathrm{B}$. Thus the dual representation is obtained by both reversing the complex structure on G/T and replacing L by the dual bundle. An explicit isomorphism

$$\bar{V} \to H^0(\mathrm{G}^c/\mathrm{B}, \bar{L}) : \phi \mapsto f_\phi$$

is given by

$$f_\phi(g) = \phi(\rho(g)v_0)$$

with $v_0 \in V$ as above. □

4.4 Weyl character formula

The **character** of a unitary representation $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ is the function $\theta_\rho : \mathrm{G} \to \mathbb{C}$ defined by

$$\theta_\rho(g) = \mathrm{trace}^c(\rho(g)).$$

This function is invariant under conjugation and, since every $g \in \mathrm{G}$ is conjugate to some element of the maximal torus T, the character is uniquely determined by its restriction to T. This restriction is still invariant under the action of the Weyl group. The map $\rho \mapsto \theta_\rho$ is a ring homomorphism, i.e.

$$\theta_{\rho_1 \oplus \rho_2} = \theta_{\rho_1} + \theta_{\rho_2}, \qquad \theta_{\rho_1 \otimes \rho_2} = \theta_{\rho_1}\theta_{\rho_2}. \tag{18}$$

Evidently, the dimension of V is the value of the character at $g = 1$.

Theorem 4.7 *Two representations of a compact Lie group* G *are isomorphic if and only if they have the same character.*

Fix a system Δ^+ of positive roots with Borel subgroup $\mathrm{B} \subset \mathrm{G}^c$. Weyl's character formula expresses the character θ_{ρ_λ} of an irreducible representation $\rho_\lambda : \mathrm{G} \to \mathrm{Aut}(V_\lambda)$ with heighest weight λ (with respect to B) as a weighted average of the characters $\chi_\lambda \circ s$ over the Weyl group W. More precisely, define $A_\lambda : T \to \mathbb{C}$

$$A_\lambda(t) = \sum_{s \in W} \det(s)\chi_\lambda \circ s(t).$$

Note that this function vanishes at the identity $t = 1$. Taking λ equal to the sum

$$\delta = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha \tag{19}$$

one finds that

$$A_\delta(t) = \sum_{s \in W} \det(s) \chi_\delta \circ s(t) = \chi_\delta(t) \prod_{\alpha \in \Delta^+} \left(1 - \chi_\alpha(t)^{-1}\right).$$

The next theorem is the required character formula.

Theorem 4.8 (Weyl character formula) *If $\rho_\lambda : \mathrm{G} \to \mathrm{Aut}(V_\lambda)$ is the irreducible representation with highest weight λ then* [1]

$$\theta_{\rho_\lambda}(t) = \frac{A_{\lambda+\delta}(t)}{A_\delta(t)}.$$

Theorem 4.9 (Weyl dimension formula) *The dimension of the irreducible representation V_λ with highes weight λ is given by*

$$\dim V_\lambda = \theta_{\rho_\lambda}(1) = \frac{\prod_{\alpha \in \Delta^+} \langle \lambda + \delta, \alpha \rangle}{\prod_{\alpha \in \Delta^+} \langle \delta, \alpha \rangle}.$$

4.5 Unitary group

Borel subgroup

Consider the unitary group $\mathrm{G} = \mathrm{U}(k)$ with maximal torus T consisting of the unitary diagonal matrices. Then $\mathfrak{t}$ is the space of diagonal matrices with imaginary entries and we denote by $\varepsilon_i : \mathfrak{t} \to \sqrt{-1}\mathbb{R}$ the evaluation of the i-th diagonal entry. The roots are the functionals $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ with $i \neq j$ and $\mathfrak{g}^{\varepsilon_i - \varepsilon_j}$ is the space of matrices whose (i,j)-entry is the only nonzero one. Let us choose

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j\}$$

as the system of positive roots. The simple roots have the form $\varepsilon_i - \varepsilon_{i+1}$ and the corresponding matrices $h_{\varepsilon_i - \varepsilon_{i+1}} \in \mathfrak{t}^c$ are given by

$$h_{\varepsilon_i - \varepsilon_{i+1}} = \mathrm{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0).$$

In this case $\mathfrak{g}^c = \mathbb{C}^{k \times k}$, $\mathrm{G}^c = \mathrm{GL}(k, \mathbb{C})$, and $B \subset \mathrm{G}^c$ is the subgroup of upper triangular matrices with nonzero diagonal entries. The dual Borel subgroup $\bar{\mathrm{B}}$ is the group of lower triangular matrices.

[1] If the center is discrete then δ is a weight despite the factor $1/2$. In general, δ may differ from a weight by a central element which cancels in the quotient.

Irreducible representations

The minimal highest weights are

$$\mu_i = \varepsilon_1 + \cdots + \varepsilon_i$$

for $i = 1, \ldots, k-1$. A general highest weight has the form

$$\lambda = \sum_{i=1}^{k-1} m_i \mu_i = \sum_{i=1}^{k-1} a_i \varepsilon_i$$

where $a_i = m_i + \cdots + m_{k-1}$ and thus $a_1 \geq \cdots \geq a_{k-1} \geq 0$. To describe representations of $\mathrm{U}(k)$ we must allow for the action of the center and obtain highest weights of the form

$$\lambda = \sum_{i=1}^{k} m_i \mu_i = \sum_{i=1}^{k} a_i \varepsilon_i$$

where $\mu_k = \varepsilon_1 + \cdots + \varepsilon_k$ and $a_i = m_i + \cdots + m_{k-1}$. Thus $a_1 \geq \cdots \geq a_k$ where the integer $a_k = m_k$ need not be positive. The action of the center is given by the sum of the a_i.

Throughout we denote by $\rho_a : \mathrm{U}(k) \to \mathrm{Aut}(V_a)$ the irreducible representation with highest weight $\lambda = \sum_i a_i \varepsilon_i$. This representation can be explicitly realized as a subspace

$$V_a \subset (\Lambda^1 \mathbb{C}^k)^{\otimes m_1} \otimes (\Lambda^2 \mathbb{C}^k)^{\otimes m_2} \otimes \cdots \otimes (\Lambda^k \mathbb{C}^k)^{\otimes m_k}$$

where $m_i = a_i - a_{i+1}$. The tensor product on the right contains a one-dimensional subspace W^λ (namely the tensor product of the subspaces $(\mathbb{C}e_1 \wedge \ldots \wedge e_i)^{\otimes m_i}$) on which $\mathrm{U}(k)$ acts with weight $\lambda = \sum_i m_i \mu_i$. The representation V can then be defined as the smallest subspace which contains W^λ and is invariant under $\mathrm{U}(k)$.

Example 4.10 Of particular interest are the special representations

$$V_{1,\ldots,1,0,\ldots,0} = \Lambda^i \mathbb{C}^k, \qquad V_{j,0,\ldots,0} = S^j \mathbb{C}^k,$$

with 1 occurring i times in the first case. In particular, $V_{0,\ldots,0} = \mathbb{C}$ is the trivial representation (the multiplicative unit in the representation ring). $\square$

Duality

Let $\mathrm{G} = \mathrm{U}(k)$ and $\mathrm{B} \subset \mathrm{GL}(k, \mathbb{C})$ be the group of upper triangular matrices. Recall from Proposition 4.1 that there exists a $g \in \mathrm{U}(k)$ such that $g^{-1}\mathrm{B}g = \bar{\mathrm{B}}$ where $\bar{\mathrm{B}}$ is the subgroup of lower triangular matrices. The required matrix $\mathfrak{g}$ is the anti-diagonal

$$g = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & & & & & 0 & 1 & 0 \\ \cdot & & & & \cdot & 1 & 0 & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & 0 & 1 & \cdot & & & & \cdot \\ 0 & 1 & 0 & & & & & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

If $V_a = V_{a_1,\ldots,a_k}$ denotes the representation of $\mathrm{U}(k)$ with heighest weight $\lambda = \sum_i a_i \varepsilon_i$ with respect to B then the dual representation is given by

$$\bar{V}_a = V_{a^*}$$

where the integers $a_1^* \geq a_2^* \geq \cdots \geq a_k^*$ are given by

$$a_i^* = -a_{k+1-i}$$

for $i = 1, \ldots, k$. Note, in particular, that the sum of the a_i changes sign under $a \mapsto a^*$ and this corresponds to the fact that in the dual representation the action of the center is reversed.

Flag manifolds and Borel-Weil theory

The Borel subgroup $B \subset \mathrm{G}^c$ (of upper triangular matrices) is the stabilizer of the standard flag $\mathbb{C}^0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^k$ (where $\mathbb{C}^i$ is identified with the subspace of all vectors of the form $(z_1, \ldots, z_i, 0, \ldots, 0) \in \mathbb{C}^k$). Hence the quotient G^c/B can be naturally identified with the flag manifold

$$F(k) = \left\{ E = \{E_i\}_{i=0}^k \,|\, E_0 \subset E_1 \subset \cdots \subset E_k,\, \dim^c E_i = i \right\}.$$

The diffeomorphism $\mathrm{G}^c/B \to F(k)$ is given by $g \mapsto \{g\mathbb{C}^i\}_i$. Consider the character $\chi_a : B \to \mathbb{C}^*$ given by $\chi_a(b) = \prod_i {b_{ii}}^{a_i}$ and associated to the highest weight $\lambda_a = \sum_i a_i\varepsilon_i$ with

$$a_1 \geq a_2 \geq \cdots \geq a_k.$$

One checks easily that the line bundle $L = L_a = \mathrm{G}^c \times_{\chi_a} \mathbb{C} \to \mathrm{G}^c/B \cong F(k)$ is the bundle whose fibre over a flag $E = \{E_i\}_i$ is the line

$$L_E = \bigotimes_{i=1}^{k} L_i{}^{\otimes a_i} \cong \bigotimes_{i=1}^{k} (\Lambda^i E_i)^{\otimes m_i}, \qquad L_i = E_i/E_{i-1}, \quad m_i = a_i - a_{i+1}.$$

Thus the representation V_a is given by

$$\bar{V}_a \cong H^0(F(k), L_a{}^*).$$

Consider, for example, the case $a = (1, \ldots, 1, 0, \ldots, 0)$ with 1 occurring i times so that $L_{aE}^* = \Lambda^i E_i^*$. Then any exterior i-form $\alpha \in \Lambda^i(\mathbb{C}^k)^*$ induces a holomorphic section of $s : F(k) \to L_a^*$ by restriction to E^i and this is the required isomorphism between $\bar{V}_a = \Lambda^i(\mathbb{C}^k)^*$ and $H^0(F(k), L_a^*)$.

Weyl character formula

Recall that the character $\theta_\rho : \mathrm{U}(k) \to \mathbb{C}$ of a representation $\rho : \mathrm{U}(k) \to \mathrm{Aut}(V)$ is uniquely determined by its restiction to the maximal torus and is invariant under the action of the Weyl group. In the case at hand the maximal torus is the group of diagonal matrices and the Weyl group is the symmetric group acting on the diagonal entries $t_1, \ldots, t_k$ by permutation. Hence the character of a finite dimensional representation of $\mathrm{U}(k)$ can be thought of a symmetric function in k variables. The Weyl character formula asserts that the character of the irreducible representation V_a with weight $\lambda = \sum_i a_i\varepsilon_i$ where $a_1 \geq \cdots \geq a_k \geq 0$ agrees with the symmetric polynomial θ_a introduced in (7).

Theorem 4.11 (Weyl) *Let $a_1 \geq \cdots \geq a_k \geq 0$. Then the character of the irreducible representation $\rho_a : \mathrm{U}(k) \to \mathrm{Aut}(V_a)$ with weight $\lambda = \sum_i a_i\varepsilon_i$ is the symmetric polynomial (7):*

$$\theta_{\rho_a} = \theta_a.$$

Proof: To apply the Weyl character formula to the unitary case note first that the linear functional $\delta : \mathfrak{t} \to i\mathbb{R}$ in (19) is given by

$$\delta = \sum_{j=1}^{k}(k-j)\varepsilon_j - \frac{k-1}{2}\sum_{j=1}^{k}\varepsilon_j, \tag{20}$$

where $\varepsilon_j : \mathfrak{t} \to i\mathbb{R}$ is given by $\varepsilon_j(t) = t_j$ Moreover, the Weyl group is the permutation group $W \cong S_k$ with $s_\sigma(t) = (t_{\sigma(1)}, \ldots, t_{\sigma(k)})$ and $\det(s_\sigma) = \mathrm{sign}(\sigma)$ is the sign of the permutation. In the quotient the last summand cancels and it is convenient to replace δ by

$$\delta_0 = \sum_{j=1}^{k}(k-j)\varepsilon_j.$$

Then $\delta_0 \circ \sigma(t) = \prod_j t_{\sigma(j)}^{k-j}$ and hence

$$A_{\delta_0}(t) = \det\left(({t_i}^{k-j})_{i,j=1}^k\right).$$

This shows that Theorem 4.8 specializes to

$$\theta_{\rho_a}(t) = \frac{\det\left(({t_i}^{a_i+k-j})_{i,j=1}^k\right)}{\det\left(({t_i}^{k-j})_{i,j=1}^k\right)} = \theta_a(t). \qquad \square$$

Of particular interest are the special characters

$$\theta_{\Lambda^i\mathbb{C}^k} = \sum_{1\le\nu_1<\cdots<\nu_i\le k} t_{\nu_1}\cdots t_{\nu_i} = \phi_i,$$

$$\theta_{S^j\mathbb{C}^k} = \sum_{1\le\nu_1\le\cdots\le\nu_j\le k} t_{\nu_1}\cdots t_{\nu_j} = \psi_j.$$

These are obvious cases of the Weyl character formula. It follows from Theorem 4.11 and the Jacobi-Trudi identity in Lemma 2.5 that the character of ρ_a can also be expressed as the relevant Schur polynomial in the ψ_j:

$$\theta_{\rho_a} = \begin{vmatrix} \psi_{a_1} & \psi_{a_1+1} & \cdots & \psi_{a_1+k-1} \\ \psi_{a_2-1} & \psi_{a_2} & \cdots & \psi_{a_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_k-k+1} & \psi_{a_k-k+2} & \cdots & \psi_{a_k} \end{vmatrix}.$$

It is important to note that each character $\theta_{\rho_a} = \theta_a$ extends uniquely to a polynomial on $\mathbb{C}^{k\times k}$ if one interprets the variables $t_1, \dots, t_k$ as the eigenvalues of the matrix $A \in \mathbb{C}^{k\times k}$. Just write θ_a as a Schur polynomial in the variables ϕ_i via Lemma 2.2 and use the fact that the ϕ_i are the coefficients of the characteristic polynomial $\det(\mathbb{1} + \lambda A) = \sum_{i=0}^{k} \phi_i \lambda^i$. Thus a symmetric polynomial in the eigenvalues is a polynomial of the same degree in the entries of the matrix.

Weyl dimension formula

The dimension formula of Theorem 4.9 takes the form

$$\dim V_a = \prod_{i<j} \frac{a_i - a_j + j - i}{j - i} = \frac{\prod_{i<j}(m_i + \cdots + m_{j-1} + j - i)}{1!\,2!\,\cdots\,(k-1)!}$$

for $a_1 \geq \cdots \geq a_k$, where $m_i = a_i - a_{i+1}$ and V_a denotes the representation with highest weight $\lambda = \sum_i a_i \varepsilon_i$. To see this just note that with $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ and δ given by (20) we have $\langle \delta, \varepsilon_{ij} \rangle = j - i$ and $\langle \lambda, \varepsilon_{ij} \rangle = a_i - a_j$. The reader may check that the dimension formula is in agreement with the identities

$$\dim \Lambda^i \mathbb{C}^k = \binom{k}{i}, \qquad \dim S^j \mathbb{C}^k = \binom{k+j-1}{j}.$$

4.6 Representation ring

Consider the representation ring

$$\mathcal{R} = \mathcal{R}(\mathrm{U}(k))$$

which is generated by the irreducible representations $\rho_a : \mathrm{U}(k) \to V_a$ with $a_1 \geq \cdots \geq a_k$. An element of $\mathcal{R}$ can be thought of as a formal linear combination of the form

$$\rho = \sum_a x^a \rho_a$$

with integer coefficients $x^a \in \mathbb{Z}$. If the x^a are all nonnegative then we may think of ρ as the direct sum of the representations ρ_a where each ρ_a appears x^a times. In the negative case an element $\rho \in \mathcal{R}^{n-k}(\mathrm{U}(k))$ should be interpreted as a difference of two representations just like in K-theory. The multiplicative structure is given by the tensor product. The ring $\mathcal{R}(\mathrm{U}(k))$ carries a natural pairing defined by

$$\langle \rho_a, \rho_b \rangle = \begin{cases} 1, & \text{if } a = b^*, \\ 0, & \text{if } a \neq b^* \end{cases}$$

where $a_i^* = -a_{k+1-i}$ as on page 28. This pairing is not positive definite but it satisfies the Frobenius condition

$$\langle \rho \otimes \rho', \rho'' \rangle = \langle \rho, \rho' \otimes \rho'' \rangle. \tag{21}$$

Let $\delta_\ell : \mathrm{U}(k) \to S^1$ denote the central representation defined by

$$\delta_\ell(U) = \det(U)^\ell$$

for $U \in \mathrm{U}(k)$ and $\ell \in \mathbb{Z}$. This representation corresponds to the highest weight $\underline{\ell} = (\ell, \dots, \ell) \in \mathbb{Z}^k$ and it satisfies

$$\rho_a \otimes \delta_\ell = \rho_{a+\underline{\ell}}. \tag{22}$$

This implies that there is a family of metrics

$$\langle \rho, \rho' \rangle_\ell = \langle \rho, \rho' \otimes \delta_{-\ell} \rangle = \langle \rho \otimes \rho', \delta_{-\ell} \rangle$$

which all satisfy the Frobenius condition.

Structure constants

The product on $\mathcal{R}$ can be expressed in terms of the structure constants N_{ab}^c defined by

$$\rho_a \otimes \rho_b =: \sum_c N_{ab}^c \rho_c.$$

The associativity law then takes the form

$$\sum_\nu N_{ab}^\nu N_{\nu c}^d = \sum_\nu N_{a\nu}^d N_{bc}^\nu$$

Moreover, the Frobenius condition can be expressed as

$$N_{abc} = N_{bca} = N_{cab}, \qquad N_{abc} := N_{ab}^{c^*}.$$

In fact, by symmetry of the tensor product, the constants N_{abc} are invariant under all permutations of the indices. Condition (22) takes the form

$$N_{a+\underline{\ell}\,b}^{c} = N_{a\,b+\underline{\ell}}^{c} = N_{a\,b}^{c-\underline{\ell}}$$

for all weights a, b, c and all integers ℓ. The next proposition shows that the structure constants N_{ab}^{c} agree with those in the ring $\mathcal{R}(k)$ of symmetric polynomials with complex coefficients in k variables. Hence these constants are given by the Littlewood-Richardson rules of Theorem 2.12.

Proposition 4.12 *The map*

$$\mathcal{R}(\mathrm{U}(k)) \mapsto \mathcal{S}(k) : \rho \mapsto \theta_\rho$$

is a ring isomorphism from the representation ring of $\mathrm{U}(k)$ *to the ring of symmetric polynomials in* k *variables.*

Proof: That the map is a ring homomorphism follows from (18). Moreover, by the Weyl character formula in Theorem 4.11, it identifies the two canonical bases $\rho_a \mapsto \theta_a$. □

Multiplicities

The structure constants can be explicitly expressed in terms of the characters as follows. For any representation $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ denote by

$$V^{\mathrm{G}} = \{v \in V \mid \rho(g)v = v \;\forall\; g \in \mathrm{G}\}$$

the subspace which is fixed under G. Then the orthogonal projection

$$\Pi_V : V \to V^{\mathrm{G}}$$

is given by

$$\Pi_V = \frac{1}{\mathrm{Vol}(\mathrm{G})} \int_{\mathrm{G}} \rho(g)\, dg$$

where dg denotes an invariant metric on G Hence the dimension of the invariant subspace is given by

$$\dim\, V^{\mathrm{G}} = \mathrm{trace}(\Pi_V) = \frac{1}{\mathrm{Vol}(\mathrm{G})} \int_{\mathrm{G}} \theta_\rho(g)\, dg.$$

Now fix some irreducible representation $\rho_\lambda : \mathrm{G} \to \mathrm{Aut}(V_\lambda)$. Then the multiplicity with which ρ_λ occurs in ρ (denoted by $\mathrm{mult}_\rho(\lambda)$) agrees with the dimension of the subspace $(V \otimes \bar{V}_\lambda)^{\mathrm{G}}$. To see this note that $(V_\lambda \otimes \bar{V}_{\lambda'})^{\mathrm{G}} = \{0\}$ unless $\lambda = \lambda'$ in which case the subspace is 1-dimensional. Thus

$$\mathrm{mult}_\rho(\lambda) = \dim (V \otimes \bar{V}_\lambda)^{\mathrm{G}} = \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta_\rho(g) \theta_{\rho_\lambda}(g)^{-1} \, dg. \tag{23}$$

This formula plays a crucial role in Witten's quantum field theory approach to the Verlinde algebra in [12]. It shows that the structure constants N_{ab}^c can be expressed in the form

$$N_{ab}^c = \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta_a(g) \theta_b(g) \theta_c(g)^{-1} \, dg, \tag{24}$$

where $\theta_a = \theta_{\rho_a} = \mathrm{trace}^c \circ \rho_a : \mathrm{G} \to \mathbb{C}$ denotes the character of the representation ρ_a. The reader may check that these constants satisfy the above conditions.

4.7 Natural isomorphism

Consider the ring $\mathcal{R}(k, n-k) = \mathcal{R}^{n-k}(\mathrm{U}(k))$ which is generated by the representations ρ_a with

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0. \tag{25}$$

In [12] Witten calls these the representations **at level** $(n-k, n)$. The multiplication is defined as the tensor product followed by the projection onto $\mathcal{R}(k, n-k)$. Note here that the tensor product of two representations ρ_a and ρ_b which both satisfy (25) is a sum of irreducible representations ρ_{a^ν}, however, the a^ν need not all satisfy (25). The product in $\mathcal{R}(k, n-k)$ is given by simply neglecting those a^ν which do not satisfy (25). Another important point to bear in mind is that the above metric $\langle \rho, \rho' \rangle$ vanishes on $\mathcal{R}(k, n-k)$. However, there is a natural nondegenerate pairing

$$\langle \rho, \rho' \rangle_{n-k} = \langle \rho \otimes \rho', \delta_{k-n} \rangle \tag{26}$$

which, by (22), determines a Frobenius structure on $\mathcal{R}(k, n-k)$.

In [12] Witten observed that there is a natural ring isomomorphism from $\mathcal{R}(k, n-k)$ to the cohomology of the Grassmannian $\mathrm{G}(k, n)$. In view of

Theorem 4.11 the character $\theta_\rho = \mathrm{trace}^c \circ \rho : \mathrm{U}(k) \to \mathbb{C}$ of any finite dimensional representation of $\mathrm{U}(k)$ is a symmetric polynomial and hence extends uniquely to a polynomial on $\mathbb{C}^{k\times k}$ which is invaraint under conjugation. This extension can then be restricted to the Lie algebra $\mathfrak{u}(k)$ and this restriction is invariant under the conjugate action of $\mathrm{U}(k)$. Now fix a connection A on the tautological bundle $E \to \mathrm{G}(k,n)$ and denote by $F_A \in \Omega^2(\mathrm{G}(k,n), \mathrm{End}(E))$ its curvature. Then $\theta_a(F_A/2\pi i)$ is a closed real valued $2|a|$-form on $\mathrm{G}(k,n)$ which represents some characteristic class of the bundle E. Witten's isomorphism is the map

$$\mathcal{R}(k, n-k) \to H^*(\mathrm{G}(k,n);\mathbb{Z}) : \rho \mapsto [\theta_\rho(F_A/2\pi i)]. \tag{27}$$

If $\rho = \sum_a x^a \rho_a$ is a virtual representation (i.e. some of the x^a are negative) then θ_ρ should be interpreted as the sum $\theta_\rho = \sum_a x^a \theta_a$. It is tempting to use (18) to show that the map $\rho \mapsto [\theta_\rho(F_A/2\pi i)]$ is a ring homomorphism. However, care must be taken with the projection onto $\mathcal{R}(k, n-k)$, i.e. one has to show that $\theta_a(F_A/2\pi i)$ represents the zero cohomology class whenever $a_1 > n-k$. The next theorem shows in fact that (27) is a ring isomorphism which idenitifies the two canonical bases.

Theorem 4.13 *Let $a \in \mathbb{Z}^k$ satisfy (25) and let A be a connection on the tautological bundle $E \to \mathrm{G}(k,n)$. Then*

$$[\theta_a(F_A/2\pi i)] = \mathrm{PD}(\sigma_a) \in H^{2|a|}(\mathrm{G}(k,n);\mathbb{Z}).$$

Moreover, (27) is a ring isomorphism.

Proof: We first prove the result for the multiplicative generators $a = (1,\dots,1,0,\dots,0)$ corresponding to the representations $V_a = \Lambda^i\mathbb{C}^k$. Namely,

$$\rho_{\Lambda^i\mathbb{C}^k}(t) = \sum_{1\le\nu_1<\cdots<\nu_i\le k} t_{\nu_1}\cdots t_{\nu_i}$$

and applying this polynomial to $\sqrt{-1}/2\pi$ times the curvature of any $\mathrm{U}(k)$-connection gives the i-th Chern class. Hence, by Proposition 3.3, we find

$$\begin{aligned} [\phi_i(F_A/2\pi\sqrt{-1})] &= [\theta_{1,\dots,1,0\dots,0}(F_A/2\pi\sqrt{-1})] \\ &= [\theta_{\Lambda^i\mathbb{C}^k}(F_A/2\pi\sqrt{-1})] \\ &= (-1)^i c_i(E) \\ &= c_i(E^*). \end{aligned}$$

This proves the result for $a = (1, \ldots, 1, 0, \ldots, 0)$. For $a = (j, 0, \ldots, 0)$ it follows from Theorem 3.4 and Lemma 2.1 that

$$\begin{aligned}
[\psi_j(F_A/2\pi i)] &= \begin{vmatrix} [\phi_1(F_A/2\pi i)] & \cdots & \cdots & [\phi_j(F_A/2\pi i)] \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & [\phi_1(F_A/2\pi i)] \end{vmatrix} \\
&= \begin{vmatrix} c_1(E^*) & \cdots & \cdots & c_j(E^*) \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & c_1(E^*) \end{vmatrix} \\
&= c_j(F).
\end{aligned}$$

Now use Theorem 4.11 to express $[\theta_a(F_A/2\pi i)]$ as a determinant in the Chern classes $c_j(F)$ and then use Theorem 3.4 to express this determinant as the Poincaré dual of the Schubert cycle σ_a. This proves the first assertion. It follows that (27) maps $\rho_a \mapsto \mathrm{PD}(\sigma_a)$ and hence is a vector space isomorphism. That this isomorphism intertwines the two product structures follows from Theorems 3.4 and 4.11 and Lemma 2.10. These results show that the structure constants are the same in both rings. $\square$

Corollary 4.14 *The map (27) is the unique ring isomorphism which maps*

$$\Lambda^i \mathbb{C}^k \mapsto c_i(E^*), \qquad S^j \mathbb{C}^k \mapsto c_j(F).$$

Corollary 4.15 *If $a_1 \geq \cdots \geq a_k \geq 0$ with $|a| = k(n-k)$ and $a_1 > n-k$ then*

$$\int_{\mathrm{G}(k,n)} \theta_a(F_A/2\pi i) = 0.$$

Proof: Let $N = a_1 + k > n$ and note that $\mathrm{G}(k,n)$ is the closure of the Schubert cycle $\Sigma_{\underline{N-n}} = \Sigma_{N-n,\ldots,N-n}$ in $\mathrm{G}(k,N)$. By Theorem 4.13, the cohomology class of $\theta_a(F_A/2\pi i)$ is Poincaré dual to the Schubert cycle $\Sigma_a \subset \mathrm{G}(k,N)$ and hence

$$\int_{\mathrm{G}(k,n)} \theta_a(F_A/2\pi i) = \Sigma_{\underline{N-n}} \cdot \Sigma_a = 0.$$

Since $a \neq \underline{n-k}$ the last equality follows from Theorem 3.1. $\square$

Corollary 4.16 *If $\theta : \mathbb{C}^{k\times k} \to \mathbb{C}$ is a polynomial which is invariant under the adjoint action of $\mathrm{G}^c = \mathrm{GL}(k, \mathbb{C})$ then*

$$\frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta(g) \det(g)^{k-n} \, dg = \int_{\mathrm{G}(k,n)} \theta(F_A/2\pi i).$$

Proof: Every invariant polynomial can be decomposed as a finite sum

$$\theta = \sum_a x^a \theta_a, \qquad x^a = \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta(g)\theta_a(g)^{-1} \, dg,$$

where θ_a denotes the character of the representation with highest weight $a_1 \geq \cdots \geq a_k \geq 0$. By Corollary 4.15, the integral of $\theta_a(F_A/2\pi i)$ over $\mathrm{G}(k, n)$ is zero unless $a = \underline{n-k}$ in which case the integral is 1 (Exercise 3.6). Hence

$$\int_{\mathrm{G}(k,n)} \theta(F_A/2\pi i) = x^{\underline{n-k}} = \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta(g) \det(g)^{k-n} \, dg. \qquad \square$$

Consider the formula of Corollary 4.16 with $\theta = \theta_a\theta_b\theta_c$, where $a, b, c \in \mathbb{Z}^k$ satisfy (25) and $|a| + |b| + |c| = k(n - k)$. In this case one obtains

$$\begin{aligned} \sigma_a \cdot \sigma_b \cdot \sigma_c &= \int_{\mathrm{G}(k,n)} \theta_a(F_A/2\pi i) \wedge \theta_b(F_A/2\pi i) \wedge \theta_c(F_A/2\pi i) \\ &= \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} \theta_a(g)\theta_b(g)\theta_c(g) \det(g)^{k-n} \, dg \\ &= N_{a\ b\ c-\underline{n-k}}. \end{aligned}$$

The last equality follows from (24). The identity $\sigma_a \cdot \sigma_b \cdot \sigma_c = N_{a\ b\ c-\underline{n-k}}$ is equivalent to the fact that the map $\rho_a \mapsto \mathrm{PD}(\sigma_a)$ is a ring homomorphism.

In [12] Witten goes further and conjectures that the above isomorphism $\mathcal{R}(k, n - k) \to H^*(\mathrm{G}(k, n); \mathbb{Z})$ should intertwine the two deformed product structures, i.e. in the case of the Grassmannian the **quantum cohomology** structure, defined in terms of J-holomorphic curves $u : \Sigma \to \mathrm{G}(k, n)$, and in the case of the representation ring the **Verlinde algebra** structure, defined in terms of holomorphic sections of certain line bundles over moduli spaces of flat $\mathrm{U}(k)$-connections with parabolic structures over a surface Σ. This conjecture was proved by Agnihotri in [1].

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