Floer homology and Novikov rings

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Abstract

We prove the Arnold conjecture for compact symplectic manifolds under the assumption that either the first Chern class of the tangent bundle vanishes over $\pi_2(M)$ or the minimal Chern number is at least half the dimension of the manifold. This includes the important class of Calabi-Yau manifolds. The key observation is that the Floer homology groups of the loop space form a module over Novikov's ring of generalized Laurent series. The main difficulties to overcome are the presence of holomorphic spheres and the fact that the action functional is only well defined on the universal cover of the loop space with a possibly dense set of critical levels.

1 Introduction

Let (M, ω) be a $2n$-dimensional compact symplectic manifold and consider the time-dependent Hamiltonian differential equation

$$\dot{x}(t) = X_H(t, x(t)). \tag{1}$$

Here the vector field $X_H : S^1 \times M \to TM$ is associated to the 1-periodic Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ via $i_{X_H}\omega = dH$. Throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$. We denote by

$$\mathcal{P}(H)$$

the set of all contractible 1-periodic solutions $x(t) = x(t+1)$ of (1). The Arnold conjecture states that if these 1-periodic solutions are all nondegenerate then their number can be estimated below by the sum of the Betti numbers of M [1], [2]. This conjecture was first proved by Conley and Zehnder for the $2n$-torus [5]. In [8] Floer proved the Arnold conjecture for **monotone** symplectic manifolds. These are manifolds for which the cohomology class of the symplectic form ω over $\pi_2(M)$ is a non-negative multiple of the first Chern class $c_1 \in H^2(M)$ of the tangent bundle (with a suitable almost complex structure)

$$\int_{S^2} v^*\omega = \lambda \int_{S^2} v^* c_1$$

for $v : S^2 \to M$. Floer constructed a chain complex from the periodic solutions of (1) and the solutions $u : \mathbb{R} \times S^1 \to M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(t,u) = 0 \tag{2}$$

which satisfy the limit condition

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t) \tag{3}$$

with $x^{\pm} \in \mathcal{P}(H)$. Here $J : TM \to TM$ is an almost complex structure on M which is compatible with ω. This means that

$$g_J(v,w) = \omega(v, Jw)$$

defines a Riemannian metric on M. We denote by $\mathcal{J}(M,\omega)$ the space of all smooth almost complex structures on M which are compatible with ω. Let

$$\mathcal{M}(x^-, x^+; H, J)$$

denote the space of all smooth solutions of (2) and (3). For a generic Hamiltonian this space is a finite dimensional manifold. It decomposes into different components and we denote the local dimension of $\mathcal{M}(x^-, x^+; H, J)$ near u by $\mu(u)$. The **minimal Chern number** of (M,ω) is the integer $N \geq 0$ defined by $N\mathbb{Z} = c_1(\pi_2(M))$. The Conley-Zehnder index defines a map $\mu : \mathcal{P}(H) \to \mathbb{Z}/2N\mathbb{Z}$ and in [21] it is shown that

$$\mu(u) = \mu(x^+) - \mu(x^-) \,(\mathrm{mod}\ 2N)$$

for $u \in \mathcal{M}(x^-, x^+; H, J)$. Floer's cochain complex is defined by

$$C^k = \bigoplus_{\mu(x)=k (\mathrm{mod}\ 2N)} \mathbb{Z}\langle x\rangle$$

and the (x,y)-entry of the coboundary operator $\delta : C^k \to C^{k+1}$ is given by the number of one dimensional components of $\mathcal{M}(x,y;H,J)$ whenever the index difference is 1 modulo $2N$. This number is finite whenever the manifold (M,ω) is monotone and the connecting orbits are counted with appropriate signs. In [8] Floer proved that $\delta^2 = 0$. The homology of this cochain complex is called the **Floer cohomology** of the pair (H,J) and will be denoted by

$$HF^*(M,H,J) = \frac{\ker\delta}{\mathrm{im}\delta}.$$

The Floer cohomology groups are graded modulo $2N$. In [8] Floer proved that these cohomology groups in fact agree with the integral cohomology of the underlying manifold M

$$HF^k(M,H,J) = \bigoplus_{j=k+n (\mathrm{mod}\ 2N)} H^j(M)$$

and this proves the Arnold conjecture.

We generalize these ideas to $2n$-dimensional compact symplectic manifolds such that for every $A \in \pi_2(M)$

$$3 - n \le c_1(A) < 0 \qquad \Longrightarrow \qquad \omega(A) \le 0.$$

Such symplectic manifolds are called **weakly monotone**. They have the property that for a generic almost complex structure there is no pseudo-holomorphic sphere with negative Chern number.

Lemma 1.1 *A comact symplectic manifold (M, ω) is weakly monotone if and only if one of the following conditions is satisfied.*

(a) $\omega(A) = \lambda c_1(A)$ *for every* $A \in \pi_2(M)$ *where* $\lambda \ge 0$ *(M is* ***monotone****).*

(b) $c_1(A) = 0$ *for every* $A \in \pi_2(M)$*.*

(c) *The minimal Chern number* $N \ge 0$ *defined by* $c_1(\pi_2(M)) = N\mathbb{Z}$ *is greater than or equal to* $n - 2$*.*

Proof: Assume M is weakly monotone but does not satisfy either of the conditions (a), (b), (c). Since (b) and (c) do not hold there there exists a homotopy class $A \in \pi_2(M)$ with

$$3 - n \le c_1(A) < 0.$$

Since M is weakly monotone this implies

$$\omega(A) \le 0.$$

Denote $\lambda = \omega(A)/c_1(A) \ge 0$. Since (a) does not hold there exists a homotopy class $A' \in \pi_2(M)$ such that $\omega(A') \ne \lambda c_1(A')$ and hence $\omega(A')c_1(A) - \omega(A)c_1(A') \ne 0$. Change the sign of A', if necessary, to obtain

$$\omega(A)c_1(A') - \omega(A')c_1(A) > 0.$$

Then the class $B = c_1(A')A - c_1(A)A'$ satisfies

$$c_1(B) = 0, \qquad \omega(B) > 0.$$

Hence for $k > 0$ sufficiently large the class $A + kB$ satisfies

$$3 - n \le c_1(A + kB) < 0, \qquad \omega(A + kB) > 0.$$

This contradicts the definition of weak monotonicity. $\square$

The three cases in the previous lemma are not disjoint and Floer proved the Arnold conjecture in the case (a). In extending Floer homology to manifolds which satisfy (b) or (c) one encounters two difficulties. The first is the presence of nonconstant J-holomorphic spheres with Chern number $c_1 \leq 0$. Such J-holomorphic spheres cannot exist in the monotone case. They obstruct the compactness of the one dimensional components of the space of connecting orbits with bounded energy. We overcome this difficulty by taking account of the fact that the space of points lying on J-holomorphic spheres of Chern number $c_1 = 0$ form roughly speaking a subset of codimension 4. Moreover, our assumptions guarantee that generically there are no J-holomorphic spheres with negative Chern number. The J-holomorphic spheres of Chern class $c_1 = 1$ also obstruct compactness if they intersect periodic solutions of (1). However such intersections do not exist generically since these spheres form a set of codimension 2. As a consequence we can prove in the weakly monotone case that the space of one dimensional connecting orbits with a bound on the energy is compact. The second difficulty arises from sequences of connecting orbits with index difference 1 whose energy converges to infinity. We take account of these sequences by constructing a suitable coefficient ring Λ_ω which algebraically incorporates the period map

$$\pi_2(M) \to \mathbb{R} : A \mapsto \int_A \omega.$$

Such a ring was used by Novikov in his generalization of Morse theory for closed 1-forms [17]. An exposition of Novikov homology with applications in symplectic geometry can be found in the thesis of Sikorav [22].

For a large class of weakly monotone symplectic manifolds we shall prove that the Floer cohomology groups agree with the cohomology of the manifold M with coefficients in Λ_ω. This class includes all manifolds which satisfy (a) or (b) but in the case (c) we must assume in addition that the minimal Chern number is $N \geq n$.[1] We conjecture that the Floer cohomology groups always agree with the cohomology of M. A generalization to arbitrary symplectic manifolds seems to require a better understanding of J-holomorphic spheres with negative Chern number. We conjecture that the presence of such spheres will have no algebraic ramifications for our setup.

The present work was inspired in part by a recent lecture of M.F. Atiyah on duality for Calabi-Yau manifolds. We also point out that the results of this paper were in principle anticipated by A. Floer. In [8] he observed the relevance of Novikov's ring for the construction of Floer cohomology and indicated a compactness argument involving the codimension of the space of holomorphic spheres.

[1] After this paper was written our proof of the Arnold conjecture has been extended by Kaoru Ono in to the case $N \geq n-2$ (cf. [18]). In particular this includes all compact symplectic manifolds of dimensions 4 and 6.

2 Holomorphic spheres

Let (M, ω) be a $2n$-dimensional compact symplectic manifold and let $J \in \mathcal{J}(M, \omega)$. A J-holomorphic sphere is a smooth map $v : S^2 \to M$ such that

$$dv \circ i = J \circ dv$$

where i is the standard complex structure on $S^2 = \mathbb{C} \cup \{\infty\}$.

Let $c_1 \in H^2(M; \mathbb{Z})$ denote the first Chern class of the tangent bundle of M considered as a complex vector bundle with a complex structure J which is compatible with ω. The complex isomorphism class of TM is independent of the choice of $J \in \mathcal{J}(M, \omega)$. Consider the homomorphisms $\phi_\omega : \pi_2(M) \to \mathbb{R}$ and $\phi_{c_1} : \pi_2(M) \to \mathbb{Z}$ defined by integration of ω and c_1 over a sphere and define the abelian group

$$\Gamma = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}.$$

In the following we shall denote by $c_1(A)$ and $\omega(A)$ the integrals of c_1 and ω over the class A. Given $A \in \Gamma$ denote by $\mathcal{M}(A; J)$ the space of all J-holomorphic spheres which represent the class A. A smooth map $v : S^2 \to M$ is called **simple** if $v = w \circ \phi$ with $\phi : S^2 \to S^2$ implies $\deg \phi = 1$. Denote by

$$\mathcal{M}_s(A; J)$$

the subspace of simple J-holomorphic spheres in the class A. This space is a finite dimensional manifold for a generic almost complex structure $J \in \mathcal{J}(M, \omega)$.

More precisely, let $\mathcal{S} = W^{1,p}(S^2, M)$ with $p > 2$ and for $v \in \mathcal{S}$ let $\Lambda_v \to S^2$ denote the vector bundle whose fibre at $z \in S^2$ is the space of complex anti-linear maps $T_z S^2 \to T_{v(z)} M$. Consider the bundle

$$\mathcal{E} \to \mathcal{S}$$

whose fibre $\mathcal{E}_v = L^p(\Lambda_v)$ at $v \in \mathcal{S}$ consists of all L^p sections η of the vector bundle Λ_v. In local coordinates such a section is a 1-form of the form $\eta = \xi ds - J(v)\xi dt$. The $\bar{\partial}$ operator defines a Fredholm section of the bundle $\mathcal{E}$ given by

$$\bar{\partial}_J(v) = dv + J \circ dv \circ i = \left(\frac{\partial v}{\partial s} + J(v) \frac{\partial v}{\partial t} \right) ds + \left(\frac{\partial v}{\partial t} - J(v) \frac{\partial v}{\partial s} \right) dt.$$

By definition the J-holomorphic spheres are the zeros of this section. It follows from elliptic regularity that every J-holomorphic sphere is smooth. Thus every J-holomorphic sphere can be represented by a smooth function $v : \mathbb{C} \to M$ which satisfies the partial differential equation

$$\frac{\partial v}{\partial s} + J(v) \frac{\partial v}{\partial t} = 0.$$

Such a function extends to S^2 if and only if it has finite energy

$$E(v) = \frac{1}{2} \int_{\mathbb{C}} \left(\left| \frac{\partial v}{\partial s} \right|^2 + \left| \frac{\partial v}{\partial t} \right|^2 \right) dsdt = \int_{\mathbb{C}} v^* \omega < \infty.$$

The linearization of $\bar{\partial}_J$ at a zero $v \in \mathcal{M}(A; J)$ defines a Fredholm operator $D\bar{\partial}_J(v) : W^{1,p}(v^*TM) \to L^p(\Lambda_v)$ whose Fredholm index, by the Riemann-Roch theorem, is given by

$$\operatorname{index} D\bar{\partial}_J(v) = 2n + 2 \int_{S^2} v^* c_1.$$

The space $\mathcal{M}(A; J)$ can only be expected to be a manifold if $\bar{\partial}$ is transversal to the zero section. This means that the operator $D\bar{\partial}_J(v)$ is onto for every $v \in \mathcal{M}(A; J)$. To achieve this we must perturb J and restrict ourselves to the class of simple curves. For the latter McDuff has proved the following result [14].

Lemma 2.1 *Assume that $v : S^2 \to M$ is a nonconstant J-holomorphic sphere. Then there are only finitely many points $z_1, \ldots, z_m \in S^2$ such that $dv(z_j) = 0$. If v is simple then there exists a point $z_0 \in S^2$ such that $dv(z_0) \neq 0$ and $v(z) \neq v(z_0)$ for every $z \neq z_0$.*

The tangent space $T_J\mathcal{J}$ to $\mathcal{J} = \mathcal{J}(M, \omega)$ is the vector space of smooth sections $X \in C^\infty(\mathrm{End}(TM, J, \omega))$ where $\mathrm{End}(TM, J, \omega) \subset \mathrm{End}(TM)$ is the bundle over M whose fibre at $x \in M$ is the space $\mathrm{End}(T_xM, J_x, \omega_x)$ of linear endomorphisms $Y : T_xM \to T_xM$ which satisfy

$$JY + YJ = 0, \qquad \omega_x(Y\xi, \eta) + \omega_x(\xi, Y\eta) = 0.$$

Following Floer [6] we choose a sufficiently rapidly decreasing sequence $\varepsilon_k > 0$ and denote by $C^\infty_\varepsilon(\mathrm{End}(TM, J, \omega))$ the subspace of those smooth sections $Y \in C^\infty(\mathrm{End}(TM, J, \omega))$ for which

$$\|Y\|_\varepsilon = \sum_{k=0}^\infty \varepsilon_k \|Y\|_{C^k(M)} \quad < \quad \infty.$$

This defines a separable Banach space of smooth sections which for a suitable choice of the sequence ε_k is dense in $L^2(\mathrm{End}(TM, J, \omega))$ [6]. For any sufficiently small section $Y \in C^\infty(\mathrm{End}(TM, J, \omega))$ we have $J \exp(-JY) \in \mathcal{J}(M, \omega)$. Given $J_0 \in \mathcal{J}(M, \omega)$ and $\delta > 0$ denote by $\mathcal{U}_\delta(J_0)$ the set of all $J = J_0 \exp(-J_0 Y) \in \mathcal{J}(M, \omega)$ with $\|Y\|_\varepsilon \leq \delta$. We also denote by

$$\mathcal{J}_{\mathrm{reg}} = \mathcal{J}_{\mathrm{reg}}(M, \omega)$$

the set of all smooth almost complex structures $J \in \mathcal{J}(M, \omega)$ such that $D\bar{\partial}_J(v)$ is onto for every simple J-holomorphic sphere $v : S^2 \to M$. Using Lemma 2.1 one can prove the following result [14].

Theorem 2.2 *For every $J_0 \in \mathcal{J}(M,\omega)$ the set $\mathcal{J}_{\mathrm{reg}}(A) \cap \mathcal{U}_\delta(J_0)$ is generic in the sense of Baire (i.e. a countable intersection of open dense sets).*

As a consequence of the implicit function theorem the space $\mathcal{M}_s(A;J)$ is a finite dimensional manifold of dimension

$$\dim \mathcal{M}_s(A;J) = 2n + 2c_1(A)$$

whenever $J \in \mathcal{J}_{\mathrm{reg}}$. The differentiable structure on this manifold is independent of the choice of the analytic setup. Moreover, it follows from Lemma 2.1 that the group $\mathrm{G} = \mathrm{PSL}(2,\mathbb{C})$ of biholomorphic maps of the sphere acts freely on $\mathcal{M}_s(A;J)$. Hence the quotient $\mathcal{M}_s(A;J)/\mathrm{G}$ is a finite dimensional manifold of dimension

$$\dim \mathcal{M}_s(A;J)/\mathrm{G} = 2n + 2c_1(A) - 6.$$

In particular this space must be empty whenever $J \in \mathcal{J}_{\mathrm{reg}}$ and $c_1(A) < 3 - n$.

Proposition 2.3 *Assume that (M,ω) is a weakly monotone $2n$-dimensional compact symplectic manifold. Let $J \in \mathcal{J}_{\mathrm{reg}}(M,\omega)$ and $A \in \Gamma$.*

(i) *If $c_1(A) < 0$ then $\mathcal{M}(A;J) = \emptyset$.*

(ii) *If $n = 2$ and $A \neq 0$ with $c_1(A) = 0$ then $\mathcal{M}(A;J) = \emptyset$.*

(iii) *If $n = 2$ and $c_1(A) = 1$ then the moduli space $\mathcal{M}(A;J)/\mathrm{G}$ is a finite set.*

Proof: To prove statement (i) note first that $\mathcal{M}_s(B;J) = \emptyset$ for every $B \in \Gamma$ with $c_1(B) < 0$. If $c_1(B) < 3 - n$ then this follows from the above dimension formula and if $c_1(B) \geq 3 - n$ then this follows from the fact that M is weakly monotone and hence $\omega(B) \leq 0$. Now assume that $c_1(A) < 0$ and $v \in \mathcal{M}(A;J)$. Then there exists a simple J-holomorphic map $w : S^2 \to M$ and a rational map $\phi : S^2 \to S^2$ of degree k such that $v = w \circ \phi$. Let $B \in \Gamma$ denote the equivalence class of w. Then $kc_1(B) = c_1(A) < 0$. Since $A \neq 0$ we have

$$0 < E(v) = \int_{S^2} v^*\omega = k \int_{S^2} w^*\omega = kE(w),$$

hence $k > 0$, and hence $c_1(B) < 0$. This implies $\mathcal{M}_s(B;J) = \emptyset$ in contradiction to $w \in \mathcal{M}_s(B;J)$.

We prove statement (ii). In the case $n = 2$ it follows from the dimension formula that $\mathcal{M}_s(B;J) = \emptyset$ for every $B \in \Gamma$ with $c_1(B) = 0$. Now the same argument as above shows that if $\mathcal{M}(A;J) \neq \emptyset$ for some nonzero $A \in \Gamma$ with $c_1(A) = 0$ then there exists a $B \in \Gamma$ with $c_1(B) = 0$ such that $\mathcal{M}_s(B;J) \neq \emptyset$.

To prove statement (iii) note first that every sphere $v : S^2 \to M$ in the homotopy class A is simple. Hence the space $\mathcal{M}(A;J)$ is a 6-dimensional manifold for $J \in \mathcal{J}_{\mathrm{reg}}$. Now every J-holomorphic curve $v \in \mathcal{M}(A;J)$ has energy $E(v) = \omega(A)$. By Gromov's compactness theorem the space $\mathcal{M}(A;J)/\mathrm{G}$ can only fail to be compact if J-holomorphic curves of nonpositive Chern number bubble off. (See for example [11], [19], and the appendix of this paper.) By (i) and (ii) such spheres do not exist. $\square$

In the following we denote by

$$M_k(c;J)$$

the set of points $x \in M$ such that there exists a non-constant J-holomorphic sphere $v : S^2 \to M$ such that $c_1(v) \le k$, $E(v) \le c$ and $x \in v(S^2)$.

Proposition 2.4 *Let (M,ω) be a weakly monotone compact symplectic manifold of dimension $2n \ge 6$ and assume that $J \in \mathcal{J}_{\mathrm{reg}}$. Then the set $M_k(c;J)$ is compact for every $c > 0$ and every integer k.*

Proof: Take a sequence $x_\nu = v_\nu(z_\nu)$ converging to $x \in M$ where $v_\nu : S^2 \to M$ is a sequence of J-holomorphic spheres with $c_1(v_\nu) \le k$ and $E(v_\nu) \le c$. We must prove that $x \in M_k(c;J)$. To see this assume without loss of generality that the v_ν all represent the same homotopy class A. By Theorem A.1 a subsequence of v_ν converges to a finite collection of J-holomorphic spheres $v^j : S^2 \to M$ for $j = 1, \ldots, \ell$ whose connected sum represents the homotopy class A. In particular

$$\sum_{j=1}^{\ell} E(v^j) = \omega(A), \qquad \sum_{j=1}^{\ell} c_1(v^j) = c_1(A)$$

and $x \in v^j(S^2)$ for some j. Since $E(v^j) > 0$ and $c_1(v^j) \ge 0$ for all j the statement follows. $\square$

For every $A \in \Gamma$ the space $\mathcal{M}_s(A;J) \times S^2$ is a manifold of dimension $2n + 2c_1(A) + 2$ on which the group G acts freely by

$$\phi^*(v,z) = (v \circ \phi, \phi^{-1}(z))$$

for $(v,z) \in \mathcal{M}_s(A;J) \times S^2$ and $\phi \in \mathrm{G}$. Hence the quotient is a manifold of dimension

$$\dim\, \mathcal{M}_s(A;J) \times_{\mathrm{G}} S^2 = 2n + 2c_1(A) - 4.$$

Note that the evaluation map

$$e_A : \mathcal{M}_s(A;J) \times_{\mathrm{G}} S^2 \to M : (v,z) \mapsto v(z)$$

is well defined. In particular the set $M_0(\infty;J)$ is a countable union of images of smooth maps defined on manifolds of dimension $2n-4$. Hence the set $M_0(\infty;J)$ is roughly speaking a subset of M of codimension 4. Similarly the set $M_1(\infty;J)$ is a subset of codimension 2.

3 Transversality and compactness

Throughout this section we assume that (M, ω) is a weakly monotone compact symplectic manifold of dimension $2n \geq 4$ and $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ is a regular almost complex structure compatible with ω. Let $C^\infty_\varepsilon(S^1 \times M)$ denote the Banach space of all smooth functions $h : S^1 \times M \to \mathbb{R}$ such that

$$\|h\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \, \|h\|_{C^k(S^1 \times M)} \quad < \quad \infty.$$

Here $\varepsilon_k > 0$ is a sufficiently rapidly decreasing sequence such that C^∞_ε is dense in L^2. The space $C^\infty_\varepsilon(S^1 \times M)$ can be viewed as a closed subspace of a suitable separable Banach space and is therefore separable. Given any smooth Hamiltonian function $H_0 : S^1 \times M \to \mathbb{R}$ we denote by

$$\mathcal{U}_\delta(H_0)$$

the set of all Hamiltonians H with $\|H - H_0\|_\varepsilon < \delta$.

Theorem 3.1 *There is a generic set $\mathcal{H}_0 \subset \mathcal{U}_\delta(H_0)$ such that the following holds for $H \in \mathcal{H}_0$.*

(i) *Every periodic solution $x \in \mathcal{P}(H)$ is non-degenerate.*

(ii) *$x(t) \notin M_1(\infty; J)$ for every $x \in \mathcal{P}(H)$ and every $t \in \mathbb{R}$.*

Proof: Let $A \in \Gamma$ with $c_1(A) \in \{0, 1\}$. Denote by $\mathcal{B}$ the Hilbert manifold of contractible $W^{1,2}$ loops $x : S^1 \to M$ and consider the bundle $\mathcal{E} \to \mathcal{B}$ whose fibre at $x \in \mathcal{B}$ is the Hilbert space of L^2-vector fields along x. Let the section $\mathcal{F} : \mathcal{B} \times \mathcal{U}_\delta(H_0) \to \mathcal{E}$ be defined by

$$\mathcal{F}(x, H) = \dot{x} - X_H(t, x).$$

The differential of this section at a zero (x, H) is the linear operator

$$D\mathcal{F}(x, H) : W^{1,2}(x^*TM) \times C^\infty_\varepsilon(S^1 \times M) \to L^2(x^*TM)$$

given by

$$D\mathcal{F}(x, H)(\xi, h) = A_x \xi - X_h(t, x)$$

where the operator $A_x : W^{1,2}(x^*TM) \to L^2(x^*TM)$ is given by

$$A_x \xi = \nabla \xi - \nabla_\xi X_H(t, x).$$

We prove that $d\mathcal{F}(x, H)$ is onto. Since A_x is a Fredholm operator (of Fredholm index zero) it suffices to show that $d\mathcal{F}(x, H)$ has a dense range. To see this note that every C^∞ vectorfield $\eta \in C^\infty(x^*TM)$ can be written in the form $\eta(t) = X_h(t, x(t))$ for some smooth function $h : S^1 \times M \to \mathbb{R}$. Hence every

L^2-vector field $\eta \in L^2(x^*TM)$ can be approximated by a sequence of vector fields of the form $X_h(t, x(t))$ with $h \in C^\infty_\varepsilon(S^1 \times M)$.

Thus we have proved that $\mathcal{F}$ intersects the zero section of $\mathcal{E}$ transversally. Hence the set

$$\mathcal{P} = \{(x, H) \in \mathcal{B} \times \mathcal{U}_\delta(H_0) : \mathcal{F}(x, H) = 0\}$$

is a separable infinite dimensional Banach manifold. A Hamiltonian $H \in \mathcal{U}_\delta(H_0)$ is a regular value of the projection $\mathcal{P} \to \mathcal{U}_\delta(H_0)$ onto the second factor if and only if every periodic solution $x \in \mathcal{P}(H)$ is non-degenerate (or equivalently the operator A_x is onto). By the Sard-Smale theorem [23] the set $\mathcal{H}_0' \subset \mathcal{U}_\delta(H_0)$ of regular values is generic in the sense of Baire.

We prove that the evaluation map

$$e_t : \mathcal{P} \to M, \qquad e_t(x, H) = x(t)$$

is a submersion for every $t \in S^1$. The tangent space $T_{(x,H)}\mathcal{P}$ is the Banach space of all pairs $(\xi, h) \in W^{1,2}(x^*TM) \times C^\infty_\varepsilon(S^1 \times M)$ such that

$$A_x\xi = X_h(t, x).$$

The differential of e_t at (x, H) is the linear functional

$$De_t : T_{(x,H)}\mathcal{P} \to T_{x(t)}M, \qquad De_t(\xi, h) = \xi(t)$$

and we must prove that this map is onto. Given $\xi_0 \in T_{x(t)}M$ choose any $\xi \in W^{1,2}(x^*TM)$ such that $\xi(t) = \xi_0$. Since A_x is a Fredholm operator the set of all vector fields $\eta \in \mathrm{range}A_x$ which are of the form form $X_h(t, x(t))$ with $h \in C^\infty_\varepsilon(S^1 \times M)$ is dense in the range of A_x. Hence there exists a sequence $(\xi_\nu, h_\nu) \in T_{(x,H)}\mathcal{P}$ such that $A_x\xi_\nu$ converges to $A_x\xi$. Since there is an estimate

$$\inf_{A_x\zeta=0} \|\xi - \zeta\|_{W^{1,2}} \le c\|A_x\xi\|_{L^2}$$

we can choose ξ_ν as to converge to ξ in the $W^{1,2}$ norm. Hence $\xi_\nu(t)$ converges to $\xi(t) = \xi_0$. This shows that De_t has a dense range. Since $T_{x(t)}M$ is finite dimensional De_t is onto.

Now consider the evaluation map

$$\mathcal{M}_s(A; J) \times_G S^2 \times S^1 \times \mathcal{P} \to M \times M : ([v, z], t, x, H) \mapsto (v(z), x(t)).$$

Since $e_t : \mathcal{P} \to M$ is a submersion this map is transversal to the diagonal Δ_M in $M \times M$. Hence the space

$$\mathcal{N} = \{([v, z], t, x, H) : \, v(z) = x(t), (x, H) \in \mathcal{P}\}$$

is an infinite dimensional Banach submanifold of $\mathcal{M}_s(A; J) \times_G S^2 \times S^1 \times \mathcal{P}$ of codimension $2n$.

The projection

$$\mathcal{N} \to \mathcal{U}_\delta(H_0) : ([v,z],t,x,H) \mapsto H$$

is a Fredholm map. The Fredholm index of this projection is $2c_1(A) - 3$. In the case $c_1(A) \le 1$ this number is negative and hence the regular values of the projection $\mathcal{N} \to \mathcal{U}_\delta(H_0)$ are those which are not in the image. Now $\mathcal{N}$ is separable and hence it follows from the Sard-Smale theorem [23] that the set $\mathcal{H}_0(A)$ of regular values is of the second category in the sense of Baire. We conclude that the 1-periodic solutions of (1) do not touch the J-holomorphic spheres in the class A for $H \in \mathcal{H}_0(A)$. The required set $\mathcal{H}_0$ is the is the intersection of $\mathcal{H}_0'$ with the sets $\mathcal{H}_0(A)$ where A ranges over the countable set of those $A \in \Gamma$ for which $c_1(A) \le 1$. □

By the previous theorem there exists a Hamiltonian function $H_0 : S^1 \times M \to \mathbb{R}$ such that all contractible 1-periodic solutions of (1) are nondegenerate and do not intersect the set $M_1(\infty; J)$. We denote by

$$\mathcal{V}_\delta(H_0)$$

the set of all Hamiltonians H which satisfy $\|H - H_0\|_\varepsilon < \delta$ and agree with H_0 up to second order on the contractible 1-periodic solutions of (1). If $\delta > 0$ is sufficiently small then the contractible 1-periodic solutions of H_0 agree with those of H for every $H \in \mathcal{V}_\delta(H_0)$.

Theorem 3.2 *There is a generic set $\mathcal{H}_1 \subset \mathcal{V}_\delta(H_0)$ such that the following holds for $H \in \mathcal{H}_1$.*

(i) *The moduli space $\mathcal{M}(x^-, x^+; H, J)$ of connecting orbits is a finite dimensional manifold for all $x^\pm \in \mathcal{P}(H)$.*

(ii) *$u(s,t) \notin M_0(\infty; J)$ for every $u \in \mathcal{M}(x^-, x^+; H, J)$ with $\mu(u) \le 2$ and every $(s,t) \in \mathbb{R} \times S^1$.*

Proof: We fix a pair $x^\pm \in \mathcal{P}(H_0)$ with index difference 1 or 2 modulo $2N$. We also choose $A \in \Gamma$ with $c_1(A) = 0$. Following the same line of argument as in the proof of Theorem 3.1 denote by $\mathcal{B}$ the Banach manifold of $W^{1,p}$ maps $u : \mathbb{R} \times S^1 \to M$ which satisfy the limit condition (3) in the $W^{1,p}$ sense with $p > 2$ (see [6] and [8]). Consider the bundle $\mathcal{E} \to \mathcal{B}$ whose fibre at $u \in \mathcal{B}$ is the Banach space of L^p-vector fields along u. Let the section $\mathcal{F} : \mathcal{B} \times \mathcal{V}_\delta(H_0) \to \mathcal{E}$ be defined by

$$\mathcal{F}(u,H) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(t,u).$$

The differential of this section at a zero (u,H) is the linear operator

$$D\mathcal{F}(u,H) : W^{1,p}(u^*TM) \times T_H\mathcal{V}_\delta(H_0) \to L^p(u^*TM)$$

given by

$$D\mathcal{F}(u,H)(\xi,h) = D_u\xi - \nabla h(t,u)$$

where the operator $D_u : W^{1,p}(u^*TM) \to L^p(u^*TM)$ is given by

$$D_u\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} - \nabla_\xi\nabla H(t,u).$$

This is a Fredholm operator of Fredholm index $\mu(u)$. In [21] using results in [10] it is proved that $\mathcal{F}$ intersects the zero section of $\mathcal{E}$ transversally. Hence the set

$$\mathcal{M}(x^-,x^+;J) = \{(u,H) \in \mathcal{B} \times \mathcal{V}_\delta(H_0) : \mathcal{F}(u,H) = 0\}$$

is a separable infinite dimensional Banach manifold.

A Hamiltonian $H \in \mathcal{U}_\delta(H_0)$ is a regular value of the projection

$$\mathcal{M}(x^-,x^+;J) \to \mathcal{V}_\delta(H_0) : (u,H) \mapsto H$$

if and only if the operator D_u is onto for every $u \in \mathcal{M}(x^-,x^+;H,J)$ and hence $\mathcal{M}(x^-,x^+;H,J)$ is a manifold whose dimension is the Fredholm index of D_u. By the Sard-Smale theorem [23] the set $\mathcal{H}_2(x^-,x^+) \subset \mathcal{V}_\delta(H_0)$ of regular values is generic in the sense of Baire.

Now as in the proof of Theorem 3.1 the evaluation map

$$e_t : \mathcal{M}(x^-,x^+;J) \to M, \qquad e_t(u,H) = u(0,t)$$

is a submersion for every $t \in S^1$. The evaluation map at the point (s,t) is also a submersion due to the obvious action of $\mathbb{R}$ on the solutions of (2). Since e_t is a submersion it follows that the evaluation map

$$\mathcal{M}_s(A;J) \times_G S^2 \times S^1 \times \mathcal{M}(x^-,x^+;J) \to M \times M$$

given by

$$([v,z],t,u,H) \mapsto (v(z),u(0,t))$$

is transversal to the diagonal Δ_M. Hence the space

$$\mathcal{N} = \left\{([v,z],t,u,H) : \ v(z) = u(0,t), (u,H) \in \mathcal{M}(x^-,x^+;J)\right\}$$

is a Banach submanifold of $\mathcal{M}_s(A;J) \times_G S^2 \times S^1 \times \mathcal{M}(x^-,x^+;J)$ of codimension $2n$.

The projection

$$\mathcal{N} \to \mathcal{V}_\delta(H_0) : ([v,z],u,t;H) \mapsto H$$

is a Fredholm map. The Fredholm index of this projection is

$$2c_1(A) + \mu(u) - 3.$$

In the case $c_1(A) = 0$ and $\mu(u) \leq 2$ this number is negative. Denote by $\mathcal{H}_3(x^-, x^+, A)$ the set of regular values of the projection $\mathcal{N} \to \mathcal{V}_\delta(H_0)$ and define

$$\mathcal{H}_1(x^-, x^+, A) = \mathcal{H}_2(x^-, x^+) \cap \mathcal{H}_3(x^-, x^+, A).$$

The required set $\mathcal{H}_1$ is the countable intersection of the sets $\mathcal{H}_1(x^-, x^+, A)$ where $x^\pm$ runs over all pairs in $\mathcal{P}(H_0)$ with index difference 1 or 2 and A runs over all spheres of Chern number 0. $\square$

For any pair (H, J) denote by

$$M_k(c, H, J)$$

the set of all point $x = u(s,t) \in M$ where $u : \mathbb{R} \times S^1 \to M$ is a contractible solution of (2) with energy

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_H(t,u) \right|^2 \right) dt ds \leq c$$

and $\mu(u) \leq k$. If the operator D_u is onto for every solution of u of (2) and (3) then $M_k(c, H, J) = \emptyset$ for $k < 0$ and the set

$$M_0(H) = M_0(c, H, J)$$

consists of all points lying on a contractible 1-periodic solution of (1). Let

$$\mathcal{H}_{\mathrm{reg}}(J)$$

denote the set of all smooth Hamiltonian functions $H : S^1 \times M \to \mathbb{R}$ such that the contractible 1-periodic solutions of (1) are nondegenerate, the operator D_u is onto for every contractible solution $u : \mathbb{R} \times S^1 \to M$ of (2) and (3), and

$$M_0(H) \cap M_1(\infty; J) = \emptyset, \qquad M_2(\infty; H, J) \cap M_0(\infty; J) = \emptyset.$$

It follows from Theorem 3.1 and Theorem 3.2 that for every $J \in \mathcal{J}_{\mathrm{reg}}$ the set $\mathcal{H}_{\mathrm{reg}}(J)$ is dense in $C^\infty(S^1 \times M)$ with respect to the topology of uniform convergence with all derivatives.

Theorem 3.3 *Assume $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ and $H \in \mathcal{H}_{\mathrm{reg}}(J)$. Then the sets $M_1(c; H, J)$ and $M_2(c; H, J)$ are compact for every $c > 0$.*

Proof: We prove that there is positive number $\hbar > 0$ such that

$$E(v) \geq \hbar, \qquad E(u) \geq \hbar$$

for every nonconstant J-holomorphic sphere $v : S^2 \to M$ and every nontrivial s-dependent solution $u : \mathbb{R} \times S^1 \to M$ of (2). For nonconstant J-holomorphic spheres this follows from Gromov's compactness. Now assume that there is a sequence of solutions u_ν of (2) with $0 \neq E(u_\nu) \to 0$.

We prove that $\partial u_\nu/\partial s$ converges to zero uniformly on $\mathbb{R}^2$ as ν tends to ∞. Otherwise there would exist a sequence (s_ν, t_ν) such that $|\partial u_\nu/\partial s(s_\nu, t_\nu)| \geq \delta > 0$. Assume without loss of generality that $s_\nu = 0$. Since $E(u_\nu)$ converges to zero no bubbling can occur and hence a subsequence of $u_\nu(s,t)$ converges with its derivatives uniformly on compact sets to a solution $u : \mathbb{R} \times S^1 \to M$ of (2) with $|\partial u_\nu/\partial s(0, t^*)| \geq \delta$ and $E(u) = 0$. But the latter implies that $u(s,t) \equiv x(t)$ in contradiction to the former.

Thus we have proved that $\partial u_\nu/\partial s$ converges to zero uniformly and, passing to a subsequence, that $u_\nu(s,t)$ converges with its derivatives uniformly on compact sets to a periodic solution $x(t)$ of (1). We prove that $u_\nu(s,t)$ converges to $x(t)$ uniformly on $\mathbb{R}^2$. To see this choose $\varepsilon > 0$ such that $d(x,y) = \sup_t d_M(x(t), y(t)) < \varepsilon$ for every $y \in \mathcal{P}(H)$, $y \neq x$. Then there exists a $\delta > 0$ such that for every C^1-function $z : S^1 \to M$ with $d(z,x) = \varepsilon$ we have $\sup_t |\dot{z}(t) - X_H(t,z)| \geq \delta$. (Otherwise, by the Arzela-Ascoli theorem, there would exist a periodic solution $y \in \mathcal{P}(H)$ with $d(x,y) = \varepsilon$.) Now choose ν sufficiently large such that $|\partial u_\nu/\partial s| < \delta$. Then it follows that $d_M(u_\nu(s,t), x(t)) < \varepsilon$ for all s and t.

Thus we have proved that $u_\nu(s,t)$ converges to $x(t)$ uniformly on $\mathbb{R}^2$. In particular this implies that u_ν satisfies the limit condition (3) with $x^- = x^+ = x$ and u_ν represents a trivial homology class. Now the energy of u_ν is given by

$$\begin{aligned} E(u_\nu) &= \int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u_\nu}{\partial s}\right|^2 dtds \\ &= \int_{-\infty}^{\infty}\int_0^1 \left\langle \frac{\partial u_\nu}{\partial s}, \nabla H(t,u_\nu) - J(u_\nu)\frac{\partial u_\nu}{\partial t}\right\rangle dtds \\ &= \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u_\nu}{\partial s}, \frac{\partial u_\nu}{\partial t}\right) dtds + \int_{-\infty}^{\infty} \frac{d}{ds}\int_0^1 H(t,u_\nu)\, dtds \\ &= 0. \end{aligned}$$

The last identity follows from Stokes' theorem and contradicts our assumption that $E(u_\nu) \neq 0$.

Now let u_ν be a sequence of solutions of (2) and (3) with fixed limits such that

$$\mu(u_\nu) = 2, \qquad E(u_\nu) \leq c.$$

Assume without loss of generality that $E(u_\nu)$ converges. Using a standard argument as in [20] one can show that there exist a subsequence (still denoted by u_ν), periodic solutions $x^- = x^0, x^1, \ldots, x^{\ell-1}, x^\ell = x^+$ (not necessarily distinct), and connecting orbits $u^j \in \mathcal{M}(x^j, x^{j-1}; H, J)$ for $j = 1, \ldots, \ell$ with total energy

$$\sum_{j=1}^{\ell} E(u^j) \leq c$$

such that the following holds. Given any sequence $s_\nu \in \mathbb{R}$ the sequence $v_\nu(s,t) = u_\nu(s+s_\nu, t)$ has a subsequence which converges modulo bubbling either to $u^j(s+s^j, t)$ for some s^j or to $x^j(t)$ for some j. Here convergence modulo bubbling means that there exist finitely many points in $\mathbb{R} \times S^1$ such that v_ν converges with its derivatives uniformly on every compact subset of the complement of these points. Moreover, every u^j is such a limit and no other connecting orbit can be approximated by u_ν in this way.

We prove that bubbling cannot occur. By Theorem A.1 there are only finitely many J-holomorphic spheres which can bubble off in our limit process. Denote these spheres by $v^1, \dots, v^m$. It follows from Theorem A.1 that

$$\sum_{j=1}^{m} E(u^j) + \sum_{j=1}^{\ell} E(v^j) = \lim_{\nu\to\infty} E(u_\nu)$$

and

$$\sum_{j=1}^{m} \mu(u^j) + \sum_{j=1}^{\ell} 2c_1(v^j) = 2.$$

Since there is no J-holomorphic sphere with negative Chern number this implies that $\mu(u^j) \le 2$ for every j. The key point in our argument is that, again by Theorem A.1, the spheres v^j together with the connecting orbits u^j and the periodic solutions x^j form a connected family. So if bubbling occurs then one of the spheres v^j must intersect one of the connecting orbits u^j or one of the periodic solutions x^j. Since $M_2(\infty; H, J) \cap M_0(\infty; J) = \emptyset$ there must be a j with $c_1(v^j) > 0$. This implies

$$\sum_{j=1}^{m} \mu(u^j) \le 0.$$

But since $H \in \mathcal{H}_{\mathrm{reg}}(J)$ there is no nonconstant connecting orbit u with $\mu(u) \le 0$. Hence $x^- = x^+$, $m = 1$, and one of the spheres v^j must intersect the periodic solution $x^\pm$ contradicting the fact that $M_0(H) \cap M_1(\infty; J, H) = \emptyset$. The same argument works for $\mu(u_\nu) = 1$ and this proves the theorem. □

4 Generalized Laurent series

Let Γ be a group with a **weight homomorphism** $\phi : \Gamma \to \mathbb{R}$ and let $\mathbb{F}$ be an integral domain. Consider the $\mathbb{F}$-module

$$\Lambda = \Lambda(\Gamma, \phi; \mathbb{F})$$

of all functions $\Gamma \to \mathbb{F} : A \mapsto \lambda_A$ such that the set

$$\{A \in \Gamma : \lambda_A \neq 0,\ \phi(A) < c\}$$

is finite for every constant $c \in \mathbb{R}$. This space is a ring with the product given by the convolution

$$(\lambda * \theta)_A = \sum_{B \in \Gamma} \lambda_B \theta_{B^{-1}A}.$$

This is a finite sum and $\lambda * \theta \in \Lambda$. The unit element is the delta function $\delta : \Gamma \to \mathbb{F}$ defined by $\delta_{\mathbb{1}} = 1$ and $\delta_A = 0$ for $A \neq \mathbb{1}$. In the case $\mathbb{F} = \mathbb{Z}$ and $\phi = 0$ the module $\Lambda(\Gamma, 0; \mathbb{Z})$ is the group ring.

Now assume that $\phi : \Gamma \to \mathbb{R}$ is injective. Then $\Lambda(\Gamma, \phi; \mathbb{F})$ is an integral domain. Moreover, the group Γ is necessarily isomorphic to a free abelian group with finitely many generators. Hence we assume $\Gamma = \mathbb{Z}^m$ and

$$\phi(k_1, \ldots, k_m) = \sum_{j=1}^{m} \omega_j k_j = \omega \cdot k$$

where the ω_j are positive and rationally independent. In this case we can identify $\Lambda(\Gamma, \phi; \mathbb{F}) = \Lambda(\omega; \mathbb{F})$ with the space of formal power series

$$f(t) = \sum_k a_k t^k$$

where $t = (t_1, \ldots, t_m)$, $k = (k_1, \ldots, k_m)$, and $t^k = t_1^{k_1} \cdots t_m^{k_m}$. The coefficients are subject to the condition

$$\#\{k : a_k \neq 0, \, \omega \cdot k \leq c\} < \infty$$

for every constant $c \in \mathbb{R}$. We call such a power series f a **generalized Laurent series**. The map $d : \Lambda \setminus \{0\} \to \mathbb{R}$ defined by

$$d(f) = \inf\{\omega \cdot k : a_k \neq 0\}$$

for $f = \sum_k a_k t^k$ is a homomorphism

$$d(fg) = d(f) + d(g).$$

Here we have used the fact that $\mathbb{F}$ has no zero divisors. Moreover, it follows from our definitions that for $f \neq 0$ there exists a unique $k = k(f) \in \mathbb{Z}^m$ such that $\omega \cdot k = d(f)$. We call the term $a_{k(f)} t^{k(f)}$ the **leading term** of f and $a_{k(f)}$ the **leading coefficient**.

Theorem 4.1 *Assume that the numbers $\omega_j > 0$ are rationally independent. Then $f \in \Lambda(\omega; \mathbb{F})$ is invertible if and only if the leading coefficient is invertible in $\mathbb{F}$. In particular, $\Lambda(\omega; \mathbb{F})$ is a field if and only if $\mathbb{F}$ is a field.*

Proof: Assume without loss of generality that the leading term is 1. We shall construct a sequence $g_\nu = 1 + q_\nu \in \Lambda$ such that $d(q_\nu) > 0$ and

$$\lim_{\nu \to \infty} d(q_\nu) = \infty, \qquad \lim_{\nu \to \infty} d(g_\nu \cdots g_2 g_1 f - 1) = \infty.$$

For any such sequence the infinite product

$$g = \prod_{\nu=1}^{\infty} g_\nu \in \Lambda$$

is well defined and is the inverse of f.

The set $\{\omega \cdot k : a_k \neq 0\}$ can be written as an increasing sequence $d_0(f) < d_1(f) < d_2(f) < \cdots$ converging to infinity with $d_0(f) = d(f) = 0$. Since the ω_j are rationally independent there exists a unique sequence $k_\nu \in \mathbb{Z}^m$ such that $\omega \cdot k_\nu = d_\nu(f)$. Assume that $f \neq 1$ and define

$$g_1(t) = 1 - a_{k_1} t^{k_1}$$

to obtain

$$d_1(g_1) = d_1(f), \qquad d_1(g_1 f) > d_1(f).$$

The last inequality follows from the fact that either $d_1(g_1 f) = d_2(f)$ or $d_1(g_1 f) = 2d_1(f)$. More generally we find that $d_\nu(g_1 f) \in \Sigma(f)$ for every ν where $\Sigma(f)$ denotes the set of finite linear combinations of the $d_\nu(f)$ with nonnegative integer coefficients. Hence

$$\Sigma(g_1 f) \subset \Sigma(f).$$

Now proceed by induction and construct a (possibly finite) sequence $g_\nu \in \Lambda$ with leading term 1 such that

$$d_1(g_\nu) = d_1(g_{\nu-1} \cdots g_1 f), \qquad d_1(g_\nu \cdots g_1 f) > d_1(g_{\nu-1} \cdots g_1 f)$$

Moreover $\Sigma(g_\nu \cdots g_1 f) \subset \Sigma(g_{\nu-1} \cdots g_1 f)$ and hence $d_1(g_\nu)$ is a strictly increasing sequence in $\Sigma(f)$. Hence $d_1(g_\nu)$ converges to infinity and this proves the theorem. □

Theorem 4.2 *Assume that the numbers $\omega_j > 0$ are rationally independent. Then $\Lambda(\omega; \mathbb{F})$ is a principal ideal domain if and only if $\mathbb{F}$ is a principal ideal domain.*

Proof: First assume that $\Lambda = \Lambda(\omega, \mathbb{F})$ is a PID. If $I \subset \mathbb{F}$ is an ideal then $I\Lambda$ is an ideal in Λ and is therefore generated by a single element f. Assume without loss of generality that $d(f) = 0$ and let a_0 be the leading coefficient of f. Then every $a \in I$ can be viewed as a generalized Laurent series in $I\Lambda$. Hence $a = gf$ for some $g \in \Lambda$ and hence $a = xa_0$ for some $x \in \mathbb{F}$.

Conversely, suppose that $\mathbb{F}$ is a PID and let $I \subset \Lambda$ be an ideal. Then the set $I_0 \subset \mathbb{F}$ of leading coefficients of elements of I is an ideal in $\mathbb{F}$. Hence I_0 is generated by a single element a_0. Choose $f_0 \in I$ with $d(f_0) = 0$ and leading coefficient a_0. We prove that I is generated by f_0. Let $f \in I$ and assume without loss of generality that $d(f) = 0$. Since the leading coefficient of f is an element of I_0 it must be a multiple of a_0. Hence there exists a $b_0 \in \mathbb{F}$ such that

$$d(f - b_0 f_0) > d(f) = 0.$$

Note in fact that either $f - b_0 f_0 = 0$ or $d(f - b_0 f_0) \in \Sigma(f) \cup \Sigma(f_0) \subset \Sigma(f) + \Sigma(f_0)$. More generally

$$\Sigma(f - b_0 f_0) \subset \Sigma(f) + \Sigma(f_0).$$

Since $f - b_0 f_0 \in I$ its leading coefficient is again a multiple of a_0 and hence there exists a $b_1 \in \mathbb{F}$ and a $k_1 \in \mathbb{Z}^m$ such that

$$d(f - b_0 f_0 - b_1 t^{k_1} f_0) > d(f - b_0 f_0) = k_1 \cdot \omega.$$

Proceed by induction and construct a sequence

$$g_\nu = b_0 + b_1 t^{k_1} + \cdots + b_\nu t^{k_\nu} \in \Lambda$$

such that

$$d(f - g_\nu f_0) > d(f - g_{\nu-1} f_0) = k_\nu \cdot \omega.$$

Then $k_\nu \cdot \omega$ is a strictly increasing sequence in $\Sigma(f) + \Sigma(f_0)$. Hence $k_\nu \cdot \omega$ converges to infinity and hence g_ν converges to $g \in \Lambda$ with $f = g f_0$. $\square$

5 Floer cohomology

Let (M, ω) be a $2n$-dimensional compact weakly monotone symplectic manifold with a regular almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ in the sense of section 2 and let $H \in \mathcal{H}_{\mathrm{reg}}(J)$ be a regular Hamiltonian in the sense of section 3. Let $\mathcal{L}$ denote the space of contractible loops $x : S^1 \to M$. The universal cover of $\mathcal{L}$ is the set of equivalence classes of pairs (x, u) where $x \in \mathcal{L}$ and $u : D \to M$ is a smooth disc such that $u(e^{2\pi i t}) = x(t)$. Two such pairs (x, u) and (y, v) are equivalent if $x = y$ and u is homotopic to v with fixed boundary. We shall, however, use a weaker equivalence relation

$$[x, u_0] \equiv [x, u_1] \qquad \Longleftrightarrow \qquad \int_D {u_0}^* c_1 = \int_D {u_1}^* c_1, \qquad \int_D {u_0}^* \omega = \int_D {u_1}^* \omega.$$

Here c_1 denotes a 2-form which represents the first Chern class of TM. The definition of the equivalence relation is independent of the choice of this form. For simplicity of notation we fix a point $x_0 \in M$ and assume that all discs satisfy $u(0) = x_0$. We denote by $\widetilde{\mathcal{L}}$ the set of all such equivalence classes. The group

$$\Gamma = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}$$

acts on the space $\widetilde{\mathcal{L}}$ via $[x,u] \mapsto [x, A\#u]$. Here $A\#u$ denotes the equivalence class of the connected sum $v\#u$ for $v \in A$. More explicitly, choose $v \in A$ with $v(0) = v(\infty) = x_0$ and define the connected sum by $v\#u(z) = v((1-2|z|)^{-1}z)$ for $|z| \leq 1/2$ and $v\#u(z) = u((2-|z|^{-1})z)$ for $|z| \geq 1/2$. Note in fact that

$$\mathcal{L} = \widetilde{\mathcal{L}}/\Gamma.$$

Let $\widetilde{\mathcal{P}}(H)$ denote the subset of those pairs $[x,u] \in \widetilde{\mathcal{L}}$ where $x \in \mathcal{P}(H)$ is a contractible 1-periodic solution of (1). These are the critical points of the **action functional** $a_H : \widetilde{\mathcal{L}} \to \mathbb{R}$ defined by

$$a_H([x,u]) = \int_D u^*\omega + \int_0^1 H(t,x(t))\,dt.$$

The solutions of (2) are the gradient flow lines of a_H. Given $[x^\pm, u^\pm] \in \widetilde{\mathcal{P}}(H)$ denote by

$$\mathcal{M}([x^-,u^-],[x^+,u^+];H,J)$$

the set of those connecting orbits $u \in \mathcal{M}(x^-,x^+;H,J)$ for which $[x^+, u^-\#u] \equiv [x^+,u^+]$. More explicitly we introduce the function $\beta(s) = (1+s^2)^{-1/2}s$. Then the connected sum $u^-\#u : D \to M$ is defined by $u^-\#u(z) = u^-(e^{4\pi}z)$ for $|z| \leq e^{-4\pi}$ and $u^-\#u(e^{2\pi(\beta(s)+it-1)}) = u(s,t)$ for $(s,t) \in \mathbb{R}^2$. It follows from the energy identity in the proof of Theorem 3.3 that

$$E(u) = a_H([x^+,u^+]) - a_H([x^-,u^-])$$

for $u \in \mathcal{M}([x^-,u^-],[x^+,u^+],H;J)$. Note that

$$a_H([x,A\#u]) - a_H([x,u]) = \omega(A).$$

for every $[x,u] \in \widetilde{\mathcal{L}}$ and every $A \in \Gamma$.

The dimension of the space $\mathcal{M}([x^-,u^-],[x^+,u^+],H;J)$ can be expressed in terms of the Conley-Zehnder index which is defined as follows. Given $[x,u] \in \widetilde{\mathcal{P}}(H)$ choose a symplectic trivialization $\Phi(t) : \mathbb{R}^{2n} \to T_{x(t)}M$ of x^*TM which extends over the disc u. Now linearize the Hamiltonian differential equation (1) along $x(t)$ to obtain a path of symplectic matrices

$$\Psi(t) = \Phi(t)^{-1} d\psi_t(x(0))\Phi(0) \in Sp(2n;\mathbb{R}).$$

Here the symplectomorphism $\psi_t : M \to M$ denotes the time-t-map of (1). Then $\Psi(0) = \mathbb{1}$ and $\Psi(1)$ is conjugate to $d\psi_1(x(0))$ so that $\det(\mathbb{1} - \Psi(1)) \neq 0$. The homotopy class of the path Ψ subject to these conditions is independent of the choice of the trivialization and is determined by the **Conley-Zehnder index** $\mu([x,u]) = \mu(\Psi)$ (see [5]). In [21] it is shown that the dimension of the space of connecting orbits is given by the formula

$$\dim \mathcal{M}([x^-,u^-],[x^+,u^+];H,J) = \mu([x^+,u^+]) - \mu([x^-,u^-]).$$

Moreover, the Conley-Zehnder index satisfies the identity

$$\mu([x, A\#u]) - \mu([x, u]) = 2c_1(A).$$

In particular the Conley-Zehnder index $\mu(x)$ of a periodic solution $x \in \mathcal{P}(H)$ of (1) is well defined modulo $2N$ where N is the minimal Chern number of (M, ω).

Denote by $\widetilde{\mathcal{P}}_k(H)$ the subset of all $[x, u] \in \widetilde{\mathcal{P}}(H)$ with $\mu([x, u]) = k$ and consider the cochain complex whose k-th cochain group

$$C^k = C^k(H)$$

consists of all functions $\widetilde{\mathcal{P}}_k(H) \to \mathbb{Z}_2 : [x, u] \mapsto \xi_{[x,u]}$ for which there are only finitely many nonzero entries in every region of finite symplectic action, i.e.

$$\#\left\{[x, u] \in \widetilde{\mathcal{P}}_k(H) : \xi_{[x,u]} \neq 0,\, a_H([x, u]) \geq c\right\} \quad < \quad \infty$$

for every constant $c > 0$. This is an infinite dimensional vector space over $\mathbb{Z}_2$ but a finite dimensional vector space over the field $\Lambda_\omega = \Lambda(\Gamma_0, \phi_\omega; \mathbb{Z}_2)$ where

$$\Gamma_0 = \frac{\ker \phi_{c_1}}{\ker \phi_{c_1} \cap \ker \phi_\omega}.$$

The scalar multiplication of $\xi \in C^k$ with $\lambda \in \Lambda_\omega$ is given by

$$(\lambda * \xi)_{[x,u]} = \sum_{A \in \Gamma_0} \lambda_A \xi_{[x,(-A)\#u]}.$$

The reader may check that this is a finite sum for every $[x, u] \in \widetilde{\mathcal{P}}(H)$ and that $\lambda * \xi \in C^k$. The dimension of C^k as a vector space over Λ_ω is precisely the number of contractible 1-periodic solutions $x \in \mathcal{P}(H)$ of (1) with Conley-Zehnder index $\mu(x) = k(\text{mod } 2N)$. An explicit basis is a set of pairs $[x, u_x]$ with one representative for each periodic solution. Thus we may identify C^k with the vector space

$$C^k = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ \mu(x) = k(\text{mod } 2N)}} \Lambda_\omega x.$$

Now it follows from Theorem 3.3 that the space $\mathcal{M}([x^-, u^-], [x^+, u^+]; H, J)$ consists of finitely many orbits (modulo time shift) whenever

$$\mu([x^+, u^+]) - \mu([x^-, u^-]) = 1.$$

We denote by $n_2([x^-, u^-], [x^+, u^+])$ the number modulo 2 of these connecting orbits. The coboundary map $\delta^k : C^k \to C^{k+1}$ is defined by

$$\delta[x, u] = \sum_{\mu([y,v])=k+1} n_2([x, u], [y, v])\, [y, v]$$

for $[x,u] \in \widetilde{\mathcal{P}}(H)$ with $\mu([x,u]) = k$. By Theorem 3.3 there are only finitely many connecting orbits from x to y in every region of finite energy and hence $\delta[x,u] \in C^k$. The reader may check that δ is a Λ_ω-linear map.

Theorem 5.1 *$\delta \circ \delta = 0$.*

Proof: The argument is the same as in Floer's original work [8], [15] and we shall only sketch the main idea. We must prove that

$$\sum_{\mu([y,v])=k} n_2([x,u],[y,v]) \cdot n_2([y,v],[z,w]) = 0$$

whenever $[x,u],[z,w] \in \widetilde{\mathcal{P}}(H)$ with

$$\mu([x,u]) = k-1, \qquad \mu([z,w]) = k+1.$$

This follows by examining the ends of the 1-manifold $\mathcal{M}([x,u],[z,w];H,J)/\mathbb{R}$. Here $\mathbb{R}$ acts by translation in the s-variable. Since all the connecting orbits in $\mathcal{M}([x,u],[z,w];H,J)$ have index $\mu = 2$ and energy $E = a_H([z,w]) - a_H([x,u])$ it follows from the proof of Theorem 3.3 that no bubbling can occur. Hence the usual glueing argument shows that the ends of the 1-manifold $\mathcal{M}([x,u],[z,w];H,J)/\mathbb{R}$ are in one-to-one correspondence with the pairs of connecting orbits from $[x,u]$ to $[y,v]$ and from $[y,v]$ to $[z,w]$ [10], [15]. Hence the number of such pairs is even and this proves the theorem. $\Box$

The cohomology groups

$$HF^k(M,\omega,H,J;\mathbb{Z}_2) = \frac{\ker \delta^k}{\operatorname{im} \delta^{k-1}}$$

are called the **Floer cohomology groups** of the quadruple (M,ω,H,J) with coefficients in $\mathbb{Z}_2$. The Floer cohomology groups are finite dimensional vector spaces over the field Λ_ω and they are graded modulo $2N$

$$HF^k = HF^{k+2N}.$$

It follows by the same arguments as in [8] that these groups are independent of the choice of the Hamiltonian H and the almost complex structure J used to define them (see also [15] and [21]). This is stated more precisely in the next theorem.

Theorem 5.2 *Given $J^\alpha, J^\beta \in \mathcal{J}_{\mathrm{reg}}(M,\omega)$ and $H^\alpha \in \mathcal{H}_{\mathrm{reg}}(J^\alpha)$, $H^\beta \in \mathcal{H}_{\mathrm{reg}}(J^\beta)$ there exists a natural vector space homomorphism*

$$HF^{\beta\alpha} : HF^*(M,\omega,H^\alpha,J^\alpha;\mathbb{Z}_2) \to HF^*(M,\omega,H^\beta,J^\beta;\mathbb{Z}_2)$$

which preserves the grading by the Conley-Zehnder index. If (H^γ, J^γ) is any other such pair then

$$HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}, \qquad HF^{\alpha\alpha} = \mathrm{id}.$$

In particular, $HF^{\beta\alpha}$ is an isomorphism.

Proof: Choose a regular homotopyy $J_s \in \mathcal{J}(M, \omega)$ of almost complex structures connecting $J_0 = J_\alpha$ to $J_1 = J_\beta$ such that the parametrized versions of the results of section 2 hold. We may assume that J_s extends to a smooth function on $s \in \mathbb{R}$ such that $J_s = J_\alpha$ for $s \leq 0$ and $J_s = J_\beta$ for $s \geq 1$. The analogue of Theorem 2.2 now states that the set

$$\mathcal{M}_s(A; \{J_s\}) = \{(s, v) : v \in \mathcal{M}_s(A; J_s)\}$$

is a manifold of dimension

$$\dim \mathcal{M}_s(A; \{J_s\}) = 2n + 2c_1(A) + 1$$

for a generic family $\{J_s\}$. This space will be empty in the case $c_1(A) < 3 - n$. In the case $c_1(A) = 0$ the set of pairs (s, p) such that p is a point on a J_s-holomorphic curve in the class A is roughly speaking a set of codimension 4 in $\mathbb{R} \times M$.

The results of section 3 now apply to the solutions of the equation

$$\frac{\partial u}{\partial s} + J(s, u)\frac{\partial u}{\partial t} - \nabla H(s, t, u) = 0 \tag{4}$$

where $H(s, t, x) = H_s(t, x)$ is a generic homotopy of Hamiltonians connecting $H_0 = H_\alpha$ to $H_1 = H_\beta$. In particular, the solutions with index difference 0 or 1 do not intersect the set $M_0(\infty; \{J_s\})$ of J_s-holomorphic curves of Chern number 0. (Note that J_s-holomorphic spheres bubble off at parameter values (s, t).) The solutions of (4) determine a cochain homomorphism from the Morse complex of the pair (H^α, J^α) to the Floer chain complex of the pair (H^β, J^β).

It follows by the same arguments as in Floer's original work that this cochain homomorphism induces an isomorphism on cohomology. For details we refer to [8], [9], [15], and [21]. $\Box$

In the terminology of Conley the above theorem states that the Floer homology goups corresponding to regular pairs (H, J) form a connected simple system. They can be viewed as the *middle dimensional cohomology* of the Γ-cover $\widetilde{\mathcal{L}}$ of the space $\mathcal{L}$ of contractible loops in M with coefficients in the field Λ_ω viewed as a representation of Γ. In section 7 we shall prove that these cohomology groups are naturally isomorphic to the homology groups of M with coefficients in Λ_ω.

6 Morse inequalities

Let $H : M \to \mathbb{R}$ be a Morse function and choose a regular almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ such that the gradient flow of H is of Morse-Smale type. If H is sufficiently small then the periodic solutions of (1) are precisely the critical points of H. For each such critical point we fix a disc $u_x(re^{i\theta}) = \gamma_x(r)$ where $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. The equivalence class $[x, u_x]$ is independent of the choice of the path γ_x. These equivalence classes form a natural basis of the cochain complex C^*. In [21] it was proved that if the second derivatives of H are sufficiently small then the Conley-Zehnder index of the pair (x, u_x) is given by

$$\mu(x) = \mathrm{ind}_H(x) - n$$

where $\mathrm{ind}_H(x)$ denotes the Morse index of x as a critical point of H. Note that every solution u of (2) which is independent of t is a gradient flow line of H

$$\frac{d}{ds} u(s) = \nabla H(u(s)). \tag{5}$$

Since (5) is a Morse-Smale flow there are only finitely many conecting orbits from x to y whenever $\mu_H(y) - \mu_H(x) = 1$. We denote the number of these connecting orbits modulo 2 by $n_2(x, y)$. Let C^k denote the $\mathbb{Z}_2$-vector space generated by the critical points x of H with Morse index $\mu_H(x) = k (\mathrm{mod}\ 2N)$ and let

$$\delta^k : C^k \to C^{k+1}$$

denote the linear map whose (x, y) entry is the number $n_2(x, y)$ whenever the index difference is 1. Then δ is a coboundary operator and its homology agrees with the cohomology of M

$$H^k(C, \delta) = \bigoplus_{j=k(\mathrm{mod}\ 2N)} H^j(M; \mathbb{Z}_2).$$

(See [24] and [20].) It follows from the universal coefficient theorem that

$$H^k(C \otimes \Lambda_\omega, \delta \otimes \mathbb{1}) = \bigoplus_{j=k(\mathrm{mod}\ 2N)} H^j(M; \mathbb{Z}_2) \otimes \Lambda_\omega.$$

Now the Morse function $H : M \to \mathbb{R}$ will in general not be regular in the sense of section 3 since there may be solutions $u : \mathbb{R}^2 \to M$ of (2) and (3) which are not independent of t. We shall, however, prove that these nontrivial solutions cannot occur with index difference 1 provided that H is sufficiently small. This leads to the following theorem which in the monotone case was proved by Floer [8]. We postpone the proof to section 7.

Theorem 6.1 *Let (M, ω) be a compact symplectic manifold of dimension $2n$. Assume either that M is monotone or $c_1(\pi_2(M)) = 0$ or the minimal Chern number is $N \geq n$. Then for every $J^\alpha \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ and every $H^\alpha \in \mathcal{H}_{\mathrm{reg}}(J^\alpha)$ there exists a natural isomorphism*

$$HF^\alpha : HF^k(M, \omega, H^\alpha, J^\alpha; \mathbb{Z}_2) \to \bigoplus_{j=k(\mathrm{mod}\ 2N)} H^{j+n}(M; \Lambda_\omega)$$

If (H^β, J^β) is any other such pair then $HF^\beta \circ HF^{\beta\alpha} = HF^\alpha$.

Remark 6.2 If the manifold M admits a Morse function which has only critical points of even index then the conclusions of Theorem 6.1 are obviously satisfied. However, in this case the Arnold conjecture follows already from the Lefschetz fixed point theorem.

Given any Hamiltonian $H : S^1 \times M \to \mathbb{R}$ with nondegenerate 1-periodic solutions we define the numbers

$$p_k = \#\left\{x \in \mathcal{P}(H) : \mu(x) = k \,(\mathrm{mod}\ 2N)\right\}.$$

Note that $p_k = p_{k+2N}$. By Theorem 6.1 these numbers are related to the $2N$-periodic Betti numbers

$$b_k = \sum_{j=k\,(\mathrm{mod}\ 2N)} \dim H^j(M;\mathbb{Z}_2).$$

via the Morse inequalities.

Theorem 6.3 *Assume that (M,ω) satisfies the requirements of Theorem 6.1 and let $H : S^1 \times M \to \mathbb{R}$ be a Hamiltonian with nondegenerate 1-periodic solutions. If $N = 0$ then we have*

$$p_\ell - p_{\ell-1} + \cdots + \pm p_{-k} \geq b_{n+\ell} - b_{n+\ell-1} + \pm b_{n+\ell-k}$$

for any two integers k and ℓ and equality holds for ℓ and $-k$ sufficiently large. In the case $N \neq 0$ the morse inequalities are satisfied when $\ell - k$ even and equality holds for $\ell - k = 2N - 1$. In both cases we have

$$p_k \geq b_{n+k}$$

for every integer k.

Remark 6.4 **(i)** The Floer homology groups can be defined with coefficients in $\mathbb{Z}$ (rather than $\mathbb{Z}_2$) or in any other principal ideal domain. In that case the number $n([x^-,u^-],[x^+,u^+])$ must be defined by counting the connecting orbits with appropriate signs as in [9].

(ii) The Conley-Zehnder index of a nondegenerate periodic solution $x \in \mathcal{P}(H)$ of (1) satisfies the identity

$$\operatorname{sign}\det(\mathbb{1} - d\psi_1(x(0)) = (-1)^{\mu(x)+n}.$$

Hence we recover the Lefschetz fixed point formula

$$(-1)^n\chi(M) = \sum_{k=0}^{2N-1} (-1)^k \dim_{\Lambda_\omega} HF^k(M,\omega,H,J;\mathbb{Z}_2) = \sum_{x\in\mathcal{P}(H)} (-1)^{\mu(x)}$$

from our Morse inequalities.

(iii) Poincaré duality can be expressed in the form

$$HF^k(M, \omega, H, J; \mathbb{Z}_2) = HF^{-k}(M, \omega, -H, J; \mathbb{Z}_2).$$

To see this replace $x \in \mathcal{P}(H)$ by $y(t) = x(-t)$ and $u \in \mathcal{M}(x^-, x^+; H, J)$ by $v(s,t) = u(-s,-t)$. This duality will remain valid in cases where the Floer cohomology groups are not isomorphic to the cohomology of M.

(iv) An interesting class of symplectic manifolds is where the first Chern class vanishes. In this case the Conley-Zehnder index of a nondegenerate periodic solution $x \in \mathcal{P}(H)$ is a well defined integer and hence the Floer homology groups are graded by the integers. If in addition the complex structure J is integrable then (M, ω, J) is called a **Calabi-Yau manifold**. A Calabi-Yau metric is one where the first Chern form of the curvature vanishes. Manifolds of this type have received considerable interest in the recent physics literature [3], [4].

(v) Our construction can be used to recover the Novikov-homology groups associated to a closed 1-form α on a compact Riemannian manifold M with nondegenerate zeros [17]. In this case the associated ring is $\Lambda_\alpha(\mathbb{F}) = \Lambda(\Gamma, \phi_\alpha; \mathbb{F})$ where $\phi_\alpha : \pi_1(M) \to \mathbb{R}$ is the homomorphism induced by α and $\Gamma = \pi_1(M)/\ker\phi_\alpha$. The Novikov homology groups can be recovered as the homology groups of a chain complex generated by the zeros of α. The boundary operator is determined by the one dimensional connecting orbits in the covering space $\tilde{M} \to M$ with fiber Γ. As in the case of Calabi-Yau manifolds the index of a zero is well defined. This construction is a natural generalization of Witten's approach to Morse theory for functions [24]. The details will be carried out elsewhere.

7 Proof of Theorem 6.1

In order to compute the Floer cohomology groups we must study the partial differential equation (2) with a time independent Hamiltonian function $H : M \to \mathbb{R}$

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(u) = 0. \tag{6}$$

The gradient flow lines $\gamma(s)$ of H appear as solutions $u(s,t) = \gamma(s)$ of (6) which are independent of t. If the gradient flow of H is of Morse-Smale type then the gradient flow lines determine a chain complex whose homology agrees with the homology of M [24], [7], [20]. As in [8] we shall use this fact to prove that the Floer cohomology groups are naturally isomorphic to the cohomology of M. The main problem is to prove that every solution of (6) with relative Morse index 1 must be independent of t. We outline a proof here and refer the reader to [10] for complete details.

Fix an almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ which is regular in the sense of section 3. For every Hamiltonian function H which is sufficiently small the 1-periodic solutions $x \in \mathcal{P}(H)$ are constant (Yorke's estimate) and hence agree with the critical points of H. Moreover, for a generic Hamiltonian function H, the gradient flow with respect to the metric induced by J is of Morse-Smale type. Now every solution $u(s,t) = u(s,t+1)$ of (6) with limits

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}, \qquad dH(x^{\pm}) = 0, \tag{7}$$

determines a homotopy class in $\pi_2(M)$. The energy and the index of u are given by

$$E(u) = H(x^+) - H(x^-) + \int u^*\omega \tag{8}$$

and

$$\mu(u) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) + 2\int u^*c_1. \tag{9}$$

Equation (8) follows from the energy identity in the proof of Theorem 3.3 and the fact that the limit orbits $x^{\pm}$ are critical points of H. Equation (9) follows from the relation between the Conley-Zehnder index and the Morse index when H is sufficiently small.

Lemma 7.1 *Fix $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ and let $H : M \to \mathbb{R}$ be a smooth Morse function whose gradient flow is of Morse-Smale type with respect to the metric induced by J. Then there exists a constant $\tau_0 > 0$ such that every solution $u(s,t) = u(s,t+1)$ of*

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \tau \nabla H(u) = 0 \tag{10}$$

and (7) with $\tau < \tau_0$, and

$$\int u^*\omega \le 0$$

is independent of t.

Proof: By Gromov's compactness there exists a constant $\hbar > 0$ such that

$$E(v) = \int v^*\omega \ge \hbar$$

for every nonconstant J_0-holomorphic sphere $v : S^2 \to M$. Now assume that the statement of the lemma were false. Then there would exist a sequence of solutions u_ν of (6) and (3) with $\tau = \tau_\nu$ converging to zero, $J = J_\nu$ converging to J_0, and with

$$\int {u_\nu}^*\omega \le 0, \qquad \frac{\partial u_\nu}{\partial t} \not\equiv 0.$$

Passing to a subsequence we may assume without loss of generality that the limit points $x^{\pm}$ are independent of ν. Choose a sequence of integers k_ν converging to infinity such that

$$k_\nu \tau_\nu \to \tau > 0.$$

Choose τ so small that the index formula (9) holds for τH_0 and that

$$\tau \left(H_0(x^+) - H_0(x^-) \right) \leq \frac{\hbar}{2}.$$

Now define

$$v_\nu(s,t) = u_\nu(k_\nu s, k_\nu t) = v_\nu(s, t + 1/k_\nu).$$

Then it follows from equation (8) that

$$\begin{aligned} E(v_\nu) &= k_\nu \tau_\nu \left(H_0(x^+) - H_0(x^-) \right) + k_\nu \int {u_\nu}^* \omega \\ &\leq k_\nu \tau_\nu \left(H_0(x^+) - H_0(x^-) \right) \\ &\leq \hbar/2. \end{aligned}$$

Hence bubbling cannot occur and it follows as in the proof of Theorem 3.3 that v_ν converges to a finite collection $v^1, \ldots, v^m$ of J_0-gradient flow lines of τH_0 connecting x^j to x^{j+1} where $x^0, \ldots, x^m$ are critical points of H_0 with $x^0 = x^-$ and $x^m = x^+$. Now in [21] it is shown that for τ sufficiently small the t-independent solutions of (10) are isolated in the space of all solutions. Hence $v_\nu(s,t) \equiv v_\nu(s)$ for ν sufficiently large in contradiction to our assumption. This proves Lemma 7.1. □

The following example shows that the statement of Lemma 7.1 becomes false without the assumption $\int u^*\omega \leq 0$.

Example 7.2 *Consider the symplectic manifold $M = S^2 = \mathbb{C} \cup \{\infty\}$ with the standard symplectic form*

$$\omega = \frac{dx \wedge dy}{\left(x^2 + y^2 + 1\right)^2}.$$

Here we denote by $z = x + iy$ a point in S^2 and by $\zeta = \xi + i\eta$ an associated tangent vector. The standard complex structure is multiplication by $i = \sqrt{-1}$ and the induced metric is

$$\langle \zeta, \zeta' \rangle = \frac{\xi\xi' + \eta\eta'}{\left(x^2 + y^2 + 1\right)^2}.$$

The gradient of the Hamiltonian function

$$H(x,y) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

with respect to this metric is $\nabla H(z) = 4z$. Hence equation (6) reads

$$\frac{\partial u}{\partial s} + i\frac{\partial u}{\partial t} = 4\tau u.$$

The critical points of H are $z = 0$ and $z = \infty$ and we require u to satisfy the limit condition

$$\lim_{s\to-\infty} u(s,t) = 0, \qquad \lim_{s\to+\infty} u(s,t) = \infty.$$

Explicit solutions are given by

$$u_k(s,t) = e^{4\tau s}e^{2\pi k(s+it)}, \qquad \pi k + 2\tau > 0.$$

This example is rather degenerate since all the integral curves of the Hamiltonian vector field $X_H(z) = -4iz$ are periodic with the same period $T = \pi/2$. The critical values $\tau = -k\pi/2$ are those where the integral curves of τX_H are of period 1. In particular, the fixed points 0 and ∞ are nondegenerate as 1-periodic solutions of τX_H if and only if $2\tau/\pi \notin \mathbb{Z}$. Their Conley-Zehnder index (with the constant disc) is

$$\mu(0) = -1 - 2[2\tau/\pi], \qquad \mu(\infty) = 1 + 2[2\tau/\pi].$$

Moreover we have

$$\int {u_k}^*\omega = \pi k, \qquad \int {u_k}^* c_1 = 2k$$

and hence

$$E(u_k) = \pi k + 2\tau, \qquad \mu(u_k) = 4[k + 2\tau/\pi] + 2$$

whenever $\pi k + 2\tau > 0$ and $2\tau/\pi \notin \mathbb{Z}$. It is also of interest to consider the solutions u_{-k} with $\pi k - 2\tau > 0$. These are connecting orbits from ∞ to 0 and they satisfy

$$\int {u_{-k}}^*\omega = \pi k, \qquad \int {u_{-k}}^* c_1 = 2k$$

and

$$E(u_{-k}) = \pi k - 2\tau, \qquad \mu(u_{-k}) = 4[k - 2\tau/\pi] + 2.$$

In particular, there is no solution with $\int u^*\omega < 0$ whenever $0 < \tau < \pi/2$. But there are always solutions with $\int u^*\omega > 0$ however small we choose τ. These solutions are stable and cannot be destroyed by a perturbation. They all have positive index $\mu(u) \geq 2$.

In the usual coordinates $(x_1, x_2, x_3) \in S^2 \subset \mathbb{R}^3$, related to $z = (1-x_3)^{-1}(x_1 + ix_2)$ via stereographic projection, the symplectic form is $\omega = (4x_3)^{-1}dx_2 \wedge dx_1$ and the Hamiltonian is the height function $H(x) = x_3$.

Our aim is to prove that the situation of the previous example is the general one, i.e. for a generic almost complex structure J every solution $u(s,t) = u(s,t+1)$ of (6) and (7) with $\mu(u) \le 1$ must be independent of t. Since the Hamiltonian function is required to be time independent it is more difficult to prove that the solutions of (6) and (7) form finite dimensional manifolds and we can only do this for simple solutions. This difficulty is similar to the one that arises in the study of J-holomorphic curves and was discussed in section 3. A function $u : \mathbb{R} \times S^1 \to M$ is called **simple** if for every integer $m > 1$ there exists a point $(s,t) \in \mathbb{R}^2$ such that $u(s,t+1/m) \ne u(s,t)$. For any two critical points $x^\pm$ of H denote by

$$\mathcal{M}_s(x^-, x^+, H, J)$$

the space of all simple solutions of (6) and (7). We point out that the gradient flow lines of H are not simple solutions of (6).

In [10] it is proved, roughly speaking, that given a sufficiently small Morse function $H : M \to \mathbb{R}$ there exists a generic set

$$\mathcal{J}_0 = \mathcal{J}_0(H) \subset \mathcal{J}(M, \omega)$$

of almost complex structures on M such that for every $J \in \mathcal{J}_0$ the simple solutions u of (6) and (7) are regular in the sense that the linearized operator D_u is surjective. (See the proof of Theorem 3.2 for the definition of D_u.) This implies that the moduli space $\mathcal{M}_s(x^-, x^+, H, J)$ is a finite dimensional manifold whose local dimension near u is the number $\mu(u)$ given by (9). This result is stated more precisely in the following theorem which is proved in [10], Theorem 7.2.

Theorem 7.3 *There exist a (sufficiently small) Morse function $H : M \to \mathbb{R}$, an open set $\mathcal{J} = \mathcal{J}(H) \subset \mathcal{J}(M, \omega)$, and a generic set $\mathcal{J}_0 = \mathcal{J}_0(H) \subset \mathcal{J}(H)$ (in the sense of Baire with respect to the C^∞-topology) such that the following holds for every $J \in \mathcal{J}_0$.*

(i) *Every nonconstant periodic solution*

$$x(t) = x(t+T)$$

of the Hamiltonian differential equation

$$\dot{x} = X_H(x)$$

has period

$$T > 1.$$

(ii) *The moduli space $\mathcal{M}_s(x^-, x^+, H/m, J)$ is a manifold of local dimension*

$$\dim_u \mathcal{M}_s(x^-, x^+, H/m, J) = \operatorname{ind}_H(x^+) - \operatorname{ind}_H(x^-) + 2 \int u^* c_1$$

near u for any two critical points $x^\pm$ of H and any integer $m \ge 1$.

The assertions of the previous theorem hold in fact for an open and dense set of sufficiently small Morse functions on M but in [10] it is not proved for all Morse functions. Also, it is not proved in [10] that the set $\mathcal{J}(H)$ can be chosen dense in $\mathcal{J}(M, \omega)$.

Now the almost complex structure $J \in \mathcal{J}_0(H)$ can be chosen such that in addition to (i) and (ii) the following conditions are satisfied.

(iii) $J \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$ is regular in the sense of section 2.

(iv) The gradient flow of H with respect to the metric induced by J is of Morse-Smale type.

These conditions can be achieved by a generic perturbation of the almost complex structure. For (iii) this follows from the results of Section 2 and for (iv) from Theorem 8.1 in [21]. Theorem 7.3 ensures that this perturbation of J can be chosen without destroying conditions (i) and (ii).

Proposition 7.4 *Let (M, ω) be a compact symplectic manifold of dimension $2n$. Assume either that M is monotone or $c_1(\pi_2(M)) = 0$ or the minimal Chern number is $N \geq n$. Assume that the Morse function $H : M \to \mathbb{R}$ and the almost complex structure $J \in \mathcal{J}(M, \omega)$ satisfy the conditions (i) and (ii) in the statement of Theorem 7.3. Then there exists a number $m_0 = m_0(H) > 0$ such that every solution u of (10) and (7) with $\mu(u) \leq 1$ and $\tau^{-1} \in \{m_0, m_0 + 1, \ldots\}$ is independent of t.*

Proof: Choose $\tau_0 > 0$ as in Lemma 7.1 and let $u(s, t) = u(s, t+1)$ be a solution of (10) and (7) with $0 < \tau < \tau_0$, $\tau^{-1} \in \mathbb{Z}$, and

$$\mu(u) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) + 2 \int u^* c_1 \leq 1.$$

Assume, by contradiction, that $u(s, t)$ is not independent of t. Then it follows from Lemma 7.1 that

$$\int u^* \omega > 0.$$

If u is simple then u must be independent of t since otherwise the functions $(s, t) \mapsto u(s_0 + s, t_0 + t)$ form a 2-dimensional family of simple solutions in contradiction with the dimension formula of (ii). If u is not simple then there exists an integer $m > 1$ such that

$$u(s, t + 1/m) \equiv u(s, t).$$

Let m be the largest such integer. (If there is no largest integer with this property then $u(s, t)$ is independent of t.) Then the function

$$v(s, t) = u(s/m, t/m) = v(s, t + 1)$$

is a simple solution of (10) with $H\tau$ replaced by τ/m and index

$$\mu(v) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) + 2\int v^*c_1.$$

If

$$\int u^*c_1 \geq 0$$

then

$$\int v^*c_1 = \frac{1}{m}\int u^*c_1 \leq \int u^*c_1.$$

and hence $\mu(v) \leq 1$. If

$$\int u^*c_1 < 0.$$

then M is not monotone and hence must have minimal Chern number $N \geq n$ or $N = 0$. In the former case

$$\int v^*c_1 \leq -N \leq -n$$

and hence $\mu(v) \leq 0$. In the latter case $\mu(v) = \mu(u) \leq 1$. In all three cases v is a simple solution of (10) and (7) with $\tau = 1/m$ and $\mu(v) \leq 1$. But this contradicts the dimension formula of (ii) because the solution v belongs to a 2-dimensional family. This proves the proposition. □

Remark 7.5 The argument in the proof of Proposition 7.4 fails for $2n$-dimensional manifolds if the minimal Chern number is $n-1$. In this case there might be a sequence of connecting orbits u_ν of period $1/2$ with $\mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) = 2n$ and $\int {u_\nu}^*c_1 = 2-2n$. Such connecting orbits would have index $\mu(u_\nu) = 4-2n \leq 0$. If the energy $E(u_\nu)$ converges to infinity then we cannot employ a compactness argument to arrive at a contradiction. Also considering the connecting orbits $v_\nu(s,t) = u_\nu(s/2, t/2)$ does not help since $\mu(v_\nu) = 2$ and therefore the dimension argument does not apply.

Proof of Theorem 6.1: In view of Theorem 5.2 the Floer cohomology groups are independent of the choice of J and H up to natural isomorphisms. Hence choose a time independent Morse function $H : M \to \mathbb{R}$ and an almost complex structure $J \in \mathcal{J}(M,\omega)$ which satisfy the conditions (i-iv) above. Replace H by $H_0 = H/m$ with $m > 0$ sufficiently large. Then every solution u of (6) and (7) with $H = H_0$ and $\mu(u) \leq 1$ is independent of t. Hence the coboundary of the Floer chain complex agrees with the coboundary operator of the Morse complex of H.

Note that our time independent Hamiltonian H need not be regular in the sense of section 3. However, by condition (iv) the gradient flow of H is of Morse-Smale type, and hence, by Corollary 4.3 and Theorem 7.3 in [21], the linearized operator D_u is onto for every solution u of (6) and (7) which is independent of t, and in particular for every solution with $\mu(u) \leq 1$. Using this one can prove directly that the continuation argument of Theorem 5.2 remains valid in the case where $H(s,t,x)$ is a homotopy from a time independent Hamiltonian function $H^\alpha = H_0$ as above to a regular Hamiltonian $H^\beta \in \mathcal{H}_{\mathrm{reg}}(J)$ in the sense of section 3. It is here that the condition (iii) is required to obtain compactness. The details are left to the reader. □

8 Examples

The product of n spheres

$$M = S^2 \times \cdots \times S^2$$

with the symplectic form $\omega = \omega_1 \times \cdots \times \omega_n$ is monotone if and only if all forms ω_j have the same volume. On the other hand the minimal Chern number of (M, ω) is 2 and hence the manifold is weakly monotone whenever $n \leq 4$. Our general theory only applies in the case $n = 1$ or $n = 2$. However, for arbitrary n an almost complex structure of the form $J = J_1 \times \cdots \times J_n$ is generic in the sense of section 2 and does not admit any J-holomorphic spheres of negative Chern number. Since there exists a Morse function whose critical points have only even indices our methods give the following refinement of the Lefschetz fixed point theorem.

Theorem 8.1 *Let (M, ω) be the n-fold product $S^2 \times \cdots \times S^2$ with any product symplectic structure. Then every 1-periodic Hamiltonian system on M with nondegenerate 1-periodic solutions has at least 2^{n-1} such solutions x with Conley Zehnder index $\mu(x) = 0 (\mathrm{mod}\ 4)$ and the same number with Conley-Zehnder index $\mu(x) = 2 (\mathrm{mod}\ 4)$.*

Example 8.2 A 4-dimensional example of a Calabi-Yau manifold is the quartic surface

$$X = \left\{ [z_0 : \ldots : z_3] \in \mathbb{C}P^3 : \sum_{j=0}^{3} {z_j}^4 = 0 \right\}.$$

This is a compact, connected, simply connected 4-dimensional Kähler manifold with $c_1 = 0$. All 4-manifolds with these properties are diffeomorphic and they are called **$K3$-surfaces**. Their second Betti number is $b_2 = 22$. $K3$-surfaces have played an important role in 4 dimensional topology.

Example 8.3 A similar example in 3 complex dimensions is the hypersurface of degree d in $\mathbb{C}P^4$

$$Z_d = \left\{ [z_0 : \ldots : z_4] \in \mathbb{C}P^4 \mid \sum_{j=0}^{4} {z_j}^d = 0 \right\}.$$

This manifold is simply connected and has Betti numbers

$$b_2 = 1, \qquad b_3 = d^4 - 5d^3 + 10d^2 - 10d + 4.$$

In particular $\pi_2(Z_d) = \mathbb{Z}$ and the symplectic form ω does not vanish over $\pi_2(Z)$. Moreover the first Chern class of Z_d is given by

$$c_1 = (5 - d){\iota_d}^* h$$

where $h \in H^2(\mathbb{C}P^4, \mathbb{Z})$ is the standard generator of the cohomology of $\mathbb{C}P^4$ and $\iota_d : Z_d \to \mathbb{C}P^4$ is the natural embedding of Z_d as a hypersurface in $\mathbb{C}P^4$. In particular the quintic hypersurface Z_5 is an example of a Calabi-Yau manifold.

Now let $A \in \pi_2(Z_d)$ be the generator of the homotopy group with $\omega(A) > 0$. An explicit representative of A is given by the holomorphic curve

$$[z_0 : z_1] \mapsto [z_0 : z_1 : iz_0 : iz_1 : 0]$$

when d is even and by $[z_0 : z_1] \mapsto [z_0 : z_1 : -z_0 : -z_1 : 0]$ when d is odd. Evaluating the first Chern class on this generator gives

$$\langle c_1(TZ_d), A\rangle = 5 - d.$$

So for $d \leq 4$ the manifold Z_d is monotone. For $d = 5$ and $d \geq 8$ the Arnold conjecture holds by Theorem 6.3 above. The manifold Z_d is always weakly monotone but our methods do not apply to the cases $d = 6$ and $d = 7$. In these cases, however, the Arnold conjecture follows from the recent work of Ono [18].

Example 8.4 An interesting example from the point of view of the Arnold conjecture is the 6-dimensional manifold

$$M = \mathbb{T}^2 \times X$$

This manifold has Euler characteristic 0 whereas the sum of the Betti numbers is 96. Thus the Lefschetz fixed point theorem does not give any periodic solution whereas our results show that a time dependent Hamiltonian flow on $\mathbb{T}^2 \times X$ with 1-periodic coefficients must have at least 96 contractible periodic solutions of period 1 provided that they are all nondegenerate.

A Bubbling off analysis

Throughout let (M, ω) be a compact symplectic manifold with an ω-tame almost complex structure J. Consider the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ covered by two charts with transition map $z \mapsto z^{-1}$. The group of biholomorphic maps of S^2 is $G = \mathrm{PSL}(2, \mathbb{C})$. It acts on S^2 by fractional linear transformations

$$\phi_A(z) = \frac{az + b}{cz + d}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

We shall use the notation $\phi \in G$ for $\phi = \phi_A$ with $A \in \mathrm{SL}(2, \mathbb{C})$.

Theorem A.1 *For every sequence $u_\nu : S^2 \to M$ of J-holomorphic spheres representing a fixed homotopy class $A \in \pi_2(M)$ there exist a subsequence (still denoted by u_ν), sequences $\phi_\nu^j \in G$ for $j = 1, \dots, \ell$, and J-holomorphic spheres $v^1, \dots, v^\ell$ such that the following holds.*

(i) *The reparametrized curves $u_\nu \circ \phi_\nu^j$ converge to v^j with all derivatives uniformly on every compact subset of $S^2 \setminus C^j$ where $C^j \subset S^2$ is a finite set.*

(ii) *The connected sum $v^1 \# v^2 \# \cdots \# v^\ell$ represents the homotopy class A. In particular*

$$\sum_{j=1}^{\ell} E(v^j) = \omega(A), \qquad \sum_{j=1}^{\ell} c_1(v^j) = c_1(A).$$

(iii) *The set $\Sigma = \bigcup_{j=1}^{\ell} v^j(S^2)$ is connected.*

(iv) *For every neighbourhood U of Σ there exists a $\nu_0 > 0$ such that $u_\nu(S^2) \subset U$ for every $\nu \geq \nu_0$.*

The importance of J-holomorphic curves in symplectic geometry was discovered by Gromov in his seminal paper [11]. He also discussed in detail the phenomenon of bubbling and gave a geometric proof of Theorem A.1. A more analytical proof was recently given by Parker and Wolfson [19]. We sketch here the main ideas of the proof. As a first step we state an a-priori estimate for the derivatives of a nonconstant J-holomorphic curve with sufficiently small energy.

Lemma A.2 (A-priori estimate) *Assume that (M, ω) is a compact symplectic manifold and J is a smooth almost complex structure on M which is ω-tame. Then there exists a constant $\hbar > 0$ such that the following holds. If $r > 0$ and $v : B_r \to M$ is a J-holomorphic curve such that*

$$E(u) = \int_{B_r} |dv|^2 < \hbar$$

then

$$|dv(0)|^2 \leq \frac{8}{\pi r^2} \int_{B_r} |dv|^2.$$

The proof relies on a partial differential inequality of the form $\Delta e \geq -Ae^2$ for the *energy density* $e = |dv|^2$. The details are carried out in [20] for example.

Remark A.3 **(i)** If $\hbar$ is as in Lemma A.2 then

$$E(v) = \int_{S^2} v^*\omega \geq \hbar$$

for every non-constant holomorphic sphere $v : S^2 \to M$. To see this identify $S^2 = \mathbb{C} \cup \{\infty\}$ and assume $E(v) < \hbar$. Since the mean value of $|dv|^2$ on the ball $B_r(z)$ converges to zero as $r \to \infty$ we have, by Lemma A.2, $dv(z) = 0$ for every $z \in \mathbb{C}$ and hence v is constant.

(ii) Given any number $c > 0$ there are only finitely many homotopy classes $A \in \pi_2(M)$ with $\omega(A) \leq c$ which can be represented by a holomorphic sphere.

The next lemma is a technical result about J-holomorphic annuli. It asserts that if the energy of the annulus is sufficiently small then it cannot be spread out uniformly but must be concentrated near the boundary circles. We denote $A(r, R) = B_R \setminus B_r \subset \mathbb{C}$ for $0 < r < R$.

Lemma A.4 *Let (M, ω) be a compact symplectic manifold and J be an ω-tame almost complex structure. Then there exist constants $c > 0$, $\hbar > 0$, and $T_0 > 0$ such that the following holds. If $u : A(r, R) \to M$ is a J-holomorphic curve such that $E(u) < \hbar$ then*

$$E_{A(e^T r, e^{-T} R)}(u) \leq \frac{c}{T} E_{A(r,R)}(u)$$

and

$$\int_0^{2\pi} d\left(u(re^{T+i\theta}), u(Re^{-T+i\theta})\right) \leq c\sqrt{\frac{E_{A(r,R)}(u)}{T}}$$

for $T \geq T_0$.

Proof: Choose the constant $\hbar > 0$ as in Lemma A.2 and consider the J-holomorphic curve $v(\tau + i\theta) = u(e^{\tau+i\theta})$ for $\log r < \tau < \log R$ and $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then for $\log r + T < \tau < \log R - T$ we have $E_{B_T(\tau+i\theta)}(v) \leq TE(v) = TE(u)$ and hence, by Lemma A.2,

$$|\mathrm{d}v(\tau + i\theta)|^2 \leq \frac{8E(u)}{\pi T}, \qquad \log r + T < \tau < \log R - T.$$

If T is sufficiently large then the loop $\gamma_\tau(\theta) = v(\tau + i\theta)$ is sufficiently short. Now for sufficiently short curves γ there is a well-defined symplectic action

$$a(\gamma) = -\int_\gamma \lambda$$

where λ is a 1-form on M such that $d\lambda = \omega$ on a geodesically convex neighbourhood of γ. This definition is independent of the choice of λ. Moreover, choosing local co-ordinates with $\gamma(0) = 0$ it is easy to see that

$$a(\gamma) \leq c_0 \int_0^{2\pi} |\dot{\gamma}(\theta)|^2 \, d\theta.$$

Hence for the above loops $\gamma_\tau(\theta) = u(e^{\tau+i\theta})$ we obtain

$$|a(\gamma_\tau)| \leq c_1 \, \ell(\gamma_\tau)^2 \leq c_2 \frac{E(u)}{T}$$

for $\log r + T < \tau < \log R - T$. Since $tau + i\theta \mapsto \gamma_\tau(\theta)$ is a J-holomorphic curve we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} a(\gamma_\tau) = \int_0^{2\pi} |\dot{\gamma}_\tau(\theta)|^2 \,\mathrm{d}\theta \geq \frac{\|\dot{\gamma}_\tau\|_{L^2}}{\sqrt{2\pi}} \ell(\gamma_\tau) \geq \frac{\|\partial_\tau \gamma_\tau\|_{L^2}}{\sqrt{2\pi c_1}} \sqrt{|a(\gamma_\tau)|}.$$

In particular, the function $\tau \mapsto a(\gamma_\tau)$ is strictly increasing. If $a(\gamma_\tau) > 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sqrt{a(\gamma_\tau)} \geq c_3^{-1} \|\partial_\tau \gamma_\tau\|_{L^2}$$

and a similar inequality holds when $a(\gamma_\tau) < 0$. Integrating these from $\tau_0 = \log r + T$ to $\tau_1 = \log R - T$ (after splitting this interval into two according to the sign of $a(\gamma_\tau)$ if necessary) we obtain

$$\int_{\tau_0}^{\tau_1} \|\partial_\tau \gamma_\tau\|_{L^2} \,\mathrm{d}\tau \leq c_3 \left(\sqrt{|a(\gamma_{\tau_0})|} + \sqrt{|a(\gamma_{\tau_1})|} \right) \leq c_4 \sqrt{\frac{E(u)}{T}}.$$

(A similar argument was used in [13].) Since $\|\partial_\tau \gamma_\tau\|_{L^2} \leq c_5 \sqrt{E(u)/T}$ for $\tau_0 \leq \tau \leq \tau_1$ this implies

$$E_{A(e^T r, e^{-T} R)}(u) = \int_{\tau_0}^{\tau_1} \|\partial_\tau \gamma_\tau\|_{L^2}^2 \,\mathrm{d}\tau \leq c_6 \frac{E(u)}{T}.$$

Moreover,

$$\begin{aligned} \int_0^{2\pi} d(\gamma_{\tau_0}(\theta), \gamma_{\tau_1}(\theta)) \,\mathrm{d}\theta &\leq \int_{\tau_0}^{\tau_1} \int_0^{2\pi} |\partial_\tau \gamma_\tau| \,\mathrm{d}\theta \mathrm{d}\tau \\ &\leq \sqrt{2\pi} \int_{\tau_0}^{\tau_1} \|\partial_\tau \gamma_\tau\|_{L^2} \,\mathrm{d}\tau \\ &\leq c_7 \sqrt{\frac{E(u)}{T}} \end{aligned}$$

and this proves the lemma. □

Let $u_\nu : S^2 = \mathbb{C} \cup \{\infty\} \to M$ be a sequence of J-holomorphic curves. A point $z \in \mathbb{C}$ is called **singular** for the sequence u_ν if there exists a sequence $z_\nu \to z$ such that $|du_\nu(z_\nu)|$ converges to ∞. The point $z = \infty$ is called singular if 0 is a singular point for the sequence $z \mapsto u_\nu(z^{-1})$. A singular point z for u_ν is called **tame** for u_ν if it is isolated and the limit

$$m_\varepsilon(z) = \lim_{\nu \to \infty} \int_{B_\varepsilon(z)} u_\nu^* \omega$$

exists for every sufficiently small $\varepsilon > 0$. In this case the **mass** of the singularity is defined to be the number

$$m(z) = \lim_{\varepsilon \to 0} m_\varepsilon(z).$$

This number exists because the function $\varepsilon \mapsto m_\varepsilon(z)$ is non-decreasing. It is always positive and in fact $m(z) \geq \hbar$.

The usual compactness argument shows that every sequence u_ν has a subsequence (still denoted by u_ν) with only finitely many singular points $z^1, \ldots, z^\ell$ which are all tame. By definition the derivatives of u_ν are uniformly bounded in every compact subset in the complement of the singular set $C = \{z^1, \ldots, z^\ell\}$. Hence it follows from elliptic bootstrapping that a further subsequence of u_ν converges with all derivatives uniformly on every compact subset of $S^2 \setminus S$ to a J-holomorphic sphere $u : S^2 \to M$. The energy of this limit satisfies the identity

$$E_{S^2 \setminus B_\varepsilon(C)}(u) + \sum_{j=1}^{\ell} m_\varepsilon(z^j) = \omega(A)$$

for every sufficiently small $\varepsilon > 0$. Take the limit $\varepsilon \to 0$ to obtain

$$E(u) + \sum_{j=1}^{\ell} m(z^j) = \omega(A).$$

Soft rescaling

We examine the behaviour of the sequence u_ν near a singularity z in more detail. Composing u_ν with a suitable element of G (independent of ν) we may assume without loss of generality that $z = 0$ and denote $m = m(z)$. For every ν there exists a unique number $\delta_\nu > 0$ such that

$$\int_{B_{\delta_\nu}} u_\nu^* \omega = m - \frac{\hbar}{2}$$

where $B_\delta = B_\delta(0)$ denotes the ball of radius delta centered at 0. By definition of the mass m the sequence $\delta_\nu > 0$ converges to 0. Consider the sequence of J-holomorphic spheres $v_\nu : S^2 \to M$ defined by

$$v_\nu(z) = u_\nu(\delta_\nu z).$$

Lemma A.5 *There exists a subsequence (still denoted by v_ν) such that the following holds.*

(i) *The singular set $C' = \{w^1, \ldots, w^k\}$ of the subsequence v_ν is finite and tame and is contained in $\mathrm{cl}(B_1) \cup \{\infty\}$.*

(ii) *The subsequence v_ν converges with all derivatives uniformly on every compact subset of $S^2 \setminus C'$ to a non-constant J-holomorphic sphere $v : S^2 \to M$.*

(iii) *The energy of v and the masses of the singularities $w^1, \ldots w^k$ satisfy*

$$\int_{S^2} v^* \omega + \sum_{j=1}^{k} m(w^j) = m.$$

(iv) $v(\infty) = u(0)$.

Proof: If follows immediately from the definitions that for every $R > 1$ and every $\varepsilon > 0$ there exists a $\nu_0 = \nu_0(R, \varepsilon) > 0$ such that

$$m - \frac{\hbar}{2} = E_{B_1}(v_\nu) \leq E_{B_R}(v_\nu) = E_{B_{R\delta_\nu}}(u_\nu) \leq m + \varepsilon$$

for $\nu \geq \nu_0$. Hence there is no bubbling outside the unit ball and this proves statement (i). We shall now prove that the limit curve $v : \mathbb{C} \to M$ satisfies

$$E_{\mathbb{C}\setminus B_R}(v) = \frac{\hbar}{2}.$$

We have already seen that $E_{\mathbb{C}\setminus B_R}(v) \leq \hbar/2$. To prove the converse choose a sequence $\varepsilon_\nu > 0$ such that $E(u; B_{\varepsilon_\nu}) = m_0$. Then it follows again from the definition of m_0 that $\varepsilon_\nu \to 0$. Now consider the sequence $w_\nu(z) = u_\nu(\varepsilon_\nu z)$. It follows as above that $E(w_\nu; B_R - B_1)$ converges to zero for any $R > 1$. This implies that $E(w_\nu; B_1 - B_\delta)$ must also converge to zero for any $\delta > 0$ since otherwise a subsequence of w_ν would converge to a nonconstant J-holomorphic curve which is constant for $|z| \geq 1$ but such a curve does not exist. Since

$$E(w_\nu; B_1 - B_{\delta_\nu/\varepsilon_\nu}) = E(u_\nu; B_{\varepsilon_\nu} - B_{\delta_\nu}) = \frac{\hbar}{2}$$

it follows that $\delta_\nu/\varepsilon_\nu$ converges to 0. Now, by Lemma A.4, there exists a $T_0 > 0$ such that for $T > T_0$

$$E(u_\nu; B_{e^{-T}\varepsilon_\nu} - B_{e^T\delta_\nu}) \leq \frac{c}{T} E(u_\nu; B_{\varepsilon_\nu} - B_{\delta_\nu}) = \frac{c}{T}\frac{\hbar}{2}.$$

Pick any number $\alpha < 1$ and choose T so large that $1 - c/T > \alpha$. Then the energy of u_ν in the union of the annuli $A(\delta_\nu, e^T\delta_\nu)$ and $A(e^{-T}\varepsilon_\nu, \varepsilon_\nu)$ must be at least $\alpha\hbar/2$. But the energy of u_ν in $A(e^{-T}\varepsilon_\nu, \varepsilon_\nu)$ converges to 0 while the energy of u_ν in $A(\delta_\nu, e^T\delta_\nu)$ converges to $E(v; A(1, e^T))$. Hence $E(v; A(1, e^T)) \geq \alpha\hbar/2$. Since $\alpha < 1$ was chosen arbitrarily it follows that $E(v, \mathbb{C} - B_1) = \hbar/2$ as claimed. In particular this implies that v is nonconstant and thus we have proved statement (ii). Statement (iii) now follows from the usual bubbling argumant already used above.

To prove statement (iv) fix $\varepsilon > 0$ and $R > 0$ and define

$$E(\varepsilon, R) = \lim_{\nu\to\infty} \int_{B_\varepsilon \setminus B_{R\delta_\nu}} u_\nu^*\omega.$$

This limit exists for $\varepsilon > 0$ sufficiently small and $R > 0$ arbitrarily large. To see this consider the annuli $B_\varepsilon \setminus B_{\delta_\nu}$ and $B_{R\delta_\nu} \setminus B_{\delta_\nu}$. The limit on the former is $\hbar/2 + E_{B_\varepsilon}(u)$ while the limit on the latter is $E_{B_R \setminus B_1}(v) = \hbar/2 - E_{\mathbb{C}\setminus B_R}(v)$. Hence

$$E(\varepsilon, R) = E_{B_\varepsilon}(u) + E_{\mathbb{C}\setminus B_R}(v), \qquad \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} E(\varepsilon, R) = 0.$$

Now $E(\varepsilon, R)$ converges to zero as $\varepsilon \to 0$ and $R \to \infty$. Hence it follows from Lemma A.4 that for $T > 0$ and $\nu > 0$ sufficiently large we have

$$\int_0^{2\pi} d(u_\nu(R\delta_\nu e^{T+i\theta}), u_\nu(\varepsilon e^{-T+i\theta}))\, \mathrm{d}\theta \le c\sqrt{E(\varepsilon, R)}.$$

Taking the limit $\nu \to \infty$ we obtain

$$\int_0^{2\pi} d(v(Re^{T+i\theta}), u(\varepsilon e^{-T+i\theta}))\, \mathrm{d}\theta \le c\sqrt{E(\varepsilon, R)}.$$

Here the constants T and c are independent of ε and R. Since $E(\varepsilon, R) \to 0$ for $\varepsilon \to 0$ and $R \to \infty$ we obtain $v(\infty) = u(0)$ as required. This completes the proof of Lemma A.5. □

Remark A.6 All the singularities of the sequence u_ν other than $z = 0$ are pushed to ∞ in the rescaled sequence v_ν. Lemma A.5 may be viewed as a *resolution* of the singularity $z = 0$.

Proof of Theorem A.1: Apply Lemma A.5 to each singular point for the sequence u_ν passing to a suitable subsequence. Now apply Lemma A.5 again to each singularity for each of the renormalized sequences v_ν obtained from the previous application of Lemma A.5 and proceed by induction. This process will terminate after finitely many steps since in each step the energy decreases by at least $\hbar$.

To prove convergence of u_ν to the collection of all bubbles take any sequence $z_\nu \in S^2$. If z_ν is uniformly bounded away from the singular set of u_ν then $u_\nu(z_\nu)$ accumulates on the limit curve of u_ν. Hence assume that z_ν converges to a singular point which we assume to be $z = 0$. There are two possibilities. If there exists a constant $\varepsilon > 0$ such that

$$\varepsilon \le \frac{|z_\nu|}{\delta_\nu} \le \frac{1}{\varepsilon}$$

where $\delta_\nu > 0$ is defined as in Lemma A.5 then $u_\nu(z_\nu)$ converges to the bubble point $u(0) = v(\infty)$. If

$$\lim_{\nu\to\infty} \frac{|z_\nu|}{\delta_\nu} = 0$$

then proceed by induction with u_ν replaced by the rescaled sequence $v_\nu(z) = u_\nu(\delta_\nu z)$.

A close examination of Lemma A.5 and the inductive procedure shows that the connected sum of all bubbles represents the original homotopy class A. This proves the theorem. □

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