Corrigendum: Gromov–Witten invariants of symplectic quotients and adiabatic limits

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Abstract

We correct an error in [1, Section 11] concerning the domain of definition for the local equivariant symplectic action.

Let (M, ω) be a symplectic manifold equipped with a Hamiltonian action by a compact Lie group G. Identify the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ with its dual by an invariant inner product and let $\mu : M \to \mathfrak{g}$ be a moment map for the action. We assume that μ is proper and G acts freely on $\mu^{-1}(0)$. Identify $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and let $(x, \eta) : S^1 \to M \times \mathfrak{g}$ be a smooth loop. The **equivariant length** of the loop (x, η) is defined by

$$\ell(x, \eta) := \int_0^{2\pi} |\dot{x} + X_\eta(x)| \, d\theta,$$

where $\mathfrak{g} \to \mathrm{Vect}(M) : \xi \mapsto X_\xi$ denotes the infinitesimal action. Fix a neighborhood U of $\mu^{-1}(0)$ with compact closure. In [1, Lemma 11.2] it is proved that, if U is sufficiently small, then there is a constant $c > 0$ such that, for every loop $(x, \eta) : S^1 \to U \times \mathfrak{g}$ there is a loop $g_0 : S^1 \to \mathrm{G}$ and an element $x_0 \in \mu^{-1}(0)$ satisfying $g_0(0) = \mathbb{1}$ and

$$\sup_{S^1} \left|\eta + \dot{g}_0 g_0^{-1}\right| \leq c\ell(x, \eta), \qquad d(x(\theta), g_0(\theta)x_0) \leq c\,(\mu(x(\theta)) + \ell(x, \eta)) \tag{1}$$

We shall define the **local equivariant symplectic action** $\mathcal{A}(x, \eta)$ under the assumption that

$$\sup_{\theta \in S^1} |\mu(x(\theta))| + \ell(x, \eta) < \delta \tag{2}$$

where δ is sufficiently small.

The error in [1] is that δ is chosen such that $\mu^{-1}((-\delta, \delta)) \subset U$ and $2c\delta$ is smaller than the injectivity radius of M. Apart from the fact that M might be noncompact and its injectivity radius could be zero, a counterexample (due to Fabian Ziltener, with injectivity radius equal to infinity) shows that, when U is too large, this choice of δ may not suffice to obtain uniqueness of the pair (x_0, g_0) up to homotopy. Instead we must choose δ as follows.

First, choose $\delta > 0$ so small that, if (x, θ) is a loop satisfying (2), then any two pairs (x_0, g_0) and (x_1, g_1) satisfying $g_0(0) = g_1(0) = \mathbb{1}$ and (1) can be connected by a homotopy (x_λ, g_λ) satisfying the same inequality with c replaced by a suitable larger constant C. (More precisely, estimate the distance of g_0 and g_1 by $c'\ell(x, \eta)$ and choose δ so small that the distance between $g_0(\theta)$ and $g_1(\theta)$ is smaller than the injectivity radius of G for all θ, and that δ is smaller than the injectivity radius of $\mu^{-1}(0)$. Let $\lambda \mapsto x_\lambda$ be the geodesic in $\mu^{-1}(0)$ from x_0 to x_1, choose $\zeta : S^1 \to \mathfrak{g}$ such that $\zeta(0) = 0$ and $g_1(\theta) = g_0(\theta)\exp(\zeta(\theta))$, and define $g_\lambda(\theta) := g_0(\theta)\exp(\lambda\zeta(\theta))$. This homotopy satisfies the required estimates.) Second, choose δ so small that, for all $x \in M$, we have

$$|\mu(x)| < \delta \qquad \Longrightarrow \qquad B_{C\delta}(x) \subset U.$$

Third, choose δ so small that $C\delta$ is smaller than the injectivity radius of M at all elements of $\mu^{-1}(0)$. Then, for every pair (x, η) satisfying (2) and every pair (x_0, g_0) satisfying (1), we have $x(\theta) \in B_{C\delta}(g_0(\theta)x_0)$ for all θ. Hence there is a unique loop $\xi_0(\theta) \in T_{g_0(\theta)x_0}M$ such that

$$x(\theta) = \exp_{g_0(\theta)x_0}(\xi_0(\theta)), \qquad |\xi_0(\theta)| < C\delta$$

for all θ. We define $u_0 : [0,1] \times S^1 \to M$ by

$$u_0(\tau, \theta) := \exp_{g_0(\theta)x_0}(\tau\xi_0(\theta)).$$

If (x_1, g_1) is another pair satisfying (1) then there is a homotopy (x_λ, g_λ) from (x_0, g_0) to (x_1, g_1) satisfying (1) with c replaced by C. Hence the resulting maps $u_\lambda : [0,1] \times S^1 \to M$ form a homotopy satisfying $u_\lambda(1, \theta) = x(\theta)$ and $\mu(u_\lambda(0,0)) = 0$, $u_\lambda(0, \theta) \in \mathrm{G}u_\lambda(0,0)$. This shows that the **local equivariant symplectic action** defined by

$$\mathcal{A}(x, \eta) := -\int u_0^*\omega + \int_0^{2\pi} \langle \mu(x(\theta)), \eta(\theta) \rangle \, d\theta$$

is independent of the choice of the pair (x_0, g_0), used to define it, provided (x, η) satisfies (2). With this understood Lemma 11.3 and the proof of Proposition 11.1 in [1] remain valid as they stand.

We would like to thank Fabian Ziltener for pointing out to us the error corrected in this note. In an interesting recent paper [2] he extended the definition of the local equivariant symplectic action to all equivariantly short loops, regardless of whether or not they are close to the zero set of the moment map. Moreover, he found a sharp constant for the isoperimetric inequality (compare with [1, Lemma 11.3]).

References

[1] A.R.Gaio, D.A.Salamon, Gromov–Witten invariants of symplectic quotients and adiabatic limits. *JSG* **1** (2005), 55-159.

[2] F. Ziltener, The invariant symplectic action and decay of vortices. To appear in *JSG*.