Notes on compact Lie groups

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1 Lie Groups

A **Lie Group** is a smooth manifold with a group structure such that the multiplication and the inverse map are smooth (C^∞). A **Lie subgroup** of a Lie group is a subgroup that is also a submanifold. Assume throughout this section that G is a Lie group. The tangent space of G at the identity element $\mathbb{1} \in \mathrm{G}$ is called the **Lie algebra** of G and will be denoted by

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) := T_{\mathbb{1}}\mathrm{G}.$$

For every $g \in \mathrm{G}$ the right and left multiplication maps $R_g, L_g : \mathrm{G} \to \mathrm{G}$ are defined by

$$R_g(h) := hg, \qquad L_g(h) := gh$$

for $h \in \mathrm{G}$. We shall denote the derivatives of these maps by

$$vg := dR_g(h)v \in T_{hg}\mathrm{G}, \qquad gv := dL_g(h)v \in T_{gh}\mathrm{G}$$

for $v \in T_h\mathrm{G}$. In particular, for $h = \mathbb{1}$ and $\xi \in T_{\mathbb{1}}\mathrm{G} = \mathfrak{g}$, we have $\xi g, g\xi \in T_g\mathrm{G}$ and hence ξ determines two vector fields $g \mapsto g\xi$ and $g \mapsto \xi g$ on G. These are called the **left-invariant** respectively **right-invariant** vector fields generated by ξ. We shall prove in Lemma 1.2 below that the integral curves of both vector fields through $g_0 = \mathbb{1}$ agree.

Exercise 1.1. (i) Prove that

$$(v_0 g_1)g_2 = v_0(g_1 g_2)$$

for $v_0 \in T_{g_0}\mathrm{G}$ and $g_1, g_2 \in \mathrm{G}$. Similarly

$$(g_0 v_1)g_2 = g_0(v_1 g_2), \qquad (g_0 g_1)v_2 = g_0(g_1 v_2).$$

(ii) Prove that with the above notation the Leibniz rule holds, i.e. if $\alpha, \beta : \mathbb{R} \to \mathrm{G}$ are smooth curves, then

$$\frac{d}{dt}\alpha(t)\beta(t) = \dot\alpha(t)\beta(t) + \alpha(t)\dot\beta(t).$$

Hint: Differentiate the map $\mathbb{R}^2 \to \mathrm{G} : (s,t) \mapsto \alpha(s)\beta(t)$.

(iii) Deduce that

$$\frac{d}{dt}\gamma(t)^{-1} = -\gamma(t)^{-1}\dot\gamma(t)\gamma(t)^{-1}$$

for every curve $\gamma : \mathbb{R} \to \mathrm{G}$.

(iv) Prove that the vector fields $g \mapsto g\xi$ and $g \mapsto \xi g$ are complete for every $\xi \in \mathfrak{g}$. **Hint:** Prove that the length of the existence interval is independent of the initial condition.

Lemma 1.2. *Let $\xi \in \mathfrak{g}$ and let $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth curve. Then the following conditions are equivalent.*

(i) *For all $s, t \in \mathbb{R}$*

$$\gamma(t+s) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot{\gamma}(0) = \xi. \tag{1.1}$$

(ii) *For all $t \in \mathbb{R}$*

$$\dot{\gamma}(t) = \xi\gamma(t), \qquad \gamma(0) = \mathbb{1}. \tag{1.2}$$

(iii) *For all $t \in \mathbb{R}$*

$$\dot{\gamma}(t) = \gamma(t)\xi, \qquad \gamma(0) = \mathbb{1}. \tag{1.3}$$

Moreover, for every $\xi \in \mathfrak{g}$ there exists a unique smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ that satisfies either of these conditions.

Proof. That (i) implies (ii) follows by differentiating the identity (1.1) with respect to s at $s = 0$. To prove that (ii) implies (i) note that, by Exercise 1.1 (i), the curves $\alpha(t) = \gamma(t+s)$ and $\beta(t) = \gamma(t)\gamma(s)$ are both integral curves of the vector field $g \mapsto \xi g$ such that $\alpha(0) = \beta(0) = \gamma(s)$. Hence they are equal. This shows that (i) is equivalent to (ii). That (i) is equivalent to (iii) follows by analogous arguments, interchanging s and t. The last assertion about the existence of γ follows from Exercise 1.1 (iv). $\square$

The **exponential map** $\exp : \mathfrak{g} \to \mathrm{G}$ is defined by

$$\exp(\xi) := \gamma_\xi(1),$$

where $\gamma_\xi : \mathbb{R} \to \mathrm{G}$ is the unique solution of (1.1). With this definition the path γ_ξ is given by

$$\gamma_\xi(t) = \exp(t\xi). \tag{1.4}$$

To see this, note that by (1.2) the curve $\alpha(s) := \gamma_\xi(ts)$ satisfies the differential equation $\dot{\alpha}(s) = t\xi\alpha(s)$, and so $\exp(t\xi) = \alpha(1) = \gamma_\xi(t)$. This proves (1.4). It follows from (1.4) and Lemma 1.2 that the exponential map satisfies the equations $\exp(0) = \mathbb{1}$ and

$$\begin{aligned} \tfrac{d}{dt}\exp(t\xi) &= \xi\exp(t\xi) = \exp(t\xi)\xi, \\ \exp((s+t)\xi) &= \exp(s\xi)\exp(t\xi) \end{aligned} \tag{1.5}$$

for all $s, t \in \mathbb{R}$ and all $\xi \in \mathfrak{g}$. The **adjoint representation** of G on its Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{Ad}(g)\eta := g\eta g^{-1} := \tfrac{d}{dt}\Big|_{t=0} g\exp(t\eta)g^{-1}.$$

In other words the linear map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the derivative of the smooth map $\mathrm{G} \to \mathrm{G} : h \mapsto ghg^{-1}$ at $h = 1\!\!1$. The map $\mathrm{G} \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}(g)$ is a group homomorphism, i.e.

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h), \qquad \mathrm{Ad}(1\!\!1) = \mathrm{id},$$

for all $g, h \in \mathrm{G}$, and is called the **adjoint action** of G on its Lie algebra. The derivative of this map at $g = 1\!\!1$ in the direction $\xi \in \mathfrak{g}$ is denoted by $\mathrm{ad}(\xi)$. The **Lie bracket** of two elements $\xi, \eta \in \mathfrak{g}$ is defined by

$$[\xi, \eta] := \mathrm{ad}(\xi)\eta = \tfrac{d}{dt}\big|_{t=0} \exp(t\xi)\eta\exp(-t\xi).$$

Lemma 1.3. *For all $\xi, \eta, \zeta \in \mathfrak{g}$ we have*

$$[\xi, \eta] = -[\eta, \xi], \tag{1.6}$$

$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0. \tag{1.7}$$

Proof. We prove that the map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto X_\xi$ defined by $X_\xi(g) := \xi g$ for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$ is a Lie algebra homomorphism. To see this, denote by $\psi_t \in \mathrm{Diff}(\mathrm{G})$ the flow generated by X_ξ, i.e. $\psi_t(g) := \exp(t\xi)g$ for $t \in \mathbb{R}$ and $g \in \mathrm{G}$. Then, by definition of the Lie bracket of vector fields,

$$\begin{aligned} [X_\xi, X_\eta](g) &= \tfrac{d}{dt}\big|_{t=0} d\psi_t(\psi_{-t}(g))X_\eta(\psi_{-t}(g)) \\ &= \tfrac{d}{dt}\big|_{t=0} \exp(t\xi)\eta\exp(-t\xi)g \\ &= [\xi, \eta]g \end{aligned}$$

Here we have used Exercise 1.1 (i). Now the assertions follow from the properties of the Lie bracket for vector fields. □

Definition 1.4. *A* **Lie algebra** *is a real vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ that satisfies the* **Jacobi identity** (1.7).

Lemma 1.5. *Let $\xi, \eta \in \mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and define $\gamma : \mathbb{R} \to \mathrm{G}$ by*

$$\gamma(t) := \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta).$$

Then $\dot\gamma(0) = 0$ and $\tfrac{d}{dt}\big|_{t=0}\gamma(\sqrt{t}) = [\xi, \eta]$.

Proof. As in the proof of Lemma 1.3, the flow of the vector field $X_\xi(g) = \xi g$ on G is given by $t \mapsto L_{\exp(t\xi)}$ and $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ for $\xi, \eta \in \mathfrak{g}$. Hence the result follows from the corresponding formula for general vector fields. □

Lemma 1.6. *Let $\mathbb{R}^2 \to \mathrm{G} : (s,t) \mapsto g(s,t)$ be a smooth map. Then*

$$\partial_s \left(g^{-1}\partial_t g\right) - \partial_t \left(g^{-1}\partial_s g\right) + \left[g^{-1}\partial_s g, g^{-1}\partial_t g\right] = 0. \tag{1.8}$$

Proof. If M is a smooth manifold, $\mathbb{R}^2 \to M : (s,t) \mapsto \gamma(s,t)$ is a smooth map, and $X(s,t), Y(s,t) \in \mathrm{Vect}(M)$ are smooth families of vector fields such that

$$\partial_s \gamma = X \circ \gamma, \qquad \partial_t \gamma = Y \circ \gamma,$$

then

$$(\partial_s Y - \partial_t X - [X,Y]) \circ \gamma = 0.$$

To obtain the formula (1.8), apply this identity to the manifold $M := \mathrm{G}$, the map $\gamma := g : \mathbb{R}^2 \to \mathrm{G}$, and the vector fields $X := X_\xi$ and $Y := X_\eta$ (as in the proof of Lemma 1.3), where $\xi := (\partial_s g)g^{-1}$ and $\eta := (\partial_t g)g^{-1}$. □

Exercise 1.7. Prove that for every $g \in \mathrm{G}$ and every $\xi \in \mathfrak{g}$

$$g \exp(\xi) g^{-1} = \exp(\mathrm{Ad}(g)\xi) \tag{1.9}$$

Hint: Consider the curve $\gamma(t) = g\exp(t\xi)g^{-1}$ and use Exercise 1.1.

Exercise 1.8. Prove that for every $\xi \in \mathfrak{g}$

$$\mathrm{Ad}(\exp(\xi)) = \exp(\mathrm{ad}(\xi)). \tag{1.10}$$

Hint: See Lemma 2.1 below.

Exercise 1.9. Prove that any two elements $\xi, \eta \in \mathfrak{g}$ satisfy $[\xi, \eta] = 0$ if and only if $\exp(s\xi)$ and $\exp(t\eta)$ commute for all $s, t \in \mathbb{R}$.

2 Lie Group Homomorphisms

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A **Lie group homomorphism** is a smooth map $\phi : \mathrm{G} \to \mathrm{H}$ which is a group homomorphism. A linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** iff

$$[\Phi(\xi), \Phi(\eta)] = \Phi([\xi, \eta])$$

for all $\xi, \eta \in \mathfrak{g}$. The next lemma asserts that the derivative of a Lie group homomorphism at the identity is a Lie algebra homomorphism. An example is the map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto X_\xi$ in the proof of Lemma 1.3; the corresponding Lie group homomorphism is the map $\mathrm{G} \mapsto \mathrm{Diff}(\mathrm{G}) : g \mapsto L_g$. (See Example 12.14 below.)

Lemma 2.1. *Let $\phi : \mathrm{G} \to \mathrm{H}$ be a Lie group homomorphism. Then its derivative $\Phi := d\phi(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}$ at the identity is a Lie algebra homomorphism.*

Proof. We show first that Φ and ϕ intertwine the exponential maps, i.e.

$$\exp(\Phi(\xi)) = \phi(\exp(\xi)) \tag{2.1}$$

for all $\xi \in \mathfrak{g}$. To see this, consider the curve $\gamma(t) := \phi(\exp(t\xi)) \in \mathrm{H}$. This curve satisfies $\gamma(s+t) = \gamma(s)\gamma(t)$ for all $s, t \in \mathbb{R}$ and $\dot\gamma(0) = \Phi(\xi)$. Hence, by Lemma 1.2, $\gamma(t) = \exp(t\Phi(\xi))$. With $t = 1$ this proves (2.1).

Next we prove that

$$\Phi(g\xi g^{-1}) = \phi(g)\Phi(\xi)\phi(g)^{-1} \tag{2.2}$$

for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$. Consider the curve $\gamma(t) := g\exp(t\xi)g^{-1}$. By (2.1), we have $\phi(\gamma(t)) = \phi(g)\exp(t\Phi(\xi))\phi(g)^{-1}$. Differentiate this curve at $t = 0$ to obtain (2.2). By (2.1) and (2.2), we have

$$\begin{aligned}
\Phi([\xi,\eta]) &= \tfrac{d}{dt}\Big|_{t=0} \Phi(\exp(t\xi)\eta\exp(-t\xi)) \\
&= \tfrac{d}{dt}\Big|_{t=0} \exp(t\Phi(\xi))\Phi(\eta)\exp(-t\Phi(\xi)) \\
&= [\Phi(\xi), \Phi(\eta)]
\end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}$. This proves Lemma 2.1. □

A **representation** of G is a Lie group homomorphism $\rho : \mathrm{G} \to \mathrm{GL}(V)$ where V is a real or complex vector space. Differentiating such a map at $g = \mathbb{1}$ gives a Lie algebra homomorphism $\dot\rho : \mathfrak{g} \to \mathrm{End}(V)$ defined by

$$\dot\rho(\xi) := \tfrac{d}{dt}\Big|_{t=0} \rho(\exp(t\xi)) \qquad \text{for } \xi \in \mathfrak{g}.$$

Examples are the obvious action of $\mathrm{U}(n)$ on $\mathbb{C}^n$ and the induced actions on spaces of symmetric polynomials or exterior forms.

Definition 2.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A* **Lie algebra automorphism of** $\mathfrak{g}$ *is a bijective Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$, so its inverse is also a Lie algebra homomorphism. The group of Lie algebra automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$. A* **derivation of** $\mathfrak{g}$ *is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ that satisfies $\delta[\xi,\eta] = [\delta\xi,\eta] + [\xi,\delta\eta]$ for all $\xi,\eta \in \mathfrak{g}$. The space of all derivations of $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$.*

Exercise 2.3. Prove that $\mathrm{Aut}(\mathfrak{g})$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ with the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$. **Hint:** Use Theorem 3.1 below.

Exercise 2.4. Let G be a Lie group with the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$. Prove that $\mathrm{Ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ is a Lie group homomorphism and $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is the corresponding Lie algebra homomorphism. **Hint:** See Lemma 1.3.

3 Closed Subgroups

Assume throughout that G is a Lie group (not necessarily compact) and denote by $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ its Lie algebra. Whenever necessary, we assume that $\mathfrak{g}$ is equipped with an inner product and define a Riemannian metric on G by $|v| := |g^{-1}v|$ for $g \in \mathrm{G}$ and $v \in T_g\mathrm{G}$. The following theorem was first proved in 1929 by John von Neumann [5] for the special case $\mathrm{G} = \mathrm{GL}(n, \mathbb{R})$ and then in 1930 by Élie Cartan [1] in full generality.

Theorem 3.1 (Closed Subgroup Theorem). *Let* G *be a Lie group and let* H *be a subgroup of* G*. Then the following are equivalent.*

(i) H *is a submanifold (and hence a Lie subgroup) of* G*.*

(ii) H *is a closed subset of* G*.*

If (i) holds, then the Lie algebra of H *is the space*

$$\mathfrak{h} = \left\{\eta \in \mathrm{G} \,\middle|\, \exp(t\eta) \in \mathrm{H} \text{ for all } t \in \mathbb{R}\right\}. \tag{3.1}$$

The proof is based on the following three lemmas.

Lemma 3.2. *Let $\xi \in \mathfrak{g}$ and let $\gamma : \mathbb{R} \to \mathrm{G}$ be a curve that is differentiable at $t = 0$ and satisfies $\gamma(0) = \mathbb{1}$ and $\dot\gamma(0) = \xi$. Then*

$$\exp(t\xi) = \lim_{k\to\infty} \gamma(t/k)^k \tag{3.2}$$

for every $t \in \mathbb{R}$.

Proof. Assume for simplicity that G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. Fix a nonzero real number t and for $k \in \mathbb{N}$ define

$$\xi_k := k\big(\gamma(t/k) - \mathbb{1}\big) \in \mathbb{R}^{n\times n}.$$

Then

$$\lim_{k\to\infty} \xi_k = t \lim_{k\to\infty} \frac{\gamma(t/k) - \gamma(0)}{t/k} = t\dot\gamma(0) = t\xi$$

and hence

$$\exp(t\xi) = \lim_{k\to\infty} \left(\mathbb{1} + \frac{\xi_k}{k}\right)^k = \lim_{k\to\infty} \gamma(t/k)^k.$$

(See [4, Satz 1.5.2].) This proves Lemma 3.2. □

Lemma 3.3. *Let* $\mathrm{H} \subset \mathrm{G}$ *be a closed subgroup. Then the set*

$$\mathfrak{h} := \left\{ \eta \in \mathfrak{g} \,\middle|\, \exp(t\eta) \in \mathrm{H} \text{ for all } t \in \mathbb{R} \right\}$$

in (3.1) *is a Lie subalgebra of* $\mathfrak{g}$

Proof. Let $\xi, \eta \in \mathfrak{h}$ and define the curve $\gamma : \mathbb{R} \to \mathrm{H}$ by

$$\gamma(t) := \exp(t\xi)\exp(t\eta)$$

for $t \in \mathbb{R}$. This curve is smooth and satisfies $\gamma(0) = \mathbb{1}$ and $\dot\gamma(0) = \xi + \eta$. Since H is closed, it follows from Lemma 3.2 that

$$\exp(t(\xi + \eta)) = \lim_{k \to \infty} \gamma(t/k)^k \in \mathrm{H}$$

for all $t \in \mathbb{R}$ and so $\xi + \eta \in \mathfrak{h}$ by definition. Thus $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$.

Now fix an element $\xi \in \mathfrak{h}$. If $h \in \mathrm{H}$, then

$$\exp(sh^{-1}\xi h) = h^{-1}\exp(s\xi)h \in \mathrm{H}$$

for all $s \in \mathbb{R}$ and hence $h^{-1}\xi h \in \mathfrak{h}$ by definition. Take $h = \exp(t\eta)$ with $\eta \in \mathfrak{h}$ to obtain $\exp(-t\eta)\xi\exp(t\eta) \in \mathfrak{h}$ for all $t \in \mathbb{R}$. Differentiating this curve at $t = 0$ gives $[\xi, \eta] \in \mathfrak{h}$ and this proves Lemma 3.3. □

Lemma 3.4. *Let* $\mathrm{H} \subset \mathrm{G}$ *be a closed subgroup and let* $\mathfrak{h} \subset \mathfrak{g}$ *be the Lie subalgebra in Lemma 3.3. Let* $\xi \in \mathfrak{g}$ *and let* $(\xi_i)_{i \in \mathbb{N}}$ *be a sequence in* $\mathfrak{g}$ *such that*

$$\exp(\xi_i) \in \mathrm{H}, \qquad \xi_i \neq 0 \tag{3.3}$$

for all $i \in \mathbb{N}$ *and*

$$\lim_{i \to \infty} \xi_i = 0, \qquad \lim_{i \to \infty} \frac{\xi_i}{|\xi_i|} = \xi. \tag{3.4}$$

Then $\xi \in \mathfrak{h}$.

Proof. Fix a real number t. Then, for each $i \in \mathbb{N}$, there exists a unique integer $m_i \in \mathbb{Z}$ such that $m_i|\xi_i| \leq t < (m_i + 1)|\xi_i|$. The sequence m_i satisfies

$$\lim_{i \to \infty} m_i|\xi_i| = t, \qquad \lim_{i \to \infty} m_i\xi_i = \lim_{i \to \infty} m_i|\xi_i|\frac{\xi_i}{|\xi_i|} = t\xi.$$

Hence

$$\exp(t\xi) = \lim_{i \to \infty} \exp(m_i\xi_i) = \lim_{i \to \infty} \exp(\xi_i)^{m_i} \in \mathrm{H}$$

for every $t \in \mathbb{R}$. Thus $\xi \in \mathfrak{h}$ by (3.1) and this proves Lemma 3.4. □

Proof of Theorem 3.1. We prove that (i) implies (ii) and (3.1). Thus assume that H is a Lie subgroup of G and let $\mathfrak{h} \subset \mathfrak{g}$ be defined by (3.1). Then $\mathfrak{h}$ is the Lie algebra of H. Namely, if $\eta \in \mathfrak{h}$, then the curve $\gamma : \mathbb{R} \to \mathrm{G}$ define by $\gamma(t) := \exp(t\eta)$ takes values in H and so $\eta = \dot\gamma(0) \in T_{\mathbb{1}}\mathrm{H} = \mathrm{Lie}(\mathrm{H})$. Conversely, if $\eta \in \mathrm{Lie}(\mathrm{H})$, then $\exp(t\eta) \in \mathrm{H}$ for all $t \in \mathbb{R}$ by Lemma 1.2 and so $\eta \in \mathfrak{h}$. Thus $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$.

To prove that H is closed, define the map $\phi : \mathrm{H} \times \mathfrak{h}^\perp \to \mathrm{G}$ by

$$\phi(h, \xi) := h\exp(\xi), \qquad h \in \mathrm{H}, \qquad \xi \in \mathfrak{h}^\perp.$$

Its derivative at $(\mathbb{1}, 0)$ is bijective. Hence ϕ restricts to a diffeomorphism from the product of two open neighborhoods $V \subset \mathrm{H}$ of $\mathbb{1}$ and $W \subset \mathfrak{h}^\perp$ of the origin onto the open neighborhood $U := \phi(V \times W) \subset \mathrm{G}$ of $\mathbb{1}$. Shrinking these neighborhoods, if necessary, we may assume that

$$U \cap \mathrm{H} = V. \tag{3.5}$$

(Otherwise there exists a sequence $(h_i, \xi_i) \in V \times W$ converging to $(\mathbb{1}, 0)$ such that $\phi(h_i, \xi_i) \in \mathrm{H} \setminus V$ for all i, contradicting the fact that $V \subset \mathrm{H}$ is a neighborhood of $\mathbb{1}$.) Also, there is an open neighborhood $U_0 \subset \mathrm{G}$ of $\mathbb{1}$ such that

$$g, g' \in U_0 \qquad \Longrightarrow \qquad g^{-1}g' \in U. \tag{3.6}$$

Now let $h_i \in \mathrm{H}$ be a sequence that converges to an element $g \in \mathrm{G}$. Then the sequence $h_i^{-1}g$ converges to $\mathbb{1}$. Choose $i_0 \in \mathbb{N}$ such that $h_i^{-1}g \in U_0$ for all $i \geq i_0$, and define $(h_i', \xi_i) := \phi^{-1}(h_i^{-1}g) \in V \times W$ for $i \geq i_0$. Then

$$h_i^{-1}g = h_i'\exp(\xi_i), \qquad h_i' \in V \subset \mathrm{H}, \qquad \xi_i \in W \subset \mathfrak{h}^\perp \tag{3.7}$$

for $i \geq i_0$ and

$$\lim_{i\to\infty} h_i' = \mathbb{1}, \qquad \lim_{i\to\infty} \xi_i = 0. \tag{3.8}$$

For $i, j \geq i_0$ this implies $h_i'\exp(\xi_i)\exp(-\xi_j)(h_j')^{-1} = h_i^{-1}h_j$. and hence

$$\exp(\xi_i) = h_{ij}\exp(\xi_j), \qquad h_{ij} := (h_i')^{-1}(h_i^{-1}h_j)h_j'. \tag{3.9}$$

Since $\lim_{i\to\infty} g^{-1}h_ih_i' = \mathbb{1}$, there is an integer $i_1 \geq i_0$ such that $g^{-1}h_ih_i' \in U_0$ for all $i \geq i_1$. By (3.5), (3.6), (3.9) this implies

$$h_{ij} = (g^{-1}h_ih_i')^{-1}(g^{-1}h_jh_j') \in U \cap \mathrm{H} = V \tag{3.10}$$

for all $i, j \geq i_1$. By (3.7), (3.9), (3.10) we have $(\mathbb{1}, \xi_i), (h_{ij}, \xi_j) \in V \times W$ and $\phi(\mathbb{1}, \xi_i) = \phi(h_{ij}, \xi_j)$ for all $i, j \geq i_1$. Since ϕ is injective on $V \times W$, this implies $h_{ij} = \mathbb{1}$ and $\xi_i = \xi_j$ for all $i, j \geq i_1$. Hence it follows from (3.8) that $\xi_i = 0$ and so by (3.7) we have $g = h_ih_i' \in \mathrm{H}$ for $i \geq i_1$. This shows that H is a closed subset of G. Thus we have proved that (i) implies (ii).

We prove that (ii) implies (i). Let $\mathrm{H} \subset \mathrm{G}$ be a closed subgroup of G and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra defined in equation (3.1) in Lemma 3.3. Define $k := \dim(\mathfrak{h})$ and $\ell := \dim(\mathfrak{g}) \geq k$, and choose a basis $e_1, \ldots, e_\ell$ of $\mathfrak{g}$ such that the vectors $e_1 \ldots, e_k$ form a basis of $\mathfrak{h}$ and $e_i \in \mathfrak{h}^\perp$ for $i > k$. Let $h_0 \in \mathrm{H}$ and define the map $\psi : \mathbb{R}^\ell \to \mathrm{G}$ by

$$\psi(x^1, \ldots, x^\ell) := h_0 \exp(x^1 e_1 + \cdots + x^k e_k) \exp(x^{k+1} e_{k+1} + \cdots + x^\ell e_\ell).$$

Then $\psi(0) = h_0$, $\psi(\mathbb{R}^k \times \{0\}) \subset \mathrm{H}$, and the derivative $d\psi(0) : \mathbb{R}^\ell \to T_{h_0}\mathrm{G}$ is bijective. Hence the inverse function theorem asserts that ψ restricts to a diffeomorphism from an open neighborhood $\Omega \subset \mathbb{R}^\ell$ of the origin to the open neighborhood $U := \psi(\Omega) \subset \mathrm{G}$ of h_0 that satisfies

$$\psi(0) = h_0, \qquad \psi(\Omega \cap (\mathbb{R}^k \times \{0\})) \subset U \cap \mathrm{H}.$$

We claim that there exists an open set $\Omega_0 \subset \mathbb{R}^\ell$ such that

$$0 \in \Omega_0 \subset \Omega, \qquad \psi(\Omega_0 \cap (\mathbb{R}^k \times \{0\})) = U_0 \cap \mathrm{H}, \qquad U_0 := \psi(\Omega_0). \tag{3.11}$$

Assume, by contradiction, that such an open set Ω_0 does not exist. Then there exists a sequence $x_i = (x_i^1, \ldots, x_i^\ell) \in \mathbb{R}^\ell$ such that

$$\lim_{i\to\infty} x_i = 0, \qquad x_i \in \Omega \setminus (\mathbb{R}^k \times \{0\}), \qquad \psi(x_i) \in \mathrm{H}.$$

Define $\eta_i := \sum_{\nu=1}^k x_i^\nu e_\nu \in \mathfrak{h}$ and $\xi_i := \sum_{\nu=k+1}^\ell x_i^\nu e_\nu \in \mathfrak{h}^\perp \setminus \{0\}$. Then

$$\lim_{i\to\infty} \xi_i = 0, \qquad \xi_i \neq 0, \qquad \exp(\xi_i) = \exp(-\eta_i) h_0^{-1} \psi(x_i) \in \mathrm{H}.$$

Passing to a subsequence, if necessary, we may assume that the sequence $\xi_i/|\xi_i|$ converges. Denote its limit by $\xi := \lim_{i\to\infty} \xi_i/|\xi_i|$. Then $\xi \in \mathfrak{h}$ by Lemma 3.4 and $\xi \in \mathfrak{h}^\perp$ by definition. Since $|\xi| = 1$, this is a contradiction. Thus there does exist an open set $\Omega_0 \subset \mathbb{R}^\ell$ that satisfies (3.11). Hence H is a submanifold of G and so is a Lie subgroup of G. This proves Theorem 3.1. □

Example 3.5. Choose a nonzero vector $(\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ such that at least one of the quotients ω_i/ω_j is irrational. Then the one-parameter subgroup

$$S_\omega := \{(e^{2\pi \mathbf{i} t\omega_1}, e^{2\pi \mathbf{i} t\omega_2}, \ldots, e^{2\pi \mathbf{i} t\omega_n}) \,|\, t \in \mathbb{R}\} \subset (S^1)^n \cong \mathbb{T}^n$$

of the torus is not closed. A similar example can be constructed in any Lie group that contains a torus of dimension at least two.

4 The Haar Measure

Let G be a compact Lie group and denote by $\mathcal{C}(\mathrm{G})$ the space of continuous functions $f : \mathrm{G} \to \mathbb{R}$ with the norm

$$\|f\| := \sup_{g \in \mathrm{G}} |f(g)| .$$

The next theorem asserts the existence of a translation invariant measure on every compact Lie group. The result extends to every compact Hausdorff group. The proof given below extends to every compact Hausdorff group that satisfies the second axiom of countability (i.e. its topology has a finite or countable basis).

Theorem 4.1. *Let* G *be a compact Lie group. Then there exists a bounded linear functional* $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ *that satisfies the following conditions.*

(i) $M(1) = 1$.

(ii) *M is left invariant, i.e.* $M(f \circ L_g) = M(f)$ *for* $f \in \mathcal{C}(G)$ *and* $g \in \mathrm{G}$.

(iii) *M is right invariant, i.e.* $M(f \circ R_g) = M(f)$ *for* $f \in \mathcal{C}(G)$ *and* $g \in \mathrm{G}$.

(iv) *If* $f \geq 0$ *and* $f \neq 0$, *then* $M(f) > 0$.

(v) *Let* $\phi : \mathrm{G} \to \mathrm{G}$ *denote the diffeomorphism defined by* $\phi(g) = g^{-1}$. *Then* $M(f \circ \phi) = M(f)$ *for every* $f \in \mathcal{C}(\mathrm{G})$.

M is uniquely determined by (i) and either (ii) or (iii). It is called the **Haar measure** *on* G.

Proof. We follow notes by Moser which in turn are based on a proof by Pontryagin. Let $\mathcal{A}$ denote the set of all measures on G of the form

$$A = \sum_{i=1}^{k} \alpha_i \delta_{a_i}$$

where $\alpha_i \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$. If $B = \sum_{j=1}^{\ell} \beta_j \delta_{b_j}$ is another such measure, denote

$$A \cdot B := \sum_{i=1}^{k} \sum_{j=1}^{\ell} \alpha_i \beta_j \delta_{a_i b_j}.$$

This defines a group structure on $\mathcal{A}$. For $A \in \mathcal{A}$ we define two linear operators $L_A, R_A : \mathcal{C}(\mathrm{G}) \to \mathcal{C}(\mathrm{G})$ by

$$(L_A f)(g) := \sum_{i=1}^m \alpha_i f(a_i g), \qquad (R_A f)(g) := \sum_{i=1}^m \alpha_i f(g a_i)$$

for $f \in \mathcal{C}(\mathrm{G})$ and $g \in \mathrm{G}$. Then

$$L_A(f \circ R_h) = (L_A f) \circ R_h, \qquad R_A(f \circ L_h) = (R_A f) \circ L_h, \tag{4.1}$$

$$L_{A \cdot B} = L_B \circ L_A, \qquad R_{A \cdot B} = R_A \circ R_B, \qquad L_A \circ R_B = R_B \circ L_A, \tag{4.2}$$

$$\min f \le L_A f \le \max f, \qquad \min f \le R_A f \le \max f. \tag{4.3}$$

We make use of the following three observations.

Observation 1: *Denote*

$$\mathrm{Osc}(f) := \max f - \min f.$$

If $f \in \mathcal{C}(\mathrm{G})$ is nonconstant, then there exists an $A \in \mathcal{A}$ such that $\min f < \min L_A f$ and hence $\mathrm{Osc}(L_A f) < \mathrm{Osc}(f)$.

Suppose f assumes its maximum at a point $g_0 \in \mathrm{G}$. Choose a neighbourhood $U \subset \mathrm{G}$ of $\mathbb{1}$ such that

$$g{g_0}^{-1} \in U \qquad \Longrightarrow \qquad f(g) > \frac{1}{2}(\max f + \min f).$$

Now $\mathrm{G} = \bigcup_{a \in \mathrm{G}} a^{-1} U$. Since G is compact, there exist finitely many points $a_1, \dots, a_m \in \mathrm{G}$ such that

$$\mathrm{G} = \bigcup_{i=1}^m {a_i}^{-1} U.$$

This means that for every $h \in \mathrm{G}$ there exists an i such that $a_i h \in U$. Consider the measure $A := m^{-1} \sum_i \delta_{a_i}$. Since, for every $g \in \mathrm{G}$, at least one of the points $a_i g {g_0}^{-1}$ lies in U we obtain

$$\begin{aligned} (L_A f)(g) &= \frac{1}{m} \sum_{i=1}^m f(a_i g) \\ &\ge \frac{m-1}{m} \min f + \frac{1}{2m}(\max f + \min f) \\ &> \min f. \end{aligned}$$

Hence $\min L_A f > \min f$ and, by (4.3), $\mathrm{Osc}(L_A f) < \mathrm{Osc}(f)$.

Observation 2: *For every* $f \in \mathcal{C}(\mathrm{G})$ *the set*

$$\mathcal{L}(f) := \{L_A f \mid A \in \mathcal{A}\}$$

is bounded and equicontinuous.

Boundedness follows from (4.3). To prove equicontinuity, note that, since G is compact and second countable, it is a metrizable topological space. Let $d : \mathrm{G} \times \mathrm{G} \to \mathbb{R}$ be a distance function which induces the given topology. Fix a function $f \in \mathcal{C}(\mathrm{G})$ and an $\varepsilon > 0$. Since G is compact, f is uniformly continuous. Hence there is a $\delta > 0$ such that, for all $g, h \in \mathrm{G}$,

$$d(g, h) < \delta \qquad \Longrightarrow \qquad |f(g) - f(h)| < \varepsilon. \tag{4.4}$$

We prove that there exists an open neighbourhood $U \subset \mathrm{G}$ of $\mathbb{1}$ such that

$$g^{-1}h \in U \qquad \Longrightarrow \qquad d(g, h) < \delta. \tag{4.5}$$

We argue by contradiction. Suppose that there exist sequences $g_\nu, h_\nu \in \mathrm{G}$ such that ${g_\nu}^{-1}h_\nu \to \mathbb{1}$ and $d(g_\nu, h_\nu) \geq \delta$. Passing to a subsequence we may assume that g_ν converges to g. Then $h_\nu = g_\nu({g_\nu}^{-1}h_\nu)$ converges also to g. Hence, for ν sufficiently large, we have $d(g_\nu, g) < \delta/2$ and $d(h_\nu, g) < \delta/2$, contradicting the assumption that $d(g_\nu, h_\nu) \geq \delta$. This proves (4.5).

A similar argument shows that there is a constant $\delta' > 0$ such that

$$d(g, h) < \delta' \qquad \Longrightarrow \qquad g^{-1}h \in U. \tag{4.6}$$

Now let $g, h \in \mathrm{G}$ such that $d(g, h) < \delta'$. Then, by (4.6), we have

$$(ag)^{-1}(ah) = g^{-1}h \in U$$

for every $a \in \mathrm{G}$, hence, by (4.5),

$$d(ag, ah) < \delta,$$

hence it follows from (4.4) that

$$|f(ag) - f(ah)| < \varepsilon$$

for every $a \in \mathrm{G}$, and this implies

$$|(L_A f)(g) - (L_A f)(h)| < \varepsilon$$

for every $A \in \mathcal{A}$. Thus we have proved equicontinuity.

Observation 3: *For every $f \in \mathcal{C}(\mathrm{G})$, $\inf_{A\in\mathcal{A}} \mathrm{Osc}(L_A f) = 0$.*

Choose a sequence $A_\nu \in \mathcal{A}$ such that

$$\lim_{\nu\to\infty} \mathrm{Osc}(L_{A_\nu} f) = \inf_{A\in\mathcal{A}} \mathrm{Osc}(L_A f).$$

By Observation 2 and the Arzéla-Ascoli theorem, the sequence $f_\nu := L_{A_\nu} f$ has a uniformly convergent subsequence (still denoted by f_ν). Let f_0 denote the limit of this subsequence. Then

$$\mathrm{Osc}(f_0) = \inf_{A\in\mathcal{A}} \mathrm{Osc}(L_A f). \tag{4.7}$$

Now, for every $B \in \mathcal{A}$,

$$\mathrm{Osc}(L_B f_0) = \lim_{\nu\to\infty} \mathrm{Osc}(L_B L_{A_\nu} f) = \lim_{\nu\to\infty} \mathrm{Osc}(L_{A_\nu\cdot B} f) \geq \mathrm{Osc}(f_0).$$

The penultimate equality follows from (4.2) and the last inequality from (4.7). By Observation 1, f_0 is constant. Hence $\mathrm{Osc}(f_0) = 0$ and so Observation 3 follows from (4.7).

Observation 3 shows that there is a sequence $A_\nu \in \mathcal{A}$ such that $L_{A_\nu} f$ converges uniformly to a constant $p \in \mathbb{R}$ (called a **left mean of** f). Similarly, there exists a sequence $B_\nu \in \mathcal{A}$ such that $R_{B_\nu} f$ converges uniformly to a constant $q \in \mathbb{R}$ (called a **right mean of** f). Since

$$\|L_A R_B f - R_B f\| \leq \mathrm{Osc}(R_B f), \qquad \|R_B L_A f - L_A f\| \leq \mathrm{Osc}(L_A f) \tag{4.8}$$

for $f \in \mathcal{C}(\mathrm{G})$ and $A, B \in \mathcal{A}$ it follows that the right and left means agree and hence are independent of the choices of the sequences A_ν and B_ν. Namely,

$$p = \lim_{\nu\to\infty} R_{B_\nu} L_{A_\nu} f = \lim_{\nu\to\infty} L_{A_\nu} R_{B_\nu} f = q.$$

Let us define the operator $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ by

$$M(f) := \lim_{\nu\to\infty} L_{A_\nu} f = \lim_{\nu\to\infty} R_{B_\nu} f,$$

where $A_\nu, B_\nu \in \mathcal{A}$ are chosen such that $\mathrm{Osc}(L_{A_\nu} f)$ and $\mathrm{Osc}(R_{B_\nu} f)$ converge to zero. Thus $M(f)$ is the left mean and the right mean of f. It is immediate from this definition that $M(1) = 1$, $M(\lambda f) = \lambda M(f)$ for $\lambda \in \mathbb{R}$, that M is left and right invariant, and

$$\min f \leq M(f) \leq \max f.$$

Now let $f, f' \in \mathcal{C}(M)$ and choose sequences $A_\nu, B_\nu \in \mathcal{A}$ such that

$$M(f) = \lim_{\nu\to\infty} L_{A_\nu} f, \qquad M(f') = \lim_{\nu\to\infty} R_{B_\nu} f'.$$

Since M is left and right invariant, we have $M(L_{A_\nu} R_{B_\nu}(f+f')) = M(f+f')$. Hence there is a sequence $C_\nu \in \mathcal{A}$ such that

$$M(f+f') = \lim_{\nu\to\infty} L_{C_\nu} R_{B_\nu} L_{A_\nu}(f+f').$$

By (4.8), the right hand side also converges to $M(f) + M(f')$ and hence

$$M(f+f') = M(f) + M(f')$$

for $f, f' \in \mathcal{C}(\mathrm{G})$. Thus we have proved that M is a nonnegative bounded linear functional that satisfies the assertions (i), (ii), and (iii) of the theorem.

We prove that M satisfies (iv). Hence let $f \in \mathcal{C}(\mathrm{G})$ be a function such that $f \geq 0$ and $f \not\equiv 0$. Then, by Observation 1, there exists an $A \in \mathcal{A}$ such that

$$\min L_A f > 0.$$

Choose $B_\nu \in \mathcal{A}$ such that $R_{B_\nu} f$ converges to $M(f)$. Then

$$M(f) = \lim_{\nu\to\infty} L_A R_{B_\nu} f = \lim_{\nu\to\infty} R_{B_\nu} L_A f \geq \min L_A f > 0$$

as claimed.

Next we prove that M is uniquely determined by conditions (i) and (ii). To see this, let M' be another bounded linear functional on $\mathcal{C}(\mathrm{G})$ that satisfies (i) and (ii). Then $M'(c) = c$ for every constant c and

$$M'(L_A f) = M'(f)$$

for every $f \in \mathcal{C}(\mathrm{G})$ and every $A \in \mathcal{A}$. Given $f \in \mathcal{C}(\mathrm{G})$ choose a sequence $A_\nu \in \mathcal{A}$ such that $L_{A_\nu} f$ converges uniformly to $M(f)$. Then

$$M(f) = M'(M(f)) = \lim_{\nu\to\infty} M'(L_{A_\nu} f) = M'(f).$$

This proves uniqueness. That M satisfies condition (v) follows from uniqueness and the fact that the map $\mathcal{C}(\mathrm{G}) \to \mathbb{R} : f \mapsto M(f \circ \phi)$ is a bounded linear functional that satisfies (i) and (ii). This proves Theorem 4.1. $\square$

5 Invariant Inner Products

An inner product $\langle .,. \rangle$ on $\mathfrak{g}$ is called **invariant** iff it is invariant under the adjoint action of G, i.e.

$$\langle g^{-1}\xi g, g^{-1}\eta g \rangle = \langle \xi, \eta \rangle$$

for $\xi, \eta \in \mathfrak{g}$ and $g \in \mathrm{G}$. A Riemannian metric on G is called **bi-invariant** iff the left and right translations L_h and R_h are isometries for every $h \in \mathrm{G}$. Every invariant inner product on $\mathfrak{g}$ determines a bi-invariant metric on G via

$$\langle v, w \rangle := \langle g^{-1}v, g^{-1}w \rangle = \langle vg^{-1}, wg^{-1} \rangle \tag{5.1}$$

for $v, w \in T_g\mathrm{G}$. In turn, such a metric determines a volume form and hence a bi-invariant measure on G. By Theorem 4.1 this agrees with the Haar measure up to a constant factor. Conversely, if G is compact, one can use the existence of a translation invariant measure to prove the existence of an invariant inner product.

Proposition 5.1. *Let* G *be a compact Lie group. Then* $\mathfrak{g}$ *carries an invariant inner product.*

Proof. Let $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ denote the Haar measure and $Q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be any inner product. For $\xi, \eta \in \mathfrak{g}$ define $f_{\xi,\eta} : \mathrm{G} \to \mathbb{R}$ by $f_{\xi,\eta}(g) := Q(g\xi g^{-1}, g\eta g^{-1})$. Then the formula $\langle \xi, \eta \rangle := M(f_{\xi,\eta})$ defines an inner product on $\mathfrak{g}$. That it is invariant follows from the formula $f_{h\xi h^{-1}, h\eta h^{-1}} = f_{\xi,\eta} \circ R_h$. $\square$

Remark 5.2. (i) The proof of Proposition 5.1 shows that the existence of a right invariant measure on G suffices to establish the existence of an invariant inner product on $\mathfrak{g}$, and hence the existence of a bi-invariant measure on G.

(ii) On any Lie group the existence of a right invariant measure is easy to prove. Choose any inner product on $\mathfrak{g}$ and extend it to a Riemannian metric on G by left translation. Then the right translations are isometries and hence the volume form defines a right invariant measure on G.

(iii) Combining (i) and (ii) gives rise to a simpler proof of the existence of a Haar measure for compact Lie groups.

(iv) Uniqueness in Theorem 4.1 implies that every left invariant measure is right invariant. Here is a direct proof for compact Lie Groups: If ω is a left invariant volume form on G, then so is ${R_g}^*\omega$. Hence there exists a group homomorphism $\lambda : \mathrm{G} \to \mathbb{R}$ such that ${R_g}^*\omega = e^{\lambda(g)}\omega$. Since G is compact, the only group homomorphism from G to $\mathbb{R}$ is $\lambda = 0$.

Lemma 5.3. *Let* G *be a compact Lie group with a bi-invariant Riemannian metric. Then the geodesics have the form*

$$\gamma(t) = \exp(t\xi)g$$

for $g \in \mathrm{G}$ *and* $\xi \in \mathfrak{g}$.

Proof. Let $I = [a,b] \subset \mathbb{R}$ be a closed interval and $\gamma_0 : I \to \mathrm{G}$ be a geodesic. Let $\xi : I \to \mathfrak{g}$ be a smooth curve such that $\xi(a) = \xi(b) = 0$ and consider the map

$$\gamma(s,t) := \gamma_0(t)\exp(s\xi(t)).$$

Then $\gamma^{-1}\partial_s\gamma = \xi$ and hence

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_a^b \langle \gamma^{-1}\partial_t\gamma, \gamma^{-1}\partial_t\gamma\rangle\,dt &= \int_a^b \langle \partial_s(\gamma^{-1}\partial_t\gamma), \gamma^{-1}\partial_t\gamma\rangle\,dt \\
&= \int_a^b \langle \partial_t(\gamma^{-1}\partial_s\gamma), \gamma^{-1}\partial_t\gamma\rangle\,dt \\
&= -\int_a^b \langle \xi, \partial_t(\gamma^{-1}\partial_t\gamma)\rangle\,dt.
\end{aligned}$$

Here the penultimate equality follows from Lemma 1.6 and Exercise 5.5. Now γ_0 is a geodesic if and only if the left hand side vanishes at $s = 0$ for every ξ, and the right hand side vanishes at $s = 0$ for every ξ if and only if $\partial_t({\gamma_0}^{-1}\partial_t\gamma_0) \equiv 0$. This proves Lemma 5.3. □

Exercise 5.4. **(i)** Prove that the group $\mathrm{GL}^+(n,\mathbb{R})$ of real $n \times n$-matrices with positive determinant is connected.

(ii) Prove that the exponential map $\exp : \mathbb{R}^{n\times n} \to \mathrm{GL}^+(n,\mathbb{R})$ is not surjective for $n > 1$. **Hint:** Every negative eigenvalue of an exponential matrix $\Phi = \exp(A)$ must have even multiplicity.

(iii) Prove that Φ^2 is an exponential matrix for every $\Phi \in \mathrm{GL}(n,\mathbb{R})$.

(iv) Prove that for every compact connected Lie group G the exponential map $\exp : \mathfrak{g} \to \mathrm{G}$ is surjective. **Hint:** Use Proposition 5.1 (existence of an invariant inner product), Lemma 5.3 (geodesics and exponential map), and the Hopf-Rinow theorem (the existance of minimal geodesics).

Exercise 5.5. Let G be a compact connected Lie group. Prove that an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ is invariant if and only if

$$\langle[\xi,\eta],\zeta\rangle = \langle\xi,[\eta,\zeta]\rangle$$

for $\xi, \eta, \zeta \in \mathfrak{g}$.

6 The Center

Let G be a connected Lie group. The subgroup

$$Z(\mathrm{G}) := \{g \in \mathrm{G} \,|\, gh = hg \,\forall\, h \in \mathrm{G}\}$$

is called the **center** of G. By Theorem 3.1 it is a Lie subgroup of G with the corresponding Lie subalgebra

$$Z(\mathfrak{g}) := \{\xi \in \mathfrak{g} \,|\, [\xi, \eta] = 0 \,\forall\, \eta \in \mathfrak{g}\}\,.$$

Note that $Z(\mathrm{G})$ is a normal subgroup and the center of the quotient $\mathrm{G}/Z(\mathrm{G})$ is trivial. The following theorem is due to Herman Weyl.

Theorem 6.1. *Let* G *be a compact connected Lie group. Then the first Betti number of* G *is given by* $\dim H^1(\mathrm{G}; \mathbb{R}) = \dim Z(\mathfrak{g})$.

Proof. The proof consists of three steps.

Step 1: *Suppose* G *is equipped with a bi-invariant Riemannian metric. Then*

$$\nabla_v X(g) = \left(d\xi(g)v + \frac{1}{2}[\xi(g), \eta]\right) g,$$

where $\xi : \mathrm{G} \to \mathfrak{g}$, $\eta \in \mathfrak{g}$, $v = \eta g \in T_g\mathrm{G}$, *and* $X(g) = \xi(g)g$.

Suppose first that $\xi(g) \equiv \xi$ is constant. Then, by Lemma 5.3, the integral curves of X_ξ are geodesics. Hence

$$\nabla_{X_\xi} X_\xi = 0$$

for every $\xi \in \mathfrak{g}$. Replace ξ by $\xi + \eta$ to obtain

$$\nabla_{X_\eta} X_\xi + \nabla_{X_\xi} X_\eta = 0$$

for all $\xi, \eta \in \mathfrak{g}$. Since

$$\nabla_{X_\eta} X_\xi - \nabla_{X_\xi} X_\eta = [X_\xi, X_\eta] = X_{[\xi,\eta]},$$

it follows that

$$\nabla_{X_\eta} X_\xi = \frac{1}{2} X_{[\xi,\eta]}.$$

This proves Step 1 in the case where $\xi : \mathrm{G} \to \mathfrak{g}$ is constant. The general case is an immediate consequence.

Step 2: *The Riemann curvature tensor of* G *is given by*

$$R(\xi g, \eta g)\zeta g = -\frac{1}{4}[[\xi, \eta], \zeta]g$$

for $g \in \mathrm{G}$ *and* $\xi, \eta, \zeta \in \mathfrak{g}$.

Consider the right invariant vector fields $X_\xi(g) = \xi g$ for $\xi \in \mathfrak{g}$. By Step 1,

$$\nabla_{X_\eta} X_\xi = \frac{1}{2} X_{[\xi,\eta]}.$$

Hence Step 2 follows by straight forward calculation from the identity

$$R(X_\xi, X_\eta)X_\zeta = \nabla_{X_\xi}\nabla_{X_\eta}X_\zeta - \nabla_{X_\eta}\nabla_{X_\xi}X_\zeta + \nabla_{[X_\xi, X_\eta]}X_\zeta.$$

Step 3: *We prove the theorem.*

Let $e_1, \ldots, e_k$ be an orthonormal basis of $\mathfrak{g}$. Then, by Step 2, the Ricci tensor of G is given by

$$\mathrm{Ric}(\xi g, \eta g) = \sum_{i=1}^k \langle R(e_i g, \xi g)\eta g, e_i g\rangle = \frac{1}{4}\sum_{i=1}^k \langle [\xi, e_i], [\eta, e_i]\rangle \tag{6.1}$$

Hence $\mathrm{Ric}(\xi g, \xi g) \geq 0$ with equality if and only if $\xi \in Z(\mathfrak{g})$. Now let $\alpha \in \Omega^1(\mathrm{G})$ and choose $\xi : \mathrm{G} \to \mathfrak{g}$ such that

$$\alpha_g(\eta g) = \langle \xi(g), \eta\rangle.$$

The Bochner–Weitzenböck formula asserts that

$$\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 = \|\nabla\alpha\|_{L^2}^2 + \int_{\mathrm{G}} \mathrm{Ric}(\alpha, \alpha)\mathrm{dvol}. \tag{6.2}$$

Since $\mathrm{Ric}(\alpha, \alpha) \geq 0$, this shows that α is harmonic if and only if $\nabla\alpha \equiv 0$ and $\mathrm{Ric}(\alpha, \alpha) \equiv 0$. By (6.1) and Step 1 this means that

$$d\xi(g)\eta g = \frac{1}{2}[\eta, \xi(g)] = 0$$

for every $g \in \mathrm{G}$ and every $\eta \in \mathfrak{g}$. Equivalently, $\xi : \mathrm{G} \to \mathfrak{g}$ is constant and takes values in the center of $\mathfrak{g}$. Thus we have proved that the space of harmonic 1-forms can be identified with $Z(\mathfrak{g})$. This proves Theorem 6.1. $\square$

Theorem 6.2. *Let* G *be a compact Lie group. Then the following holds.*

(i) *The fundamental group of* G *is abelian.*

(ii) *If* $Z(\mathrm{G})$ *is finite, then so is* $\pi_1(\mathrm{G})$*.*

Proof. Assertion (i) holds for every topological group. To see this, choose two curves $\alpha, \beta : [0,1] \to \mathrm{G}$ with the endpoints

$$\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = \mathbb{1}.$$

Denote

$$\alpha\#\beta(t) := \begin{cases} \alpha(2t), & \text{if } 0 \le t \le 1/2, \\ \alpha(1)\beta(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

(Here the term $\alpha(1)$ can be dropped, but the more general form will be needed below.) Define

$$\alpha_s(t) := \begin{cases} \alpha(2t-s), & \text{if } s/2 \le t \le (s+1)/2, \\ \mathbb{1}, & \text{otherwise,} \end{cases}$$

and

$$\beta_s(t) := \begin{cases} \beta(2t+s-1), & \text{if } (1-s)/2 \le t \le 1-s/2, \\ \mathbb{1}, & \text{otherwise,} \end{cases}$$

for $0 \le s, t \le 1$. Then $\gamma_s(t) = \alpha_s(t)\beta_s(t)$ is a homotopy from $\gamma_0 = \alpha\#\beta$ to $\gamma_1 = \beta\#\alpha$. This proves (i).

To prove (ii), note that by (i) the fundamental group

$$\Gamma := \pi_1(\mathrm{G})$$

is abelian, and by Theorem 6.1

$$\mathrm{Hom}(\Gamma, \mathbb{R}) \cong H^1(\mathrm{G}; \mathbb{R}) = 0.$$

This implies that Γ is finite. To see this, note first that Γ is finitely generated. Let $\gamma_1, \dots, \gamma_n \in \Gamma$ be generators. Since Γ is abelian, the set $R \subset \mathbb{Z}^n$ of all integer vectors $m = (m_1, \dots, m_n)$ that satisfy

$$\gamma_1{}^{m_1} \cdots \gamma_n{}^{m_n} = 1$$

form a subgroup of $\mathbb{Z}^n$ and there is a natural isomorphism

$$\Gamma \cong \mathbb{Z}^n / R.$$

Since $\mathrm{Hom}(\Gamma, \mathbb{R}) = R^\perp = \{0\}$, it follows that R spans $\mathbb{R}^n$. Hence the quotient $\mathbb{R}^n/R$ is compact, so $\Gamma \cong \mathbb{Z}^n/R$ is a finite set. This proves Theorem 6.2. □

Now let us denote by $\widetilde{\mathrm{G}}$ the universal cover of G. In explicit terms,

$$\widetilde{\mathrm{G}} = \{\gamma : [0,1] \to \mathrm{G} \,|\, \gamma(0) = \mathbb{1}\} \,/ \sim$$

where $\sim$ denotes homotopy with fixed endpoints. The projection

$$\pi : \widetilde{\mathrm{G}} \to \mathrm{G}$$

is given by $\pi([\gamma]) = \gamma(1)$.

Proposition 6.3. *Let* G *be a connected Lie group. Then*

$$Z(\widetilde{\mathrm{G}}) = \pi^{-1}(Z(\mathrm{G})).$$

Proof. Let $\alpha, \beta : [0,1] \to \mathrm{G}$ be smooth curves such that

$$\alpha(0) = \beta(0) = \mathbb{1}, \qquad \alpha(1) \in Z(\mathrm{G}).$$

Define

$$\alpha_s(t) := \begin{cases} \alpha((1+s)t), & \text{if } 0 \le t \le 1/(s+1), \\ \alpha(1), & \text{otherwise,} \end{cases}$$

and

$$\beta_s(t) := \begin{cases} \beta((2t-s)/(2-s)), & \text{if } s/2 \le t \le 1, \\ \mathbb{1}, & \text{otherwise,} \end{cases}$$

for $0 \le s, t \le 1$. Since $\alpha(1) \in Z(\mathrm{G})$, we have

$$\alpha_1\beta_1 = \beta_1\alpha_1 = \alpha\#\beta.$$

Moreover $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Hence both $\alpha\beta$ and $\beta\alpha$ are homotopic to $\alpha\#\beta$. This proves that

$$\pi^{-1}(Z(\mathrm{G})) \subset Z(\widetilde{\mathrm{G}}).$$

The converse inclusion is obvious. □

The **commutator subgroup**

$$[\mathrm{G}, \mathrm{G}] \subset \mathrm{G}$$

is defined as the smallest subgroup of G that contains all commutators $[a, b] := aba^{-1}b^{-1}$ for $a, b \in \mathrm{G}$. Thus $[\mathrm{G}, \mathrm{G}]$ is the subset of all products of finitely many such commutators. It is a normal subgroup of G.

Proposition 6.4. *Let* G *be a compact connected Lie group. Then* $Z(\mathrm{G})$ *is finite if and only if* $[\mathrm{G},\mathrm{G}] = \mathrm{G}$.

Proof. Choose an invariant inner product on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ (Proposition 5.1) and consider the subbundle

$$E := \{(g, \xi g) \,|\, \xi \perp Z(\mathfrak{g})\} \subset T\mathrm{G}.$$

By Exercise 5.5, the Lie bracket of any two right invariant vector fields $X_\xi(g) = \xi g$ and $X_\eta(g) = \eta g$ is contained in E. Hence, by Frobenius' theorem, E is integrable. Let H be the leaf of E through $\mathbb{1}$, i.e.

$$\mathrm{H} := \left\{\gamma(1) \,|\, \gamma : [0,1] \to \mathrm{G},\, \gamma(0) = \mathbb{1},\, \gamma(t)^{-1}\dot\gamma(t) \perp Z(\mathfrak{g})\right\}.$$

If $\alpha, \beta : [0,1] \to \mathrm{G}$ are paths that are tangent to E, then so are $\alpha\beta$ and α^{-1}. Hence H is a subgroup of G. Next we prove that

$$[\mathrm{G},\mathrm{G}] \subset \mathrm{H}.$$

To see this note that, for every pair $\xi, \eta \in \mathfrak{g}$ the curve

$$\gamma(t) := \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta)$$

is tangent to E. Since the exponential map is surjective it follows that every commutator $[a,b] = aba^{-1}b^{-1}$ of two elements in G lies in H. Hence

$$[\mathrm{G},\mathrm{G}] \subset \mathrm{H}.$$

Next we prove that

$$\mathrm{H} \subset [\mathrm{G},\mathrm{G}]$$

To see this note that, by Exercise 5.5, the orthogonal complement of $Z(\mathfrak{g})$ is spanned by vectors of the form $[\xi,\eta]$ for $\xi,\eta \in \mathfrak{g}$. Choose $\xi_1,\dots,\xi_k,\eta_1,\dots,\eta_k$ such that the vectors $[\xi_i,\eta_i]$ form a basis of $Z(\mathfrak{g})^\perp$. For $i = 1,\dots,k$ define the curve $\gamma_i : \mathbb{R} \to [\mathrm{G},\mathrm{G}]$ by

$$\gamma_i(t) := \exp(\sqrt{t}\xi_i)\exp(\sqrt{t}\eta_i)\exp(-\sqrt{t}\xi_i)\exp(-\sqrt{t}\eta_i)$$

for $t \geq 0$ and $\gamma_i(t) := \gamma_i(-t)^{-1}$ for $t < 0$. Then γ_i is continuously differentiable and $\dot\gamma_i(0) = [\xi_i,\eta_i]$. Define the map $\phi : \mathbb{R}^k \to [\mathrm{G},\mathrm{G}]$ by

$$\phi(t_1,\dots,t_k) := \gamma_1(t_1)\cdots\gamma_k(t_k).$$

This map is a continuously differentiable embedding near $t = 0$ and it is everywhere tangent to E. Hence the image U_0 of a sufficiently small neighbourhood of $0 \in \mathbb{R}^k$ under ϕ is a neighbourhood of $\mathbb{1}$ in H with respect to the intrinsic topology of H and it is contained in $[\mathrm{G}, \mathrm{G}]$. More generally, for every $h \in \mathrm{H}$ the set $U = U_0 h \subset \mathrm{H}$ is a neighbourhood of h with respect to the intrinsic topology and $Uh^{-1} \subset [\mathrm{G}, \mathrm{G}]$. Hence the sets $\mathrm{H} \cap [\mathrm{G}, \mathrm{G}]$ and $\mathrm{H} \setminus [\mathrm{G}, \mathrm{G}]$ are both open with respect to the intrinsic topology of H. Since $\mathrm{H} \cap [\mathrm{G}, \mathrm{G}] \neq \emptyset$ it follows that $\mathrm{H} \subset [\mathrm{G}, \mathrm{G}]$, as claimed.

Thus we have proved that

$$[\mathrm{G}, \mathrm{G}] = \mathrm{H}$$

is a leaf of the foliation determined by E. Hence $[\mathrm{G}, \mathrm{G}] = \mathrm{G}$ if and only if $E = T\mathrm{G}$ if and only if $Z(\mathrm{G})$ is finite. This proves Proposition 6.4. □

Corollary 6.5. *Let* G *be a compact connected Lie group with finite center. Then every principal* G*-bundle* $P \to \Sigma$ *over a compact oriented Riemann surface of sufficiently large genus carries a flat connection.*

Proof. By Theorem 6.2, $\pi_1(\mathrm{G})$ is finite, and hence $\widetilde{\mathrm{G}}$ is compact. By Proposition 6.3, we have

$$\pi_1(\mathrm{G}) = \pi^{-1}(\mathbb{1}) \subset Z(\widetilde{\mathrm{G}}).$$

There is a one-to-one correspondence between isomorphism classes of principal G-bundles over a Riemann surface and elements $\gamma \in \pi_1(\mathrm{G})$. Suppose that Σ is a Riemann surface of genus g and let $P_\gamma \to \Sigma$ be the principal bundle corresponding to $\gamma \in \pi_1(\mathrm{G})$. Then a gauge equivalence class of flat connection on P_γ (with respect to the identity component of the gauge group) can be represented by elements

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in \widetilde{\mathrm{G}}$$

that satisfy

$$\prod_{j=1}^{g} [\alpha_j, \beta_j] = \gamma.$$

By Proposition 6.4, every element $\gamma \in \widetilde{\mathrm{G}}$ can be expressed in this form whenever $Z(\widetilde{\mathrm{G}})$ is finite. This proves Corollary 6.5. □

Exercise 6.6. The nontrivial SO(3)-bundle over the 2-sphere does not carry a flat connection.

7 Inner Automorphisms

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. An automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ is called an **inner automorphism** iff there exists a smooth path $\eta : [0,1] \to \mathfrak{g}$ such that $\Phi(\xi(0)) = \xi(1)$ for every solution $\xi : [0,1] \to \mathfrak{g}$ of the ordinary differential equation $\dot{\xi} + [\xi, \eta] = 0$. The inner automorphisms of $\mathfrak{g}$ form a group which will be denoted by $\mathrm{Inn}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$. If $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ is the Lie algebra of a connected Lie group G, then $\mathrm{Inn}(\mathfrak{g})$ is the image of the adjoint representation $\mathrm{Ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$. The following example shows that $\mathrm{Inn}(\mathfrak{g})$ need not be a closed subset of $\mathrm{Aut}(\mathfrak{g})$.

Example 7.1. Let $A \in \mathrm{GL}(V)$ be an automorphism of a real or complex vector space V. Then $\mathrm{G} = \mathbb{R} \times V$ is a Lie group with the product

$$(s, x) \cdot (t, y) := (s + t, x + \exp(sA)y) \tag{7.1}$$

for $s, t \in \mathbb{R}$ and $x, y \in V$. The unit is the origin in $\mathbb{R} \times V$, the inverse of $(s, x) \in \mathrm{G}$ is the pair $(s, x)^{-1} = (-s, -\exp(-sA)x)$, the adjoint action of G on itself is given by $(s, x) \cdot (t, y) \cdot (s, x)^{-1} = (t, x + \exp(sA)y - \exp(tA)x)$, so the adjoint action of $(s, x) \in \mathrm{G}$ on the Lie algebra $\mathfrak{g} = \mathbb{R} \times V$ is given by

$$\mathrm{Ad}(s, x)(\tau, \eta) = (\tau, \exp(sA)\eta - \tau Ax) \tag{7.2}$$

for $(\tau, \eta) \in \mathfrak{g}$, and hence the Lie bracket on $\mathfrak{g}$ is given by

$$[(\sigma, \xi), (\tau, \eta)] = (0, \sigma A\eta - \tau A\xi) \tag{7.3}$$

for $(\sigma, \xi), (\tau, \eta) \in \mathfrak{g}$. Since A is nonsingular, the center of $\mathfrak{g}$ is trivial.

Consider the case $V = \mathbb{C}^2$ and $A = \mathrm{diag}(\mathbf{i}\alpha, \mathbf{i}\beta)$, where α, β are nonzero real numbers such that α/β is irrational. Then

$$\mathrm{Ad}(s, x)(\tau, \eta) = (\tau, e^{s\mathbf{i}\alpha}\eta_1 - \tau\mathbf{i}\alpha x_1, e^{s\mathbf{i}\beta}\eta_2 - \tau\mathbf{i}\beta x_2)$$

for $s, \tau \in \mathbb{R}$ and $x_1, x_2, \eta_1, \eta_2 \in \mathbb{C}$ by (7.2). Now choose a pair $\lambda_1, \lambda_2 \in S^1$ such that $(\lambda_1, \lambda_2) \neq (e^{s\mathbf{i}\alpha}, e^{s\mathbf{i}\beta})$ for all $s \in \mathbb{R}$ and define the linear map $\Phi_\lambda : \mathfrak{g} \to \mathfrak{g}$ by $\Phi_\lambda(\tau, \eta) := (\tau, \lambda_1\eta_1, \lambda_2, \eta_2)$. Then $\Phi_\lambda \in \mathrm{Aut}(\mathfrak{g}) \setminus \mathrm{Inn}(\mathfrak{g})$. Since α/β is irrational, there exists a sequence $s_i \in \mathbb{R}$ such that $\lim_{i\to\infty}(e^{s_i\mathbf{i}\alpha}, e^{s_i\mathbf{i}\beta}) = (\lambda_1, \lambda_2)$. Hence $\lim_{i\to\infty} \mathrm{Ad}(s_i, 0) = \Phi_\lambda$ and so $\mathrm{Inn}(\mathfrak{g})$ is not closed in $\mathrm{Aut}(\mathfrak{g})$.

For every finite-dimensional Lie algebra $\mathfrak{g}$ the group $\mathrm{Inn}(\mathfrak{g})$ is the leaf through the identity of a foliation on $\mathrm{Aut}(\mathfrak{g})$ with $T_{\mathbb{1}}\mathrm{Inn}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$. Thus it carries an intrinsic smooth structure which turns it into a Lie group with the Lie algebra $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$. This shows that every finite-dimensional Lie algebra with a trivial center is the Lie algebra of a connected Lie group.

8 Maximal Toral Subgroups

Let G be a compact connected Lie group. A **Lie subgroup** of G is a closed subgroup H which is a submanifold. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** iff it is invariant under the Lie bracket. If $\mathrm{H} \subset \mathrm{G}$ is a Lie subgroup, then by definition of the Lie bracket $\mathfrak{h} := T_{\mathbb{1}}\mathrm{H}$ is a Lie subalgebra of $\mathfrak{g}$. A **maximal torus** in G is a connected abelian subgroup $T \subset \mathrm{G}$ which is not properly contained in any other connected abelian subgroup. The fundamental example is the subgroup of diagonal matrices in $\mathrm{U}(n)$ or $\mathrm{SU}(n)$.

Exercise 8.1. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus with Lie algebra $\mathfrak{t} := \mathrm{Lie}(\mathrm{T})$. Let $\eta \in \mathfrak{g}$ such that $[\eta, \tau] = 0$ for every $\tau \in \mathfrak{t}$. Prove that $\eta \in \mathfrak{t}$.

Lemma 8.2. *Let* G *be a compact connected Lie group and* $\mathrm{T} \subset \mathrm{G}$ *be a maximal torus. Then every element in* G *is conjugate to an element in* T.

Proof. Given $h \in \mathrm{G}$ choose $\xi \in \mathfrak{g}$ with $\exp(\xi) = h$. Such an element exists by Exercise 5.4 (iv). Then, by Exercise 1.7, $ghg^{-1} = \exp(g\xi g^{-1})$ for every $g \in \mathrm{G}$. Hence we must find $g \in \mathrm{G}$ such that $g\xi g^{-1} \in \mathrm{Lie}(\mathrm{T}) = \mathfrak{t}$. Choose an invariant inner product on $\mathfrak{g}$ and fix a generator $\tau \in \mathfrak{t}$ such that $\{\exp(s\tau) \,|\, s \in \mathbb{R}\}$ is dense in T. Since the orbit of ξ under the adjoint action of G is compact there is an $\eta \in \mathfrak{g}$, conjugate to ξ, which minimizes the distance to τ in this conjugacy class, i.e.

$$|\eta - \tau|^2 = \inf_{g \in \mathrm{G}} |g\eta g^{-1} - \tau|^2.$$

We must prove that $\eta \in \mathfrak{t}$. To see this differentiate the map

$$\mathrm{G} \to \mathbb{R} : g \mapsto |g\eta g^{-1} - \tau|^2$$

at $g = \mathbb{1}$ to obtain $\langle \eta - \tau, [\zeta, \eta] \rangle = 0$ for all $\zeta \in \mathfrak{g}$. This implies $\langle \zeta, [\eta, \tau] \rangle = 0$ for all $\zeta \in \mathfrak{g}$ and hence $[\eta, \tau] = 0$. By Exercise 1.9, $\exp(t\eta)$ commutes with $\exp(s\tau)$ for all s and t. Since τ generates the torus, it follows that $\exp(t\eta)$ commutes with T for every t and hence $[\eta, \mathfrak{t}] = 0$. By Exercise 8.1, this implies $\eta \in \mathfrak{t}$. □

Lemma 8.3. *Any two maximal tori in* G *are conjugate.*

Proof. Let $\mathrm{T}_1, \mathrm{T}_2 \subset \mathrm{G}$ be two maximal tori and choose an element $g_2 \in \mathrm{T}_2$ such that $\mathrm{T}_2 = \mathrm{cl}\left(\left\{{g_2}^k \,|\, k \in \mathbb{Z}\right\}\right)$. By Lemma 8.2, there exists a $g \in \mathrm{G}$ such that $g_2 \in g\mathrm{T}_1 g^{-1}$. Hence $\mathrm{T}_2 \subset g\mathrm{T}_1 g^{-1}$, and hence $\mathrm{T}_2 = g\mathrm{T}_1 g^{-1}$. □

Lemma 8.3 shows that any two maximal tori in G have the same dimension. This dimension is called the **rank** of G. The rank of G agrees with the dimension of a maximal abelian Lie subalgebra of $\mathfrak{g}$. (Prove this!)

Lemma 8.4. *Let* $\mathrm{T} \subset \mathrm{G}$ *be a maximal torus. Then every element of the Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is conjugate to an element of* $\mathfrak{t} := \mathrm{Lie}(\mathrm{T})$.

Proof. Let $\xi \in \mathfrak{g}$. The set $\overline{\{\exp(s\xi) \,|\, s \in \mathbb{R}\}}$ is a torus and hence is contained in a maximal torus T'. By Lemma 8.3, there exists a $g \in \mathrm{G}$ such that $g\mathrm{T}'g^{-1} = \mathrm{T}$. Hence $\exp(sg\xi g^{-1}) \in \mathrm{T}$ for every $s \in \mathbb{R}$, and hence $g\xi g^{-1} \in \mathfrak{t}$. This proves Lemma 8.4. □

Lemma 8.5. *Let* G *be a compact connected Lie group and let* $\mathrm{T} \subset \mathrm{G}$ *be a maximal torus. Then* T *is a maximal abelian subgroup of* G.

Proof. We follow the argument of Frank Adams in *Lectures on Lie groups.* Let $h \in \mathrm{G}$ be an element that commutes with T. We shall prove that $h \in \mathrm{T}$. To see this let $\mathrm{S} \subset \mathrm{G}$ be a maximal torus containing h and denote by

$$\mathrm{H} := \mathrm{cl}\left(\left\{h^k \,|\, k \in \mathbb{Z}\right\}\right)$$

the subgroup of G generated by h. Examining closed subgroups of tori we see that H is a Lie subgroup of S. Moreover, the Lie algebra $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$ commutes with $\mathfrak{t} = \mathrm{Lie}(\mathrm{T})$ and hence must be contained in $\mathfrak{t}$. Hence the identity component of H is equal to $\mathrm{H} \cap \mathrm{T}$ and the quotient $\mathrm{H}/(\mathrm{H} \cap \mathrm{T})$ is a finite group. This finite group is generated by a single element $[h] \in \mathrm{G}/(\mathrm{T} \cap \mathrm{H})$ and hence is isomorphic to $\mathbb{Z}_m$ for some integer m. This implies $h^m \in \mathrm{T}$. Hence the set

$$\widehat{\mathrm{T}} := \{h^i t \,|\, t \in \mathrm{T},\, 1 \le i \le m-1\}$$

is a Lie subgroup of G such that

$$\widehat{\mathrm{T}}/\mathrm{T} \cong \mathbb{Z}_m.$$

Any such group is generated by a single element $\widehat{h}$. To see this, choose an isomorphism $\phi : \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{T}$, a vector $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$ such that $\phi(\tau) = h^m$, and a vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ such that the numbers $1, \omega_1, \ldots, \omega_n$ are rationally independent. Then the element

$$\widehat{h} := h\phi((\omega - \tau)/m) \in \mathrm{G}$$

generates $\widehat{\mathrm{T}}$. By Lemma 8.2, there exists a maximal torus containing $\widehat{h}$ and hence both h and T. Since T is a maximal torus it follows that $h \in \mathrm{T}$. This proves Lemma 8.5. □

Example 8.6. In general, a maximal abelian subgroup need not be a torus. For example the $n \times n$-matrices with diagonal entries ± 1 and determinant 1 form a maximal abelian subgroup of $\mathrm{G} = \mathrm{SO}(n)$.

For every maximal torus $\mathrm{T} \subset \mathrm{G}$ denote

$$\mathrm{G_T} := \left\{ g \in \mathrm{G} \,|\, g^{-1}\mathrm{T}g = \mathrm{T} \right\}.$$

The quotient $\mathrm{W} := \mathrm{G_T}/\mathrm{T}$ is called the **Weyl group** of T. The next lemma shows that every adjoint orbit in $\mathfrak{g}$ intersects $\mathfrak{t}/\mathrm{W}$ in precisely one point.

Lemma 8.7. *Let* G *be a compact connected Lie group and* $\mathrm{T} \subset \mathrm{G}$ *be a maximal torus. Let* $\xi, \eta \in \mathfrak{t}$. *Then the following are equivalent.*

(i) *There exists a* $g \in \mathrm{G}$ *such that* $g^{-1}\xi g = \eta$.

(ii) *There exists a* $g \in \mathrm{G_T}$ *such that* $g^{-1}\xi g = \eta$.

Proof. That (ii) implies (i) is obvious. Hence suppose that ${g_0}^{-1}\xi g_0 = \eta$ for some $g_0 \in \mathrm{G}$. Choose sequences $\xi_\nu, \eta_\nu \in \mathfrak{t}$ such that $\xi_\nu \to \xi$, $\eta_\nu \to \eta$, and

$$\mathrm{cl}(\{\exp(s\xi_\nu) \,|\, s \in \mathbb{R}\}) = \mathrm{cl}(\{\exp(t\eta_\nu) \,|\, t \in \mathbb{R}\}) = T$$

for every ν. Choose $g_\nu \in \mathrm{G}$ such that

$$\left|{g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right| = \inf_{g \in \mathrm{G}} \left|g^{-1}\xi_\nu g - \eta_\nu\right|. \tag{8.1}$$

Since $|{g_0}^{-1}\xi_\nu g_0 - \eta_\nu|$ converges to zero it follows that

$$\lim_{\nu\to\infty} \left|{g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right| = 0. \tag{8.2}$$

Now differentiate the map $g \mapsto |g^{-1}\xi_\nu g - \eta_\nu|^2$ at $g = g_\nu$. Then, by (8.1), we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \left|\exp(-t\zeta){g_\nu}^{-1}\xi_\nu g_\nu \exp(t\zeta) - \eta_\nu\right|^2 \\ &= \langle {g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu, [{g_\nu}^{-1}\xi_\nu g_\nu, \zeta]\rangle \\ &= \langle [{g_\nu}^{-1}\xi_\nu g_\nu, \eta_\nu], \zeta\rangle \end{aligned}$$

for every $\zeta \in \mathfrak{g}$. Hence $[{g_\nu}^{-1}\xi_\nu g_\nu, \eta_\nu] = 0$. Since η_ν generates the torus T this implies ${g_\nu}^{-1}\xi_\nu g_\nu \in \mathfrak{t}$. Since ξ_ν generates the torus, this implies $g_\nu \in \mathrm{G_T}$. Passing to a convergent subsequence, we may assume that g_ν converges to some element $g \in \mathrm{G_T}$. By (8.2), we have

$$g^{-1}\xi g - \eta = \lim_{\nu\to\infty} \left({g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right) = 0$$

and this proves Lemma 8.7. $\square$

9 Isotropy Subgroups

Let G be a compact connected Lie group and M be a compact smooth manifold equipped with a left action of G. The action will be denoted by

$$\mathrm{G} \times M \to M : (g, x) \mapsto gx.$$

The **isotropy subgroup** of an element $x \in M$ is defined by

$$\mathrm{G}_x := \{h \in \mathrm{G} \,|\, hx = x\}.$$

Since $\mathrm{G}_{gx} = g\mathrm{G}_x g^{-1}$ the set of isotropy subgroups is invariant under conjugation. The next theorem asserts that the set of conjugacy classes of isotropy subgroups is finite.

Theorem 9.1. *There exist finitely many Lie subgroups* $\mathrm{H}_1, \ldots, \mathrm{H}_N$ *of* G *such that for every* $x \in M$ *there exists a* j *such that* G_x *is conjugate to* H_j*.*

Proof. The proof is by induction on the dimension of M. If M is zero-dimensional, then the result is obvious. Now assume that dim $M = n > 0$ and that the result has been proved for all manifolds of dimensions less than n. We prove that every point $x_0 \in M$ has a neighbourhood U in which only finitely many isotropy subgroups occur up to conjugacy. To see this, let $\mathrm{G}_0 := \mathrm{G}_{x_0}$ choose a G-invariant metric on M, denote by $L_x : \mathfrak{g} \to T_xM$ the infinitesimal action, and define $H_0 := \ker\, L_{x_0}^* \subset T_{x_0}M$. Then the exponential map

$$\mathrm{G} \times H_0 \to M : (g, v_0) \mapsto g \exp_{x_0}(v_0)$$

descends to a map

$$\phi_0 : \mathrm{G} \times_{\mathrm{G}_0} H_0 \to M,$$

where $(g, v_0) \sim (gg_0, g_0^{-1}v_0)$ for $g \in \mathrm{G}$, $v_0 \in H_0$, and $g_0 \in \mathrm{G}_0$. The restriction of ϕ_0 to a sufficiently small neighbourhood of the zero section in the vector bundle $\mathrm{G} \times_{\mathrm{G}_0} H_0 \to \mathrm{G}/\mathrm{G}_0$ is a G-equivariant diffeomorphism onto a neighbourhood of the G-orbit of x_0. It follows that the isotropy groups of points $x \in M$ belonging to this neighbourhood are all conjugate to subgroups of G_0 that appear as isotropy subgroups of the action of G_0 on H_0. By considering the action of G_0 on the unit sphere in H_0 we obtain from the induction hypothesis that there are only finitely many such isotropy subgroups. This proves the local statement. Cover M by finitely many such neighbourhoods to prove the global assertion of Theorem 9.1. $\square$

10 Centralizers

Let G be a compact connected Lie group. For any subset $\mathrm{H} \subset \mathrm{G}$ the **centralizer** of H is defined by

$$C(\mathrm{H}) := C_{\mathrm{G}}(\mathrm{H}) := \{g \in \mathrm{G} \,|\, gh = hg \,\forall\, h \in \mathrm{H}\}$$

By Theorem 3.1 this set is a Lie subgroup of G with Lie algebra

$$\mathrm{Lie}(C(\mathrm{H})) = \left\{\xi \in \mathfrak{g} \,|\, h\xi h^{-1} = \xi \,\forall\, h \in \mathrm{H}\right\} = \bigcap_{h \in \mathrm{H}} \ker(\mathbb{1} - \mathrm{Ad}(h)).$$

Moreover, $C(\mathrm{G}) = Z(\mathrm{G})$ is the center of G and $C(Z(\mathrm{G})) = \mathrm{G}$. A subgroup $\mathrm{H} \subset \mathrm{G}$ is abelian if and only if $\mathrm{H} \subset C(\mathrm{H})$ and it is maximal abelian if and only if $\mathrm{H} = C(\mathrm{H})$. For a singleton $\mathrm{H} = \{h\}$ we denote $C(h) := C(\{h\})$.

Lemma 10.1. *Let* G *be a group. Then, for every subset* $\mathrm{H} \subset \mathrm{G}$*,*

$$\mathrm{H} \subset C(C(\mathrm{H})), \qquad C(C(C(\mathrm{H}))) = C(\mathrm{H}).$$

Proof. The first assertion follows directly from the definition and the second assertion follows from the first. Namely, the inclusion $\mathrm{H} \subset C(C(\mathrm{H}))$ implies that $C(C(C(\mathrm{H}))) \subset C(\mathrm{H})$, and the converse inclusion follows by applying the first assertion to $C(\mathrm{H})$ instead of H. This proves Lemma 10.1. □

A subgroup $\mathrm{H} \subset \mathrm{G}$ is called a **centralizer subgroup** iff there exists a subset of G whose centralizer is equal to H. By Lemma 10.1 this condition is equivalent to

$$\mathrm{H} = C(C(\mathrm{H})). \tag{10.1}$$

A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **centralizer subalgebra** iff there exists a centralizer subgroup $\mathrm{H} \subset \mathrm{G}$ such that $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$. Let us denote by $\mathcal{C} \subset 2^{\mathrm{G}}$ the set of all centralizer subgroups. By (10.1), the map

$$\mathcal{C} \to \mathcal{C} : \mathrm{H} \mapsto C(\mathrm{H}) \tag{10.2}$$

is an involution. Moreover the group G acts on $\mathcal{C}$ by conjugation and the involution $\mathrm{H} \mapsto C(\mathrm{H})$ is equivariant under this action, i.e.

$$C(g\mathrm{H}g^{-1}) = gC(\mathrm{H})g^{-1}.$$

The fixed points of the involution (10.2) are the maximal abelian subgroups of G. Consider the equivalence relation on $\mathcal{C}$ defined by $\mathrm{H} \sim \mathrm{H}'$ iff there exists an element $g \in \mathrm{G}$ such that $\mathrm{H}' = g\mathrm{H}g^{-1}$. The following theorem asserts that the quotient $\mathcal{C}/{\sim}$ is a finite set.

Theorem 10.2. *Let* G *be a compact connected Lie group. Then there exist finitely many centralizer subgroups* $\mathrm{H}_1, \dots, \mathrm{H}_m$ *of* G *such that every centralizer subgroup* $\mathrm{H} \subset \mathrm{G}$ *is conjugate to one of the* H_i*.*

Proof. Since G is compact it admits a faithful representation $\rho : \mathrm{G} \to \mathrm{U}(n)$. Now let $\mathrm{H} \subset \mathrm{G}$ be a subgroup and let $g \in C(\mathrm{H})$. Then $\rho(g)$ commutes with all matrices in the span of $\rho(\mathrm{H})$. Choose elements $h_1, \dots, h_{n^2} \in H$ such that $\rho(\mathrm{H})$ is contained in the span of the matrices $\rho(h_i)$. Then $C(\mathrm{H})$ can be characterized as the set of all $g \in \mathrm{G}$ such that $\rho(g)$ commutes with $\rho(h_i)$ for $i = 1, \dots, n^2$. In other words, if the group G acts on the vector space $V := (\mathbb{C}^{n\times n})^{n^2}$ by $g \cdot A_i := \rho(g)A_i\rho(g)^{-1}$ for $i = 1, \dots, n^2$, then every centralizer subgroup of G is the isotropy subgroup of some element of V. By Theorem 9.1, the set of conjugacy classes of such isotropy subgroups is finite. This proves Theorem 10.2. □

11 Simple Groups

A Lie algebra $\mathfrak{g}$ is called **abelian** iff the Lie bracket vanishes. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an **ideal** iff $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. The Lie algebra of a normal Lie subgroup of G is necessarily an ideal. A Lie algebra $\mathfrak{g}$ is called **simple** iff it is not abelian and has no nontrivial ideals (that is $\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$). It is called **semi-simple** iff it is a direct sum of simple Lie algebras. A Lie group is called **simple** (respectively **semi-simple**) iff its Lie algebra is simple (respectively semi-simple).

Theorem 11.1. *Every compact connected simply connected simple Lie group is isomorphic to one in the following list*

$$\begin{array}{rcll} A_n & := & \mathrm{SU}(n+1), & n \geq 1, \\ B_n & := & \mathrm{Spin}(2n+1), & n \geq 2, \\ C_n & := & \mathrm{Sp}(n), & n \geq 3, \\ D_n & := & \mathrm{Spin}(2n), & n \geq 4, \end{array}$$

or to one of the exceptional groups G_2*,* F_4*,* E_6*,* E_7*,* E_8*.*

There are relations such as $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$, $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$, and $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. The Lie groups $D_1 \cong \mathrm{Spin}(2) \cong \mathrm{U}(1) \cong S^1$ and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ are not simple. (See Exercises 12.10 and 12.11 below for Spin(3) and Spin(4).)

The Killing form

Every Lie algebra carries a natural pairing

$$\kappa(\xi,\eta) := \mathrm{trace}(\mathrm{Ad}(\xi)\mathrm{Ad}(\eta))$$

called the **Killing form**. On $\mathfrak{su}(n)$ this form is negative definite. In general the Killing form may have a kernel and/or be indefinite.

Theorem 11.2 (**Cartan**). *The Killing form is nondegenerate (and negative definite) if and only if* G *is semisimple.*

Exercise 11.3. Prove that the Killing forms on $\mathfrak{su}(n)$ and $\mathfrak{so}(2n)$ are given by

$$\begin{aligned}\kappa(\xi,\eta) &= -(2n-1)\,\mathrm{trace}(\xi^*\eta), &\qquad \xi,\eta\in\mathfrak{su}(n),\\ \kappa(\xi,\eta) &= -(n-2)\,\mathrm{trace}(\xi^T\eta), &\qquad \xi,\eta\in\mathfrak{so}(2n).\end{aligned}$$

Exercise 11.4. Prove that the Killing form on $\mathrm{SL}(2,\mathbb{R})$ is indefinite.

Root systems

Let G be a compact Lie group with maximal torus T. The exponential map is onto by Exercise 5.4 (iv). It determines an isomorphism

$$\mathrm{T}\cong\mathfrak{t}/\Lambda$$

where $\mathfrak{t}=\mathrm{Lie}(\mathrm{T})$ and

$$\Lambda := \{\tau\in\mathfrak{t}\,|\,\exp(\tau)=\mathbb{1}\}$$

is a lattice which spans $\mathfrak{t}$. A one-dimensional complex representation is a homomorphism $\mathrm{T}\to S^1$. Under the identification $\mathrm{T}\cong\mathfrak{t}/\Lambda$ any such homomorphism is of the form $\tau\mapsto e^{2\pi\mathrm{i}\mathrm{w}(\tau)}$ where

$$\mathrm{w}:\mathfrak{t}\to\mathbb{R}$$

is a linear map that satisfies

$$\mathrm{w}(\Lambda)\subset\mathbb{Z}.$$

Any such map $\mathrm{w}\in\mathfrak{t}^*$ is called a **weight**.

Now consider the adjoint representation of T on $\mathfrak{g}$. Since the action preserves any invariant inner product, the commuting Automorphisms $\mathrm{ad}(\tau)$ for $\tau \in \mathfrak{t}$ are simultaneously diagonalizable (over $\mathbb{C}$). It follows that there exists a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} V_\alpha$$

where $V_\alpha \subset \mathfrak{g}$ are two-dimensional representations of T. In other words there exists a complex structure J_α on V_α and weights $\mathrm{w}_\alpha \in \mathfrak{t}^*$ such that

$$[\tau, \xi] = 2\pi J_\alpha \mathrm{w}_\alpha(\tau)\xi, \qquad \tau \in \mathfrak{t}, \qquad \xi \in V_\alpha.$$

The weights w_α are called the **roots** of the Lie algebra $\mathfrak{g}$. For each α define $\tau_\alpha \in \mathfrak{t}$ to be the dual element with respect to the Killing form, i.e.

$$\kappa(\tau_\alpha, \sigma) = \mathrm{w}_\alpha(\sigma), \qquad \sigma \in \mathfrak{t}.$$

The **length** of the root w_α is defined by

$$\ell(\alpha) := \sqrt{-\kappa(\tau_\alpha, \tau_\alpha)}.$$

The length of the longest root is an important invariant of the Lie group G. We denote the square of its inverse by

$$a(\mathrm{G}) := \frac{1}{\sup_\alpha \ell(\alpha)^2}.$$

Here is a list of these invariants for the simple groups.

G	dim(G)	a(G)
$\mathrm{SU}(n)$	n^2-1	n
$\mathrm{Spin}(n)$	$\frac{1}{2}n(n-1)$	$n-2$
$\mathrm{Sp}(n)$	$n(2n+1)$	$n+1$
G_2	14	4
F_4	52	9
E_6	78	12
E_7	133	18
E_8	248	30

12 Examples

Example 12.1 (General linear group). The group $\mathrm{GL}(n, \mathbb{R})$ of invertible real $n \times n$-matrices is a Lie group. This space is an open set in $\mathbb{R}^{n \times n}$ and hence is obviously a manifold. Its Lie algebra is the vector space $\mathbb{R}^{n \times n}$ of all real $n \times n$ matrices with Lie bracket operation

$$[A, B] := AB - BA.$$

In this case the exponential map $\exp : \mathbb{R}^{n \times n} \to \mathrm{GL}(n, \mathbb{R})$ is the usual exponential map for matrices and the expressions gv and vg for $v \in T_h G$ are given by matrix multiplication. The example $\mathrm{GL}(n, \mathbb{C})$ of invertible complex $n \times n$-matrices is similar. However, the group $\mathrm{GL}(n, \mathbb{C})$ is connected while the group $\mathrm{GL}(n, \mathbb{R})$ has two components distinguished by the sign of the determinant.

Example 12.2 (Special linear group). The determinant map

$$\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$$

is a Lie group homomorphism and its kernel is a Lie group denoted by

$$\mathrm{SL}(n, \mathbb{C}) := \left\{ \Phi \in \mathbb{C}^{n \times n} \,|\, \det \Phi = 1 \right\}.$$

The formula

$$\det(\exp(A)) = \exp(\mathrm{trace}(A))$$

shows that the Lie algebra of $\mathrm{SL}(n, \mathbb{C})$ is given by

$$\mathfrak{sl}(n, \mathbb{C}) := \left\{ A \in \mathbb{C}^{n \times n} \,|\, \mathrm{trace}\, A = 0 \right\}.$$

The Lie group $\mathrm{SL}(n, \mathbb{R})$ with Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is defined analogously.

Example 12.3 (Circle). The unit circle

$$S^1 := \{ z \in \mathbb{C} \,|\, |z| = 1 \}$$

in the complex plane is a Lie group (under multiplication of complex numbers). Its Lie algebra is the space $\mathbf{i}\mathbb{R}$ of imaginary numbers with zero Lie bracket. (See Exercise 1.9.) The exponential map $\mathbf{i}\mathbb{R} \to S^1$ descends to a Lie group isomorphism $\mathbb{R}/\mathbb{Z} \to S^1 : t \mapsto e^{2\pi \mathbf{i} t}$.

Example 12.4 (Torus). Let V be an n-dimensional real vector space and $\Lambda \subset V$ be a lattice (a discrete additive subgroup) which spans V. Then

$$T := V/\Lambda$$

is a compact abelian Lie group (the group operation is the addition in V) with Lie algebra V. The exponential map is the projection $V \to V/\Lambda$. Any such Lie group is called a **torus**. Tori can be characterized as compact connected finite-dimensional abelian Lie groups. The basic example is the standard torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ and every n-dimensional torus is isomorphic to $\mathbb{T}^n$.

Example 12.5 (Orthogonal group). The orthogonal $n \times n$-matrices form a Lie group

$$\mathrm{O}(n) := \left\{\Phi \in \mathbb{R}^{n\times n} \,|\, \Phi^T\Phi = \mathbb{1}\right\}.$$

This group has two components distinguished by the determinant $\det \Phi = \pm 1$ and the component of the identity is denoted by

$$\mathrm{SO}(n) := \{\Phi \in \mathrm{O}(n) \,|\, \det\,\Phi = 1\}.$$

Its Lie algebra is the space of antisymmetric matrices

$$\mathfrak{so}(n) := \left\{A \in \mathbb{R}^{n\times n} \,|\, A^T + A = 0\right\}.$$

The group $\mathrm{SO}(n)$ is compact and connected and the exponential map is surjective (see Exercise 5.4).

Example 12.6 (Unitary group). The unitary $n \times n$-matrices form a Lie group

$$\mathrm{U}(n) := \left\{U \in \mathbb{C}^{n\times n} \,|\, U^*U = \mathbb{1}\right\}$$

where U^* denotes the conjugate transpose of U. This group is connected and its Lie algebra is given by

$$\mathfrak{u}(n) := \left\{A \in \mathbb{C}^{n\times n} \,|\, A^* + A = 0\right\}.$$

The case $n = 1$ corresponds to the circle $S^1 = \mathrm{U}(1)$. The subgroup of unitary matrices of determinant 1 is denoted by

$$\mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$$

and its Lie algebra by

$$\mathfrak{su}(n) := \{A \in \mathfrak{u}(n) \,|\, \mathrm{trace}(A) = 0\}.$$

Both groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact and connected.

Example 12.7 (Unit quaternions). Denote by

$$\mathbb{H} = \mathbb{R}^4$$

the space of quaternions

$$x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$$

with (noncommutative) multiplicative structure

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The norm of $x \in \mathbb{H}$ is defined by

$$|x|^2 := x\bar{x} = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2, \qquad \bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3,$$

and satisfies the rule $|xy| = |x| \cdot |y|$. Hence the unit quaternions form a group

$$\mathrm{Sp}(1) := \{x \in \mathbb{H} \,|\, |x| = 1\}$$

with unit 1 and inverse map $x \mapsto \bar{x}$. Its Lie algebra consists of the imaginary quaternions

$$\mathfrak{sp}(1) := \{x \in \mathbb{H} \,|\, x_0 = 0\}.$$

The exponential map is given by the usual formula $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. The quaternion multiplication defines a group structure on $S^3 = \mathrm{Sp}(1)$ and a Lie algebra structure on $\mathbb{R}^3 \simeq \mathfrak{sp}(1)$. This Lie algebra structure corresponds to the vector product.

Example 12.8. The quaternion matrices $\Phi \in \mathbb{H}^{n \times n}$ with $\Phi^*\Phi = \mathbb{1}$ form a compact connected group denoted by $\mathrm{Sp}(n)$. Its Lie algebra $\mathfrak{sp}(n)$ consists of the quaternion matrices $A \in \mathbb{H}^{n \times n}$ with $A^* + A = 0$. Here A^* denotes the conjugate transpose as in the complex case.

Exercise 12.9. (i) Prove that the map

$$\mathrm{Sp}(1) \to \mathrm{SU}(2) : x \mapsto U(x)$$

defined by

$$U(x) := \begin{pmatrix} x_0 + \mathbf{i}x_1 & x_2 + \mathbf{i}x_3 \\ -x_2 + \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}$$

is a Lie group isomorphism.

(ii) Prove that the corresponding Lie algebra homomorphism

$$\mathfrak{sp}(1) \to \mathfrak{su}(2) : \xi \mapsto u(\xi)$$

is given by

$$u(\xi) := \begin{pmatrix} \mathbf{i}\xi_1 & \xi_2 + \mathbf{i}\xi_3 \\ -\xi_2 + \mathbf{i}\xi_3 & -\mathbf{i}\xi_1 \end{pmatrix}.$$

Show that the matrices

$$I = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

satisfy the quaternion relations. In other words, the Lie algebra $\mathfrak{su}(2)$ is isomorphic to the imaginary quaternions and the isomorphism is given by $\mathbf{i} \mapsto I$, $\mathbf{j} \mapsto J$, $\mathbf{k} \mapsto K$. The natural orientation of $\mathrm{SU}(2)$ is determined by the irdered basis I, J, K of $\mathfrak{su}(2)$.

(iii) Prove that

$$[u(\xi), u(\eta)] = 2u(\xi \times \eta), \qquad \mathrm{trace}(u(\xi)^* u(\eta)) = 2\langle \xi, \eta \rangle$$

for $\xi, \eta \in \mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$.

Exercise 12.10 (Spin(3))**.** The unit quaternions act on the imaginary quaternions by conjugation. This determines a homomorphism

$$\mathrm{Sp}(1) \to \mathrm{SO}(3) : x \mapsto \Phi(x)$$

defined by

$$\Phi(x)\xi := x\xi\bar{x}$$

for $x \in \mathrm{Sp}(1)$ and $\xi \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. On the left the multiplication is understood as a product of matrix and vector and on the right as a product of quaternions.

(i) Prove that

$$\Phi(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_0x_2 - x_1x_3) \\ 2(x_0x_3 - x_1x_2) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_0x_1 - x_2x_3) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

(ii) Verify that the map $\mathrm{SU}(2) \to \mathrm{SO}(3) : U(x) \mapsto \Phi(x)$ is a Lie group homomorphism and a double cover. Deduce that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.

(iii) Let $\mathfrak{su}(2) \to \mathfrak{so}(3) : u(\xi) \mapsto A(\xi)$ denote the corresponding Lie algebra homomorphism. Prove that

$$A(\xi) = 2 \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Prove that $[A(\xi), A(\eta)] = 2A(\xi \times \eta)$ and $\mathrm{trace}(A(\xi)^T A(\eta)) = 8\langle \xi, \eta \rangle$.

Exercise 12.11 (Spin(4))**.** The group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts on $\mathbb{H}$ by the orthogonal transformations $x \mapsto ux\bar{v}$ for $(u, v) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Prove that this action determines a double cover

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$$

and find an explicit formula for the matrix $\Psi(u, v) \in \mathbb{R}^{4\times 4}$ defined by

$$\Psi(u, v)x = ux\bar{v}.$$

Lemma 12.12. **(i)** $\mathrm{SO}(n)$ *is connected and in the case* $n \geq 3$ *its fundamental group is isomorphic to* $\mathbb{Z}_2$*. Hence for* $n \geq 3$ *the universal cover of* $\mathrm{SO}(n)$ *is a compact group (with the same Lie algebra). It is denoted by* $\mathrm{Spin}(n)$*.*

(ii) $\mathrm{SU}(n)$ *is connected and simply connected and* $\pi_2(\mathrm{SU}(n)) = 0$*.*

(iii) *The fundamental group of* $\mathrm{U}(n)$ *is isomorphic to the integers. The determinant homomorphism* $\det : \mathrm{U}(n) \to S^1$ *induces an isomorphism of fundamental groups.*

Proof. The subgroup of all matrices $\Phi \in \mathrm{SO}(n)$ whose first column is the first unit vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ is isomorphic to $\mathrm{SO}(n-1)$. Hence there is a fibration

$$\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \to S^{n-1}$$

where the second map sends a matrix in $\mathrm{SO}(n)$ to its first column. The homotopy exact sequence of this fibration has the form

$$\pi_{k+1}(S^{n-1}) \to \pi_k(\mathrm{SO}(n-1)) \to \pi_k(\mathrm{SO}(n)) \to \pi_k(S^{n-1})$$

By Exercise 12.10, $\pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}_2$. For $n \geq 4$ this follows from the exact sequence with $k = 1$. The connectedness of $\mathrm{SO}(n)$ is obvious for $n = 1, 2$. For $n \geq 3$ it follows from the exact sequence with $k = 0$. This proves (i).

To prove (ii) consider the fibration

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \to S^{2n-1}$$

where the last map sends $U \in \mathrm{SU}(n)$ to the first column of U. The homotopy exact sequence of this fibration has the form

$$\pi_{k+1}(S^{2n-1}) \to \pi_k(\mathrm{SU}(n-1)) \to \pi_k(\mathrm{SU}(n)) \to \pi_k(S^{2n-1}).$$

For $n = 1$ the group $\mathrm{SU}(1) = \{1\}$ is obviously connected and simply connected. For $n \geq 2$ use the exact sequence inductively (over n) with $k = 0, 1$. The statement about π_2 is proved similarly with $k = 2$.

To prove (iii) consider the fibration

$$\mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \to S^1.$$

The homotopy exact sequence of this fibration has the form

$$1 = \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(S^1) \to \pi_0(\mathrm{SU}(n)) = 1.$$

In view of statement (ii) this shows that $\pi_1(\mathrm{U}(n)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. □

Let Y be a compact oriented smooth 3-manifold, and recall from Exercise 12.9 that $\mathrm{SU}(2)$ is diffeomorphic to S^3 and carries a natural orientation. Hence every smooth map $g : Y \to \mathrm{SU}(2)$ has a well defined degree. The next proposition shows that this degree can be expressed as as the integral of natural 3-form over Y.

Lemma 12.13. *For every compact oriented smooth* 3*-manifold* Y *and every smooth map* $g : Y \to \mathrm{SU}(2)$ *we have*

$$\int_Y \mathrm{trace}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) = -24\pi^2 \deg(g).$$

Proof. Denote

$$\omega_g := \mathrm{trace}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \in \Omega^3(Y).$$

If $f : Y' \to Y$ is a smooth map, then

$$\omega_{g\circ f} = f^*\omega_g.$$

In particular, with $\omega_0 := \omega_{\mathrm{id}} \in \Omega^3(\mathrm{SU}(2))$, we have $\omega_g = g^*\omega_0$ and hence

$$\int_Y \omega_g = \deg(g) \int_{\mathrm{SU}(2)} \omega_0. \tag{12.1}$$

To compute the integral of ω_0 consider the diffeomorphism $U : S^3 \to \mathrm{SU}(2)$ defined in Exercise 12.9. With the standard orientations of S^3 and $\mathrm{SU}(2)$ this map has degree 1. Moreover, it follows from the symmetry of this map that ω_U is a constant multiple of the volume form on S^3. To find out the factor we compute the form on the tangent space T_xS^3 for $x = (1, 0, 0, 0)$. On this space

$$U^{-1}dU = Idx_1 + Jdx_2 + Kdx_3,$$

hence

$$U^{-1}dU \wedge U^{-1}dU = 2Idx_2 \wedge dx_3 + 2Jdx_3 \wedge dx_1 + 2Kdx_1 \wedge dx_2,$$

and hence

$$U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU = 2(I^2 + J^2 + K^2)dx_1 \wedge dx_2 \wedge dx_3$$

This implies $\omega_U = -12\,\mathrm{dvol}_{S^3}$ and hence, by (12.1) with $g = U$,

$$\int_{\mathrm{SU}(2)} \omega_0 = \int_{S^3} \omega_U = -12\,\mathrm{Vol}(S^3) = -24\pi^2.$$

This proves Lemma 12.13. □

Example 12.14 (**Diffeomorphisms).** Let M be a compact manifold. Then the diffeomorphisms of M form an *infinite-dimensional Lie group* $\mathrm{Diff}(M)$ with group multiplication given by composition $(f, g) \mapsto f \circ g$. Its Lie algebra is the space $\mathrm{Vect}(M)$ of vector fields on M. The Lie algebra structure on $\mathrm{Vect}(M)$ is the usual one if the sign in the definition of the Lie bracket of two vector fields is chosen appropriately. The one-parameter subgroup generated by a vector field $X \in \mathrm{Vect}(M)$ is its flow $R \to \mathrm{Diff}(M) : t \mapsto \phi_t$ defined by

$$\frac{d}{dt}\phi_t = X \circ \phi_t, \qquad \phi_0 = \mathrm{id}.$$

Note also that the inverse of the adjoint action of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ is given by pullback, i.e. $\mathrm{Ad}(\phi^{-1})X = \phi^*X = d\phi \circ X \circ \phi^{-1}$. Interesting subgroups are given by the (exact) volume preserving diffeomorphisms or by the isometries of a Riemannian manifold, or by the (Hamiltonian) symplectomorphisms of a symplectic manifold.

Example 12.15 (Invertible linear operators). For invertible operators on an infinite-dimensional Hilbert space H the relation between Lie-group and Lie-algebra is somewhat subtle. Not every one parameter group $t \mapsto S(t)$ of invertible linear operators is differentiable. Such groups can be generated by unbounded operators and this leads to the theory of semigroups of linear operators.

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